

1. Consider the following vectors in  $\mathbb{R}^4$ :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set  $\mathcal{B} = \{v_1, v_2, v_3\}$  is a basis of some subspace  $V$  of  $\mathbb{R}^4$ .

a) Find an orthogonal basis  $\mathcal{D} = \{w_1, w_2, w_3\}$  of the subspace  $V$ .

b) Compute the vector  $\text{proj}_V u$ , the orthogonal projection of  $u$  on  $V$ .

$$w_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$w_2 = v_2 - \left( \frac{w_1 \cdot v_2}{w_1 \cdot w_1} \right) w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{(1 \cdot 2 + 0 \cdot 1 + (-1) \cdot (-1) + 1 \cdot 0)}{(1 \cdot 1 + 0 \cdot 0 + (-1) \cdot (-1) + 1 \cdot 1)} w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \checkmark$$

$$w_3 = v_3 - \left( \frac{w_1 \cdot v_3}{w_1 \cdot w_1} \right) w_1 - \left( \frac{w_2 \cdot v_3}{w_2 \cdot w_2} \right) w_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{(2 + 0 + 1 + 3)}{(1 + 0 + 1 + 1)} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{(2 - 2 + 0 - 3)}{(1 + 1 + 0 + 1)} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \checkmark$$

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \checkmark$$

$$\text{proj}_V u = \left( \frac{u \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \left( \frac{u \cdot w_2}{w_2 \cdot w_2} \right) w_2 + \left( \frac{u \cdot w_3}{w_3 \cdot w_3} \right) w_3 = \frac{(3 + 0 - 3 + 3)}{(1 + 0 + 1 + 1)} w_1 + \frac{(3 + 3 + 0 - 3)}{(1 + 1 + 0 + 1)} w_2 + \frac{(3 - 3 + 3 + 0)}{(1 + 1 + 1 + 0)} w_3$$

$$= w_1 + w_2 + w_3 = \begin{bmatrix} 1 + 3 + 1 \\ 0 + 3 - 1 \\ -1 + 0 + 1 \\ 1 + 3 + 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 4 \end{bmatrix}$$

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