MTH-309T-F19-EX1-035-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	1	7	2	0	5

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
17	8	3	20	20	3	5	--	--	76	B

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\left(\begin{matrix} 1 & & \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{matrix}\right) \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

(-2) $\left[\begin{matrix} 1 & -1 & 1 & 2 \\ 2 & -3 & 0 & b \\ 0 & 1 & 2 & 2 \end{matrix}\right]$

$= \left[\begin{matrix} 1 & -1 & 1 & 2 \\ 0 & -1 & -2 & b+4 \\ 0 & 1 & 2 & 2 \end{matrix}\right]$

$\left[\begin{matrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 4+b \\ 0 & 1 & 2 & 2 \end{matrix}\right]$

$= \left[\begin{matrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 4-b \\ 0 & 0 & 0 & b-2 \end{matrix}\right]$

$\because$ the last row is 0.

$\therefore b-2=0$. $\quad b+6=0$

$\therefore \cancel{b=2}$. $\quad b=-6$

$\therefore b = 2$ such that $w \in \text{Span}(v_1, v_2, v_3)$

b) the set $[v_1, v_2, v_3]$ is not linearly dependant because every column is not a pivot column ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A \cdot A^{-1} = I$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$(-1)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\Rightarrow (-2)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\Rightarrow (-2)\ (1)\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & \cancel{0} \\ 0 & 1 & -1 & -1 & 1 & \cancel{0} \\ 0 & 0 & 1 & 2 & -2 & \cancel{0} \end{array}\right]$$

$$\Downarrow$$

$$\Rightarrow (1)\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \therefore A^{-1} = \begin{bmatrix} -2 & 3 & \cancel{0} \\ 1 & -1 & \cancel{0} \\ 2 & -2 & \cancel{0} \end{bmatrix}$$

0	1	2	3	4	5	6	7	8	9	10
								X		

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ $\therefore$ $A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$ ✓

$\therefore \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \cancel{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

Let $C = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

?

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\therefore$ The Standard Matrix of T.

$T : c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ $\therefore T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\therefore c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\begin{bmatrix} c_1 - 2c_2 \\ c_1 + c_2 \\ c_1 - 3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$. $\therefore T(u) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ ✓

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

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5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a). ~~for T_A to be~~

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

∴ There is a Pivot Pos in every column.

$T_A(v) = Av$ is one to one ✓

for $T_A(v)$ to be one to one the $Nul\{A\} = \{0\}$

$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 4 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$Nul(A) = \{0\}$

$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$

$\therefore \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓

∴ Every column does not have pivot pos. $T_A(v) = Av$ is not one to one.

If $V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 2 & 4 & 16 \\ 3 & 4 & 2 & 17 \end{array}\right] = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\therefore V_2 = \begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix}$ ✓

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True ✓

?

$T(w+u) = T(w) + T(u) = T(u) + T(v)$.

$\therefore T(w) = T(u) + T(v)$.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True: ✓

If $c_1 u + c_2 v + c_3 w = 0$ is linearly independant

it means $\exists u, v, w = 0$. ?

$\therefore$ The set $\{u, v\}$ has to be linearly independent ← why?

because $x_1 u + x_2 v = 0$.

0	1	2	3	4	5	6	7	8	9	10
			✗							

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False ✓

$\therefore Au = A[u] + A[v] = x_1 u_1 + x_2 v_2$

can be linearly dependent if u, v $x_1 \& x_2 \neq 0$.

~~Also $c_1 u + c_2 v$ can also be~~

?

$\therefore c_1 u + c_2 v$ need not be linearly dependant because.

$u, v = 0$.

$\therefore c_1 u + c_2 v$ becomes linearly independent.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True. $T(u) = T(c_1 v + c_2 w) = c_1 T(v) + c_2 T(w)$

$T(v) = T(v + w)$

$= T(u) + T(w)$.

0 1 2 3 4 5 6 7 8 9 10