MTH 309Y LINEAR ALGEBRA

Exam 2

April 15, 2014

NAME: SAMPLE v. 1

ID NUMBER: ______________ **RECITATION:** __________

- **Books and electronic devices (calculators, cellphones etc.) are not permitted.**
- **You may use one sheet of notes.**
- **For full credit explain your answers fully, showing all work.**
- **Each problem is worth 20 points.**

1	
2	
3	
4	
5	
TOTAL:	

1. Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ (you do not need to verify it).

a) Compute $[w]_{\mathcal{B}}$, the coordinate vector of w relative to the basis $\mathcal{B}$.

b) Let $u \in \mathbb{R}^3$ be a vector such that

$$[u]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Compute the vector u.

a)

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -5 \\ 1 & 0 & 1 & 1 \end{array}\right] (-1) \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & -2 & 1 & 0 \end{array}\right] 2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 5 & -10 \end{array}\right] \left(\tfrac{1}{5}\right) \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right] (-2)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] (-2) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

$$\boxed{[w]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}}$$

b) $u = 3v_1 + 2v_2 + 1 \cdot v_3 = 3 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix}$

$$\boxed{u = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix}}$$

2. Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 2 & 9 & 4 & 17 \end{bmatrix} \qquad v = \begin{bmatrix} 5 \\ -8 \\ 0 \end{bmatrix}$$

a) Find a basis of the null space Nul(A).

b) Find a basis of the column space Col(A).

c) Find an orthogonal basis of Col(A).

d) Compute $\text{proj}_{\text{Col}(A)}\, v$, i.e. the orthogonal projection of the vector v onto the column space of A.

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 2 & 9 & 4 & 17 \end{bmatrix} (-2) \rightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 5 & 0 & 15 \end{bmatrix} (-5) \rightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} (-2)$$

$$\rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

a) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 + 5x_4 \\ -3x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix} x_4 \qquad \left(\begin{array}{c}\text{basis of} \\ \text{Nul}(A)\end{array}\right) = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b) basis of Col(A) = (pivot columns of A) = $\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix}} \right\}$

c) orthogonal basis of Col(A):

$$w_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad w_2 = v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 = \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix} - \frac{20}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

(orthogonal basis) = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

d) $$\text{proj}_{\text{Col}(A)} v = \frac{w_1 \cdot v}{w_1 \cdot w_1} w_1 + \frac{w_2 \cdot v}{w_2 \cdot w_2} w_2 = \frac{5}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \frac{(-18)}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -1 \end{bmatrix}$$

$$\boxed{\text{proj}_{\text{Col}(A)} v = \begin{bmatrix} 7 \\ -3 \\ -1 \end{bmatrix}}$$

3. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{b}$$

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A^T A \cdot \begin{bmatrix} a \\ b \end{bmatrix} = A^T b$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \cdot \left(\tfrac{1}{6}\right) \longrightarrow \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 2 & 3 & 3 \end{array}\right] (-2) \longrightarrow$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 7/3 & 3 \end{array}\right] \cdot (3/7) \longrightarrow \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 1 & 9/7 \end{array}\right] (-1/3)$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 3/7 \end{array}\right]$$

$a = -3/7$

$b = 9/7$

$$\boxed{f(x) = -\frac{3}{7}x + \frac{9}{7}}$$

4. Decide which of the following sets of vectors are subspaces of $\mathbb{R}^2$. Justify your answers.

a) The set S_1 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ such that $a_1 + a_2 = 1$.

b) The set S_2 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ such that $a_1 - a_2 = 0$.

c) The set S_3 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ such that $a_1 \cdot a_2 = 0$.

d) The set S_4 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ such that $a_1 \cdot a_2 \geqslant 0$.

a) Not subspace (e.g. no zero vector)

b) Subspace (students should show work)

c) Not subspace (e.g. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in S_3$, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S_3$)

d) Not subspace (e.g. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S_4$ but $(-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin S_4$)

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are vectors in $\mathbb{R}^3$ such that $\mathbf{u}_1$ is orthogonal to $\mathbf{u}_2$, and $\mathbf{u}_2$ is orthogonal to $\mathbf{u}_3$ then $\mathbf{u}_1$ must be orthogonal $\mathbf{u}_3$.

b) If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$ and $\mathbf{u}$ is a vector in $\mathbb{R}^3$ such that

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

then $\mathbf{u}$ must be in $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

c) If V is a subspace of $\mathbb{R}^3$ and $\mathbf{u}$ is a vector in $\mathbb{R}^3$ such that $\mathrm{proj}_V \mathbf{u} = \frac{1}{2}\mathbf{u}$ then $\mathbf{u} = \mathbf{0}$.

d) If A is a 2×4 matrix then $\operatorname{rank} A = 2$.

a) FALSE: e.g. $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

b) TRUE: $u = 1 \cdot v_1 + 5 v_2$ so $u \in \mathrm{Span}(v_1, v_2)$

c) FALSE: if $\mathrm{proj}_V u = \frac{1}{2} u$ then $\frac{1}{2} u \in V$

and $u - \mathrm{proj}_V u = u - \frac{1}{2} u = \frac{1}{2} u \in V^\perp$

so $\frac{1}{2} u$ is orthogonal to itself and so $\frac{1}{2} u = 0$. Thus $u = 0$.

d) FALSE: e.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ rank$(A) = 0$.