MTH-309T-F19-EX1-074-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	3	8	7	0	6
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $\cancel{V_1 + V_3 = w \mid b = 2}$

$\cancel{-V_2 + V_3 = w}$

$2V_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}, b = -6$ $\boxed{b = -6}$

$-3V_1 + V_3 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}, b = -6$

b) The set $\{V_1, V_2, V_3\}$ is linearly independent, because the vectors cannot linearly combine to cancel one another

cannot cancel all terms

$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$

cannot cancel all terms

no scalar multiple

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$[A \mid I] \to [I \mid A^{-1}]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$C = (A^T)^{-1} B \qquad (A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (-2+4+6) & \overset{-4+5+4}{(\cancel{\quad})} & (-6+4+2) \\ (3-4-6) & (6-5-4) & (9-4-2) \\ (-1+4+3) & (-2+5+2) & (-3+4+1) \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\boxed{T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

b) $1 = x_1 - 2x_2$

$10 = x_1 + x_2$

$-2 = x_1 - 3x_2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 7$

$x_2 = 3$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) One to one -> pivot in every column

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ A is one-to-one

b) $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ not one-to-one

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 + 4x_3 \\ 3x_1 + 4x_2 + 2x_3 \end{bmatrix}$$

$=$

no such vectors exist, since $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ are linearly independent

$x_2 = x_3$

$\begin{bmatrix} 1 & 1 \\ 0 & 6 \\ 3 & 6 \end{bmatrix}$ $x_1 = 1$, $x_2 = 1$ $\begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$

$x_1 = 2$, $x_2 = 1$ $\begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True

For any equation $W + u = a_1 u + a_2 V$, $W = a_1 u - u + a_2 V = cu + a_2 V$

$w \in \mathrm{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True

If $\{u, V\}$ were dependent, then $a_1 u + a_2 V + 0W = 0$ wouldprove $\{u, V, w\}$ to be dependent

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u},\mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u},\mathbf{v}$ also must be linearly dependent.

False

$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$AV_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\quad AV_2 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

V_1, V_2 are independent

AV_1, AV_2 are dependent

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v},\mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

$u = a_1 V + a_2 w$ $\qquad$ True

$\cancel{T(u) = a_1 T(v) + a_2 T(w)}$

u, $a_1 V + a_2 w$

are equivolent

vectors

$\therefore$

Their transformations

are the same

0 1 2 3 4 5 6 7 8 9 10