



MTH 309T LINEAR ALGEBRA
EXAM 1
 October 3, 2019

Name:

Gerra Gamaro

UB Person Number:

5	0	1	9	7	7	7	0
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1 2 3 4 5 6 7 TOTAL GRADE

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1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{w} \end{array} \right]$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array} \right] \xrightarrow{\cdot(-2)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b \end{array} \right]$$

$$\downarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 4+b \end{array} \right] \xrightarrow{\cdot(1)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4+b \end{array} \right]$$

$$\downarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & 4+b \end{array} \right] \xrightarrow{\cdot(2)} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2+b \end{array} \right]$$

$$\downarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2+b \end{array} \right] \xrightarrow{\cdot(3)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2+b \end{array} \right] \xrightarrow{\text{true for } 2+b=0 \text{ to have a solution}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2+b \end{array} \right]$$

b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent because x_3 is a free variable indicating that the system has infinitely many solutions, making it linearly dependent.

$b = -6$



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1} .

$$A^{-1} \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\cdot(-1)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{\cdot(-2)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\cdot(1)}$$

$$\xrightarrow{\cdot(-1)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right] \xrightarrow{\cdot(1)}$$

$$\downarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right] \xrightarrow{\cdot(1)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right] \xrightarrow{\cdot(-1)}$$

$$\downarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right] \underbrace{\qquad}_{A^{-1}}$$

$$A^{-1} = \boxed{\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}}$$



$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T C = B \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \rightarrow x_1 = \begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \rightarrow x_2 = \begin{bmatrix} 13 \\ -11 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \rightarrow x_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$C = \boxed{\begin{bmatrix} 8 & 13 & 0 \\ -7 & -11 & 3 \\ 6 & 9 & 2 \end{bmatrix}}$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = [T(e_1) \ T(e_2)]$
 $T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$Au = T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array} \right] \xrightarrow{\cdot(-1)}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array} \right] \xrightarrow{\cdot(-1)}$$

$$\boxed{U = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array} \right] \xrightarrow{\cdot(1/3)}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array} \right] \xrightarrow{\cdot(1)}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{array} \right] \xrightarrow{\cdot(2)}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \boxed{\begin{cases} x_1 = 7 \\ x_2 = 3 \end{cases}}$$



one to one \rightarrow pivot in every column

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\cdot(-3)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot(1/2)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\cdot(1/2)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot(-2)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot(2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a) Pivot in every column,
therefore it is one to one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\downarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot(1/2)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{free}}$

b) Not one to one because
there is a free variable
in the last column.

$$T_A(v_1) = T_A(v_2)$$

$$Av_1 = Av_2$$

$$\text{Nul}(A) = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right] \xrightarrow{\substack{x_1 x_2 \\ x_3}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\substack{x_1 \\ x_2 \\ x_3}} \begin{array}{l} x_1 = 2x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{array}$$

$$\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$v_2 = v_1 + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \rightarrow \boxed{v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}$$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True because u is in the span of itself, making it equal to $w \in \text{Span}(u, v)$

b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

False \rightarrow two vectors can be linearly independent but the third can be linearly dependent and vice versa.

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 3 \\ -12 \end{bmatrix} \quad v_1, v_2, v_3 \rightarrow \text{linearly dependent}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -2 & 4 & -12 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

↑ free

However, v_1 and v_2 are linearly independent.

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 5 & 0 \\ -2 & 4 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \left. \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array} \right\} \text{trivial solution.}$$



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

- a) If A is a 2×2 matrix and u, v are vectors in \mathbb{R}^2 such that Au, Av are linearly dependent then u, v also must be linearly dependent.

False \rightarrow (Handwritten)

- b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

$$u \in \text{Span}(v, w)$$

$$T(u) \in \text{Span}(T(v), T(w))$$

True because the transformation is applied for each vector, providing the same vectors at different positions



MTH 309T LINEAR ALGEBRA
EXAM 1

October 3, 2019

Name:

Avery Weissman

UB Person Number:

5	0	1	3	9	1	3	3
0	●	0	0	0	0	0	0
1	1	●	1	1	●	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	●	●
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	●	9	9	9

Instructions:

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1 2 3 4 5 6 7 TOTAL GRADE

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1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

- a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.
 b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

$$\text{a)} \begin{array}{l} x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w} \\ x_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array} \right] \xrightarrow{(1-2)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array} \right] \xrightarrow{(0+2)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array} \right] \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array} \right] \xrightarrow{(1)} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array} \right]$$

For $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, must have a sol'n. $\therefore b+6=0$

$$\boxed{b=-6}$$

- b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not linearly independent because all columns of $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ are not pivot columns.



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1} .

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{(C-1)} \left(\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{(C2)} \left(\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{\text{R2} \rightarrow R2} \left(\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right) \xrightarrow{(C-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right) \xrightarrow{(C2)} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right)$$

$$\xrightarrow{\text{R3} \rightarrow R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right) \xrightarrow{(C-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right) \xrightarrow{(C1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right)$$

$$\xrightarrow{\text{R2} \rightarrow R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right)$$

$$A^{-1} = \boxed{\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$



3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^T C = B \Rightarrow (A^T)^{-1} (A^T) C = B (A^T)^{-1} \Rightarrow C = B (A^T)^{-1}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ -3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 3 & 1 & -1 \\ -2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$C = \boxed{\begin{bmatrix} -11 & 3 & 1 & -1 \\ -2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}}$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{pmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors \mathbf{u} satisfying $T(\mathbf{u}) = \begin{pmatrix} 1 \\ 10 \\ -2 \end{pmatrix}$.

a) $\left[T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \quad T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \right] = \left[T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \quad T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \right] = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix}$

$\boxed{\begin{pmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{pmatrix}}$

b) $T(\mathbf{u}) = \begin{pmatrix} 1 \\ 10 \\ -2 \end{pmatrix}$

$$A \mathbf{x} = \begin{pmatrix} 1 \\ 10 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ -2 \end{pmatrix} \quad \begin{cases} x_1 = 1 + 2x_2 \\ x_1 = 10 - x_2 \\ x_1 = 3x_2 - 2 \end{cases} \quad \begin{cases} 10 - x_2 = 3x_2 - 2 \\ 12 + x_2 = x_2 + 10 \\ 4x_2 = 12 \\ x_2 = 3 \end{cases}$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad \boxed{\mathbf{u} = \begin{pmatrix} 7 \\ 10 \\ 3 \end{pmatrix}}$$

This is one solution.



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\text{a)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{pmatrix} \xrightarrow{(-1)\downarrow} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{R}_3 \rightarrow R_3/3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(-1)\downarrow} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(1,-2)\leftrightarrow(2,3)}$$

$$\xrightarrow{\text{R}_2 \rightarrow R_2/2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{R}_1 \rightarrow R_1 - R_2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Pivot position in every column} \Rightarrow A \text{ is one to one.}$$

$$\text{b)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{pmatrix} \xrightarrow{(1,-3)\downarrow} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{\text{R}_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(1,-1)\downarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Not one to one, no pivot position in every column.

$$T_A(v_1) = T_A(v_2)$$

$$T_A(v_1 - v_2) = 0$$

$$v_1 - v_2 \in \text{Null}(A)$$

$$\left\{ \begin{array}{l} \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \\ \text{free} \end{array} \right.$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_3 &= x_3 \end{aligned}$$

$$\begin{aligned} x_3 &= 1 \\ &\xrightarrow{\text{R}_2 \rightarrow R_2 - R_1} x_2 &= -2 \\ &\xrightarrow{\text{R}_1 \rightarrow R_1 - R_2} x_1 &= -2 \end{aligned} \quad \begin{aligned} x_1 &= -x_2 \\ x_2 &= -2 \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_3 &= 2 \\ &\xrightarrow{\text{R}_2 \rightarrow R_2 - R_1} x_2 &= -2 \\ &\xrightarrow{\text{R}_1 \rightarrow R_1 - R_2} x_1 &= -2 \end{aligned} \quad \begin{aligned} x_1 &= -x_2 \\ x_2 &= -4 \\ x_3 &= 2 \end{aligned}$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

- a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

False. If u, v, w are vectors in

\mathbb{R}^3 s.t. $w \notin \text{Span}(u, v)$, w must be a linear combination of $u+v$ not $w+u$.

- b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

True. If ~~$\text{Span}\{u, v, w\}$~~ is If
 $\{u, v, w\}$ is linearly independent, only have
 trivial solution as the solution to homogeneous
 equations i.e., $c_1u + c_2v = 0$ must also
 only have the trivial solution as the
 answer.



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

- a) If A is a 2×2 matrix and u, v are vectors in \mathbb{R}^2 such that Au, Av are linearly dependent then u, v also must be linearly dependent.

True. For Au and Av to be linearly dependent, u and v must have infinitely many solutions. Some combination of u & v other than the trivial solution will result in the 0 vector.

- b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

True. If u is in $\text{Span}(v, w)$, then it is a linear combination of v & w . Therefore its transformation must be in the span as well.



MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Chenhao Zou

UB Person Number:

5	0	2	1	8	3	8	0
0	0	0	0	0	0	0	1
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1 2 3 4 5 6 7 TOTAL GRADE

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1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

- a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.
 b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

$$\mathbf{w} = C_1 \mathbf{v}_1 + C_2 \mathbf{v}_2 + C_3 \mathbf{v}_3$$

$$C_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & \frac{b}{3} \end{bmatrix}$$

$\frac{b}{3} = 2 \Rightarrow b = 6$ so that the matrix is consistent and has solution that makes $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

b)

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$x_1 + 3x_3 = 0 \quad x_1 = -3x_3$$

$$x_2 + 2x_3 = 0 \quad x_2 = -2x_3$$

$$x_2 + 2x_3 = 0 \quad x_2 = -2x_3 \quad x_3 = x_3$$

$$x = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} x_3$$

so that there are other solutions other than $x_1 = x_2 = x_3 = 0$

$\therefore \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1} .

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \\ \Rightarrow & \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \\ \Rightarrow & \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \times -2 + 0 \\ \Rightarrow & \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \\ \Rightarrow & \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 \end{array} \right] \\ \therefore A^{-1} = & \left[\begin{array}{cc} 0 & 1 \\ 1 & -1 \\ 2 & -2 \end{array} \right] \end{aligned}$$



3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A^T C = B$$

$$\text{set } C = \begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & C_9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 0 & 2 & 5 \\ 2 & 1 & -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) the standard matrix of T

is $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} u = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 11 \\ 0 & 4 & 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$u_1 = 7$$

$$u_2 = 3$$

$$u_3 = u_3$$

$$u = \begin{bmatrix} 7 \\ 3 \\ u_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_3.$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

T_A is not one-to-one.

$$T_A(v_1) = T_A(v_2)$$

$$T_A(v_1 - v_2) = 0$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

$$x_1 = 2x_3$$

$$x_2 = -2x_3$$

$$x_2 = -4x_3$$

there is no solution.

∴ there is no two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

T_A is not one-to-one

$$T_A(v_1) = T_A(v_2)$$

$$T_A(v_1 - v_2) = 0$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2x_3 & x &= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x_3 \\ x_2 &= -2x_3 & & \\ x_3 &= x_3 & & \end{aligned}$$

$$v_1 - v_2 = x$$

$$\text{assume } x_3 = 1 \quad x = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{assume } v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{then } v_1 = x + v_2$$

$$= \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore v_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

~~False~~

~~set $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$~~

~~$w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$~~

~~$w+u = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \notin \text{Span}(u, v)$~~

~~$w \notin \text{Span}(u, v)$~~

True
 $w+u \in \text{Span}(u, v)$
 then $w \in \text{Span}(u, v)$

b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

~~False~~

~~assume $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$~~

~~$\{u, v, w\}$ is linearly independent.~~

~~while $\{u, v\} = \{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$ is linearly dependent.~~

True.

$\{u, v, w\}$ is linearly independent.

then $x_1u + x_2v + x_3w = 0$. has only one solution $x_1 = x_2 = x_3 = 0$

$$x_1u + x_2v = -x_3w$$

$$x_1u + x_2v = 0$$

\Rightarrow every column of the matrix is a pivot column.

since $\{u, v, w\}$ every column of u, v, w is pivot column.

$\{u, v\}$ also has pivot column is every column.



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

- a) If A is a 2×2 matrix and u, v are vectors in \mathbb{R}^2 such that Au, Av are linearly dependent then u, v also must be linearly dependent.

False.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

True.

$$u \in \text{Span}(v, w) \quad u = c_1v + c_2w$$

$$T(u) = Au$$

$$T(v) = Av$$

$$T(w) = Aw$$

$$Au = c_1Av + c_2Aw$$



MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

John Stone

UB Person Number:

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
 - For full credit solve each problem fully, showing all relevant work.

1 2 3 4 5 6 7 TOTAL GRADE



1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$b = -6$

a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array} \right] \xrightarrow{(-1)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array} \right] \xrightarrow{(1)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array} \right] \quad 0 = b + 6 \quad \underline{b = -6}$$

b) Yes.

Not every column in the matrix is a pivot column.

Therefore, independent.



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1} .

$$\begin{array}{c}
 \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{+(-1)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{+(-2)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \\
 \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \xrightarrow{\cdot(-1)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \xrightarrow{\cdot(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \\
 A^{-1} = \begin{pmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{pmatrix}
 \end{array}$$



3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} A & Q & X \\ B & R & Y \\ C & S & Z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} A_1 + B_1 + C_1 &= 1 \\ A_1 + B_2 + C_2 &= 4 \\ A_2 + B_1 + C_1 &= 3 \end{aligned}$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 1 \end{array} \right) \xrightarrow{(1)(-1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 1 \end{array} \right)$$

$$\begin{aligned} Q(1) + R(1) + S(1) &= 2 \\ Q(1) + R(2) + S(2) &= 5 \\ Q(2) + R(1) + S(1) &= 2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -5 \end{array} \right) \xrightarrow{(1)(-1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -2 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right) \xrightarrow{(2)(-1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 5 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right) \xrightarrow{(1)(-1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array} \right) \xrightarrow{(1)(-1)}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right) \xrightarrow{x} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array} \right) \xrightarrow{R_2+3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array} \right) \xrightarrow{S=5}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 3 \end{array} \right) \xrightarrow{(1)(-1)}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 2 & 1 & -1 & 3 \end{array} \right) \xrightarrow{(1)(-2)}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & 1 \end{array} \right) \xrightarrow{(1)(-1)}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 6 \end{array} \right) \xrightarrow{(1)(-1)}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 6 \end{array} \right) \xrightarrow{(1)(-1)}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 6 \end{array} \right) \xrightarrow{A=8}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 6 \end{array} \right) \xrightarrow{B=-7}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 6 \end{array} \right) \xrightarrow{C=6}$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors \mathbf{u} satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

c) $\left[\begin{array}{cc} T(e_1) & T(e_2) \end{array} \right] \quad T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Standard Matrix $= \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \quad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & -2 \\ 0 & 1 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

d)

$x_1 - 2x_2 = 1$	$\left \begin{array}{cc c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array} \right $
$x_1 + x_2 = 10$	
$x_1 - 3x_2 = -2$	



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array} \right) \xrightarrow{\text{R}_3 - 3\text{R}_1} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array} \right) \xrightarrow{\text{R}_2 - 2\text{R}_1} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 4 \end{array} \right) \xrightarrow{\text{R}_3 - 4\text{R}_2} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{array} \right) \xrightarrow{\text{R}_2 \leftrightarrow \text{R}_3} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \xrightarrow{\text{R}_3 - 3\text{R}_1} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & -2 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & -2 \end{array} \right) \xrightarrow{\text{R}_2 - 2\text{R}_1} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & -2 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & -2 \end{array} \right) \xrightarrow{\text{R}_3 + 2\text{R}_2} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{array} \right)$$

No p.v. pos

Not one-to-one

One-to-one

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \mathbf{v}_1 = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \mathbf{v}_2$$

$$\begin{array}{r} -2 & 2 & -1 \\ -2 & 2 & 0 \\ 0 & 4 & -4 \\ -6 & 8 & -2 \end{array} \quad \begin{array}{r} -4 & 4 & -2 \\ -4 & -4 & 0 \\ 0 & 8 & -8 \\ -2 & 16 & 4 \end{array} \quad \begin{array}{l} \Rightarrow \\ \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{array}$$

$$\boxed{\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}}$$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True

Since $w + u$ is in the span of u, v ,

b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

True

To be a linearly dependent set, for $x_1u + x_2v + x_3w = 0$

$x_1, x_2, x_3 = 0$ must be the only solution.

If $\{u, v\}$ was a linearly dependent set, $\{u, v, w\}$ would also

be linearly dependent, since x_3w could be set $x_3 = 0$.

Therefore, since $\{u, v, w\}$ is linearly independent, $\{u, v\}$ must be, too.



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

- a) If A is a 2×2 matrix and u, v are vectors in \mathbb{R}^2 such that Au, Av are linearly dependent then u, v also must be linearly dependent.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

linearly independent

$$\left\{ \begin{array}{l} Au = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ Av = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array} \right. \quad \left. \begin{array}{l} \text{linearly dependent} \\ \text{linearly dependent} \end{array} \right\}$$

False.

- b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

False



**MTH 309T LINEAR ALGEBRA
EXAM 1**

October 3, 2019

Name:

DANIEL WALSH

UB Person Number:

5	0	2	4	4	7	5	2
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1 2 3 4 5 6 7 TOTAL GRADE

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1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

- a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.
 b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array} \right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array} \right]$$

$\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ for $b = -6$

b)

$$\left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{array} \right] \xrightarrow[\text{reduction}]{\text{row}} \left[\begin{array}{ccc|c} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array} \right]$$

$$x_3 = 3x_1 + 2x_2$$

The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not linearly independent



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1} .

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 6 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$



3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad A^T C = B$$

$$(A^T)^{-1} \cdot A^T C = (A^T)^{-1} \cdot B$$

$$C = (A^T)^{-1} \cdot B$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$C = (A^{-1})^T \cdot B$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = (A^{-1})^T \cdot B = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (-2+4+6) & (-4+5+4) & (-6+4+2) \\ (3-4-6) & (6-5-4) & (9-4-2) \\ (-1+4+3) & (-2+5+2) & (-3+4+1) \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors \mathbf{u} satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & -2(0) \\ 1 & +0 \\ 1 & -3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 & -2(1) \\ 0 & +1 \\ 0 & -3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

Standard matrix of $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\left\{ \begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & 1 & 10 & -2 \\ 1 & -3 & -2 & 0 \end{array} \right\} \quad | \quad \left\{ \begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 \end{array} \right\}$

$$\left\{ \begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 3 & 9 & 0 \\ 0 & -1 & -3 & 0 \end{array} \right\} \quad | \quad \left\{ \begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right\}$$

$$\left\{ \begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right\} \quad | \quad \left\{ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right\}$$

$$x_1 = 1 + 2x_2$$

$$x_2 = 3$$

$$\begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} x_1 = 7 \\ x_2 = 3 \end{array}$$

$$\text{Null}(A) = \{0\}$$

$$\mathbf{u} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

all vectors satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$: $\left\{ \begin{bmatrix} 7 \\ 3 \end{bmatrix} \right\}$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

$$\text{a) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Let } v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = v_1 + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Not 1-to-1; not all columns are pivot columns

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$T_4(v)$ is 1-to-1; pivot columns in all columns.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x_1 = 2x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

$$w + u = c_1 u + c_2 v$$

$$w = (c_1 - 1)u + c_2 v$$

TRUE, w is a linear combination of u and v

b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

By definition,
 u, v, w are not constant multiples of each other: $u \neq c_1 v$ for
any c_1
 u, v are not constant multiples of each other

$\therefore u, v$ are linearly independent

TRUE



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

- a) If A is a 2×2 matrix and u, v are vectors in \mathbb{R}^2 such that Au, Av are linearly dependent then u, v also must be linearly dependent.

FALSE

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Au = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad Av = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Au, Av are linearly dependent

- u, v are not linearly dependent

- b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

$$u = c_1 v + c_2 w$$

$$\begin{aligned} \text{By linearity: } T(u) &= T(c_1 v + c_2 w) = T(c_1 v) + T(c_2 w) \\ &= c_1 T(v) + c_2 T(w) \end{aligned}$$

$T(u)$ is a constant multiple of $T(v)$ and $T(w)$

$\therefore T(u) \in \text{Span}(T(v), T(w))$

TRUE



MTH 309T LINEAR ALGEBRA
EXAM 1

October 3, 2019

Name:

Albi Tocilla

UB Person Number:

5	0	1	5	3	4	7	8
0	1	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1 2 3 4 5 6 7 TOTAL GRADE

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$$\begin{array}{r} -3(6-b) \\ -18+3b \\ \hline 8 \\ 16-12=4 \end{array}$$

$$2 - 2\left(\frac{6-b}{8}\right)$$

$$2 - \frac{12+6-b}{8}$$

$$2 - 2\left(\frac{6-b}{8}\right)$$

$$2 - \frac{12+2b}{8}$$



1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$$\begin{aligned} \frac{-1}{8}(6-b) &= \frac{-b+6}{8} \\ 6-b &\stackrel{?}{=} 0 \end{aligned}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & 6 \end{array} \right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 5 & -2 & b+4 \end{array} \right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -8 & b-6 \end{array} \right] \xrightarrow{R_3 \cdot \frac{1}{8}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{b-6}{8} \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{b-6}{8} \end{array} \right] \xrightarrow{R_1 - 3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-18+3b}{8} \\ 0 & 1 & 0 & 2 - \frac{12+2b}{8} \\ 0 & 0 & 1 & \frac{b-6}{8} \end{array} \right] \xrightarrow{\therefore} \begin{aligned} x_1 &= \frac{-18+3b}{8} \\ x_2 &= \frac{4+2b}{8} \\ x_3 &= \frac{b-6}{8} \end{aligned}$$

Since b is in the numerator of all solutions \therefore if $b \in \mathbb{R}$ then $w \in \text{Span}(v_1, v_2, v_3)$

- b) The set $\{v_1, v_2, v_3\}$ is linearly independent because there is a pivot column in every column as shown in part A where we reduced the matrix in row echelon form.



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad \begin{array}{l} 6 - 2(-1) = 2 \\ 6 - 2(1) = -2 \end{array}$$

Compute A^{-1} .

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$\rightarrow R_1 - R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$

$\rightarrow R_2 + R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$

$\therefore A^{-1} = \boxed{\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$



3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ -2 & -2 & 1 \end{bmatrix} \quad (\text{calculated in problem 2})$$

Using the fact that: $(A^T)^{-1} = (A^{-1})^T$

$$\therefore (A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \rightarrow (A^T)^{-1}(A^T)C = B(A^T)^{-1}$$

$$C = B(A^T)^{-1}$$

$$\therefore C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1(-2) + 3(2) - 3, & -2 + 3, & 2 - 4 + 3 \\ 4(-2) + 5(3) - 4, & 4 - 5 + 4, & 4(2) - 5(2) + 4 \\ 3(-2) + 2(3) - 1, & 3 - 2 + 1, & 3(2) - 2(2) + 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 + 6 - 3, & -1 + 3, & -2 + 3 \\ -8 + 15 - 4, & -5, & 8 - 10 + 4 \\ -1, & 0, & 6 - 4 + 1 \end{bmatrix} = \begin{bmatrix} 1, & 2, & 1 \\ 3, & -5, & 2 \\ -1, & 0, & 3 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 1, & 2, & 1 \\ 3, & -5, & 2 \\ -1, & 0, & 3 \end{bmatrix}$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors \mathbf{u} satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \therefore e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, e_2 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

\therefore Std Matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $A(\mathbf{u}) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array} \right] \rightarrow R_2 - R_1 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array} \right] \rightarrow \frac{R_3}{3} \rightarrow$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \end{array} \right] \rightarrow R_1 + 2R_2 \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \end{array} \right] \rightarrow \begin{array}{l} x_1 = 7 \\ x_2 = 3 \\ x_3 = -1 \end{array}$$

Since $\mathbb{R}^2 \rightarrow \mathbb{R}^3$

$\mathbf{u} = \begin{bmatrix} 7 \\ 3 \\ -1 \end{bmatrix}$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{array}{l} 2 \\ 6 \\ 4 \end{array}$$

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since A has a pivot position in every column it is 1 to 1

$$\text{b)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A is not one to one since no pivot position in every column



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

False

$w + u \in \text{Span}(u, v)$
That means w is linearly dependent for
 $w \in \text{Span}(u, v) \wedge w \notin \text{Span}(u, v)$, a vector

b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

False

$\{u, v, w\}$ is only linearly independent if it has only one solution to the homogeneous equation $x_1u + x_2v + x_3w = 0$

Counter Example:

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & * & * \end{array} \right]$$

$\therefore \{u, v\}$ is not necessarily linearly independent



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

- a) If A is a 2×2 matrix and u, v are vectors in \mathbb{R}^2 such that Au, Av are linearly dependent then u, v also must be linearly dependent.

If Au, Av are linearly dependent that means the homogeneous equation has infinite solutions, for both $u \neq v$ that means u or v are multiples of one another \therefore not linearly independent

\therefore true

- b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

(True), transformations are linear operators

$$T(u+v) = T(u) + T(v)$$

\therefore if u is in $\text{Span}(v, w)$

the $T(u)$ is in $\text{Span}(T(v), T(w))$



MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Marissa Loniewski

UB Person Number:

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
 - For full credit solve each problem fully, showing all relevant work.

5	0	2	3	3	5	4	7
0	●	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	●	2	2	2	2	2
3	3	3	3	●	3	3	3
4	4	4	4	4	4	●	4
5	●	5	5	5	5	●	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

1

2

3

4

5

6

7

TOTAL

GRADE



1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

- a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.
 b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

$$\left[\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \right] \Rightarrow \mathbf{w} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3$$

$$\begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\xrightarrow{\substack{R1 \cdot -2 \\ R3 \cdot -1}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & b-4 \end{array} \right] \xrightarrow{\substack{R2+R1 \\ R3-R2}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -b-4-2 \end{array} \right] \xrightarrow{\substack{R4 \cdot -1 \\ If b=-6}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$-b-6=0$$

$$\boxed{-6=b}$$

$$c_1 - c_2 + c_3 = -2$$

$$c_2 + 2c_3 = 2$$

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not linearly independent

because c_3 is a free variable.

Infinite # of sltns if $b = -6$.

No sltns if $b \neq -6$.



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1} .

$$\xrightarrow{\substack{R1 \leftarrow R1 - R2 \\ +R3}} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{R2 \leftarrow R2 - R3 \\ +R3}} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{R2 \leftarrow R2 + R1 \\ R3 \leftarrow R3 - 2R1}} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$\xrightarrow{\substack{R1 \leftarrow R1 - R2 \\ R3 \leftarrow R3 - 2R2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \xrightarrow{\substack{R3 \leftarrow R3 - 2R1 \\ +R1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \xrightarrow{\substack{R1 \leftarrow R1 - R2 \\ R2 \leftarrow R2 - R3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$



3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad (A^T)C = AB \\ A \qquad \qquad \qquad I \quad C = AB \\ C = AB$$

$$\boxed{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}} \quad \boxed{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}} = \boxed{\begin{bmatrix} 3 & 1 & 1 \\ 4 & 4 & 4 \\ 5 & 8 & 7 \end{bmatrix}} = C$$

$$1(1) - 1(4) + 2(3) \\ = 1 - 4 + 6 \\ = 3$$

$$1(2) - 1(5) + 2(2) \\ = 2 - 5 + 4 \\ = 1$$

$$1(3) - 1(4) + 2(1) \\ = 3 - 4 + 2 \\ = 1$$

$$1(1) + 0 + 1(3) = 4 \\ 1(2) + 0 + 1(2) = 4 \\ 1(3) + 0 + 1(1) = 4$$

$$0 + 2(4) - 1(3) = 8 - 3 = 5$$

$$0 + 2(5) - 1(2) = 10 - 2 = 8$$

$$0 + 2(4) - 1(1) = 8 - 1 = 7$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{pmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors \mathbf{u} satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) Standard Matrix:

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

b) $T(\vec{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$A\vec{u} = b$$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \vec{u} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$\begin{array}{l}
 \text{R2} \cdot -1 \\
 + \text{R2} \\
 \hookrightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array} \right] \xrightarrow{\substack{\text{R1} \leftrightarrow \text{R3} \\ \text{R2} \leftrightarrow \text{R3}}} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array} \right] = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array} \right] \xrightarrow{\text{R3} \cdot -1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{array} \right]
 \end{array}$$

$$= \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array} \right] \Rightarrow \begin{array}{l} u_1 - 2u_2 = 1 \\ u_2 = 3 \\ u_1 - 6 = 1 \end{array} \quad \begin{array}{l} u_1 - 2(3) = 1 \\ u_1 = 7 \end{array}$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$$\text{a)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\text{b)} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

1-1:

Pivot pos. in every column

(a)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\substack{R_2 \leftrightarrow R_3 \\ R_2 - \frac{1}{2}R_1}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\substack{R_2 - R_3 \\ R_3 - R_2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

T_A is one-to-one because A has a pivot position in every column.

$$\text{Let } \vec{w} = \vec{v}_1 - \vec{v}_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\substack{R_2 - \frac{1}{2}R_1 \\ R_3 - 3R_1}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\substack{R_3 - R_2 \\ R_2 - R_3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\text{Null}(A) = \text{Span} \left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right)$$

$$\begin{aligned} \vec{v}_1 &= \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \\ \vec{v}_2 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\text{Let } \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{b)} A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{\substack{R_1 \leftrightarrow R_3 \\ R_2 - \frac{3}{2}R_1 \\ R_3 - R_2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \end{aligned}$$

T_A is not one-to-one because A does not have a pivot position in every column.

$$T_A(\vec{v}_1) = T_A(\vec{v}_2)$$

$$T_A(\vec{v}_1) - T_A(\vec{v}_2) = 0$$

$$T_A(\vec{v}_1 - \vec{v}_2) = 0$$

$$\begin{aligned} A\vec{w} &= 0 \\ \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} &= 0 \\ \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

$$\text{Let } \vec{w} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \vec{v}_1 - \vec{v}_2$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \vec{v}_2 = \vec{v}_1$$

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \vec{v}_1$$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True.

$w \in \text{Span}(u, v) \Leftrightarrow \begin{cases} w + u = c_1 u + c_2 v \\ w = c_1 u + c_2 v - u \\ w = (c_1 - 1)u + c_2 v \text{ where } c_1, c_2 \in \mathbb{R} \end{cases}$

If $w + u \in \text{Span}(u, v)$
because w can be written as
linear combination
of u and v .

b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

False.

Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
 $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
 $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
 $A \vec{x} = \begin{bmatrix} b \\ b \\ b \end{bmatrix}$
 $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$A \vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \\ b \\ b \end{bmatrix}$

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be exact soltn. Then
 $\{\vec{u}, \vec{v}\}$ is not a sltn to the eqn and \therefore not linearly independent.

True., $\vec{u}, \vec{v}, \vec{w}$ cannot be written as a linear combination of each other so \vec{u}, \vec{v} cannot be written as a linear combination of each other.
 $\therefore \{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent
 $\Rightarrow \{\vec{u}, \vec{v}\}$ ~~*~~



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and \mathbf{u}, \mathbf{v} are vectors in \mathbb{R}^2 such that $A\mathbf{u}, A\mathbf{v}$ are linearly dependent then \mathbf{u}, \mathbf{v} also must be linearly dependent.

False.

Counterexample :

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\vec{u} = T(\vec{u}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\vec{v} = T(\vec{v}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\vec{u} = A\vec{v}$$

$$T(\vec{u}) = T(\vec{v})$$

$$T(\vec{u}) - T(\vec{v}) = \mathbf{0}$$

$$T(\vec{u} - \vec{v}) = \mathbf{0}$$

$$\vec{u}, \vec{v} \in \text{Null}(A)$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that \mathbf{u} is in $\text{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\text{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

$$\vec{u} \in \text{Span}(\vec{v}, \vec{w}) \Rightarrow T(\vec{u}) \in \text{Span}(T(\vec{v}), T(\vec{w}))$$

$$\vec{u} = c_1 \vec{v} + c_2 \vec{w}$$

$$T(\vec{u}) = T(c_1 \vec{v} + c_2 \vec{w})$$

$$= T(c_1 \vec{v}) + T(c_2 \vec{w})$$

$$= c_1 T(\vec{v}) + c_2 T(\vec{w}) \Rightarrow T(\vec{u}) \in \text{Span}(T(\vec{v}), T(\vec{w}))$$

True.



MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Judy Mei

UB Person Number:

5	0	2	3	7	0	3	2
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
 - For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE
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1. (20 points) Consider the following vectors in \mathbb{R}^3 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array} \right] \xrightarrow{R_1 \times -1} \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array} \right] \xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array} \right] \xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array} \right] \xrightarrow{\text{swap } R_1 \text{ and } R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array} \right]$$

$$\xrightarrow{R_1 + (-R_3)} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{\text{restore } R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \text{free}$$

b) It is linearly dependent since the reduced form contains a free variable meaning it has infinitely many solutions.

a) $b = 2$



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1} .

$$\begin{array}{l}
 \xrightarrow{\quad\quad\quad} \\
 \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & x_1 & y_1 & z_1 \\ 1 & 0 & 1 & x_2 & y_2 & z_2 \\ 0 & 2 & -1 & x_3 & y_3 & z_3 \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \\
 -1+1 \\
 1=1=0 \\
 0+0=0
 \end{array}$$

$$\begin{array}{l}
 x_1 - x_2 + 2x_3 \quad | \quad 1 \\
 x_1 + 0x_2 + x_3 \quad | \quad 0 \\
 0x_1 + 2x_2 + 2x_3 \quad | \quad 0
 \end{array} \xrightarrow{\quad\quad\quad} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2 + (-R_1)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right] \xrightarrow{-1+1=0}$$

$$\xrightarrow{R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 + (-2R_2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 1 \end{array} \right] \xrightarrow{*1/3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 & 0 & 1/3 \end{array} \right] \xrightarrow{-1+1=0}$$

$$\xrightarrow{+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5/3 & 0 & 0 \\ 0 & 0 & 1 & -2/3 & 0 & -2/3 \end{array} \right] \xrightarrow{-R_1 + (E R_3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & -5/3 & 0 & 0 \\ 0 & 0 & 1 & -2/3 & 0 & -2/3 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -5/3 & -5/3 & 1/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$



3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad Ax = B$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$



4. (20 points) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{pmatrix}$$

a) Find the standard matrix of T .

b) Find all vectors \mathbf{u} satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = [T(e_1) \ T(e_2)]$ where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{aligned} T(e_1) &= T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ T(e_2) &= T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \end{aligned}$$

$$\boxed{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

b)



5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors \mathbf{v}_1 and \mathbf{v}_2 such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a)

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$



6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

- a) If u, v, w are vectors in \mathbb{R}^3 such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True,

- b) If u, v, w are vectors in \mathbb{R}^3 such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

True. In order to be linearly independent, each column is a pivot column. So if you do set $\{u, v\}$, they will both still have a pivot column in each column.



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

- a) If A is a 2×2 matrix and u, v are vectors in \mathbb{R}^2 such that Au, Av are linearly dependent then u, v also must be linearly dependent.

False,

- b) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in $\text{Span}(v, w)$ then $T(u)$ must be in $\text{Span}(T(v), T(w))$.

True,