MTH-309T-F19-EX1-069-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	1	7	1	3	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
19	9	7	19	20	3	2	--	10	87	A-

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a. $b_1V_1 + b_2V_2 + b_3V_3 = W$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$$

$3 - 2 \times 1$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & \cancel{b-4} \end{array}\right] \quad b+4$$

$3 + 2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & \cancel{b-2} \end{array}\right] \quad b+6$$

$b+6 \quad \cancel{b-2} = 0$

$\boxed{\cancel{b=2}} \quad b = -6$

b.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$1 \times 2 - 2$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2x_1 - 3x_2 = 0$ $\quad 2x_1 = 3x_2$

$x_2 - 2x_3 = 0$ $\quad x_1 = \frac{3}{2}x_2$

$x_1 = \frac{3}{2}x_2$ $\quad x_2 + 2x_3 = 0$

$x_2 = x_2$ $\quad 2x_3 = -x_2$

$x_3 = -\frac{1}{2}x_2$ $\quad x_3 = -\frac{1}{2}x_2$

$$= \begin{bmatrix} \frac{3}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} x_2 \rightarrow \text{Infinity solutions}$$

Set $\{V_1, V_2, V_3\}$ not linear independent

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = ② - ① =$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = ③ - ② \times 2$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$0 - -1 \times 2$
$= 0 + 2$

$R_2 = ② + ③$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$R_1 = ① - ③ \times 2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$R_1 = ① + ②$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & \cancel{3}\ -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & \cancel{3}\ -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 10 \\ 4 & 5 & 5 & 5 & -9 & 9 \\ 3 & 3 & 2 & 7 & 7 & 1 \end{matrix}$ $5-14=-9$

MTH-309T-F19-EX1-069-P03

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \checkmark \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \checkmark$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right]$$

This would work if done correctly.

$R_2 = ② + ①$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right]$$

$1-6=5$

$R_3 = ③ - ① \times 2$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & -1 & -1 & 2 & -2 & -5 \end{array}\right]$$

$R_3 = ③ + ②$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 7 & 5 & 2 \end{array}\right]$$

$R_2 = R_2 - 2R_3$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & -9 & -3 & 3 \\ 0 & 0 & 1 & 7 & 5 & 2 \end{bmatrix}$$

$R_1 = R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & 5 & 6 \\ 0 & 1 & 0 & 9 & -3 & 3 \\ 0 & 0 & 1 & 7 & 5 & 2 \end{array}\right]$$

$$C = \begin{bmatrix} 10 & 5 & 0 \\ 9 & -3 & 3 \\ 7 & 5 & 2 \end{bmatrix}$$

Simpler: $C = (A^T)^{-1} \cdot B$

$= (A^{-1})^T \cdot B$

Then use A^{-1} from problem 2.

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0	1	2	3	4	5	6	7	8	9	10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a. $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ✓

b. $\begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix}$ ✓

② - ①

$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{bmatrix}$

③ - ①

$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{bmatrix}$

③ x 2 - ①

$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{bmatrix}$

$3x_2 = 9$
$x_2 = 3$

$x_1 - 2x_2 = 1$
$x_1 - 6 = 1$
$x_1 = 7$

$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

answer ?

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right]$

$R_3 = R_3 - 3R_1$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$

$R_3 = 2R_3 - R_2$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$

$x_3 = 4$

$x_2 = 2$

$x_1 = -2$

It is one to one ✓

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$

$R_3 = R_3 - 3R_1$

$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -x_2$

$x_2 = x_2$

$x_3 = -\frac{1}{2}x_2$

$x_2 + 2x_3 = 0$

$2x_3 = -x_2$

$x_3 = \frac{1}{2}x$

$\begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} x_2$

$V_1 = \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix}$ $V_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ ✓

It is not one to one ✓

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \operatorname{Span}(\mathbf{u}, \mathbf{v})$.

Ture ✓

u. v are in span

w+u is span

w in ∈ span

← why ?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~False~~

$$u \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} v \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} w \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

u v w are linearly independent

But u v are linear dependent.

✓

$x_1 u + x_2 v_2 + x_3 w = 0$ has solution of 0

$x_1 u + x_2 v_2$ will have solution.

0	1	2	3	4	5	6	7	8	9	10
☐	☐	☐	☒	☐	☐	☐	☐	☐	☐	☐

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~Ture~~

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot u + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot v$

$X_1 U_1 + X_2 V_2$ are linearly dependent

$U_1 V_2$ are also linear dependent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

Ture ← why ?

$T(v) = T(w) = T(u)$

$AV = AW = AU$

?

0 1 2 3 4 5 6 7 8 9 10