MTH-309T-F19-EX1-045-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

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Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
16	10	10	20	13	2	1	2	10	81	B

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$c_1 v_1 + c_2 v_2 + c_3 v_3 = w$

(b) RREF :- $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_2 + R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \boxed{0} \end{bmatrix}$

~~So every~~ Since every column, can not be a pivot column. ✓

$\boxed{\text{Set } \{v_1, v_2, v_3\} \text{ is not linearly independent}}$

(a) RREF: $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right]$

$\xrightarrow{R_3 \to R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & \cancel{\boxed{b+4}} \end{array}\right]$ — (1)

$b + 6$

Now, there exists sol^n only if is not a pivot column.

~~So, Sol^n exists only when, $b+$~~

Since, $W \in \text{Span}(v_1, v_2, v_3)$.

Here, sol^n exists only when, $b + \cancel{4}\,6 \neq \cancel{1}\,0$

$\Rightarrow b \neq \cancel{-4}\ -6$

$\Rightarrow \cancel{b \neq -3}$

$\therefore b \in \mathbb{R}$, except $b = -3$.

~~From (1),~~ ~~$R_2 \to R_1 + R_2$~~ $\begin{bmatrix} 1 & -1 & 1 & -2 \end{bmatrix}$

$R_2 \to$

Also, sol^n exist, if $b + 4 = 0$

$\therefore b = -4$.

For, $b = -4$, there is infinite sol^n.

$\therefore$ ~~$b = \{b \in \mathbb{R}^3$~~

~~See next Page~~

Ans :-

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
																✗				

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$\det(A) = 1(0-2) - (-1)(-1-0) + 2(2-0)$

$= -2 - 1 + 4 = 4 - 3 = \boxed{1} \rightarrow A^{-1}$ is possible

$a_{11} = \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = \boxed{-2}$ $\quad a_{12} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{13} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = \boxed{2}$

$a_{21} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 4 - 1 = \boxed{3}$ $\quad a_{22} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{23} = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = \boxed{2}$

$a_{31} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = \boxed{-1}$ $\quad a_{32} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = \boxed{-1}$ $\quad a_{33} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 0 - (-1) = \boxed{1}$

Now, $\operatorname{adj}(A) = \begin{bmatrix} -2 & -1 & 2 \\ 3 & -1 & 2 \\ -1 & -1 & 1 \end{bmatrix} \times [$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$a_{11} = \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = \boxed{-2}$ $\quad a_{12} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1 - 4 = \boxed{-3}$ $\quad a_{13} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = \boxed{-1}$

$a_{21} = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{22} = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = \boxed{-1}$ $\quad a_{23} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = \boxed{-1}$

$a_{31} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = \boxed{2}$ $\quad a_{32} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = \boxed{2}$ $\quad a_{33} = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -(-1) = \boxed{1}$

$\therefore \operatorname{adj}(A) = \begin{bmatrix} -2 & -3 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

$\therefore A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)} = \dfrac{1}{1} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$ ✓

Ans

Quick check: $\left[(1 \ -1 \ 2) \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right] = [-2 - 1 + 4] = [-3 + 4] = [1]$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Now, $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$

Here, $[A^T]^{-1}[A]^T[C] = [I][C] = [C] = [A^T]^{-1}[B]$

$$\therefore [C] = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (-2+4+6) & (-4+5+4) & (-6+4+2) \\ (3-4-6) & (6-5-4) & (9-4-2) \\ (-1+4+3) & (-2+5+2) & (-3+4+1) \end{bmatrix}$$

$$\therefore \boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

Ans:-

Check:- $[1\ 1\ 0]\begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix} = [8-7] = [1]$

$[2\ 1\ -1]\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = [0+2-3] = [1]$

$[2\ 1\ -1]\begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix} = \begin{bmatrix} 16-\cancel{8}-6 \\ 7 \end{bmatrix}$.

$\Rightarrow [16-7\ \cancel{+6}] = \underline{3}$ $\quad [16-13] = 3?$

$24-7$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

(a) $\cancel{T(e_1)}$ $[e_1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$// \quad T([e_1]) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad // \quad T([e_2]) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

So, Standard matrix, $A = [T(e_1) \quad T(e_2)] = \boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ ✓ Ans:-

(b) From definition, $T(u) = A \cdot u = A \cdot \begin{bmatrix} u_1 \\ u_2 \\ \cancel{\ } \end{bmatrix}$, as $T: R^2 \to R^3$

Now, $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

Since, $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 - 2u_2 \\ u_1 + u_2 \\ \cancel{u_4}\ -3u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\Downarrow$

$$\begin{cases} u_1 - 2u_2 = 1 & -(1) \\ u_1 + u_2 = 10 & -(2) \\ u_1 - 3u_2 = -2 & -(3) \end{cases}$$

from (2), $u_1 = 10 - u_2$

$\therefore$ " (1), $10 - u_2 - 2u_2 = 1$

$\Rightarrow 10 - 3u_2 = 1$

$\Rightarrow 10 - 1 = 3u_2$

$\therefore u_2 = \frac{9}{3} = 3$, $u_1 = 10 - 3 = 7$

$\therefore \boxed{\vec{u} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} = 7\hat{u}_1 + 3\hat{u}_2}$ ✓

Ans:-

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) Aug. mat. of A: RREF

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_1 \to -R_2 + R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3/2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

✓

Every column is a pivot column.

$\therefore$ $\boxed{T_A(v) \text{ is one-to-one}}$

b)

Aug. mat: RREF $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & \cancel{2}\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to \frac{R_3}{2}]{\cancel{R_1 \to R_2 + R}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & \cancel{1}\ 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \cancel{\boxed{1}} \end{bmatrix} \to \text{every column is a pivot column.}$$

$\therefore$ $\boxed{T_A(v) \text{ in one-to-one}}$ ✗

Ans:

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

~~False.~~ ← why?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True. ✓

Since, in set $\{u, v, w\}$ are already indep.

as subset $\{u, v\}$ will also be indep.

why?

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False. ← why?

$$\cancel{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$A\cancel{(u-v) = 0, \quad u-v}$

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

~~False.~~

Transformation doesnt guarantee being in the span..

~~If $T(v)x_1 + T(w)x_2 = 0$, doesn't have sol'n,~~

~~they~~

0 1 2 3 4 5 6 7 8 9 10