MTH-309T-F19-EX1-005-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	1	7	1	8	3	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

12	1	0	5	5	9	--	32	nan
1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\underline{w} \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$V_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\quad (-2) \cdot \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}$ $\quad 2-3$ $=2+6$

$V_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$

$V_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\quad (1) \cdot \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$ $\quad 2-2$

$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow (1) \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad 2-(-1)$

$x_1 = -x_3 \rightarrow -2$

$x_2 = -2x_3 \rightarrow (2)$

$x_3 = \text{free} \rightarrow b$

(a) $\boxed{b = -1}$ $\quad (-1).$

(b) No, dependent because it has infinite solutions x_3 is a free variable

0 1 2 3 4 5 6 7 8 9 10 11 **12** 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$(-1)\quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$-2+1$

$1-1=-1$

$$(-2)\quad \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$(1)\cdot\quad \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = A^{-1}}$$

$-1+2$

$-1+1$

$$(1)\cdot\quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

$$(-1)\cdot\quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

-2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$\cancel{(A^T)^{-1} A^T} C \ \ (A^T)^{-1} B$

$C = (A^{-1})^T \cdot B$

$$C = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$3 \times 3 \qquad 3 \times 3$

$2+4$

$-2+4+6=$

$1+5$

$-4+5+4$

$-6+4+2$

$-2+2$

$3+4-6$

$-1+6$

$6-5-4$

$1-4$

$9-4-2$

$5-2$

$-1+4+3$

$3+3$

$-2+5+2$

$-3+4+1$

$$\boxed{\begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$7 - 6 = 1$
$7 + 3 = 10$
$7 - 9 = -2$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

(a) $\boxed{T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

(b) $\boxed{T(u) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

(b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$(-1)\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$

$-1+10$

$(-1)\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{bmatrix}$

$(-3)\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{array}\right]$ $(2)\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$2 \cdot 3$

$-1+2$

$-9+$

$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right]$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right]$ (-3)

(-1) $\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$

$-2+4$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\boxed{\text{(a) ONE-TO-ONE}}$

-3 $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$

(-1) $\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$-2+2 =$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\boxed{\text{(b) NOT ONE-TO-ONE}}$

$x_1 = -x_2$

$x_2 = -2x_3$

$x_3 = \text{free}$

$$\boxed{V_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

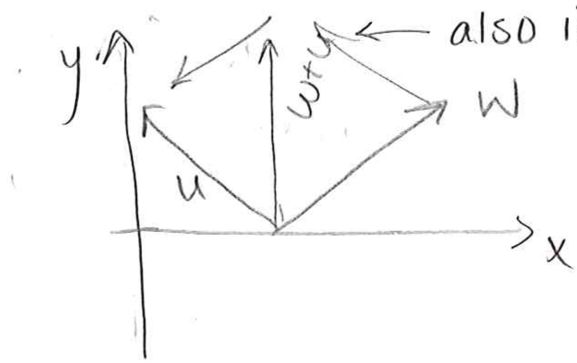
6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True:

if $x = w$, then $x_0 = w + u$

graphical interpretation



b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False

$$A = \begin{array}{c} \begin{matrix} u & v & w & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \end{array} \quad \text{but} \quad A = \left[\begin{array}{cc|c} 1 & 1 & 9 \\ 0 & 2 & 3 \\ 3 & 4 & 5 \end{array}\right]$$

is linear independent

[after row reduction each value of u,v,w corresponds to a unique solution]

is inconsistent after row reduction because last row is

$$\left[\begin{array}{cc|c} x & y & z \\ a & b & c \\ \boxed{0 \quad 0} & & \# \end{array}\right]$$

No solution; linearly dependent

0 1 2 3 4 5 6 7 8 9 10

9

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$$\begin{array}{c} \quad A \qquad\quad u \\ -2 \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 6 & 10 \end{array}\right] \end{array}$$

FALSE,

just because Au, Av are linearly dependent does not mean that u, v will also be linearly dependent

$$\begin{array}{c} \quad A \qquad\quad v \\ (-2) \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 6 & 7 \end{array}\right] \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 0 & 3 \end{array}\right]$$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True

Col(A) = set of values for T_A
so if u is in span(v, w) for matrix A then it will also be in the span of the transformation for A.

0 1 2 3 4 5 6 7 8 9 10