MTH-309T-F19-EX1-007-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	3	1	7	0	3
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

10	4	9	9	5	4	7	48	nan
1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$b=6 \quad b=-6$

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

$x_1 v_1 + x_2 v_2 + x_3 v_3 = w$

$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

let $x_1 = 0$, $x_2 = 2$, $x_3 = 0$

let $x_1 = -3$, $x_2 = 0$, $x_3 = 1$

$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & b \end{array}\right] \to \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right]$

$x_1 = -3x_3$ $\quad x_3 = \text{free}$

$x_2 = 2 - 2x_3$ $\quad$ infinite soln

$b = -6$

$\boxed{b = -6, 6}$

b) $x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

since $x_3 = \text{free}$, there are infinitely many solutions

$\Rightarrow$ set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is linearly independent

?

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & -2 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \cdot 1 +$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \cdot -2 \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \cdot 1 + \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \cdot -1$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix} -2 & 2 & -1 \\ -3 & 3 & -1 \\ 2 & -2 & 1 \end{bmatrix}}$$

0 1 2 3 **4** 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad A^T C = B \quad \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 2 & 5 & 0 \\ -1 & -3 & 3 \\ 0 & 5 & 2 \end{bmatrix}}$$

$c_{11} + c_{21} = 1 \qquad c_{11} = 1 - c_{21}$

$-c_{11} + 2c_{31} = 4$

$2c_{11} + c_{21} - c_{31} = 3 \qquad 2(1-c_{21}) + c_{21} + c_{31} = 3$

$2 - c_{21} + c_{31} = 3$

$c_{31} = 1 + c_{21}$

$2(1-c_{21}) + c_{21} - (1 + c_{21}) = 3$

$2 - 2c_{21} + c_{21} - 1 - c_{21} = 3$

$1 - 2c_{21} = 3$

$-2c_{21} = 2 \quad c_{21} = -1 \Rightarrow \boxed{c_{11} = 2, \; c_{21} = -1, \; c_{31} = 0}$

$c_{12} + c_{22} = 2 \qquad c_{12} = 2 - c_{22}$

$-c_{12} + 2c_{32} = 5 \qquad -(2 - c_{22}) + 2(2 - c_{22}) = 5$

$2c_{12} + c_{22} - c_{32} = 2 \qquad -2 + c_{22} + 4 - 2c_{22} = 5$

$2(2 - c_{22}) + c_{22} - c_{32} = 2 \qquad 2 - c_{22} = 5$

$-c_{22} = 3$

$4 - 2c_{22} + c_{22} - c_{32} = 2 \qquad c_{22} = -3$

$4 - c_{22} - c_{32} = 2 \qquad \Rightarrow c_{12} = 5$

$-c_{22} - c_{32} = -2 \qquad c_{22} = -3$

$-c_{32} = -2 + c_{22} \qquad c_{32} = 5$

$c_{32} = 2 - c_{22}$

$c_{13} + c_{23} = 3 \qquad c_{13} = 3 - c_{23}$

$-c_{13} + 2c_{33} = 4 \qquad -(3 - c_{23}) + 2(5 - c_{23}) = 4$

$2c_{13} + c_{23} - c_{33} = 1 \qquad -3 + c_{23} + 10 - 2c_{23} = 4$

$7 + c_{23} - 2c_{23} = 4$

$2(3 - c_{23}) + c_{23} - c_{33} = 1 \qquad -c_{23} = -3$

$6 - 2c_{23} + c_{23} - c_{33} = 1 \qquad c_{23} = 3$

$6 - c_{23} - c_{33} = 1 \qquad \Rightarrow c_{13} = 0$

$-c_{23} - c_{33} = -5 \qquad c_{23} = 3$

$-c_{33} = -5 + c_{23} \qquad c_{33} = 2$

$c_{33} = 5 - c_{23}$

0 1 2 3 4 5 6 7 8 9 10

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4. **(20 points)** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$ $\quad T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ $\quad u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 0 & -1 & -3 \end{array}\right] \cdot 1/4$

$\to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \to \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \end{array}$ $\quad x_1 - 2x_2 = 1$, $x_2 = 3$ $\quad x_1 = 1 + 2x_2 = 1 + 2(3) = 7$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$ or any multiple of $u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

check $T\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7-6 \\ 7+3 \\ 7-9 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $T = \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & 2x_2 & 4x_3 \\ 3x_1 & 4x_2 & 2x_3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot \tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 4 \end{bmatrix} \cdot -3$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \cdot -1 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot \tfrac{1}{2}$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot -2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot -1$

$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ pivot position in every col. so YES, one-to-one

$x_1 + x_2 =$

$x_2 + 2x_3 =$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 32 \\ 28 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ \\ \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \cdot -3 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \cdot -1$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Pivot position NOT in every column, so NOT one-to-one

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+1+0 \\ 0+2+0 \\ 6+4+0 \end{bmatrix} \begin{matrix} 3 \\ 2 \\ 10 \end{matrix}$

$x_1 + x_2 = 2$ $\quad x_1 = 2 - x_2$

$2x_2 + 4x_3 = 6$

$3x_1 + 4x_2 + 2x_3 = 9$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 4 & 6 \\ 3 & 4 & 2 & 9 \end{array}\right] \cdot \tfrac{1}{2} \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 2 & 9 \end{array}\right]$

$3(2 - x_2) + 4x_2 + 2x_3 = 9$

$6 - 3x_2 + 4x_2 + 2x_3 = 9$

$x_2 + 2x_3 = 3$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 9 \end{array}\right]$

$\frac{2x_3}{2} = \frac{3 - x_2}{2}$

$x_3 = \frac{3 - x_2}{2}$

0 1 2 3 4 **5** 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True, linear combination

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True, if set of vectors is linearly independent,
then $\{u, v\}$ are also linearly independent as long
as u or v are not a scaler multiple of the other

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent. False

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$\{u\}, \{v\}$ are only linearly dependent if $u = \vec{0}$, $v = \vec{0}$

Matrix A can be linearly dependent even if u, v are not zero-vectors

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span(T(v), T(w)).

True by linear combination

0 1 2 3 4 5 6 7 8 9 10