MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	1	7	9	9	7	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

20	10	10	20	14	7	6	2	--	89	A-
1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$. → w can be written as a linear combination of v_1, v_2, v_3

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

(a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] (R1\times -2)+R3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] R2+R3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right] R2+R1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right]$$

$\boxed{b=-6 \text{ for } w \in \mathrm{span}(v_1, v_2, v_3)}$ ✓

(otherwise get no solution due to pivot position in column of constants)

(b)

$$c_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

→ this set in linearly independent if homo. eqn. only has trivial solution ∴

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] (R1\times -2)+R3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] R2+R3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable! not every column of the matrix is a pivot therefore the set $\{v_1, v_2, v_3\}$ are not linearly independent ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ☒

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$(R1 \times -1) + R3$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$(R3 \times -1) + R2$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right]$$

$(R2 \times -1) + R3$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right]$$

$R2 + R1$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right] \; 1 \; -1$$

$R3 \times -2 + R1$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-2 - 1 + 4 = 1$

$3 + 1 - 4 = 0$

$-1 - 1 \; 2 = 0$

$-2 \; 0 \; 2 = 0$

(-2×-2)
$4 - 1 = 3$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right]$$ R1+R2

get this to RRE

this matrix is solution

Simpler:

$$C = (A^T)^{-1} \cdot B$$
$$= (A^{-1})^T \cdot B$$

Then use A^{-1} from problem 2.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right]$$ (R1x-2)+R3

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & -1 & -1 & 1 & -2 & -5 \end{array}\right]$$ R2+R3=R3

2x2 + -4
4 + -4 = 0

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$ R2x-1+R1

(6x2)-4
12-4
8

(5x2)+-5
10-5
5

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -4 & -5 & -4 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$ R3X-2+R2

(6x-2)+5
-12+5
-7

(2x-2)+7
-4+7
3

R3x2+R1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -4 & -5 & -4 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

(5x-2)+7
-10+7
-3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$Ax =$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T. $\rightarrow \boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}} \Rightarrow$ standard matrix of $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ ✓

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$(R1 \times -1) + R2 \quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$$

$$(R1 \times -1) + R3 \quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] /3$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] /-1$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] (R3 \times 2) + R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] (R2 \times -1) + R3$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 7$

$x_2 = 3$

$$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ✗

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

R1x-3 +R3 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$ /2

R2x-1 +R3 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ /2

R3x-2 +R2 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

↓

one-to-one and onto b/c pivot pos. in every column and row, respectively. ✓

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ (R1x-3)+R3

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ /2

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ (R3x-1)+R2

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

↓

not one-to one b/c can only have a pivot position in the first two columns, here we have only two. ✓

v_1, v_2 ?

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w}+\mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

$\Downarrow$ w+u can be written as a linear comb. of u and v

w can be written as a linear comb. of u and v

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c_3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$\Downarrow$ then...

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \cancel{c_1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \cancel{c_2} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$(c_1 - 1)$ c_3

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = c_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c_3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$\therefore$ $\boxed{\text{true}}$ ✓ w is in the span (u,v)

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$\boxed{\text{True,}}$ ✓ if u, v, and w are already linearly independent and are not linear combinations of eachother, the the set $\{u, v\}$ must also already be linearly independent

$$\left\{ \underset{u}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{v}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{w}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\} \Rightarrow \text{linearly independent}$$

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow \text{still linearly independent}$$

This could be explained more clearly...

0	1	2	3	4	5	6	7	8	9	10
☐	☐	☐	☐	☐	☐	☐	☒	☐	☐	☐

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u},\mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u},\mathbf{v}$ also must be linearly dependent.

$\Downarrow$ u and v are linear combinations of eachother

$R2\times{}^{1}/_{2}+R1$

$$\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}\begin{matrix}/2\\/2\end{matrix}\quad \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix}\begin{matrix}/2\\/2\end{matrix}\quad \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}$$

$$\begin{bmatrix}2&1\\0&2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\overset{Au}{\begin{bmatrix}1\\2\end{bmatrix}}\Rightarrow u_1=\begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}2&1\\0&2\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\overset{Au}{\begin{bmatrix}2\\4\end{bmatrix}}$$

$\rhd$ ~~true~~, solutions are still linearly dependent

The question was if independent vectors can become dependent after matrix multiplication.

b) If $T\colon\mathbb{R}^2\to\mathbb{R}^2$ is a linear transformation and $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v},\mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}),T(\mathbf{w}))$.

$$T\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=T\left(c_1\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+c_2\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\right)$$

$$T(u)=c_1T(v)+c_2T(w)$$

$$T\left(c_1\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+c_2\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\right)=T\left(\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\right)+T\left(\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\right)$$

✓

$\Downarrow$ $\boxed{\text{true}}$

For linear transformations, T(u+v) can be written as T(u) + T( ), and here, because u is in the span of v and w, its transformation can be written as a combination of the transformations for v and w

0	1	2	3	4	5	6	7	8	9	10
☐	☐	☐	☐	☐	☐	☒	☐	☐	☐	☐