MTH-309T-F19-EX1-006-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Eithne Amos

UB Person Number:

5	0	2	0	9	9	0	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)
$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \xrightarrow{r2+r1} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \xrightarrow[+r3]{(-2)\cdot r1} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & b \end{bmatrix}.$$

$$\xrightarrow{(-\frac{1}{3})\cdot r3} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & b \end{bmatrix} \xrightarrow[+r3]{(-1)\cdot r2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-2 \end{bmatrix}$$

$0 = b-2$

$b = 2$

b)
$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\{v_1, v_2, v_3\}$ is not linearly independent because every column of $[v_1\ v_2\ v_3]$ is not a pivot column

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{(-1)\cdot r1 \\ +r2}} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r2+r1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{(-2)\cdot r2 \\ +r3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{r3+r2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{-r3+r1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$(A^T)^{-1} A^T C = B\,(A^T)^{-1}$$

$$C = B\,(A^T)^{-1}$$

$$C = B\,(A^{-1})^T$$

$$C = B \cdot \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = [T(e_1)\ T(e_2)]$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

standard matrix:

$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $T_A(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$Au = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{(-1)r1+r2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{r2 \cdot \frac{1}{3}} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 1 & -3 & -2 \end{bmatrix}$

$\xrightarrow{-r1+r3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{2r2+r1} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{r2+r3} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = 7$
$x_2 = 3$
$x_3 = x_3$

$v = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A has pivot in every column $\to$ A is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

not one-to-one

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

true

$\mathrm{Span}(u, v) = \underbrace{c_1}_{w} u + c_2 v$

if $w+u \in \mathrm{Span}(u, v)$, then

w must be lin. combination of u, v

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

true - each vector must be linearly independent from each other vector for the whole set to be linearly independent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

false

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v})$, $T(\mathbf{w})$).

true

MTH-309T-F19-EX1-058-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Matthew Simkulet

UB Person Number:

5	0	2	8	0	5	1	2
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

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1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) w in span if $\underset{x_1}{c_1}v_1 + \underset{x_2}{c_2}v_2 + \underset{x_3}{c_3}v_3 = w$

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ x_2 + 2x_3 &= 2 \\ 2x_1 - 3x_2 &= b \end{aligned} \rightarrow \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix}$$

any integer can be reduced to a leading 1 except when $b=-6$

a) $b=-6$

b) $x_1v_1 + x_2v_2 + x_3v_3 = 0$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_3 = 0$

$x_2 + 2x_3 = 0$

x_3 is a free variable $\therefore$ the equation has infinitely many solutions $\therefore$ the set is linearly dependent.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A \cdot A^{-1} = I$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_1^{-1} & A_2^{-1} & A_3^{-1} \\ A_4^{-1} & A_5^{-1} & A_6^{-1} \\ A_7^{-1} & A_8^{-1} & A_9^{-1} \end{bmatrix}$$

$A_1^{-1} - A_4^{-1} + 2A_7^{-1} = 1$

$A_1^{-1} + A_7^{-1} = 0$

$2A_4^{-1} - A_7^{-1} = 0$

$2A_1^{-1} + 3A_7^{-1} = 1$

$A_1^{-1} + A_7^{-1} = 0$

$A_7^{-1} = 1$

$\cancel{A_7^{-1} = \frac{1}{2}}$

$A_1^{-1} + 1 = 0$

$A_1^{-1} = -1$

$2A_4^{-1} - 1 = 0$

$A_4^{-1} = \frac{1}{2}$

$A_2^{-1} - A_5^{-1} + 2A_8^{-1} = 0$

$A_2^{-1} + A_8^{-1} = 1$

$2A_5^{-1} - A_8^{-1} = 0$

$2A_2^{-1} + 3A_8^{-1} = 0$

$A_2^{-1} + A_8^{-1} = 1$

$A_8^{-1} = -2$

$A_2^{-1} - 2 = 1$

$A_2^{-1} = 3$

$2A_5^{-1} + 2 = 0$

$A_5^{-1} = -1$

$A_3^{-1} - A_6^{-1} + 2A_9^{-1} = 0$

$A_3^{-1} + A_9^{-1} = 0$

$2A_6^{-1} - A_9^{-1} = 1$

$2A_3^{-1} + 3A_9^{-1} = 1$

$A_3^{-1} + A_9^{-1} = 0$

$A_9^{-1} = 1$

$A_3^{-1} + 1 = 0$

$A_3^{-1} = -1$

$2A_6^{-1} - 1 = 1$

$A_6^{-1} = 1$

$$A^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ \frac{1}{2} & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$-1 \cdot \begin{cases} c_1 + c_4 = 1 \\ -c_1 + 2c_7 = 4 \\ 2c_1 + c_4 + c_7 = 3 \end{cases}$ $\quad -1 \cdot \begin{cases} c_2 + c_5 = 2 \\ -c_2 + 2c_8 = 5 \\ 2c_2 + c_5 + c_8 = 2 \end{cases}$ $\quad -1 \begin{cases} c_3 + c_6 = 3 \\ -c_3 + 2c_9 = 4 \\ 2c_3 + c_6 + c_9 = 1 \end{cases}$

$c_1 + c_7 = 2$
$-c_1 + 2c_7 = 4$
$3c_7 = 6$
$c_7 = 2$
$-c_1 + 4 = 4$
$c_1 = 0$
$0 + c_4 = 1$
$c_4 = 1$

$c_2 + c_8 = 0$
$-c_2 + 2c_8 = 5$
$3c_8 = 5$
$c_8 = \frac{5}{3}$
$-c_2 + \frac{10}{3} = \frac{15}{3}$
$c_2 = -\frac{5}{3}$
$-\frac{5}{3} + c_5 = \frac{6}{3}$
$c_5 = \frac{11}{3}$

$c_3 + c_9 = -2$
$-c_3 + 2c_9 = 4$
$3c_9 = 2$
$c_9 = \frac{2}{3}$
$-c_3 + \frac{4}{3} = \frac{12}{3}$
$c_3 = -\frac{8}{3}$
$-\frac{8}{3} + c_6 = \frac{3}{3}$
$c_6 = \frac{11}{3}$

$$C = \begin{bmatrix} 0 & -\frac{5}{3} & -\frac{8}{3} \\ 1 & \frac{11}{3} & \frac{11}{3} \\ 2 & \frac{5}{3} & \frac{2}{3} \end{bmatrix}$$

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T. T is 3×2 $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$-1\left(\begin{array}{l} u_1 - 2u_2 = 1 \\ u_1 + u_2 = 10 \\ u_1 - 3u_2 = -2 \end{array}\right.$$

$-u_2 = -3$

$u_2 = 3 \quad u_1 = 7$

$u = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & \cancel{2}\,1 \end{bmatrix}$$

Pivot position every column

T_A is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} v_1 - 2v_5 = v_2 - 2v_6 \\ v_3 + 2v_5 = v_4 + 2v_6 \end{array}$$

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Not every column has a pivot position

T_A is not one-to-one

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ v_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} v_2 \\ v_4 \\ v_6 \end{bmatrix}$$

$\cancel{v_1 + v_3 = v_2 + v_4}$

$\cancel{2v_3 + 4}$

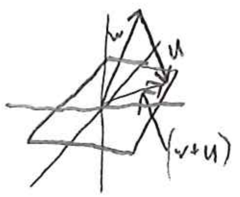
$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

~~[illegible] comb[illegible] of two vectors with coefficients of 1 [illegible] combination with a [illegible] would [illegible] fall [illegible] span.~~

True; since the span of two vectors in $\mathbb{R}^3$ can be visualized as a plane in 3D space, and u is in the span of (u, v), then w must also lie in that plan if w+u is to be in the span as well.



w+u is in +

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True; $\{u, v\}$ is linearly independent only if u and v are scalar multiples, which would not allow $\{u, v, w\}$ to be linearly independent.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$A_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $A_v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$2u_1 + u_2 = 1$ $2u_1 + u_2 = 2$

$u_1 + u_2 = 1$ $u_1 + u_2 = 2$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

True; since all matrix transformations are linear transformations, the matrix transformation T_A preserves linear dependence between $\{u, v\}$, and $\{Au, Av\}$.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

True: since all matrix transformations are linear transformations, applying the same transformation to all three vectors preserves the column space and T(u)'s existence in it.

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Yanyan Li

UB Person Number:

5	0	2	7	5	1	3	0

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2+\frac{b}{3} \end{array}\right]$

$\because w \in \text{span}(v_1, v_2, v_3)$

$\therefore 2 + \frac{b}{3} = 0$

$\therefore b = -6$

b) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

there is no pivot position in last column.

Thus, the Set is Linearly dependent.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$AA^{-1} = I$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \times 2 \\ )+ \end{array} \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right])- \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right])+ \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 2 & 0 & 2 & -2 & 2 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

the standard matrix $= \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} u = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 7 \\ x_2 = 3 \end{array} \Rightarrow v = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $Nul(A) = \{0\}$

every column has pivot position

Thus, It's one-to-one

b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

the third column has no pivot postion

Thus, It's not one-to-one

$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T_A(v_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T_A(v_2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True

$w + u = c_1 u + c_2 v \quad (c_1, c_2 \in \mathbb{R})$

$w = (c_1 - 1)u + c_2 v$

$w \in \mathrm{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True.

$\{u, v, w\}$ is Linearly independent means they're not multiple of each other.

then $\{u, v\}$: u, v won't be multiple of each other

then they're Lineearly independent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True

$u = c_1 v + c_2 w$

$T(u) = T(c_1 v + c_2 w)$

$\because$ T is Linear transformation

$\therefore T(c_1 v_1) + T(c_2 w) = T(c_1 v + c_2 w)$

$\therefore T(u) = T(c_1 v_1) + T(c_2 w_1)$

$\therefore T(u) \in \text{span}(T(v), T(w))$

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Connor Wilson

UB Person Number:

5	0	2	5	4	9	2	2
0	●	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	●	2	2	2	●	●
3	3	3	3	3	3	3	3
4	4	4	4	●	4	4	4
●	5	5	●	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	●	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$x_1 - x_2 + x_3 = -2$

$x_2 + 2x_3 = 2$

$2v_2 \rightarrow b = -6$

$b = 2x_1 - 3x_2$

$\boxed{b = -6 \text{ only}}$

$x_1 - (2 - 2x_3) + x_3 = -2$

$x_1 + 3x_3 = 0$

$b = 2(-3x_3) - 3(2 - 2x_3)$

$b = -6x_3 - 6 + 6x_3$

$\boxed{b = -6}$

~~b) $\{v_1, v_2, v_3\}$ is linearly independent because~~

~~$v_1 \neq c_1 v_2$ for any $c_1 \in \mathbb{R}$;~~

~~$v_2 \neq c_2 v_3$ for any $c_2 \in \mathbb{R}$;~~

~~$v_3 \neq c_3 v_1$ for any $c_3 \in \mathbb{R}$.~~

$\boxed{\text{b) } \{v_1, v_2, v_3\} \text{ is not linearly independent because } 3v_1 = v_3 - 2v_2.}$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

$a_1 - a_4 + 2a_7 = 1$

$a_1 + a_7 = 0$

$2a_4 - a_7 = 0$

$-a_7 - a_4 + 2a_7 = 1$

$1 + a_4 = a_7$

$1 + a_4 = 2a_4$

$a_4 = 1 \quad a_7 = 2 \quad a_1 = -2$

$a_2 - a_5 + 2a_8 = 0$

$a_2 + a_8 = 1$

$2a_5 - a_8 = 0$

$2a_5 = a_8$

$a_8 = 1 - a_2$

$1 - a_8 - \frac{1}{2}a_8 + 2a_8 = 0$

$\frac{1}{2}a_8 = -1$

$a_8 = -2$

$a_5 = -1 \quad a_2 = 3$

$a_3 - a_6 + 2a_9 = 0$

$a_3 + a_9 = 0$

$2a_6 - a_9 = 1$

$a_3 = -a_9$

$a_6 = \frac{1}{2}(1 + a_9)$

$-a_9 - \frac{1}{2}(1 + a_9) + 2a_9 = 0$

$-\frac{1}{2} + \frac{1}{2}a_9 = 0$

$a_9 = 1$

$a_3 = -1$

$a_6 = \frac{1}{2}(1 + 1) = 1$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad (A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\cancel{(A^T)}(A^{-1})^T C = B(A^{-1})^T$$

$$C = B(A^{-1})^T$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 2 \\ -1 & 2 & 3 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 2 \\ -1 & 2 & 3 \end{bmatrix}}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

std. matrix of $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

~~$x_1 - 2x_2 = 1$~~

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & 3 & -2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 5 & -3 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 5 & -3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & -18 \end{array}\right]$$

$\therefore$ There are no vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

1-1 if $Nul(A) = 0$ $\longrightarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

∴ only solution to $Av = 0$ is $v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

∴ $Nul(A) = 0$

$\boxed{\therefore A \text{ is one-to-one}}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 2x_3$
$x_2 = -2x_3$

∴ $Nul(A) \neq 0$

∴ A is not one-to-one.

ex. for vectors $v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$, $T_A(v_1) = T_A(v_2)$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$.

This is true. By def. of span, $w+u = c_1u + c_2v$ for some $c_1, c_2 \in \mathbb{R}$. Therefore $w = c_1u + c_2v - u$, and combining gives $w = (c_1 - 1)u + c_2v$. However, $c_1 - 1$ is just some other constant in $\mathbb{R}$, so let $c_3 = c_1 - 1$. Therefore $w = c_3u + c_2v$, meaning $w \in \operatorname{Span}(u, v)$ by definition.

QED.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

This is true. If $\{u, v, w\}$ is linearly independent, then

$u \neq c_1v + c_4w$ for any $c_1, c_4 \in \mathbb{R}$

$v \neq c_2w + c_5u$ for any $c_2, c_5 \in \mathbb{R}$

$w \neq c_3u + c_6v$ for any $c_3, c_6 \in \mathbb{R}$

Since $u \neq c_1v$ and $v \neq c_5u$ for any $c_1, c_5 \in \mathbb{R}$, then $\{u, v\}$ is linearly independent by definition.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

This is False. For example, let $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $Au = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $Av = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Au and Av are the same vector, so they are clearly linearly dependent, however, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

This is true. If $u \in \text{Span}(v, w)$, then $u = c_1 v + c_2 w$ for some $c_1, c_2 \in \mathbb{R}$. So $T(u) = T(c_1 v + c_2 w)$. In order to show that $T(u) \in \text{Span}(T(v), T(w))$, we must show that $T(u) = c_3 T(v) + c_4 T(w)$ for some $c_3, c_4 \in \mathbb{R}$. Let $c_1 = c_3$ and $c_2 = c_4$. By def. of linear transformation, $c_1 T(v) + c_2 T(w) = T(c_1 v) + T(c_2 w)$. Again, by def. of linear transformation, $T(c_1 v) + T(c_2 w) = T(c_1 v + c_2 w)$. Since we already had that $T(u) = T(c_1 v + c_2 w)$, then $T(u) \in \text{Span}(T(v), T(w))$.

QED.

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Kristiyan Aleksandrov

UB Person Number:

5	0	2	3	9	2	0	9
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$-3 \leq b \leq 2$

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{+(\cdot -2)} \left[\begin{array}{cccc} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b-2 \end{array}\right] +$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array}$$

$$\left[\begin{array}{cccc} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b \end{array}\right]$$

$b = 0$

b) $\left[\begin{array}{cccc} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \xrightarrow{\cdot -2} \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] + \rightarrow \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot -1$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_3$
$x_2 = 0$
$x_3 = x_3$

$x_3 = \text{free}$
So infinite Solutions
Which means that it is linearly dependent.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2\cdot -1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2\cdot -2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \; 3\cdot +1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{+} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \; e\cdot -1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

8
0
6

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \qquad B \cdot (A^T)^{-1} = C$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right] \leftarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array}\right] \leftarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right] \div 3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \rightarrow$$

$$(A^T)^{-1} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 4 \\ 5 & 5 & 5 \\ -2 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -7 & -7 \\ 8 & 9 & 8 \\ 5 & 7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -5 & -7 & -7 \\ 8 & 9 & 8 \\ 5 & 7 & 7 \end{bmatrix} =$$

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

$$e_1 = \begin{matrix} 1 \\ 0 \end{matrix} \quad e_2 = \begin{matrix} 0 \\ 1 \end{matrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \quad T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$\begin{array}{cc|c} c_1 & c_2 & \\ \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot(-1) \to \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -1 & -2 \end{bmatrix} \cdot(-1)$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -2 \end{bmatrix} \div 3$$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\cdot -3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \cdot -\frac{1}{2}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \to \div 2 \\ \to \div 2 \end{matrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot -1$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot 2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot -2$

A is One to one

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \cdot -\frac{1}{3}$

Also one to one

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Pivot Position in every column

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True because U is not needed for W to be in the span if it was given that all 3 were in $\mathbb{R}^3$.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False it would depend on what vectors U and V are because the homogeneous solution may be different when using only the 2 vectors out of the 3.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in $\mathrm{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\mathrm{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

True

MTH-309T-F19-EX1-015-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Alexander T. Nowasell

UB Person Number:

5	0	2	9	0	3	2	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)

$\boxed{b=0}$

only solution

$$\begin{matrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{matrix}$$

$$\begin{matrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b-2 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b \end{matrix}$$

for $\begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$ in span

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{matrix}$$

~~reduces~~ ~~b)~~ ~~linearly independent~~

reduced matrix

$$\begin{matrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) No

$\boxed{\text{linearly dependent}}$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

Reduction

$$\begin{matrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 0 & -3 & 4 & 2 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & -2 & 3 & 2 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{matrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

Check:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{matrix}$$

$$\begin{matrix} -2 \\ 1 \\ 2 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

$$\begin{matrix} -1 \\ 1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{matrix}$$

12
34
56

1 3 5
2 4 6

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A^{T-1} \cdot B = C$

$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$A^{T-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$

$-2+4+6$
$3-4-6=-7$
$-1+4+3=6$

$\begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

$=$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

$\begin{matrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 2 & 1 & -1 & 0 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -1 & -2 \end{matrix}$

$-4+5+\cancel{4}=5$
$6-5-4=-3$
$-2+5+2=5$

$\begin{matrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{matrix}$

$\begin{matrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \end{matrix}$

$-6+4+2=0$
$9-4-2=3$
$-3+4+1=2$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

$\begin{matrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{matrix}$

b) $x_1 = 7$
$x_2 = 3$

$\begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$\begin{matrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{matrix}$

$\begin{matrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{matrix}$

$\begin{matrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{matrix}$

$\begin{matrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{matrix}$

$\begin{matrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

1 1 0
0 2 4
3 4 4

1 1 0
0 2 4
0 1 4

1 1 0
0 1 2
0 0 2

1 1 0
0 1 0
0 0 1

1 0 0] Pivot
0 1 0] Pivot
0 0 1] Pivot

one to one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

0
0
0

-2
-2
1

1 1 0
0 2 4
3 4 2

1 1 0
0 1 2
0 1 2

$x_1 = 2x_3$
$x_2 = -2x_3$

1 0 -2
0 1 2
0 0 0

1 1 0
0 1 2
0 0 0] - Not Pivot

Not one-to-one

$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$V_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$$\begin{array}{c}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ u \quad v \quad w\end{array}$$

$$w + u \in \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

True,
if double plus u kept it the span of u, v, the w by it self should be in the span.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True,
because if, $\{u, v, w\}$ was linerly independent, then making w zero didn't change it to be dependent, So removing w entirely changes nothing.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True, ~~The transformation~~ the 2x2 Matrix keeps the

True, they are dependent after ~~the transformation,~~ ~~then putting~~

$\boxed{\text{True}}$,

Putting to linearly dependant vectors through transformation, they must stay dependent when put together with more dependents.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span(T(v), T(w)).

True, the linear transformation must be one to one and onto, so the span would be kept it tact.

MTH-309T-F19-EX1-052-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Abigail DeSantis

UB Person Number:

5	0	2	5	9	7	3	8
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1.5 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1/2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

b: $v_3 - v_1 = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$

$2v_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$

b: $\{-2, -6\}$

the set $\{v_1, v_2, v_3\}$ is linearly independent because there is only one solution to $x_1v_1 + x_2v_2 + x_3v_3 = 0$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$(A^T)^{-1} A^T C = B (A^T)^{-1}$$

$$I C = B (A^{-1})^T$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} (-2+6-3) & (1-2+3) & (2-4+3) \\ (-8+15-4) & (4-5+4) & (8-10+4) \\ (-6+6-1) & (3-2+1) & (6-4+1) \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 2 \\ -1 & 2 & 3 \end{bmatrix}$$

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-2(0) \\ 1+(0) \\ 1-3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \left[T(e_1) \;\; T(e_2)\right]$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-2(1) \\ 0+1 \\ 0-3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

one to one

b/c $\mathrm{Nul}(A) = \{0\}$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{Nul}(A) = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

Not one to one

$x_1 = 2x_3$

$x_2 = -2x_3$

$x_3 = x_3$

$v_1 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True because if $w + 1(u)$ is a linear combo of span(u,v), then $w + 0(u)$ is also a linear combo of span(u,v) $\Rightarrow$ therefore w is an element of span(u,v)

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True

$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

because the only way for 2 vectors to be linearly dependent is if they are scalar multiples of one another, but if 2 of the 3 vectors are scalar multiples $Ax = 0$ will have a free variable therefore it is not linearly independent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

true because $T(cv) = cT(v)$

if $u = cv$ then $T(u) = cT(v)$

$T(u) = T(cv)$

$T(u) = T(u)$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True

if u is a linear combo of v & w

$u = v + w$

then $T(u) = T(v+w)$

$T(u) = T(v) + T(w)$

$T(u)$ is a linear combo of $T(v)$ & $T(w)$

therefore

$T(u) \in \text{span}(T(v), T(w))$

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Jessica Zuritis

UB Person Number:

5	0	1	7	9	9	7	6
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$. → w can be writen as a linear combination of v_1, v_2, v_3

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

(a)
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] (R1 \times -2) + R3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] R2+R3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right] R2+R1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right]$$

$\boxed{b = -6 \text{ for } w \in \text{span}(v_1, v_2, v_3)}$

(otherwise get no solution) due to pivot position in column of constents

(b)
$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

→ this set in linearly independent if homo. eqn. only has trivial solution ∴

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] (R1 \times -2) + R3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] R2+R3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable! not every column of the matrix is a pivot therefore the set $\{v_1, v_2, v_3\}$ are not linearly independent

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$(R1x-1)+R3 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$(R3x-1)+R2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right]$$

$$(R2x-1)+R3 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right]$$

$$R2+R1 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right] \; 1\;-1$$

$$\begin{matrix} R3x-2 \\ +R1 \end{matrix} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-2-1+4=1$

$3+1-4=0$

$-1-1\ 2=0$

$-2\ 0\ 2=0$

$(-2x-2)$

$4-1=3$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \quad R1+R2$$

get this to RRE

this matrix is solution

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \quad (R1 \times -2) + R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & -1 & -1 & 1 & -2 & -5 \end{array}\right] \quad R2+R3=R3$$

$2 \times 2 + -4$

$4 + -4 = 0$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \quad R2 \times -1 + R1$$

$(6 \times 2) - 4$

$12 - 4$

8

$(5 \times 2) + -5$

$10 - 5$

5

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -4 & -5 & -4 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \quad R3 \times -2 + R2$$

$(6 \times -2) + 5$

$-12 + 5$

-7

$(5 \times -2) + 7$

$-10 + 7$

-3

$(2 \times -2) + 7$

$-4 + 7$

3

$$R3 \times 2 + R1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -4 & -5 & -4 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$Ax = \quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T. $\longrightarrow \boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}} \Rightarrow$ standard matrix of $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$(R1 \times -1) + R2 \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$$

$$(R1 \times -1) + R3 \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] /3$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \cdot -1$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] (R3 \times 2) + R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] (R2 \times -1) + R3$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 7$
$x_2 = 3$

$$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

R1x-3 +R3 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} 1/2$

R2x-1 +R3 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} 1/2$

R3x-2 +R2 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\downarrow$

one-to-one and onto b/c pivot pos. in every column and row, respectively.

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ (R1x-3)+R3

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} 1/2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ (R3x-1)+R2

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\downarrow$

Not one-to one b/c can only have a pivot position in the first two columns, here we have only two.

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

w+u can be written as a linear comb. of u and v

w can be written as a linear comb. of u and v

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c_3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$\Downarrow$ then...

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = c_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c_2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = c_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c_3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$\therefore$ $\boxed{\text{true}}$ w is in the span (u, v)

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$\boxed{\text{True,}}$ if u, v, and w are already linearly independent and are not linear combinations of eachother, the the set $\{u, v\}$ must also already be linearly independent

$$\left\{ \underset{u}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{v}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{w}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\} \Rightarrow \text{linearly independent}$$

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow \text{still linearly independent}$$

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\Downarrow$

u and v are linear combinations of eachother

$$\overset{Au}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \Rightarrow u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\overset{Au}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

$R2 \times \frac{-1}{2} + R1$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{matrix} /2 \\ /2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} \begin{matrix} /2 \\ /2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\triangleright$ true, solutions are still linearly dependent

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

$$T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = T\left(c_1\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + c_2\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right)$$

$$T(u) = c_1 T(v) + c_2 T(w)$$

$\Downarrow$

$$T\left(c_1\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + c_2\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right)$$

$\Downarrow$

$\boxed{\text{true}}$

For linear transformations, $T(u+v)$ can be written as $T(u) + T(v)$, and here, because u is in the span of v and w, its transformation can be written as a combination of the transformations for v and w

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Brandon Hosken

UB Person Number:

5	0	1	8	4	5	0	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathrm{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathrm{w} \in \mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$.

b) Is the set $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \cdot(-2)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4b \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6b \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6b \end{array}\right]$$

x_3 is free

~~b=0~~

a) The only value of b that allows w to be in Span(v_1, v_2, v_3) is 0 because if it isn't zero there is no solution to the augmented matrix after row reduction.

b) The set $\{v_1, v_2, v_3\}$ is not linearly independent because x_3 is a free variable meaning it has infinitely many solutions.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$\cancel{A^{-1} = [w_1 \ w_2 \ w_3]}$

$\cancel{w_1 = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \cdot -1$$

$\cancel{\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right]}$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \cdot -1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right] \cdot 1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \cdot 1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \cdot -1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \cdot -2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \cdot 1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$C = B \cdot (A^T)^{-1}$

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$(A^T)^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \cdot -2$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right] \cdot 1$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right] \cdot 1$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \cdot -2$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \cdot -1$

$\cancel{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{array}\right]}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$

$C = B \cdot (A^T)^{-1}$

$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$

$C = [Bv_1 \ Bv_2 \ Bv_3]$

$Bv_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2+6+-3 \\ -8+15+-8 \\ -6+6+-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$Bv_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+-2+3 \\ -4+-5+4 \\ -3+-2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -4 \end{bmatrix}$

$Bv_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+-4+3 \\ 8+-10+4 \\ 6+-4+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\boxed{C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -5 & 2 \\ -1 & -4 & 3 \end{bmatrix}}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$A = [T(e_1) \; T(e_2)]$ $\quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-2(0) \\ 1+0 \\ 1-3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-2(1) \\ 0+1 \\ 0-3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

b)

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot -1$

$\downarrow$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -4 & -12 \end{array}\right] \cdot -1$

$\downarrow$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -4 & -12 \end{array}\right] \cdot -2$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 0 & -14 \end{array}\right] \cdot 1/3$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -14 \end{array}\right] \cdot 2$

$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & -14 \end{array}\right]$

$0 \neq -14$

$\therefore$ There are no solutions for u that ~~that~~ satisfies $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot -3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \cdot 1/2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \cdot -1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot 1/2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ S-2

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ S $\cdot -1$

$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$ $\therefore A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot -3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \cdot 1/2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \cdot -1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ S $\cdot -1$

$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ is not one-to-one as it does not have pivot position in every row.

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

~~False, because~~
True, because w+u is some linear combination of u that results in the ~~Span~~ the new ~~m~~ vector being in the span, for that to occur by the definition of a linear combination w must also be in the span(u,v).

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~True,~~ False, because the row reduction of the vectors in an augmented matrix require n x n to be linearly independent so without the 3rd vector in $\mathbb{R}^3$, it is impossible to have only 1 solution.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True, because Au & Av is just a linear transformation so it will still retain the properties of having infinitely many solutions to the vector.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True, because ~~all~~ you are applying the same transformation to ~~both~~ all 3 vectors so the span will stay the same as before meaning T(u) is in Span((T(v), T(w))

MTH-309T-F19-EX1-023-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Mohammedanas Tai

UB Person Number:

5	0	2	3	2	6	6	5
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{(2)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -5 & 2 & -4+b \end{array}\right] \xrightarrow{5}$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{6+b}{12} \end{array}\right] \leftarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 12 & 6+b \end{array}\right] \frac{1}{12}$$

Span (b) = $\mathbb{R}^3$ since b'd be a solution to any real number

b)

$$\left.\begin{array}{l} x_1 - x_2 + x_3 = -2 \\ x_2 + 2x_3 = 2 \\ x_3 = \frac{6+b}{12} \end{array}\right\} \quad \begin{array}{l} x_1 = -2 - x_2 + x_3 \\ x_2 = 2 - 2x_3 \\ x_3 = \frac{6+b}{12} \end{array}$$

Since the equation has a solution any value ~~be~~ of b given

it is ~~a~~ not linearly indedepant set

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3×3 3×3

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A^{-1}$$

$$A^{-1} \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we needed to find a matrix s.t
it is equal to the Identiy matrix.

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$2 \times 2 \times 2 \times 3$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$A^T \cdot B = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = C$$

$$\left(\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix} \right) + \left(\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & -2 \end{bmatrix} \right) + \left(\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix} \right)$$

$$\left(1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right) + \left[2\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right]$$

$$+ \left[3\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right]$$

$$C = \begin{bmatrix} 7 & 7 & 1 \\ -1 & 6 & 5 \\ 9 & 7 & 3 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$A = [T(x_1), T(x_2)]$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

Standard Matrix = $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b)

$x_1 - 2x_2 = 1$

$x_1 + x_2 = 10$

$x_1 - 3x_2 = -2$

$$\Rightarrow \begin{array}{ccc} x_1 & x_2 & \\ \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 9 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{bmatrix}$$

$t(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -19 \\ 11 \\ 29 \end{bmatrix}$ ✓

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} (-3)$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -10 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -2 \end{bmatrix}$

Both is not one to one since every row is not a pivot ~~column~~ row

$A \cdot \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$

$\mathrm{v}_1 \rightarrow$

$= \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$

$A \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot 2$

two =

The vectors are such when the last number makes both of them equal,

$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\mathrm{v}_2 = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix}$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True because if $w+u$ is in the Span(u,v) then $w \in \text{Span}(u,v)$ shows it is a part of that span. Graphically it would be on the same line because w is only being added by u

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False you can have $\left\{ \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 10 \\ 14 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix} \right\}$

when it is dependant while these two are m dependant.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True because for what A you are given it'll be on the span of u,v assumming they are dependent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

~~Tr~~ True because when ~~we are~~ we are doing linear transformati we are just multiplying vectors ~~with~~ which makes it a linear ~~trans~~ combination and a span is a set of liner combination.

MTH-309T-F19-EX1-013-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

PRANSHU TEOTIA

UB Person Number:

5	0	2	1	8	5	6	0

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{bmatrix}$$

$$R_1 = R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{bmatrix}$$

$\underline{\underline{b = -6}}$ → For all other values of b there will not be a solution.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = R_1 + R_2 \rightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

↳ Free variable

~~Since col.~~ Since, X_3 is a free variable, the set $\{V_1, V_2, V_3\}$ is <u>NOT</u> linearly independent

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = -R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = -2R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_1 = R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 2 \\ 3 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$\Rightarrow$

$a + 4d + 3g = 1$	$b + 4e + 3h = 2$	$c + 4f + 3i = 3$
$2a + 5d + 2g = 4$	$2b + 5e + 2h = 5$	$2c + 5f + 2i = 4$
$3a + 4d + g = 3$	$3b + 4e + h = 4$	$3c + 4f + i = 1$
$\cancel{a = \frac{7}{6}, d = 0, g}$	$b = \frac{-3}{4}, e = 2, h = \frac{-7}{4}$	$c = -2, f = 2, i = -1$
$a = -1, d = 2, g = -2$		

$$C = \begin{bmatrix} -1 & -\frac{3}{4} & -2 \\ 2 & 2 & 2 \\ -2 & -\frac{7}{4} & -1 \end{bmatrix}$$

4. **(20 points)** Let $T \colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$x_1 - 2x_2 = 1$

$x_1 + x_2 = 10$ We ge $x_1 = 7,\ x_2 = 3$

$x_1 - 3x_2 = -2$

(b) $u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} \to \begin{matrix} 7a + 3b = 1 \\ 7c + 3d = 10 \\ 7e + 3f = -2 \end{matrix}$$

3×2 $\quad 2 \times 1$

Standard Matrix of $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

(a)

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_3 \text{ and } R_3 = R_3/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We have pivot pos. in every row so matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ is one-to one.

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Since every col is not a pivot col matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ is not one-to one.

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $V_2 = \begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix}$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

TRUE

If $w + u = v$, then $w = v - u$

v is in the span(u, v) and $(v - u)$ is also in the span(u, v)

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

TRUE

The set of u, v, w, in that each column is a pivot column

So, in the set of $\{u, v\}$ each column will also be a pivot column.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

FALSE

If $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

then $Au \to \begin{bmatrix} 1 & 2 & \cancel{1} \\ 0 & 0 & 0 \end{bmatrix}$ is linearly dependent ~~but u is linearly independent.~~

~~TRUE, the~~ but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is linearly independent.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then T(u) must be in Span(T(v), T(w)).

TRUE

$u = v + w$

$T(u) = T(v + w)$

$\to T(v) + T(w)$

So, $T(u)$ is in Span $(T(V), T(w))$