MTH-309T-F19-EX1-034-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Justin W. Allen

UB Person Number:

5	0	2	9	4	9	4	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\begin{array}{cccc} v_1 & v_2 & v_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_1} \begin{array}{cccc} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & (4+b) \end{array}$$

$$R_1 = R_1 + R_2 \quad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & (4+b) \end{bmatrix} \quad R_3 = R_3 + R_2 \quad \begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & (6+b) \end{vmatrix}$$

a. $w \in \text{Span}(v_1, v_2, v_3)$ when $b = -6$

b. $\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{vmatrix} \longrightarrow$ $1 \quad -1$

The set is not linearly independent, because the V_3 column is a free ~~fa~~ variable, so there is not a leading one in each column

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}$$

$R_1 = R_1 + R_2$

$$\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \xrightarrow{R_3 = R_3 - 2R_2} \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}$$

$R_2 = R_2 + R_3$

$$\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}$$

$R_1 = R_1 - R_3$

$$\left|\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right|$$

$$\boxed{A^{-1} = \begin{vmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix}}$$

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \qquad \left|\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right|$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right] \qquad \begin{array}{cccccc} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}$$

$$\left.\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right. \qquad (A^T)^{-1} = \begin{vmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{vmatrix}$$

$$(A^T)^{-1} A^T C = (A^T)^{-1} B \qquad \begin{vmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$$

$(-2 + 4 + 6)$ $(-4 + 5 \; 4)$ $(-6 + 4 + 2)$

$(3 + -4 - 6)$ $(6 - 5 - 4)$ $(9 - 4 - 2)$

$(-1 + 4 + 3)$ $(2 + 5 + 2)$ $(-3 + 4 + 1)$

$$\begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 9 & 2 \end{bmatrix}$$

$$\boxed{C = \begin{vmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 9 & 2 \end{vmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

1 0

0 1

0 0

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 & -2x_2 \\ x_1 & +x_2 \\ x_1 & -3x_2 \end{bmatrix}$ $\quad x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{array}{cc} T(e_1) & T(e_2) \end{array}$$
$$\begin{vmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{vmatrix} \qquad \begin{vmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{vmatrix}$$

$$A = \begin{vmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{vmatrix} \qquad \begin{matrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{matrix} \qquad \begin{matrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{matrix}$$

$$\begin{matrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{matrix} \qquad \begin{matrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{matrix}$$

$x_1 = 7$

$x_2 = 3$

$$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$　　b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{matrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{matrix} \quad \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{matrix} \quad \begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{matrix}$$

$$\begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix} \quad \begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix}$$

A is one-to one, because there is a leading one in each column.

b. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \quad \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{matrix} \quad \begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix}$

The matrix is not one-to-one, because the third column does not have a leading one.

$T_A(V_1) = T_A(V_2) \qquad T_A(V_1 - V_2) = 0$

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{matrix} \quad \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \quad \begin{matrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$x_1 - 2x_3 = 0$　　$x_1 = 2x_3$

$x_2 + 2x_3 = 0$　　$x_2 = -2x_3$

$x_3 = x_3$　　$x_3 = x_3$

x_3 Nul $A \in$ Span $\left(\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}\right)$

$x_3 \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$

$$\boxed{V_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \qquad V_2 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

This statement is false, because $w + u$ will only span (u, v) if $w + u$ is in the null set of (u, v)

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$~~

This statement is true.

If $\{u, v, w\}$ is linearly independent then ~~th~~ each column in the aug matrix must have a leading one. If you remove one vector (column), each column will still have a leading one

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

This statement is false

$$\left|\begin{matrix} 1 & 1 & 2 \\ 0 & 2 & 4 \end{matrix}\right| \qquad \left|\begin{matrix} 1 & 1 & 2 \\ 0 & 2 & 9 \end{matrix}\right|$$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

T(u) must be in Span(T(v), T(w)), be

0 1 2 3 4 5 6 7 8 9 10