MTH-309T-F19-EX1-064-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

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Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
20	9	10	18	16	5	6	--	10	92	A-

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $W \in Span(V_1, V_2, V_3)$

$W = C_1V_1 + C_2V_2 + C_3V_3$

$C_1\begin{bmatrix}1\\0\\2\end{bmatrix} + C_2\begin{bmatrix}-1\\1\\-3\end{bmatrix} + C_3\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}-2\\2\\b\end{bmatrix}$

$\begin{bmatrix}1 & -1 & 1 & -2\\0 & 1 & 2 & 2\\2 & -3 & 0 & b\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & 2 & 2\\2 & -3 & 0 & b\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & 2 & 2\\0 & -3 & -6 & b\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & 2 & 2\\0 & 1 & 2 & \frac{b}{-3}\end{bmatrix}$

$\frac{b}{-3} = 2 \quad b = -6$ so that the matrix is consistant and has solution that makes $W \in Span(V_1, V_2, V_3)$

b)
$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\begin{bmatrix}1 & -1 & 1 & 0\\0 & 1 & 2 & 0\\2 & -3 & 0 & 0\end{bmatrix}$$

$\Rightarrow \begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & 2 & 0\\2 & -3 & 0 & 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & 2 & 0\\0 & -3 & -6 & 0\end{bmatrix}$

$\begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & 2 & 0\\0 & 1 & 2 & 0\end{bmatrix}$

$X_1 + 3X_3 = 0 \qquad X_1 = -3X_3$

$X_2 + 2X_3 = 0 \qquad X_2 = -2X_3$

$X_2 + 2X_3 = 0 \qquad \cancel{X_2 = -2X_3}\ X_3 = X_3$

$X = \begin{bmatrix}-3\\-2\\1\end{bmatrix} X_3$

so that there are other ~~soth~~ solutions other that $X_1 = X_2 = X_3 = 0$

$\therefore \{V_1, V_2, V_3\}$ is linearly dependent.

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \times -2 + ③$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

$$(-1) \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} \cancel{0} & \cancel{1} & \cancel{0} \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A: \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad B: \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A^T C = B$$

$$\text{set } C = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 0 & 2 & 5 \\ 2 & 1 & -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\therefore C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix} \checkmark$$

Simpler: $C = (A^T)^{-1} \cdot B = (A^{-1})^T \cdot B$

Then use A^{-1} from problem 2.

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) the standard matrix of T

is $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} u = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 10 \\ 1 & -2 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 10 \\ 0 & 3 & 9 \\ 0 & 4 & 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 10 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$u_1 = 7$

$u_2 = 3$

$u_3 = u_3$ ← There is no u_3

$$u = \begin{bmatrix} 7 \\ 3 \\ u_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_3.$$

$$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$A: \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 4 \\ 0 & 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -4 \\ 0 & 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

pivot in every column

T_A is ~~not~~ one-to-one.

$T_A(V_1) = T_A(V_2)$

$T_A(V_1 - V_2) = 0$

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix}$

$x_1 = 2x_3$
$x_2 = -2x_3$
$x_2 = -4x_3$

there is no solution.

$\Rightarrow$ there is no two vectors V_1 and V_2 such that $T_A(V_1) = T_A(V_2)$.

Which means that T_A is one-to-one.

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$A: \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ ✓

T_A is not one-to-one

$T_A(V_1) = T_A(V_2)$

$T_A(V_1 - V_2) = 0$

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

$x_1 = 2x_3$
$x_2 = -2x_3$
$x_3 = x_3$

$X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x_3$

$V_1 - V_2 = X$

assume $x_3 = 1$ $X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ ✓

assume $V_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

then $V_1 = X + V_2$

$= \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ ✓

$\therefore V_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ~~16~~ 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \mathrm{Span}(u, v)$ then $w \in \mathrm{Span}(u, v)$.

~~False~~

~~set $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$~~

~~$w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$~~

~~$w + u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathrm{span}(u, v)$~~

~~$w \notin \mathrm{Span}(u, v)$~~

True ✓

$w + u \in \mathrm{Span}(u, v)$

then $w \in \cancel{\mathrm{Span}}\ \mathrm{Nul}(u, v)$?

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

~~False~~

~~assume $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$~~

~~$\{u, v, w\}$ is linearly independent.~~

~~while $\{u, v\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$ is linearly dependent.~~

True.

$\{u, v, w\}$ is linearly independent

then $x_1 u + x_2 v + x_3 w = 0$. has only one solution $x_1 = x_2 = x_3 = 0$ ✓

$x_1 u + x_2 v = -x_3 w$

$x_1 u + x_2 v = 0$

$\Rightarrow$ every column of the matrix is a pivot column.

since $\{u, v, w\}$ every column of u, v, w is pivot column.

$\{u, v\}$ also has pivot column is every column. ✓

0	1	2	3	4	5	6	7	8	9	10
☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False. ✓

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

$[\quad]\cdot\begin{bmatrix}1\\0\end{bmatrix}\cdot[\ = [$

$[\quad]\cdot\begin{bmatrix}0\\1\end{bmatrix}$

?

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True.

$u \in Span(v, w)$ $\quad u = c_1 v + c_2 w$ ✓

$T(u) = Au$

$T(v) = Av$

$T(w) = Aw$

$Au = c_1 Av + c_2 Aw.$ ✓

0 1 2 3 4 5 6 7 8 9 10