MTH-309T-F19-EX1-044-P00

MTH 309T LINEAR ALGEBRA
EXAM 1
October 3, 2019

Name:

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UB Person Number:

5 0 2 2 7 3 4 2

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
9	3	6	12	16	6	2	--	--	54	D+

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) -6 ? ← where did this come from?

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & 6 \end{array}\right] -2 \qquad \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & \cancel{5} & 0 & -12 \end{array}\right] -5 \qquad \cancel{\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 60 \end{array}\right]}$$

-1 ?

This does not come from the preceding matrix

$$\cancel{\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 60 \end{array}\right]}$$

free variable

b) No Because the set $\{V_1, V_2, V_3\}$ when ~~row redu~~ in reduced row echelon form dosent have a pivot column for every column and a pivot position in every row. ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] R_2 = -R_1 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & \cancel{0} & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \leftarrow -1 \quad R_3 = 2R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \cancel{0} & \cancel{0} & 1 \end{array}\right] \quad \begin{array}{c} \\ \\ 2 \;\; -2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_2 = R_3 + R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_1 = -R_3 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\boxed{\cancel{A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$

You need to perform reduction of the whole matrix, not just of its left hand side.

☐	☐	☐	☒	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$3\times 3 \cdot 3\times 3 = 3\times 3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] R_2 = R_2 + R_1 \qquad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 5 & 7 & 7 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] R_3 = -2R_1 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 5 & 7 & 7 \\ 0 & -1 & -1 & 1 & -2 & -5 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & -1 & -1 & 1 & -2 & -5 \\ 0 & 0 & 2 & 5 & 7 & 7 \end{array}\right] R_1 = -R_2 + R_1$$

This would work, if done correctly.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 4 & 8 \\ 0 & -1 & -1 & 1 & -2 & -5 \\ 0 & 0 & 2 & 5 & 7 & 7 \end{array}\right] R_2 = -R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 4 & 8 \\ 0 & 1 & 1 & 1 & -2 & -5 \\ 0 & 0 & 2 & 5 & 7 & 7 \end{array}\right] R_3 = \tfrac{1}{2} \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 4 & 8 \\ 0 & 1 & 1 & 1 & -2 & -5 \\ 0 & 0 & 1 & 5/2 & 7/2 & 7/2 \end{array}\right] R_2 = -R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 4 & 8 \\ 0 & 1 & 0 & -3/2 & -11/2 & -17/2 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} \end{array}\right] \qquad C = \begin{bmatrix} 0 & 4 & 8 \\ -\frac{3}{2} & -\frac{11}{2} & -\frac{17}{2} \\ \frac{5}{2} & \frac{7}{2} & \frac{7}{2} \end{bmatrix}$$

$\frac{2}{2} - \frac{5}{2} = -\frac{3}{2}$

$-\frac{4}{2} - \frac{7}{2}$

Simpler: $C = (A^T)^{-1} \cdot B$

$= (A^{-1})^T \cdot B$

then use A^{-1} from problem 2.

Score: 0 1 2 3 4 5 **6** 7 8 9 10

4. **(20 points)** Let $T\colon \overset{n}{\mathbb{R}^2} \to \overset{m}{\mathbb{R}^3}$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $m \times n$ matrix

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \qquad \begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array} \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix}$$

free

b) $x_1 - 2x_2$
$x_1 + x_2$
$x_1 - 3x_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 10 \\ 1 & -3 & 0 & -2 \end{array}\right]$$

$u = ?$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐	☐	☐	☐

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot -3$ $\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{matrix} \cdot -\frac{1}{2}$ $\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{matrix} \cdot 1/2$ $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix} \cdot 1/2$ $\rightarrow$ $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix} \cdot -1$ $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix} \cdot -1$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

T_A is One to one b/c there is a piv position in every column of A

✓

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot -3$ $\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{matrix} \cdot -\frac{1}{2}$ $\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{matrix} \cdot 1/2$ $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix} \cdot -1$ $\begin{array}{ccc} x_1 & x_2 & x_3 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}$ free

Not One to one ✓

$x_1 = 0$
$x_2 = -2x_3$
$x_3 = x_3$

$x_3 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 11 \end{bmatrix}$

$3 \times 3 \cdot 3 \times 1$

$V_2 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

$T_A(v_1) \neq T_A(v_2)$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{matrix} 1 \\ 1 \\ 4 \end{matrix} = \begin{bmatrix} 2 \\ 18 \\ 15 \end{bmatrix}$

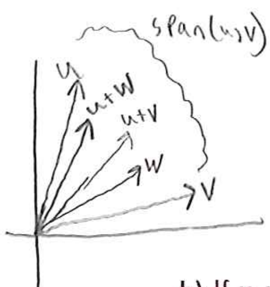
7+6

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True Because Span(u,v) is the span of all vectors inbetween vector u and vector v. In $\mathbb{R}^2$ the graph below explains this concept which still holds true for an extra dimension. To scale this problem up a dimension we could just make the 3rd row of values in u,v,w = 1



therefore if
w+u ∈ Span(u,v)
then w ∈ Span(u,v)

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True because in order for {u v w} to be linear independent every column must be a pivot column therefore the subset {u,v} must also have every column be a pivot column.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$$\cancel{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \cdot \begin{matrix} 3 \\ 4 \end{matrix} = \begin{matrix} 3 \\ 4 \end{matrix}}$$

False ✓

linear dept

$$\begin{matrix} -1 & 1 \\ 1 & -1 \end{matrix} \cdot \begin{matrix} -4 \\ -4 \end{matrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leftarrow \text{linear dependent}$$

But $\begin{bmatrix} -4 \\ -4 \end{bmatrix}$ = is not linear dep b/c $u \neq 0$

What are A, u, and v here?

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

~~False~~

$$\{T(v_1), T(v_2)\} \neq T(v_1)\ T(v_2)$$

?

$$\begin{matrix} 4 \\ 2 \end{matrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{matrix} 8 \\ 2 \end{matrix}$$

0 1 2 3 4 5 6 7 8 9 10