MTH-309T-F19-EX1-069-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	1	7	1	3	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	TOTAL	GRADE
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1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a. $b_1 V_1 + b_2 V_2 + b_3 V_3 = W$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$$

③ - ② × 1

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b-4 \end{array}\right]$$

③ + ②

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-2 \end{array}\right]$$

$b-2=0$

$\boxed{b=2}$

b. $$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

① × ② - ②

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2x_1 - 3x_2 = 0$ $\qquad$ $2x_1 = 3x_2$

$x_2 - 2x_3 = 0$ $\qquad$ $x_1 = \frac{3}{2}x_2$

$x_1 = \frac{3}{2}x_2$ $\qquad$ $x_2 + 2x_3 = 0$

$x_2 = x_2$ $\qquad$ $2x_3 = -x_2$

$x_3 = -\frac{1}{2}x_2$ $\qquad$ $x_3 = -\frac{1}{2}x_2$

$$= \begin{bmatrix} \frac{3}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} x_2 \rightarrow \text{Infinity solutions}$$

Set $\{v_1, v_2, v_3\}$ not linear independent

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R2 = ② - ①$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = ③ - ② \times 2$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$0 - -1 \times 2$
$= 0 + 2$

$R_2 = ② + ③$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$R_1 = ① - ③ \times 2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & 2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$A^{-1} = \begin{bmatrix} -2 & 3 & 3 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

$R_1 = ① + ②$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

0 1 2 3 4 5 6 7 8 9 10

$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 10 \\ 4 & 5 & 5 & 5 & -9 & 9 \\ 3 & 3 & 2 & 7 & 7 & 7 \end{matrix}$ $5-14=-9$

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3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right]$$

$R_2 = ② + ①$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right]$$

$1-6=-5$

$R_3 = ③ - ① \times 2$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & -1 & -1 & 2 & -2 & -5 \end{array}\right]$$

$R_3 = ③ + ②$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 7 & 5 & 2 \end{array}\right]$$

$R_2 = R_2 - 2R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & -9 & -3 & 3 \\ 0 & 0 & 1 & 7 & 5 & 2 \end{array}\right]$$

$R_1 = R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & 5 & 0 \\ 0 & 1 & 0 & 9 & -3 & 3 \\ 0 & 0 & 1 & 7 & 5 & 2 \end{array}\right]$$

$$C = \begin{bmatrix} 10 & 5 & 0 \\ 9 & -3 & 3 \\ 7 & 5 & 2 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a. $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

b. $\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix}$ ② - ①

$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}.$

③ - ①

$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{bmatrix}$ 7-

③ × 2 - ①

$= \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{array}\right]$

$3x_2 = 9$

$x_2 = 3$

$x_1 - 2x_2 = 1$

$x_1 - 6 = 1$

$x_1 = 7$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right]$

$R_3 = R_3 - 3R_1$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$

$R_3 = 2R_3 - R_2$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$

$x_3 = 4$

$x_2 = 2$

$x_1 = -2$

It is one to one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$

$R_3 = R_3 - 3R_1$

$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -x_2$

$x_2 = x_2$

$x_3 = -\frac{1}{2}x_2$

$x_2 + 2x_3 = 0$

$2x_3 = -x_2$

$x_3 = \frac{1}{2}x$

$\begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} x_2$

$V_1 = \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix}$ $V_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

It is not one to one

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

Ture

u. v are in span

w+u is span

w in ∈ span

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False

$u \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} v \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} w \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

u v w are linearly independent

But u v are linear dependent.

✓

$x_1 u + x_2 v_2 + x_3 w = 0$ has solution of 0

$x_1 u + x_2 v_2$ will have solution.

0 1 2 3 4 5 6 7 8 9 10

7. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

Ture

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot U + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} V$

$X_1 U_1 + X_2 V_2$ are linearly dependent

$U_1 V_2$ are also linear dependent.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

Ture

$T(v) = T(w) = T(u)$

$AV = AW = AU$

0 1 2 3 4 5 6 7 8 9 10