MTH-309T-F19-EX1-027-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

SM Hoque

UB Person Number:

5	0	2	2	8	1	0	8
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right] R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

$x_1 - x_2 + x_3 = -2$

$x_2 + 2x_3 = 2$

$0 = b+6$

$b = -6$

b)

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1 - c_2 + c_3 = 0 \rightarrow c_1 = 0$

$c_2 + 2c_3 = 0 \rightarrow c_2 = 0$

$c_3 = 0$

Yes because all c_n values are equal to 0.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] R_2 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] R_3 - 2R_2$$

$$A^{-1} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] R_2 + R_1 : R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] R_1 - R_3 : R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] R_2 : R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 2 & 3 \\ -3 & -1 & -2 & 4 & 5 & 4 \\ 1 & 1 & 1 & 3 & 2 & 1 \end{array}\right] -R_2 - R_1 : R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ -3 & -1 & -2 & 4 & 5 & 4 \\ 1 & 1 & 1 & 3 & 2 & 1 \end{array}\right] 3R_1 + R_2 : R_2 \qquad C = \begin{bmatrix} -5 & -7 & -7 \\ 5 & 2 & -1 \\ 3 & 7 & 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & -1 & -2 & -11 & -16 & -17 \\ 1 & 1 & 1 & 3 & 2 & 1 \end{array}\right] \begin{array}{l} R_3 : R_3 - R_1 \\ R_2 : -R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & +1 & +2 & +11 & +16 & +17 \\ 0 & 1 & 1 & 8 & 9 & 8 \end{array}\right] R_3 : R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & 1 & 2 & 11 & 16 & 17 \\ 0 & 0 & +1 & +3 & +7 & +9 \end{array}\right] \begin{array}{l} R_3 = -R_3 \\ R_2 = R_2 - 2R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & 1 & 0 & 5 & 2 & -1 \\ 0 & 0 & 1 & 3 & 7 & 9 \end{array}\right]$$

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{array}{cc} \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \\ x_1 \quad x_2 \end{array}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$T(u) = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \quad R_2 : R_2 - R_1$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \quad R_3 : R_3 + R_2$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & 0 & 7 \end{array}\right] \quad R_3 : R_3 - R_1$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 6 \end{array}\right] \quad R_2 : R_2 \cdot \tfrac{1}{3}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -5 \\ 0 & 2 & 6 \end{array}\right]$$

$R_1 : R_1 + 2R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -5 \\ 0 & 2 & 6 \end{array}\right]$$

$R_3 : R_3 - 2R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -5 \\ 0 & 0 & 16 \end{array}\right]$$

$$u = \begin{bmatrix} 7 \\ -5 \\ \underline{16} \end{bmatrix}$$

free variable

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ Not one-to-one **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ one-to-one

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ $R_3: R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$ $R_2: R_2 - R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ $R_1: -R_2 + R_1$

$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $R_3: R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ $R_1: R_1 - R_3$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ $R_2: R_2 - R_3$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False

$w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

w + u is a linear combination so just w itself may not be in span

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True

L. I. depends on the values $c_1, c_2, \ldots c_n$

Set $\{u, v, w\} \in \mathbb{R}^3$
$c_1 = c_2 = c_3 = 0$

Set $\{u, v\} \in \mathbb{R}^2$
$c_1 = c_2 = 0$

Set gets dropped by one dimension solution remains the same

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

u & v need to be set to 0 and checked to see if a solution exists. i.e. $c_1, c_2, \ldots c_n$ all equal 0.

$Au = 0$

$Av = 0$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True

If the Span(v, w) contains u then the span of (T(v), T(w)) contains T(u)

0 1 2 3 4 5 6 7 8 9 10