MTH-309T-F19-EX1-070-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	3	0	4	7	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

19	10	7	10	20	5	4	--	--	75	B-
1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & \cancel{5+b} \end{array}\right]$$

$6+b$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & \cancel{5+b} \end{array}\right]$$

free var

$6+b$

~~Must equal~~

$\boxed{a)\ b = \cancel{-5}}$ -6

b) The set is $\boxed{\text{linearly dependent}}$ because there is not a pivot column in every column

✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 2 & -2 & 4 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 2 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \checkmark$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \to \begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 0 & 1 & -5 & -5 & -8 & -7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2 & 4 & 5 & 4 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 1 & -5 & -5 & -8 & -7 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 2 & 4 & 5 & 4 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & -3 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -4 & -5 & -4 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -5+\frac{1}{3} & -4 \\ 0 & 1 & 0 & 5 & 7+\frac{1}{3} & 7 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \end{array}\right]$$

$$-\frac{15}{3} - \frac{1}{3} = -\frac{16}{3}$$

$$\frac{21}{3} + \frac{1}{3}$$

This would work if done correctly.

$$C = \begin{bmatrix} -4 & -\frac{16}{3} & -4 \\ 5 & \frac{22}{3} & 7 \\ 0 & -\frac{1}{3} & 0 \end{bmatrix}$$

Simpler: $C = (A^T)^{-1} \cdot B$

$= (A^{-1})^T \cdot B$

Then use A^{-1} from problem 2.

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

3x2 2x2

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

?

$$\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

a) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $\cancel{u = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}}$ ← why ?

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

leading ones

pivot columns

* $\boxed{\text{a) one to one}}$ ✓

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \boxed{\text{not one to one}}$$ * ✓

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

b) $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

These should be <u>different</u> vectors but since it was not explicitly stated I will accept it.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ☒

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

~~False~~

* $\boxed{\text{False}}$

w could be projected outside the span when not added to u

?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$\boxed{\text{True}}$ ✓ because you will still have every column as a pivot column ✓

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}, A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False 2 ✓

this may simply mean that the matrix A is linearly dependent not necessarilly u or v

example ?

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

~~True because~~

True ✓ because if the same transformation is performed on some u that's in the span of some v and w, ~~it will~~ u will be moved into that new span of T(v), T(w) by the transformation T(u).

why?

0 1 2 3 4 5 6 7 8 9 10