MTH-309T-F19-EX1-038-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

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Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

20	10	5	20	20	6	2	--	--	83	B+
1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $x_1 v_1 + x_2 v_2 + x_3 v_3 = w$

$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] (0 \cdot -2) \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right] (0\,1)$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right] (0\,1) \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$

For $w \in \text{Span}(v_1, v_2, v_3)$, must have a sol'n. $\therefore$ $b+6=0$

$\boxed{b=-6}$ ✓

b) $\{v_1, v_2, v_3\}$ is not linearly independent because all columns of $[v_1\ v_2\ v_3]$ are not pivot columns. ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ✗

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (\cdot -1) \Rightarrow \left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (\cdot 2)$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] (\cdot -1)$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] (\cdot -1) \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \cdot(-1)$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad (A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A^T C = B \Rightarrow (A^T)^{-1}(A^T) C = B(A^T)^{-1} \Rightarrow C = B(A^T)^{-1}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ -3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 31 & -1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} -11 & 31 & -1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}}$$

There are mistakes in matrix multipl. too.

0	1	2	3	4	5	6	7	8	9	10
☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $[T(e_1) \quad T(e_2)] = [T(\begin{bmatrix}1\\0\end{bmatrix}) \quad T(\begin{bmatrix}0\\1\end{bmatrix})] = \begin{bmatrix} 1-2(0) & 0-2(1) \\ 1+0 & 0+1 \\ 1-3(0) & 0-3(1) \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

$\boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ ✓

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$Ax = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$x_1 = 1 + 2x_2$
$x_1 = 10 - x_2$
$x_1 = 3x_2 - 2$

$10 - x_2 = 3x_2 - 2$
$+2 \; +x_2 \quad +x_2 \; +2$
$4x_2 = 12$
$x_2 = 3$

$x_1 = 1 + 2(3)$
$x_1 = 7$

$\begin{bmatrix} 7 - 2(3) \\ 7 + 3 \\ 7 - 3(3) \end{bmatrix} = \begin{bmatrix} 7-6 \\ 10 \\ 7-9 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ ✓

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix} \cdot(-1) \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 0 & 0 \end{bmatrix} \cdot 1/3 \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & 0 \end{bmatrix} \cdot(-1) \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & 0 \end{bmatrix} (\cdot -2)$

$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 1 & 0 & 0 \end{bmatrix} \cdot \frac{1}{4} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Pivot position in evry column $\therefore$ A is one to one.

✓

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b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} (\cdot -3) \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \cdot 1/2 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} (\cdot -1) \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Not one to one, no pivot position in every column.

✓

$T_A(v_1) = T_A(v_2)$
$T(v_1 - v_2) = 0$
$v_1 - v_2 \in Nul(A)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ free

$x_1 + x_2 = 0$
$x_2 + 2x_3 = 0$
$x_3 = x_3$

$x_3 = 1$ $x_2 = -2(1)$ $x_1 = -x_2$
$x_2 = -2$ $x_1 = 2$

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$x_3 = 2$ $x_2 = -2(2)$ $x_1 = -(2)$
$x_2 = -4$ $x_1 = -2$

$\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$

$\boxed{v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$.

False. If u,v,w are vectors in $\mathbb{R}^3$ s.t. w+u ∈ Span, w must be a linear combination of u+v not w+u.

?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True. ~~If Span(u, v, w)~~ If $\{u, v, w\}$ is linearly independent, only have trivial solution as the solution to homogeneous equation. ∴, $c_1u + c_2v = 0$ must also only have the trivial solution as the answer.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~True.~~ For Au and Av to be linearly dependent, u and v must have infinitely many solutions. ? Some combination of u & v other than the trivial solution will result in the 0 vector.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True. If u is in Span(v,w), then it is a linear combination of v & w. Therefore its transformation must be in the span as well. ← why?

0 1 2 3 4 5 6 7 8 9 10