MTH-309T-F19-EX1-025-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

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Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
20	8	10	17	20	5	4	2	--	86	B+

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$. $A\vec{x} = b$

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $[v_1 \; v_2 \; v_3 \,|\, w]$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & b \\ 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \end{array}\right] \cdot(1)$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & b \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right] \cdot(-2) \rightarrow \left[\begin{array}{ccc|c} 0 & -3 & -6 & b \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right] \cdot(3)$$

$\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ $\boxed{b = -6}$ ✓

b) $$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \cdot(-2) \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \cdot(1) \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot(-1)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ if b $\boxed{\text{Not linearly independent}}$ ✓

x_3 is a free variable so there are infinite solutions. this means that they are linearly dependant.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a-d+2g & b-e+2h & c-f+2i \\ a+g & b+h & c+i \\ 2d-g & 2e-h & 2f-i \end{bmatrix} = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$$

$a-d+2g=1$ | $b-e+2h=0$ | $c-f+2i=0$

$a+g=0$ | $b+h=1$ | $c+i=0$

$2d-g=0$ | $2e-h=0$ | $2f-i=1$

$a=-g$ | $b=1-h$ | $c=-i$

$d=\frac{1}{2}g$ | $e=\frac{1}{2}h$ | $f=\frac{1}{2}+\frac{i}{2}$

$-g+\frac{1}{2}g+2g=1$ | $1-h-\frac{1}{2}h+2h=0$ | $-i-\frac{1}{2}-\frac{i}{2}+2i=0$

$\frac{3}{2}g=1$ | $\frac{1}{2}h=-1$ | $\frac{1}{2}i=\frac{1}{2}$

$g=\frac{2}{3}$ | $h=-2$ | $i=1$

$a=-\frac{2}{3}$ | $b=3$ | $c=-1$

$d=\frac{4}{3}$ | $e=-1$ | $f=1$

$$\boxed{A^{-1} = \begin{bmatrix} \cancel{-2/3} & 3 & -1 \\ \cancel{4/3} & -1 & 1 \\ \cancel{-2/3} & -2 & 1 \end{bmatrix}}$$

Not the best method, but it (mostly) worked.

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

Simpler: $C = (A^T)^{-1} \cdot B = (A^{-1})^T \cdot B$

and A^{-1} was computed in problem 2.

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$C_{11} + C_{21} = 1$

$-C_{11} + 2C_{13} = 4$

$2C_{11} + C_{12} - C_{13} = 3$

$C_{12} = 1 - C_{11}$

$C_{13} = 2 + \frac{1}{2}C_{11}$

$2C_{11} + 1 - C_{11} - 2 - \frac{1}{2}C_{11} = 3$

$-1 + \frac{1}{2}C_{11} = 3$

$\frac{1}{2}C_{11} = 4$

$C_{11} = 8$

$C_{21} = -7$

$C_{31} = 6$

$C_{21} + C_{22} = 2$

$-C_{21} + 2C_{23} = 5$

$2C_{21} + C_{22} - C_{23} = 2$

$C_{22} = 2 - C_{21}$

$C_{23} = \frac{5}{2} + \frac{1}{2}C_{21}$

$2C_{21} + \overset{\frac{4}{2}}{2} - C_{21} - \frac{5}{2} - \frac{1}{2}C_{21} = 2$

$\frac{1}{2}C_{21} - \frac{1}{2} = 2$

$\frac{1}{2}C_{21} = \frac{5}{2}$

$C_{12} = 5$

$C_{22} = -3$

$C_{32} = 5$

$C_{13} + C_{23} = 3$

$-C_{13} + 2C_{33} = 4$

$2C_{13} + C_{23} - C_{33} = 1$

$C_{23} = 3 - C_{13}$

$C_{33} = 2 + \frac{1}{2}C_{13}$

$2C_{13} + 3 - C_{13} - 2 - \frac{1}{2}C_{13} = 1$

$\frac{1}{2}C_{13} = 0$

$C_{13} = 0$

$C_{23} = 3$

$C_{33} = 2$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$ ✓

0 1 2 3 4 5 6 7 8 9 10 X

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$

$ax_1 + c\cancel{x_2} = x_1 - 2x_2$

$cx_1 + dx_2 = x_1 + x_2$

$ex_1 + fx_2 = x_1 - 3x_2$

$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ ✓

b) $Au = b$

$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot u = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ ✓ $\quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot (-1)$ ✓ $\rightarrow$

$\rightarrow \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot (-1) \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{array}\right] \cdot \frac{1}{4} = \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right]$

$\cancel{\left[\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]}$

$x_1 = 3x_2 - 2$

$x_2 = x_2$

$\boxed{\cancel{\begin{bmatrix} -2 \\ 0 \end{bmatrix} + \text{span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}}}$ $\quad \cancel{\begin{bmatrix} -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$ $\quad u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$T_A(v) = T_A(\omega)$

if $\omega = v + n$

$n \in Nul(A)$

$Nul(A) = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \cdot(-3)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \cdot(-1)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \cdot(-1)$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$ $\boxed{\text{One}}$

$\boxed{\text{One to - One}}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \cdot(-3)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 2x_3$

$x_2 = -2x_3$ infinite

$x_3 = x_3$

$x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$Nul(A) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\boxed{V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}$ $V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$\cancel{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)}$

$\boxed{\text{True}}$ ✓ Because if $w+u$ is in the span (u,v), then w must be some linear combination of u or v.

why?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True. ✓ if the whole set is linearly independent, then its parts must hold the relation.

why?

0 1 2 3 4 5 6 7 8 9 10

7. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ← good matrix

$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ dependent but $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ - independent.

$\boxed{\text{False}}$ ✓ ran out of time

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True, ✓ transformations are a linear operator so their original relations hold true to the outcomes of a transformation

why?

0 1 2 3 4 5 6 7 8 9 10