MTH-309T-F19-EX1-085-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	1	2	9	4	6	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
13	10	1	13	14	1	1	2	10	61	C

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1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \overset{+R2}{=} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \overset{-2R1}{=} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & b \end{array}\right] \overset{+3R2}{=} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$

$x_1 = -3x_3$

$x_2 = 2 - 2x_3$

$x_3 = 0 = b+6$

$b = -6$ ✓

b) ~~Yes~~, it is linearly independent because v_1, v_2, v_3 are not multiples of each other. There is no way to get anything but 0 in second row of v_1 and therefore we can't get such multiple of v_1 so that we obtain v_2, v_3.

?

This does not suffice to check if three vectors are independent

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

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2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{-R2} \left[\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{+2R1} \left[\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-R3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] = \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]}_{1}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{-R2} \left[\begin{array}{ccc|c} 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{+2R1} \left[\begin{array}{ccc|c} 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{-R3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] = \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]}_{2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \xrightarrow{-R2} \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \xrightarrow{+2R1} \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-R3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] = \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]}_{3}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}}_{} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

This is not A^T.

$$\begin{vmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

?

0 1 2 3 4 5 6 7 8 9 10

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4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] = \Big[$ ✓

. . .

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{array}{c} \\ \\ -3R1 \end{array}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \overset{:R3}{=} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}_{-R2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{:4} \overset{-R2}{=} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a) is one-to-one ✓

b) $\begin{array}{c} \\ \\ -3R1 \end{array}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \overset{-2R3}{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

b) is not one-to-one ✓

$\underline{v_1, v_2}$?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☒ 14 ☐ 15 ☐ 16 ☐ 17 ☐ 18 ☐ 19 ☐ 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

~~FALSE~~ ← why ?

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

~~FALSE~~

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right) ?$$

☐	☒	☐	☐	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~TRUE~~

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \cancel{0} \\ 0 & \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Because if Au and AV yield the linearly dependent result, then u v are also dependent.

not so.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

~~True. Because applying the~~

~~FALSE~~

0 1 2 3 4 5 6 7 8 9 10