MTH-309T-F19-EX1-022-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	2	8	4	5	4

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathrm{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathrm{w} \in \mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$. b = any number but 4

b) Is the set $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & b \end{array}\right] \times 2 \quad 3-2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 5 & 0 & b-4 \end{array}\right] \times 5$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -10 & 10(b-4) \end{array}\right] \times -\frac{1}{10}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \underbrace{-(b-4)}_{\downarrow\ -b+4} \end{array}\right]$$

$-b+4=0$

$-b=-4$

$b=4$

$\therefore$ If $b=4$, then $W \notin \mathrm{Span}(V_1, V_2, V_3)$

The set $\{V_1, V_2, V_3\}$ is linearly independent because each vector is not a scalar multiple of each other

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \times 1$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \times 2$$

$-1 \cdot 2 = -2$

$1 - -2 = 3$

$-1 \cdot 2 = -2$

$0 - -2 = 2$

$1 \cdot 2 = 2$

$0 - 2 = -2$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & -2 & 1 \end{array}\right] \times \frac{1}{3}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] -1$$

$\frac{2}{3} \cdot {}^{+}1 = +\frac{2}{3}$

$-\frac{3}{3} + +\frac{2}{3} = -\frac{1}{3}$

$-\frac{3}{3} + -\frac{2}{3} = -\frac{5}{3}$

$\frac{3}{3} - \frac{4}{3} = -\frac{1}{3}$

$0 - -\frac{4}{3} = \frac{4}{3}$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] \times 2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -\frac{1}{3} & \frac{4}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] \times -1$$

$\frac{1}{3} \times -1 = -\frac{1}{3}$

$-\frac{1}{3} + +\frac{1}{3} = 0$

$\frac{5}{3} \times -1 = -\frac{5}{3}$

$\frac{4}{3} + -\frac{5}{3} = \frac{9}{3}$

$\frac{1}{3} \times -1 = -\frac{1}{3}$

$-\frac{2}{3} + +\frac{1}{3} = -\frac{1}{3}$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right]$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} 0 & 3 & -\frac{1}{3} \\ \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}}$$

0	1	2	3	4	5	6	7	8	9	10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \underset{3\times 3}{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}} \qquad B = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}}$$

Find a matrix C such that $A^T\boxed{C} = B$ (where A^T is the transpose of A).

$$\frac{3}{1} - \frac{4}{3} \qquad \frac{9}{3} - \frac{4}{3} \qquad \frac{3}{3} - \frac{4}{3} = \frac{-1}{3}$$

$$\frac{15}{1} - \frac{4}{3} = \frac{45}{3} - \frac{4}{3} \qquad \frac{2}{1} + \frac{10}{3}$$

$$\frac{6}{1} - \frac{1}{3} = \frac{18}{3} - \frac{1}{3} \qquad \frac{6}{3} + \frac{10}{3}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T(A^T)^{-1}C = B(A^T)^{-1}$$

$$C = B(A^T)^{-1}$$

$$(A^T)^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ 3 & \frac{5}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$B(A^T)^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ -\frac{1}{3} \end{bmatrix} = 0 + 3\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + -\frac{1}{3}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ -\frac{4}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{41}{3} \\ \frac{17}{3} \end{bmatrix}$$

$$B \cdot \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + \frac{5}{3}\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{4}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{10}{3} \\ \frac{25}{3} \\ \frac{10}{3} \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{4}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ 11 \\ \frac{10}{3} \end{bmatrix}$$

$$B \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + -\frac{2}{3}\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{8}{3} \\ \frac{2}{3} \end{bmatrix} + \begin{bmatrix} -\frac{4}{3} \\ -\frac{10}{3} \\ -\frac{4}{3} \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$\boxed{\therefore C = \begin{bmatrix} 5 & \frac{10}{3} & \frac{5}{3} \\ \frac{41}{3} & 11 & \frac{2}{3} \\ \frac{17}{3} & \frac{10}{3} & -\frac{1}{3} \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T. $\longrightarrow T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$\left.\begin{array}{l} e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \end{array}\right\} T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\vec{u})$$

$$\begin{cases} x_1 - 2x_2 = 1 \\ x_1 + x_2 = 10 \\ x_1 - 3x_2 = -2 \end{cases}$$

$\downarrow$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \times 1$$

$-2 \times 1 = -2$
$1 - -2 = 3$
$1 \times 1 = 1$
$10 - 1 = 9$

$\downarrow$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \times 1$$

$-2 \times 1 = -2$
$-3 + 2 = -1$
$1 \times 1 = 1$
$-2 + 1 = -3$

$\downarrow$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \times \frac{1}{3} \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \times -1 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{array}\right] \Rightarrow \text{Inconsistent system}$$

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14 ☐ 15 ☐ 16 ☐ 17 ☐ 18 ☐ 19 ☐ 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ $\times 3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \times 2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix} \times -\tfrac{1}{4}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ A is one-to-one because there is a pivot position in every column

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $\times 3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \times 2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ B is not one-to-one

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$. True

$$U = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad W = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$W + U = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \therefore W \in \text{Span}(U)$$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent. True. Removing a vector will still make the set linearly independent

$$U = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad V = \begin{bmatrix} 17 \\ 19 \\ 21 \end{bmatrix} \quad W = \begin{bmatrix} 33 \\ 33 \\ 33 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

$A\bar{u} \neq A\bar{v}$

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent. False

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\bar{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $A\bar{u} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

$\bar{v} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ $A\bar{v} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$). False. $T(\bar{u})$ may no longer be in Span$(T(\bar{v}), T(\bar{w}))$ after transformation

$\bar{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\bar{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\bar{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10