MTH-309T-F19-EX1-024-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	1	5	3	4	7	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
18	10	5	18	14	3	3	--	10	78	B

$-3(6-b)$

$\frac{-18+36}{8}$

$16-12=4$

$2-2\left(\frac{6-b}{8}\right)$

$2-\frac{12+6}{8}$

$2-2\left(\frac{6-b}{8}\right)$

$2-\frac{12+2b}{8}$

MTH-309T-F19-EX1-024-P01

$\frac{-1}{8}\,6-6=\frac{-6+6}{8}$

$\frac{6+6}{8}$

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1=\begin{bmatrix}1\\0\\2\end{bmatrix},\quad v_2=\begin{bmatrix}-1\\1\\-3\end{bmatrix},\quad v_3=\begin{bmatrix}1\\2\\0\end{bmatrix},\quad w=\begin{bmatrix}-2\\2\\b\end{bmatrix}$$

a) Find all values of b such that $w\in\operatorname{Span}(v_1,v_2,v_3)$.

b) Is the set $\{v_1,v_2,v_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c}1&-1&1&-2\\0&1&2&2\\2&-3&0&6\end{array}\right]\xrightarrow{R_3-2R_1}\left[\begin{array}{ccc|c}1&-1&1&-2\\0&1&2&2\\0&\cancel{5}\ -1&-2&6+4\end{array}\right]\xrightarrow[R_1-5R_2]{R_1+R_2}\left[\begin{array}{ccc|c}1&0&3&0\\0&1&2&2\\0&0&-8&6-6\end{array}\right]\xrightarrow{R_3\cdot\frac{-1}{8}}$

$\left[\begin{array}{ccc|c}1&0&3&0\\0&1&2&2\\0&0&1&\frac{6-6}{8}\end{array}\right]\xrightarrow[R_2-2R_3]{R_1-3R_3}\cancel{\left[\begin{array}{ccc|c}1&0&0&\frac{-18+36}{8}\\0&1&0&2-\frac{12+2b}{8}\\0&0&1&\frac{6-b}{8}\end{array}\right]}\to\ \therefore\ x_1=\frac{-18+3b}{8}$

$x_2=\frac{4+2b}{8}$

$x_3=\frac{6-b}{8}$

Since b is in the numerator of all solutions $\therefore$ if $b\in\mathbb{R}$ then $w\in\operatorname{Span}(v_1,v_2,v_3)$

b) The set $\{v_1,v_2,v_3\}$ is linearly inde~~pendent~~dent because there is a pivot column in every column as shown in part A where we reduced the matrix in row echelon form.

Correct conclusion, but it comes from an incorrect row reduction.

Again, this is correct given the reduced matrix you obtained, but the matrix itself is wrong.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$0-2(-1)=2$
$0-2(1)=-2$

$1-2$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to R_2 - R_1 \to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \to R_1 + R_2 \\ \\ \to R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} \to R_1 - R_3 \to \\ \to R_2 + R_3 \to \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

✓

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}} \qquad A = \underset{3\times 3}{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \text{ (calculated in problem 2)}$$

Using the fact that: $(A^T)^{-1} = (A^{-1})^T$ ✓ $\quad (A^T)^{-1} \cdot C$

$$\therefore (A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \rightarrow \begin{array}{l} (A^T)^{-1}(A^T)C = B(A^T)^{-1} \\ C = B(A^T)^{-1} \end{array}$$

$$\therefore C = \underset{B}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}} \underset{(A^T)^{-1}}{\begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}} = \begin{bmatrix} 1(-2)+3(2)-3, & 1-2+3, & 2-4+3 \\ 4(-2)+5(3)-4, & 4-5+4, & 4(2)-5(2)+4 \\ 3(-2)+2(3)-1, & 3-2+1, & 3(2)-2(2)+1 \\ -6+6 & & 6-4+1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+6-3, & -1+3, & -2+3 \\ -8+15-4, & -5, & 8-10+4 \\ -1, & 0, & 6-4+1 \end{bmatrix} = \begin{bmatrix} 1, & 2, & 1 \\ 3, & -5, & 2 \\ -1, & 0 & 3 \end{bmatrix}$$

$$\therefore C = \cancel{\begin{bmatrix} 1, & 2, & 1 \\ 3, & -5, & 2 \\ -1, & 0, & 3 \end{bmatrix}}$$

There are mistakes in the matrix multiplication too...

☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$, $\therefore e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $e_2 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\therefore$ Std Matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $A(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \to \frac{R_3}{3} \to$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow[R_3 + R_2]{R_1 + 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \to$ $x_1 = 7$, $x_2 = 3$, $x_3 = x_3$

Since $\mathbb{R}^2 \to \mathbb{R}^3$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ ✓

There is no x_3

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{matrix} = 2 \\ = 6 \\ = 9 \end{matrix}$

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{matrix} \to R_1 - R_2 \to \\ \\ \to R_3 - R_2 \to \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \to R_1 + R_3 \\ \to R_2 - R_3 \end{matrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since A has a pivot position in every column it is 1 to 1 ✓

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \to R_1 - R_2 \to \\ \to R_3 - R_2 \to \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

A is not one to one since no pivot position in every column ✓

$\underline{v_1, v_2}$?

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐	☐

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False is $w + u \in \text{Span}(u, v)$

that means w is linearly dependent for

u & $\therefore$ $w \notin \text{Span}(u, v)$

?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

False, $\{u, v, w\}$ is only linearly independent if it has only one solution to the homogenus equation $x_1 u + x_2 v + x_3 w = 0$ ✓

Counter Example: $\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \to \left[\begin{array}{cc|c} u_1 & 0 & * \\ 0 & v_2 & * \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} u_1 & 0 & * \\ 0 & v_1 & * \\ 0 & * & * \end{array}\right]$?

$\therefore \{u, v\}$ is ~~not~~ neccacirly linearly independent

0	1	2	3	4	5	6	7	8	9	10
			✗							

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

If Au, Av are linearly dependent that means the homogenus equation has infinite solutions, for both u & V that means u or v are multiples of one another & ∴ not linearly independent

∴ ~~true~~

Not necessarily.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v})$, $T(\mathbf{w})$).

True, ✓ transformations are linear operators

$T(u+v) = T(u) + T(V)$

∴ if u is in Span(v, w) the T(u) is in Span(T(v), T, w)

why?

0 1 2 3 4 5 6 7 8 9 10