MTH 309T LINEAR ALGEBRA
EXAM 1
October 3, 2019

Name:

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UB Person Number:

5	0	2	8	6	2	3	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.

- For full credit solve each problem fully, showing all relevant work.

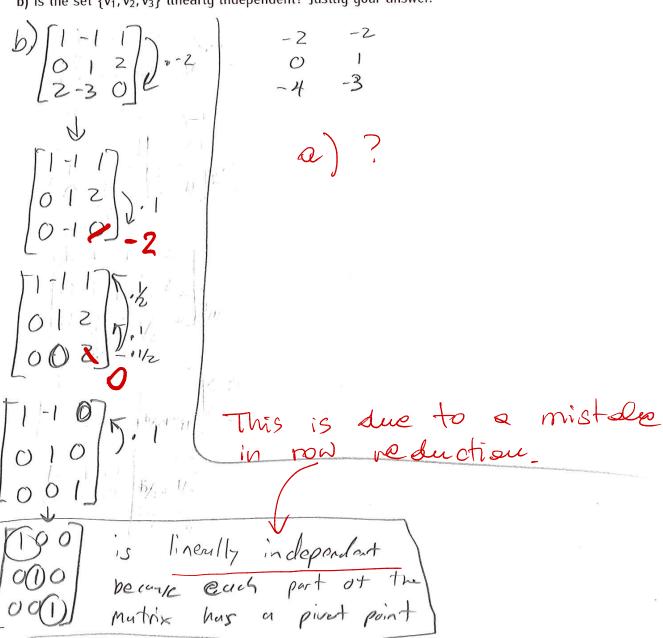| 1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \; v_1 \\ 2 \; v_2 \\ b \; v_3 \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

b) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \;\cdot -2$

$\begin{array}{cc} -2 & -2 \\ 0 & 1 \\ -4 & -3 \end{array}$

$\downarrow$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} \;\cdot 1 \quad \mathbf{-2}$

a) ?

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \cdot \frac{1}{2} \\ \cdot 1 \\ \cdot \frac{1}{2} \end{matrix} \quad \mathbf{0}$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;\cdot 1$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

is linearly independent because each part of the matrix has a pivot point

This is due to a mistake in row reduction.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \end{bmatrix} R_2 - 1 \quad \checkmark$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \end{bmatrix} R_3 \cdot -2$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix} R \cdot 1 \quad \cdot -2$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 0 & | & -3 & 4 & -2 \\ 0 & 1 & 0 & | & 1 & -2 & 1 \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix} R \cdot 1$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -2 & 2 & -1 \\ 0 & 1 & 0 & | & 1 & -2 & 1 \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & \cancel{2} & -1 \\ 1 & \cancel{-2} & 1 \\ 2 & -2 & 1 \end{bmatrix} = A^{-1}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \checkmark$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} & \\ & C \\ & \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \curvearrowright \cdot 1 \\ \curvearrowright \cdot -2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right] \curvearrowright \cdot 1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \curvearrowright -2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \curvearrowright -1$$

$$A^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

(in red:) $C = (A^T)^{-1} \cdot B$, not $B \cdot (A^T)^{-1}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & -6 \\ 13 & -13 & 13 \\ 12 & -12 & 6 \end{bmatrix} = C$$

(crossed out in red)

$$C = \begin{bmatrix} 1 & 2 & -6 \\ 13 & -13 & 13 \\ 12 & -12 & 6 \end{bmatrix}$$
(crossed out in red)

(in red:) Simpler: $(A^T)^{-1} = (A^{-1})^T$ and then you can use problem 2.

(answer bubble at bottom: × marked on 7)

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

$\begin{cases} x_1 = 2x_2 + 1 \\ x_2 = 3 \end{cases}$

$\downarrow$

$\begin{bmatrix} x_1 = 7 \\ x_2 = 3 \end{bmatrix}$

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \begin{array}{l} \\ \swarrow^{-1} \\ \swarrow^{-1} \end{array}$

$\downarrow$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \cdot \frac{1}{3}$

$\downarrow$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \swarrow \cdot 1$

$\begin{array}{cc} x_1 & x_2 \end{array}$
$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ ✓

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14 ☐ 15 ☐ 16 ☐ 17 ☐ 18 ☐ 19 ☒ 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot -3$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \cdot \frac{1}{2}$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \cdot -1$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot -1 \quad \cdot \frac{1}{2}$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot -1$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

is one to one

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot -3$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \cdot \frac{1}{2}$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad -\cdot\frac{1}{2} \quad \cdot\frac{1}{2}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\underbrace{}_{free}$

$x_1 = 2x_3$

$x_2 = -2x_3$

$x_3 = x_3$

$\bigg| \quad 3$

Not one to one ✓

b $\downarrow$

Two vectors ✓

$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ✗

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

False,

Counter example

$$u = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \quad w = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$u + w = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \notin \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \right)$$

(in red:) $\in$

$$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

True ✓ because in order to be independent there must be no dependency on another vector, so if a vector were to leave the group that vector would still be independent.

(in red:) This could be explained more clearly...

| 0 | 1 | 2 | 3 | 4 | 5 | 6 ✗ | 7 | 8 | 9 | 10 |

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and u, v are vectors in $\mathbb{R}^2$ such that Au, Av are linearly dependent then u, v also must be linearly dependent.

False ← why?

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

False

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad T = \begin{bmatrix} 5 & 7 \end{bmatrix} \quad ?$$

0 1 2 3 4 5 6 7 8 9 10