MTH-309T-F19-EX1-031-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Kyle Drapikowski

**UB Person Number:**

| 5 | 0 | 1 | 8 | 2 | 2 | 1 | 4 |
|---|---|---|---|---|---|---|---|

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

| -- | -- | -- | -- | -- | -- | -- | 0 | nan |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |

**1. (20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathrm{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

**a)** Find all values of $b$ such that $\mathrm{w} \in \mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$.

**b)** Is the set $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ linearly independent? Justify your answer.

a) if w span $(v_1, v_2, v_3)$ th $w = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$w = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$w = -2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} + \begin{bmatrix} b \\ 2b \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} b-4 \\ 2b+2 \\ -10 \end{bmatrix} \quad \begin{matrix} b-4=0 \Rightarrow b=4 \\ 2b+2=0 \quad b=-1 \end{matrix}$$

b)
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{2R_1 - R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-3R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yes $\{v_1, v_2, v_3\}$ is linearly independent because every column of the ~~m~~ reduced matrix is a pivot column.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} (r_1c_1) & (r_1c_2) & (r_1c_3) \\ (r_2c_1) & (r_2c_2) & (r_2c_3) \\ (r_3c_1) & (r_3c_2) & (r_3c_3) \end{bmatrix}$$

Compute $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$3 \times 3 \quad \times \quad 3 \times 3$

$$C = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix}$$

$r_1 c_1 :$ $2x_1 + x_2 - x_3 = 1$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{RR} \begin{bmatrix} C \end{bmatrix}$$

$r_2 c_1 :$ $-x_1 + 2x_3 = 4$

$r_3 c_1 :$ $x_1 + x_3 = 3$

$x_1 = 3 - x_3$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of $T$. im defining this as A.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$

$T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 - 2(0) \\ 1 + 0 \\ 1 - 3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2(1) \\ 0 + 1 \\ 0 - 3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

Standard matrix of T: $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} * \overset{u}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$\left.\begin{aligned} 1x_1 - 2x_2 &= 1 \\ x_1 + x_2 &= 10 \\ x_1 - 3x_2 &= -2 \end{aligned}\right\}$$

$x_1 = 1 + 2x_2$

$1 + 2x_2 + x_2 = 10 \Rightarrow 1 + 3x_2 = 10$

$3x_2 = 9$

$x_2 = 3$

$\downarrow$ backsub

$x_1 - 2(3) = 1$

$x_1 = 7$

$7 - 3(3) = -2$

$7 - 9 = -2$ ✓

so $u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

or any multiple of it.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

**a)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$\xrightarrow{-\frac{1}{4}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ it is one-to-one as every column has 1 pivot position.

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_2-R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

not going to be one to one because column 3 will not have a pivot.

$v_1 = \begin{bmatrix} \; \end{bmatrix}$ $v_2 = \begin{bmatrix} \; \end{bmatrix}$

$T_A(v_1) = T_A(v_2)$ if $v_2 = v_1 + n$

So if $v_1 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ and $n = 2$

$v_2 = \begin{bmatrix} 5 \\ 9 \\ 6 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True

Span $(u, v)$ = all vectors / linear combinations $c_1u + c_2v$

or $c_1w + c_2u$ spanning $(u, v)$

therefor $c_1 w$ span $(u, v)$

**b)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False because if $\{u, v\}$ are scalar multiples ie $c_1v_1 + c_2v_2 = 0$ then it is linearly dependent.

$1u_1 + (-1)u_2 = 0$

~~$R^3 \to 3 = n$ rows~~

~~u v w~~
~~1 2 3 → 3 = p colums~~

~~by definition if $p > n \Rightarrow$ linearly dependent~~

~~but $p = n$ so it is linearly independent~~

~~but if u v 1 2 $p = 2$ and $R^3$ $n = 3$~~

~~$p \not> n$~~

~~$2 \not> 3$ so it is linearly independent as well~~

~~* does not apply for $p = 2$~~

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2 \times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True.

Au, Av indicates that after multiplication there was a free variable causing $\infty$ solutions, that makes Au, Av linearly dependent

This means that the vectors u and v must have also had free variables when multiplied by some x value.

**b)** If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

False, if the transformation changes u, v, w to where u no longer

Span ( v, w) that would make T(u) not Span (T(v), T(w))

$\overset{u}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}$ $u_1 + 3u_2$ span (u) but $T(u) = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$ then it may not span

for two other vectors

if it is not a

scalar multiple

0 1 2 3 4 5 6 7 8 9 10