MTH-309T-F19-EX1-071-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	4	5	4	2	4
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

(a)

$$C_1V_1 + C_2V_2 + C_3V_3 = C_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + C_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + C_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 \\ 0 \\ 2C_1 \end{bmatrix} + \begin{bmatrix} -C_2 \\ C_2 \\ -3C_2 \end{bmatrix} + \begin{bmatrix} C_3 \\ 2C_3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 - C_2 + C_3 \\ C_2 + 2C_3 \\ 2C_1 - 3C_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$C_1 - C_2 + C_3 = -2$

$C_2 + 2C_3 = 2$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & \end{array} \\ \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix} \end{array}$$

$C_1 + 3C_3 = 0$

$C_2 + 2C_3 = 2$

$$\begin{bmatrix} -3C_3 \\ 2-2C_3 \\ C_3 \end{bmatrix} \Rightarrow 2C_1 - 3C_2 = -6C_3 - 3(2-2C_3)$$

$$= -6C_3 - 6 + 6C_3 = -6$$

$b = -6$

(b).

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\downarrow$ infinite

no. it is dependent

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] = \left[\begin{array}{cccccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad (A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & -1 & -1 & 1 & -2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{bmatrix} \quad \begin{bmatrix} -8x_4 - 5x_5 \\ 3 + 7x_4 + 3x_5 \\ 2 - 6x_4 - 5x_5 \\ x_4 \\ x_5 \end{bmatrix} \quad \begin{aligned} x_3 + 6x_4 + 5x_5 &= 2 \\ x_2 - 7x_4 - 3x_5 &= 3 \\ x_1 + 8x_4 + 5x_5 &= 0 \end{aligned}$$

$$(A^T)^{-1} A^T C = (A^T)^{-1} B \qquad (A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$(A^T)^{-1} B = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = (T(e_1), T(e_2))$ $\quad T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $T(u) = Au = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & -3 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ $\quad \begin{array}{l} x_2 = 3 \\ x_1 = 7 \end{array}$ $\quad \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes. it is one to one.

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

no. one to one.

$T_A(V_1) = AV_1 \quad T_A(V_2) = AV_2$

$AV_1 = AV_2$

$V_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$

$$AV_1 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$AV_2 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$T_A(V_1) = T_A(V_2)$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$$\begin{array}{ccc} u & v & w \end{array}$$
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \quad u+w = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$$

True

$w+u \in \text{Span}(u,v)$. suppose that $w+u = a\,u + bv$ a.b are sclar.

$$w = (a-1)u + bv$$

$$w \in \text{Span}(u\,v)$$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

if it is linearly independent.
they must have pivot in every column.
it also in $\mathbb{R}^3$. so the reduce form should be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
so the set $\{u.v\}$ must be linerly independent.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False.

Au will be 2×1 Av will be 2×1.

so Au Av will be 2×2.

only infinate number or no solution will be dependent

$A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $Av = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 11 & 18 \\ 0 & -5 \end{bmatrix}$ dependent.

$u = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $Au = \begin{bmatrix} 18 \\ -5 \end{bmatrix}$

$u, v = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to$ independ

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True.

u can be represent by v and w.

$u = av + bw$ $T(u) = Au = A(av + bw)$

$= aAv + bAw$

$= aT(v) + bT(w)$

so Tu must be in Span $(T(v), T(w))$

0 1 2 3 4 5 6 7 8 9 10