MTH-309T-F19-EX1-078-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	3	1	1	6	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	TOTAL	GRADE
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1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & 0 & b+4 \end{array}\right] +$

$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -2 \\ 0 & 0 & 2 & b+6 \\ 0 & -1 & 0 & b+4 \end{array}\right] + \xrightarrow{R_1 = R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -b-6 \\ 0 & 0 & 2 & b+6 \\ 0 & -1 & 0 & b+4 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -b-6 \\ 0 & -1 & 0 & b+4 \\ 0 & 0 & 2 & b+6 \end{array}\right] \xrightarrow{R_1 = R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & b+4 \\ 0 & 0 & 2 & b+6 \end{array}\right] \begin{array}{l} \\ \cdot(-1) \\ \cdot(\frac{1}{2}) \end{array}$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -b-4 \\ 0 & 0 & 1 & \frac{b+6}{2} \end{array}\right]$

a) All values of b are $c_1(2) + c_2(-3) + c_3(0)$ that $\vec{w} \in \mathrm{Span}(v_1, v_2, v_3)$

b) Yes, the set $\{v_1, v_2, v_3\}$ is linearly independent because after row reduction every column is a pivot column.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$ ← this is the answer

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 + 2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right]$$

$\downarrow R_1 = R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right]$$

$R_3 = R_3 - 3R_2$ ←

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 2 & -3 & 1 \end{array}\right]$$

$\downarrow R_2 = R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ -1 & 0 & 0 & 2 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ -1 & 0 & 0 & 2 & -3 & 1 \end{array}\right] \begin{array}{l} \cdot(-1) \\ \\ \cdot(-1) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ 1 & 0 & 0 & -2 & 3 & -1 \end{array}\right]$$

1 $R_3 \leftrightarrow R_1$
2 $R_2 \leftrightarrow R_3$

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T C = B \qquad A^T \qquad A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\cancel{(A^T)^{-1} A^T} C = (A^T)^{-1} B$$

$$C = (A^{-1})^T B \qquad (A^{-1})^T = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$3 \times 3 \qquad 3 \times 3 \quad = 3 \times 3$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = 1\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+8+6=12 \\ 3-4-6=-7 \\ -1+4+3=6 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+10+4=10 \\ 6-5-4=-3 \\ -2+5+2=5 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = 3\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} +2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+8+2=4 \\ 9-4-2=3 \\ -3+4+1=2 \end{bmatrix}$$

$$C = \begin{bmatrix} 12 & 10 & 4 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(u) = \begin{bmatrix} a_1 - 2a_2 \\ a_1 + a_2 \\ a_1 - 3a_2 \end{bmatrix}$ $T(v) = \begin{bmatrix} b_1 - 2b_2 \\ b_1 + b_2 \\ b_1 - 3b_2 \end{bmatrix}$

$T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

standard matrix of $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $R_3 = R_3 - R_1$ $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$

$+2R_2$ $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & -3 \end{array}\right]$

$\downarrow$

$\begin{bmatrix} 3 & 0 & 21 \\ 1 & 1 & 10 \\ 0 & 1 & -3 \end{bmatrix} * \left(\frac{1}{3}\right) \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 1 & 1 & 10 \\ 0 & 1 & -3 \end{array}\right]$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 1 & -3 \end{bmatrix}$

$x_1 = 7$
$x_2 = x_2$

all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ are: $\begin{bmatrix} 7 \\ x_2 \end{bmatrix}$ where x_2 is free

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a)

$$-(3R_1)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} - \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & -1 & 0 \end{bmatrix} \cdot\left(\tfrac{1}{2}\right)$$

$$\downarrow$$

$$\cdot\left(\tfrac{1}{2}\right)\cdot,\ (-1)\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \leftarrow \quad + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} +$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yes A is one-to-one

b)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} -2R_2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 0 & -6 \end{bmatrix} \begin{matrix} \cdot\frac{1}{2} \\ \cdot -\frac{1}{3} \end{matrix} \to + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} -(\cdot 2) \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} -$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yes, A is one to one.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True, because since $u+w$ is in the span (u,v) then it must be true that $w \in \text{Span}(u,v)$ because the sum of the two vectors $(u+w)$ are in the span (u,v).

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True

~~False~~, for example

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then the set $\{u,v\}$ $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ is also linearly independent because it has only one solution and pivot col in every column. If u,v,w are independent then it has pivot col. in every column that means with two vectors it still holds true.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

u and v could still be independent and be multiplied with A to make it linearly dependant.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

Yes, true because if u is in the span of (v, w) then the transformation could just be the scalar of u vector and it would still let T(u) be in the span of T(v), T(w)

0 1 2 3 4 5 6 7 8 9 10