MTH-309T-F19-EX1-031-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	1	8	2	2	1	4

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
9	10	3	20	15	4	3	2	--	66	C+

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) if w span (v_1, v_2, v_3) th $w = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$w = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$w = -2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$?

$$\cancel{w = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} + \begin{bmatrix} b \\ 2b \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} b-4 \\ 2b+2 \\ -10 \end{bmatrix}}$$

$b - 4 = 0 \Rightarrow \cancel{b = 4}$
$2b + 2 = 0 \quad \cancel{b = -1}$

$b = -6$

b)
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -2 & 0 \end{bmatrix} \xrightarrow{2R_1 - R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & \cancel{1}\ 0 & \cancel{1}\ 0 \end{bmatrix} \xrightarrow{R_2 - R_3} \cancel{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\xrightarrow{-3R_3 + R_1} \cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}} \xrightarrow{-2R_3 + R_2} \cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Yes $\{v_1, v_2, v_3\}$ is linearly independent because every column of the ~~row~~ reduced matrix is a pivot column.

This comes from a mistake in the row reduction

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} (r_1c_1) & (r_1c_2) & (r_1c_3) \\ (r_2c_1) & (r_2c_2) & (r_2c_3) \\ (r_3c_1) & (r_3c_2) & (r_3c_3) \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$ ✓

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0	1	2	3	4	5	6	7	8	9	10

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3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$3\times 3 \quad \times \quad 3\times 3$

This is not A^T

$$C = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix}$$

$r_1 c_1 : \; 2x_1 + x_2 - x_3 = 1$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{RR} \begin{bmatrix} C \end{bmatrix}$$

$r_2 c_1 : \; -x_1 + 2x_3 = 4$

$r_3 c_1 : \; x_1 + x_3 = 3$

$x_1 = 3 - x_3$

Where did this come from?

0	1	2	3	4	5	6	7	8	9	10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T. im defining this as A.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$

$T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-2(0) \\ 1+0 \\ 1-3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-2(1) \\ 0+1 \\ 0-3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

Standard matrix of T: $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} * \overset{u}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$\left.\begin{aligned} x_1 - 2x_2 &= 1 \\ x_1 + x_2 &= 10 \\ x_1 - 3x_2 &= -2 \end{aligned}\right\}$$

$x_1 = 1 + 2x_2$

$1 + 2x_2 + x_2 = 10 \Rightarrow 1 + 3x_2 = 10$

$3x_2 = 9$

$x_2 = 3$

$\downarrow$ backsub

$x_1 - 2(3) = 1$

$x_1 = 7$

$7 - 3(3) = -2$

$7 - 9 = -2$ ✓

so $u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ ✓

or any multiple of it.

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$\xrightarrow{-\frac{1}{4}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ it is one-to-one as every column has 1 pivot position.

✓

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_2-R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

not going to be one to one because column 3 will not have a pivot.

$v_1 = \begin{bmatrix} \ \end{bmatrix}$ $v_2 = \begin{bmatrix} \ \end{bmatrix}$

$T_A(v_1) = T_A(v_2)$ if $v_2 = v_1 + n$

So if $v_1 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ and $n = 2$

$v_2 = \begin{bmatrix} 5 \\ 9 \\ 6 \end{bmatrix}$

n must be a vector in Nul(A), not a number

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True ✓

Span (u, v) = all vectors / linear combinations $c_1 u + c_2 v$,

or $c_1 w + c_2 u$ spanning (u, v)

therefor $c_1 w$ span (u, v)

why ?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~False~~ because if $\{u, v\}$ are scalar multiples ie $c_1 v_1 + c_2 v_2 = 0$ then it is linearly dependent.

$1 u_1 + (-1) u_2 = 0$

so ?

$R^3 \rightarrow 3 = n$ rows

$u \; v \; w$
$1 \; 2 \; 3 \rightarrow 3 = p$ colums

by definition if $p > n \Rightarrow$ linearly dependent

but $p = n$ so it is linearly independent

but if $u \; v$
$1 \; 2$ $p = 2$ and $R3$ $n = 3$

$p \not> n$ *does not apply for $p = 2$

$2 \not> 3$ so it is linearly independent as well

0	1	2	3	4	5	6	7	8	9	10
☐	☐	☐	☐	☒	☐	☐	☐	☐	☐	☐

7. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~True.~~

Au, Av indicates that after multiplication there was a free variable causing ∞ solutions, that makes Au, Av linearly dependent

This means that the vectors u and v must have also had free variables when multiplied by some x value.

Not true.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

~~False~~, if the transformation changes u, v, w to where u no longer Span(v, w) that would make T(u) not Span(T(v), T(w))

u
$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $u_1 + 3u_2$ span (u) but $T(u) = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$ then it may not span for two other vectors if it is not a scalar multiple

What is T here?

0 1 2 3 4 5 6 7 8 9 10