MTH-309T-F19-EX1-039-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	3	8	3	1	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
12	4	8	20	20	5	3	2	10	81	B

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

1a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cccc} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & \cancel{2(b+4)} \end{array}\right]$ ← $b+6$

2b+4 has to equal 0 or no solution

How did you get this?

$2b+8=0$

$-8 \quad -8$

$\frac{2b}{2} = \frac{-8}{2}$

$\boxed{\cancel{b=-4}}$

1b) No since there is not a pivot column in every column when reduced ✓

also there is a free variable w/ x_3 so infinite solutions and dependent.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

pivot in each row and column so one-to-one and onto

A is invertible

$$\boxed{A^{-1} = \begin{bmatrix} \cancel{0} & \cancel{1} & \cancel{0} \\ 1 & -1 & \cancel{0} \\ 2 & -2 & \cancel{0} \end{bmatrix}}$$

Where did this come from?

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

3×3 , C 3×3 , $B = 3 \times 3$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right]$$

$\downarrow$ row reduction ← calculations?

$$AC = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right]$$

$A^T x = B$

$C = 3 \times 3$

$\underbrace{A^T \cdot C}_{\text{defined}}$ (3×3, 3×3)

$B = 3 \times 3$ ✓

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$ ✓

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $\begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

$$\boxed{\text{standard matrix of } T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$ ✓

b) $T(u) = \begin{matrix} 1 \\ 10 \\ -2 \end{matrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \text{ ✓ } \Rightarrow \text{row reduction } \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 7$

$x_2 = 3$

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

b) $\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a)

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\Downarrow$ row reduction

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

pivot in every column so one-to-one

b)

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\Downarrow$ row reduction

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

not one-to-one since pivot not in every column

$\text{Nul}(A) = \text{span}\left\{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$

$T_A(v_1) = T_A(v_2)$

$v_2 = v_1 + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\boxed{v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$

$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\boxed{v_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False $w = \begin{bmatrix} \\ \\ \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$u = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

w is only in the Span(u, v) - u ?

$w + u \begin{bmatrix} \\ \\ \end{bmatrix}$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True if u, v, w is linearly independent the only solution is zero vector, and has a pivot ~~column~~ position in every column, u, v would be independent as well w/ pivot position in each column and only solution would be zero vector.

This could be explained more clearly...

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~True~~, Au and Av have infinite solutions ?
so u, and v will also be dependent as they will be scaler multiples of each other.

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\underbrace{u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

↑ independent

but $\underbrace{Au = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad Av = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

↑ dependent

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True if u is a linear combination of v, w. Tu is a linear combination of T(v), T(w).
In homework problems T(v) = T(w₁) + T(w₂) + T(w₃) ?
meaning it is a linear combination.

why ?

0 1 2 3 4 5 6 7 8 9 10