MTH-309T-F19-EX1-020-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

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Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	TOTAL	GRADE
10	--	--	--	--	--	--	10	nan

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}.$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \xrightarrow{R_3+(-2)R_1} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4b \end{bmatrix} \xrightarrow[R_3+(1)R_2]{R_1+(1)R_2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4b+2 \end{bmatrix}$$

$0 = 4b+2$

$4b = -2$

$b = -\frac{2}{4}$

$\boxed{b = -1/2}$

b) No,

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{R_3+(-2)R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$a_1 v_1 + \dots + a_n v_n \neq 0$

$x_1 - x_2 = 1$

$x_2 = 2$

$x_1 = 3$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

10

$(ad-bc)$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2+(-1)R_1} \left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3+(-2)R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_1+(1)R_2} \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ & & & & & \\ & & & & & \end{array}\right]$$

$(ad-bc)$

$$A = \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \end{bmatrix} \rightarrow \begin{bmatrix} -2 & -1 & 2 \\ -3 & -1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightarrow \boxed{A^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

$A^T C = B$ $\quad$ $AC = \frac{B}{A^{-1}}$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3+(-3)R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{R_3+(-1/2)R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

$\boxed{0 \neq 2}$

Not one-to-one

no solution

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3+(-3)R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_3+(-1/2)R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_2 = 0$

$2x_2 = 4$

$x_2 = 2$ $\quad T_A$ = one-to-one

$x_1 = -2$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

$$\begin{array}{ccc} w & u & v \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \end{array}$$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

false,

$w+u = c_1 u + c_2 v$

$w \neq d_1 u + d_2 v$

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$w+u = \begin{bmatrix} 2+1 \\ 2+1 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = c_1 \underset{u}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \checkmark \quad \begin{matrix} c_1 = 3 \\ c_2 = 0 \end{matrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \neq d_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False, u, v, w can all be ~~linearly~~ linearly independent but if you form the matrix of vectors, $\text{Nul}(A) \neq \{0\}$

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True, $a_1 v_1, \ldots, a_n v_n$ = non-zero therefore the vectors u,v. must also be linearly dependent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True, there can be any T(u) within the span of T(v) and T(w) because you can transform T(u) into either T(v, w).

0 1 2 3 4 5 6 7 8 9 10