MTH 309T LINEAR ALGEBRA
EXAM 1
October 3, 2019

Name:

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UB Person Number:

5	0	2	5	0	9	7	2

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.

- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

4	10	10	0	3	7	4	38	nan
1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & b \end{array}\right] \xrightarrow{\cdot(-2)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 5 & 0 & b-4 \end{array}\right] \xrightarrow{\cdot(-5)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -10 & b-14 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{b-14}{-10} \end{array}\right]$$

$$b = 4$$

$Ax = b \rightarrow Av = b$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = w$

b) a set is linearly independent if it has only one solution. Since there is a pivot position in every column it is linearly independent

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

let

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} A & | & e_1 e_2 e_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 & 2 & 0 \\ 0 & 1 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & 1 & | & 2 & -1 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$?\;\; \underbrace{A^T C = B}$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$(AB)^T = B^T A^T$

$$C = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

This is good!

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T(c_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

$[Te_1][Te_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} C = \begin{bmatrix} 1 \\ 10 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 20 \\ 4 \end{bmatrix}$

b) $T(u) = c_1 x_1 + c_2 x_2$

$u = c_1 e_1 + c_2 e_2$

$T(u) = T(c_1 e_1) + T(c_2 e_2)$

$= c_1 T(e_1) + c_2 T(e_2)$

$= [T(e_1) T(e_2)] \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A \cdot U$

$\begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \end{bmatrix}$

$\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A \cdot U$

$U = \text{Span} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

one-to-one pivot
position in every column

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

↑ no pivot position

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Pivot position in every column

One-to-one ✓

$T_A(v_1) = T_A(v_2)$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$T_A(v) = Av$

$T_A\left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}\right) \qquad T\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\right)$

$c\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

False

$w + u \in \text{span}(u, v)$

$w \in \text{Span}(u, v)$
$w = cu + cv$

$w + u \in \text{Span}(u, v)$
$w + u = cu + cv$

$A(v + w) = Av + Aw$

$w + u \in \text{Span}(u, v) \neq w \in \text{Span}(u, v)$

·Counter example

$w \in \text{Span}(u, v) + u \notin \text{Span}(u, v)$

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

True

$\{u, v, w\}$ is linearly independent, ∴ every column is a pivot column and there is only one solution

$\{u, v\}$ must also be linearly independent because without w there will still be a pivot column in every column

ex) $\{u, v, w\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
pivot columns

$\{u, v\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
pivot columns

MTH-309T-F19-EX1-002-P07

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and u, v are vectors in $\mathbb{R}^2$ such that Au, Av are linearly dependent then u, v also must be linearly dependent.

True - Linearly dependent sets have infinitely many solutions a set of two vectors is linearly independent if and only if one vector is a scalar multiple of the other. Since $A(v+u) = Au + Av$, if Au and Av are linearly dependent u, v must also be linearly dependent.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in Span(v, w) then $T(u)$ must be in Span$(T(v), T(w))$.

True!

$T(u+v) = T(u) + T(v)$
$T(cv) = cT(v)$

$u \in$ Span(v, w)

$T(u) \in$ Span$(T(v), T(w))$?

Span holds though transformations.

$T \cdot S(v) = (A \cdot B)v = A(B(v))$
$col(A) = row(B)$