MTH-309T-F19-EX1-047-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	3	2	0	2	5	3
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

(a) $\begin{matrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{matrix}$ $(-\infty, \infty)$ – because V_1, V_2, V_3 can be multiplied by an infinite amount of different scalars

$$V_1 \begin{vmatrix} 1 \\ 0 \\ 2 \end{vmatrix} + V_2 \begin{vmatrix} -1 \\ 1 \\ -3 \end{vmatrix} + V_3 \begin{vmatrix} 1 \\ 2 \\ 0 \end{vmatrix} = \begin{matrix} -2 \\ 2 \\ b \end{matrix}$$

(B) $V_1 \begin{vmatrix} 1 \\ 0 \\ 2 \end{vmatrix} + V_2 \begin{vmatrix} -1 \\ 1 \\ -3 \end{vmatrix} + V_3 \begin{vmatrix} 1 \\ 2 \\ 0 \end{vmatrix}$ → the set $\{V_1, V_2, V_3\}$ is not linearly independant because V_1, V_2, and V_3 are scalar multiples of each other and therefore a vector other than $\{0\}$ can be in thier Nul space

$$\begin{vmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{vmatrix}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{vmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{vmatrix}$$

$R_1 \times -1 + R_2$ $\quad$ $R_2 \times -2 + R_3$ $\quad$ $R_3 \times 1 + R_2$

$$\begin{vmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{vmatrix}$$

$R_2 \times 1 + R_1$

$R_3 \times -2 + R_1$

$$A^{-1} = \begin{vmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$A^T C = B \to C = B(A^T)^{-1}$$

$$C = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$\begin{vmatrix} (-2+2+6), & (3-2-6), & (-1+2+3) \\ (-8+5+8), & (12-5-8), & (-4+5+4) \\ (-6+2+2), & (9-2-2), & (-3+2+1) \end{vmatrix}$$

$$\begin{vmatrix} 6 & -5 & 4 \\ 5 & -1 & 5 \\ -2 & 5 & 0 \end{vmatrix}$$

$$C = \begin{vmatrix} 6 & -5 & 4 \\ 5 & -1 & 5 \\ -2 & 5 & 0 \end{vmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $\begin{vmatrix} 1 & 2 \\ 1 & 1 \\ 1 & -3 \end{vmatrix}$

b) $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{vmatrix} \xrightarrow{R_1 \times -1 + R_2 + R_2 \times -1} \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & -9 \\ 1 & -3 & -2 \end{vmatrix} \xrightarrow{R_1 \times -1 + R_3} \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & -9 \\ 0 & -5 & -3 \end{vmatrix} \xrightarrow{R_2 \times 5 + R_3}$

$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & -9 \\ 0 & 0 & -48 \end{vmatrix} \to \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

There are no vectors (u) that satisfy

$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{vmatrix}$$

$R_1 \times -3 + R_3$ $R_2 \times \frac{1}{2}$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{vmatrix} \quad \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \checkmark$$

$R_2 \times -1$ $R_3 \times \frac{1}{2}$

a) A is one to one because it has a pivot column in every column and the Null of A is equal to zero

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$\ast$ Nul(A) = 0 proof

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$

$R_1 \times -3 + R_3$ R_2

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} \to$$ b) $T_A(v) = Av$

is not one to one because it doesn't have a pivot in every column and it has infinite solutions for it's Nul space

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 2 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{vmatrix} \cdot \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 2 \\ 8 \\ 10 \end{vmatrix} = \begin{vmatrix} 2 \\ 6 \\ 9 \end{vmatrix}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

False, counter example

$u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$w + \begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ but $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~False - Counter example~~

~~$u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v =$~~

True, because something in linerly independent when the vectors are not scalar multiples of each others and taking out one vector of a set wont make it that the other 2 are now scalar multpoles of each other.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False

$u = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $v = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$ $A = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

$A \cdot u = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$ $A \cdot v = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

False

True, because if $u = c_1 v + c_2 w$ then $T(u) = c_1 T(v) + c_2 T(w)$ because the transfomation is linear

0 1 2 3 4 5 6 7 8 9 10