MTH-309T-F19-EX1-010-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Zachary Ross

UB Person Number:

5	0	1	7	8	1	2	7

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

13	7	8	12	20	8	6	74	nan
1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $w \notin cv_1$ since 2nd row of cv_1 is always 0.
$w \notin cv_3$ since 2nd row is same for both, but first row is different.
$w \in cv_2$ when $c = 2$ and $b = -6$ ✓

$$w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} = 2v_2 = 2 \cdot \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$$

$$\boxed{b = -6}$$

b) Yes, since there is only 1 solution!

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & c_1 \\ 0 & 1 & 2 & c_2 \\ 2 & 3 & 0 & c_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & c_1 \\ 0 & 1 & 2 & c_2 \\ 0 & 5 & -2 & c_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & c_1 \\ 0 & 1 & 2 & c_2 \\ 0 & 0 & -12 & c_3 \end{array}\right] \to$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right]$$

$$\boxed{\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ x_3 &= c_3 \end{aligned}}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

13

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A^{-1} \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad r_2 = r_2 - r_1$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad r_1 = r_1 + r_2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad r_3 = r_3 - 2r_2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \quad \begin{array}{l} r_1 = r_1 - r_3 \\ r_2 = r_2 + r_3 \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}_{A^{-1}}$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

0 1 2 3 4 5 6 **7** 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$3 \times 3 \cdot 3 \cdot 3$ A B C

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$x_1 = 1 - y_1$

$z_1 = 1 + \frac{x_1}{2}$

$z_1 = 1 + \frac{1-y_1}{2}$

$x_1 + y_1 = 1$

$-x_1 + 2z_1 = 2$

$2x_1 + y_1 - z_1 = 3$

$2(1-y_1) + y_1 - \left(1 + \frac{(1-y_1)}{2}\right) = 3$

$2 - 2y_1 + y_1 - 1 - \frac{1}{2} + \frac{y_1}{2} = 3$

$1 = 2y_1 + y_1 = 6$

$y_1 = -5$

$x_1 = 6$

$z_1 = 4$

$x_2 = 4 - y_2$

$z_2 = \frac{5}{2} + \frac{x_2}{2}$

$x_2 + y_2 = 4$

$-x_2 + 2z_2 = 5$

$2x_2 + y_2 - z_2 = 4$

$4 = 2(4-y_2) + y_2 - \left(\frac{5}{2} + \frac{(4-y_2)}{2}\right)$

$8 = 4(4-y_2) + 2y_2 - 5 - 4 + y_2$

$8 = 16 - y_2 + 2y_2 - 9 + y_2$

$y_2 = 3$

$x_2 = 1$

$z_2 = 1$

$x_3 + y_3 = 3$

$-x_3 + 2z_3 = 2$

$2x_3 + y_3 - z_3 = 1$

$$\boxed{C = \begin{bmatrix} 6 & 3 \\ -5 & 1 \\ 4 & 1 \end{bmatrix}}$$

$x_3 = 3 - y_3$

$z_3 = 1 - \frac{3}{2} + \frac{y_3}{2}$

$1 = 2(3-y_3) + y_3 - \left(1 - \frac{3}{2} + \frac{y_3}{2}\right)$

$2 = 12 - 4y_3 + 2y_3 - 2 + 3 - y_3$

$2 = 13 - 3y_3 = \frac{11}{3}$

Score: 0 1 2 3 4 5 6 7 **8** 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = 3\times 2$ since $3\times 2 \cdot 2\times 1$

$$\begin{bmatrix} c_1 & c_4 \\ c_2 & c_5 \\ c_3 & c_6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$c_1 x_1 + c_4 x_2 = 1x_1 - 2x_2$ $\quad c_1 = 1 \quad c_4 = -2$

$c_2 x_1 + c_5 x_2 = 1x_1 + 1x_2$ $\quad c_2 = 1 \quad c_5 = 1$

$c_3 x_1 + c_6 x_2 = 1x_1 - 3x_2$ $\quad c_3 = 1 \quad c_6 = -3$

Standerd Matrix: $\boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

b)
$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 7$

$x_2 = 3$

$\boxed{T(u) = \text{Span}\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right)}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\downarrow$ $r_3 = r_3 - 3r_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

All column are pivot columns, therefore

$\boxed{\text{all } T_A(v) \text{ are one-to-one.}}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & \boxed{0} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

not one to one

$$\boxed{v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix}}$$

$$T_A(v_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T_A(v_2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1+1 \\ 2-2 \\ -3+4-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{T_A(v_1) = T_A(v_2)}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

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6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False.

$$u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad w + u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \in \text{Span}(v)$$

$\in \text{Span}(u, v)$

$$w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$w \neq cu$, $w \neq cv$, so $w \notin \text{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True. If set $\{u, v, w\}$ are indepentant, that means $u \neq cv$, $u \neq cw$, $v \neq cu$, $v \neq cw$, $w \neq cu$, $w \neq cv$.

Show linear independence between u and v.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True. If Au and Av are linearly dependent, then Au = cAv. Therefore u = cv, so u and v must be linearly dependent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True. If u ∈ Span(v, w) then u = cv or u = cw. If your take T(u), you get A · u = A · cv, which shows T(u) ∈ Span(T(v), T(w)).

0 1 2 3 4 5 6 7 8 9 10