MTH-309T-F19-EX1-077-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Gerra Gamaro

UB Person Number:

5	0	1	9	7	7	7	0

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
20	8	7	20	20	3	2	--	--	80	B

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $[V_1 \ V_2 \ V_3 \mid W]$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \cdot(-2)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] \cdot(1)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] \cdot(1)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & \underbrace{0}_{\text{free}} & \underbrace{6+b}_{6+b=0} \end{array}\right] \rightarrow 0x_1 + 0x_2 + 0x_3 = 6+b$$

$0 = 6+b$

free $\quad 6+b=0$ for it to have a solution

$\boxed{b = -6}$ ✓

b) $\{V_1, V_2, V_3\}$ is linearly dependent because x_3 is a free variable indicating that the system has infinitely many solutions, making it linearly dependent ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A^{-1} \to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \cdot(-1)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & \cancel{0}\ 1 & 0 \\ 0 & 2 & -1 & 0 & \cancel{0}\ -2 & 1 \end{array}\right] \cdot(-2)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \cdot(1)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \cdot(1)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \cdot(-1)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & \cancel{0} & \cancel{1}\ -1 \\ 0 & 1 & 0 & 1 & \cancel{0} & 1 \\ 0 & 0 & 1 & 2 & \cancel{0} & 1 \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{A^{-1}}$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & \cancel{0} & -1 \\ 1 & \cancel{0} & 1 \\ 2 & \cancel{0} & 1 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^TC = B \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} & C & \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \rightarrow x = \begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \rightarrow x = \begin{bmatrix} 13 \\ -11 \\ 9 \end{bmatrix}$$

Where did this come from?

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \rightarrow x = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & \cancel{13} & 0 \\ -7 & \cancel{-11} & 3 \\ 6 & \cancel{9} & 2 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = [T(e_1) \;\; T(e_2)]$

$T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$ ✓

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$Au = T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot(-1)$$ ✓

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \cdot(-1)$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \cdot(1/3)$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \cdot(1)$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \cdot(2)$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \boxed{\begin{array}{l} x_1 = 7 \\ x_2 = 3 \end{array}}$$

$$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$ ✓

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

one to one → pivot in every column

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot(-3)$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \cdot(1/2) \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \cdot(-1) \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \cdot(-1) \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot(1/2)$$

$$\to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot(-2) \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot(2) \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Pivot in every column, therefore it is one to one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot(-3)$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \cdot(1/2) \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \cdot(-1) \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \cdot(-1) \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & \underbrace{0}_{\text{free}} \end{bmatrix}$$

b.) Not one to one because there is a free variable on the last column.

$T_A(V_1) = T_A(V_2)$

$AV_1 = AV_2$

$$\mathrm{Nul}(A) = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \to \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \qquad \begin{aligned} x_1 &= 2x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$$

$$V_2 = V_1 + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \to \boxed{V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w}+\mathbf{u} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$.

? True ✓ because u is in the span of it self, making it equal to $w \in \operatorname{Span}(u,v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~False~~ → two vectors can be linearly independent but the third can be linearly dependent and vise versa.

$V_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$ $V_3 = \begin{bmatrix} 1 \\ 3 \\ -12 \end{bmatrix}$ $V_1, V_2, V_3 \to$ linearly dependent

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -2 & 4 & -12 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free

However, V_1 and V_2 are linearly independent

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 5 & 0 \\ -2 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$, $x_2 = 0$ } trivial solution.

This is not what this problem was about...

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False →

why ?

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

$u \in \text{Span}(v, w)$

$T(u) \in \text{Span}(T(v), T(w))$

True ✓ because the transformation is applied for each vector, providing the same vectors at different positions

so ?

0 1 2 3 4 5 6 7 8 9 10