MTH 309T LINEAR ALGEBRA
EXAM 1
October 3, 2019

Name:

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UB Person Number:

5 0 2 6 3 5 0 9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.

- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = {}^4\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\quad -2v_1 + 4v_2 - v_3 = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

$$-2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -16 \end{bmatrix}$$

$-4 -12$

b) aug. matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ The set $\{v_1, v_2, v_3\}$ is lin. independent. b/c there is only one solution to the homogeneous matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}(-1)$$

$$(-2)\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 3 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}\left(\tfrac{1}{3}\right)$$

$$(-1)\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}(1)$$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$(-1)\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & -5 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$(-\tfrac{1}{2})\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -2 & -5 & -1 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix}$$

$$(\tfrac{1}{2})\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & -1 & | & 0 & 0 & 1 \\ 0 & 0 & -\tfrac{1}{2} & | & -1 & 1 & -\tfrac{1}{2} \end{bmatrix}$$

$$(-2)\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -\tfrac{1}{2} & | & 0 & 0 & \tfrac{1}{2} \\ 0 & 0 & -\tfrac{1}{2} & | & -1 & 1 & -\tfrac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -3 & -4 & -2 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix}$$

$$(\tfrac{1}{2})\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -\tfrac{1}{2} & | & 0 & 0 & \tfrac{1}{2} \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & -1 & 1 \\ 0 & 0 & 1 & | & 2 & -2 & 1 \end{bmatrix}(-2)$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$3 \times 3 \qquad 3 \times 3$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

3×3

$x_1 \; x_2 \; x_3$

3×3

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} ( ) & ( ) & ( ) \\ ( ) & ( ) & ( ) \\ ( ) & ( ) & ( ) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$x_1 = 2x_2 + 1$

a) $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

b) $\lceil$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 - 0 \\ 1 + 0 \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2 \\ 0 + 1 \\ 0 - 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$(-1) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 2 & 0 \end{bmatrix}$

$(-3) \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

$(2) \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

$(1) \; (\tfrac{1}{4}) \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$(-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is one-to-one

$(\tfrac{1}{2}) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} (\;)$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} (-1)$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ - not one-to-one

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -3 \\ -16 \\ -17 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2+0 \\ 0+4+12 \\ 3+8+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$

$T_A(v_1) - T_A(v_2) = 0$

$\begin{bmatrix} 3 \\ 16 \\ 17 \end{bmatrix}$

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

$$\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$$

False

$u \quad v$

$w = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ $\text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \end{bmatrix}\right)$
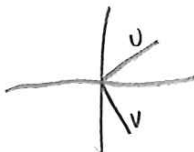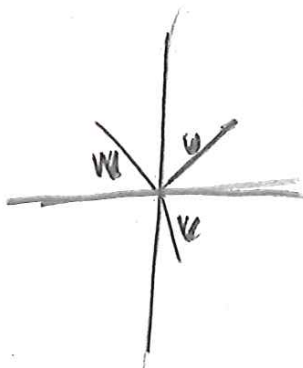
$W + U = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

R is in $\text{Span}(U,V)$, but W not is Span

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

True.   U, V, W are 3 unique vectors, so even if only 2 of them were in the set, it would still be linearly independent

still independent

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}, A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

False. A single vector such as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is linearly independent. If $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then it is linearly dependent.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in $\text{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\text{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

True. All 3 vectors are transformed by the same transformation, so properties are not changed. · They are still scalar multiples.