MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

Ayesha Khatun

UB Person Number:

5	0	2	4	3	5	1	4

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$c_1 v_1 + c_2 v_2 + c_3 v_3 = w$

(b) RREF :-
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_2 + R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

~~So every~~ Since every column, can not be a pivot column,

$\boxed{\text{Set } \{v_1, v_2, v_3\} \text{ is not linearly independent}}$

(a) RREF:
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right] \xrightarrow{R_3 \to R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & \boxed{b+4} \end{array}\right] \quad \text{— (1)}$$

Now, there exists sol^n only if is not a pivot column.

~~So, sol^n exists only when, b+~~

Since, $w \in \mathrm{Span}(v_1, v_2, v_3)$.

Here, sol^n exists only when, $b + 4 \neq 1$

$\Rightarrow b \neq 1 - 4$

$\Rightarrow b \neq -3$

$\therefore b \in \mathbb{R}$, except $b = -3$.

Ans :-

~~From (1),~~ ~~$R_2 \to R_3 + R_2$~~ $\left[\begin{array}{ccc} 1 & -1 & 1 & -2 \end{array}\right.$

~~$R_2 \to$~~

Also, sol^n exist, if $b + 4 = 0$

$\therefore b = -4$.

For, $b = -4$, there is infinite sol^n.

$\therefore$ ~~$b = \{b \in \mathbb{R}\}$~~

~~See next Page~~

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$\det(A) = 1(0-2) - (-1)(-1-0) + 2(2-0)$

$= -2 - 1 + 4 = 4 - 3 = \boxed{1} \rightarrow A^{-1}$ is possible

$a_{11} = \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = \boxed{-2}$ $\quad a_{12} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{13} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = \boxed{2}$

$a_{21} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 4-1 = \boxed{3}$ $\quad a_{22} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{23} = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2-0 = \boxed{2}$

$a_{31} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = \boxed{-1}$ $\quad a_{32} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1-2 = \boxed{-1}$ $\quad a_{33} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 0-(-1) = \boxed{1}$

Now, $\text{adj}(A) = \begin{bmatrix} -2 & -1 & 2 \\ 3 & -1 & 2 \\ -1 & -1 & 1 \end{bmatrix} \times [$

$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$a_{11} = \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = \boxed{-2}$ $\quad a_{12} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1-4 = \boxed{-3}$ $\quad a_{13} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = \boxed{-1}$

$a_{21} = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{22} = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{23} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1-2 = \boxed{-1}$

$a_{31} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = \boxed{2}$ $\quad a_{32} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = \boxed{2}$ $\quad a_{33} = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -(-1) = \boxed{1}$

$\therefore \text{adj}(A) = \begin{bmatrix} -2 & -3 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

$\therefore A^{-1} = \dfrac{\text{adj}(A)}{\det(A)} = \dfrac{1}{1} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$ Ans.

Quick check: $\left[(1 \ -1 \ 2)\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}\right] = [-2-1+4] = [-3+4] = [1]$

0 1 2 3 4 5 6 7 8 9 10

✓

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Now, $$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

Here, $$[A^T]^{-1}[A^T][C] = [I][C] = [C] = [A^T]^{-1}[B]$$

$$\therefore [C] = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (-2+4+6) & (-4+5+4) & (-6+4+2) \\ (3-4-6) & (6-5-4) & (9-4-2) \\ (-1+4+3) & (-2+5+2) & (-3+4+1) \end{bmatrix}$$

$$\therefore \boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

Ans:-

Check:- $[1\ 1\ 0]\begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix} = [8-7] = [1]$ ✓

$[2\ 1\ -1]\begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix} = [16 - \cancel{8} - 6]$, 7

$\Rightarrow [16 - 7 + 6] = \underline{3}$ $\quad$ $[16 - 13] = 3$ ✓

24 - 7

$[2\ 1\ -1]\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = [0+2-3] = [1]$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

(a) $\cancel{T(e_1)}$ $[e_1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T([e_1]) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad T([e_2]) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

So, standard matrix, $A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ Ans:-

(b) From definition, $T(u) = A \cdot u = A \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, as $T: R^2 \to R^3$

Now, $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

Since, $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 - 2u_2 \\ u_1 + u_2 \\ u_1 - 3u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\Downarrow$

$$\begin{cases} u_1 - 2u_2 = 1 & -(1) \\ u_1 + u_2 = 10 & -(2) \\ u_1 - 3u_2 = -2 & -(3) \end{cases}$$

from (2), $u_1 = 10 - u_2$

$\therefore$ " (1), $10 - u_2 - 2u_2 = 1$

$\Rightarrow 10 - 3u_2 = 1$

$\Rightarrow 10 - 1 = 3u_2$

$\therefore u_2 = \frac{9}{3} = 3$, $u_1 = 10 - 3 = 7$

$\therefore \boxed{\vec{u} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} = 7\hat{u}_1 + 3\hat{u}_2}$

Ans:-

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) Aug. mat. of A: RREF

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_1 \to -R_2 + R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3/2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

Every column is a pivot column.

$\therefore$ $\boxed{T_A(v) \text{ is one-to-one}}$

b)

Aug. mat: RREF $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 \to \frac{R_3}{2}]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \to \text{every column is a pivot column.}$$

$\therefore$ $\boxed{T_A(v) \text{ in one-to-one}}$

Ans:

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

False.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True.

Since, in set $\{u, v, w\}$ are already indep.

as subset $\{u, v\}$ will also be indep.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False.

$\cancel{\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$A\cancel{(u-v) = 0, \quad u = v}$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

False.

Transformation doesnt guarantee being in the span..

~~If $T(v)x_1 + T(w)x_2 = 0$, doesn't have~~ sol'n,

~~they~~