1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

$\cancel{v_1 \cdot v_2 = (1)(2) + (0)(1) + (-1)(-1) + 0}$

a.) Use Gram-Smit:

$\langle w_1, w_1 \rangle = 3$

$w_1 = v_1$

$\langle w_1, v_2 \rangle = (1)(2) + 0 + 1 + 0 = 3$

$$w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - 1 w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = w_2$$

$$w_3 = v_3 - \frac{\langle w_1, v_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_3 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 - 2 w_1 + w_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\langle w_1, v_3 \rangle = 2 + 0 + 1 + 3 = 6$

$\langle w_2, v_3 \rangle = 2 - 2 + 0 - 3 = -3$

$\langle w_2, w_2 \rangle = 3$

$$D = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} ✓$$

b.) $$\text{proj}_V u = \frac{\langle u, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle u, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \frac{\langle u, w_3 \rangle}{\langle w_3, w_3 \rangle} w_3$$

$\langle u, w_1 \rangle = 3 + 0 - \cancel{3} + \cancel{3} = 3$

$\langle u, w_2 \rangle = 3 + \cancel{3} + 0 - \cancel{3} = 3$

$\langle u, w_3 \rangle = \cancel{3} - \cancel{3} + 3 + 0 = 3$

$$= w_1 + w_2 + w_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \text{proj}_V u \; ✓$$

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