MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Emery Comstock

UB Person Number:

5	0	2	5	2	7	7	2
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	TOTAL	GRADE
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1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathrm{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathrm{w} \in \mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$.

b) Is the set $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ linearly independent? Justify your answer.

$$\left(\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = W \right) = \begin{array}{cccc} x_1 & x_2 & x_3 & c \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] -2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & \end{array}\right] \begin{matrix} 2 \\ 1 \end{matrix} \quad \cancel{\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 12 \end{array}\right]} \quad \cancel{\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 12 \end{array}\right]} \quad \cancel{x_1 = -3x_3 \quad x_2 = 2 - 2x_3 \quad x_3 = x_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2+b \end{array}\right] \qquad \begin{aligned} x_1 &= -3x_3 \qquad x_3 = x_3 \\ x_2 &= 2 - 2x_3 \\ 0 &= 2+b \\ b &= -2 \end{aligned}$$

a) $\boxed{b = -2}$

b) Linearly $\boxed{\text{dependent}}$ because x_3 is a free variable in the matrix $[x_1V_1 \ x_2V_2 \ x_3V_3]$ resulting in a null space with infinite solutions

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$m = n \checkmark$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] -1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] -2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] 1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] 1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] -1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} = B \qquad \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix}$$

$C_{11} = 0.5 \quad 4 = 2c_{12} - c_{12} \quad 4 = c_{12} \quad 3 = 2c_{13} + \cancel{c_{13}} - c_{13}$

$\frac{3}{2} = c_{13}$

$$C_1 = \begin{bmatrix} 0.5 \\ 4 \\ 1.5 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix}$$

$2 = 2c_{21} \quad c_{21} = 1$

$5 = 1c_{22} \quad c_{22} = 5$

$2 = 2c_{23} \quad c_{23} = 1$

$$C_2 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_{31} \\ c_{32} \\ c_{33} \end{bmatrix}$$

$3 = 2c_{31} \quad c_{31} = 1.5$

$4 = 1c_{32} \quad c_{32} = 4$

$1 = 2c_{33} \quad c_{33} = 0.5$

$$C_3 = \begin{bmatrix} 1.5 \\ 4 \\ 0.5 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 0.5 & 1 & 1.5 \\ 4 & 5 & 4 \\ 1.5 & 1 & 0.5 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

Standard Matrix of $T = \left[T(e_1) \; T(e_2) \; [redacted] \right]$

$$\left[T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-2(0) \\ 1+\cancel{0} \\ 1-3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-2(1) \\ 0+1 \\ 0-3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \right]$$

a) $\boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

b) $\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 11 \\ 0 & -1 & -3 \end{bmatrix}$ (-1)

$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 11 \end{bmatrix} \; 2-3 \; \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

b) $\boxed{\text{No vectors}}$ satisfy that condition

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} -3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} (\tfrac{1}{2})$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} 2 -1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} (\tfrac{1}{2})$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{\text{a) One to one}}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} -3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} (\tfrac{1}{2})$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} 2 -1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Not one to one

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] -1$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 2x_3$
$x_2 = -2x_3$
$x_3 = x_3$

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x_3$

b) Not one to one

$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True, vectors encompassed by a ~~elements of a~~ span are linear combinations of the vectors in that span, so the linear combination of some vector and a vector in some span can only be encompassed by that span if the other vector is also a linear combination of the vectors in the span.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~False~~

$c_1 U + c_2 V + c_3 W = 0$

$c_1 U + c_2 V = 0$

~~False~~

True, because if $\{U, V\}$ is not linearly dependent then $\{U, V, W\}$ cannot be linearly independent as multiplying W by a constant of zero would yield all solutions of $\{U, V\}$

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\boxed{\text{True,}}$ Au and Av can be seen as ~~linear~~ ~~linear transformations where $T_A(u)=Au$ and $T_A(v)=Av$ and $T(u)+T(v)$ [illegible]~~ ~~So if~~ multiplying by the same matrix causes a linear transformation on the solutions but will not modify the amount of solutions

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

$\boxed{\text{True,}}$ because ~~linear transformations ensure that the transformation of the sum of 2 vectors~~

$\text{Span}(V,W) = C_1V + C_2W$

if: $U = C_1V + C_2W$

then:

$T(U) = T(C_1V + C_2W) =$

✓ $T(U) = T(C_1V) + T(C_2W)$

$\text{Span}(T(V), T(W)) = C_1T(V) + C_2T(W) = T(C_1V) + T(C_2W)$

~~[illegible]~~

0 1 2 3 4 5 6 7 8 9 10