MTH-309T-F19-EX1-030-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	3	1	7	4	5	0

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathrm{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathrm{w} \in \mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$.

b) Is the set $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ linearly independent? Justify your answer.

(a) W is a linear combination of V_1, V_2, V_3

$$W = X_1V_1 + X_2V_2 + X_3V_3$$

$$= X_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + X_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + X_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} X_1 - X_2 + X_3 \\ 0 + X_2 + 2X_3 \\ 2X_1 - 3X_2 + 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$\Rightarrow$ augmented matrix

$$\begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \qquad\qquad\qquad \begin{array}{ccc} X_1 & X_2 & X_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{\text{Row Reduction}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

X_3 : free variable

Sol
$$\begin{cases} X_1 = -3X_3 \\ X_2 = -2X_3 + 2 \\ X_3 = X_3 \end{cases}$$

$$\therefore b = 2X_1 - 3X_2$$
$$= -6X_3 + 6X_3 - 6$$
$$= -6$$

$$\boxed{b = -6}$$

(b) $[\,V_1 \;\; V_2 \;\; V_3 \mid 0\,]$

$$\begin{array}{ccc} & & X_1 \;\; X_2 \;\; X_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable

$\Rightarrow$ X_3 is a free variable, so $X_1V_1 + X_2V_2 + X_3V_3 = 0$ has infinetely many solutions.

$$\begin{cases} X_1 = -3X_3 \\ X_2 = -2X_3 \\ X_3 = X_3 \end{cases}$$

$\Rightarrow \{V_1, V_2, V_3\}$ is linearly dependent.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 + 2 \cdot R_2]{(-1) \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right].$$

$$\xrightarrow[R_2 + R_3]{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] = [I_3 \mid A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A^T C = B$

$\Leftrightarrow (A^T)^{-1} \cdot A^T \cdot C = (A^T)^{-1} \cdot B$ $\quad$ ⌜$\because (A^T)^{-1} \cdot A^T = I$⌟

$\Leftrightarrow C = (A^T)^{-1} \cdot B$

$\Leftrightarrow C = (A^{-1})^T \cdot B$ $\quad$ ⌜$\because (A^T)^{-1} = (A^{-1})^T$⌟

$$\therefore C = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+4+6 & -4+5+4 & -6+4+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ -1+4+3 & -2+5+2 & -3+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find $\overset{A}{\underset{\|}{\text{the standard matrix of } T}}$.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) A is a 3X2 matrix.

$A = [T(e_1) \;\; T(e_2)]$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\therefore\ A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ is the standard matrix of T.

(b) $T(u) = T_A(u) = A \cdot u = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$, $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

augmented matrix

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

Sol

$$\begin{cases} x_1 = 7 \\ x_2 = 3 \end{cases}$$

$\therefore\ u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

If T_A is one-to-one, A has a pivot position in every column.

$\Leftrightarrow \text{Nul}(A) = \{0\}$

(a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \xrightarrow[R_3-3R_1]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$

$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$

$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\therefore$ A has a pivot position in every column,

So $\underline{T_A(v) \text{ is one-to-one}}$.

(b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \xrightarrow[R_3-3R_1]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$\xrightarrow{R_3-R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \quad\quad \text{free} \end{array}$

$\therefore$ A doesn't have a pivot position in 3rd column,

So $\underline{T_A(v) \text{ is not one-to-one}}$.

sol $\begin{cases} x_1 = -x_2 = 2x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases} \to x = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x_3 \to \text{Nul}(A) = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

• $T_A(v_1) = T_A(v_2)$ if and only if $v_1 - v_2 \in \text{Nul}(A)$

if, $v_1 = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then

$T_A(v_1) = T_A(v_2)$. $\because$ $v_1 - v_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \in \text{Nul}(A)$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

$w+u$ is a linear combination of u, v.

$\Leftrightarrow$ $w+u = C_1 u + C_2 v$

$w = (C_1 - 1)u + C_2 v$ is a linear combination of u, v.

$\therefore$ $w \in \mathrm{Span}(u, v)$

True

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

① $x_1 u + x_2 v + x_3 w = 0$ has only one and trivial solution $x_1 = x_2 = x_3 = 0$.

Then, $x_1 u + x_2 v = 0$ also has only one and trivial solution $x_1 = x_2 = 0$.

② $A = [u \ v \ w]$

A has a pivot position in every column.

$B = [u \ v]$

Then, B also has a pivot position in every column.

True

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then

$Au = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $Av = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\rightarrow$ $Au = Av$, so Au, Av are linearly dependent

but u and v are linearly Independent.

F: $[u \ v \mid 0]$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow[\text{reduction}]{\text{Row}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

False

So $x_1u + x_2v = 0$ has only one solution $x_1 = x_2 = 0$ $\Leftrightarrow$ u, v are linearly independent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in $\mathrm{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\mathrm{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

$u = c_1 v + c_2 w$

$T(u) = T(c_1 v + c_2 w)$

$= T(c_1 v) + T(c_2 w)$

$= c_1 \cdot T(v) + c_2 \cdot T(w)$ is a linear combination of $T(v), T(w)$

$\therefore$ $T(u) \in \mathrm{Span}(T(v), T(w))$ True

0 1 2 3 4 5 6 7 8 9 10