MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Umar Ahmed

UB Person Number:

5 0 2 6 3 5 0 9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.

- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

8	9	3	10	15	5	5	2	5	60	C-
1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\quad -2v_1 + 4v_2 - v_3 = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$?

$$-2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -16 \end{bmatrix}$$

$-4 -12$) ?

b) aug. matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}$$

?

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}$$

$(-2)\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 3 & 6 \end{bmatrix}$

$(-1)\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}(1)$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}(-1)$

$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}(\frac{1}{3})$

∴ The set $\{v_1, v_2, v_3\}$ is lin. independent b/c there is only one solution to the homogeneous matrix

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-1)} \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{2})} \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{(\frac{1}{2})} \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} & | & -1 & 1 & -\frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{(-2)} \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & | & -1 & 1 & -\frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{(\frac{1}{2})} \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix} \xleftarrow{(-2)}$$

$$\begin{bmatrix} 1 & -1 & 0 & | & -3 & 4 & -2 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix} \xrightarrow{}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -2 & 3 & -1 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$3 \times 3 \qquad 3 \times 3$

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$3 \times 3 \qquad\qquad\qquad x_1 \; x_2 \; x_3 \qquad\qquad 3 \times 3$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad \checkmark$$

$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad \checkmark$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

?

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$x_1 = 2x_2 + 1$

a) $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 - 0 \\ 1 + 0 \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2 \\ 0 + 1 \\ 0 - 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) [?

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$(-1) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 2 & 0 \end{bmatrix}$

$(-3) \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

$(2) \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

$(1) \, (\tfrac{1}{4}) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$(-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is one-to-one ✓

$(\tfrac{1}{2}) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} (\ )$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} (-1)$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ · not one-to-one ✓

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} -3 \\ -16 \\ -17 \end{bmatrix}$ ← $T_A(v_1) \neq T_A(v_2)$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2+0 \\ 0+4+12 \\ 3+8+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$

$T_A(v_1) - T_A(v_2) \neq 0$

$\begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

$$\begin{bmatrix} 1 \\ \end{bmatrix}\begin{bmatrix} \\ \end{bmatrix}\begin{bmatrix} \\ \end{bmatrix}$$

False

$$U \quad V$$

$$W = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \end{bmatrix}\right)$$

it is: $w = v - u$

$$W + U = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$
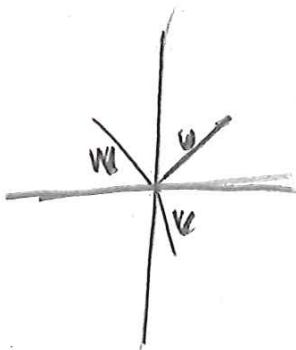
$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

is in $\text{Span}(U, V)$, but $\underline{W \text{ not is } \text{Span}}$

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

True! ✓ | U, V, W are 3 unique vectors, so even if only 2
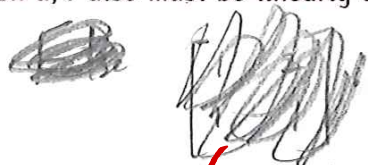? | of them were in the set, it would still be linearly
   | independent

still independent

7. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and u, v are vectors in $\mathbb{R}^2$ such that Au, Av are linearly dependent then u, v also must be linearly dependent.

$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

False ✓   A single vector such as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

is linearly independent. If $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

then it is linearly dependent.

ok, but this problem asks about (in)dependence of __two__ vectors.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $u, v, w \in \mathbb{R}^2$ are vectors such that u is in Span(v, w) then $T(u)$ must be in Span$(T(v), T(w))$.

True. ✓   All 3 vectors are transformed by the same transformation, so properties are not changed. -They are still scalar multiples

being in the span does not mean that one vector is a multiple of another

0 1 2 3 4 5 6 7 8 9 10