MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Fahim Noor

UB Person Number:

5	0	2	8	0	2	1	1
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{\cdot -1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & b \end{array}\right]$

$\xrightarrow{\cdot 3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right]$ $\therefore$ for all values of b greater than -5, $w \in \text{Span}(v_1, v_2, v_3)$

b) The $\{v_1, v_2, v_3\}$ is linearly independent because no vector in that set is a multiple of another.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

Identity Matrix

$$[\,A \mid I\,] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

Identity Matrix

$\therefore$ A is invertible and

$$A^{-1} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\underset{3\times 3}{A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}} \cdot \underset{3\times 3}{C} = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}}$$

$$C = B \cdot \frac{1}{A^T} = B \cdot A^{T^{-1}}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^{T^{-1}} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 2 & 6 & 2 & 2 & 0 \\ 0 & 0 & 6 & -3 & 3 & 3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 5 & -1 & -3 \\ 0 & 0 & 6 & -3 & 3 & 3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 6 & 0 & -6 & -6 & 0 & 6 \\ 0 & 2 & 0 & 5 & -1 & -3 \\ 0 & 0 & 6 & -3 & 3 & 3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 6 & 0 & 0 & -3 & -3 & 3 \\ 0 & 2 & 0 & 5 & -1 & -3 \\ 0 & 0 & 6 & -3 & 3 & 3 \end{array}\right] \begin{array}{c} \cdot\frac{1}{6} \\ \cdot\frac{1}{2} \\ \cdot\frac{1}{6} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 5/2 & -\frac{1}{2} & -3/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

Identity Matrix $\qquad A^{T^{-1}}$

$$C = B \cdot A^{T^{-1}} = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}} \cdot \underset{3\times 3}{\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 5/2 & -\frac{1}{2} & -3/2 \\ -\frac{1}{2} & \frac{1}{2} & 1/2 \end{bmatrix}}$$

$$\boxed{C = \begin{bmatrix} -\frac{1}{2} & 5 & -3/2 \\ -2 & -5/2 & 2 \\ 3/2 & -3 & 1/2 \end{bmatrix}}$$

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = [\,T(e_1),\ T(e_2)\,]$ $\quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(e_1) = \begin{bmatrix} 1 - 2(0) \\ 1 + 0 \\ 1 - 3(0) \end{bmatrix} = \begin{bmatrix} 1-0 \\ 1+0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 0 - 2(1) \\ 0 + 1 \\ 0 - 3(1) \end{bmatrix} = \begin{bmatrix} 0-2 \\ 0+1 \\ 0-3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

b)

$$\begin{array}{cc} x_1 & x_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right]$$

$$\boxed{\begin{aligned} x_1 &= 2x_2 \\ x_2 &= x_2 \end{aligned}}$$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ $\cdot -3$

$\cdot \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$\cdot -1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$\cdot -1 \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

$\cdot 1 \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\cdot \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ A has a pivot position in every column so it is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $\cdot -3$

$\cdot \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\cdot -1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\cdot -1 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 2x_3$

$x_2 = -2x_3$

$x_3 = x_3$

vector form

$\begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} \to x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ A is not one-to-one since there is not a pivot position in every column, so

$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ & $V_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$

s.t $T_A(v_1) = T_A(v_2)$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False $w+u$ can be in the span of u, v but $w \in$ has no correlation with the span of u, v since w is added to u.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True since $\{u, v, w\}$ in $\mathbb{R}^3$ are all vectors with leading ones and in row reduced e form to be linearly independent, also to be linearly independent u, v, w cannot be multiples of each other so $\{u, v\}$ must be linearly independent because they are not multiples of each other as proven by $\{u, v, w\}$ linear independence

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $Au = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$ $Av = \begin{bmatrix} 10 \\ 22 \end{bmatrix}$

2×1 2×1

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 2×2 $v = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $Au = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$ $Av = \begin{bmatrix} 13 \\ 29 \end{bmatrix}$

$Au = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $Av = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$

True

If Au and Av are linearly dependent the u, v must be linearly dependent since the scaler multiple multiplied to either Au or Av to get the other is the same scaler multiple to get c·u to v or c·v to u

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True since T(u) the u is a vector $\mathbb{R}^2$ and v, w is a vector in $\mathbb{R}^2$ ∴ T(u) must be in the span of T(v), T(w).

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MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Vincent Zheng

UB Person Number:

5	0	2	5	8	5	3	9
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

The value of b is -6

because $2v_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}$$

It is independent because Null(A) is {0}

2-
-3-2

$$\begin{matrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -5 & -2 & 0 \end{matrix}$$

~~Is it linearly independent?~~

-5+5
-2+10

$$\begin{matrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 8 & 0 \end{matrix}$$

$x_1 = 0$
$x_2 = -2$
$x_3 = 0$

~~It is not linearly independent because x_2 can be ... when set to 0~~

$$\begin{matrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & \\ 0 & 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$1-1 \quad 0--1 \quad 1-2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & +2 & -2 & 0 \end{array}$$

-2

$2-2 \quad 2-2$

$-1+2 \quad 2\cdot \quad -1-2$

$$\begin{array}{cccccc} 1 & 0 & 0 & -2 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}$$

$1-1$

$1-2$

$0-2$

$$A^{-1} = \begin{bmatrix} -2, & 3, & -1 \\ 1, & -1, & 1 \\ 2, & -2, & 1 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix} \quad x_1=0,\ x_2=3,\ x_3=7$$

$x_1 = x_2\ 1-x_2$

$x_2 = 5 - x_3$

$x_3 = 7$

$5-7=-2$

$1+2=1$

1
-2
7

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$B = [$

B

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

0 2
4 1
2 0

$3\times3\ \ 3\times3 = 3\times3$

3 -2 0 1
-3 0 2 3
6 -2 -1 = 3

3
0
1

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 0 & 2 & 5 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

1 1 0 2
0 1 2 7
0 -1 1 -2

1 1 0 2 $x_1 = $
0 1 2 7 $x_2 = 5$
0 0 3 5 x_3

$$x_1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \quad x_1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

0 1 2

$$x_1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} x_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

2 0 2
-2 6 = 4
4 -2 -2

2
-3
6

7
-2
9

1 1 1
-1 6 = 5
2 3 5

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

A) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

B) $V = C_p \cancel{\begin{bmatrix} 3 \\ 7 \end{bmatrix}} \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

Anything multiple of this vector.

$[1\,2\,3$ $\begin{bmatrix} x_1 & x_2 \\ x_1 & x_2 \end{bmatrix}$ $[3$

2×1 2×1

2×3 $3\times 1 = 2\times 1$

$2\times 3\,[$

3×2 2×1

$3-1$

$7-3$

$7-$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$x_1 = -x_2$
$x_2 = -2x_3$
$x_3 = \text{free}$

2
-2
1

-4
2

a)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix}$$

4-3

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

4-2

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

This is one to one because there is a pivot in every ~~...~~ col.

b)

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix}$$

This isn't one to one cuse there isn't a pivot in every col.

$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ $v_2 \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$U = \begin{bmatrix}1\\1\end{bmatrix}$ $v = \begin{bmatrix}5\\5\end{bmatrix}$

$w = \begin{bmatrix}2\\2\end{bmatrix}$

$\begin{bmatrix}3\\3\end{bmatrix} = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}5\\5\end{bmatrix}\right)$

The statement is true

False

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = 0$

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ This is false beause Just becuse

$u+v+w = 0$

doe,n't men

$u+v = 0$

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

True because if you have inf solution and then transform it is still the same.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True

$u: \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ Span$\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix}\right)$

$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ $\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix}\right)$

2×1 1×1

2×1 2+2

True because of the example of a linear transformation of $[2]$

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

SM Hoque

UB Person Number:

5	0	2	2	8	1	6	8
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right] R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

$x_1 - x_2 + x_3 = -2$

$x_2 + 2x_3 = 2$

$0 = b + 6$

$b = -6$

b)

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1 - c_2 + c_3 = 0 \rightarrow c_1 = 0$

$c_2 + 2c_3 = 0 \rightarrow c_2 = 0$

$c_3 = 0$

Yes because all c_n values are equal to 0.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] R_2 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] R_3 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] R_2 + R_1 : R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] R_1 - R_3 : R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] R_2 : R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 2 & 3 \\ -3 & -1 & -2 & 4 & 5 & 4 \\ 1 & 1 & 1 & 3 & 2 & 1 \end{array}\right] \quad -R_2 - R_1 : R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ -3 & -1 & -2 & 4 & 5 & 4 \\ 1 & 1 & 1 & 3 & 2 & 1 \end{array}\right] \quad 3R_1 + R_2 : R_2$$

$$C = \begin{bmatrix} -5 & -7 & -7 \\ 5 & 2 & -1 \\ 3 & 7 & 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & -1 & -2 & -11 & -16 & -17 \\ 1 & 1 & 1 & 3 & 2 & 1 \end{array}\right] \quad \begin{array}{l} R_3 : R_3 - R_1 \\ R_2 : -R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & +1 & +2 & +11 & +16 & +17 \\ 0 & 1 & 1 & 8 & 9 & 8 \end{array}\right] \quad R_3 : R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & 1 & 2 & 11 & 16 & 17 \\ 0 & 0 & +1 & +3 & +7 & +9 \end{array}\right] \quad \begin{array}{l} R_3 : -R_3 \\ R_2 : R_2 - 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & 1 & 0 & 5 & 2 & -1 \\ 0 & 0 & 1 & 3 & 7 & 9 \end{array}\right]$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{array}{c} \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \\ x_1 \quad x_2 \end{array}$$

$$u = \begin{bmatrix} 7 \\ -5 \\ \underline{16} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & | & 7 \\ 0 & 1 & | & -5 \\ 0 & 0 & | & 16 \end{bmatrix}$$

free variable

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$T(u) = \begin{bmatrix} 1 & -2 & | & 1 \\ 1 & 1 & | & 10 \\ 1 & -3 & | & -2 \end{bmatrix} \quad R_2 : R_2 - R_1$$

$$\begin{bmatrix} 1 & -2 & | & 1 \\ 0 & 3 & | & 9 \\ 1 & -3 & | & -2 \end{bmatrix} \quad R_3 : R_3 + R_2$$

$$\begin{bmatrix} 1 & -2 & | & 1 \\ 0 & 3 & | & 9 \\ 1 & 0 & | & 7 \end{bmatrix} \quad R_3 : R_3 - R_1$$

$$\begin{bmatrix} 1 & -2 & | & 1 \\ 0 & 3 & | & 1 \\ 0 & 2 & | & 6 \end{bmatrix} \quad R_2 : R_2 - R_3$$

$$\begin{bmatrix} 1 & -2 & | & 1 \\ 0 & 1 & | & -5 \\ 0 & 2 & | & 6 \end{bmatrix} \quad R_1 : R_1 + R_3$$

$$\begin{bmatrix} 1 & 0 & | & 7 \\ 0 & 1 & | & -5 \\ 0 & 2 & | & 6 \end{bmatrix} \quad R_3 : R_3 - 2R_2$$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ Not one-to-one **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ one-to-one

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ $R_3: R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$ $R_2: R_2 - R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ $R_1: -R_3 + R_1$

$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $R_3: R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ $R_1: R_1 - R_3$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ $R_2: R_2 - R_3$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False

$w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

w + u is a linear combination so just w itself may not be in span

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True

L. I. depends on the values $c_1, c_2, \ldots c_n$

set $\{u, v, w\} \in \mathbb{R}^3$

$c_1 = c_2 = c_3 = 0$

set $\{u, v\} \in \mathbb{R}^2$

$c_1 = c_2 = 0$

set gets dropped by one dimension solution remains the same

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

u & v need to be set to 0 and checked to see if a solution exists. i.e. $c_1, c_2, \ldots c_n$ all equal 0.

$Au = 0$

$Av = 0$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True

If the Span(v, w) contains u then the span of (T(v), T(w)) contains T(u)

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Raymond Groesbeck

UB Person Number:

5	0	2	8	6	2	3	9
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \; v_1 \\ 2 \; v_2 \\ b \; v_3 \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

b) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \cdot -2$

$\downarrow$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} \cdot 1$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot \frac{1}{2} \quad \cdot 1 \quad \cdot 1/2$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot 1$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is linearly independant because each part of the matrix has a pivot point

$$\begin{matrix} -2 & -2 \\ 0 & 1 \\ -4 & -3 \end{matrix}$$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \cdot -1$$

$$\downarrow$$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \cdot -2$$

$$\downarrow$$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \cdot 1 \quad \cdot -2$$

$$\downarrow$$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \cdot 1$$

$$\downarrow$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\boxed{\begin{bmatrix} -2 & 2 & -1 \\ 1 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} = A^{-1}}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \cdot 1 \\ \cdot -2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & -6 \\ 13 & -13 & 13 \\ 12 & -12 & 6 \end{bmatrix} = C$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right] \cdot 1$$

$$\boxed{C = \begin{bmatrix} 1 & 2 & -6 \\ 13 & -13 & 13 \\ 12 & -12 & 6 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \cdot -2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \cdot -1$$

$$A^{T^{-1}} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \begin{array}{l} \\ \cdot(-1) \\ \cdot(-1) \end{array}$

$\downarrow$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \begin{array}{l} \\ \cdot\frac{1}{3} \\ \\ \end{array}$

$\downarrow$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \begin{array}{l} \\ \\ \cdot 1 \end{array}$

$\downarrow$

$\begin{array}{cc} x_1 & x_2 \end{array}$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x_1 = 2x_2 + 1 \\ x_2 = 3 \end{cases}$

$\downarrow$

$\begin{bmatrix} x_1 = 7 \\ x_2 = 3 \end{bmatrix}$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot -3 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \cdot 1/2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \cdot -1 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot -1 \;\; \cdot 1/2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot -1 \to$

$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is one to one}}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot -3 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \cdot 1/2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} - \cdot 1/2 \;\; \cdot 1/2 \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

free

$x_1 = 2x_3$

$x_2 = -2x_3$

$x_3 = x_3$

1

3

Not one to one

b

Two vectors

$\mathrm{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \mathrm{v}_2 = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False,

Counter example

$$u = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \quad w = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$u + w = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}\right)$$

$$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \notin \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}\right)$$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True because in order to be independent there must be no dependency on another vector, so if a vector were to leave the group that vector would still be independent.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

False

$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $T = \begin{bmatrix} 5 & 7 \end{bmatrix}$

MTH-309T-F19-EX1-074-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Daniel Kane

UB Person Number:

5	0	2	3	8	7	0	6
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\cancel{V_1 + V_3 = w \mid b = 2}$

$\cancel{-V_1 + V_3 = w}$

$2V_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}, b = -6$ $\boxed{b=-6}$

$-3V_1 + V_3 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}, b = -6$

b) The set $\{V_1, V_2, V_3\}$ is linearly independent, because the vectors cannot linearly combine to cancel one another

cannot cancel all terms

$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$

cannot cancel all terms

no scaler multiple

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$[A \mid I] \to [I \mid A^{-1}]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$C = (A^T)^{-1} B \qquad (A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (-2+4+6) & \overset{-4+5+4}{(\cancel{\quad})} & (-6+4+2) \\ (3-4-6) & (6-5-4) & (9-4-2) \\ (-1+4+3) & (-2+5+2) & (-3+4+1) \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\boxed{T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

b) $1 = x_1 - 2x_2$

$10 = x_1 + x_2$

$-2 = x_1 - 3x_2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$ $x_1 = 7$, $x_2 = 3$ $\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) One to one -> pivot in every column

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ A is one-to-one

b) $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ not one-to-one

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 + 4x_3 \\ 3x_1 + 4x_2 + 2x_3 \end{bmatrix}$$

=

no such vectors exist, Since $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ are linearly independent

$x_2 = x_3$

$\begin{bmatrix} 1 & 1 \\ 0 & 6 \\ 3 & 6 \end{bmatrix}$ $x_1 = 1$, $x_2 = 1$ $\begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$

$x_1 = 2$, $x_2 = 1$ $\begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True

For any equation $W+u = a_1 u + a_2 V$, $W = a_1 u - u + a_2 V = cu + a_2 V$

$w \in \text{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True

If $\{u, V\}$ were dependent, then $a_1 u + a_2 V + 0W = 0$ wouldprove $\{u, V, w\}$ to be dependent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad V_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$AV_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad AV_2 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

V_1, V_2 are independent

AV_1, AV_2 are dependent

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$u = a_1 V + a_2 w$ True

$\cancel{T(u) \neq a_1 T(v) + a_2 T(w)}$

u, $a_1 V + a_2 w$
are equivolent
vectors

$\therefore$

Their transformations
are the same

MTH-309T-F19-EX1-068-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Jon Yaeger

UB Person Number:

5	0	2	3	6	2	3	5
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

$$V = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{-2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] \xrightarrow{+} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & \underbrace{0}_{\text{free}} & 6+b \end{array}\right] = \begin{array}{l} x_1 = -3x_3 \\ x_2 = 2 - x_3 \\ x_3 = x_3 \end{array} \quad \boxed{b = -6}$$

b) $$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{array}$$

$x_3 = 1$

$\boxed{\text{Yes linear independent}}$

$$-3\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = B \cdot (A^T)^{-1}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 3 & 2 \\ -1 & 2 & 3 \end{bmatrix}}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{matrix}$$

$$\boxed{T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

b) $$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1 = 7$

$x_2 = 3$

$$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ - pivot in every column

$\boxed{T_A(v) = Av \text{ is one-to-one}}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \left(\begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right)$

T_A is not one-to-one

$x_3 = \text{free}$

$x_1 = 2x_3$

$x_2 = -2x_3$

$\boxed{V_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}}$

$x_3 = 1$ $x_3 = -1$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{\text{False}}$

$$w + u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ but there is no x value where xw = Span(u, v)

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{\text{True}}$ linear independent means vectors are not multiples so removing one will not affect linear independency

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\boxed{\text{false}}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ Au and Av are dependent

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are independent

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in $\mathrm{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\mathrm{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

$\boxed{\text{True}}$ a transform will only move a vector, not change it so if u is in the span it will also be in the transform span

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

MTH-309T-F19-EX1-075-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Westley Burke

UB Person Number:

5	0	2	9	0	3	5	0
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \xrightarrow{-2R_1+R_3 = R_3} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b-4 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-2 \end{bmatrix}$$

$w \in span(v_1, v_2 v_3)$ aslong as $b=2$ if $b \neq 2$ then nosoultion

the set of vectors is linearly dependent because even if you had $b=2$ then there would still be a free variable

if $w \in \{v_1 v_2 v_3\}$ then $w = c_1 v_1 + c_2 v_2 + c_3 v_3$ antlere is a nontrivial answer to this equation

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = A^{-1}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

2 x 3

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2+4+3 & -4+5+4 & -6+4+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ -1+4+3 & -2+5+2 & -3+4+1 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$$\boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}} u = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

b)

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{-R_1+R_3} = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 1 & 1 \end{array}\right]$$

$=$

$$\boxed{\begin{aligned} X_1 &= 7 \\ X_2 &= 3 \end{aligned}}$$

$7 - 2(3) = 1$

$7 + 3 = 10$

$7 - (3)(3) = -2$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$. $V_1 \neq V_2$

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{2R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$-2R_3+R_2$ $-R_3+R_1$

$-3R_1+R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow[-R_2+R_1]{-R_2+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\frac{R_3}{4}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

ⓐ is onto one

ⓑ is not one to one

$X_1 = -2$

$X_2 = 2$

$V_1 = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w}+\mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True if u is in the span as well then $w+u \in span(u,v)$

Because

$w = -u + v$

$w+u = v$

B $-u, v$ are in the span of u, v

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

false

$$\begin{array}{ccc} u & v & w \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

u v are dependent

u v w are independent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True Because A Stay the same So inorder to keep Au, Av dependent u V must be dependent

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True if T is a traslation done to all Vectors then the vectors will change in the. Same magnitude

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Lauren Kim

UB Person Number:

5	0	2	2	0	6	9	9
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$[\, v_1 \quad v_2 \quad v_3 \,] = \vec{w}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-6 \end{array}\right]$$

$b-6=0$

if $b=6$, then $\vec{w} \in \text{span}(\vec{v_1}, \vec{v_2}, \vec{v_3})$

$x_1 - x_2 + x_3 = -2$

x_2

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

x_3 is free

to be linearly independent, every column after row reduction must be a pivot column. Therefore, the set is not linearly independent.

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1/2 & 1 & -1 & 1/2 \end{array}\right]$$

$$\xrightarrow{2R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow[-2R_3+R_1]{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \checkmark$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$(A^T)^{-1} = (A^{-1})^T$

$AA^{-1} = I$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

$x_1 + y_1 = 1$

$-x_1 + 2z_1 = 4$

$2x_1 + y_1 - z_1 = 3$

$(A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & 1 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right]$ $z_1 = 6$

$y_1 = 5 - 2(6)$

$y_1 = -7$

$x_1 = 1 - -7$

$x_1 = 8$

$C_1 = \begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix}$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -1 & 0 & 2 & 5 \\ 2 & 1 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 5 \end{array}\right]$

$z_2 = 5$

$y_2 = 7 - 10$

$y_2 = -3$

$x_2 = 2 - -3$

$x_2 = 5$

$C_2 = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$

check

$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix} = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{matrix}$ ✓

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$z_3 = 2$

$y_3 = 7 - 4 = 3$

$x_3 = 3 - 3 = 0$

$C_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$Tx = Ax$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{matrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$. $= A\vec{u}$ $\quad \vec{u}$ is 2×1 $= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}\vec{u} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} \qquad \boxed{\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$u_1 - u_2 = 1$

$u_2 = 3$

$u_1 = 4$

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

One to one if pivot position in every column.

a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to$

$\boxed{\text{A is one to one}}$

b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

not a pivot column

A is <u>not</u> 1-1.

$T_A(v_1) = A \cdot v_1$

$A \cdot v_1 = A \cdot v_2$

let $x_3 = 2$

$x_3 = 2$

$x_2 = -2(2) = -4$

$x_1 = 0 - -4 = 4$

$\boxed{\begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = v_1 \qquad v_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}$

$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array}\right] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array}\right] \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = x_3$ $\quad x_3 = 1$

$x_2 + 2x_3 = 0 \quad x_2 = 0 - 2$

$x_1 + x_2 = 0 \quad = -2$

$x_1 = 0 - -2$

let $x_3 = 1 \quad = 2$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True. lin. combinations of vectors in a span are still within a span and vice versa.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True. In the set of 3, they are all lin. ind. from each other, meaning if one vector was taken away, the remaining two would still be lin. ind.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

true. A matrix is a linear transform, so anything that is linear dependent that is linearly transformed stays linearly dependent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

true. linear transforms will not change is something is within a span.

$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \overset{u}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$