MTH-309T-F19-EX1-024-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5 0 1 5 3 4 7 8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

$-3(6-b)$

$\frac{-18+36}{8}$

$16-12=4$

$2-2\left(\frac{6-b}{8}\right)$

$2-\frac{12+b}{8}$

$2-2\left(\frac{6-b}{8}\right)$

$2-\frac{12+2b}{8}$

$\cdot \frac{-1}{8}\, 6-6 = \frac{-b+6}{8}$

$\frac{6+b}{8}$

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & 6 \end{array}\right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 5 & -2 & b+4 \end{array}\right] \begin{matrix} \xrightarrow{R_1+R_2} \\ \\ \xrightarrow{R_1 - 5R_2} \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -8 & b-6 \end{array}\right] \xrightarrow{R_3 \cdot \frac{-1}{8}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{6-b}{8} \end{array}\right] \begin{matrix} \xrightarrow{R_1 - 3R_3} \\ \xrightarrow{R_2 - 2R_3} \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-18+3b}{8} \\ 0 & 1 & 0 & 2-\frac{12+2b}{8} \\ 0 & 0 & 1 & \frac{6-b}{8} \end{array}\right] \rightarrow \therefore$$

$x_1 = \frac{-18+3b}{8}$

$x_2 = \frac{4+2b}{8}$

$x_3 = \frac{6-b}{8}$

Since b is in the numerator of all solutions $\therefore$ if $b \in \mathbb{R}$ then $w \in \text{Span}(v_1, v_2, v_3)$

b) The set $\{v_1, v_2, v_3\}$ is linearly independent because there is a pivot column in every column as shown in part A where we reduced the matrix in row echelon form.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$0-2(-1)=2$
$0-2(1)=-2$

$1-2$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to R_2 - R_1 \to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \to R_1 + R_2 \\ \\ \to R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} \to R_1 - R_3 \to \\ \to R_2 + R_3 \to \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \overset{3\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}} \qquad A = \underset{3\times 3}{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \quad \text{(calculated in problem 2)}$$

Using the fact that: $(A^T)^{-1} = (A^{-1})^T$

$$\therefore (A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \rightarrow \begin{aligned} (A^T)^{-1}(A^T)C &= B(A^T)^{-1} \\ C &= B(A^T)^{-1} \end{aligned}$$

$$\therefore C = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}}_{(A^T)^{-1}} = \begin{bmatrix} 1(-2)+3(2)-3, & 1-2+3, & 2-4+3 \\ 4(-2)+5(3)-4, & 4-5+4, & 4(2)-5(2)+4 \\ \underset{-6+6}{3(-2)+2(3)-1}, & 3-2+1, & \underset{6-4+1}{3(2)-2(2)+1} \end{bmatrix}$$

$$= \begin{bmatrix} -2+6-3, & -1+3, & -2+3 \\ -8+15-4, & -5, & 8-10+4 \\ -1, & 0, & 6-4+1 \end{bmatrix} = \begin{bmatrix} 1, & 2, & 1 \\ 3, & -5, & 2 \\ -1, & 0 & 3 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 1, & 2, & 1 \\ 3, & -5, & 2 \\ -1, & 0, & 3 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \quad \therefore e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, e_2 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\therefore$ Std Matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $A(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \begin{array}{l} \\ \to R_2 - R_1 \to \\ \to R_3 - R_1 \to \end{array} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \to \frac{R_3}{3} \to$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \begin{array}{l} \to R_1 + 2R_2 \to \\ \\ \to R_3 + R_2 \to \end{array} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 6 & 0 \end{array}\right] \to \begin{array}{l} X_1 = 7 \\ X_2 = 3 \\ X_3 = X_3, \end{array}$ Since $\mathbb{R}^2 \to \mathbb{R}^3$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $\begin{matrix} = 2 \\ = 6 \\ = 1 \end{matrix}$

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \to R_3 - 3R_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \to R_2 \cdot \frac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{matrix} \to R_1 - R_2 \to \\ \\ \to R_3 - R_2 \to \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \to R_1 + R_3 \\ \to R_2 - R_3 \end{matrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \to R_3 \cdot \frac{1}{2} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since A has a pivot position in every column it is 1 to 1

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \to R_3 - 3R_1 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to R_2 \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \to R_1 - R_2 \to \\ \\ \to R_3 - R_2 \to \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

A is not one to one since no pivot position in every column

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

False is $w + u \in Span(u, v)$
that means w is linearly dependent for
$u \notin$, $W \notin Span(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False, $\{u, v, w\}$ is only inlinear independent is it has only one solution to the homogenus equation $x_1 u + x_2 v + x_3 w = 0$

Counter Example: $\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} u_1 & 0 & * \\ 0 & v_2 & * \\ 0 & 0 & 0 \end{array}\right]$

$$\left[\begin{array}{cc|c} u_1 & 0 & * \\ 0 & v_1 & * \\ 0 & * & * \end{array}\right]$$

$\therefore \{u, v\}$ is not neccacirly linearly independent

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0	1	2	3	4	5	6	7	8	9	10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

If Au, Av are linearly dependent that means the homogenus equation has infinite solutions, for both u & v that means u or v are multiples of one another & ∴ not linearly independent

∴ true

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v})$, $T(\mathbf{w})$).

True, transformations are linear operators

$T(u+v) = T(u) + T(v)$

∴ if u is in Span(v, w)
the T(u) is in Span(T(v), T, w)

0 1 2 3 4 5 6 7 8 9 10