



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

SINCE $\mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$v = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$T_A(v) = Av$$

$$Av = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + y_1 + 0 & x_2 + y_2 + 0 & x_3 + y_3 + 0 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 4z_1 & 3x_2 + 4y_2 + 4z_2 & 3x_3 + 4y_3 + 4z_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 4z_1 & 3x_2 + 4y_2 + 4z_2 & 3x_3 + 4y_3 + 4z_3 \end{bmatrix}$$

$$A.v = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + y_1 + 0 & x_2 + y_2 + 0 & x_3 + y_3 + 0 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 2z_1 & 3x_2 + 4y_2 + 2z_2 & 3x_3 + 4y_3 + 2z_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 2z_1 & 3x_2 + 4y_2 + 2z_2 & 3x_3 + 4y_3 + 2z_3 \end{bmatrix}$$

$$\text{FOR } T_A(v_1) = T_A(v_2)$$

$$3x_{31} + 4y_{31} + 4z_{31} = 3x_{32} + 4y_{32} + 2z_{32}$$

$$4z_{31} = 2z_{32}$$

$$2z_{31} = 1z_{32}$$

$$\frac{z_{31}}{z_{32}} = \frac{1}{2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 12 & 16 & 16 \\ 15 & 22 & 24 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Pivot position
in every
column

One-to-one ✓

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

no pivot position

one-to-one pivot
position in every column

$$T_A(v_1) = T_A(v_2)$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$T_A(v) = Av$$

$$T_A\left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}\right) \quad T\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\right)$$

$$c\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$



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$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$A = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 0 \end{array} \right] \xrightarrow{\substack{R_2 \leftarrow R_2 \cdot \frac{1}{2} \\ R_3 \leftarrow -3R_1 + R_3}} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 4 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 4 & 0 \end{array} \right]$$

$$\xrightarrow{R_3 \leftarrow -3R_1 + R_3} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{array} \right] \xrightarrow{\substack{R_3 \leftarrow R_3 - R_2 \\ R_3 \leftarrow R_3 - R_1}} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

$$\text{b)} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

$$\text{a)} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{R_3 \leftarrow R_2 + R_1} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{array} \right]$$

Onto Pivot column

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{array} \right]$$

One to one



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$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

one to one if pivot position in every column.

$$\text{a)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right) \xrightarrow{\text{Row 3} - 3\text{Row 1}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right) \xrightarrow{\text{Row 2} - 2\text{Row 1}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right) \xrightarrow{\text{Row 3} - 4\text{Row 2}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

A is one to one

$$\text{b)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right) \xrightarrow{\text{Row 3} - 3\text{Row 1}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \xrightarrow{\text{Row 3} - \text{Row 2}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

↑
not a pivot column

A is not 1-1.
 $T_A(v_1) = A \cdot v_1$

$$A \cdot v_1 = A \cdot v_2$$

$$\begin{aligned} &\text{let } x_3 = 2 \\ &x_3 = 2 \\ &x_2 = -2(2) = -4 \\ &x_1 = 0 - -4 = 4 \end{aligned}$$

$$\boxed{\begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = v_1}$$

$$v_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \xrightarrow{\text{Row 3} - 3\text{Row 1}} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right) \xrightarrow{\text{Row 3} - 4\text{Row 2}} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{array} \right)$$

$$\begin{aligned} &\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right) \xrightarrow{\text{Row 3} - 3\text{Row 1}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \\ &x_3 = x_3 \quad x_3 = 1 \\ &x_2 + 2x_3 = 0 \quad x_2 = 0 - 2 \\ &x_1 + x_2 = 0 \quad = -2 \\ &\text{let } x_3 = 1 \quad x_1 = 0 - 2 \\ & \quad \quad \quad = 2 \end{aligned}$$



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$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$(-3) \leftarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \quad -3 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right)$$

$$(-1) \leftarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \quad (-1) \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right)$$

$$-2+4 \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

(a) ONE-TO-ONE

(b) Not ONE-TO-ONE

$$x_1 = -x_2$$

$$x_2 = -2x_3$$

x_3 = free

$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$
$v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$



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a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A has pivot in every column $\rightarrow A$ is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

not one-to-one



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \quad T = \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & 2x_2 & 4x_3 \\ 3x_1 & 4x_2 & 2x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\text{R}_2 \leftrightarrow \text{R}_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\text{R}_3 - 3\text{R}_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{R}_3 - \text{R}_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\text{R}_2 \leftrightarrow \text{R}_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{R}_2 - \text{R}_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{R}_3 - \text{R}_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{R}_2 \leftrightarrow \text{R}_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{pivot position in every col. so } \underline{\text{YES, one-to-one}}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\text{R}_2 \leftrightarrow \text{R}_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\text{R}_3 - 3\text{R}_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{R}_3 - \text{R}_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{pivot position NOT in every column, so } \underline{\text{NOT one-to-one}}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 2+1+0 \\ 0+2+0 \\ 6+4+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 10 \end{bmatrix}$$

$$x_1 + x_2 =$$

$$x_2 + 2x_3 =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 32 \\ 28 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 32 \\ 28 \end{pmatrix}$$

$$x_1 + x_2 = 2$$

$$x_1 = 2 - x_2$$

$$2x_2 + 4x_3 = 6$$

$$3x_1 + 4x_2 + 2x_3 = 9$$

$$3(2-x_2) + 4x_2 + 2x_3 = 9$$

$$6 - 3x_2 + 4x_2 + 2x_3 = 9$$

$$x_2 + 2x_3 = 3$$

$$\frac{8x_3 = 3 - x_2}{2}$$

$$x_3 = \frac{3 - x_2}{2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\text{R}_2 \leftrightarrow \text{R}_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\text{R}_3 - 3\text{R}_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$



Come back

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right]$$

$\downarrow R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \leftarrow -3R_1 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \leftarrow R_3 - R_2$$

Pivot column
because column
without a
leading 1.

NOT one-to-one.

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \leftarrow R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \leftarrow -3R_1 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \leftarrow R_2 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \leftarrow \text{One-to-one because no pivot columns?}$$

→ One-to-one
pivot columns
→ Null(A)?



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$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{(-3)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{(-2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is one-to one

because every column has a pivot position and the matrix is a homogeneous equation

$$\text{b)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{(-3)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A is one-to one

because every column has a pivot position and the matrix is a homogeneous equation.



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$\downarrow \quad \downarrow$
 $r_3 \Rightarrow r_3 - 3r_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

\downarrow

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All columns are
pivot columns, therefore

all $T_A(v)$ are one-to-one.

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$\downarrow \quad \downarrow$
 $\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$

not one to one

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ -1 \\ -\frac{1}{2} \end{bmatrix}$$

$$T_A(v_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T_A(v_2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1+1 \\ 2-2 \\ -3+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$T_A(v_1) = T_A(v_2)$



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a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\cdot -3}$

\downarrow
 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}}$

\downarrow
 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot -1}$

\downarrow
 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[-1]{\cdot \frac{1}{2}}$

\downarrow
 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[-1]$

$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$
 is one to one

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\cdot -3}$

\downarrow
 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[\cdot \frac{1}{2}]{} \downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[-1]{\cdot \frac{1}{2}}$

\downarrow
 $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$
 free

$x_1 = 2x_3$
 $x_2 = -2x_3$
 $x_3 = x_3$

Not one to one
 b
 Two vectors
 $v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \rightarrow R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array} \right] \xrightarrow{\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 + (-R_2)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Matrix is one to one because there is a pivot position in every column.

$$\text{b)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

not one-to-one

$$\text{using } v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad T_A(v_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ 0+0+0 \\ 3+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\text{now: } T_A(v_2) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} v_2 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \text{following same row reduce seen above}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} v_1 &= -v_2 + 1 \\ v_2 &= -2v_3 \\ v_3 &= v_3 \end{aligned}$$

$$v_3 = 1 \Rightarrow \begin{aligned} v_3 &= 1 \\ v_2 &= -2 \\ v_1 &= -1 \end{aligned} \Rightarrow v_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$R_3 = R_3 - R_2 \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\substack{R_2 = R_2 - R_3 \\ \text{and} \\ R_3 = R_3/2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have pivot pos. in every row so matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ is one-to one.

$$\text{b)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_2 \neq R_2/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = R_1 - R_2 \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Since every col is not a pivot col matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \text{ is not one-to one.}$$

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \Rightarrow R_3 - 3R_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \Rightarrow R_2 - R_3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 \Rightarrow R_3 - 3R_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \Rightarrow R_2 - R_3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \Rightarrow R_3 - R_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

\therefore a is one-to-one, but b is not one-to-one.

$$\text{Now, } T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$$

$$\therefore T_A(\mathbf{v}_1 - \mathbf{v}_2) = 0$$

$$\therefore \begin{bmatrix} v_1 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} - \begin{bmatrix} v_2 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} v_1 \\ 6 \\ 5 \\ 4 \end{bmatrix} - \begin{bmatrix} v_2 \\ 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array}$$

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array}$$

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array}$$

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

1 0 0 pivot
0 1 0 pivot
0 0 1 pivot

one to one

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array}$$

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}$$

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}$$

$$\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}$$

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -2x_3 \end{aligned}$$

Not one-to-one

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{array} \right) \xrightarrow{x_2 \leftrightarrow x_3} \left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{array} \right) \xrightarrow{x_3 - 3x_1} \downarrow$$

It's not one to one.

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{array} \right) \xrightarrow{x_3 - x_2} \downarrow$$

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{array} \right) \xrightarrow{x_2 \leftrightarrow x_3} \downarrow$$

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array} \right) \xrightarrow{x_2 \leftrightarrow x_3} \downarrow$$

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = -2$$

$$x_2 + x_5 = 2$$

$$x_3 = 0$$

$$x_4 = \text{free}$$

$$x_5 = \text{free}$$

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array} \right) \xrightarrow{-1} \downarrow$$

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

\Leftrightarrow pivot in each column. One-to-one

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b). } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2x_3 & &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\ x_2 &= -2x_3 & \text{Free} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\ x_3 &= x_3. \end{aligned}$$

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

$$T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2).$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{+} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1}$

T_A is one-to-one because
 $\text{Null}(A) = \{0\}$.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)}$

(b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{+} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1}$

T_A is not one-to-one because
 $\text{Null}(A) \neq \{0\}$.

$T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$

$T_A(\mathbf{v}_1) - T_A(\mathbf{v}_2) = 0$

$T_A(\mathbf{v}_1) - T_A(\mathbf{v}_2) = T_A(\mathbf{x})$

$T_A(\mathbf{v}_1 - \mathbf{v}_2) = T_A(\mathbf{x})$

$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{x}$

$\mathbf{v}_1 - \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \left| \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right.$

$T_A(\mathbf{x}) = A\mathbf{x} = 0$

$\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}$

$x_1 = 2x_3$

$x_2 = -2x_3$

$x_3 = x_3$

$\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x_3$

Let $x_3 = 1$

$\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{L_3 - 3L_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - L_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \leftarrow \frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since A has a pivot position in every column,
 $T_A(\mathbf{v})$ is one-to-one.

$$\text{b)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{L_3 - 3L_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

since A does not have a pivot position in every column, $T_A(\mathbf{v})$ is not one-to-one.

$$T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$$

$$\text{let } T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x_1 = 1 + 2x_3 \\ x_2 = 2 - 2x_3 \\ x_3 = x_3 \end{cases}$$

$$\begin{aligned} \text{if } x_3 = 1: \\ x_1 &= 1 + 2(1) = 3 \\ x_2 &= 2 - 2(1) = 0 \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} \text{if } x_3 = -1: \\ x_1 &= 1 + 2(-1) = -1 \\ x_2 &= 2 - 2(-1) = 4 \\ x_3 &= -1 \end{aligned}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left. \begin{array}{l} \text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\text{R}_3 + (-3)\text{R}_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \\ \text{R}_3 + (-\frac{1}{2})\text{R}_2 \xrightarrow{\text{R}_3 + (-\frac{1}{2})\text{R}_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \\ \text{rank } A = 2 \\ \text{rank } T_A = 2 \\ \text{rank } A \neq \text{rank } T_A \end{array} \right\} \text{Not one-to-one}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\text{R}_3 + (-3)\text{R}_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{R}_3 + (-\frac{1}{2})\text{R}_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_2 &= 0 \end{aligned}$$

$$\boxed{\begin{aligned} x_2 &= 0 & T_A &= \text{one-to-one} \\ x_1 &= -2 \end{aligned}}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\frac{1}{2}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{-3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \xrightarrow{-1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \xrightarrow{\frac{1}{2}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{-2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{-1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Since every column of matrix A has pivot position

T_A is one-to one

$$\text{Nul}(A) = \left\{ \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{\frac{1}{2}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{-3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{-1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} x_1 + x_2 &= 0 & x_1 &= -x_2 \\ x_2 + 2x_3 &= 0 & x_2 &= -2x_3 \\ x_3 &= x_3 & x_3 &= x_3 \end{aligned}$$

$$\text{Nul}(A) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} x_3$$

Since not all columns of matrix A has pivot column
 T_A is not one-to one.

$$\text{since } \text{Nul}(A) \text{ in part a) is } \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\text{and } \text{Nul}(A) \text{ in part b) is } \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

and, $T_A(v_1) = T_A(v_2)$

v_1 has to equal v_2

which v_1 and v_2 are both $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

because each Null space contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{x_3}$$

$$\begin{bmatrix} 1 & 1 & c \\ c & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\quad}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ c & 1 & 4 \\ c & 2 & 4 \end{bmatrix} \xrightarrow{x_2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ c & 0 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{4}}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

i. A is one-to-one
because there is a
pivot position in
every column

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{x_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ c & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ c & 1 & 2 \\ c & 2 & 4 \end{bmatrix} \xrightarrow{x_2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore B$ is not
one-to-one



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{array} \right] \xrightarrow{(-3)} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -4 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -4 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -2 \end{array} \right]$$

Both ~~is~~ is not
one to one
since every row
is not a pivot column

$$A \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot 2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Two = 1
The vectors are such where the last number marks both
of them equal.

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{array}{l} 2 \\ 6 \\ 9 \end{array}$$

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since A has a pivot position in every column it is 1 to 1

$$\text{b)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A is not one to one since no pivot position in every column



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$T_A(v) = T_A(w)$$

$$\text{if } w = v + n \\ n \in \text{Nul}(A)$$

$$\text{Nul}(A) = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \middle| \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \xrightarrow{(-1)}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \xrightarrow{(-1)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \boxed{\text{One}}$$

$$\boxed{\text{One to One}}$$

$$\text{Nul}(A) = \left\{ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \right\}$$

$$\left(\begin{array}{c} v \\ 0 \\ 0 \end{array} \right)$$

$$\left(\begin{array}{c} w \\ 0 \\ 0 \end{array} \right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{(-3)}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 = 2x_3$$

$$x_2 = -2x_3 \quad \text{infinitely}$$

$$x_3 = x_3$$

$$x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Nul}(A) = \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\left| \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right.$$

$$\begin{aligned} x_1 &= -x_2 & x_1 &= 2 \\ x_2 &= -2x_3 & x_2 &= -2 \\ x_3 &= x_3 & x_3 &= 1 \end{aligned}$$

a) One-to-one

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} 2 \\ -2 \\ 1 \end{array} \right] =$$

b) not one-to-one

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ *Not one-to-one*

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ *one-to-one*

a) $\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array} \right] R_3: R_3 - 3R_1$

$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{array} \right] R_3: R_3 - 3R_1$

$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right] R_2: R_2 - R_3$

$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array} \right] R_1: R_1 - R_3$

$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{array} \right] R_1: -R_1, R_1$

$\left[\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array} \right] R_2: R_2 - R_3$

$\left[\begin{array}{ccc} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{array} \right]$

$\left[\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \right]$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$R_3 \leftarrow -3R_1 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right]$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$R_3 \leftarrow -3R_1 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right]$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right]$$

$$R_3 \leftarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$$R_3 \leftarrow -2R_2 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array} \right]$$

$$R_3 \leftarrow -2R_2 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array} \right]$$

$$R_3 \leftarrow -\frac{1}{4}R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{array} \right]$$

$$R_1 \leftarrow -R_2 + R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

T_A is not one-to-one
because only columns 1
and 2 have pivot positions

T_A is one-to-one
because there is a pivot
position in every column

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x_1 = 2x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix}$$

4-3 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

4-2 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This is
one to one
because

there is
a pivot in every
column.

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{array}{r} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \quad \begin{array}{l} \text{x}_1 = -x_2 \\ \text{x}_2 = -2x_3 \\ \text{x}_3 = \text{free} \\ 2 \\ -2 \\ 1 \end{array} \quad \begin{array}{r} 0 \\ 2 \\ -2 \\ 1 \end{array}$$

This isn't one to
one cause there isn't
a pivot in every col.

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

If T_A is one-to-one, A has a pivot position in every column.

$$\Leftrightarrow \text{Nul}(A) = \{0\}$$

$$(a) \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$\therefore A$ has a pivot position in every column,

So $T_A(v)$ is one-to-one.

$$(b) \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{so } \begin{cases} x_1 = -x_2 = 2x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases} \rightarrow x = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x_3 \rightarrow \text{Nul}(A) = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$\bullet T_A(v_1) = T_A(v_2)$ if and only if $v_1 - v_2 \in \text{Nul}(A)$

If, $v_1 = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then

$$T_A(v_1) = T_A(v_2), \text{ i.e. } v_1 - v_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \in \text{Nul}(A)$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{3R_1+R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\text{b)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{3R_1+R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{4}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

it is one-to-one as every column has 1 pivot position.

not going to be one-to-one because column 3 will not have a pivot.

$$v_1 = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \quad v_2 = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$T_A(v_1) = T_A(v_2) \text{ if } v_2 = v_1 + n$$

so if $v_1 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ and $n = 2$

$$v_2 = \begin{bmatrix} 5 \\ 9 \\ 6 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

(-1 if $\text{Nul}(A) \neq \emptyset$)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\downarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\downarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\downarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

\therefore only solution to
 $Av = \emptyset$ is $v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \text{Nul}(A) = \emptyset$

$\boxed{\therefore A \text{ is one-to-one}}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\downarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 = 2x_3$$

$$x_2 = -2x_3$$

$\therefore \text{Nul}(A) \neq \emptyset$

$\therefore A$ is not one-to-one.

ex. for vectors $v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$,
 $v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$, $T_A(v_1) = T_A(v_2)$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$(a) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{since } \text{Nul}(A) = \{0\}$$

A is one to one

$$(b) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} x_3$$

$$\text{Nul}(A) = \text{Span}\left(\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}\right)$$

$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$	$v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$T_A(v_1) = 0$
--	---	----------------

$$T_A(v_2) = 0$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 2 & 4 & \\ 3 & 4 & 4 & \end{array} \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 2 & 4 & \\ 0 & 1 & 4 & \end{array}$$

$$\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 1 & 4 & \end{array}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

A is one-to one, because there is a leading one in each column.

$$\text{b). } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is not one-to-one, because the third column does not have a leading one.

$$T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2) \quad T_A(\mathbf{v}_1 - \mathbf{v}_2) = 0$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 & 2 & 0 \end{array}$$

$$\begin{aligned} x_1 - 2x_3 &= 0 & x_1 &= 2x_3 \\ x_2 + 2x_3 &= 0 & x_2 &= -2x_3 \\ x_3 &= x_3 & x_3 &= x_3 \end{aligned}$$

$$x_3 \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \quad \text{Nul } A \in \text{Span} \left(\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \right)$$

$$\boxed{V_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

a) for T_A to be
 $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

There is a
pivot pos in
every column.
 $\therefore T_A(v) = Av$ is one to one

for $T_A(v)$ to be one to one the $\text{Nul}(A) = \{0\}$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right]$$

$$\xrightarrow{(-3)\downarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 4 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array} \right]$$

$$\xrightarrow{(-2)\downarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array} \right]$$

$$\xrightarrow{(-4)\downarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{array} \right]$$

$$\xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{Nul}(A) = \{0\}$$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right]$$

$$\xrightarrow{0\downarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Every column does not have
pivot pos. $T_A(v) = Av$ is not
one to one.

$$\text{If } v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 2 & 4 & 16 \\ 3 & 4 & 2 & 17 \end{array} \right] = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\therefore v_2 = \begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{-3}$

$$\begin{array}{r} -3 -3 0 \\ 3 4 4 \\ \hline 0 1 4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{array}{r} 0 -1 -2 \\ 0 1 4 \\ \hline 0 0 2 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{array}{r} 1 1 0 \\ 0 -1 -2 \\ \hline 1 0 -2 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

A has a pivot pos. in every column
 $\therefore T_A(v)$ is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A does not have a pivot pos. in every column.
 $\therefore T_A(v)$ is not one-to-one

$\text{Null}(A) = T_A(v) = 0$

$$Av = 0 \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

free var.

$x_1 = 2x_3$

$x_2 = -2x_3$

$x_3 = x_3$

$x_3 = 1$

$$\begin{aligned} x_1 &= 2 \\ x_2 &= -2 \\ x_3 &= 1 \end{aligned} \rightarrow v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$x_3 = 2$

$$\begin{aligned} x_1 &= 4 \\ x_2 &= -4 \\ x_3 &= 2 \end{aligned} \rightarrow v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right) \xrightarrow{\cdot \frac{1}{2}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 6 & 0 \end{array} \right) \xrightarrow{R_3 - 4R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \text{a) not one to one}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right) \xrightarrow{\cdot \frac{1}{2}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right) \xrightarrow{R_3 - 4 \cdot R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & -6 & 0 \end{array} \right) \xrightarrow{R_3 + R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 \end{array} \right) \xrightarrow{-\frac{1}{4}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \quad \text{b) one to one} \quad \checkmark$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{PS } \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 2 & 4 \end{array} \right) \xrightarrow{(-1)} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 0 & 0 \end{array} \right) \xrightarrow{1/3} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & 0 \end{array} \right) \xrightarrow{(-1)} \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & 0 \end{array} \right) \xrightarrow{(2,-1)} \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 1 & 0 & 0 \end{array} \right) \xrightarrow{1/4} \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \Rightarrow \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

Pivot position in
every column $\therefore A$ is one to
one.

$$\text{PS } \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \xrightarrow{(a-3)} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{1/2} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{(1,2)} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

Not one to one, no pivot position in every column.

$$T_A(v_1) = T_A(v_2)$$

$$T_A(v_1 - v_2) = 0$$

$$v_1 - v_2 \in \text{Null}(A)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

free

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + 2x_3 &= 0 \\ y_3 &= x_3 \end{aligned}$$

$$\begin{aligned} x_3 &= 1 \\ x_2 &= -2x_1 \\ x_1 &= 1 \end{aligned}$$

$$x_1 = -x_2$$

$$x_1 = 2$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$x_3 = 2$$

$$x_2 = -2x_1$$

$$x_1 = -(2)$$

$$x_2 = -4$$

$$x_1 = -2$$

$$\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\textcircled{a)} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

\downarrow row reduction

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot in every column
so one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\textcircled{b)} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

\downarrow row reduction

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

not one-to-one
since pivot not in
every column

$$\text{Nul}(A) = \text{span} \left[\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right]$$

$$T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$$

$$\mathbf{v}_1 = \mathbf{b} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\begin{array}{l} R_3 - 3R_1 \\ R_2 - 2R_1 \end{array}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right] \xrightarrow{R_3 - 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Pivot position in every col.,

so $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ is one-to-one.

$$\text{b)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \leftarrow \text{Not one-to-one}$$

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} C_1 \\ C_2 \\ C_3 \end{array} \right] \Rightarrow \left[\begin{array}{c} C_1 + C_2 \\ C_2 + 2C_3 \\ 0 \end{array} \right]$$

$$\begin{aligned} \text{Let } C_1 &= 1 \\ C_2 &= 2 \\ C_3 &= 3 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$$

$$\boxed{v_1 = \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 9 \\ 24 \\ 0 \end{bmatrix}}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a. } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Since after row reduction, matrix A has a pivot position in every row, it is one-to-one therefore T_A is one-to-one

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since A does not have a pivot position in every row, T_A is not one-to-one

$$A\mathbf{v}_1 = A\mathbf{v}_2$$

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 + 4x_3 \\ 3x_1 + 4x_2 + 2x_3 \end{bmatrix}$$

$$\text{If } x_1 = x_2 = 1 \text{ then } \mathbf{v} = \begin{bmatrix} 2 \\ 2+4x_3 \\ 3+4x_2+2x_3 \end{bmatrix}$$

$$\text{Setting } x_3 = 0 \quad \text{Setting } x_3 = 1$$

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot -3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\cdot 2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot 2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot -2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

A is
one-to-one

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{\downarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{\downarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{\downarrow} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

Also
one-to-one
pivot position
in every column

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{\cdot \frac{1}{3}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{-3}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{1/2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{1/2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1}$$

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \therefore A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \text{ is one-to-one}$$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{-3}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{1/2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{5}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \text{ is not one-to-one as it does not have pivot position in every row.}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array} \right) \xrightarrow{-3} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & -4 \end{array} \right) \xrightarrow{\frac{1}{2}} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{array} \right) \xrightarrow{\frac{1}{2}} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right)$$

$$\text{b)} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \xrightarrow{-3} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & -2 \end{array} \right) \xrightarrow{\frac{1}{2}} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

T_A is one to one b/c there
is a pivot position in every column of
 A

$$\text{b)} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \xrightarrow{-3} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{\frac{1}{2}} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \text{free} & & \end{array} \quad \begin{array}{c} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}$$

Not One to one

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ 11 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ 15 \end{pmatrix}$$

$$\begin{array}{l} x_1 = 0 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{array} \quad x_3 \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

a) Aug. mat. of A : RREF

$$\begin{array}{l} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array} \right] \\ \xrightarrow{R_2 \rightarrow R_2/2} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{array} \right] \\ \xrightarrow{R_3 \rightarrow R_3 - 3R_1} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{array} \right] \\ \xrightarrow{R_1 \rightarrow -R_1 + R_1} \left[\begin{array}{ccc} 0 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{array} \right] \\ \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc} 0 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array} \right] \\ \xrightarrow{R_3 \rightarrow R_3/2} \left[\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right] \\ \xrightarrow{R_1 \rightarrow R_1 + 2R_3} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right] \\ \xrightarrow{R_2 \rightarrow R_2 - 2R_3} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array}$$

Every column is a pivot column.

$\therefore [T_A(v) \text{ is one-to-one}]$

b)

$$\begin{array}{l} \text{Aug. mat: } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \\ \text{RREF} \end{array}$$

$$\begin{array}{l} \xrightarrow{R_2 \rightarrow R_2/2} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{array} \right] \\ \xrightarrow{R_3 \rightarrow R_3 - 3R_1} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \right] \end{array}$$

$$\begin{array}{l} \xrightarrow{\cancel{R_1 + R_2 + R_3}} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right] \\ \xrightarrow{R_3 \rightarrow R_3/2} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right] \end{array}$$

$$\xrightarrow{R_1 \rightarrow R_1 - R_2} \left[\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} \xrightarrow{R_1 \rightarrow R_1 + 2R_3} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \\ \xrightarrow{R_2 \rightarrow R_2 - 2R_3} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array} \rightarrow \text{every column is a pivot column.}$$

$\therefore [T_A(v) \text{ is one-to-one}]$

Ans:



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \text{ Null}(A) = \{0\}$$

- every column has pivot position

Thus, It's one-to-one

$$\text{b)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

the third column has no pivot position

Thus, It's not one-to-one

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T_A(\mathbf{v}_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T_A(\mathbf{v}_2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left| \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array} \right| \xrightarrow{\substack{R_1 \times 3 + R_3 \\ R_2 \times 2}} \left| \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right|$$

$$R_1 \times 3 + R_3 \quad R_2 \times 2$$

$$\left| \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right| \xrightarrow{\substack{R_1 \times 3 + R_3 \\ R_2 \times 2}} \left| \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \right| \xrightarrow{\substack{R_2 \times 2 \\ R_3 \times 2}} \left| \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right|$$

$$R_1 \times 3 + R_3 \quad R_2$$

$$\left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right| \rightarrow \text{b)} T_A(v) = Av$$

It's not one to one
because it doesn't have
a pivot in every column
and it has infinite solutions
for its Null space

$\text{a)} A$ is one to one because
it has a pivot column in every
column and the Null of A
is equal to zero

$$\left| \begin{array}{ccc} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right| \xrightarrow{\substack{R_1 \times 3 + R_3 \\ R_2 \times 2}} \left| \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right| \xrightarrow{\substack{R_1 \times 2 \\ R_2 \times 2}} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right|$$

$$\left| \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right| = \left| \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right|$$

(*) $\text{Null}(A) = 0$ proof



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 2 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 2 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\text{b)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 3 & 0 & 0 \end{array} \right] \xrightarrow{\quad}$$

one-to-one

Not one to one.

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

110
624
0-14

116
624
0-12

110
6-3 0
0-14

110
016
6-12

110
616
604

110
016
008

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

110
016
00-2

a-i

One to one

210

610

pg-2

16

610

卷之三

66-2

160
010
001



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{Q)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\cdot -3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{S_{\cdot 2}}$$

$$\downarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{S_{\cdot \cdot 1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{pivot in all columns} \rightarrow \text{one-to-one}$$

$$\text{b)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\cdot -3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot -1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{S_{\cdot \cdot 1}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Some columns missing pivots → not one-to-one

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases} \quad N_{\text{rl}}(A) = \text{span}\left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right)$$

Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T_A(v_1) = T_A(v_2)$ if and only if $v_1 = v_2 + n$
for $n \in N_{\text{rl}}(A)$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \quad \begin{array}{l} v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ v_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \end{array}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] R_3 \rightarrow -3R_1 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] R_2 \rightarrow \frac{R_2}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] R_1 \rightarrow -R_2 + R_1 \\ R_3 \rightarrow -R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] R_3 \rightarrow \frac{R_3}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] R_2 \rightarrow -2R_3 + R_2 \\ R_1 \rightarrow 2R_3 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \leftarrow \begin{array}{l} \text{A IS} \\ \text{one to} \\ \text{one} \\ \text{w/ a} \\ \text{pivot pos.} \\ \text{in every} \\ \text{column} \end{array}$$

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] R_3 \rightarrow -\frac{1}{2}R_1 + R_3$$

$$\downarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] R_1 = -\frac{1}{2}R_2 + R_1 \\ R_3 = -\frac{1}{2}R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 = 2x_3$$

$$x_2 = -4x_3$$

$$x_3 = x_3$$

$$\text{Null}(A) = \text{Span} \left\{ \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} \right\}$$

$T(A)$ is not one to one
because there is not
a pivot position in

every column

$$V_1 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} V_2 = \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} T(\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}) = 0$$

$$T(A(V_1)) = T(A(V_2))$$

$$T(A(\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix})) = T(A(\begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}))$$

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \text{ is in Null}(A)$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

one to one

$$\text{so } \text{Nul}(A) = \{0\}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Nul}(A) = \left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Not one to one

$$x_1 = 2x_3$$

$$x_2 = -2x_3$$

$$x_3 = x_3$$

$$v_1 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

1-1:
Pivot Pos. in every
column

(a)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\substack{R2 \leftrightarrow R3 \\ R1 - 3R2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R1 - 3R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\substack{R2 - R3 \\ R3 - R2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

T_A is
one-to-one because
 A has a pivot position
in every column.

$$\text{Let } \vec{w} = \vec{v}_1 - \vec{v}_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{\substack{R2 \leftrightarrow R3 \\ R1 - 3R2}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right]$$

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\text{Null}(A) = \text{Span} \left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right)$$

$$\begin{cases} \vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{cases}$$

$$\text{Let } \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(b) A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\substack{R2 \leftrightarrow R3 \\ R1 - 3R2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\substack{R1 - 3R2 \\ R3 - R2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

T_A is not one-to-one
because A does not have
a pivot position in
every column.

$$T_A(\vec{v}_1) = T_A(\vec{v}_2)$$

$$T_A(\vec{v}_1) - T_A(\vec{v}_2) = 0$$

$$T_A(\vec{v}_1 - \vec{v}_2) = 0$$

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

$$\text{Let } \vec{w} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \vec{v}_1 - \vec{v}_2$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \vec{v}_2 = \vec{v}_1$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \vec{v}_1$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\xrightarrow{\text{row } 1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{row } 1 - \text{row } 2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{row } 2 - \text{row } 3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{row } 1 - \text{row } 2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{row } 1 / 4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A$ has a pivot position in every column

so it is one-to-one

$$\xrightarrow{\text{row } 1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{row } 1 - \text{row } 2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{row } 2 - \text{row } 3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{row } 1 - \text{row } 2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 = 2x_3 \quad \text{vector form}$$

$$x_2 = -2x_3$$

$$x_3 = x_3$$

$$\left[\begin{array}{c} 2x_3 \\ -2x_3 \\ x_3 \end{array} \right] \rightarrow x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore A$ is not one-to-one since there is not a pivot position in every column, so

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \& \quad v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

$$\text{s.t. } T_A(v_1) = T_A(v_2)$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

* One-to-one

Pivot pos in every col

$$a) A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\sim}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{\quad 2 \rightarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

a) is one-to-one

$$b) A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\sim}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{\quad 2 \rightarrow -2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{1 \rightarrow -1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_3 = 0$$

$$x_2 + 2x_3 = 0$$

$$x_3 = x_3$$

$$\text{Nu}(A)$$

b) not one-to-one

$$T\left(\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right) = T\left(\begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}\right)$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{array}{l} (-1) \circledast \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \\ \downarrow \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 2 & 0 \end{bmatrix} \end{array}$$

~~Handwritten notes:~~

- Row 3 - Row 1

$$\begin{array}{l} (-3) \circledast \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix} \\ \downarrow \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{l} (2) \circledast \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \\ \downarrow \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{l} (0) \circledast \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ (\frac{1}{4}) \circledast \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\ \downarrow \end{array}$$

$$\begin{array}{l} (-1) \circledast \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \downarrow \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is one-to-one}$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{(-3)}$$

$$\begin{array}{l} \downarrow \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

$$\begin{array}{l} \cancel{\text{Row 3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{(-1)} \\ \downarrow \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ - not one-to-one}$$

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad v_2 = \begin{bmatrix} -3 \\ -16 \\ -17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2+0 \\ 0+4+12 \\ 3+8+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$$

$$T_A(v_1) - T_A(v_2) = 0$$

$$\begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix} \begin{array}{|c|c|c|} \hline \cancel{1} & \cancel{2} & \cancel{3} \\ \hline \cancel{4} & \cancel{5} & \cancel{6} \\ \hline \cancel{7} & \cancel{8} & \cancel{9} \\ \hline \end{array}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{2 \cdot -1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

a) One to one

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{2 \cdot -1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Not one to one

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{-1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{aligned} x_1 &= 2x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x_3$$

b) Not one to one

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right)$$

Pivot position
every column

T_A is one-to-one

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

Not every column has a pivot position

T_A is not one-to-one

$$\begin{aligned} v_1 - 2v_5 &= v_2 - 2v_6 \\ v_3 + 2v_5 &= v_4 + 2v_6 \end{aligned}$$

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & v_1 \\ 0 & 2 & 4 & v_2 \\ 3 & 4 & 2 & v_3 \\ 0 & 0 & 0 & v_4 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & v_1 \\ 0 & 2 & 4 & v_2 \\ 0 & 1 & 2 & v_3 \\ 0 & 0 & 0 & v_4 \end{array} \right]$$

$$v_1 + v_3 = v_2 + v_4$$

$$2v_5 + 4v_6 = 0$$

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$A \cdot v = b$$

$$T_A(v_1) = T_A(v_2)$$

$$a) A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$b) A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

one-to-one pivot position in every column

$$T_A(v_1) - T_A(v_2) = 0$$

$$T_A(v_1 - v_2) = 0$$

$$A) A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} R_3 \rightarrow R_3 - 3R_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} R_1 \rightarrow R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} R_3 \rightarrow R_2 - R_3 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} R_3 \rightarrow -\frac{1}{2}R_3 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} R_2 \rightarrow 2R_3 + R_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 \rightarrow 2R_3 + R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is one-to-one

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} x_1=0 \\ x_2=0 \\ x_3=0 \end{matrix} \text{ Null } A: \text{Span}(0)$$

$$\text{so } T_A(v_1) - T_A(v_2) = 0$$

$$v_1 \text{ can be } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_2 \text{ can be } \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$B) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} R_3 \rightarrow R_3 - 3R_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} R_1 \rightarrow R_1 - R_3 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} R_2 \rightarrow \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} R_3 \rightarrow R_2 - R_3 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see not every column will be a pivot so $T_A(v) = Av$ is not one-to-one.



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \quad (\text{row } 3 - 3\text{ times row } 2)$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \quad (\text{row } 3 - \text{row } 2)$$

$$\begin{array}{ccc}
 \text{a)} &
 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right) \xrightarrow{\text{(r3)} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)} &
 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right) \xrightarrow{\text{(r3)} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right)} \\
 &
 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right) \xrightarrow{\text{(r1)} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)} &
 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \xrightarrow{\text{(r1)} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)} \\
 &
 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right) \xrightarrow{\text{(r2)} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)} &
 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{\text{not one-to-one}}
 \end{array}$$

One-to-one

$$\text{b)} T_A(v_1) = T_A(v_2)$$

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

$$\text{Try} \\ v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$Av_1 = Av_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 2 + 0 \\ 0 - 4 + 4 \\ 6 - 8 + 2 \end{bmatrix} = \begin{bmatrix} 4 - 4 + 0 \\ 0 - 8 + 8 \\ 12 - 16 + 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot position in
every column, so
it is one-to-one

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

every column doesn't
have pivot position
so not one-to-one

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_3 &= x_3 \end{aligned}$$

if $x_3 \neq 1$

$$\begin{aligned} x_1 - 2 &= 0 \\ x_1 &= 2 \\ x_2 + 2 &= 0 \\ x_2 &= -2 \end{aligned}$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right) = T_A\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\cdot -3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 4 & 4 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \xrightarrow{\cdot -1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \xrightarrow{2 \cdot -1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right] \xrightarrow{5 \cdot -1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right] \xrightarrow{\cdot \frac{1}{4}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\text{Nul}(A) = \{0\}$$

A has a pivot
in every col

So T_A is one-to-one

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{\cdot -3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 4 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{\cdot \frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{2 \cdot -1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{5 \cdot -1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

A does not have
a pivot in every
col, so T_A is not
one-to-one

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

$$Av_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} \xrightarrow{\cdot -2} \begin{bmatrix} 2 \\ 0 \\ -8 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix} + \begin{bmatrix} -4 \\ -8 \\ -16 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array} \right) \xrightarrow{\text{R}_3 - 3\text{R}_1} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \xrightarrow{\text{R}_3 - 3\text{R}_1} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right) \xrightarrow{\text{R}_2 - 2\text{R}_1} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{\text{R}_3 - \text{R}_2} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array} \right) \xrightarrow{\text{R}_3 - 2\text{R}_2} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

No p.v.p.s

Not one-to-one

One-to-one

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \mathbf{v}_1 = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \mathbf{v}_2$$

$$\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \xrightarrow{\text{R}_3 - 3\text{R}_1} \begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \xrightarrow{\text{R}_2 - 2\text{R}_1} \begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \xrightarrow{\text{R}_2 - \frac{1}{4}\text{R}_3} \begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{c} \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \end{array} \right]$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

T_A is not one-to-one.

$$T_A(v_1) = T_A(v_2)$$

$$T_A(v_1 - v_2) = 0$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

$$x_1 = 2x_3$$

$$x_2 = -2x_3$$

$$x_2 = -4x_3$$

there is no solution.

→ there is no two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

T_A is not one-to-one

$$T_A(v_1) = T_A(v_2)$$

$$T_A(v_1 - v_2) = 0$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$x_1 = 2x_3 \quad x = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x_3$$

$$x_2 = -2x_3$$

$$x_3 = x_3$$

$$v_1 - v_2 = x$$

$$\text{assume } x_3 = 1 \quad x = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{assume } v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{then } v_1 = x + v_2$$

$$= \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore v_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a) } A = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array} \right] \xrightarrow{-3} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right] \xrightarrow{-1} \quad \text{b) } A = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right] \xrightarrow{-3} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array} \right] \quad \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array} \right]$$

yes
pivot position
in every col.

$$A = \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{array} \right] \quad \text{yes one to one}$$

pivot position
in every col.



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{array}{l} R1x-3 \\ + R3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 12 \\ 0 & 1 & 4 & 12 \end{array} \right]$$

$$\begin{array}{l} R2x-1 \\ + R3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 12 \\ 0 & 1 & 4 & 12 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 2 & 12 \end{array} \right]$$

$$\begin{array}{l} R3x-2 \\ + R2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 1 & 12 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 12 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 12 \end{array} \right]$$

↓
one-to-one
and onto b/c
pivot pos. in every
column and row,
respectively.

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} (R1x-3) + R3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 12 \\ 0 & 1 & 2 & 12 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 12 \\ 0 & 1 & 2 & 12 \end{array} \right] (R3x-1) + R2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 0 & 0 \end{array} \right]$$



not one-to-one
b/c can only have
a pivot position in
the first two
columns, here
we have only
two.



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$n = 3 = m = 3$$

$T_A(v) = Av$ is not one-to-one.



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 4 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] - \text{pivot in every column}$$

$T_A(v) = Av$ is one-to-one

T_A is not one-to-one

$$\begin{aligned} x_3 &= \text{free} & v_1 &= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} & v_2 &= \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \\ x_1 &= 2x_3 & & & & \\ x_2 &= -2x_3 & x_3 &= 1 & & x_3 &= -1 \end{aligned}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 3 & 4 & 4 & | & 0 \end{bmatrix}$$

$$R_3 = R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 0 & 1 & 4 & | & 0 \end{bmatrix}$$

$$R_3 = R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 0 & 0 & 4 & | & 0 \end{bmatrix}$$

$$x_3 = 4$$

$$x_2 = 2$$

$$x_1 = -2$$

It is one to one

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 3 & 4 & 2 & | & 0 \end{bmatrix}$$

$$R_3 = R_3 - 3R_1$$

$$= \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2$$

$$x_2 = x_2$$

$$x_3 = -\frac{1}{2}x_2$$

$$x_2 + 2x_3 = 0$$

$$2x_3 = -x_2$$

$$x_3 = -\frac{1}{2}x_2$$

$$\begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} x_2$$

$$V_1 = \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} \quad V_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

It is not one to one



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot columns

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

~~not~~

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

* \rightarrow not one to one

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

* a) one to one

pivot columns

b)

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes, it is one to one.

no. one to one.

$$T_A(\mathbf{v}_1) = A\mathbf{v}_1 \quad T_A(\mathbf{v}_2) = A\mathbf{v}_2$$

$$A\mathbf{v}_1 = A\mathbf{v}_2$$

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

$$A\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$A\mathbf{v}_2 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$\downarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$\downarrow R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix}$$

$\downarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

This is not
one to one
because there
is not a pivot
in every
column

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$\downarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$\downarrow R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$\downarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is not
one to one
because there
is not a pivot
in every column

$$V_1 = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

One to one - pivot position in every column
onto-pivot position in every row

$$\text{a) } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

↓

$$R_3 = R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

$$R_2 = R_2 + R_3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$R_3 \div 4$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b) } \left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 4 & 0 & 2 \\ 3 & 4 & 4 & 3 & 4 \end{array} \right]$$

↓

$$R_3 = R_3 - 3R_1$$

$$\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 4 & 0 & 2 \\ 0 & 1 & 4 & 0 & 1 \end{array} \right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 4 & 0 & 1 \\ 0 & 2 & 4 & 0 & 2 \end{array} \right]$$

$R_3 \rightarrow R_3 - 2R_2$

$$\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & -4 & 0 & 0 \end{array} \right]$$

$R_2 \rightarrow R_2 + R_3$

$$\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -4 & 0 & 0 \end{array} \right]$$

$$R_1 = R_1 - R_2$$

$R_3 \div (-4)$

$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right]$$

not one to one

$$Av_1 = Av_2$$

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array} \right] \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right] = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right] \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right]$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

a) One to one \rightarrow pivot in every column

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A \text{ is one-to-one}$$

$$\text{b)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{not one-to-one}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 + 4x_3 \\ 3x_1 + 4x_2 + 2x_3 \end{bmatrix}$$

$$\leftarrow \qquad \qquad \qquad \rightarrow$$

No such vectors exist, since $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$
are linearly independent

$$x_2 = x_3$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 6 \\ 3 & 6 \end{bmatrix} \quad \begin{array}{l} x_1 = 1 \\ x_2 = 1 \end{array} \quad \begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$$

$$\begin{array}{l} x_1 = 2 \\ x_2 = 1 \end{array} \quad \begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$. $\mathbf{v}_1 \neq \mathbf{v}_2$

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$\cancel{3R_1 + R_3}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{R_3}{4}$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\begin{array}{l} R_3 \\ -2R_3+R_2 \end{array}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\leftarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\downarrow} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Ⓐ is one-to-one

Ⓑ is not one-to-one

$$x_1 = -2$$

$$x_2 = 2$$

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$



$$\text{Nul}(A) = \{0\}$$

pivot position in every column

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{\frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array} \right] \xrightarrow{R_3 - 3R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \xrightarrow{R_3 - R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \xrightarrow{R_2 - R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \therefore \text{one-to-one}$

b) $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - 3R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{(-1)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

not one-to-one

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \mathbf{v}_1 = T_A(\mathbf{v}_1)$$

$$\mathbf{v}_1 + \mathbf{v}_2 = 0$$

$$\mathbf{v}_2 + 2\mathbf{v}_3 = 0$$

\therefore when $\mathbf{v}_2 = 2$

$$\mathbf{v}_1 = -2, \mathbf{v}_3 = -1$$

when $\mathbf{v}_2 = 4$

$$\mathbf{v}_1 = -4, \mathbf{v}_2 = -2$$

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}$$



one to one \rightarrow pivot in every column

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{(-3)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{(1/2)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{(1/2)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\downarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a) Pivot in every column,
therefore it is one to one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\downarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{(1/2)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{free}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

b) Not one to one because
there is a free variable
(in the last column).

$$T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$$

$$A\mathbf{v}_1 = A\mathbf{v}_2$$

$$\text{Nul}(A) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 2 \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \xrightarrow{\text{free}} \left\{ \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \begin{aligned} x_1 &= 2x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$\mathbf{v}_2 = \mathbf{v}_1 + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \rightarrow \boxed{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$-(3R_1) \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \right) \xrightarrow{\text{RREF}} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \right) \xrightarrow{\cdot \frac{1}{2}} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} \right) \xrightarrow{\cdot \left(\begin{array}{c} 1 \\ 2 \\ 1 \end{array}\right)}$$

↓

$$\begin{aligned} & (-1) \cdot \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \right) \leftarrow + \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \right) \leftarrow \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} \right) \xrightarrow{\cdot \left(\begin{array}{c} 1 \\ 2 \\ 1 \end{array}\right)} \\ & \downarrow \quad \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right) \xrightarrow{\text{RREF}} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \quad \boxed{\text{yes } A \text{ is one-to-one}} \end{aligned}$$

$$\text{b)} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \right) \xrightarrow{-2R_2} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 0 & -6 \end{bmatrix} \right) \xrightarrow{\cdot \frac{1}{2}} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 0 & -6 \end{bmatrix} \right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix} \right) \xrightarrow{\text{RREF}}$$

↓

$$\begin{aligned} & \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \xrightarrow{\cdot(-1)} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \right) \xrightarrow{\cdot(-1)} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \right) \xrightarrow{\text{RREF}} \\ & \downarrow \quad \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \quad \boxed{\text{yes, } A \text{ is one to one}} \end{aligned}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right) \xrightarrow{\text{R}_3 - \frac{1}{3}\text{R}_1} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{array} \right)$$

$$\xrightarrow{\text{R}_1 - \text{R}_2} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{array} \right) \xrightarrow{\text{R}_2 - \frac{1}{2}\text{R}_1} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{array} \right) \xrightarrow{\text{R}_3 \cdot \frac{1}{4}} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \xrightarrow{\text{R}_1 \leftrightarrow \text{R}_2} \left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

Solution is
as every row is one to one
a pivot position or column has leading one



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{-3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\text{C1/2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{-1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{\text{C1/2}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\cdot 2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

One-to-one

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{-3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{C1/2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{-1} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \right) \xrightarrow{\text{R2-R3}} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \right) \xrightarrow{\text{R1-R2}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Not one-to-one: pivot position not present in every row.

Find the null space \Rightarrow set of vectors \mathbf{v} such that

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \mid \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{to create}} T_A(\mathbf{v}) = 0$$

$$x_1 - 2x_3 = 0 \quad \text{scalar: } 2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$x_2 + 2x_3 = 0$$

$$\Rightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$x_3 = x_3$$

$$x_1 = 2x_3 \Rightarrow \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \{ v_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \}$$

$$x_2 = -2x_3 \Rightarrow \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} \quad \{ v_2 = \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} \}$$

$$x_3 = x_3 \quad \{ \}$$

$$\text{Null space: } \left\{ x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\} \subset \text{span} \left(\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} \right)$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T_A(v)$ is 1-to-1; pivot columns
in all columns.

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 6 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Not 1-to-1; not all columns are pivot columns

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{Let } v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = v_1 + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{aligned} x_1 &= 2x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$(a) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \times -3$$

↓

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \times \frac{1}{2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \times -1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \times \frac{1}{2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix has pivot column

~~here~~ in every column.

It's one to one.

$$(b) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \times -3$$

↓

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \times \frac{1}{2}$$

$$\begin{array}{c|ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \times -2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

It's not one to one.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a_1 \\ 0 & 1 & 2 & a_2 \\ 0 & 0 & 0 & a_3 \end{array} \right]$$

$$x_1 + x_2 = a_1$$

$$x_2 + 2x_3 = a_2$$

$$a_3 = 0$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{a)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 2 & 4 & \\ 3 & 4 & 4 & \end{array} \right] \xrightarrow{(3)-3(1)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 2 & 4 & \\ 0 & 1 & 0 & \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 2 & 4 & \\ 0 & 1 & 0 & \end{array} \right] \xrightarrow{(2)\div 2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 1 & 0 & \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 1 & 0 & \end{array} \right] \xrightarrow{(3)-(2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 0 & 2 & \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 0 & 2 & \end{array} \right] \xrightarrow{0\div 2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 0 & 1 & \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 0 & 1 & \end{array} \right]$$

Since A has a pivot position in every column,
 $T_A(v) = Av$ is one to one.

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{b)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 2 & 4 & \\ 3 & 4 & 2 & \end{array} \right] \xrightarrow{(3)-3(1)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 2 & 4 & \\ 0 & 1 & 2 & \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 2 & 4 & \\ 0 & 1 & 2 & \end{array} \right] \xrightarrow{(2)\div 2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 1 & 2 & \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 1 & 2 & \end{array} \right] \xrightarrow{(2)-(3)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 0 & 0 & \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & \\ 0 & 0 & 0 & \end{array} \right] \xrightarrow{0\div 0} \left[\begin{array}{ccc|c} 1 & 0 & -2 & \\ 0 & 1 & 2 & \\ 0 & 0 & 0 & \end{array} \right]$$

Since A does not have a pivot column in every column,
 $T_A(v) = Av$ is not one to one

From the matrix, we can get $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \in \text{Nul}(A)$

Let v_1 be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$v_2 = v_1 + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ ✓

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

one to one if all $\text{Col}(A)$ are lin. dep.

① $x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ NO
 $x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ NO
 $x_3 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ NO
 So lin. dep. ✓
 So One to ONE

② $x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ NO
 $x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ NO
 $x_3 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ NO
 - All lin. dep
 So One to ONE



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow[-3R1]{R3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow[R2]{R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow[4]{R3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow[m]{R1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) is one-to-one

$$\text{b)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow[-3R1]{R3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[2R3]{R3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

b) is not one-to-one



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

One-to-one pivot position
in every column

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Not one-to-one

$$T_A(v_1) = T_A(v_2)$$

$$T_A(v_1) - T_A(v_2) = 0$$

$$Av_1 - Av_2 = 0$$

$$A(v_1 - v_2) = 0$$

$$\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to-one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\substack{R3 - 3R1 \\ R2 \rightarrow \frac{1}{2}R2 \\ R3 \rightarrow R3 - R2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R3 \rightarrow R3 - 3R1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

T_A not one-to-one



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a)

$$\text{a)} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$