



# MTH 309T LINEAR ALGEBRA

## EXAM 1

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Name:

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UB Person Number:

5	0	1	7	8	1	2	7
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

14

10

4

18

17

7

6

2

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78

B

1

2

3

4

5

6

7

PIAZZA

HILL

TOTAL

GRADE



1. (20 points) Consider the following vectors in  $\mathbb{R}^3$ :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of  $b$  such that  $w \in \text{Span}(v_1, v_2, v_3)$ .

b) Is the set  $\{v_1, v_2, v_3\}$  linearly independent? Justify your answer.

a)  $w \notin \text{CV}_1$  since 2nd row of  $\text{CV}_1$  is always 0.  
 $w \notin \text{CV}_3$  since 2nd row is same for both, but first row is different.

$w \in \text{CV}_2$  when  $c = 2$  and  $b = -6$  ✓

$$w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} = 2v_2 = 2 \cdot \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$$

$$\boxed{b = -6}$$

← how do you know that this is the only value of

b) ~~Yes~~ since there is only 1 solution!  $b$  which works?

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & c_1 \\ 0 & -1 & 2 & c_2 \\ 0 & -3 & 0 & c_3 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & c_1 \\ 0 & -1 & 2 & c_2 \\ 0 & -3 & 0 & c_3 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & c_1 \\ 0 & 1 & 2 & c_2 \\ 0 & 0 & -12 & c_3 \end{array} \right]$$

$$\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ x_3 &= c_3 \end{aligned}$$



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute  $A^{-1}$ .

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \quad r_2 = r_2 - r_1$$

$$\downarrow$$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \quad r_1 = r_1 + r_2$$

$$\downarrow$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \quad r_3 = r_3 - 2r_2$$

$$\downarrow$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \quad \begin{array}{l} r_1 = r_1 - r_3 \\ r_2 = r_2 + r_3 \end{array}$$

$$\downarrow$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$A^{-1}$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \quad \checkmark$$



3. (10 points) Let  $A$  be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix  $C$  such that  $A^T C = B$  (where  $A^T$  is the transpose of  $A$ ).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$3 \times 3 = 3 \times 3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$x_1 = 1 - y_1$$

$$z_1 = 1 + \frac{x_1}{2}$$

$$z_1 = 1 + \frac{1 - y_1}{2}$$

$$x_2 = 4 - y_2$$

$$z_2 = \frac{5}{2} + \frac{x_2}{2}$$

$$x_1 + y_1 = 1$$

$$-x_1 + 2z_1 = 2$$

$$2y_1 + y_1 - z_1 = 3$$

$$x_2 + y_2 = 4$$

$$-x_2 + 2z_2 = 5$$

$$2(4 - y_2) + y_2 - z_2 = 4$$

$$x_3 + y_3 = 3$$

$$x_3 + 2z_3 = 2$$

$$16 - y_2 + 2y_2 - 9 + y_2 + 2x_3 + y_3 - z_3 = 1$$

$$y_2 = 3$$

$$x_2 = 1$$

$$z_2 = 1$$

$$x_3 = 3 - y_3$$

$$z_3 = 1 - \frac{3}{2} + \frac{y_3}{2}$$

$$1 = 2(3 - y_3) + y_3 - \left(1 - \frac{3}{2} + \frac{y_3}{2}\right)$$

$$2 = 12 - 4y_3 + 2y_3 - 2 + 3 - y_3$$

$$2 = 13 - 3y_3 = \frac{11}{3}$$

$$2(1 - y_1) + y_1 - \left(1 + \frac{1 - y_1}{2}\right) = 3$$

$$2 - 2y_1 + y_1 - 1 - \frac{1}{2} + \frac{y_1}{2} = 3$$

$$1 = 2y_1 + y_1 = 6$$

$$y_1 = -5$$

$$x_1 = 6$$

$$z_1 = 4$$

$$C = \begin{bmatrix} 6 & 3 \\ -5 & 1 \\ 4 & 1 \end{bmatrix}$$



4. (20 points) Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of  $T$ .

b) Find all vectors  $u$  satisfying  $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ .

a)  $A = 3 \times 2$  since  $3 \times 2 = 2 \times 1$

$$\begin{bmatrix} c_1 & c_4 \\ c_2 & c_5 \\ c_3 & c_6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$c_1 x_1 + c_4 x_2 = 1x_1 - 2x_2$$

$$c_2 x_1 + c_5 x_2 = 1x_1 + 1x_2$$

$$c_3 x_1 + c_6 x_2 = 1x_1 - 3x_2$$

$$c_1 = 1 \quad c_4 = -2$$

$$c_2 = 1 \quad c_5 = 1$$

$$c_3 = 1 \quad c_6 = -3$$

Standard Matrix:  $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$  ✓

b)

$$\left[ \begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array} \right] \xrightarrow{\checkmark} \left[ \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right]$$

$$\cancel{T(u) = \text{Span}\left(\left[\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right]\right)}$$

$$\left. \begin{array}{l} x_1 = 7 \\ x_2 = 3 \end{array} \right\}$$

$$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \text{ (only)}$$





5. (20 points) For each matrix  $A$  given below determine if the matrix transformation  $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T_A(v) = Av$  is one-to-one or not. If  $T_A$  is not one-to-one, find two vectors  $v_1$  and  $v_2$  such that  $T_A(v_1) = T_A(v_2)$ .

a)  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\downarrow \quad r_3 = r_3 - 3r_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All columns are  
pivot columns, therefore ✓

all  $T_A(v)$  are one-to-one.

b)  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\downarrow$  where did this  
come from? It  
should be  $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

not one to one

$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  ✓

$v_2 = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$  ✓

$$T_A(v_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

✓  $T_A(v_2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1+1 \\ 2-2 \\ -3+4-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T_A(v_1) = T_A(v_2)$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If  $u, v, w$  are vectors in  $\mathbb{R}^3$  such that  $w + u \in \text{Span}(u, v)$  then  $w \in \text{Span}(u, v)$ .

False.

$$u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$w + u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \in \text{Span}(v)$$

$\in \text{Span}(u, v)$

$$\frac{w \neq cu}{w \neq cv}, \text{ so } w \notin \text{Span}(u, v)$$

$\in$

$$w = v - u$$

b) If  $u, v, w$  are vectors in  $\mathbb{R}^3$  such that the set  $\{u, v, w\}$  is linearly independent then the set  $\{u, v\}$  must be linearly independent.

True. If set  $\{u, v, w\}$  are independent, that means  
 $\underline{u \neq cv}, \underline{u \neq cw}, \underline{v \neq cu}, \underline{v \neq cw}, \underline{w \neq cu}, \underline{w \neq cv}.$

✓

→ Show linear independence between  $u$  and  $v$ .



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If  $A$  is a  $2 \times 2$  matrix and  $u, v$  are vectors in  $\mathbb{R}^2$  such that  $Au, Av$  are linearly dependent then  $u, v$  also must be linearly dependent.

~~True~~: If  $Au$  and  $Av$  are linearly dependent, then  $Au = cAv$ . Therefore  $u = cv$ , so  $u$  and  $v$  must be linearly dependent. ↑

not necessarily:  
take  $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$   
 $Au = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $Av = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ , so  $Au = cAv$  but  $u \neq cv$ .

b) If  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear transformation and  $u, v, w \in \mathbb{R}^2$  are vectors such that  $u$  is in  $\text{Span}(v, w)$  then  $T(u)$  must be in  $\text{Span}(T(v), T(w))$ .

True. If  $u \in \text{Span}(v, w)$  then  $u = cv$  or  $u = cw$ .  
If you take  $T(u)$ , you get  $A \cdot u = A \cdot cv$ ,  
which shows  $T(u) \in \text{Span}(T(v), T(w))$ . ↑

No:  
 $u = c_1v + c_2w$   
but a similar argument still works.