MTH-309T-F19-EX1-049-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Patrick Hull

UB Person Number:

5	0	2	9	9	7	8	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
8	7	5	15	13	4	6	2	--	60	C

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$b \neq 0$

$b > 0$

$b < 0$

← why?
Also, it should be
$b = -6$

b)

$$\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{matrix}$$

b is linearly independent as the reduced form has exactly one solution when A = 0

$$\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & \cancel{0} \end{matrix}$$

-2

$$\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix} \qquad \begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{matrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{matrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ +1 & -1 & +1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

MTH-309T-F19-EX1-049-P03

$B \cdot \frac{1}{A^T}$

$B \cdot {A^T}^{-1}$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A^T C = B \rightarrow C = \frac{B}{A^T}$ ~~$C = B \cdot (A^T)^{-1}$~~

~~$C = B \cdot (A^{-1})^T$~~

$C = (A^T)^{-1} \cdot B$

$= (A^{-1})^T \cdot B$

$A^{-1} = \begin{matrix} 2 & 3 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 1 \end{matrix}$

$(A^{-1})^T = \begin{bmatrix} 2 & -1 & 2 \\ 3 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 3 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

	c_1	c_2	c_3
c_1	11	-6	9
c_2	27	-13	22
c_3	13	-6	11

$c_{11} = \overset{2}{1 \cdot 2} + \overset{6}{2 \cdot 3} + \overset{3}{1 \cdot 3}$

$c_{12} = \overset{-1}{-1 \cdot 1} + \overset{-2}{2 \cdot -1} + \overset{-3}{3 \cdot -1}$

$c_{13} = \overset{2}{2 \cdot 1} + \overset{4}{2 \cdot 2} + \overset{3}{3 \cdot 1}$

$c_{21} = \overset{8}{4 \cdot 2} + \overset{15}{5 \cdot 3} + \overset{4}{4 \cdot 1}$

$c_{22} = \overset{-4}{4 \cdot -1} + \overset{-5}{5 \cdot -1} + \overset{-4}{4 \cdot -1}$

$c_{23} = \overset{8}{4 \cdot 2} + \overset{10}{5 \cdot 2} + \overset{4}{4 \cdot 1}$

$c_{31} = \overset{6}{3 \cdot 2} + \overset{6}{3 \times 2} + \overset{1}{1 \cdot 1} = 13$

$c_{32} = -3 + -2 + -1 = -6$

$c_{33} = 6 + 4 + 1 = 11$

~~$C = \begin{bmatrix} 11 & -6 & 9 \\ 27 & -13 & 22 \\ 13 & -6 & 11 \end{bmatrix}$~~

Matrix multiplication is incorrect too.

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$A = [T(e_1), T(e_2)]$

where $e_1 = \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ $e_2 = \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$

a)

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ ✓

b)

$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

Where did this come from ?

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a)

$\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & -1 & 4 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & -1 & 4 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{matrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a is
One to one ✓

b)

$\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & -1 & 2 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 8 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{matrix}$

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{matrix}$

~~b is one to one~~

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

~~$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$~~

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

~~False~~ take $W = \begin{matrix}1\\0\\0\end{matrix}$ $U = \begin{matrix}1\\1\\0\end{matrix}$ $V = \begin{matrix}2\\1\\0\end{matrix}$

$W + U = \begin{matrix}2\\1\\0\end{matrix} \in \text{Span}(U, V)$, but

$W = \begin{matrix}1\\0\\0\end{matrix} \notin \text{Span}(U, V)$

$w = v - u$, so $w \in \text{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

TRUE / FALSE ?

$X_1 V + X_2 V + X_3 W = 0$

Let $X_1 V = 3$

$X_2 V = 3$

$X_3 W = -6$

$3 + 3 + -6 = 0$

$X_1 U + X_2 V = 0$

$X_1 U = X_2 V$, so $X_1 U$ is a scalar multiple of $X_2 V$

?

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Au, Av are dependent

but u, v are independent

✓

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

~~False~~

$u = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(u) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $T(v) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ $T(w) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

What is T here?

$u \in \text{Span}(v, w)$

$T(u) \notin \text{Span}(T(v), T(w))$

0 1 2 3 4 5 6 7 8 9 10