MTH 309T LINEAR ALGEBRA
EXAM 1
October 3, 2019

Name:

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UB Person Number:

3 7 5 8 1 0 7 3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.

- For full credit solve each problem fully, showing all relevant work.

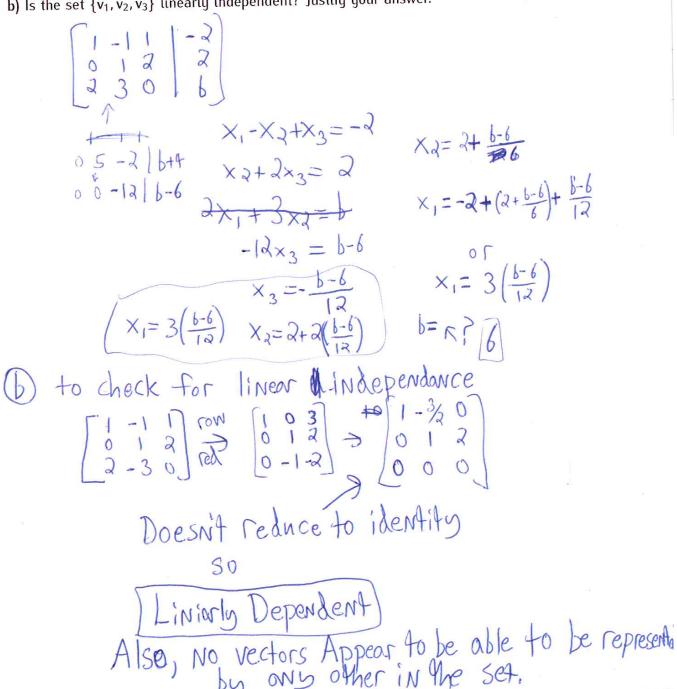| 1	2	3	4	5	6	7	TOTAL	GRADE

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1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & b \end{array} \right]$$

$$\begin{array}{ccc|c} 0 & 5 & -2 & b+4 \\ 0 & 0 & -12 & b-6 \end{array}$$

$x_1 - x_2 + x_3 = -2$

$x_2 + 2x_3 = 2$

$2x_1 + 3x_2 = b$

$-12x_3 = b-6$

$x_3 = -\dfrac{b-6}{12}$

$\boxed{x_1 = 3\left(\dfrac{b-6}{12}\right) \quad x_2 = 2 + 2\left(\dfrac{b-6}{12}\right)}$

$x_2 = 2 + \dfrac{b-6}{6}$

$x_1 = -2 + \left(2 + \dfrac{b-6}{6}\right) + \dfrac{b-6}{12}$

or

$x_1 = 3\left(\dfrac{b-6}{12}\right)$

$b = \mathbb{R}? \boxed{b}$

(b) to check for linear independance

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Doesn't reduce to identity

so

$\boxed{\text{Liniarly Dependent}}$

Also, No vectors Appear to be able to be representa
by any other in the set.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

There is also another way that requires you to find a cofactor by doing

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} [\ \ ]$$

ect.

finding the cross product of each box to find a ~~Mult~~ factor, X then

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$X (\text{this}) = A^{-1}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

SO

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$C = BA^{-1T}$$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -5 & 24 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

SO

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array} \right]$$

SO

$$\begin{bmatrix} (-5+6-3) & (2-2+3) & (4-4+3) \\ (-20+15-4) & (8-5+4) & (16-10+4) \\ (-15+6-1) & (6-2+1) & (12-4+1) \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array} \right]$$

SO

$$\boxed{C = \begin{bmatrix} -2 & 3 & 3 \\ -9 & 7 & 10 \\ -10 & 5 & 9 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 4 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array} \right] \leftarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -1 & -2 & 0 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array} \right]$$
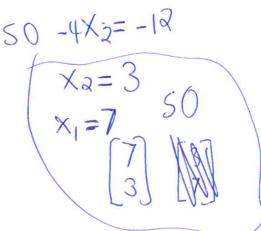
4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

(a) $\begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = \left(\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}\right)$

(b) $\begin{bmatrix} 1 & -2 & | & 1 \\ 1 & 1 & | & 10 \\ 1 & -3 & | & -2 \end{bmatrix}$

$x_1 - 2x_2 = 1$ $x_1 = 1 + 2x_2$

SO $x_1 + x_2 = 10$

$x_1 - 3x_2 = -2$ $x_1 = 1 + 6 = 7$

or

$\begin{array}{ccc} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -4 & -12 \end{array}$

SO $-4x_2 = -12$

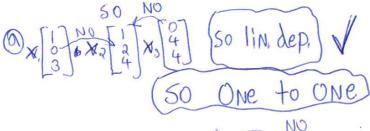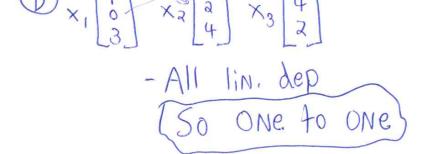$\boxed{\begin{array}{l} x_2 = 3 \\ x_1 = 7 \end{array} \quad SO \quad \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ ✓

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

one to one if all Col(A) are lin, dep.

SO NO

ⓐ $x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ NO $x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $x_3 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ $\boxed{\text{So lin, dep,}}$ ✓

$\boxed{\text{So One to One}}$

ⓑ $x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ NO $x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $x_3 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ NO

− All lin. dep

$\boxed{\text{So One to One}}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

$w = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

False,

$W + u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ is valid

but $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ is invalid

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

False, $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $w = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$

$u + v = w$ so $\{u, v, w\}$ is linearly independent

but u and v alone $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ are

linearly dep. because Neither vector can represent the other.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and u, v are vectors in $\mathbb{R}^2$ such that Au, Av are linearly dependent then u, v also must be linearly dependent.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

~~False~~

True, thought it was false but couldn't disprove

$u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then $Au = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $Av = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

but $u \neq v$ are

~~Not linearly~~ dep.

so $Au = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $Av = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

but

u and v

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in Span(v, w) then $T(u)$ must be in Span($T(v), T(w)$).

$u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

False

so $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is valid

but

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

is Not

0 1 2 3 4 5 6 7 8 9 10