MTH-309T-F19-EX1-084-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

3	7	5	8	1	0	7	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
10	10	5	20	6	3	3	--	--	57	C-

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$$

$-1 \quad \cancel{x} \quad -2 \mid b+4$

$0 \quad 0 \quad -12 \mid b-6$

$x_1 - x_2 + x_3 = -2$

$x_2 + 2x_3 = 2$

$\cancel{2x_1 + 3x_2 = b}$

$-12x_3 = b-6$

$x_3 = -\frac{b-6}{12}$

$x_1 = 3\left(\frac{b-6}{12}\right) \quad x_2 = 2 + 2\left(\frac{b-6}{12}\right)$

$x_2 = 2 + \frac{b-6}{6}$

$x_1 = -2 + \left(2 + \frac{b-6}{6}\right) + \frac{b-6}{12}$

or

$x_1 = 3\left(\frac{b-6}{12}\right)$

$b = \mathbb{R}$? $\cancel{6}$ -6

?

why?

(b) to check for linear independance

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & \cancel{-3}\,0 & \cancel{8}\,3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Doesn't reduce to identity

so

$\boxed{\text{Liniarly Dependent}}$

This would mean that they are independent

? [Also, no vectors Appear to be able to be representa by any other in the set.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

There is also another way that requires you to find a cofactor by doing

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}$$

ect.

finding the cross product of each box to find a ~~mult~~ factor, X then

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$X(\text{this}) = A^{-1}$

0 1 2 3 4 5 6 7 8 9 10

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3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \checkmark$$

Simpler: $(A^T)^{-1} = (A^{-1})^T$ then use problem 2

so $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \checkmark$

$$C = \cancel{BA^{-1}} \; (A^T)^{-1} \cdot B$$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

so

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \downarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right] \downarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \downarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -1 & -2 & 0 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \cancel{-5} & \cancel{2} & \cancel{4} \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} \cancel{-5} & \cancel{2} & \cancel{4} \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

so

$$\begin{bmatrix} (-5+6-3) & (2-2+3) & (4-4+3) \\ (-20+15-4) & (8-5+4) & (16-10+4) \\ (-15+6-1) & (6-2+1) & (12-4+1) \end{bmatrix}$$

so

$$C = \cancel{\begin{bmatrix} -2 & 3 & 3 \\ -9 & 7 & 10 \\ -10 & 5 & 9 \end{bmatrix}}$$

☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

(a) $\begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

(b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$ ✓ so $x_1 - 2x_2 = 1$, $x_1 + x_2 = 10$, $x_1 - 3x_2 = -2$

$x_1 = 1 + 2x_2$

$x_1 = 1 + 6 = 7$

or
$$\begin{array}{ccc} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -4 & -12 \end{array}$$

so $-4x_2 = -12$

$x_2 = 3$

$x_1 = 7$

so $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ V b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

one to one if all Col(A) are lin. dep.

SO NO

ⓐ $x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ NO $x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $x_3 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ $\boxed{\text{So lin. dep.}}$ ✓

? $\boxed{\text{So One to One}}$

NO

ⓑ $x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ NO $x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $x_3 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

? - All lin. dep

$\boxed{\text{So One to One}}$

Linear independence does not always mean that one vector is a multiple of another — this works only for linear independence of <u>two</u> vectors (not three).

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☒ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14 ☐ 15 ☐ 16 ☐ 17 ☐ 18 ☐ 19 ☐ 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

* a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w}+\mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

~~False~~,

$w = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$w + u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ is valid

but $\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{array}\right]$ is invalid

What do you mean by valid / invalid ? ?

Both of these matrices represent consistent equations.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~False~~, $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \; v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$

$u + v = w$ so $\{u, v, w\}$ is linearly independe

but u and v alone $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ are

linearly dep. because neither vector can represent the other.

not true

☐	☐	☐	☒	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

~~$u = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$~~

~~$u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then $Au = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ $Av = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$~~

$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

~~but u & v are not linearly dep.~~

~~False~~

~~True~~, thought it was false but couldn't disprove

it is

so $Au = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $Av = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$?

but u and v

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

~~False~~

so $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ is valid

what do you mean by "valid"?

but $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not

?

0 1 2 3 4 5 6 7 8 9 10