MTH-309T-F19-EX1-032-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

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Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$x_1 - x_2 + x_3 = -2$

$x_2 + 2x_3 = 2$

$2v_2 \to b = -6$

$b = 2x_1 - 3x_2$

$\boxed{b = -6 \text{ only}}$

$x_1 - (2 - 2x_3) + x_3 = -2$

$x_1 + 3x_3 = 0$

$b = 2(-3x_3) - 3(2 - 2x_3)$

$b = -6x_3 - 6 + 6x_3$

$\boxed{b = -6}$

~~b) $\{v_1, v_2, v_3\}$ is linearly independent because~~

~~$v_1 \neq c_1 v_2$ for any $c_1 \in \mathbb{R}$;~~

~~$v_2 \neq c_2 v_3$ for any $c_2 \in \mathbb{R}$;~~

~~$v_3 \neq c_3 v_1$ for any $c_3 \in \mathbb{R}$.~~

b) $\{v_1, v_2, v_3\}$ is not linearly independent because $3v_1 = v_3 - 2v_2$.

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

$a_1 - a_4 + 2a_7 = 1$

$a_1 + a_7 = 0$

$2a_4 - a_7 = 0$

$-a_7 - a_4 + 2a_7 = 1$

$1 + a_4 = a_7$

$1 + a_4 = 2a_4$

$a_4 = 1 \quad a_7 = 2 \quad a_1 = -2$

$a_2 - a_5 + 2a_8 = 0$

$a_2 + a_8 = 1$

$2a_5 - a_8 = 0$

$2a_5 = a_8$

$a_8 = 1 - a_2$

$1 - a_8 - \frac{1}{2}a_8 + 2a_8 = 0$

$\frac{1}{2}a_8 = -1$

$a_8 = -2$

$a_5 = -1 \quad a_2 = 3$

$a_3 - a_6 + 2a_9 = 0$

$a_3 + a_9 = 0$

$2a_6 - a_9 = 1$

$a_3 = -a_9$

$a_6 = \frac{1}{2}(1 + a_9)$

$-a_9 - \frac{1}{2}(1 + a_9) + 2a_9 = 0$

$-\frac{1}{2} + \frac{1}{2}a_9 = 0$

$a_9 = 1$

$a_3 = -1$

$a_6 = \frac{1}{2}(1+1) = 1$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad (A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\cancel{(A^T)}(A^{-1})^T C = B(A^{-1})^T$$

$$C = B(A^{-1})^T$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 2 \\ -1 & 2 & 3 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 2 \\ -1 & 2 & 3 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

std. matrix of $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

$\cancel{x_1 - 2x_2 = 1}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & 3 & -2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 5 & -3 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 5 & -3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & -18 \end{array}\right]$$

$\boxed{\therefore \text{There are no vectors } u \text{ satisfying } T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

1-1 if $Nul(A) = 0$ $\longrightarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ only solution to $Av = 0$ is $v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore Nul(A) = 0$

$\boxed{\therefore A \text{ is one-to-one}}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 2x_3$
$x_2 = -2x_3$

$\therefore Nul(A) \neq 0$

$\boxed{\begin{array}{l} \therefore A \text{ is not one-to-one.} \\ \text{ex. for vectors } v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \\ v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}, \ T_A(v_1) = T_A(v_2) \end{array}}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

This is true. By def. of span, $w + u = c_1 u + c_2 v$ for some $c_1, c_2 \in \mathbb{R}$. Therefore $w = c_1 u + c_2 v - u$, and combining gives $w = (c_1 - 1)u + c_2 v$. However, $c_1 - 1$ is just some other constant in $\mathbb{R}$, so let $c_3 = c_1 - 1$. Therefore $w = c_3 u + c_2 v$, meaning $w \in \mathrm{Span}(u, v)$ by definition.

QED.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

This is true. If $\{u, v, w\}$ is linearly independent, then

$$u \neq c_1 v + c_4 w \text{ for any } c_1, c_4 \in \mathbb{R}$$
$$v \neq c_2 w + c_5 u \text{ for any } c_2, c_5 \in \mathbb{R}$$
$$w \neq c_3 u + c_6 v \text{ for any } c_3, c_6 \in \mathbb{R}$$

Since $u \neq c_1 v$ and $v \neq c_5 u$ for any $c_1, c_5 \in \mathbb{R}$, then $\{u, v\}$ is linearly independent by definition.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

This is False. For example, let $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $Au = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $Av = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Au and Av are the same vector, so they are clearly linearly dependent, however, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

This is true. If $u \in \text{Span}(v, w)$, then $u = c_1 v + c_2 w$ for some $c_1, c_2 \in \mathbb{R}$. So $T(u) = T(c_1 v + c_2 w)$. In order to show that $T(u) \in \text{Span}(T(v), T(w))$, we must show that $T(u) = c_3 T(v) + c_4 T(w)$ for some $c_3, c_4 \in \mathbb{R}$. Let $c_1 = c_3$ and $c_2 = c_4$. By def. of linear transformation, $c_1 T(v) + c_2 T(w) = T(c_1 v) + T(c_2 w)$. Again, by def. of linear transformation, $T(c_1 v) + T(c_2 w) = T(c_1 v + c_2 w)$. Since we already had that $T(u) = T(c_1 v + c_2 w)$, then $T(u) \in \text{Span}(T(v), T(w))$.

QED.

0 1 2 3 4 5 6 7 8 9 10