MTH-309T-F19-EX1-072-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	1	8	8	7	3	0

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
15	10	10	17	14	7	2	2	--	77	B

MTH-309T-F19-EX1-072-P01

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) If: $w = 2v_2$, $\boxed{b = -6}$ ← How do you know that there are no other values of b which work?

$w = v_1 + v_3$, $\boxed{\cancel{b = 2}}$

$w = 3v_1 + 2v_2$, $\boxed{\cancel{b = 0}}$

b) The set is not linearly independent because there is a linear combination of vectors v_1 and v_2 which give v_3.

$$3v_1 + 2v_2 = v_3$$

$$3\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$ ✓

$$\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = v_3$$

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

First, append Identity matrix, I, then find rref(A)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2,\ (-1)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$C_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

$x_1 + x_2 + 0x_3 = 1$
$-x_1 + 0x_2 + 2x_3 = 4$
$2x_1 + x_2 - x_3 = 3$

$\to$ $-x_1 + x_3 = -2$
$-x_1 + 2x_3 = 4$

$x_3 = 6 \therefore x_1 = 8$ and $x_2 = -7$

$C_1 = \begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix}$

$C_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

$x_1 + x_2 + 0x_3 = 2$
$-x_1 + 0x_2 + 2x_3 = 5$
$2x_1 + x_2 - x_3 = 2$

$-x_1 + x_3 = 0$
$-x_1 + 2x_3 = 5$

$x_1 = x_3 = 5$
$\therefore x_2 = -3$

$C_2 = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$

$C_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$x_1 + x_2 + 0_3 = 3$
$-x_1 + 0x_2 + 2x_3 = 4$
$2x_1 + x_2 - x_3 = 1$

$-x_1 + x_3 = 2$
$-x_1 + 2x_3 = 4$

$x_3 = 2$ and $x_1 = 4$
$\therefore x_2 = -1$

$C_3 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$\therefore \boxed{C = \begin{bmatrix} 8 & 5 & 4 \\ -7 & -3 & -1 \\ 6 & 5 & 2 \end{bmatrix}}$ ✓

Simpler:

$C = (A^T)^{-1} \cdot B$

$= (A^{-1})^T \cdot B$

Then use A^{-1} from problem 2.

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $\boxed{T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ ✓

b) $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ ✓

$x_1 - \overset{2}{\cancel{}}x_2 = 1$ $\quad\to\quad$ $2x_1 = 11 \to x_1 = 4.5$ or $9/2$

$x_1 + x_2 = 10$ $\quad\to\quad$ $9/2 - 3x_2 = -2$

$x_1 - 3x_2 = -2$ $\quad\quad$ $-3x_2 = 13/2$

$x_2 = -13/6$

But $9/2 - 13/6 \neq 10$ $\therefore$ $\boxed{\text{There are no vectors, } u \text{, which satisfy } \cancel{T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}}}$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\downarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\downarrow R_3 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\downarrow R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix} \cdot \left(-\frac{1}{4}\right)$$

$$\downarrow$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}}$$

pivot in every column

This is not one to one because there is not a pivot in every column

?

$V_1 = \begin{bmatrix} \; \end{bmatrix}$

$V_2 = \begin{bmatrix} \; \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\downarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\downarrow R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

✓

This is not one to one because there is not a pivot in every column

$V_1 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$V_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

← $T_A(v_1) \neq T_A(v_2)$

Try $v_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ instead.

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True; If there is a linear combination of vectors v and u that equal w, w must be in the span of u and v. Since addition and subtraction of one vector from another is a linear operator, the problem can be considered as the following:

$w = (w+u) - u$ ∴ If $w+u = x_1 v + x_2 u$, then $w = x_1 v + (x_2 - 1)u$, which is a linear combination of u and v ∴ $w \in \mathrm{Span}(u, v)$

✓

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True, any subset of a linearly independent set of vectors must also be linearly independent

why ?

0 1 2 3 4 5 6 7 8 9 10

7. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False; Linear dependence/independence is only guaranteed to be preserved if A is a matrix which defines a linear transformation. Since this condition is not specified, then linear dependence cannot be guaranteed after transformation.

Every matrix defines a linear transformation.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True; ✓ Since T is a linear operator whose operation preserves the dimensions of the original vectors, then any vector $u \in \text{Span}(v, w)$ must be in the span of $T(v), V(w)$. Additionally, since there are 3 vectors in 2 space, and it is known that $u \in \text{Span}(v, w)$, v and w either are linearly dependent on one another, and u, or are linearly independent. Since T is a linear transformation, these properties are maintained, meaning $T(u) \in \text{Span}\, T(v), T(w)$ by definition (3 vectors in 2 space, 2 are linearly independent; v and w) or because all 3 vectors were linearly dependent to begin with.

This just states that this property is true, without explaining why.

0 1 2 3 4 5 6 7 8 9 10