MTH-309T-F19-EX1-019-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	2	8	8	3	7

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
18	10	10	20	20	7	2	2	10	98	A

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{\cdot(-2)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right] \Longrightarrow b = \cancel{6}$$

-6

b) The set $\{v_1, v_2, v_3\}$ is linearly dependent because x_3 is a free variable meaning there are infinitely many solutions. ✓

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒	☐	☐
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\cdot(-1)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\cdot 1\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\cdot(-2)\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\cdot 1\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\cdot(-1)\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \implies A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$ ✔

check: $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☒ 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$Ax = b$

$x = A^{-1} \cdot b$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A^T C = B \Rightarrow C = (A^T)^{-1} \cdot B \qquad (A^T)^{-1} = (A^{-1})^T$ ✓

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad (A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$(A^{-1})^T \cdot B = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2+4+6 & -4+5+4 & -6+4+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ -1+4+3 & -2+5+2 & -3+4+1 \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$ ✓

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$ $\qquad T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-2(0) \\ 1+0 \\ 1-3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-2(1) \\ 0+1 \\ 0-3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ $\qquad$ Standard matrix of $T = A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $A \cdot u = T(u)$

$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot(-1) \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \cdot(-1) \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 9 \end{array}\right] \cdot(-1) \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{array}\right] \cdot 2$

$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{array}\right] \cdot(-3) \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$ $\qquad x_1 = 7$, $x_2 = 3$ $\qquad u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ ✓

Check: $T\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7-2(3) \\ 7+3 \\ 7-3(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\cdot(-3)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot\frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\cdot\frac{1}{2}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot(-2)$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot 2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓ since A has a pivot position in every column, $T_A(v)$ is one-to-one.

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\cdot(-3)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot 1/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

since A does not have a ✓ pivot position in every column, $T_A(v)$ is not one-to-one.

$T_A(v_1) = T_A(v_2)$

let $T_A(v_1) = T_A(v_2) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ $\begin{cases} x_1 = 1 + 2x_3 \\ x_2 = 2 - 2x_3 \\ x_3 = x_3 \end{cases}$

if $x_3 = 1$:
$x_1 = 1 + 2(1) = 3$
$x_2 = 2 - 2(1) = 0$
$x_3 = 1$

if $x_3 = -1$:
$x_1 = 1 + 2(-1) = -1$
$x_2 = 2 - 2(-1) = 4$
$x_3 = -1$

$V_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$.

True. Vector u must be in the span(u,v), and given that w ∈ span(u,v) then w+u must also be in the span(u,v).

True, but this is not what this statement says.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True. In order for {u,v,w} to be linearly independent, it must have a leading one in every column. This means {u,v} also has a leading one in every column, so {u,v} is linearly independent.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\{Au, Av\}$ linearly dependent

False. ✓ Multiplying matrix A by vectors u and v does not necessarily preserve linear dependence.

Why? example?

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in $\mathrm{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\mathrm{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

True. ✓ If vector $u \in \mathrm{span}(v, w)$ and transformation T is applied to u, v, w then $T(u) \in \mathrm{span}(T(v), T(w))$.

why?

0 1 2 3 4 5 6 7 8 9 10