MTH-309T-F19-EX1-073-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	1	7	7	9	5	8
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$$

$R_3 = R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right]$$

$R_3 = R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$

$b+6=0$

$\boxed{b=-6}$

b) $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

aug matrix

$[\, v_1 \;\; v_2 \;\; v_3 \mid 0 \,]$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right]$$

Reducing rows

$R_3 = R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$R_3 = R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable means infinitly many solutions

the set $\{v_1, v_2, v_3\}$ is not linearly indipendent

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_2 = R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_2 = R_2(-1)$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_1 = R_1 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_1 = R_1 - 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & -3 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

3x3

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

B
3x3

$$C = BA^{-1}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -2+2+6 & 3+3+6 & 0 \\ -8+5+8 & 12+5+8 & 0 \\ -6+2+2 & 9+3+2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 12 & 0 \\ 5 & 25 & 0 \\ -2 & 14 & 0 \end{bmatrix}$$

C is a 3x3 matrix

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}}_{B}$$

for Matrix division

1) $a/b = a \cdot b^{-1}$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$$[3 \times 2][2 \times 1] = [3 \times 1]$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$c_1 - 2c_2 = 1$

$c_1 + c_2 = 10$

$c_1 - 3c_2 = -2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix}$$

rref

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -7 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_2 = 3$

$c_2 = 7$

$c_1 - 2c_2 = 1$

$c_1 = +2(7) + 1$

$c_1 = 15$

$15 - 14$

b)

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$3 \times 2 \quad 2 \times 1 \quad 3 \times 1$

$\text{rref}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{pmatrix}$ would solve it

$$\boxed{U = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

One to one - pivot position in every column
onto - pivot position in every row

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$R_3 = R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$R_2 \Leftrightarrow R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$

$R_2 = R_2 + R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$R_3 \cdot \frac{-1}{4}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{array}\right]$

$R_3 = R_3 - 3R_1$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{array}\right]$

$R_2 \Leftrightarrow R_3$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & 4 & 0 & 2 & 4 \end{array}\right]$

$R_3 \to R_3 - 2R_2$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{array}\right]$

$R_2 \to R_2 + R_3$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{array}\right]$

$R_1 = R_1 - R_2$

$R_3 \cdot \frac{-1}{4}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$

not one to one

$Av_1 = Av_2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

3x3 3x1 3x3 3x1

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True, u is in the span of u

So adding it does not take it out

of span.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

true. there would be no free variables

when reducing the set to $\{u, v, w\}$ to $\{u, v\}$

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$A = \begin{bmatrix} \# & \# \\ \# & \# \end{bmatrix}$

true, same transformation done on u and v

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}$, $\mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v})$, $T(\mathbf{w})$).

False

if transformation is 2×2 matrix it could take u out of span of v and w

0 1 2 3 4 5 6 7 8 9 10