MTH-309T-F19-EX1-053-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	3	3	5	4	7

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
20	10	3	20	20	10	8	2	10	102	A

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$[w \in \mathrm{Span}(v_1, v_2, v_3)] \Rightarrow w = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccc} c_1 & c_2 & c_3 & w \end{array}$$

$$\xrightarrow{R1\cdot -2 + R3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right] R4\cdot -1$$

$$\xrightarrow{R2\cdot -1 + R3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -b-4 \end{array}\right] \xrightarrow{R2+R1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -b-\cancel{4-2} \\ & & & -6 \end{array}\right]$$

If $b=-6$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-b-6=0$

$-6=b$ ✓

$c_1 - c_2 + c_3 = -2$

$c_2 + 2c_3 = 2$

$\{v_1, v_2, v_3\}$ is not linearly independent because c_3 is a free variable. Infinite # of sltns if $b=-6$. ✓

No sltns if $b \neq -6$.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ✗

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{array}{c} \begin{array}{cc} A & I \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \end{array} \xrightarrow[R2\cdot -2 + R3]{R1\cdot -1 + R2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 + R1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R3 + R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R3\cdot -1 + R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$ ✓

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$(A A^T) C = AB$$
$$I C = AB$$
$$C = AB$$

?

$A \cdot A^T \neq I$

$A \cdot A^{-1} = 1$

$$\overset{A}{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}} = \boxed{\begin{bmatrix} 3 & 1 & 1 \\ 4 & 4 & 4 \\ 5 & 8 & 7 \end{bmatrix} = C}$$

$1(1) - 1(4) + 2(3)$
$= 1 - 4 + 6$
$= 3$

$1(2) - 1(5) + 2(2)$
$= 2 - 5 + 4$
$= 1$

$1(3) - 1(4) + 2(1)$
$= 3 - 4 + 2$
$= 1$

$1(1) + 0 + 1(3) = 4$
$1(2) + 0 + 1(2) = 4$
$1(3) + 0 + 1(1) = 4$

$0 + 2(4) - 1(3) = 8 - 3 = 5$

$0 + 2(5) - 1(2) = 10 - 2 = 8$

$0 + 2(4) - 1(1) = 8 - 1 = 7$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) Standard Matrix:

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \checkmark$$

b) $T(\vec{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$A\vec{u} = b$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \vec{u} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

R1 · -1 + R2

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{R1 \cdot -1 + R3} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{R3 \cdot -1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] \Rightarrow \begin{cases} u_1 - 2u_2 = 1 \\ u_2 = 3 \end{cases}$$

$u_1 - 2(3) = 1$

$u_1 - 6 = 1$

$u_1 = 7$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \checkmark$$

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

1-1: Pivot Pos. in every column

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \overset{R2\cdot\frac{1}{2}}{=} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix}$ R1·-3 +R3

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ R2·-1 +R3 $= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot\frac{1}{2}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

T_A is one-to-one because A has a pivot position in every column.

(b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ R2·$\frac{1}{2}$ $= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix}$ R1·-3 +R3

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

T_A is not one-to-one because A does not have a pivot position in every column.

$T_A(\vec{v}_1) = T_A(\vec{v}_2)$

$T_A(\vec{v}_1) - T_A(\vec{v}_2) = 0$

$T_A(\vec{v}_1 - \vec{v}_2) = 0$

$A\vec{w} = 0$

Let $\vec{w} = \vec{v}_1 - \vec{v}_2$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$ R2·-1 +R1 $= \begin{array}{c} x_1 \; x_2 \; x_3 \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} x_1 \; x_2 \; x_3 \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{l} x_1 - 2x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array}$

$x_1 = 2x_3$
$x_2 = -2x_3$
$x_3 = x_3$

$\text{Nul}(A) = \text{Span}\left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right)$

Let $\vec{w} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \vec{v}_1 - \vec{v}_2$

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \vec{v}_2 = \vec{v}_1$

Let $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \vec{v}_1$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

$\boxed{\text{True.}}$

$$\begin{cases} w + u = c_1 u + c_2 v \\ w = c_1 u + c_2 v - u \\ w = (c_1 - 1)u + c_2 v \end{cases}$$

$\Leftarrow$ $w \in \mathrm{Span}(u, v)$ if $w + u \in \mathrm{Span}(u + v)$ because w can be written as linear combination of u and v.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$\boxed{\cancel{\text{False.}}}$

Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{array}{cccc} u & v & w & b \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{ccc} u & v & b \end{array} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$A\vec{x} = [b]$

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be exact sltn. Then $\{\vec{u}, \vec{v}\}$ is not a sltn to the eqn and $\therefore$ not linearly independent.

True, $\vec{u}, \vec{v}, \vec{w}$ cannot be written as a linear combination of each other so $\vec{u}, \vec{v}$ cannot be written as a linear combination of each other. $\therefore \{u, v, w\}$ is linearly independent $\Rightarrow \{\vec{u}, \vec{v}\}$

0 1 2 3 4 5 6 7 8 9 10

7. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False ✓

Counter example:

$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A\vec{u} = T(\vec{u}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A\vec{v} \quad T(\vec{v}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

What is A here?

$A\vec{u} = A\vec{v}$

$T(\vec{u}) = T(\vec{v})$

$T(\vec{u}) - T(\vec{v}) = 0$

$T(\vec{u} - \vec{v}) = 0$

$\vec{u}, \vec{v} \in Nul(A)$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$\vec{u} \in Span(\vec{v}, \vec{w}) \Rightarrow T(\vec{u}) \in Span(T(\vec{v}), T(\vec{w}))$

$\vec{u} = c_1\vec{v} + c_2\vec{w}$

$T(\vec{u}) = T(c_1\vec{v} + c_2\vec{w})$

$= T(c_1\vec{v}) + T(c_2\vec{w})$

$= c_1 T(\vec{v}) + c_2 T(\vec{w}) \Rightarrow T(\vec{u}) \in Span(T(\vec{v}), T(\vec{w}))$

$\boxed{\therefore \text{True.}}$ ✓

0 1 2 3 4 5 6 7 8 9 10