MTH-309T-F19-EX1-025-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Jared Scott

UB Person Number:

5	0	2	2	3	9	7	1
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

3

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. $A\vec{x} = b$

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $[v_1 \; v_2 \; v_3 \mid w]$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -3 & 0 & b \\ 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \end{array}\right] \cdot (1)$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & b \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right] \cdot (-2) \to \left[\begin{array}{ccc|c} 0 & -3 & -6 & b \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right] \cdot (3)$$

$\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ $\boxed{b = -6}$

b) $$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \cdot (-2) \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \cdot (1) \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot (1)$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{\text{Not linearly independent}}$

x_3 is a free variable so there are infinite solutions, this means that they are linearly dependant.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a-d+2g & b-e+2h & c-f+2i \\ a+g & b+h & c+i \\ 2d-g & 2e-h & 2f-i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$a-d+2g=1$ $\quad$ $b-e+2h=0$ $\quad$ $c-f+2i=0$

$a+g=0$ $\quad$ $b+h=1$ $\quad$ $c+i=0$

$2d-g=0$ $\quad$ $2e-h=0$ $\quad$ $2f-i=1$

$a=-g$ $\quad$ $b=1-h$ $\quad$ $c=-i$

$d=\frac{1}{2}g$ $\quad$ $e=\frac{1}{2}h$ $\quad$ $f=\frac{1}{2}+\frac{i}{2}$

$-g+\frac{1}{2}g+2g=1$ $\quad$ $1-h-\frac{1}{2}h+2h=0$ $\quad$ $-i-\frac{1}{2}-\frac{i}{2}+2i=0$

$\frac{3}{2}g=1$ $\quad$ $\frac{1}{2}h=-1$ $\quad$ $\frac{1}{2}i=\frac{1}{2}$

$g=\frac{2}{3}$ $\quad$ $h=-2$ $\quad$ $i=1$

$a=-\frac{2}{3}$ $\quad$ $b=3$ $\quad$ $c=-1$

$d=\frac{4}{3}$ $\quad$ $e=-1$ $\quad$ $f=1$

$$\boxed{A^{-1} = \begin{bmatrix} -2/3 & 3 & -1 \\ 4/3 & -1 & 1 \\ -2/3 & -2 & 1 \end{bmatrix}}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$C_{11} + C_{21} = 1$

$-C_{11} + 2C_{13} = 4$

$2C_{11} + C_{12} - C_{13} = 3$

$C_{12} = 1 - C_{11}$

$C_{13} = 2 + \frac{1}{2} C_{11}$

$2C_{11} + 1 - C_{11} - 2 - \frac{1}{2} C_{11} = 3$

$-1 + \frac{1}{2} C_{11} = 3$

$\frac{1}{2} C_{11} = 4$

$C_{11} = 8$

$C_{21} = -7$

$C_{31} = 6$

$C_{21} + C_{22} = 2$

$-C_{21} + 2C_{23} = 5$

$2C_{21} + C_{22} - C_{23} = 2$

$C_{22} = 2 - C_{21}$

$C_{32} = \frac{5}{2} + \frac{1}{2} C_{31}$

$\frac{4}{2}$

$2C_{21} + 2 - C_{21} - \frac{5}{2} - \frac{1}{2} C_{21} = 2$

$\frac{1}{2} C_{21} - \frac{1}{2} = 2$

$\frac{1}{2} C_{21} = \frac{5}{2}$

$C_{12} = 5$

$C_{22} = -3$

$C_{32} = 5$

$C_{13} + C_{23} = 3$

$-C_{13} + 2C_{33} = 4$

$2C_{13} + C_{23} - C_{33} = 1$

$C_{23} = 3 - C_{13}$

$C_{33} = 2 + \frac{1}{2} C_{13}$

$2C_{13} + 3 - C_{13} - 2 - \frac{1}{2} C_{13} = 1$

$\frac{1}{2} C_{13} = 0$

$C_{13} = 0$

$C_{23} = 3$

$C_{33} = 2$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$

$a x_1 + c \cancel{x} x_2 = x_1 - 2x_2$

$c x_1 + d x_2 = x_1 + x_2$

$e x_1 + f x_2 = x_1 - 3x_2$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

b) $Au = b$

$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot u = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ $\quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot (-1) \to$

$\to \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot (-1) \to \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{array}\right] \cdot \frac{1}{4} = \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x_1 = 3x_2 - 2$
$\quad x_2 = x_2$

$\boxed{\begin{bmatrix} -2 \\ 0 \end{bmatrix} + \text{span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}}$ $\quad \begin{bmatrix} -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$T_A(v) = T_A(\omega)$

if $\omega = v + n$
$n \in Nul(A)$

$Nul(A): \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \cdot(-3)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \cdot(-1)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \cdot(-1)$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$ $\boxed{\text{One}}$

$\boxed{\text{One to - One}}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \cdot(-3)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 2x_3$

$x_2 = -2x_3$ infinite

$x_3 = x_3$

$x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$Nul(A) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\boxed{v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}$

$V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$$\cancel{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)}$$

$\boxed{\text{True}}$ because if w+u is in the span(u,v), then w must be some linear combination of u or v.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True, if the whole set is linearly independent, then its parts must hold the relation.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\boxed{\text{False}}$ ran out of time

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True, transformations are a linear operator so their original relations hold true to the outcomes of a transformation

MTH-309T-F19-EX1-019-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Emily Blanchard

UB Person Number:

5	0	2	2	8	8	3	7
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{\cdot(-2)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right] \Longrightarrow \quad b = 6$$

b) The set $\{v_1, v_2, v_3\}$ is linearly dependent because x_3 is a free variable meaning there are infinitely many solutions.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\cdot(-1) \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\cdot 1 \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\cdot(-2) \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\cdot 1 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\cdot(-1) \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \Longrightarrow A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

check: $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$Ax = b$

$x = A^{-1} \cdot b$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A^T C = B \Rightarrow C = (A^T)^{-1} \cdot B \qquad (A^T)^{-1} = (A^{-1})^T$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad (A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$(A^{-1})^T \cdot B = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2+4+6 & -4+5+4 & -6+4+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ -1+4+3 & -2+5+2 & -3+4+1 \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-2(0) \\ 1+0 \\ 1-3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-2(1) \\ 0+1 \\ 0-3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

Standard matrix of $T = A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $A \cdot u = T(u)$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{\cdot(-1)} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{\cdot(-1)} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 9 \end{array}\right] \xrightarrow{\cdot(-1)} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{array}\right] \cdot 2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{array}\right] \xrightarrow{\cdot(-3)} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 7$
$x_2 = 3$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

check: $T\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7-2(3) \\ 7+3 \\ 7-3(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ ✓

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \qquad \text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

a)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\cdot(-3)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot\frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\cdot\frac{1}{2}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\cdot(-2)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot 2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since A has a pivot position in every column, $T_A(v)$ is one-to-one.

b)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\cdot(-3)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot 1/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

since A does not have a pivot position in every column, $T_A(v)$ is not one-to-one.

$T_A(v_1) = T_A(v_2)$

let $T_A(v_1) = T_A(v_2) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} x_1 = 1 + 2x_3 \\ x_2 = 2 - 2x_3 \\ x_3 = x_3 \end{cases}$$

if $x_3 = 1$:
$x_1 = 1 + 2(1) = 3$
$x_2 = 2 - 2(1) = 0$
$x_3 = 1$

if $x_3 = -1$:
$x_1 = 1 + 2(-1) = -1$
$x_2 = 2 - 2(-1) = 4$
$x_3 = -1$

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True. Vector u must be in the span(u,v), and given that $w \in$ span(u,v) then w+u must also be in the span(u,v).

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True. In order for $\{u,v,w\}$ to be linearly independent, it must have a leading one in every column. This means $\{u,v\}$ also has a leading one in every column, so $\{u,v\}$ is linearly independent.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\{Au, Av\}$ linearly dependent

False. Multiplying matrix A by vectors u and v does not necessarily preserve linear dependence.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True. If vector $u \in \text{span}(v,w)$ and transformation T is applied to u, v, w then $T(u) \in \text{span}(T(v), T(w))$.

MTH-309T-F19-EX1-065-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Bella Esposito

UB Person Number:

5	0	2	2	4	3	9	8
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$2v_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} \overset{b=-6}{\Rightarrow} w = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} \qquad \boxed{b=-6}$$

$$v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad -3v_1 = \begin{bmatrix} -3 \\ 0 \\ -6 \end{bmatrix}$$

$$-2v_1 = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{(-2)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_1 = R_2 + R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & -2 \end{bmatrix} \xrightarrow{R_3 = R_2 + R_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

yes, linearly independent
because every col. is a pivot column

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-2)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{R_1 = R_1 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -1 & -1 \\ 2 & -2 & -1 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -1 & -1 \\ 2 & -2 & -1 \end{bmatrix} \qquad (A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T C = B \qquad \underline{(A^T)^{-1} = (A^{-1})^T}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$(A^T)^{-1} A^T C = (A^T)^{-1} B$$

$$C = (A^T)^{-1} B = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2+4+6 & -4+5+4 & -6+4+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ 1-4-3 & 2-5-2 & 3-4-1 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & +3 \\ 6 & +5 & +2 \end{bmatrix}}$$

Check $A^T C = B$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & +3 \\ 6 & +5 & 2 \end{bmatrix} = \begin{bmatrix} 8-7+0 & 5-3+0 & 0+3+0 \\ -8+0+12 & -5+0+10 & 4 \\ 16-7-6 & 10-3-5 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{-2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{array}\right] \begin{cases} x_1 = 7 \\ x_2 = 3 \end{cases}$

$T\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7-6 \\ 7+3 \\ 7-9 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array}\right] \xrightarrow{-3} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{-1}$

$A = \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{array}\right]$

yes one to one

pivot position in every col.

b) $A = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array}\right] \xrightarrow{-3} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array}\right]$

yes

pivot position in every col.

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$W \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \underset{W}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + \underset{u}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \in \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

false

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True, if u, v, w are all lin. independent, u, v must be linearly independent

$$\underbrace{\underbrace{u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{u,\, v \Rightarrow \text{lin. ind.}} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{all 3 lin. ind.}}$$

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True, since multiplying by the same matrix give linear dependence

b) If $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True since every matrix transformation is a linear transformation

MTH-309T-F19-EX1-008-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Romika Sairam

UB Person Number:

5	0	3	1	1	0	9	7

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer. C_1, C_2, C_3

RREF & See if each column has leading one?

a) $C_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + C_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + C_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

$\boxed{b = 0}$

$0\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$

$\boxed{b = -6}$

$y(2) + x(-3) + 0$

(b)
$$\begin{array}{cccc} v_1 & v_2 & v_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \leftarrow -2R_1 + R_3 \to R_3$$
$$\begin{array}{cccc} -2 & +2 & -2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \leftarrow R_2 + R_3 \to R_3$$
$$\begin{array}{cccc} +0 & +1 & +2 & +0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$ Vectors V_1, V_2 & V_3 are Linearly dependant (NOT Linearly Independent) because, V_2 & V_3 are free variables.

$V_1 - V_2 + V_3 = 0$

$V_2 + 2V_3 = 0$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

3x3

A^T or A^{-1}?

Compute $\boxed{A^{-1}.}$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

3x3.

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\underset{3\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}} = B = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}}$$

$A^T C$

$$\cancel{\underset{3\times 2}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{2\times}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}}$$

Colums have to be same ?

Come back!

$1x_1 - 1x_2 + 2x_3 = 1$

4. **(20 points)** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

2 rows to 3 columns?

a) Find the $\boxed{\text{standard matrix of } T.}$

b) Find all vectors u satisfying $\boxed{T(\mathbf{u})} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

Standard

(a) $T = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

(b) $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \begin{bmatrix} x_1 - 2x_2 = 1 \\ x_1 + x_2 = 10 \\ x_1 - 3x_2 = -2 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

$x_1 = 1 + 2x_2 \Rightarrow 1 + 2(3) = \boxed{7}\ x_1 = 7$

$1 + 2x_2 + x_2 = 10 \Rightarrow 1 + 3x_2 = 10 \Rightarrow 3x_2 = 9 \Rightarrow \boxed{x_2 = 3}$

$1 + 2x_2 - 3x_2 = -2 \Rightarrow 1 - x_2 = -2$

$\boxed{x_2 = 3}$

Come back

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

→ One - to - One
pivot columns
→ Nul(A) ?

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right]$$

$\tfrac{1}{2}R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \leftarrow -3R_1 + R_3 \to R_3$$

$-3 \quad -3 \quad +0 \quad +0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

pivot column because column without a leading 1.

$\boxed{\text{NOT one-to-one}}$.

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \leftarrow \tfrac{1}{2}R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \leftarrow -3R_1 + R_3 \to R_3$$

$-3 \quad -3 \quad +0 \quad +0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \leftarrow -R_2 + R_3 \to R_3$$

$+0 \quad -1 \quad -2 \quad +0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{One to one because no pivot columns?}$$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True because, w+u consists of w as well and when $w+u \in \mathrm{Span}(u,v)$; w must be in $\mathrm{Span}(u,v)$.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

Counterexample:
let $\{u,v,w\}$ be set $\{5,6,8\}$ respectively Then, $\{5,6,8\}$ doesn't mean $\{5,8\}$ is also linearly independent.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and **u**, **v** are vectors in $\mathbb{R}^2$ such that A**u**, A**v** are linearly dependent then $\boxed{\mathbf{u}, \mathbf{v}}$ also must be linearly dependent.

let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$Au = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$Av = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

linearly independent

True because, Since Au & Av are linearly dependent, u & v must also be.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that **u** is in $\boxed{\text{Span}(\mathbf{v}, \mathbf{w})}$ then $T(\mathbf{u})$ must be in $\boxed{\text{Span}(T(\mathbf{v}), T(\mathbf{w}))}$.

Come back.

$u \in \text{Span}(v, w)$

then

$T(u) \in \text{Span}(T(v), T(w))$?!

MTH-309T-F19-EX1-020-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

William Hiltz

UB Person Number:

5	0	1	9	3	1	7	7
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \xrightarrow{R_3+(-2)R_1} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4b \end{bmatrix} \xrightarrow[R_3+(1)R_2]{R_1+(1)R_2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4b+2 \end{bmatrix}$$

$0 = 4b+2$

$4b = -2$

$b = -\frac{2}{4}$

$\boxed{b = -1/2}$

b) No,

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{R_3+(-2)R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$a_1v_1 + \dots + a_nv_n \neq 0$

$x_1 - x_2 = 1$

$x_2 = 2$

$x_1 = 3$

MTH-309T-F19-EX1-020-P02

(ad - bc)

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + (-1)R_1} \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 + (-2)R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_1 + (1)R_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ & & & & & \\ & & & & & \end{bmatrix}$$

(ad+bc)

$$A = \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \end{bmatrix} \rightarrow \begin{bmatrix} -2 & -1 & 2 \\ -3 & -1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightarrow \boxed{A^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}}$$

$A^T C = B$ $\quad$ $AC = \frac{B}{A^{-1}}$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3+(-3)R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{R_3+(-1/2)R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

$0 \neq 2$

Not one-to-one
no solution

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3+(-3)R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_3+(-1/2)R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_2 = 0$
$2x_2 = 4$

$x_2 = 2$ $\quad T_A$ = one-to-one
$x_1 = -2$

$$\begin{array}{ccc} w & u & v \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 6 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \end{array}$$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

false,

$w + u = c_1 u + c_2 v$

$w \neq d_1 u + d_2 v$

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$w+u = \begin{bmatrix} 2+1 \\ 2+1 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{u} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \checkmark \quad \begin{matrix} c_1 = 3 \\ c_2 = 0 \end{matrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \neq d_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

Fake, u, v, w can all be ~~linearly~~ linearly independent but if you form the matrix of vectors, $\text{Nul}(A) \neq \{0\}$

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True, $a_1 v_1, \ldots, a_n v_n$ = non-zero therefore the vectors u, v. must also be linearly dependent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True, there can be any T(u) within the span of T(v) and T(w) because you can transform T(u) into either T(v, w).

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

David Palumbo

UB Person Number:

5	0	2	6	2	8	0	3
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \cdot -2 \rightarrow \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -5 & -2 & 4+b \end{bmatrix} \cdot 5 \rightarrow \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 8 & 14+b \end{bmatrix}$$

$8 = 14 + b$

a) $\boxed{b = -6}$

b) $$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 8 \end{bmatrix} \cdot \tfrac{1}{8} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot -2 \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot -1 \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot 1$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) The set is linearly independent b/c the homogenous vector equation $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ will only have one solution.

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot -1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot -2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \begin{matrix} \cdot -2 \\ \cdot 1 \end{matrix}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$(A^T)^{-1} = (A^{-1})^T \qquad C = (A^{-1})^T B$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$(A^T)^{-1} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+4+6 \\ 3-4-6 \\ -1+4+3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix} \quad (A^T)^{-1} \cdot \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -4+5+4 \\ 6-5-4 \\ -2+5+2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$$

$$(A^T)^{-1} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+4+2 \\ 9-4+2 \\ -3+4+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

a) Find the standard matrix of T.

$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$\boxed{a)\ \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2(1) \\ 1 \\ -3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

b) $Tu = Au = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot -1 \cdot -1 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 8 \\ 0 & -1 & -4 \end{array}\right] \rightarrow$

$\rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & -1 & -4 \\ 0 & 3 & 8 \end{array}\right] \cdot -1 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 3 & 8 \end{array}\right] \cdot -3 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{array}\right]$

no solutions for u

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

* one-to-one
pivot pos in ever col

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot -3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \cdot -2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

a) is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot -3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \cdot -2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot -1 \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 2x_3 = 0$
$x_2 + 2x_3 = 0$
$x_3 = x_3$
Nul(A)

b) not one-to-one

$$\boxed{T\left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}\right)}$$

$\uparrow v_1$ $\qquad$ $\uparrow v_2$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$\boxed{\text{True}}$ Since $u \in \text{span}(u, v)$
for $u + w \in \text{span}(u, v)$ w would have to be some combination of $cu + dv$ where c & d are some constants.
Therefore $w \in \text{Span}(u+v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent. *Linearly dependent - homogenous vector equation has more than 1 solution

$\boxed{\text{False}}$ let $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

~~$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent~~

~~$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly dependent~~

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is linearly dependent

~~$\boxed{\text{True}}$ For a set of p vectors (2 in this case) of $\mathbb{R}^n$ ($\mathbb{R}^3$ in this case) can only be linearly dependent if $p > n$. $2 \not> 3$ so the set $\{u, v\}$ would be linearly independent.~~

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\boxed{\text{False}}$ Counterexample: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $v = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

Au & Av both $= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is linearly dependent

but u and v are not linearly dependent

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$\boxed{\text{True}}$ If u is in Span (v, w) it is a combination of $cv + dw$ where c and d are some constants

So $T(u) = T(cv + dw) = T(cv) + T(dw) = cT(v) + dT(w)$

Which means that $T(u)$ is a combination of constant values of $T(v)$ & $T(w)$ which is what it means to be in Span$(T(v), T(w))$.

MTH-309T-F19-EX1-021-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Kunjie Lin

UB Person Number:

5	0	2	3	6	7	8	7

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

(a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$$

$$\frac{1}{2}\times \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 2 & -3 & 0 & b \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$-1\times \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 1 & -\frac{3}{2} & 0 & \frac{b}{2} \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -\frac{1}{2} & -1 & \frac{b}{2}+2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\times -2 \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -\frac{1}{2} & -1 & \frac{b}{2}+2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\times -1 \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & -2(\frac{b}{2}+2) = -b-4 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & -b-4 \\ 0 & 0 & 0 & b+6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & -b-4 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$

Since $b+6$ is in last colum $b+6$ has to be equal to 0 which $b=-6$ to make zero otherwise $b+6$ will be 1 which is undefind. So $\boxed{b=-6}$

(b)

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & -\frac{1}{2} & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = -x_2 + x_3 \\ x_2 = x_3 \\ x_3 = x_3 \end{array}$$

No the set is not linear Independent

Some Scalar mutiple of v_1 add to Scalar multiple of v_2 will product v_3.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{-1}\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 4 & -2 & 1 \end{bmatrix}$$

$A - D + 2G = 1$
$A + G = 0$
$2D - G = 0$

$-D + G = 1$
$2D - G = 0$

$-2D + 2G = 2$
$2D - G = 0$

$G = 2$
$A = -2$
$D = 1$

$B - E + 2H = 0$
$B + H = 1$
$2E - H = 0$

$2B - 2E + 4H = 0$
$2E - H = 0$

$2B + 3H = 0$
$2B + 2H = 2$

$-H = 2$
$H = -2$
$B = 3$
$E = -1$

$C - F + 2I = 0$
$C + I = 0$
$2F - I = 1$

$2C - 2F + 4I = 0$
$2F - I = 1$

$2C + 3I = 1$
$2C + 2I = 0$

$I = 1$
$C = -1$
$F = 1$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

$A+D=1$

$12 \quad -A+2G=4$

$16 \cdot 7 \cdot 6 \quad 2A+D-G=3$

$2A+D-G=3$

$A+D=1$

$A-G=2$

$-A+2G=4$

$G=6$

$A=8$

$D=-7$

$B+E=2$

$-B+2H=5$

$10 \; -3 \; -5$

$2B+E-H=2$

$2B+E-H=2$

$B+E=2$

$B-H=0$

$-B+2H=5$

$H=5$

$B=5$

$E=-3$

$C+F=3$

$-C+2I=4$

$2C+F-I=1$

$2C+F-I=1$

$C+F=3$

$C-I=-2$

$-C+2I=4$

$I=2$

$-C+4=4$

$-C=0$

$C=0$

$F=3$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-2\cdot 0 \\ 1+0 \\ 1-3\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0-2\cdot 1 \\ 0+1 \\ 0-3\cdot 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

Standard matrix of T is equal

$T\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b)

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \times -1$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \times -1$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \times \frac{1}{3}$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \times -1$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] \times -1$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \times 2$

$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 = 7$

$x_2 = 3$

$x_3 = x_3$

So when u equal $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$ will satisfaing $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \times\frac{1}{2}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \times -3$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \times -1$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \times\frac{1}{2}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \times -2$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \times -1$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Since every column of matrix A has pivot position

TA is one to one

$Nul(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \times\frac{1}{2}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \times -3$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \times -1$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$Nul(A) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

$x_1 + x_2 = 0 \qquad x_1 = -x_2$

$x_2 + 2x_3 = 0 \qquad x_2 = -2x_3$

$x_3 = x_3 \qquad x_3 = x_3$

$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} x_3$

Since not all column of matrix A has pivot column

TA is not one to one.

Since $Nul(A)$ in part a is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

and $Nul(A)$ in part b is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

and, $TA(v_1) = TA(v_2)$

v_1 has to equal v_2

which v_1 and V_2 are both $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

because each Null space contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

false

$w+u = x_1 u + x_2 v$

$w+u-u = x_1 u + x_2 v - u$

$w = x_1 u + x_2 v - u$

since $w+u \in \mathrm{span}(u,v)$

so $w+u = x_1 u + x_2 v$

and $w = x_1 u + x_2 v - u$

does not equal to $w = x_1 u + x_2 v$

which w is not linear combination of $x_1 u + x_2 v$

which w not in Span (u,v)

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$$\begin{array}{ccc} u & v & w \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$ True

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$ because u v have pivot position at every column

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

false

A $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ u $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ v $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Au \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Av \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

u and v are not linearly dependent
no scalar multiple of u could product v

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

A $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ u $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ v $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ w $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

True
Yes because linear transformation has some algebra properties as Vector addiction with $u = x_1 v + x_2 w$

$T(u) = x_1 T(v) + x_2 T(w)$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

MTH-309T-F19-EX1-007-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Mallory Karlik

UB Person Number:

5	0	2	3	1	7	0	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$b=6 \quad b=-6$

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

$x_1 v_1 + x_2 v_2 + x_3 v_3 = w$

$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

let $x_1 = 0$, $x_2 = 2$, $x_3 = 0$

let $x_1 = -3$, $x_2 = 0$, $x_3 = 1$

$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right]$

$x_1 = -3x_3$ $\quad x_3 = \text{free}$

$x_2 = 2 - 2x_3$ $\quad$ infinite soln

$b = -6$

$\boxed{b = -6, 6}$

b) $x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

since $x_3 = \text{free}$, there are infinitely many solutions

$\Rightarrow$ set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is linearly independent

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & -2 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix} -2 & 2 & -1 \\ -3 & 3 & -1 \\ 2 & -2 & 1 \end{bmatrix}}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad A^T C = B \qquad \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 2 & 5 & 0 \\ -1 & -3 & 3 \\ 0 & 5 & 2 \end{bmatrix}}$$

$c_{11} + c_{21} = 1$ $\qquad c_{11} = 1 - c_{21}$

$-c_{11} + 2c_{31} = 4$

$2c_{11} + c_{21} - c_{31} = 3$ $\qquad 2(1-c_{21}) + c_{21} + c_{31} = 3$

$2 - c_{21} + c_{31} = 3$

$c_{31} = 1 + c_{21}$

$2(1-c_{21}) + c_{21} - (1 + c_{21}) = 3$

$2 - 2c_{21} + c_{21} - 1 - c_{21} = 3$

$1 - 2c_{21} = 3$

$-2c_{21} = 2 \quad c_{21} = -1 \Rightarrow \boxed{c_{11} = 2,\ c_{21} = -1,\ c_{31} = 0}$

$c_{12} + c_{22} = 2$ $\qquad c_{12} = 2 - c_{22}$

$-c_{12} + 2c_{32} = 5$ $\qquad -(2-c_{22}) + 2(2-c_{22}) = 5$

$2c_{12} + c_{22} - c_{32} = 2$ $\qquad -2 + c_{22} + 4 - 2c_{22} = 5$

$2 - c_{22} = 5$

$2(2-c_{22}) + c_{22} - c_{32} = 2$ $\qquad -c_{22} = 3$

$4 - 2c_{22} + c_{22} - c_{32} = 2$ $\qquad c_{22} = -3$

$4 - c_{22} - c_{32} = 2$ $\qquad \Rightarrow c_{12} = 5$

$-c_{22} - c_{32} = -2$ $\qquad c_{22} = -3$

$-c_{32} = -2 + c_{22}$ $\qquad c_{32} = 5$

$c_{32} = 2 - c_{22}$

$c_{13} + c_{23} = 3$ $\qquad c_{13} = 3 - c_{23}$

$-c_{13} + 2c_{33} = 4$ $\qquad -(3-c_{23}) + 2(5-c_{23}) = 4$

$2c_{13} + c_{23} - c_{33} = 1$ $\qquad -3 + c_{23} + 10 - 2c_{23} = 4$

$7 + c_{23} - 2c_{23} = 4$

$2(3-c_{23}) + c_{23} - c_{33} = 1$ $\qquad -c_{23} = -3$

$6 - 2c_{23} + c_{23} - c_{33} = 1$ $\qquad c_{23} = 3$

$6 - c_{23} - c_{33} = 1$ $\qquad \Rightarrow c_{13} = 0$

$-c_{23} - c_{33} = -5$ $\qquad c_{23} = 3$

$-c_{33} = -5 + c_{23}$ $\qquad c_{33} = 2$

$c_{33} = 5 - c_{23}$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$ $\quad T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ $\quad u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 0 & -1 & -3 \end{array}\right] \tfrac{1}{4}$$

$$\to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \to \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 - 2x_2 = 1$
$x_2 = 3$

$x_1 = 1 + 2x_2 = 1 + 2(3) = 7$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$ or any multiple of $u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

check $T\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7-6 \\ 7+3 \\ 7-9 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ ✓

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $\quad T = \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & 2x_2 & 4x_3 \\ 3x_1 & 4x_2 & 2x_3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \cdot(-3) \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \cdot(-1) \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \tfrac{1}{2}$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot(-2) \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot(-1) \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

pivot position in every col. so YES, one-to-one

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \cdot(-3) \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \cdot(-1)$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Pivot position NOT in every column, so NOT one-to-one

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+1+0 & 3 \\ 0+2+0 & 2 \\ 6+4+0 & 10 \end{bmatrix} \quad [\ ]$

$x_1 + x_2 =$

$x_2 + 2x_3 =$

$x_1 + x_2 = 2 \qquad x_1 = 2 - x_2$

$2x_2 + 4x_3 = 6$

$3x_1 + 4x_2 + 2x_3 = 9$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 4 & 6 \\ 3 & 4 & 2 & 9 \end{array}\right] \tfrac{1}{2} \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 2 & 9 \end{array}\right]$

$3(2-x_2) + 4x_2 + 2x_3 = 9$

$6 - 3x_2 + 4x_2 + 2x_3 = 9$

$x_2 + 2x_3 = 3$

$\cancel{\frac{2x_3}{2}} = \frac{3-x_2}{2}$

$x_3 = \frac{3-x_2}{2}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 9 \end{array}\right]$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 32 \\ 28 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ \\ \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True, linear combination

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True, if set of vectors is linearly independent, then $\{u, v\}$ are also linearly independent as long as u or v are not a scalar multiple of the other

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent. False

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$\{u\}, \{v\}$ are only linearly dependent if $u = \vec{0}$, $v = \vec{0}$

Matrix A can be linearly dependent even if u, v are not zero-vectors

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True by linear combination

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Miguel Sanz

UB Person Number:

5	0	2	5	9	1	5	6
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 \to R_3 + (-2R_1)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right]$$

$$c_1 V_1 + c_2 V_2 + c_3 V_3 = W \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 2 \\ c_3 = 0 \end{array} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$$\therefore \boxed{b = -6}$$

b)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + (-2R_1)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

not linearly independent because every column of the matrix is not a pivot column.

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A \cdot I = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & -1 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + (-R_1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + (-R_3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\therefore C = B(A^{-1})^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{C = \begin{bmatrix} -2 & 2 & 6 \\ 12 & -5 & -8 \\ -3 & 2 & 1 \end{bmatrix}}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(e_1) = \begin{bmatrix} 1 - 2(0) \\ 1 + 0 \\ 1 - 3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Standard Matrix of } \boxed{T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

$$T(e_2) = \begin{bmatrix} 0 - 2(1) \\ 0 + 1 \\ 0 - 3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

b)

$$\left[\begin{array}{c|c} x_1 - 2x_2 & 1 \\ x + x_2 & 10 \\ x_1 - 3x_2 & -2 \end{array}\right] \Rightarrow \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \end{array} \xrightarrow{R_2 \to R_2 + (-R_3)} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 1 & -3 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + (-R_1)} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{R_2 \to R_2 + (4R_3)} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 7 \\ x_2 = 3 \end{array}$$

$$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + (-3R_1)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + (-2R_2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + (R_3)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \xrightarrow{R_3 \to \frac{-1}{4} R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + (-R_2)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Matrix is one to one because there is a pivot position in every column.

---

b) $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + (-3R_1)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + (-2R_3)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

not one-to-one

using $V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad T_A(V_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ 0+0+0 \\ 3+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

now: $T_A(V_2) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 3 \end{array}\right] \Rightarrow$ following same row reduce seen above

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} V_1 = -V_2 + 1 \\ V_2 = -2V_3 \\ V_3 = V_3 \end{array}$$

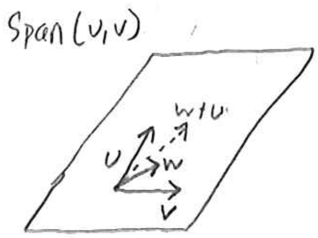
$V_3 = 1 \Rightarrow \begin{array}{l} V_3 = 1 \\ V_2 = -2 \\ V_1 = -1 \end{array} \Rightarrow V_2 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$$\Rightarrow \boxed{V_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}}$$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True, If w was not within the span(u,v) then the resultant vector would not be within the plane



b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False, only linearly independent if u is a scalar multiple of v, or vice versa.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$$\begin{array}{c} A \\ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \end{array} \quad \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

$$\begin{array}{cc} A & u \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{array} \text{ (L.I)}$$

$$\begin{array}{cc} A & v \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{array} \text{ (L.I)}$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \to \text{L I}$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \text{LI}$$

$$\begin{array}{cc} u & v \end{array} \\ \begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix}$$

False

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to$$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True, every matrix transformation is a linear transformation, making the statement true

MTH-309T-F19-EX1-072-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Kyle Williams

UB Person Number:

5	0	1	8	8	7	3	0

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) If: $W = 2V_2$, $\boxed{b = -6}$

$W = V_1 + V_3$, $\boxed{b = 2}$

$W = 3V_1 + 2V_2$, $\boxed{b = 0}$

b) The set is not linearly independent because there is a linear combination of vectors V_1 and V_2 which give V_3.

$3V_1 + 2V_2 = V_3$

$$3\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = V_3$$

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

First, append Identity matrix, I, then find rref(A)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2,\ (-1)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \checkmark$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$C_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

$x_1 + x_2 + 0x_3 = 1$
$-x_1 + 0x_2 + 2x_3 = 4$
$2x_1 + x_2 - x_3 = 3$

$\to -x_1 + x_3 = -2$
$-x_1 + 2x_3 = 4$
$x_3 = 6 \therefore x_1 = 8$ and $x_2 = -7$

$C_1 = \begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix}$

$C_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

$x_1 + x_2 + 0x_3 = 2$
$-x_1 + 0x_2 + 2x_3 = 5$
$2x_1 + x_2 - x_3 = 2$

$-x_1 + x_3 = 0$
$-x_1 + 2x_3 = 5$
$\therefore x_1 = x_3 = 5$, $x_2 = -3$

$C_2 = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$

$C_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$x_1 + x_2 + 0_3 = 3$
$-x_1 + 0x_2 + 2x_3 = 4$
$2x_1 + x_2 - x_3 = 1$

$-x_1 + x_3 = 2$
$-x_1 + 2x_3 = 4$
$x_3 = 2$ and $x_1 = 4$
$\therefore x_2 = -1$

$C_3 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$$\therefore \boxed{C = \begin{bmatrix} 8 & 5 & 4 \\ -7 & -3 & -1 \\ 6 & 5 & 2 \end{bmatrix}}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $\boxed{T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

b) $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$x_1 - x_2 = 1$
$x_1 + x_2 = 10$
$x_1 - 3x_2 = -2$

$\rightarrow$

$2x_1 = 11 \rightarrow x_1 = 4.5$ or $9/2$
$9/2 - 3x_2 = -2$
$-3x_2 = 13/2$
$x_2 = -13/6$

But $9/2 - 13/6 \neq 10$ $\therefore$ $\boxed{\text{There are no vectors, } u \text{, which satisfy } T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}}$

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$$\xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

This is not one to one because there is not a pivot in every column

$V_1 = \begin{bmatrix} \ \end{bmatrix}$

$V_2 = \begin{bmatrix} \ \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is not one to one because there is not a pivot in every column

$V_1 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$V_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True; If there is a linear combination of vectors V and u that equal W, W must be in the span of u and V. Since addition and subtraction of one vector from another is a linear operator, the problem can be considered as the following:

$w = (w+u) - u$ $\therefore$ If $w+u = x_1 V + x_2 u$, then $w = x_1 V + (x_2 - 1) u$, which is a linear combination of u and V $\therefore$ $w \in \text{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True, any subset of a linearly independent set of vectors must also be linearly independent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False; Linear dependence/independence is only guaranteed to be preserved if A is a matrix which defines a linear transformation. Since this condition is not specified, then linear dependence cannot be guaranteed after transformation.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True; Since T is a linear operator whose operation preserves the dimensions of the original vectors, then any vector $u \in \text{Span}(v, w)$ must be in the span of $T(v), V(w)$. Additionally, since there are 3 vectors in 2 space, and it is known that $u \in \text{Span}(v, w)$, v and w either are linearly dependent on one another, and u, or are linearly independent. Since T is a linear transformation, these properties are maintained, meaning $T(u) \in \text{Span}\, T(v), T(w)$ by definition (3 vectors in 2 space, 2 are linearly independent; v and w) or because all 3 vectors were linearly dependent to begin with.

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Devraj Chowdhary

UB Person Number:

5	0	2	1	7	2	0	5
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

MTH-309T-F19-EX1-035-P01

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\left(\cancel{\begin{matrix} 1 & & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{matrix}}\right) \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

$(-2)\left[\begin{matrix} 1 & -1 & 1 & 2 \\ 2 & -3 & 0 & b \\ 0 & 1 & 2 & 2 \end{matrix}\right]$

$= \left[\begin{matrix} 1 & -1 & 1 & 2 \\ 0 & -1 & -2 & b-4 \\ 0 & 1 & 2 & 2 \end{matrix}\right]$

$\left[\begin{matrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 4-b \\ 0 & 1 & 2 & 2 \end{matrix}\right]$

$= \left[\begin{matrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 4-b \\ 0 & 0 & 0 & b-2 \end{matrix}\right]$

$\because$ the last row is 0.

$\therefore b-2=0.$

$\therefore b=2.$

$\therefore b=2$ such that $w \in \text{span}(v_1, v_2, v_3)$

b) the set $\{v_1, v_2, v_3\}$ is not linearly dependant because every column is not a pivot column

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A \cdot A^{-1} = I$$

$$\therefore \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$(-1)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\Rightarrow (-2)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\Rightarrow (-2)\ (1)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\Downarrow$$

$$\Rightarrow (1)\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \therefore A^{-1} = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ $\therefore$ $A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$\therefore \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \cancel{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

Let $C = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\therefore$ The Standard Matrix of T.

$T = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ $\therefore T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\therefore c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\begin{bmatrix} c_1 - 2c_2 \\ c_1 + c_2 \\ c_1 - 3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ $\therefore T(u) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

MTH-309T-F19-EX1-035-P05

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) ~~For T_A to be~~

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

∴ There is a Pivot Pos in every column. $T_A(v) = Av$ is one to one

for $T_A(v)$ to be one to one the $Nul\{A\} = \{0\}$

$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 4 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$Nul(A) = \{0\}$

$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$

$\therefore \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

∴ Every column does not have pivot pos. $T_A(v) = Av$ is not one to one.

If $V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 2 & 4 & 16 \\ 3 & 4 & 2 & 17 \end{array}\right] = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\therefore V_2 = \begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix}$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True. $\therefore$ $T(w+u) = T(w) + T(u) = T(u) + T(v)$.

$\therefore$ $T(w) = T(u) + T(v)$.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True: If ~~u,v,w~~ $c_1 u + c_2 v + c_3 w = 0$ is linearly independant it means $\exists u, v \& w = 0$.

$\therefore$ the set $\{u, v\}$ has to be linearly independent

because $x_1 u + x_2 v = 0$.

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False.

$\therefore Au = A[u] + A[v] = x_1 u + x_2 v$

can be linearly dependent if also x_1 & $x_2 \neq 0$.

~~Also $c_1 u + c_2 v$ can also be~~

$\therefore c_1 u + c_2 v$ need ~~be~~ not be linearly dependant because.

$u, v = 0$.

$c_1 u + c_2 v$ becomes linearly independent.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True.

$$T(u) = T(v + w)$$
$$= T(v) + T(w).$$