MTH-309T-F19-EX1-088-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Shawn Mammen

**UB Person Number:**

| 5 | 0 | 1 | 9 | 0 | 2 | 2 | 7 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

**1. (20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

**a)** Find all values of $b$ such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

**b)** Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a)
$$\begin{array}{cccc} 2 & -2 & 2 & -4 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R3 \to R3 - 2R1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b-4 \end{array}\right] \xrightarrow{R3 \to R3 + R2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-2 \end{array}\right]$$

when $b = 2$

b) The set isn't linearly independent because since the span is made up of linear combinations of the vectors all the vectors in the span are linearly dependent

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \to R2 - R1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \to R1 + R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3 \to R3 - 2R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xleftarrow[R1 \to R1 - R3]{R2 \to R2 + R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

**3. (10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \xrightarrow{R2 \to R2+R1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \xrightarrow{R3 \to R3-2R1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & -1 & -1 & 1 & -2 & -5 \end{array}\right]$$

$$\xrightarrow{R3 \to R3+R2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

12 10 4

$$\xrightarrow{R2 \to R2-2R3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \xrightarrow{R1 \to R1-R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R3-3R1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[R3 \to R3 - R2]{R2 \to \frac{1}{2}R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$T_A$ not one-to-one

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R3 \to R3-3R1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True, because if we add the vectors together in a vector equation an the resulting vector is in the span, then that means that the vector that was added is in the span as it is a linear combination.

**b)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

true, because if one of the vectors is linearly dependent then the whole set would have been linearly dependent but it's not.

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2 \times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

true

**b)** If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

true, if all the vectors undergo a linear transformation then they will still shared the relationship of being linear combinations.

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Robert Ann

**UB Person Number:**

| 5 | 0 | 2 | 6 | 4 | 6 | 1 | 1 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

**Instructions:**

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| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

**a)** Find all values of $b$ such that $w \in \text{Span}(v_1, v_2, v_3)$.

**b)** Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \cdot -2 \rightarrow \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{bmatrix} \cdot 1 \rightarrow \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix} \cdot 1 \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix}$

$w \in \text{span}(v_1, v_2, v_3)$ if and only if $b+6=0$ $\quad b=-6$

b) set is linearly independent if and only if

$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ has only trivial solution $(x_1 = x_2 = x_3 = 0)$

$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix} \cdot -2 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix} \cdot 1 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot 1 \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases}$

infinite solutions $x_1 \neq x_3 \neq x_2$

so set is not linearly independent

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot -1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot -2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\cdot 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\cdot -2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 1 \Rightarrow \quad 2c_1$$

$$A^T C = B$$

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ = standard matrix of $T$

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-2(0) \\ 1+0 \\ 1-3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-2(1) \\ 0+1 \\ 0-3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

b)

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{\cdot -1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{\cdot -1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{\cdot 1/3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{bmatrix} \cdot 1/3$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\cdot 2} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = 7 \\ x_2 = 3 \end{cases} \qquad u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

chk $\begin{bmatrix} 7-2(3) \\ 7+3 \\ 7-3(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot -3 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \cdot -\tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \cdot \tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot \tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot -2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot -1 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ pivot in all columns - one-to-one

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot -3 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \cdot \tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \cdot -1 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot -1 \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

some columns missing pivots - not one-to-one

$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{cases} x_1 = 2x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases}$ $Nul(A) = span\left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right)$

let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T_A(v_1) = T_A(v_2)$ if and only if $v_1 = v_2 + n$ for $n \in Nul(A)$

$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

**b)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2 \times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

**b)** If $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Sang Hyun Park

**UB Person Number:**

| 5 | 0 | 3 | 1 | 7 | 4 | 5 | 0 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
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**Instructions:**

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- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of $b$ such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

(a) W is a linear combination of $V_1, V_2, V_3$

$W = X_1V_1 + X_2V_2 + X_3V_3$

$= X_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + X_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + X_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} X_1 - X_2 + X_3 \\ 0 + X_2 + 2X_3 \\ 2X_1 - 3X_2 + 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

$\Rightarrow$ augmented matrix $\begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \end{array}\right] \end{array} \xrightarrow{\text{Row Reduction}} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right] \end{array}$

$X_3$ : free variable

Sol $\begin{cases} X_1 = -3X_3 \\ X_2 = -2X_3 + 2 \\ X_3 = X_3 \end{cases}$

$\therefore b = 2X_1 - 3X_2$

$= -6X_3 + 6X_3 - 6$

$= -6$

$\boxed{b = -6}$

(b) $[\, V_1 \;\; V_2 \;\; V_3 \mid 0 \,]$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_3 + R_2} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

free variable

$\Rightarrow$ $X_3$ is a free variable, so $X_1V_1 + X_2V_2 + X_3V_3 = 0$ has infinetely many solutions.

$\begin{cases} X_1 = -3X_3 \\ X_2 = -2X_3 \\ X_3 = X_3 \end{cases}$

$\Rightarrow \{V_1, V_2, V_3\}$ is linearly dependent.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 + 2 \cdot R_2]{(-1) \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3]{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] = [I_3 \mid A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. **(10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$A^T C = B$

$\Leftrightarrow (A^T)^{-1} \cdot A^T \cdot C = (A^T)^{-1} \cdot B$ 「$\because (A^T)^{-1} \cdot A^T = I$」

$\Leftrightarrow C = (A^T)^{-1} \cdot B$

$\Leftrightarrow C = (A^{-1})^T \cdot B$ 「$\because (A^T)^{-1} = (A^{-1})^T$」

$$\therefore C = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+4+6 & -4+5+4 & -6+4+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ -1+4+3 & -2+5+2 & -3+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

**4. (20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find $\overset{A}{\underset{\|}{\text{the standard matrix}}}$ of $T$.

**b)** Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A$ is a $3 \times 2$ matrix.

$A = [T(e_1) \;\; T(e_2)]$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ is the standard matrix of $T$.

(b) $T(u) = T_A(u) = A \cdot u = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$, $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

augmented matrix

$$\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \end{array} \xrightarrow{\text{row reduction}} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

Sol

$\begin{cases} x_1 = 7 \\ x_2 = 3 \end{cases}$

$\therefore u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

If $T_A$ is one-to-one, $A$ has a pivot position in every column.

$\Leftrightarrow \mathrm{Nul}(A) = \{0\}$

(a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \xrightarrow[R_3 - 3R_1]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$

$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\therefore$ A has a pivot position in every column,

So $T_A(v)$ is one-to-one.

(b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \xrightarrow[R_3 - 3R_1]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$\xrightarrow{R_3 - R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \qquad\quad \text{free} \end{array}$

$\therefore$ A doesn't have a pivot position in 3rd column,

So $T_A(v)$ is not one-to-one.

sol $\begin{cases} x_1 = -x_2 = 2x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases} \rightarrow x = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x_3 \rightarrow \mathrm{Nul}(A) = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

- $T_A(v_1) = T_A(v_2)$ if and only if $v_1 - v_2 \in \mathrm{Nul}(A)$

If, $v_1 = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then

$T_A(v_1) = T_A(v_2)$. $\because$ $v_1 - v_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \in \mathrm{Nul}(A)$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$w+u$ is a linear combination of $u, v$.

$\Leftrightarrow$ $w+u = C_1 u + C_2 v$

$w = (C_1 - 1) u + C_2 v$ is a linear combination of $u, v$.

$\therefore$ $w \in \text{Span}(u, v)$

True

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

① $x_1 u + x_2 v + x_3 w = 0$ has only one and trivial solution $x_1 = x_2 = x_3 = 0$.

Then, $x_1 u + x_2 v = 0$ also has only one and trivial solution $x_1 = x_2 = 0$.

② $A = [u \ v \ w]$

A has a pivot position in every column.

True

$B = [u \ v]$

Then, B also has a pivot position in every column.

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2\times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then

$Au = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $Av = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\rightarrow$ $Au = Av$, so $Au, Av$ are linearly dependent

but $u$ and $v$ are linearly independent.

$\because$ $[u \; v \,|\, 0]$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow{\text{Row reduction}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

So $x_1u + x_2v = 0$ has only one solution $x_1 = x_2 = 0$ $\Leftrightarrow$ $u, v$ are linearly independent.

False

**b)** If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$u = c_1 v + c_2 w$

$T(u) = T(c_1 v + c_2 w)$

$= T(c_1 v) + T(c_2 w)$

$= c_1 \cdot T(v) + c_2 \cdot T(w)$ is a linear combination of $T(v), T(w)$

$\therefore$ $T(u) \in \text{Span}(T(v), T(w))$

True

MTH-309T-F19-EX1-059-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

KARRAR ALJANAHI

**UB Person Number:**

| 5 | 0 | 2 | 6 | 4 | 8 | 1 | 0 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of $b$ such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} = A$

A) $c_1 v_1 + c_2 v_2 + c_3 v_3 = w$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

Aug.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] R_3 \to 2R_1 - R_3 \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -4-b \end{bmatrix} R_1 \to R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -4-b \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -6-b \end{bmatrix}$$

For w to be in the Span $(v_1, v_2, v_3)$ b must equal $-6$.

B) The solution to $Ax = 0$ is

A is $\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

$x_1 = -3x_3$

$x_2 = -2x_3$

$x_3 = x_3$

The presence of a free variable suggests infinitely many solutions to $Ax = 0$ since the homogenous equation does not have one trivial solution it is Linearly Dependent

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$A \cdot A^{-1} = I_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] R_2 \to R_2 - R_1 \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] R_1 \to R_2 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] R_3 \to R_3 - 2R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & -2 & 1 \end{array}\right] R_3 \to \tfrac{1}{3} R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & -2/3 & 1/3 \end{array}\right] R_2 \to R_2 + R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\tfrac{1}{3} & \tfrac{1}{3} & \tfrac{1}{3} \\ 0 & 0 & 1 & 2/3 & -2/3 & 1/3 \end{array}\right] R_1 \to R_1 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & 5/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 2/3 & -2/3 & 1/3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2/3 & 5/3 & -1/3 \\ -1/3 & 1/3 & 1/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

3. **(10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$(A^T)^{-1} A^T C = B (A^T)^{-1}$$

$$C = B (A^T)^{-1}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Inverse of $A^T$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] R_2 \to R_1 + R_2 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] R_2 \to R_3 + R_2 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] R_1 \to R_3 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] R_1 \to R_1 + R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] R_3 \to R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 2 \end{array}\right] R_2 \longleftrightarrow R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] R_2 \to -R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \quad \text{Inverse } A^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$3-4-2$
$1-10$
$-7-2$

$$(A^T)^{-1} \cdot B = C$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0+4+6 & 0+5+4 & 0+4+2 \\ 1-4-6 & 2-5-4 & 3-4-2 \\ 0+4+3 & 0+5+2 & 0+4+1 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 10 & 9 & 6 \\ -9 & 7 & -9 \\ 7 & 7 & 5 \end{bmatrix}}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of $T$.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

A) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\boxed{\text{Standard matrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

B) $x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix}$ $R_2 \to R_1 - R_2$ $\begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & -9 \\ 1 & -3 & -2 \end{bmatrix}$ $R_3 \to R_1 - R_3$

$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & -9 \\ 0 & 1 & 3 \end{bmatrix}$ $R_2 \to 3R_3 + R_2$ $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ $R_1 \to 2R_3 + R_1$

$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ $R_3 \longleftrightarrow R_2$ $\begin{array}{cc|c} x_1 & x_2 & \\ \hline 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}$

$x_1 = 7$
$x_2 = 3$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

**5. (20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$A \cdot V = b$

$T_A(V_1) = T_A(V_2)$

$T_A(V_1) - T_A(V_2) = 0$

$T_A(V_1 - V_2) = 0$

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

one-to-one Pivot position in every column

A) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_2 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 \to -\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 2R_3 + R_2}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to 2R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ This is one-to-one

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ Nul $A$: Span$(0)$

So $T_A(V_1) - T_A(V_2) = 0$

$V_1$ can be $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $V_2$ can be $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

B) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_2 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

We can see not every column will be a pivot so $T_A(v) = AV$ is not one-to-one.

**6. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\boxed{\text{True}}$

$w + u \in \text{Span}(u, v)$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ True

$w \in \text{Span}(u, v)$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ True

Since w + u is in the span of u, v their combination of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ can be made from $c_1 u + c_2 v = w + u$. If you let $c_1 = 1$ and $c_2 = 1$

$\cancel{u} + v = w + \cancel{u}$

$v = w$

Since $v = w$ w is in span of v which makes $w \in \text{Span}(u, v)$ True

**b)** If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Lin independent means one trivial solution To $Ax = b$

$\boxed{\text{TRUE}}$

Since u, v, and w are linearly independent that means all columns on A are pivot columns. So if you were to remove a vector w for example and solve for independence with u and v you are left with a 3×2 matrix with pivot positions in every column so they are Independent.

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2\times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}, A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\boxed{\text{FALSE}}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Au = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Av = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

then $AV = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Aw = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Since these two vectors are NOT Scalar multiples of each other this statement is false.

**b)** If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$\boxed{\text{TRUE}}$ If U is in Span of V, W this means

$C_1V + C_2W = U$

A linear transformation is applied to both sides $T_A(C_1 V + C_2 W) = T_A(u)$

$C_1T_A(V) + C_2T_A(W) = T_A(u)$

Any linear combination of $T(V)$ and $T(W)$ will get you $T_A(u)$

MTH-309T-F19-EX1-045-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Ayesha Khatun

**UB Person Number:**

| 5 | 0 | 2 | 4 | 3 | 5 | 1 | 4 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of $b$ such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$c_1 v_1 + c_2 v_2 + c_3 v_3 = w$

(b)

RREF :- $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_2 + R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \boxed{0} \end{bmatrix}$

~~So even~~ Since every column, can not be a pivot column,

$\boxed{\text{Set } \{v_1, v_2, v_3\} \text{ is not linearly independent}}$

(a)

RREF: $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right] \xrightarrow{R_3 \to R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & \boxed{b+4} \end{array}\right]$ — (1)

Now, there exists soln only if is not a pivot column.

~~So, Sol^n exists Only when, b+~~

Since, $w \in \mathrm{Span}(v_1, v_2, v_3)$.

Here, soln exists only when, $b+4 \neq 1$

$\Rightarrow b \neq 1-4$

$\Rightarrow b \neq -3$

$\therefore b \in \mathbb{R}$, except $b = -3$.

~~From (1),~~ ~~$R_2 \to R_1 + R_2$~~ $\left[\begin{array}{cccc} 1 & -1 & 1 & -2 \end{array}\right]$ $R_2 \to$

Also, soln exist, if $b+4=0$

$\therefore b = -4$.

$\therefore$ ~~$b = \{b \in \mathbb{R}^3$~~

Ans :-

For, $b=-4$, there is infinite soln.

~~See next Page~~

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$\det(A) = 1(0-2) - (-1)(-1-0) + 2(2-0)$$

$$= -2 - 1 + 4 = 4 - 3 = \boxed{1} \rightarrow A^{-1} \text{ is possible}$$

$a_{11} = \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = \boxed{-2}$ $\quad a_{12} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{13} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = \boxed{2}$

$a_{21} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 4-1 = \boxed{3}$ $\quad a_{22} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{23} = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2-0 = \boxed{2}$

$a_{31} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = \boxed{-1}$ $\quad a_{32} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1-2 = \boxed{-1}$ $\quad a_{33} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 0-(-1) = \boxed{1}$

Now, $\operatorname{adj}(A) = \begin{bmatrix} -2 & -1 & 2 \\ 3 & -1 & 2 \\ -1 & -1 & 1 \end{bmatrix} \times \Big[$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$a_{11} = \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = \boxed{-2}$ $\quad a_{12} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1-4 = \boxed{-3}$ $\quad a_{13} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = \boxed{-1}$

$a_{21} = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{22} = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{23} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1-2 = \boxed{-1}$

$a_{31} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = \boxed{2}$ $\quad a_{32} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = \boxed{2}$ $\quad a_{33} = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -(-1) = \boxed{1}$

$$\therefore \operatorname{adj}(A) = \begin{bmatrix} -2 & -3 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{1}{1}\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}} \text{ Ans.}$$

Quick check: $\left[(1\ -1\ 2)\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}\right] = [-2-1+4] = [-3+4] = [1]$

3. **(10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}.$$

Now, $$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

Here, $$[A^T]^{-1}[A^T][C] = [I][C] = [C] = [A^T]^{-1}[B]$$

$$\therefore [C] = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (-2+4+6) & (-4+5+4) & (-6+4+2) \\ (3-4-6) & (6-5-4) & (9-4-2) \\ (-1+4+3) & (-2+5+2) & (-3+4+1) \end{bmatrix}$$

$$\therefore \boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

Ans:-

Check:- $[1\ 1\ 0]\begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix} = [8-7] = [1]$ ✓

$[2\ 1\ -1]\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = [0+2-3] = [1]$

$[2\ 1\ -1]\begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix} = [16 - 7 - 6],$

$\Rightarrow [16 - 7 + 6] = 3$ $\qquad [16 - 13] = 3$ ✓

$24 - 7$

**4. (20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

(a) $\cancel{T(e_1)}$ $[e_1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$// \quad T([e_1]) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad // \quad T([e_2]) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

So, standard matrix, $A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ Ans:-

(b) From definition, $T(u) = A \cdot u = A \cdot \begin{bmatrix} u_1 \\ u_2 \\ \cancel{\phantom{x}} \end{bmatrix}$, as $T: \mathbb{R}^2 \to \mathbb{R}^3$

Now, $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

Since, $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 - 2u_2 \\ u_1 + u_2 \\ \cancel{u_1} - 3u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\Downarrow$

$$\begin{cases} u_1 - 2u_2 = 1 & \text{—(1)} \\ u_1 + u_2 = 10 & \text{—(2)} \\ u_1 - 3u_2 = -2 & \text{—(3)} \end{cases}$$

from (2), $u_1 = 10 - u_2$

$\therefore$ " (1), $10 - u_2 - 2u_2 = 1$

$\Rightarrow 10 - 3u_2 = 1$

$\Rightarrow 10 - 1 = 3u_2$

$\therefore u_2 = \frac{9}{3} = 3$, $u_1 = 10 - 3 = 7$

$\therefore \boxed{\vec{u} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} = 7\hat{u}_1 + 3\hat{u}_2}$

Ans:-

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

**a)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) Aug. mat. of A : RREF

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_1 \to -R_2 + R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3/2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

Every column is a pivot column.

$\therefore$ $\boxed{T_A(v) \text{ is one-to-one}}$

b)

Aug. mat: RREF $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 \to \frac{R_3}{2}]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \to$$ every column is a pivot column.

$\therefore$ $\boxed{T_A(v) \text{ in one-to-one}}$

Ans:

**6. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

False.

**b)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True.

Since, in set $\{u, v, w\}$ are already indep.

~~as~~ subset $\{u, v\}$ will also be indep.

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2\times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False.

$\cancel{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$A\cancel{(u-v)=0, \quad u=v}$

**b)** If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

False.

Transformation doesnt guarantee being in the span..

~~If $T(v)x_1 + T(w)x_2 = 0$, doesn't have sol'n,~~

~~they~~

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Daniel Maas

**UB Person Number:**

| 3 | 7 | 5 | 8 | 1 | 0 | 7 | 3 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of $b$ such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & b \end{array}\right]$$

$0 \ 5 \ -2 \mid b+4$

$0 \ 0 \ -12 \mid b-6$

$x_1 - x_2 + x_3 = -2$

$x_2 + 2x_3 = 2$

~~$2x_1 + 3x_2 = b$~~

$-12x_3 = b-6$

$x_3 = -\frac{b-6}{12}$

$x_1 = 3\left(\frac{b-6}{12}\right) \quad x_2 = 2 + 2\left(\frac{b-6}{12}\right)$

$x_2 = 2 + \frac{b-6}{6}$

$x_1 = -2 + \left(2 + \frac{b-6}{6}\right) + \frac{b-6}{12}$

or

$x_1 = 3\left(\frac{b-6}{12}\right)$

$b = \mathbb{R}?$ $\boxed{6}$

(b) to check for linear independance

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Doesn't reduce to identity

so

$\boxed{\text{Liniarly Dependent}}$

Also, no vectors appear to be able to be represente[d] by any other in the set.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

There is also another way that requires you to find a Cofactor by doing

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} & \end{bmatrix}$$

ect.

finding the cross product of each box to find a ~~mult~~ factor, X then

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$X(\text{this}) = A^{-1}$$

3. **(10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

so $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

$$C = BA^{-1T}$$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

so

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \downarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right] \downarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\downarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -1 & -2 & 0 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 4 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -5 & 2 & 4 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

so

$$\begin{bmatrix} (-5+6-3) & (2-2+3) & (4-4+3) \\ (-20+15-4) & (8-5+4) & (16-10+4) \\ (-15+6-1) & (6-2+1) & (12-4+1) \end{bmatrix}$$

so

$$C = \begin{bmatrix} -2 & 3 & 3 \\ -9 & 7 & 10 \\ -10 & 5 & 9 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

(a) $\begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

(b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$ SO $x_1 - 2x_2 = 1$ $\quad x_1 = 1 + 2x_2$

$x_1 + x_2 = 10$

$x_1 - 3x_2 = -2$ $\quad x_1 = 1 + 6 = 7$

or

$$\begin{matrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -4 & -12 \end{matrix}$$

SO $-4x_2 = -12$

$x_2 = 3$

$x_1 = 7$

SO

$\begin{bmatrix} 7 \\ 3 \end{bmatrix}$

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ V

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

one to one if all Col(A) are lin. dep.

SO NO

(a) $\cancel{x_1}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \xrightarrow{\text{NO}} \cancel{x_2}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \cancel{x_3}\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ $\boxed{\text{So lin. dep.}}$ ✓

$\boxed{\text{SO One to ONE}}$

NO

(b) $x_1\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \xrightarrow{\text{NO}} x_2\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad x_3\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

- All lin. dep

$\boxed{\text{So One to One}}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

* **a)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

~~$u=\begin{bmatrix}1\\2\\3\end{bmatrix}$ $v=\begin{bmatrix}4\\2\\7\end{bmatrix}$ $w=$~~ $w=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$ $u=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $v=\begin{bmatrix}1\\1\\1\end{bmatrix}$

False,

$w+u = \begin{bmatrix}0\\0\\0\end{bmatrix}$ so

$\left[\begin{array}{cc|c}1&1&0\\1&1&0\\1&1&0\end{array}\right]$ is valid

but $\left[\begin{array}{cc|c}1&1&-1\\1&1&-1\\1&1&-1\end{array}\right]$ is invalid

**b)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False, $u=\begin{bmatrix}1\\2\\3\end{bmatrix}$ $v=\begin{bmatrix}3\\2\\1\end{bmatrix}$ $w=\begin{bmatrix}4\\4\\4\end{bmatrix}$

$u+v = w$ so $\{u, v, w\}$ is linearly independe

but u and v alone $u=\begin{bmatrix}1\\2\\3\end{bmatrix}$ $v=\begin{bmatrix}3\\2\\1\end{bmatrix}$ are

linearly dep. because neither vector can represent the other.

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2\times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

~~$u = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $v$~~

~~$u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then $Au = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ $Av = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$~~

~~but u & v are not linearly dep.~~

~~False~~

True, thought it was false but couldn't disprove

$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so $Au = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $Av = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

but u and v

**b)** If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

False

so $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ is valid

but

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

is not

MTH-309T-F19-EX1-051-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Julia Shapiro

**UB Person Number:**

| 5 | 0 | 2 | 6 | 0 | 8 | 1 | 3 |
|---|---|---|---|---|---|---|---|

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of $b$ such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 \to -2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right]$$

$R_1 \to R_1 + R_2$
$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$

$b = -6$ If b equals anything other than $-6$, there will be a pivot posinon in the last column leading to no solutions

b)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \Rightarrow -2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$R_3 \Rightarrow R_2 + R_3$
$R_1 \to R_2 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The set $\{v_1, v_2, v_3\}$ is not linearly dependent because $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ does not have only the trivial solution. $x_3$ is a free variable giving infinite solutions making the set linearly dependent

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad R2 \leftrightarrow R3$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R3 \to -R1+R3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right]$$

$R2 = \frac{R2}{2}$ $\downarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{-1}{2} & 0 & 0 & -\frac{1}{2} \end{array}\right] \xleftarrow[R3 \to -R2+R3]{R1 \to R2+R1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right]$$

$R3 \to \frac{-2}{1} \cdot R3$ $\downarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R1 \to \frac{-3}{2}R3+R1]{R2 \to \frac{1}{2}R3+R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

**3. (10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A^{T-1} \cdot A^T C = A^{T-1} \cdot B$$

$$C = A^{T-1} \cdot B$$

$$A^{T-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

$R2 \to -2R1 + R2$
$R3 \to R1 + R3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \end{array}\right]$$

$R2 \to -R2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \end{array}\right]$$

$R1 \to -R2 + R1$
$R3 \to -R2 + R3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$R2 \to -R3 + R2$
$R1 \to R3 + R1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$A^{T-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$A^{T-1} \cdot B =$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+0-3 \\ 0+0+6 \\ 0+4-3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+0-2 \\ 0+0+4 \\ 0+5-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+0-1 \\ 0+0+2 \\ 0+4-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$A^{T-1} \cdot B = \begin{bmatrix} -2 & 0 & 2 \\ 6 & 4 & 2 \\ 1 & 3 & 3 \end{bmatrix} = C$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) Standard Matrix of T:

$$A = [T(e_1)\ T(e_2)] = \left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right] = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

b)

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow[R_3 \to -R_1 + R_3]{R_2 \to -R_1 + R_2} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] R_2 \to \frac{R_2}{3}$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \begin{array}{l} R_3 \to R_2 + R_3 \\ R_1 \to 2R_2 + R_1 \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$$u = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$$

there is a unique solution

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathrm{v}_1$ and $\mathrm{v}_2$ such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right]$ $R3 \to -3R1+R3$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$ $R2 \to \frac{R2}{2}$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$ $R1 \to -R2+R1$, $R3 \to -R2+R3$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$ $R3 \to \frac{R3}{2}$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $R2 \to -2R3+R2$, $R1 \to 2R3+R1$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\leftarrow$ A IS one to one w/ a pivot pos, in every column

$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$ $R3 \to -3R1+R3$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$ $R1 = -\frac{1}{2}R2+R1$, $R3 = -\frac{1}{2}R2+R3$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 2x_3$
$x_2 = -4x_3$
$x_3 = x_3$

$\mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}\right\}$

T(A) is not one to one because there is not a pivot position in every column

$V_1 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}\right) = 0$

$T\left(\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}\right) = 0$

$TA(V_1) = TA(V_2)$
$TA\left(\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}\right) = TA\left(\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}\right)$

$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$ is in Nul(A)

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

$w + u \in \mathrm{Span}(u, v) \quad w \in \mathrm{Span}(u, v)$

$W = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$W + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$W = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 0 \end{bmatrix} \in \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$w + \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \in \mathrm{span}\left(\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$

$w = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \in \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$

$\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \in \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$

$$\left[\begin{array}{cc|c} 1 & 4 & 2 \\ 2 & 5 & 2 \\ 3 & 6 & 3 \end{array}\right] \xrightarrow[R2 \to -2R1+R2]{R3 \to -3R1+R3} \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & -3 & -2 \\ 0 & -6 & -3 \end{array}\right] \xrightarrow{R3 = -2R2+R3} \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{R2}{-3}} \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{array}\right]$$

True because w + u linear combination of u (w) to be in span, therefore it will be in span(u, v)

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$u = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$

$\boxed{\text{True}}$ because if set $\{u, v, w\}$ is all linearly independent from each other, then u and v has to be linearly independent

$A\begin{bmatrix}-3\\1\end{bmatrix} = \cancel{\begin{bmatrix}1&2\\2&4\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}4+2\\6+4\end{bmatrix} = \begin{bmatrix}-2\\-2\end{bmatrix}}$

$A\cdot\begin{bmatrix}1\\0\end{bmatrix} = \cancel{\begin{bmatrix}1&2\\2&4\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1+0\\2+0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}}$ } dependent

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2\times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1&2\\2&4\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1&2\\2&4\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$

$A = \left[\begin{array}{cc|c}1&2&0\\2&4&0\end{array}\right]$ $R_2 \to -2R_1+R_2$

$\begin{bmatrix}1&2&0\\0&0&0\end{bmatrix}$

$\cancel{v = \begin{bmatrix}1\\1\end{bmatrix} \quad u = \begin{bmatrix}2\\2\end{bmatrix}}$

$\cancel{A\cdot\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}1&2\\2&4\end{bmatrix}\cdot\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}-2+2\\-4+4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}}$

$\cancel{A\cdot\begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}1&2\\2&4\end{bmatrix}\cdot\begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}-4+4\\-8+8\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}}$

$x_1 = -2x_2$ $\quad Nul(A) = Span\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$

$x_2 = x_2$

False $v = \begin{bmatrix}1\\0\end{bmatrix}$ $u = \begin{bmatrix}0\\1\end{bmatrix}$

$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1&1\\1&1\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1&1\\1&1\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

2 examples $\begin{bmatrix}-3\\2\end{bmatrix}, [$

A(v), A(u) are linearly dependent, but v, u is linearly independent

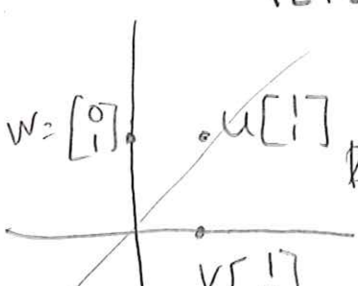
**b)** If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$u = \begin{bmatrix}1\\1\end{bmatrix}$ $v = \begin{bmatrix}1\\0\end{bmatrix}$ $w = \begin{bmatrix}0\\1\end{bmatrix}$ $u \in Span(v, w)$

$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-5\\-5\end{bmatrix}$ $\quad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \to \begin{bmatrix}-5\\0\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \to \begin{bmatrix}0\\-5\end{bmatrix}$

True because with any reflection, T(u) will be a linear combination of T(v) and T(w) if u is in span(v,w)




| Reflect over x-axis | reflect over line y=x | reflect over y-axis |
|---|---|---|
| $T(u) = \begin{bmatrix}1\\-1\end{bmatrix}$ | $T(u) = \begin{bmatrix}1\\1\end{bmatrix}$ | $T(u) = \begin{bmatrix}-1\\1\end{bmatrix}$ |
| $T(w) = \begin{bmatrix}0\\-1\end{bmatrix}$ | $T(w) = \begin{bmatrix}1\\0\end{bmatrix}$ | $T(w) = \begin{bmatrix}0\\1\end{bmatrix}$ |
| $T(v) = \begin{bmatrix}1\\0\end{bmatrix}$ | $T(v) = \begin{bmatrix}0\\1\end{bmatrix}$ | $T(v) = \begin{bmatrix}-1\\0\end{bmatrix}$ |

MTH-309T-F19-EX1-016-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Shuoling Li

**UB Person Number:**

| 5 | 0 | 2 | 2 | 0 | 5 | 2 | 0 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

**a)** Find all values of $b$ such that $w \in \text{Span}(v_1, v_2, v_3)$.

**b)** Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a). $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]^{-2} \quad \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{bmatrix}^{+} \quad \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix}$

When $b = -6$, it has infinite solutions.

$x_1 + 3x_3 = 0$

$x_2 + 2x_3 = 2$

b) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix}^{\times -2} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & -2 \end{bmatrix}^{+} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -5 & -2 \end{bmatrix}^{\times 5}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}^{\times -\frac{1}{2}} \leftarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}^{+}$

It's linearly independent.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Compute $A^{-1}$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix} \times -2$$

$$\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times -1 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ A is inverstible.

$$C_1: \begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \cdot -2 \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} -1 \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$C_2: \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & -1 \\ 2 & 0 & 3 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ \cancel{2} & \cancel{0} & \cancel{3} & \cancel{0} \end{bmatrix} - \to \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & \\ \cancel{1} & \cancel{0} & \cancel{2} & \cancel{-1} & \\ 0 & 0 & 1 & 0 & \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] -1$$

$C_2$

$$C_3 \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} \times -2$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} + \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \times -2 \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

**3. (10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad A = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A^T = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$A^T \cdot A^{T^{-1}} C = B A^{T^{-1}}$$

$$C = B A^{T^{-1}}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \quad \therefore \; A^T \text{ inverse} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \overset{-1}{-1\times 1} + \overset{4}{1\times 4} + 0\times 3 & \overset{-2}{-1\times 2} + \overset{5}{1\times 5} + 0\times 2 & -1\times 3 + 1\times 4 + 0\times 1 \\ \overset{-1}{-1\times 1} + 0\times 4 + 2\times 3 & -1\times 2 + 0\times 5 + 2\times 2 & -1\times 3 + 0\times 4 + -1\times 1 \\ \overset{2}{2\times 1} + 1\times 4 + -1\times 3 & 2\times 2 + 1\times 5 + -1\times 2 & 2\times 3 + 1\times 4 + -1\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 1 \\ 5 & 2 & -4 \\ 3 & 7 & 9 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a). $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -1 & -3 \end{bmatrix}_{\times -1} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\times 2} \begin{bmatrix} 2 & 0 & 21 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix}_{\times -1}$

$-4 \quad -1 \quad 0 \quad \times -\frac{1}{2}$

$\begin{matrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{matrix}$ $\times -2$

$\begin{matrix} 0 & +1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{matrix}$ $\times -1$

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{matrix}$ $-1$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**5. (20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathrm{v}_1$ and $\mathrm{v}_2$ such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{bmatrix} \xrightarrow{\times\frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{bmatrix} \times -3$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{bmatrix} \times -1$$

It's not one to one.

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{bmatrix} \times -2$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \times \tfrac{1}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} -1$$

$$V_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$x_1 = -2$

$x_2 + x_5 = 2$

$x_3 = 0$

$x_4 = \text{free}$

$x_5 = \text{free}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

F

$u \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad w = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

**b)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

linearly independent

F

linearly independent mean have exactly one solution.

if $\{u, v, w\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\{u, v\} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ ← have infinite solution.

∴ It's False

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2\times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent. T

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

**b)** If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

T

MTH-309T-F19-EX1-073-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

John koszela

**UB Person Number:**

| 5 | 0 | 1 | 7 | 7 | 9 | 5 | 8 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

**a)** Find all values of $b$ such that $w \in \text{Span}(v_1, v_2, v_3)$.

**b)** Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

$R_3 = R_3 - 2R_1$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right]$

$R_3 = R_2 + R_3$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$

$b+6=0$

$\boxed{b=-6}$

b) $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

aug matrix

$[v_1 \ v_2 \ v_3 | 0]$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right]$

Reducing rows

$R_3 = R_3 - 2R_1$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$

$R_3 = R_3 + R_2$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free variable means infinity many solutions

the set $\{v_1, v_2, v_3\}$ is not linearly indipendent

**2. (10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_2 = R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_2 = R_2(-1)$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_1 = R_1 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_1 = R_1 - 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & -3 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$



3. **(10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

3x3

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

B
3x3

$C = BA^{-1}$

$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$

$C = \begin{bmatrix} -2+2+6 & 3+3+6 & 0 \\ -8+5+8 & 12+5+8 & 0 \\ -6+2+2 & 9+3+2 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 6 & 12 & 0 \\ 5 & 25 & 0 \\ -2 & 14 & 0 \end{bmatrix}$

C is a 3x3 matrix

$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

$A^T$     C     B

for Matrix division

1) $a/b = a \cdot b^{-1}$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$$[3 \times 2][2 \times 1] = [3 \times 1]$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

b)

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$3 \times 2$ $\quad 2 \times 1$ $\quad 3 \times 1$

$\text{rref}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{pmatrix}$ would solve it

$$\boxed{U = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$c_1 - 2c_2 = 1$

$c_1 + c_2 = 10$

$c_1 - 3c_2 = -2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix}$$

rref

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -7 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_2 = 3$

$c_2 = 7$

$c_1 - 2c_2 = 1$

$c_1 = +2(7) + 1$

$c_1 = 15$

15-14

**5. (20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathrm{v}_1$ and $\mathrm{v}_2$ such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

one to one - pivot position in every column
onto - pivot position in every row

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\downarrow$

$R_3 = R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$\downarrow$

$R_2 \Leftrightarrow R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$

$R_2 = R_2 + R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$R_3 \cdot \frac{-1}{4}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{array}\right]$

$\downarrow$

$R_3 = R_3 - 3R_1$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{array}\right]$

$R_2 \Leftrightarrow R_3$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & 4 & 0 & 2 & 4 \end{array}\right]$

$R_3 \to R_3 - 2R_2$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{array}\right]$

$R_2 \to R_2 + R_3$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{array}\right]$

$R_1 = R_1 - R_2$

$R_3 \cdot \frac{-1}{4}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$

not one to one

$A v_1 = A v_2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$3 \times 3$ $3 \times 1$ $3 \times 3$ $3 \times 1$

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

**6. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True, u is in the span of u

So adding it does not take it out

of span.

**b)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

true. there would be no free variables

when reducing the set to $\{u, v, w\}$ to $\{u, v\}$

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2 \times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$A = \begin{bmatrix} \# & \# \\ \# & \# \end{bmatrix}$

true, same transformation done on u and v

**b)** If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

False

if transformation is $2 \times 2$ matrix it could take u out of span of v and w

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# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Fuming Zhao

**UB Person Number:**

| 5 | 0 | 3 | 2 | 0 | 7 | 9 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

**a)** Find all values of $b$ such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

**b)** Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $w \in \mathrm{Span}(v1, v2, v_3)$ means that $c_1v_1 + c_2v_2 + c_3v_3 = w$ has solution

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \text{①×2−③}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -4-b \end{bmatrix} \text{②−③}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{bmatrix}$$

Since $6+b$ is in constant column.

$\therefore 6+b=0$

$b=-6$

$\therefore b = -6$

b) $x_1v_1 + x_2v_2 + x_3v_3 = 0$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix} \text{①×2−③}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \text{②−③}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $x_3$ is free variable, this homogenous equation has many solutions.
This means that the set $\{v_1, v_2, v_3\}$ is not linearly independent

**2. (10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (2)-(1)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (3)-(2)\times 2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] (2)+(3)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] (1)+(2)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] (1)-(3)\times 2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

**3. (10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{bmatrix} \begin{matrix} \\ ①+② \\ ①\times 2-③ \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 1 & 1 & -1 & 2 & 5 \end{bmatrix} \begin{matrix} \\ \\ ②-③ \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{bmatrix} \begin{matrix} \\ ②-③\times 2 \\ \\ \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{bmatrix} \begin{matrix} ①-② \\ \\ \\ \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) Let T be 3x2 matrix

$$T = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \\ C_5 & C_6 \end{bmatrix}$$

$$T\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) = \begin{bmatrix} X_1 - 2X_2 \\ X_1 + X_2 \\ X_1 - 3X_2 \end{bmatrix}$$

$$C_1X_1 + C_2X_2 = X_1 - 2X_2$$
$$C_3X_1 + C_4X_2 = X_1 + X_2$$
$$C_5X_1 + C_6X_2 = X_1 - 3X_2$$

$\therefore C_1 = 1, C_2 = -2, C_3 = 1, C_4 = 1, C_5 = 1, C_6 = -3$

$$\therefore T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

Let $u$ be vector $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

aug. matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{(1)-(3)} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{(2)-(1)} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{(2)\div 3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{(2)-(3)} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(1)+(2)\times 2} \begin{array}{c} C_1 \ C_2 \\ \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\therefore \begin{cases} C_1 = 7 \\ C_2 = 3 \end{cases} \qquad \therefore u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

**5. (20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

**a)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ ③-①×3

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$ ②÷2

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ ③-②

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ ③÷2

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Since A has a pivot position in every column, $T_A(v)=Av$ is one to one.

**b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ ③-①×3

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ ②÷2

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ ②-③

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ ①-②

$\downarrow$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Since A does not have a pivot column in every column, $T_A(v)=Av$ is not one to one

From the matrix, we can get $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \in \mathrm{Nul}(A)$

Let $v_1$ be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$v_2 = v_1 + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

Yes, Since $w+u \in \mathrm{Span}(u,v)$, then $w+u = c_1 u + c_2 v$ has solutions, $w = c_1 u + c_2 u - u = (c_1 - 1)u + c_2 u$ has solutions

**b)** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False

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**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2 \times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False

$(c_1 A u + c_2 A v) = 0$

$\downarrow$

$A(c_1 u) + c_2$

**b)** If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

Yes. Since $u$ is in Span $(v, w)$,

then $u = c_1 v + c_2 w$ has solution.

Also, $T(u) = T(c_1 v + c_2 w)$ has solution

Also, $T(u) = T(c_1 v) + T(c_2 w)$ has solution.

This means that $T(u)$ is in Span $(T(v), T(w))$.