MTH-309T-F19-EX1-065-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Bella Esposito

UB Person Number:

5	0	2	2	4	3	9	8
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
15	9	10	20	13	3	2	2	10	81	B

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$2v_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} \xrightarrow{b=-6} w = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$$

$$\boxed{b=-6}$$

How do you know that there are no other values of b which work?

$$v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad -3v_1 = \begin{bmatrix} -3 \\ 0 \\ -6 \end{bmatrix}$$

$$-2v_1 = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{(-2)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_1 = R_2 + R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 = R_2 + R_3} \begin{bmatrix} 1 & 0 & \boxed{3} \\ 0 & 1 & \boxed{2} \\ 0 & 0 & \boxed{0} \end{bmatrix}$

this is not a pivot column!

~~yes,~~ linearly independent because every col. is a pivot column

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R(2)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & +1 \end{array}\right] \rightarrow R_1 = R_1 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & -1 \\ 2 & -2 & +1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A=\begin{bmatrix}1&-1&2\\1&0&1\\0&2&-1\end{bmatrix} \qquad B=\begin{bmatrix}1&2&3\\4&5&4\\3&2&1\end{bmatrix} \qquad A^T=\begin{bmatrix}1&1&0\\-1&0&2\\2&1&-1\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-2&3&1\\1&-1&-1\\2&-2&-1\end{bmatrix} \qquad (A^{-1})^T=\begin{bmatrix}-2&1&2\\3&-1&-2\\1&-1&-1\end{bmatrix}$$

Find a matrix C such that $A^TC=B$ (where A^T is the transpose of A).

$$(A^T)^{-1}=(A^{-1})^T$$

$$A^T C = B$$

$$\begin{bmatrix}1&1&0\\-1&0&2\\2&1&-1\end{bmatrix} C = \begin{bmatrix}1&2&3\\4&5&4\\3&2&1\end{bmatrix}$$

$$(A^T)^{-1}A^T C = (A^T)^{-1}B$$

$$C=(A^T)^{-1}B = \begin{bmatrix}-2&1&2\\3&-1&-2\\1&-1&-1\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&4\\3&2&1\end{bmatrix} = \begin{bmatrix}-2+4+6&-4+5+4&-6+4+2\\3-4-6&6-5-4&9-4-2\\1-4-3&2-5-2&3-4-1\end{bmatrix}$$

$$\boxed{C=\begin{bmatrix}8&5&0\\-7&-3&+3\\6&+5&+2\end{bmatrix}}$$

check $A^TC=B$

$$\begin{bmatrix}1&1&0\\-1&0&2\\2&1&-1\end{bmatrix}\begin{bmatrix}8&5&0\\-7&-3&+3\\6&+5&7\end{bmatrix} = \begin{bmatrix}8-7+0&5-3+0&0+3+0\\-8+0+12&-5+0+10&4\\16-7-6&10-3-5&3-2\end{bmatrix} = \begin{bmatrix}1&2&3\\4&5&4\\3&2&1\end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{-2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{array}\right] \Big\} \begin{array}{l} x_1 = 7 \\ x_2 = 3 \end{array}$

$T\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7-6 \\ 7+3 \\ 7-9 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array}\right]$ -3, -1

$A = \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{array}\right]$

yes one to one
pivot position
in every col.

✓

b) $A = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array}\right]$ -3 $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array}\right]$

~~yes~~

pivot position
in every col.

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

no pivot !

so not one-to-one

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

$w \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{w} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{u} \in \mathrm{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

~~false~~

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \mathrm{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathrm{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True, ✓ if u, v, w are all lin. independent, u, v must be linearly independent

$\underbrace{u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{all 3 lin. ind.}}$

$u, v \Rightarrow$ lin. ind.

← an example does not show that this is always true.

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0	1	2	3	4	5	6	7	8	9	10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~True~~, since multiplying by the same matrix give linear dependence

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True ✓ since every matrix transformation is a linear transformation

so ?

0 1 ~~2~~ 3 4 5 6 7 8 9 10