MTH-309T-F19-EX1-059-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	6	4	8	1	0
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} = A$$

A) $c_1 v_1 + c_2 v_2 + c_3 v_3 = w$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

Aug.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 \to 2R_1 - R_3} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -4-b \end{bmatrix} \xrightarrow{R_1 \to R_2 + R_1}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -4-b \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -6-b \end{bmatrix}$$

For w to be in the span(v_1, v_2, v_3) b must equal -6.

B) The solution to $Ax = 0$ is

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = -3x_3$

$x_2 = -2x_3$

$x_3 = x_3$

The presence of a free variable suggests infinitely many solutions to $Ax = 0$. Since the homogenous equation does not have one trivial solution it is Linearly Dependent

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$A \cdot A^{-1} = I_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] R_2 \to R_2 - R_1 \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] R_1 \to R_2 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] R_3 \to R_3 - 2R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & -2 & 1 \end{array}\right] R_3 \to \frac{1}{3} R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & -2/3 & 1/3 \end{array}\right] R_2 \to R_2 + R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 2/3 & -2/3 & 1/3 \end{array}\right] R_1 \to R_1 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & 5/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 2/3 & -2/3 & 1/3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2/3 & 5/3 & -1/3 \\ -1/3 & 1/3 & 1/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$(A^T)^{-1} A^T C = B(A^T)^{-1}$

$C = B(A^T)^{-1}$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Inverse of A^T

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_3 + R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_2 + R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to -R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \qquad \text{Inverse } A^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

3-4-2
-10
-7-2

$(A^T)^{-1} \cdot B = C$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0+4+6 & 0+5+4 & 0+4+2 \\ 1-4-6 & 2-5-4 & 3-4-2 \\ 0+4+3 & 0+5+2 & 0+4+1 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 10 & 9 & 6 \\ -9 & 7 & -9 \\ 7 & 7 & 5 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

A) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$$\boxed{\text{Standard matrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

B) $x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} R_2 \to R_1 - R_2 \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & -9 \\ 1 & -3 & -2 \end{bmatrix} R_3 \to R_1 - R_3$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & -9 \\ 0 & 1 & 3 \end{bmatrix} R_2 \to 3R_3 + R_2 \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} R_1 \to 2R_3 + R_1$$

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} R_3 \longleftrightarrow R_2 \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = 7$
$x_2 = 3$

$$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

$A \cdot V = b$ $\quad T_A(V_1) = T_A(V_2)$

$T_A(V_1) - T_A(V_2) = 0$

$T_A(V_1 - V_2) = 0$

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

one-to-one pivot position in every column

A) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_2 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 \to -\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 2R_3 + R_2}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to 2R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ This is one-to-one

So $T_A(V_1) - T_A(V_2) = 0$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ $\quad$ Nul A: Span(0)

V_1 can be $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ V_2 can be $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

B) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_2 - R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

We can see not every column will be a pivot so $T_A(v) = AV$ is not one-to-one.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\boxed{\text{True}}$

$w + u \in \text{Span}(u, v)$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ True

$w \in \text{Span}(u, v)$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ True

Since $w + u$ is in the Span of u, v their combination of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ can be made from $c_1 u + c_2 v = w + u$. If you let $c_1 = 1$ and $c_2 = 1$

$\cancel{u} + v = w + \cancel{u}$

$v = w$

Since $v = w$, w is in span of v which makes $w \in \text{Span}(u, v)$ True

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Lin independent means one trivial solution To $Ax = b$

$\boxed{\text{TRUE}}$

Since u, v, and w are linearly independent that means all columns on A are pivot columns. So if you were to remove a vector w for example and solve for independence with u and v you are left with a 3×2 matrix with pivot positions in every column so they are Independent.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\boxed{\text{FALSE}}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Au = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Av = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

then $Av = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Aw = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Since these two vectors are NOT scalar multiples of each other this statement is false.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

$\boxed{\text{TRUE}}$ If U is in span of V, W this means

$c_1 V + c_2 W = U$

A linear transformation $T_A(c_1 V + c_2 W) = T_A(u)$

is applied to both sides

$c_1 T_A(V) + c_2 T_A(W) = T_A(u)$

Any linear combination of $T(V)$ and $T(W)$ will get you $T_A(u)$

0 1 2 3 4 5 6 7 8 9 10