



# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	1	7	2	0	5
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1 2 3 4 5 6 7 TOTAL GRADE

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TOTAL

GRADE



1. (20 points) Consider the following vectors in  $\mathbb{R}^3$ :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of  $b$  such that  $w \in \text{Span}(v_1, v_2, v_3)$ .

b) Is the set  $\{v_1, v_2, v_3\}$  linearly independent? Justify your answer.

a)  $\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{bmatrix}$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-2 \end{bmatrix}$$

$$\xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-2 \end{bmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -3 & -6 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-2 \end{bmatrix}$$

$$\xrightarrow{R_1 + 3R_2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-2 \end{bmatrix}$$

$\therefore$  the last row is 0.

$$b-2=0.$$

$$b=2.$$

$\therefore b=2$  such that  $w \in \text{Span}(v_1, v_2, v_3)$

b) the set  $\{v_1, v_2, v_3\}$   
is not linearly dependent  
because every column  
is not a pivot column



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute  $A^{-1}$ .

$$A \cdot A^{-1} = I$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x & y & z \\ u & v & w \\ p & q & r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(-1)R_1 \rightarrow \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$(-2)R_2 \rightarrow \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$(-2)R_2 \rightarrow \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & -2 & 0 \end{bmatrix}$$

$$(1)R_2 \rightarrow \begin{bmatrix} 1 & -1 & 0 & | & -3 & 4 & 0 \\ 0 & 1 & 0 & | & 1 & -1 & 0 \\ 0 & 0 & 1 & | & 2 & -2 & 0 \end{bmatrix}$$

$$(1)R_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & -2 & 3 & 0 \\ 0 & 1 & 0 & | & 1 & -1 & 0 \\ 0 & 0 & 1 & | & 2 & -2 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$



3. (10 points) Let  $A$  be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix  $C$  such that  $A^T C = B$  (where  $A^T$  is the transpose of  $A$ ).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad \therefore \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\text{Let } C = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$



4. (20 points) Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of  $T$ .

b) Find all vectors  $u$  satisfying  $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ .

$$a) \quad T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$\therefore$  The Standard Matrix of  $T$ .

$$T = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \quad \text{so } T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$b) \quad T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 - 2c_2 \\ c_1 + c_2 \\ c_1 - 3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} \quad \text{so } T(u) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$





5. (20 points) For each matrix  $A$  given below determine if the matrix transformation  $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T_A(v) = Av$  is one-to-one or not. If  $T_A$  is not one-to-one, find two vectors  $v_1$  and  $v_2$  such that  $T_A(v_1) = T_A(v_2)$ .

a)  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b)  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) For  $T_A$  to be one-to-one  
 $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$   
 There is a pivot pos in every column.  
 $T_A(v) = Av$  is one-to-one

for  $T_A(v)$  to be one-to-one the  $\text{Nul}(A) = \{0\}$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

b)  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\text{Nul}(A) = \{0\}$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Every column does not have pivot pos.  $T_A(v) = Av$  is not one-to-one.

If  $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 3 \\ 0 & 2 & 4 & | & 16 \\ 3 & 4 & 2 & | & 17 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\therefore v_2 = \begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix}$$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If  $u, v, w$  are vectors in  $\mathbb{R}^3$  such that  $w + u \in \text{Span}(u, v)$  then  $w \in \text{Span}(u, v)$ .

True.

$$\begin{aligned} T(w+u) &= T(w) + T(u) = T(u) + T(v) \\ \therefore T(w) &= T(u) + T(v) \end{aligned}$$

b) If  $u, v, w$  are vectors in  $\mathbb{R}^3$  such that the set  $\{u, v, w\}$  is linearly independent then the set  $\{u, v\}$  must be linearly independent.

True:

If  $c_1 u + c_2 v + c_3 w = 0$  is linearly independent

it means  $\exists u, v, w = 0$ .

$\therefore$  the set  $\{u, v\}$  has to be linearly independent

because

$$c_1 u + c_2 v = 0$$



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If  $A$  is a  $2 \times 2$  matrix and  $u, v$  are vectors in  $\mathbb{R}^2$  such that  $Au, Av$  are linearly dependent then  $u, v$  also must be linearly dependent.

False.  $Au = A[u] + A[v] = x_1 u + x_2 v$   
 can be linearly dependent of  $u, v$   $x_1, x_2 \neq 0$ .

~~Also  $c_1 u + c_2 v$  can also be~~  
 $c_1 u + c_2 v$  need ~~not~~ be linearly dependent because.

$$u, v = 0.$$

$c_1 u + c_2 v$  becomes linearly independent.

b) If  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear transformation and  $u, v, w \in \mathbb{R}^2$  are vectors such that  $u$  is in  $\text{Span}(v, w)$  then  $T(u)$  must be in  $\text{Span}(T(v), T(w))$ .

True.

$$\begin{aligned} T(u) &= T(v+w) \\ &= T(v) + T(w). \end{aligned}$$