MTH-309T-F19-EX1-037-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

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**Name:**

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**UB Person Number:**

| 5 | 0 | 2 | 6 | 4 | 1 | 5 | 8 |
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**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
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| 1 | 2 | 3 | 4 | 5 | 6 | 7 | PIAZZA | HILL | TOTAL | GRADE |
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| 8 | 6 | 5 | 15 | 12 | 2 | 2 | -- | -- | 50 | D+ |

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

**a)** Find all values of $b$ such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

**b)** Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] R3 \cdot R2 \; ? \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & 0 & 2b \end{array}\right] R3 - 3 \cdot R2$$

Row multiplication is not an elementary row operation!

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -6 & 2b-6 \end{array}\right] \cdot -\frac{1}{6} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -\frac{1}{3}b+1 \end{array}\right] R1 + R2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -\frac{1}{3}b+1 \end{array}\right] -3 \cdot R3 \quad 3(-\tfrac{1}{3}b+1) = -b+3 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -b+3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -\frac{1}{3}b+1 \end{array}\right] R2 - 2 \cdot R3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -b+3 \\ 0 & 1 & 0 & \frac{2}{3}b \\ 0 & 0 & 1 & -\frac{1}{3}b \end{array}\right]$$

$x_1 = -b+3$
$x_2 = \frac{2}{3}b$
$x_3 = -\frac{1}{3}b$

?

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] R3 + 3 \cdot R2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_2 - x_3$ (−2)
$x_2 = -3x_3$

$x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

It is not linearly independent bc there are infinite solutions ✓

0 1 2 3 4 5 6 7 ~~8~~ 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] R2 \cdot \tfrac{1}{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] R3 \cdot -2 \cdot R2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] + \tfrac{1}{2}R3$$

?

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] R1 + R2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] R1 - 2R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \quad A^{-1} = \cancel{\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}}$$

0 1 2 3 4 5 **6** 7 8 9 10

3. **(10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

✓

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}^{-1}$$

simpler: $(A^T)^{-1} = (A^{-1})^T$, then use problem 2.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 2 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R2+2R3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \end{array}\right] +R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R3-R2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 \end{array}\right] \cdot -\tfrac{1}{2} \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{array}\right] -4 \cdot R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{array}\right] -R2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 0 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{array}\right]$$

$$\tfrac{1}{2}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \quad C = \cancel{\begin{bmatrix} -1 & 4 & 0 \\ -4 & 10 & 0 \\ -3 & 4 & 0 \end{bmatrix}}$$

$$C = \boxed{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}} \cdot \begin{bmatrix} 3 & -4 & 0 \\ -2 & 4 & 0 \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{bmatrix}$$

$$3\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + -4\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix}$$

$$-2\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + 0 = \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of $T$.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Standard matrix $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 0 & -3 \end{bmatrix}$ ✓

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -3 & -2 \end{array}\right] \checkmark \quad R2 \cdot -\tfrac{1}{3}R3 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 20/3 \\ 0 & -3 & -2 \end{array}\right] R3+3R2$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 20/3 \\ 0 & 0 & 18 \end{array}\right] R1+2R2 \quad \left[\begin{array}{cc|c} 1 & 0 & 10/3 \\ 0 & 1 & 20/3 \\ 0 & 0 & 18 \end{array}\right]$$

$$u = \cancel{\begin{bmatrix} 10/3 \\ 20/3 \\ 18 \end{bmatrix}} \qquad u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

This would mean: no solutions

| | | | | | | | | | | | | | | | | | | | | |
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| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

**a)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right) \cdot \frac{1}{2} \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right) R_3 \cdot R_2 \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right) R_3 - \tfrac{1}{2}R$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \cancel{2}\ 1 & 0 \end{array}\right)$$

a) ~~not~~ one to one ← if not one-to-one, then what are $v_1$ and $v_2$?!

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right) \cdot \frac{1}{2} \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right) R3 - 4 \cdot R2 \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & -2 & 6 \end{array}\right)$$

R3 · R2

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \cdot -\frac{1}{4} \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & \cancel{1}\ 0 & 0 \end{array}\right)$$

b) one ~~to~~ one ✓

| | | | | | | | | | | | | | | | | | | | | |
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| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

✓

True, Vo tn is in the span

?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~false~~, $\{u, v\}$ could have a differenti solution with nothing trivial

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| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

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**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2\times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~false~~ false ✓, u, v can be linearly independent before being multiplied by A.

why?

**b)** If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T$(u) must be in Span($T$(v), $T$(w)).

True ✓, T(u) is in span(T(v), T(w)) if u is in span(v, w)

why?

0 1 2 3 4 5 6 7 8 9 10