



MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

OZLEM SAHAN

UB Person Number:

5	0	2	5	0	9	7	2
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1 2 3 4 5 6 7 TOTAL GRADE

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7

2

0

12

20

5

7

53

nan

1

2

3

4

5

6

7

TOTAL

GRADE



1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\text{SPAN}(v_1, v_2, v_3) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{2R_1 - R_3 = R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 + R_2 = R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 - R_3 = R_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

↑ ↑ ↑
[2, 1-2] [2, 1-(-3)]
 2, 1-0

CONTINUED HERE

A AFTER
ROW REDUCTION

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} v_1 + 3v_3 &= -2 \quad / \cdot (-2) & -2v_1 - 6v_3 &= 4 \\ v_2 + 2v_3 &= 2 \quad / \cdot (3) & 3v_2 + 6v_3 &= 6 \Rightarrow -2v_1 + 3v_2 = 10 \\ & & & \downarrow \\ & & & 2v_1 - 3v_2 = -10 \end{aligned}$$

CONTINUED HERE

$$2v_1 = 3v_2 - 10$$

$$v_1 = \frac{3v_2 - 10}{2}$$

$$v_1 = \frac{3}{2}v_2 - 5$$

FOR $v_1 + 3v_3 = -2$

$$\frac{3}{2}v_2 - 5 + 3v_3 = -2$$

$$\frac{3}{2}v_2 + 3v_3 = 3$$

$$\frac{1}{2}v_2 + v_3 = 1$$

$$v_2 = 2 - 2v_3$$

$$v_1 = \frac{3}{2} \cdot 2(1 - v_3) - 5$$

$$v_1 = 3 - 3v_3 - 5$$

$$v_1 = -3v_3 - 2$$

$$v_1 \rightarrow -3v_3 - 2$$

$$v_2 \rightarrow 2 - 2v_3$$

$$v_3 \rightarrow v_3$$

• b HAS INFINITE MANY
OF VALUES

• SET OF $\{v_1, v_2, v_3\}$ IS
LINEARLY DEPENDENT SINCE

v_3 IS A FREE VARIABLE.



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1} .

$$A^{-1} = \frac{1}{|A|} \cdot A^T$$

$$|A| = 1 \cdot [0 \cdot (-1) - 2 \cdot 1] - (-1) [(1)(-1) - 1 \cdot 0] + 2 [1 \cdot 2 - 0 \cdot 0]$$

$$= -2 - 1 + 4 = 1$$

DETERMINANT $A \Rightarrow |A| = 1$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

TO VERIFY $A \cdot A^{-1} = I$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+1+2 & 1+0+2 & 0-2-2 \\ 1+0+2 & 1+0+1 & 0+0-1 \\ 0+0-1 & 0+4+1 & 0+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & -4 \\ 3 & 2 & -1 \\ -4 & -1 & 5 \end{bmatrix} \xrightarrow[\text{(ON SCRAP PAPER)}]{\text{ROW REDUCTION}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$



3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$A^T \cdot C = \begin{bmatrix} x_1 + y_1 + 0 & x_2 + y_2 + 0 & x_3 + y_3 + 0 \\ -x_1 + 0 + 2z_1 & -x_2 + 0 + 2z_2 & -x_3 + 0 + 2z_3 \\ 2x_1 + y_1 - z_1 & 2x_2 + y_2 - z_2 & 2x_3 + y_3 - z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$x_1 + y_1 = 1 \rightarrow y_1 = 1 - x_1$$

$$-x_1 + 2z_1 = 4$$

$$2x_1 + y_1 - z_1 = 3 \rightarrow 2x_1 + 1 - x_1 - z_1 = 3$$

$$\begin{cases} z_1 = 6 \\ x_1 = 8 \\ y_1 = -7 \end{cases} \quad \begin{cases} x_1 - z_1 = 2 \\ -x_1 + 2z_1 = 4 \end{cases}$$

$$\begin{aligned} x_2 + y_2 &= 2 \rightarrow y_2 = 2 - x_2 \\ -x_2 + 2z_2 &= 5 \\ 2x_2 + y_2 - z_2 &= 2 \\ \begin{cases} x_2 - z_2 = 1 \\ -x_2 + 2z_2 = 5 \end{cases} &\Rightarrow \begin{cases} z_2 = 6 \\ x_2 = 7 \\ y_2 = -6 \end{cases} \end{aligned}$$

$$\begin{aligned} x_3 + y_3 &= 3 \rightarrow y_3 = 3 - x_3 \\ -x_3 + 2z_3 &= 4 \\ 2x_3 + y_3 - z_3 &= 1 \\ \begin{cases} x_3 - z_3 = -2 \\ -x_3 + 2z_3 = 4 \end{cases} &\Rightarrow \begin{cases} z_3 = 2 \\ x_3 = 0 \\ y_3 = 3 \end{cases} \end{aligned}$$

$$\left. \begin{array}{lll} x_1 = 8 & x_2 = 7 & x_3 = 0 \\ y_1 = -7 & y_2 = -6 & y_3 = 3 \\ z_1 = 6 & z_2 = 6 & z_3 = 2 \end{array} \right\} C = \begin{bmatrix} 8 & 7 & 0 \\ -7 & -6 & 3 \\ 6 & 6 & 2 \end{bmatrix}$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$T(v) = T \cdot v$$

$$v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad T(v) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} \quad \checkmark \quad T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$T(u) = T \cdot u$$

$$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot u = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

3x2

3x1

$$2 \times 1 \rightarrow u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 - 2u_2 \\ u_1 + u_2 \\ u_1 - 3u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$u_1 - 2u_2 = 1$$

$$u_1 + u_2 = 10$$

$$u_1 - 3u_2 = -2$$

$$\begin{cases} u_1 - 2u_2 = 1 \\ -u_1 - u_2 = -10 \end{cases}$$

$$-3u_2 = -9$$

$$u_2 = 3$$

$$u_1 + 3 = 10 \quad | \quad u_1 = 7$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow[\text{REDUCTION}]{\text{ROW } \downarrow}$$

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$$

$$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

SCRAP PAPER



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

SINCE $\mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$v = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$T_A(v) = A \cdot v$$

$$A \cdot v = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + y_1 + 0 & x_2 + y_2 + 0 & x_3 + y_3 + 0 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 4z_1 & 3x_2 + 4y_2 + 4z_2 & 3x_3 + 4y_3 + 4z_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 4z_1 & 3x_2 + 4y_2 + 4z_2 & 3x_3 + 4y_3 + 4z_3 \end{bmatrix}$$

$$A \cdot v = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + y_1 + 0 & x_2 + y_2 + 0 & x_3 + y_3 + 0 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 2z_1 & 3x_2 + 4y_2 + 2z_2 & 3x_3 + 4y_3 + 2z_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 2z_1 & 3x_2 + 4y_2 + 2z_2 & 3x_3 + 4y_3 + 2z_3 \end{bmatrix}$$

FOR $T_A(v_1) = T_A(v_2)$

$$\rightarrow 3x_{31} + 4y_{31} + 4z_{31} = 3x_{32} + 4y_{32} + 2z_{32}$$

$$4z_{31} = 2z_{32}$$

$$2z_{31} = 1z_{32} \quad \frac{z_{31}}{z_{32}} = \frac{1}{2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 12 & 16 & 16 \\ 15 & 22 & 24 \end{bmatrix}$$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

TRUE \rightarrow IF u, v, w ARE VECTORS IN \mathbb{R}^3
 THEN $\text{SPAN}(u, v, w) = \left\{ \begin{array}{l} \text{SET OF ALL} \\ \text{LINEAR COMBINATIONS} \end{array} \right\}$

2

b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

FALSE \rightarrow ONLY IF $u, x_1 + v, x_2 + w, x_3 = 0 \rightarrow \text{LINEARLY INDEPENDENT}$
 $u, x_1 + v, x_2 = 0$
 $\{u, v\}$ LINEARLY INDEPENDENT? \rightarrow THIS EQUATION DOES NOT GUARANTEED
 $u, x_1 + v, x_2 = 0$

?



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and u, v are vectors in \mathbb{R}^2 such that Au, Av are linearly dependent then u, v also must be linearly dependent.

TRUE \rightarrow IF A IS 2×2 MATRIX THEN
 $\text{NUL}(A)$ IS SPAN OF SOME
 VECTORS IN \mathbb{R}^2 .

$Au \neq 0$ FOR DEPENDENCE
 $Av \neq 0$

$\text{NUL}(A) = (\text{SET OF SOLUTIONS OF } A \cdot v = 0)$

\hookrightarrow LINEARLY
 INDEPENDENT

IF $Au \neq 0$
 $Av \neq 0$ THEN u, v LINEARLY DEPENDENT.

b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

TRUE

