MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	3	1	7	0	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
14	8	8	20	15	4	4	2	8	81	B

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$b=6$ $\cancel{b=-6}$

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

$x_1 v_1 + x_2 v_2 + x_3 v_3 = w$

$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

let $x_1 = 0$, $x_2 = 2$, $x_3 = 0$

let $x_1 = -3$, $x_2 = 0$, $x_3 = 1$

$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & b \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right] \end{array}$

$x_1 = -3x_3$
$x_2 = 2 - 2x_3$
$b = -6$ ✓

$x_3 =$ free
infinite soln

$\boxed{b = -6 \quad 6}$?

b) $x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

since $x_3 =$ free, there are infinitely many solutions

$\Rightarrow$ set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is linearly ~~x~~dependent

infinitely many solutions means dependent

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

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2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & -2 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \cdot 1 +$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \cdot(-2) \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \cdot 1 + \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \cdot(-1)$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix} -2 & \cancel{2} & -1 \\ \cancel{-3} & \cancel{3} & \cancel{-1} \\ 2 & -2 & 1 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad A^T C = B \qquad \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} \cancel{2} & 5 & 0 \\ \cancel{-1} & -3 & 3 \\ \cancel{0} & 5 & 2 \end{bmatrix}}$$

$c_{11} + c_{21} = 1 \qquad c_{11} = 1 - c_{21}$

$-c_{11} + 2c_{31} = 4$

$2c_{11} + c_{21} - c_{31} = 3 \qquad 2(1-c_{21}) + c_{21} + c_{31} = 3$

$2 - c_{21} + c_{31} = 3$

$c_{31} = 1 + c_{11}$

$2(1-c_{21}) + c_{21} - (1+c_{11}) = 3$

$2 - 2c_{21} + c_{21} - 1 - c_{21} = 3$

$1 - 2c_{21} = 3$

$-2c_{21} = 2 \quad c_{21} = -1 \Rightarrow \boxed{\begin{matrix} c_{11} = 2 \\ c_{21} = -1 \\ c_{31} = 0 \end{matrix}}$

This is a long way to do it, but it (mostly) worked.

$c_{12} + c_{22} = 2 \qquad c_{12} = 2 - c_{22}$

$-c_{12} + 2c_{32} = 5$

$2c_{12} + c_{22} - c_{32} = 2$

$2(2-c_{22}) + c_{22} - c_{32} = 2$

$4 - 2c_{22} + c_{22} - c_{32} = 2$

$4 - c_{22} - c_{32} = 2$

$-c_{22} - c_{32} = -2$

$-c_{32} = -2 + c_{22}$

$c_{32} = 2 - c_{22}$

$-(2-c_{22}) + 2(2-c_{22}) = 5$

$-2 + c_{22} + 4 - 2c_{22} = 5$

$2 - c_{22} = 5$

$-c_{22} = 3$

$c_{22} = -3$

$\Rightarrow c_{12} = 5$

$c_{22} = -3$

$c_{32} = 5$

$c_{13} + c_{23} = 3 \qquad c_{13} = 3 - c_{23}$

$-c_{13} + 2c_{33} = 4$

$2c_{13} + c_{23} - c_{33} = 1$

$2(3-c_{23}) + c_{23} - c_{33} = 1$

$6 - 2c_{23} + c_{23} - c_{33} = 1$

$6 - c_{23} - c_{33} = 1$

$-c_{23} - c_{33} = -5$

$-c_{33} = -5 + c_{23}$

$c_{33} = 5 - c_{23}$

$-(3-c_{23}) + 2(5-c_{23}) = 4$

$-3 + c_{23} + 10 - 2c_{23} = 4$

$7 + c_{23} - 2c_{23} = 4$

$-c_{23} = -3$

$c_{23} = 3$

$\Rightarrow c_{13} = 0$

$c_{23} = 3$

$c_{33} = 2$

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0	1	2	3	4	5	6	7	8	9	10

4. (20 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$ $T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ ✓

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 0 & -1 & -3 \end{array}\right] \cdot 1/4$

$\to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \to \begin{array}{ccc} x_1 & x_2 & \\ \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \end{array}$ $\begin{aligned} x_1 - 2x_2 &= 1 \\ x_2 &= 3 \end{aligned}$ $x_1 = 1 + 2x_2 = 1 + 2(3) = 7$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$ ✓ or any multiple of $u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

check $T\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7-6 \\ 7+3 \\ 7-9 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $T = \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & 2x_2 & 4x_3 \\ 3x_1 & 4x_2 & 2x_3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \cdot -3 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \cdot -1 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \tfrac{1}{2}$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot -2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot -1$

$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ pivot position in every col. so YES, one-to-one ✓

$x_1 + x_2 =$

$x_2 + 2x_3 =$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 32 \\ 28 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \cdot -3 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \cdot -1$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Pivot position NOT in every column, so NOT one-to-one ✓

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+1+0 & 3 \\ 0+2+0 & 2 \\ 6+4+0 & 10 \end{bmatrix}$

$x_1 + x_2 = 2$ $\qquad x_1 = 2 - x_2$

$2x_2 + 4x_3 = 6$

$3x_1 + 4x_2 + 2x_3 = 9$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 4 & 6 \\ 3 & 4 & 2 & 9 \end{array}\right] \tfrac{1}{2} \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 2 & 9 \end{array}\right]$

$3(2 - x_2) + 4x_2 + 2x_3 = 9$

$6 - 3x_2 + 4x_2 + 2x_3 = 9$

$x_2 + 2x_3 = 3$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 9 \end{array}\right]$

$\frac{\cancel{2}x_3}{\cancel{2}} = \frac{3 - x_2}{2}$

$x_3 = \frac{3 - x_2}{2}$

v_1, v_2 ?

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True ✓, linear combination ?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True ✓, if set of vectors is linearly independent, then $\{u, v\}$ are also linearly independent is long is u or v are not a scaler multiple of the other ?

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent. False ✓

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\{v\}, \{v\}$ are only linearly dependent if $v=\vec{0}$, $v=\vec{0}$ ← but $\{u, v\}$ can be linearly dep. even if $u \neq 0$ and $v \neq 0$ ...

? Matrix A can be linearly dependent even if v, v are not zero-vectors

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span(T(v), T(w)).

True ✓ by linear combination ?

0 1 2 3 ~~4~~ 5 6 7 8 9 10