MTH-309T-F19-EX1-041-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Jacob Vitko

UB Person Number:

5	0	2	1	0	1	8	7
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a. Span is all linear combinations of $c_1v_1 + c_2v_2 + c_3v_3$

$$c_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 + c_3 \\ c_2 + 2c_3 \\ 2c_1 - 3c_2 \end{bmatrix}$$

$c_1 - c_2 + c_3 = -2$
$c_2 + 2c_3 = 2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{+(1)} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

free

$c_1 = -3c_3$
$c_2 = 2 - 2c_3$
$c_3 = c_3$

$$\begin{bmatrix} -3c_3 - (2 - 2c_3) + c_3 \\ 2 - 2c_3 + 2c_3 \\ 2(-3c_3) - 3(2 - 2c_3) \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$$

$b = -6$

$w \in \text{Span}(v_1, v_2, v_3)$ if $b = -6$

b. $x_1v_1 + x_2v_2 + x_3v_3 = 0$

$$x_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

the equation has one solution therefore the set $\{v_1, v_2, v_3\}$ is linearly independent

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$A \cdot A^{-1} = I$ where I is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\leftarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -4 & -5 & -4 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & -1 & -1 & 1 & -2 & -5 \end{array}\right] \leftarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & -1 & -1 & 1 & -2 & -5 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -4 & -5 & -4 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

standard matrix of T = A

a. A is 3x2 matrix where $A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$

$$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

b. Find $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot u = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \leftarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right]$$

$x_1 = 7 \quad x_2 = 3$

$$u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since after row reduction, matrix A has a pivot position in every row, it is one-to-one therefore T_A is one-to-one

b. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Since A does not have a pivot position in every row, T_A is not one-to-one

$Av_1 = Av_2$

$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 + 4x_3 \\ 3x_1 + 4x_2 + 2x_3 \end{bmatrix}$

If $x_1 = x_2 = 1$ then $v = \begin{bmatrix} 2 \\ 2 + 4x_3 \\ 7 + 2x_3 \end{bmatrix}$

Setting $x_3 = 0$ Setting $x_3 = 1$

$v_1 = \begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$

$v_1 = \begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$.

True

if $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ then $\operatorname{Span}(u, v)$ is all vectors of the form $\begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}$ where c_1 and c_2 are any constants.

If $w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ then $w + u = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ therefore w and $w + u$ are both in the $\operatorname{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True

If $x_1 u + x_2 v + x_3 w = 0$ has one solution, then $x_1 u + x_2 v$ also has to have one solution, therefore $\{u, v\}$ is also linearly independent

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then $x_1 = x_2 = x_3 = 0$

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ then $x_1 = x_2 = 0$

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True

If Au and Av are linearly dependent, then they have infinitely many solutions. Therefore u and v would have to also have infinitely many solutions

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True

Since u is in the span(v, w) taking the linear transformation of each vector is just taking the product of each vector with A. Therefore T(u) would still be in Span(T(v), T(w))

MTH-309T-F19-EX1-031-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Kyle Drapikowski

UB Person Number:

5	0	1	8	2	2	1	4
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) if w span (v_1, v_2, v_3) then $w = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$w = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$w = -2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} + \begin{bmatrix} b \\ 2b \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} b-4 \\ 2b+2 \\ -10 \end{bmatrix}$$

$b - 4 = 0 \Rightarrow b = 4$

$2b + 2 = 0 \quad b = -1$

b)

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{2R_1 - R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-3R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yes $\{v_1, v_2, v_3\}$ is linearly independent because every column of the ~~m~~ reduced matrix is a pivot column.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} (r_1c_1) & (r_1c_2) & (r_1c_3) \\ (r_2c_1) & (r_2c_2) & (r_2c_3) \\ (r_3c_1) & (r_3c_2) & (r_3c_3) \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$3\times 3 \quad \times \quad 3\times 3$

$$C = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix}$$

$r_1 c_1 :$ $2x_1 + x_2 - x_3 = 1$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{RR} \left[\; C \; \right]$$

$r_2 c_1 :$ $-x_1 + 2x_3 = 4$

$r_3 c_1 :$ $x_1 + x_2 = 3$

$x_1 = 3 - x_2$

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T. im defining this as A.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$$

$$T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$$

a) $$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 - 2(0) \\ 1 + 0 \\ 1 - 3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2(1) \\ 0 + 1 \\ 0 - 3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

standard matrix of T: $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} * \overset{u}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 - 2x_2 &= 1 \\ x_1 + x_2 &= 10 \\ x_1 - 3x_2 &= -2 \end{aligned}\right\}$$

$x_1 = 1 + 2x_2$

$1 + 2x_2 + x_2 = 10 \Rightarrow 1 + 3x_2 = 10$

$3x_2 = 9$

$x_2 = 3$

↓ backsub

$x_1 - 2(3) = 1$

$x_1 = 7$

$7 - 3(3) = -2$

$7 - 9 = -2$ ✓

so $u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

or any multiple of it.

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$\xrightarrow{-\frac{1}{4}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ it is one-to-one as every column has 1 pivot position.

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_2-R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

not going to be one to one because column 3 will not have a pivot.

$v_1 = \begin{bmatrix} \ \end{bmatrix}$ $v_2 = \begin{bmatrix} \ \end{bmatrix}$

$T_A(v_1) = T_A(v_2)$ if $v_2 = v_1 + n$

So if $v_1 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ and $n = 2$

$v_2 = \begin{bmatrix} 5 \\ 9 \\ 6 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True

Span (u, v) = all vecks / linear combinations $c_1 u + c_2 v_2$

or $c_1 w + c_2 u$ spanning (u, v)

therefor $c_1 w$ span (u, v)

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

False because if $\{u, v\}$ are scalar multiples ie $c_1 v_1 + c_2 v_2 = 0$ then it is linearly dependent.

$1 u_1 + (-1) u_2 = 0$

~~$R^3 \rightarrow 3 = n$ rows~~

~~u v w~~

~~1 2 3 $\rightarrow$ 3 = p colums~~

~~by definition if $p > n \Rightarrow$ linearly dependent~~

~~but $p = n$ so it is linearly independent~~

~~but if u v~~

~~1 2 $p = 2$ and $R3$ $n = 3$~~

~~$p \not> n$~~

~~$2 \not> 3$ so it is linearly independent as well~~

~~* does not apply for $p = 2$~~

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True.

Au, Av indicates that after multiplication there was a free variable causing ∞ solutions, that makes Au, Av linearly depedent

This means that the vectors u and v must have also had free variables when multiplied by some x value.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

False, if the transformation changes u, v, w to where u no longer Span(v, w) that would make T(u) not Span(T(v), T(w))

$$\begin{array}{c} u \\ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{array} \quad u_1 + 3u_2 \quad \text{span}(u) \quad \text{but} \quad T(u) = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

then it may not span for two other vectors if it is not a scalar multiple

MTH-309T-F19-EX1-037-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Michael Leishear

UB Person Number:

5	0	2	6	4	1	5	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & b \end{array}\right] R3 \cdot R2 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & 0 & 2b \end{array}\right] R3 - 3 \cdot R2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -6 & 2b-6 \end{array}\right] \cdot -\tfrac{1}{6} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -\tfrac{1}{3}b+1 \end{array}\right] R1 + R2$$

$3(-\tfrac{1}{3}b+1)$

$-b+3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -\tfrac{1}{3}b+1 \end{array}\right] -3 \cdot R3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -b+3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -\tfrac{1}{3}b+1 \end{array}\right] R2 - 2 \cdot R3$$

$$\begin{array}{cccc} 1 & 0 & 0 & -b+3 \\ 0 & 1 & 0 & \tfrac{2}{3}b \\ 0 & 0 & 1 & -\tfrac{1}{3}b \end{array}$$

$x_1 = -b+3$

$x_2 = \tfrac{2}{3}b$

$x_3 = -\tfrac{1}{3}b$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] R3 + 3 \cdot R2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

-2

$x_1 = x_2 - x_3$

$x_2 = -3x_3$

$x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

It is not linearly independent bc there are infinite solutions

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] R2 \cdot \tfrac{1}{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] R3 \cdot -2 \cdot R2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] + \tfrac{1}{2} R3$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] R1 + R2 \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] R1 - 2R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \quad A^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 2 & 0 & 1 & 0 \end{array}\right] R2+2R3 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \end{array}\right] +R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \end{array}\right] R3-R2 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 \end{array}\right] \cdot -\tfrac{1}{2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{array}\right] -4\cdot R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{array}\right] -R2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 0 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{array}\right] \quad \tfrac{1}{2}\begin{bmatrix}3\\4\\1\end{bmatrix} - \tfrac{1}{2}\begin{bmatrix}3\\4\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 4 & 0 \\ -4 & 16 & 0 \\ -3 & 4 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -4 & 0 \\ -2 & 4 & 0 \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{bmatrix}$$

$$3\begin{bmatrix}1\\4\\3\end{bmatrix} + -4\begin{bmatrix}1\\4\\3\end{bmatrix} + 0\begin{bmatrix}1\\4\\3\end{bmatrix} = \begin{bmatrix}-1\\-4\\-3\end{bmatrix}$$

$$-2\begin{bmatrix}2\\5\\2\end{bmatrix} + 4\begin{bmatrix}2\\5\\2\end{bmatrix} + 0 = \begin{bmatrix}4\\10\\4\end{bmatrix}$$

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$e1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Standard matrix $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 0 & -3 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -3 & -2 \end{array}\right] R2 \cdot -\tfrac{1}{3} R3 \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 20/3 \\ 0 & -3 & -2 \end{array}\right] R3+3R2$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 20/3 \\ 0 & 0 & 18 \end{array}\right] R1+2R2 \quad \left[\begin{array}{cc|c} 1 & 0 & 10/3 \\ 0 & 1 & 20/3 \\ 0 & 0 & 18 \end{array}\right]$$

$$\boxed{u = \begin{bmatrix} 10/3 \\ 20/3 \\ 18 \end{bmatrix}}$$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \cdot \frac{1}{2} \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] R_3 \circ R_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right] R_3 - 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ a) not one to one

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \cdot \frac{1}{2} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] R3 - 4 \cdot R2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & -2 & 0 \end{array}\right] R3 \circ R2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \cdot -\frac{1}{4} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$ b) one to one

✓

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True, V_0 tn is in the span

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

false, $\{u, v\}$ could have a differenti solution with nothing trivial

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~false~~ false, u, v can be linearly independent before being multiplied by A.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True, T(u) is in span (T(v), T(w)) if u is in span(v, w)

MTH-309T-F19-EX1-010-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Zachary Ross

UB Person Number:

5	0	1	7	8	1	2	7
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $w \notin cv_1$ since 2nd row of cv_1 is always 0.
$w \notin cv_3$ since 2nd row is same for both, but first row is different.
$w \in cv_2$ when $c = 2$ and $b = -6$ ✓

$$w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} = 2v_2 = 2 \cdot \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$$

$$\boxed{b = -6}$$

b) Yes, since there is only 1 solution!

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & c_1 \\ 0 & 1 & 2 & c_2 \\ 2 & 3 & 0 & c_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & c_1 \\ 0 & 1 & 2 & c_2 \\ 0 & 5 & -2 & c_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & c_1 \\ 0 & 1 & 2 & c_2 \\ 0 & 0 & -12 & c_3 \end{array}\right] \to$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right] \end{array}$$

$$\boxed{\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ x_3 &= c_3 \end{aligned}}$$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad r_2 = r_2 - r_1$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad r_1 = r_1 + r_2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad r_3 = r_3 - 2r_2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \quad \begin{array}{l} r_1 = r_1 - r_3 \\ r_2 = r_2 + r_3 \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$3 \times 3 \cdot 3, \cdot 3$

A B C

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$x_1 = 1 - y_1$

$z_1 = 1 + \frac{x_1}{2}$

$z_1 = 1 + \frac{1-y_1}{2}$

$x_1 + y_1 = 1$

$-x_1 + 2z_1 = 2$

$2x_1 + y_1 - z_1 = 3$

$2(1-y_1) + y_1 - \left(1 + \frac{(1-y_1)}{2}\right) = 3$

$2 - 2y_1 + y_1 - 1 - \frac{1}{2} + \frac{y_1}{2} = 3$

$1 = 2y_1 + y_1 = 6$

$y_1 = -5$

$x_1 = 6$

$z_1 = 4$

$x_2 = 4 - y_2$

$z_2 = \frac{5}{2} + \frac{x_2}{2}$

$x_2 + y_2 = 4$

$-x_2 + 2z_2 = 5$

$2x_2 + y_2 - z_2 = 4$

$4 = 2(4-y_2) + y_2 - \left(\frac{5}{2} + \frac{(4-y_2)}{2}\right)$

$8 = 4(4-y_2) + 2y_2 - 5 - 4 + y_2$

$8 = 16 - y_2 + 2y_2 - 9 + y_2$

$y_2 = 3$

$x_2 = 1$

$z_2 = 1$

$x_3 + y_3 = 3$

$-x_3 + 2z_3 = 2$

$2x_3 + y_3 - z_3 = 1$

$x_3 = 3 - y_3$

$z_3 = 1 - \frac{3}{2} + \frac{x_3}{2}$

$1 = 2(3-y_3) + y_3 - \left(1 - \frac{3}{2} + \frac{y_3}{2}\right)$

$2 = 12 - 4y_3 + 2y_3 - 2 + 3 - \cancel{y_3}$

$2 = 13 - 3y_3 = \frac{11}{3}$

$$\boxed{C = \begin{bmatrix} 6 & 3 \\ -5 & 1 \\ 4 & 1 \end{bmatrix}}$$

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = 3 \times 2$ since $3\times 2 \cdot 2\times 1$

$$\begin{bmatrix} C_1 & C_4 \\ C_2 & C_5 \\ C_3 & C_6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$C_1 x_1 + C_4 x_2 = 1x_1 - 2x_2$ $\quad C_1 = 1 \quad C_4 = -2$

$C_2 x_1 + C_5 x_2 = 1x_1 + 1x_2$ $\quad C_2 = 1 \quad C_5 = 1$

$C_3 x_1 + C_6 x_2 = 1x_1 - 3x_2$ $\quad C_3 = 1 \quad C_6 = -3$

Standerd Matrix: $\boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

b)

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$\boxed{T(u) = \text{Span}\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right)}$

$x_1 = 7$

$x_2 = 3$

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\downarrow$ $r_3 = r_3 - 3r_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

All column are pivot columns, therefore $\boxed{\text{all } T_A(v) \text{ are one-to-one.}}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & \boxed{0} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

not one to one

$\boxed{v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix}}$

$T_A(v_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T_A(v_2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1+1 \\ 2-2 \\ -3+4-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{T_A(v_1) = T_A(v_2)}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False.

$u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad w + u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \in \text{Span}(v)$

$\in \text{Span}(u, v)$

$w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\quad w \neq cu$, $w \neq cv$, so $w \notin \text{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True. If set $\{u, v, w\}$ are indepentant, that means $u \neq cv$, $u \neq cw$, $v \neq cu$, $v \neq cw$, $w \neq cu$, $w \neq cv$.

Show linear independence between u and v.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True: If Au and Av are linearly dependent, then $Au = cAv$. Therefore $u = cv$, so u and v must be linearly dependent.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True. If $u \in \mathrm{Span}(v, w)$ then $u = cv$ or $u = cw$. If your take $T(u)$, you get $A \cdot u = A \cdot cv$, which shows $T(u) \in \mathrm{Span}(T(v), T(w))$

MTH-309T-F19-EX1-086-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Alvin Tsang

UB Person Number:

5	0	2	5	6	5	4	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 1/2 & 1 & \frac{-2-b}{2} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 2+\frac{b}{2} \end{array}\right]$$

$2+\frac{b}{2}=0$

$\frac{b}{2}=-2$

$b=-4$

The set $\{v_1, v_2, v_3\}$ are linearly independent because v_1, v_2, v_3 are not scalar multiples of each other.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(A^{-1})^T C = B$$

$$(A^T)^{-1} \cdot A^T C = (A^T)^{-1} B$$

$$C = (A^{-1})^T B$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 5 & 4 \\ 0 & -1 & 0 \\ -3 & -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ -8 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ -5 \\ -10 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ -8 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

Standard Matrix $T = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & -3 \end{bmatrix}$

$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 2 & 10 \\ 1 & -3 & -2 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 9 \\ 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 4 & 9 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$$

$x_1 - 2x_2 = 1$

$x_1 + x_2 = 10$

$x_1 - 3x_2 = -2$

$-3x_2 = -9$

$x_2 = 3$

$x_1 + 3 = 10$

$x_1 = 7$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1/4 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

One-to-one b/c pivot position in every column

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Not one-to-one

$T_A(v_1) = T_A(v_2)$

$T_A(v_1) - T_A(v_2) = 0$

$Av_1 - Av_2 = 0$

$A(v_1 - v_2) = 0$

$A(v_1 - v_2) = 0$

$$\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$v_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$w+u = c_1 u + c_2 v$

$w = c_1 u + c_2 v$

True because if $w \in \text{Span}(u,v)$, according to $x_1 v_1 + \dots + x_p v_p = w$, the set of vectors is linearly dependent. All the vectors are just linear combinations of each other so w is in the span of u, v!

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True because if they are all linearly independent, the vectors can't be changed to be linearly dependent. There is no relationship between u, v, w, so if you take one out, it would not change the linear independently.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

false

$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Not linearly dependent

$Au = \begin{bmatrix} 1 + 0 \\ 1 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Av = \begin{bmatrix} 0 + 1 \\ 0 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Linearly dependent

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True $c_1 v + c_2 w = u$

$cT(v) + cT(w) = T(u)$

$c(Av) + c(Aw) = T(u)$

$A(cv) + A(cw) = T(u)$

$T_A(cv) + T_A(cw) = T(u)$

$T(cv) + T(cW) = T(u)$; $c_1 v + c_2 w = u$

MTH-309T-F19-EX1-005-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Purvi Patel

UB Person Number:

5	0	1	7	1	8	3	3
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\underline{w} \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$V_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\quad (-2)\cdot \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right]$ $\quad 2-3$ $\quad = 2+6$

$V_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$

$V_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\quad (1)\cdot \left[\begin{array}{cccc} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$ $\quad 2-2$

$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow (1)\cdot \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad 2-(-1)$

$X_1 = \;\; -X_3 \rightarrow -2$

$X_2 = -2X_3 \rightarrow (2)$

$X_3 = \text{free} \rightarrow b$

(-1).

(a) $\boxed{b=-1}$

(b) No, dependent because it has infinite solutions X_3 is a free variable

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

(-1)
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$-2+1$

$1-1=-1$

(-2)
$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$(1)\cdot$
$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

$-1+2$

$-1+1$

(1)
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

$(-1)\cdot$
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

-2
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\boxed{\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = A^{-1}}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\cancel{(A^T)^{-1} A^T} C = (A^T)^{-1} B$$

$$C = (A^{-1})^T \cdot B$$

$$C = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

3×3 $\quad$ 3×3

$$\boxed{\begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

$2+6$

$-2+4+6$

$1+4$

$-4+5+4$

$-6+4+2$

$-2+2$

$3+4-6$

$-1+6$

$6-5-4$

$1-4$

$9-4-2$

$5-2$

$-1+4+3$

$3+3$

$-2+5+2$

$-3+4+1$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$7 - 6 = 1$
$7 + 3 = 10$
$7 - 9 = -2$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a $\boxed{T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

b $\boxed{T(u) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

b $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$(-1)\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$

$-1+10$

$(-1)\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right]$

$(-3)\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{array}\right] (2) \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$2-3$

$-1+2$

$-9+$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \quad (-3)$$

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \quad (-1)$$

$$-2+4 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\boxed{\text{(a) ONE-TO-ONE}}$

$$-3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$$

$$(-1) \quad \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$-2+2 =$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{\text{(b) NOT ONE-TO-ONE}}$

$x_1 = -x_2$

$x_2 = -2x_3$

$x_3 = \text{free}$

$$\boxed{V_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}}$$

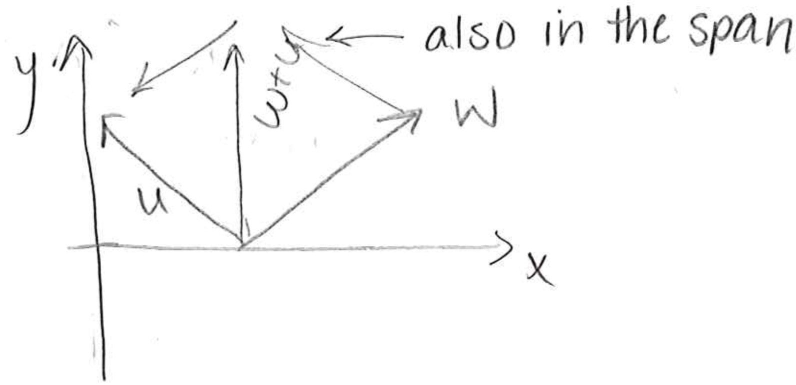
6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True:

if $x = w$, then $x_0 = w + u$

graphical interpretation



b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False

$$A = \begin{array}{cccc} u & v & w & \\ \end{array}$$
$$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \quad \text{but} \quad A = \left[\begin{array}{cc|c} 1 & 1 & 9 \\ 0 & 2 & 3 \\ 3 & 4 & 5 \end{array}\right]$$

is linear independent

[after row reduction each value of u,v,w corresponds to a unique solution]

is inconsistent after row reduction because last row is

$$\left[\begin{array}{cc|c} x & y & z \\ a & b & c \\ \boxed{0 & 0 & \#} \end{array}\right]$$

No solution; linearly dependent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$$\begin{array}{cc} & \begin{array}{ccc} A & & u \end{array} \\ -2 & \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 6 & 10 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} A & & v \end{array} \\ (-2) & \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 6 & 7 \end{array}\right] \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 0 & 3 \end{array}\right]$$

FALSE,

just because Au, Av are linearly dependent does not mean that u, v will also be linearly dependent

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True

Col(A) = set of values for T_A so if u is in span(v, w) for matrix A then it will also be in the span of the transformation for A.

MTH-309T-F19-EX1-078-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Ricky Chen

UB Person Number:

5	0	2	3	1	1	6	8
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \rightarrow R_3 = R_3 - 2R_1 \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & 0 & b+4 \end{array}\right)$ +

$\left[\begin{array}{cccc} 1 & -1 & -2 & -2 \\ 0 & 0 & 2 & b+6 \\ 0 & -1 & 0 & b+4 \end{array}\right]$ + $\xrightarrow{R_1 = R_1 - R_3}$ $\left[\begin{array}{cccc} 1 & 0 & -2 & -b-6 \\ 0 & 0 & 2 & b+6 \\ 0 & -1 & 0 & b+4 \end{array}\right]$

$\left[\begin{array}{cccc} 1 & 0 & -2 & -b-6 \\ 0 & -1 & 0 & b+4 \\ 0 & 0 & 2 & b+6 \end{array}\right] \xrightarrow{R_1 = R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & b+4 \\ 0 & 0 & 2 & b+6 \end{array}\right] \begin{array}{l} \\ \cdot(-1) \\ \cdot(\frac{1}{2}) \end{array}$

$\downarrow$

$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -b-4 \\ 0 & 0 & 1 & \frac{b+6}{2} \end{array}\right]$

a) All values of b are $c_1(2) + c_2(-3) + c_3(0)$ that $w \in \text{Span}(v_1, v_2, v_3)$

$\cancel{x_1 = 0}$ $\cancel{x_2 = -b-4}$ $\cancel{x_3 = \frac{b+6}{2}}$

b) Yes, the set $\{v_1, v_2, v_3\}$ is linearly independent because after row reduction every column is a pivot column.

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$ ← this is the answer

~~$R_2 = R_1 - R_2$~~

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 + 2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right]$$

$\downarrow R_1 = R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 2 & -3 & 1 \end{array}\right]$$

$\downarrow R_2 = R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ -1 & 0 & 0 & 2 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ -1 & 0 & 0 & 2 & -3 & 1 \end{array}\right] \begin{array}{l} \cdot(-1) \\ \\ \cdot(-1) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ 1 & 0 & 0 & -2 & 3 & -1 \end{array}\right]$$

(1) $R_3 \leftrightarrow R_1$

(2) $R_2 \leftrightarrow R_3$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T C = B$$

$$\cancel{(A^T)^{-1} A^T} C = (A^T)^{-1} B$$

$$C = (A^{-1})^T B$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$3 \times 3 \qquad 3 \times 3 \quad = 3 \times 3$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = 1\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+8+6=12 \\ 3-4-6=-7 \\ -1+4+3=6 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+6+4=10 \\ 6-5-4=-3 \\ -2+5+2=5 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = 3\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} +2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+8+2=4 \\ 9-4-2=3 \\ -3+4+1=2 \end{bmatrix}$$

$$C = \begin{bmatrix} 12 & 10 & 4 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(u) = \begin{bmatrix} a_1 - 2a_2 \\ a_1 + a_2 \\ a_1 - 3a_2 \end{bmatrix}$ $T(v) = \begin{bmatrix} b_1 - 2b_2 \\ b_1 + b_2 \\ b_1 - 3b_2 \end{bmatrix}$

$T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

standard matrix of $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $R_3 = R_3 - R_1$ $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$

$+2R_2$ $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & -3 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{cccc} 3 & 0 & 21 \\ 1 & 1 & 10 \\ 0 & 1 & -3 \end{array}\right] * \left(\frac{1}{3}\right) \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 1 & 1 & 10 \\ 0 & 1 & -3 \end{array}\right]$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 1 & -3 \end{bmatrix}$

$x_1 = 7$
$x_2 = x_2$

all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ are: $\begin{bmatrix} 7 \\ x_2 \end{bmatrix}$ where x_2 is free

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a)

$$-(3R_1)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} 2- \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & -1 & 0 \end{bmatrix} \cdot\left(\tfrac{1}{2}\right)$$

$$\downarrow$$

$$\cdot\left(\tfrac{1}{2}\right)\cdot (-1)\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \leftarrow + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} +$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yes A is one-to-one

b)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} -2R_2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 0 & -6 \end{bmatrix} \begin{matrix} \cdot \frac{1}{2} \\ \cdot -\frac{1}{3} \end{matrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix} +$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} -(\cdot 2) \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} -$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yes, A is one to one.

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True, because since $u+w$ is in the span (u,v) then it must be true that $w \in \text{Span}(u,v)$ because the sum of the two vectors $(u+w)$ are in the span (u,v).

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

True
~~False~~, for example

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then the set $\{u,v\}$ $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ is also linearly independent because it has only one solution and pivot col in every column. If u, v, w are independent then it has pivot col. in every columnn that means with two vectors it still holds true.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

u and v could still be independent and be multiplied with A to make it linearly dependant.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v})$, $T(\mathbf{w})$).

Yes, true because if u is in the span of (v, w) then the transformation could just be the scalar of u vector and it would still let T(u) be in the span of T(v), T(w)

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MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Umar Ahmed

UB Person Number:

5	0	2	6	3	5	0	9
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $-2v_1 + 4v_2 - v_3 = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

$$-2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -16 \end{bmatrix}$$

$-4 \; -12$

b) aug. matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \to (-2)\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

$$\to (-1)\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right](1)$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]\left(\tfrac{1}{3}\right) \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right](-1) \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore$ The set $\{v_1, v_2, v_3\}$ is lin. independent b/c there is only one solution to the homogeneous matrix

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$(-1)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left(-\tfrac{1}{2}\right)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left(\tfrac{1}{2}\right)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -\tfrac{1}{2} & -1 & 1 & -\tfrac{1}{2} \end{array}\right]$$

$$\downarrow$$

$$(-2)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\tfrac{1}{2} & 0 & 0 & \tfrac{1}{2} \\ 0 & 0 & -\tfrac{1}{2} & -1 & 1 & -\tfrac{1}{2} \end{array}\right]$$

$$\downarrow$$

$$\left(\tfrac{1}{2}\right)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\tfrac{1}{2} & 0 & 0 & \tfrac{1}{2} \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right](-2)$$

$$\uparrow$$

$$(1)\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & -4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\uparrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -5 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$A^{-1} = \begin{bmatrix} -2 & -5 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

3x3 3x3

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

3x3 3x3

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$2x_2 + 1$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

5 5

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$x_1 = 2x_2 + 1$

a) $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 - 0 \\ 1 + 0 \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2 \\ 0 + 1 \\ 0 - 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$$A = \boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

b) [

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5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$(-1)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 2 & 0 \end{bmatrix}$

$\downarrow$

$(-3)\begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

$\downarrow$

$(2)\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

$\downarrow$

$\begin{matrix} (1) \\ (\frac{1}{4}) \end{matrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\downarrow$

$(-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}(-3)$

$\downarrow$

$(\frac{1}{2})\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}(-1)$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ - not one-to-one

$V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} -3 \\ -16 \\ -17 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2+0 \\ 0+4+12 \\ 3+8+6 \end{bmatrix} = \begin{pmatrix} 3 \\ 16 \\ 17 \end{pmatrix}$

$T_A(V_1) - T_A(V_2) = 0$

$\begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False

$$w = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad \text{Span}\left(\overset{u}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \overset{v}{\begin{bmatrix} 7 \\ 4 \end{bmatrix}}\right)$$

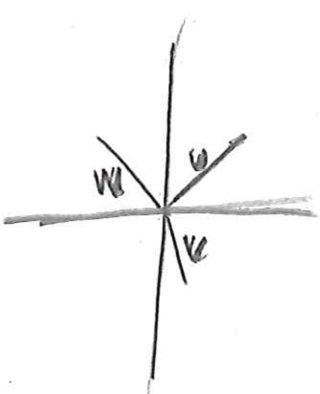
$$w + u = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

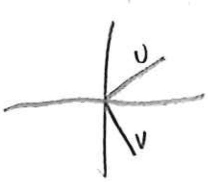
$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

is in Span(u, v), but w $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ not is Span

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True. U, V, W are 3 unique vectors, so even if only 2 of them were in the set, it would still be linearly independent





still independent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

False A single vector such as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is linearly independent. If $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then it is linearly dependent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v})$, $T(\mathbf{w})$).

True. All 3 vectors are transformed by the same transformation, so properties are not changed. -They are still scalar multiples.

MTH-309T-F19-EX1-022-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Cortlandt Chin

UB Person Number:

5	0	2	2	8	4	5	4
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$. b = any number but 4

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & b \end{array}\right] \times 2 \quad 3-2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 5 & 0 & b-4 \end{array}\right] \times 5$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -10 & 10(b-4) \end{array}\right] \times -\frac{1}{10}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \underbrace{-(b-4)}_{-b+4} \end{array}\right]$$

$-b+4=0$

$-b=-4$

$b=4$

$\therefore$ If $b=4$, then $w \notin \mathrm{Span}(v_1, v_2, v_3)$

The set $\{v_1, v_2, v_3\}$ is linearly independent because each vector is not a scalar multiple of each other

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \times 1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \times 2$$

$-1 \cdot 2 = -2$

$1 - -2 = 3$

$-1 \cdot 2 = -2$

$0 - -2 = 2$

$1 \cdot 2 = 2$

$0 - 2 = -2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & -2 & 1 \end{array}\right] \times \frac{1}{3}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] -1$$

$\frac{2}{3} \cdot 1 = +\frac{2}{3}$

$-\frac{3}{3} + +\frac{2}{3} = \frac{1}{3}$

$-\frac{3}{3} + \frac{2}{3} = \frac{5}{3}$

$\frac{3}{3} - \frac{4}{3} = -\frac{1}{3}$

$0 - -\frac{4}{3} = \frac{4}{3}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] \times 2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -\frac{1}{3} & \frac{4}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] \times -1$$

$\frac{1}{3} \times -1 = -\frac{1}{3}$

$-\frac{1}{3} + +\frac{1}{3} = 0$

$\frac{5}{3} \times -1 = -\frac{5}{3}$

$\frac{4}{3} + \frac{5}{3} = \frac{9}{3}$

$\frac{1}{3} \times -1 = -\frac{1}{3}$

$-\frac{2}{3} + +\frac{1}{3} = -\frac{1}{3}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right]$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} 0 & 3 & -\frac{1}{3} \\ \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$\underset{3\times 3}{A} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad \underset{3\times 3}{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T \boxed{C} = B$ (where A^T is the transpose of A).

$$\frac{3}{1} - \frac{4}{3} \qquad \frac{9}{3} - \frac{4}{3} \qquad \frac{3}{3} - \frac{4}{3} = \frac{-1}{3}$$

$$\frac{15}{1} - \frac{4}{3} = \frac{45}{3} - \frac{4}{3} \qquad \frac{2}{1} + \frac{10}{3}$$

$$\frac{6}{1} - \frac{1}{3} = \frac{18}{3} - \frac{1}{3} \qquad \frac{6}{3} + \frac{10}{3}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T (A^T)^{-1} C = B (A^T)^{-1}$$

$$C = B (A^T)^{-1}$$

$$(A^T)^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ 3 & \frac{5}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$B(A^T)^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ -\frac{1}{3} \end{bmatrix} = 0 + 3\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + -\frac{1}{3}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ -\frac{4}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{41}{3} \\ \frac{17}{3} \end{bmatrix}$$

$$B \cdot \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + \frac{5}{3}\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{4}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{10}{3} \\ \frac{25}{3} \\ \frac{10}{3} \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{4}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ 11 \\ \frac{10}{3} \end{bmatrix}$$

$$B \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{8}{3} \\ \frac{2}{3} \end{bmatrix} + \begin{bmatrix} -\frac{4}{3} \\ -\frac{10}{3} \\ -\frac{4}{3} \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$\boxed{\therefore C = \begin{bmatrix} 5 & \frac{10}{3} & \frac{5}{3} \\ \frac{41}{3} & 11 & \frac{2}{3} \\ \frac{17}{3} & \frac{10}{3} & -\frac{1}{3} \end{bmatrix}}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T. $\longrightarrow T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$\left.\begin{array}{l} e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \end{array}\right\} T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\vec{u})$$

$$\begin{cases} x_1 - 2x_2 = 1 \\ x_1 + x_2 = 10 \\ x_1 - 3x_2 = -2 \end{cases}$$

$$\Downarrow$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \times 1$$

$-2 \times 1 = -2$
$1 - -2 = 3$
$1 \times 1 = 1$
$10 - 1 = 9$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \times 1$$

$-2 \times 1 = -2$
$-3 + 2 = -1$
$1 \times 1 = 1$
$-2 + 1 = -3$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \times \frac{1}{3} \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \times -1 \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{array}\right] \Rightarrow \text{Inconsistent system}$$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \times 3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \times 2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix} \times -\tfrac{1}{4}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ A is one-to-one because there is a pivot position in every column

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \times 3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \times 2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ B is not one-to-one

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$. True

$U = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $W = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, $V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$W + U = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ $\therefore W \in \text{Span}(U)$

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent. True. Removing a vector will still make the set linearly independent

$U = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $V = \begin{bmatrix} 17 \\ 19 \\ 21 \end{bmatrix}$ $W = \begin{bmatrix} 33 \\ 33 \\ 33 \end{bmatrix}$

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

$A\bar{u} \neq A\bar{v}$

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent. False

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\bar{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $A\bar{u} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

$\bar{v} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ $A\bar{v} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$). False. $T(\bar{u})$ may no longer be in Span$(T(\bar{v}), T(\bar{w}))$ after transformation

$\bar{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\bar{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\bar{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

MTH-309T-F19-EX1-009-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Matthew Cho

UB Person Number:

5	0	1	5	2	5	8	1
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

$1\ -1\ 1\ -2$

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $V_1 - V_2 + V_3 = -2$

$V_1 + 2V_2 = 2$

$2V_1 - 3V_2 = b$

~~$V_1 = 2 - 2V_2$~~

~~$2 - 2V_2 - V_2 + V_3 = -2$~~

~~$-3V_2 + V_3 = -4$~~

~~$V_3 = -4 - 3V_2$~~

~~$2 - 2V_2 - V_2 - 4 - 3V_2 = -2$~~

~~$2 - 3V_2 - 4 - 3V_2 = -2$~~

~~$-3V_2 - 4 - 3V_2 = -4$~~

~~$V_2 = 0$~~

~~$V_1 = 2$~~

~~$b = 4$~~

$V_1 = 2 - 2V_2$

$2 - 2V_2 + V_3 = -2$

$-3V_2 + V_3 = -4$

$V_3 = -4 - 3V_2$

$= 2 - 2V_2 - V_2 - 4 - 3V_2 = -2$

$\Rightarrow \underline{V_2 = 0}$

$V_1 + 2(0) = 2$

$\underline{V_1 = 2}$

$2V_1 - 3V_2 = b$

$2(2) - 3(0) = b$

$4 = b$

$\boxed{b = 4}$

b) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & 4 \end{array}\right] (-2) = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & 0 & 8 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -1 & 0 & 8 \\ 0 & 1 & 2 & 2 \end{array}\right] (1)$

$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & 3 & 2 \end{array}\right] \cdot \tfrac{1}{3} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & 1 & 2/3 \end{array}\right] \cdot(-1) \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 2/3 \end{array}\right] (1) \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$

$\Downarrow$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2/3 \end{array}\right] (-1) \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$

The set $\{V_1, V_2, V_3\}$ is linearly independent because the the set contains only 1 solution, thus being a homogeneous equation which proves that the set is linearly independent.

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (-1) = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (-2) = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= (1) \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = (-2) \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = (1) \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$x_1 - x_2 = 1$

$-x_1 + x_2 = 0$

$x_2 = x_1$

$x_1 - x_1 = 1$

$0 \neq 1$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T \cdot B$$

$$C = (A^T)^{-1} \cdot B$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 0 & 0 \\ 6 & 13 & 0 \\ 0 & 0 & 6 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$T(u+v) = Tu + Tv$

A) $T(u) = \begin{bmatrix} u_1 - 2u_2 \\ u_1 + u_2 \\ u_1 - 3u_2 \end{bmatrix}$ $\quad T(u+v) = \begin{bmatrix} u_1+v_1 - 2u_2 - 2v_2 \\ u_1+v_1 + u_2 + v_2 \\ u_1+v_1 - 3u_2 - 3v_2 \end{bmatrix}$ =

$T(v) = \begin{bmatrix} v_1 - 2v_2 \\ v_1 + v_2 \\ v_1 - 3v_2 \end{bmatrix}$ $\quad = \begin{bmatrix} (u_1+v_1) - 2(u_2+v_2) \\ (u_1+v_1) + (u_2+v_2) \\ (u_1+v_1) - 3(u_2+v_2) \end{bmatrix}$

standard matrix

$A\left[T(e_1)\ T(e_2)\right] =$

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

standard matrix

$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} \Rightarrow A = \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 10 \\ 1 & 3 & 0 & -2 \end{array}\right] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 3 \\ -17 \end{bmatrix}}$

$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 10 \\ 1 & 3 & 0 & -2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 3 & 0 & 9 \\ 0 & 5 & 0 & -2 \end{array}\right] = \left[\begin{array}{cccc} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 5 & 0 & -2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & -17 \end{array}\right]$

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} (\cdot 3) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \cdot 1/2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} (-1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \cdot (-1/4) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$(-2) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A is one to one
because every column has a pivot position and the matrix is a homogeneous equation

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} (-3) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \cdot 1/2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} (-1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot 1/2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} -2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} -2$

A is one to one
because every column has a pivot position and the matrix is a homogeneous equation.

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}$

True

$V_1 + 0V_2 = 1$
$0V_1 + V_2 = 2$ ✓
$0V_1 + 0V_2 = 0$

$u + w = \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$

$V_1 + 0V_2 = 4$
$0V_1 + V_2 = 8$
$0V_1 + 0V_2 = 0$

$V_1 + 0V_2 = 4$
$0V_1 + V_2 = 6$ ✓
$0V_1 + 0V_2 = 0$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True because in order for a set to be linearly independent they must have 1 solution (Homogeneous equation) so that means that u, v, w all only have 1 solution which ~~means~~ means that the set {u, v} is linearly independent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $u = [\ ]$, $v = [\ ]$

True

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

This is true

MTH-309T-F19-EX1-062-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Andrew Woloszyn

UB Person Number:

5	0	1	8	4	7	8	1
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

MTH-309T-F19-EX1-062-P01

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \cdot 1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \cdot -1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & -3 & -3 & b \end{array}\right] \cdot 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 1 & 0 & 5 & 2 \\ 1 & -3 & -3 & b \end{array}\right] \cdot -1 \quad \left[\begin{array}{ccc|c} 0 & 0 & -2 & -2 \\ 1 & 0 & 5 & 2 \\ 1 & -3 & -3 & b \end{array}\right] \cdot \frac{-1}{2} \quad \left[\begin{array}{ccc|c} 1 & -3 & -3 & b \\ 1 & 0 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \cdot -1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -3 & b \\ 0 & 3 & 8 & 2-b \\ 0 & 0 & 1 & 1 \end{array}\right] \cdot -1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 3 & 8 & 2-b \\ 0 & 0 & 1 & 1 \end{array}\right] \cdot -5, \ \cdot -8 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 3 & 0 & -6-b \\ 0 & 0 & 1 & 1 \end{array}\right] \cdot \frac{1}{3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & \frac{-6-b}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

b) They are linearly independent.
If we create an augmented matrix of v_1, v_2, v_3 and augment with 0 instead of w, we see that the matrix still reduces to the identity matrix, meaning the only solution to that equation is the trivial solution.

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot -1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot -2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\cdot -2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

Add R2 to R1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \xrightarrow{①\cdot -1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 1 & 0 & -1 & 2 & 0 & -2 \end{array}\right] \xrightarrow{②\cdot 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \xrightarrow{②\cdot 1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 & 5 & 4 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \xrightarrow{③\cdot -2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & -8 & -5 & 0 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \xrightarrow{②\cdot -1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 7 & 3 \\ 0 & 1 & 0 & -8 & -5 & 0 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

$$C = \begin{bmatrix} 9 & 7 & 3 \\ -8 & -5 & 0 \\ 6 & 5 & 2 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b)

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \cdot -3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \cdot -1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \cdot -1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \cdot -1 \quad \cdot \tfrac{1}{4}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\mathrm{Nul}(A) = \{\vec{0}\}$

A has a pivot in every col

So T_A is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$x_1 = 2x_3$
$x_2 = -2x_3$
$x_3 = x_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \cdot -3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \cdot \tfrac{1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \cdot -1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot -1 \qquad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A does not have a pivot in every col, so T_A is not one-to-one

$x_1 = 2x_3$
$x_2 = -2x_3$
$x_3 = x_3$

$$V_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

$$Av_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ -8 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix} + \begin{bmatrix} -4 \\ -8 \\ -16 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False,

if $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True,

since every matrix transformation is a linear transformation, then both of the resulting vectors were linearly transformed. Since they were linearly dependent before transformation, then transforming the line they form does not break their linear dependence since linear transformations keep straight lines intact.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True,

if the whole span is linearly transformed, then $T(u)$ will be transformed in the same way, preserving its linearity and remaining on the plane of the span.