



MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	4	4	7	5	2
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1 2 3 4 5 6 7 TOTAL GRADE

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20

10

10

20

20

10

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2

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112

A

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3

4

5

6

7

PIAZZA

HILL

TOTAL

GRADE



1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+b \end{array} \right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+b \end{array} \right]$$

$w \in \text{Span}(v_1, v_2, v_3)$ for $b = -6$ ✓

b)
$$\left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{array} \right] \xrightarrow[\text{reduction}]{\text{row}} \begin{array}{c} x_1 \quad x_2 \quad x_3 \\ \left[\begin{array}{ccc} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right] \end{array}$$

$$x_3 = 3x_1 + 2x_2$$

The set $\{v_1, v_2, v_3\}$ is not linearly independent ✓



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1} .

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$(A^T)^{-1} \cdot A^T C = (A^T)^{-1} \cdot B$$

$$C = (A^T)^{-1} \cdot B$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$C = (A^{-1})^T \cdot B$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = (A^{-1})^T \cdot B = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (-2+4+6) & (-4+5+4) & (-6+4+2) \\ (3-4-6) & (6-5-4) & (9-4-2) \\ (-1+4+3) & (-2+5+2) & (-3+4+1) \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$a) \quad T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 - 2(0) \\ 1 + 0 \\ 1 - 3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2(1) \\ 0 + 1 \\ 0 - 3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$$\text{Standard matrix of } T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \quad \checkmark$$

$$b) \quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array} \right] \quad \checkmark \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & -3 & 0 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array} \right] \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & -1 & 0 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right] \quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$x_1 = 1 + 2x_2$$

$$x_2 = 3$$

$$\begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right]^T$$

$$x_1 = 7$$

$$x_2 = 3$$

$$\text{Null}(A) = \{0\}$$

$$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\text{all vectors satisfying } T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} : \left\{ \begin{bmatrix} 7 \\ 3 \end{bmatrix} \right\} \quad \checkmark$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$T_A(v)$ is 1-to-1; pivot columns in all columns.

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Not 1-to-1; not all columns are pivot columns

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 2x_3$$

$$x_2 = -2x_3$$

$$x_3 = x_3$$

$$x = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$





6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

$$w + u = c_1 u + c_2 v$$

$$w = (c_1 - 1)u + c_2 v$$

TRUE; w is a linear combination of u and v



b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

By definition,
 u, v, w are not constant multiples of each other: $u \neq c_1 v$ for any c_1

u, v are not constant multiples of each other

$\therefore u, v$ are linearly independent

TRUE





7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and u, v are vectors in \mathbb{R}^2 such that Au, Av are linearly dependent then u, v also must be linearly dependent.

FALSE

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Au = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad Av = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Au, Av are linearly dependent
- u, v are not linearly dependent



b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

$$u = c_1 v + c_2 w$$

By linearity: $T(u) = T(c_1 v + c_2 w) = T(c_1 v) + T(c_2 w)$

$$= c_1 T(v) + c_2 T(w)$$

$T(u)$ is a constant multiple of $T(v)$ and $T(w)$

$$\therefore T(u) \in \text{Span}(T(v), T(w))$$

TRUE

