MTH-309T-F19-EX1-073-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

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**Name:**

John koszela

**UB Person Number:**

| 5 | 0 | 1 | 7 | 7 | 9 | 5 | 8 |
|---|---|---|---|---|---|---|---|

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | PIAZZA | HILL | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 7 | 5 | 20 | 15 | 4 | 1 | 2 | 10 | 81 | B |

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

**a)** Find all values of $b$ such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

**b)** Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

$R_3 = R_3 - 2R_1$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right]$

$R_3 = R_2 + R_3$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$

$b+6=0$

$\boxed{b=-6}$ ✓

b) $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

aug matrix

$[\, v_1 \;\; v_2 \;\; v_3 \,|\, 0 \,]$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right]$

Reducing rows

$R_3 = R_3 - 2R_1$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$

$R_3 = R_3 + R_2$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free variable means infinity many solutions ✓

the set $\{v_1, v_2, v_3\}$ is not linearly indipendent

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

**2. (10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_2 = R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_2 = R_2(-1)$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_1 = R_1 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_1 = R_1 - 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & +3 & \cancel{0} \\ 1 & -1 & \cancel{0} \\ 2 & -2 & \cancel{0} \end{bmatrix}$$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$



**3. (10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \checkmark$$

3x3

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

B
3x3

$C = \cancel{BA}^{-1}$

$C = (A^T)^{-1} \cdot B$

$$\cancel{C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}}$$

C is a 3x3 matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$A^T \quad C \quad B$

$$C = \begin{bmatrix} -2+2+6 & 3+3+6 & 0 \\ -8+5+8 & 12+5+8 & 0 \\ -6+2+2 & 9+3+2 & 0 \end{bmatrix}$$

$$\cancel{C = \begin{bmatrix} 6 & 12 & 0 \\ 5 & 25 & 0 \\ -2 & 14 & 0 \end{bmatrix}}$$

for Matrix division

1) $a/b = a \cdot b^{-1}$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$$[3 \times 2][2 \times 1] = [3 \times 1]$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$ ✓

b)

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$3 \times 2$ $\quad 2 \times 1$ $\quad 3 \times 1$

$\text{rref}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{pmatrix}$ would solve it

$$\boxed{U = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$ ✓

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$ ✓

$c_1 - 2c_2 = 1$

$c_1 + c_2 = 10$

$c_1 - 3c_2 = -2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix}$$

rref

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -7 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_2 = 7$

$c_1 - 2c_2 = 1$

$c_1 = +2(7) + 1$

$c_1 = 15$

$15 - 14$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathrm{v}_1$ and $\mathrm{v}_2$ such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

One to one - pivot position in every column

onto - pivot position in every row

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\downarrow$

$R_3 = R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$\downarrow$

$R_2 \leftrightarrow R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$

$R_2 = R_2 + R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$R_3 \div \frac{-1}{4}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← one-to-one

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{array}\right]$

$\downarrow$

$R_3 = R_3 - 3R_1$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{array}\right]$

$R_2 \leftrightarrow R_3$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & 4 & 0 & 2 & 4 \end{array}\right]$

$R_3 \to R_3 - 2R_2$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{array}\right]$

$R_2 \to R_2 + R_3$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{array}\right]$

$R_1 = R_1 - R_2$

$R_3 \div \frac{-1}{4}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$

not one to one ✓

$A v_1 = A v_2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ✗ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

3x3 3x1 3x3 3x1

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$T_A(v_1) \neq T_A(v_2)$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☒ | ☐ | ☐ | ☐ | ☐ | ☐ |

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $u, v, w$ are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True, ✓ u is in the span of u
So adding it does not take it out of span.

?

b) If $u, v, w$ are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

true. ✓ there would be no free variables when reducing the set to $\{u, v, w\}$ to $\{u, v\}$

?

0 1 2 3 4 5 6 7 8 9 10

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2 \times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$A = \begin{bmatrix} \# & \# \\ \# & \# \end{bmatrix}$

true , same transformation done on u and v

**b)** If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T$(u) must be in Span($T$(v), $T$(w)).

False

if transformation is 2x2 matrix it could take u out of span of v and w

0 1 2 3 4 5 6 7 8 9 10