MTH-309T-F19-EX1-040-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

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Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

10	10	10	14	16	6	4	--	--	70	B-
1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $x_1 v_1 + x_2 v_2 + x_3 v_3 = w$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \cdot(1) \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 1 & -2 & -1 & 2+b \end{array}\right] \cdot(-1)$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & \cancel{-2-b} \end{array}\right] \cdot 1 \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & \cancel{-b} \\ 0 & -1 & -2 & \cancel{-2-b} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -1 & -2 & -2-b \\ 0 & 0 & 0 & -b \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2+b \\ 0 & 0 & 0 & b \end{array}\right]$$

4+b

6+b

4+b

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & b \\ 0 & 1 & 2 & 2+b \\ 0 & 0 & 0 & \cancel{b} \end{array}\right]$$

b+6

answer ?

b) The set $\{v_1, v_2, v_3\}$ is not lin. independent; it has more than one, trivial solution. ?

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{+\,\cdot 2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{+\,\cdot(-1)} \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & -2 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{+\,\cdot 2}$$

$$= \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & -2 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\cdot(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{+\,\cdot(-1)}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{+\,\cdot(-1)} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow (A^{-1})^T = \begin{bmatrix} -2 & \cancel{2}\ 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$(A^{-1})^T \cdot B = C \qquad \begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2+8+6 & -4+10+4 & -6+8+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ -1+4+3 & -2+5+2 & -3+4+1 \end{bmatrix}$$

✓

$$\boxed{C = \begin{bmatrix} \cancel{12} & \cancel{10} & \cancel{4} \\ -7 & -3 & 3 \\ 6 & \cancel{2} & 2 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

a) $A = [T(e_1) \quad T(e_2)]$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$ ✓

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$ ✓ $\cdot(-1) \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -1 & -3 \end{array}\right] \cdot(-1) \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{array}\right]$

$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 1 & 1 & 10 \end{array}\right] \cdot -1 \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$

$\cancel{u = \text{Span}\left(\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}\right)}$

This would mean: no solutions

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a)
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 3 & 4 & 4 & 0 \\ 0 & 2 & 4 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \cdot(-3)$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \cdot(-1) \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \cdot(-1) \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Pivot position in every col., so $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ is one-to-one. ✓

b)
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \cdot(-3) \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \cdot(-2) \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

not one-to-one ✓

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 + c_2 \\ c_2 + 2c_3 \\ 0 \end{bmatrix}$$

Let $c_1 = 1$, $c_2 = 2$, $c_3 = 3$

$$\Rightarrow \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$$

$v_1 = \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 9 \\ 24 \\ 0 \end{bmatrix}$

you can plug arbitrary values into free variables, not all variables.

$T_A(v_1) \neq T_A(v_2)$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True; since $u \in Span(u,v)$, then if $w+u \in Span(u,v)$ then w must be a multiple of a vector in Span(u,v). Therefore, $w \in Span(u,v)$.

why?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

A set is lin. ind. when $x_1 u + x_2 v + x_3 w = 0$ has only one solution, where $u, v, w = 0$. If the set $\{u, v, w\}$ is lin. ind., then either combination of two of those vectors must also be lin. ind. Therefore, this statement is true. ✓

This could be explained more clearly...

0 1 2 3 4 5 6 7 8 9 10

7. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False; ✓ the matrix A can cause Au, Av to become lin. dependent even if u,v are lin. independent.

Example?

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True; ✓ let $u = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$T(u) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$

$T(v) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$T(w) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

T(u) is in Span(T(v), T(w))

So, T(u) must be in Span(T(v), T(w))

An example does not show that this is always true.

0 1 2 3 4 5 6 7 8 9 10