MTH 309T LINEAR ALGEBRA
EXAM 1
October 3, 2019

Name:

Dylan Tua

UB Person Number:

5	0	2	3	8	1	2	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.

- For full credit solve each problem fully, showing all relevant work.

	1	2	3	4	5	6	7	TOTAL	GRADE
	--	--	--	--	--	--	--	0	nan

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $x_1 v_1 + x_2 v_2 + x_3 v_3 = w$

$$\begin{bmatrix} 1 & -1 & 1 & | & -2 \\ 0 & 1 & 2 & | & 2 \\ 2 & -3 & 0 & | & b \end{bmatrix} +\cdot(1) \Rightarrow \begin{bmatrix} 1 & -1 & 1 & | & -2 \\ 0 & 1 & 2 & | & 2 \\ 1 & -2 & -1 & | & 2+b \end{bmatrix} +\cdot(-1)$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & | & -2 \\ 0 & 1 & 2 & | & 2 \\ 0 & -1 & -2 & | & -2-b \end{bmatrix} +\cdot 1 \Rightarrow \begin{bmatrix} 1 & -1 & 1 & | & -2 \\ 0 & 0 & 0 & | & -b \\ 0 & -1 & -2 & | & -2-b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & | & -2 \\ 0 & -1 & -2 & | & -2-b \\ 0 & 0 & 0 & | & -b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & | & -2 \\ 0 & 1 & 2 & | & 2+b \\ 0 & 0 & 0 & | & b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 & | & b \\ 0 & 1 & 2 & | & 2+b \\ 0 & 0 & 0 & | & b \end{bmatrix}$$

b) The set $\{v_1, v_2, v_3\}$ is __not__ lin. independent; it has more than one, trivial solution.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] +).2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right] +).(-1) \Rightarrow \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & -2 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & -2 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \cdot (1) \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \cdot (-1) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow (A^{-1})^T = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$(A^{-1})^T \cdot B = C$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2+8+6 & -4+10+4 & -6+8+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ -1+4+3 & -2+5+2 & -3+4+1 \end{bmatrix}$$

$$C = \begin{bmatrix} 12 & 10 & 4 \\ -7 & -3 & 3 \\ 6 & 2 & 2 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

a) $A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -2 & | & 1 \\ 1 & 1 & | & 10 \\ 1 & -3 & | & -2 \end{bmatrix} \cdot (-1) \Rightarrow \begin{bmatrix} 1 & -2 & | & 1 \\ 1 & 1 & | & 10 \\ 0 & -1 & | & -3 \end{bmatrix} (-1) \Rightarrow \begin{bmatrix} 1 & -2 & | & 1 \\ 1 & 1 & | & 10 \\ 0 & 1 & | & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & | & 6 \\ 1 & 1 & | & 10 \\ 0 & 1 & | & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & 6 \\ 0 & 1 & | & 3 \\ 1 & 1 & | & 10 \end{bmatrix} \cdot (-1) \Rightarrow \begin{bmatrix} 1 & 0 & | & 6 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 1 \end{bmatrix}$

$u = Span\left(\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}\right)$

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors $\underline{v_1}$ and $\underline{v_2}$ such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a)

$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 3 & 4 & 4 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 4 & 4 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 1 & 1 & 0 & | & 0 \end{bmatrix} \overset{+}{\underset{(\cdot)(3)}{}}$

$\Rightarrow \begin{bmatrix} 0 & 1 & 4 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 1 & 1 & 0 & | & 0 \end{bmatrix} \overset{+\cdot(-1)}{\Rightarrow} \begin{bmatrix} 0 & 1 & 4 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 1 & 1 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 1 & 4 & | & 0 \end{bmatrix} \overset{\cdot(-1)}{\Rightarrow} \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 4 & | & 0 \end{bmatrix}$

$= \begin{bmatrix} \boxed{1} & 1 & 0 & | & 0 \\ 0 & \boxed{1} & 0 & | & 0 \\ 0 & 0 & \boxed{1} & | & 0 \end{bmatrix}$

Pivot position in every col., so $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ is one-to-one.

b) $\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 3 & 4 & 2 & | & 0 \end{bmatrix} \overset{+\cdot(-3)}{\Rightarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \overset{+\cdot(-2)}{\Rightarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$ not one-to-one

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 + c_2 \\ c_2 + 2c_3 \\ 0 \end{bmatrix}$

Let $c_1 = 1$
$c_2 = 2$
$c_3 = 3$

$\Rightarrow \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$ $v_1 = \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 9 \\ 24 \\ 0 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True; since $u \in \text{Span}(u, v)$, then if $w + u \in \text{Span}(u, v)$ then w must be a multiple of a vector in $\text{Span}(u, v)$. Therefore, $w \in \text{Span}(u, v)$.

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

A set is lin. ind. when $x_1 u + x_2 v + x_3 w = 0$ has only one solution, where $u, v, w = 0$. If the set $\{u, v, w\}$ is lin. ind., then either combination of two of those vectors must also be lin. ind. Therefore, this statement is true.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and u, v are vectors in $\mathbb{R}^2$ such that Au, Av are linearly dependent then u, v also must be linearly dependent.

False; the matrix A can cause Au, Av to become lin. dependent even if u, v are lin. independent.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

True; let $u = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$T(u) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$

$T(v) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$T(w) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$T(u)$ is in $\text{Span}(T(v), T(w))$

So, $T(u)$ must be in $\text{Span}(T(v), T(w))$

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$$