MTH-309T-F19-EX1-058-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	8	0	5	1	2
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) w in span if $c_1v_1 + c_2v_2 + c_3v_3 = w$

$\quad\quad\quad\quad\quad\quad x_1 \quad x_2 \quad x_3$

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ x_2 + 2x_3 &= 2 \\ 2x_1 - 3x_2 &= b \end{aligned} \rightarrow \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix}$$ any integer can be reduced to a leading 1 except when $b=-6$

a) $b = -6$

b) $x_1v_1 + x_2v_2 + x_3v_3 = 0$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_3 = 0$

$x_2 + 2x_3 = 0$

x_3 is a free variable $\therefore$ the equation has infinitely many solutions $\therefore$ the set is linearly dependent.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A \cdot A^{-1} = I$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_1^{-1} & A_2^{-1} & A_3^{-1} \\ A_4^{-1} & A_5^{-1} & A_6^{-1} \\ A_7^{-1} & A_8^{-1} & A_9^{-1} \end{bmatrix}$$

$$A_1^{-1} - A_4^{-1} + 2A_7^{-1} = 1$$
$$A_1^{-1} + A_7^{-1} = 0$$
$$2A_4^{-1} - A_7^{-1} = 0$$

$$2A_1^{-1} + 3A_7^{-1} = 1$$
$$A_1^{-1} + A_7^{-1} = 0$$
$$A_7^{-1} = 1$$
$$\cancel{A_7^{-1} = \tfrac{1}{2}}$$
$$A_1^{-1} + 1 = 0$$
$$A_1^{-1} = -1$$
$$2A_4^{-1} - 1 = 0$$
$$A_4^{-1} = \tfrac{1}{2}$$

$$A_2^{-1} - A_5^{-1} + 2A_8^{-1} = 0$$
$$A_2^{-1} + A_8^{-1} = 1$$
$$2A_5^{-1} - A_8^{-1} = 0$$

$$2A_2^{-1} + 3A_8^{-1} = 0$$
$$A_2^{-1} + A_8^{-1} = 1$$
$$A_8^{-1} = -2$$
$$A_2^{-1} - 2 = 1$$
$$A_2^{-1} = 3$$
$$2A_5^{-1} + 2 = 0$$
$$A_5^{-1} = -1$$

$$A_3^{-1} - A_6^{-1} + 2A_9^{-1} = 0$$
$$A_3^{-1} + A_9^{-1} = 0$$
$$2A_6^{-1} - A_9^{-1} = 1$$

$$2A_3^{-1} + 3A_9^{-1} = 1$$
$$A_3^{-1} + A_9^{-1} = 0$$
$$A_9^{-1} = 1$$
$$A_3^{-1} + 1 = 0$$
$$A_3^{-1} = -1$$
$$2A_6^{-1} - 1 = 1$$
$$A_6^{-1} = 1$$

$$A^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ \tfrac{1}{2} & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$-1 \cdot \begin{cases} c_1 + c_4 = 1 \\ -c_1 + 2c_7 = 4 \\ 2c_1 + c_4 + c_7 = 3 \end{cases} \qquad -1 \cdot \begin{cases} c_2 + c_5 = 2 \\ -c_2 + 2c_8 = 5 \\ 2c_2 + c_5 + c_8 = 2 \end{cases} \qquad -1 \begin{cases} c_3 + c_6 = 3 \\ -c_3 + 2c_9 = 4 \\ 2c_3 + c_6 + c_9 = 1 \end{cases}$$

$c_1 + c_7 = 2$
$-c_1 + 2c_7 = 4$
$3c_7 = 6$
$c_7 = 2$
$-c_1 + 4 = 4$
$c_1 = 0$
$0 + c_4 = 1$
$c_4 = 1$

$c_2 + c_8 = 0$
$-c_2 + 2c_8 = 5$
$3c_8 = 5$
$c_8 = \frac{5}{3}$
$-c_2 + \frac{10}{3} = \frac{15}{3}$
$c_2 = \frac{-5}{3}$
$\frac{-5}{3} + c_5 = \frac{6}{3}$
$c_5 = \frac{11}{3}$

$c_3 + c_9 = -2$
$-c_3 + 2c_9 = 4$
$3c_9 = 2$
$c_9 = \frac{2}{3}$
$-c_3 + \frac{4}{3} = \frac{12}{3}$
$c_3 = \frac{-8}{3}$
$\frac{-8}{3} + c_6 = \frac{3}{3}$
$c_6 = \frac{11}{3}$

$$C = \begin{bmatrix} 0 & \frac{-5}{3} & \frac{-8}{3} \\ 1 & \frac{11}{3} & \frac{11}{3} \\ 2 & \frac{5}{3} & \frac{2}{3} \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T. T is 3x2 $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$-1\left(\begin{aligned} u_1 - 2u_2 &= 1 \\ u_1 + u_2 &= 10 \\ u_1 - 3u_2 &= -2 \end{aligned}\right.$$

$-u_2 = -3$

$u_2 = 3 \quad u_1 = 7$

$u = \left(\begin{bmatrix} 3 \\ 7 \end{bmatrix}\right)$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & \cancel{8}\,1 \end{bmatrix}$$

Pivot position every column

T_A is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$v_1 - 2v_5 = v_2 - 2v_6$

$v_3 + 2v_5 = v_4 + 2v_6$

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Not every column has a pivot position

T_A is not one-to-one

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_3 \\ v_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} v_2 \\ v_4 \\ v_6 \end{bmatrix}$$

$v_1 + v_3 = v_2 + v_4$

$2v_3 + 4$

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

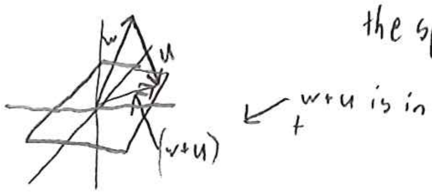
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

~~combination of two vectors with coefficients of 2 combination with a span.~~

True; since the span of two vectors in $\mathbb{R}^3$ can be visualized as a plane in 3D space, and u is in the span of (u, v), then w must also lie in that plan if w+u is to be in the span as well.



b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True; $\{u, v\}$ is linearly independent only if u and v are scalar multiples, which would not allow $\{u, v, w\}$ to be linearly independent.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$A_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $A_v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$2u_1 + u_2 = 1$ $2u_1 + u_2 = 2$

$u_1 + u_2 = 1$ $u_1 + u_2 = 2$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

True; since all matrix transformations are linear transformations, the matrix transformation T_A preserves linear dependence between $\{u, v\}$, and $\{Au, Av\}$.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span(T(v), T(w)).

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

True: since all matrix transformations are linear transformations, applying the same transformation to all three vectors preserves the column space and $T(u)$'s existence in it.

0 1 2 3 4 5 6 7 8 9 10