MTH-309T-F19-EX1-014-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	1	8	4	9	9	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	TOTAL	GRADE
--	10	--	16	--	--	--	26	nan

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) as $W \in$ span (V_1, V_2, V_3) $\therefore V_1 + V_2 + V_3 = W$

$$-4 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$\therefore b = (-8)$.

b) $C_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

① $C_1 - C_2 + C_3 = 0$ $\therefore C_1 = C_2 - C_3$ $\therefore C_3 = 0$ $[\because C_2 = 0$

② $C_2 + 2C_3 = 0$ $\therefore C_3 = -\frac{C_2}{2}$

③ $2C_1 - 3C_2 = 0$ $\therefore C_1 = \frac{3C_2}{2}$ $\therefore C_1 = 0$ $[\because C_1 = 0$

$\therefore C_2 - C_3 = \frac{3C_2}{2}$

$\therefore 2C_2 - 2C_3 = 3C_2$ $\therefore 2C_2 + 2\left(\frac{C_2}{2}\right) = 3C_2$ $\therefore 3C_2 = 3C_2$

$\therefore C_2 = 0$

as, $\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so, we can say the set $\{V_1, V_2, V_3\}$ is linearcly independent.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

$-1-2(-1)$
$=-1+2$
$=1$
$0-2(-1)$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_1} = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} = \begin{bmatrix} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + R_3} = \begin{bmatrix} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad \therefore A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & c_4 & c_7 \\ c_2 & c_5 & c_8 \\ c_3 & c_6 & c_9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

$\frac{2}{3}$ 3

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) Standard matrix of T :- $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) as linear transformation,

$Au = T(u)$

$\therefore \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

① $u_1 - 2u_2 = 1$
② $u_1 + u_2 = 10$
③ $u_1 - 3u_2 = -2$

$\therefore u_1 = 1 + 2u_2$ $\quad \therefore u_1 = 1 + 2(3) = 1 + 6 = 7$

$\therefore 1 + 2u_2 + u_2 = 10$ $[\because u_1 = 1 + 2u_2]$ $\therefore 3u_2 = 9$ $\therefore u_2 = 3$

$\therefore u_1 - 3(3) = -2$ $[\because u_2 = 3]$

$\therefore u_1 - 9 = -2$ $\quad \therefore u_1 = 7$

$\therefore u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

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5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \Rightarrow R_3 - 3R_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 - R_3 - 3R_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ a is one-to one. but b is not one-to-one

Now, $T_A(V_1) = T_A(V_2)$

$\therefore T_A(V_1 - V_2) = 0$

$\therefore \underset{V_1}{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}} - \underset{V_2}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \underset{V_1}{\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}} - \underset{V_2}{\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False. because, $w + u = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \in$ span (u, v).

not necessary, $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \in$ span (u, v).

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False.
$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $W = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

the set $\{u, v, w\}$ is linearly independent. ~~be~~ cause, $x_1 u + x_2 v + x_3 w = 0$ has only one, trivial solution. on the other, $\{u, v, w\}$ is linearly dependent because $x_1 u + x_2 v + x_3 w = 0$ has non-trivial solution

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in $\text{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\text{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

~~False.~~

True.

0 1 2 3 4 5 6 7 8 9 10