MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	2	0	5	2	0
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

16	6	10	14	2	3	1	--	10	57	C-
1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a). $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{\cdot -2} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{bmatrix} \xrightarrow{+} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix}$

When $b = -6$, it has infinite solutions.

$x_1 + 3x_3 = 0$

$x_2 + 2x_3 = 2$

so ?

Is $w \in \text{Span}(v_1, v_2, v_3)$ if $b = -6$?

b) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{\times -2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & \cancel{-5} & -2 \end{bmatrix} \xrightarrow{+} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -5 & -2 \end{bmatrix} \times 5$

-1

$\downarrow$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \times -\frac{1}{2} \leftarrow \cancel{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}} +$

It's linearly ~~independent~~.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ A is invertible.

$$C_1 : \begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \checkmark$$

$$C_2 : \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & -1 \\ 2 & 0 & 3 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ \cancel{2} & \cancel{0} & \cancel{3} & \cancel{0} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ \cancel{1} & \cancel{0} & \cancel{2} & \cancel{-1} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\cancel{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]}$$

C_2

$$C_3 \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \checkmark$$

$A^{-1} = ?$ This is a long way to do it...

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$A^T \cdot A^{T-1} C = B A^{T-1}$$

$$C = B A^{T-1}$$

$$A^{-1} = \begin{bmatrix} -2 & \overset{3}{\cancel{1}} & -1 \\ 1 & -1 & 1 \\ 2 & \underset{-2}{\cancel{0}} & 1 \end{bmatrix} \qquad \therefore \quad A^T \text{ inverse} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad \checkmark$$

$$= \begin{bmatrix} \overset{-1}{-1\times 1} + \overset{4}{1\times 4} + 0\times 3 & \overset{-2}{-1\times 2} + \overset{5}{1\times 5} + 0\times 2 & -1\times 3 + 1\times 4 + 0\times 1 \\ \overset{-1}{-1\times 1} + 0\times 4 + 2\times 3 & -1\times 2 + 0\times 5 + 2\times 2 & -1\times 3 + 0\times 4 + -1\times 1 \\ \overset{2}{2\times 1} + 1\times 4 + -1\times 3 & 2\times 2 + 1\times 5 + -1\times 2 & 2\times 3 + 1\times 4 + -1\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 1 \\ 5 & 2 & -4 \\ 3 & 7 & 9 \end{bmatrix}$$

A^{-1} incorrect, but fine otherwise.

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a). $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{-1} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -1 & -3 \end{bmatrix}_{\times -1} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 21 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix}_{\times -1}$

$-4 \ \ -1 \ \ 0 \quad x - \frac{1}{2}$

$\begin{matrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{matrix} \quad x\,-2$

$\begin{matrix} 0 & +1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{matrix} \quad x-1$

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{matrix} \quad -1$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

?

$u = ?$

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{bmatrix} \xrightarrow{\times\frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{bmatrix} \times -3$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{bmatrix} \times -1$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{bmatrix} \times -2$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \times\frac{1}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} -1$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

?

It's not one to one.

which part ? a), b) ?

$V_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$x_1 = -2$
$x_2 + x_5 = 2$
$x_3 = 0$
$x_4 = \text{free}$
$x_5 = \text{free}$

?

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

F

$$u\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad w = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

linearly independent → $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

F

linearly independent mean have exactly one solution. ✓

if $\{u, v, w\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\{u, v\} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ ← have infinite solution.

only one

∴ It's False.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~T~~

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

T $\leftarrow$ why?

0 1 2 3 4 5 6 7 8 9 10