MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Fuming Zhao

UB Person Number:

5	0	3	2	0	7	9	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
20	10	10	20	20	5	6	2	10	102	A

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $w \in \operatorname{Span}(v_1, v_2, v_3)$ means that $c_1v_1 + c_2v_2 + c_3v_3 = w$ has solution

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \text{①×2−③}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -4-b \end{bmatrix} \text{②−③}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{bmatrix}$$

Since $6+b$ is in constant column.

$\therefore$ $6+b=0$

$b=-6$

$\therefore$ $b = -6$ ✓

b) $x_1v_1 + x_2v_2 + x_3v_3 = 0$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix} \text{①×2−③}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \text{②−③}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is free variable, this homogenous equation has many solutions.

This means that the set $\{v_1, v_2, v_3\}$ is not linearly independent ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (2)-(1)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (3)-(2)\times 2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] (2)+(3)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] (1)+(2)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] (1)-(3)\times 2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \checkmark$$

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒
0	1	2	3	4	5	6	7	8	9	10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \begin{array}{l} \\ ①+② \\ ①\times 2-③ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 1 & 1 & -1 & 2 & 5 \end{array}\right] \begin{array}{l} \\ \\ ②-③ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \begin{array}{l} \\ ②-③\times 2 \\ \\ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \begin{array}{l} ①-② \\ \\ \\ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \boxed{\begin{matrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{matrix}} \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right] \longleftarrow C$$

Simpler:

$$C = (A^T)^{-1} \cdot B$$

$$= (A^{-1})^T \cdot B$$

Then use A^{-1} from problem 2.

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☒ 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) Let T be 3x2 matrix

$$T = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \\ c_5 & c_6 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$c_1x_1 + c_2x_2 = x_1 - 2x_2$$
$$c_3x_1 + c_4x_2 = x_1 + x_2$$
$$c_5x_1 + c_6x_2 = x_1 - 3x_2$$

$\therefore c_1 = 1, c_2 = -2, c_3 = 1, c_4 = 1, c_5 = 1, c_6 = -3$

$$\therefore T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$ ✓

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

Let u be vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

aug. matrix ✓

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{①-③} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{②-①} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{②\div 3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{②-③} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{①+②\times 2} \begin{array}{c} c_1 \; c_2 \quad \\ \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\therefore \begin{cases} c_1 = 7 \\ c_2 = 3 \end{cases} \qquad \therefore u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ ③-①×3

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$ ②÷2

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ ③-②

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ ③÷2

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Since A has a pivot position in every column, $T_A(v) = Av$ is one to one. ✓

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ ③-①×3

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ ②÷2

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ ②-③

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ ①-②

$\downarrow$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Since A does not have a pivot column in every column, $T_A(v) = Av$ is not one to one ✓

From the matrix, we can get $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \in \mathrm{Nul}(A)$

Let v_1 be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ✓

$v_2 = v_1 + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

Yes, Since $w+u \in \mathrm{Span}(u,v)$, then $w+u = c_1u + c_2v$ has solutions,
$w = c_1u + c_2u - u = (c_1-1)u + c_2u$ has solutions

✓

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~False~~ ← true

(why?)

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}, A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False ← why?

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in $\text{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\text{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

Yes. ✓ Since u is in Span (v, w),

then $u = c_1 v + c_2 w$ has solution. ✓

Also, $T(u) = T(c_1 v + c_2 w)$ has solution

Also, $T(u) = T(c_1 v) + T(c_2 w)$ has solution.

This means that $T(u)$ is in Span $(T(v), T(w))$.

I guess you mean

$T(u) = c_1 T(v) + c_2 T(w)$

0 1 2 3 4 5 6 7 8 9 10