MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

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Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	TOTAL	GRADE
10	6	4	9	12	6	--	47	nan

1 -1 1 -2

MTH-309T-F19-EX1-009-P01

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

a) $V_1 - V_2 + V_3 = -2$

$V_1 + 2V_2 = 2$

$2V_1 - 3V_2 = b$

$\cancel{V_1 = 2 - 2V_2}$

$\cancel{2 - 2V_2 - V_2 + V_3 = -2}$

$\cancel{-3V_2 + V_3 = -4}$

$\cancel{V_3 = -4 - 3V_2}$

$\cancel{2 - 2V_2 - V_2 - 4 - 3V_2 = -2}$

$\cancel{2 - 3V_2 - 4 - 3V_2 = -2}$

$\cancel{-3V_2 - 4 - 3V_2 = -4}$

$\cancel{V_2 = 0}$

$V_1 = 2$

$b = 4$

$V_1 = 2 - 2V_2$

$2 - 2V_2 + V_3 = -2$

$-3V_2 + V_3 = -4$

$V_3 = -4 - 3V_2$

$= 2 - 2V_2 - V_2 - 4 - 3V_2 = -2$

$\Rightarrow \underline{V_2 = 0}$

$V_1 + 2(0) = 2$

$\underline{V_1 = 2}$

$2V_1 - 3V_2 = b$

$2(2) - 3(0) = b$

$4 = b$

$\boxed{b = 4}$

b) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & 4 \end{array}\right] (-2) = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & 0 & 8 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -1 & 0 & 8 \\ 0 & 1 & 2 & 2 \end{array}\right] (1)$

$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & 3 & 2 \end{array}\right] \overset{\div 3}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & 1 & 2/3 \end{array}\right] \overset{\div(-1)}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 2/3 \end{array}\right] (1) \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$

$\Downarrow$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2/3 \end{array}\right] (-1) \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$

The set $\{V_1, V_2, V_3\}$ is linearly independent because the the set contains only 1 solution, thus being a homogeneous equation which proves that the set is linearly independent.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

10

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (-1) = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] (-2) = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= (1) \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = (-2) \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = (1) \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$x_1 - x_2 = 1$

$-x_1 + x_2 = 0$

$x_2 = x_1$

$x_1 - x_1 = 1$

$0 \neq 1$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

0 1 2 3 4 5 **6** 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A: \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T \cdot B$$

$$C = (A^T)^{-1} \cdot B$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 6 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

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4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$T(u+v) = Tu + Tv$

A)

$$T(u) = \begin{bmatrix} u_1 - 2u_2 \\ u_1 + u_2 \\ u_1 - 3u_2 \end{bmatrix} \qquad T(u+v) = \begin{bmatrix} u_1 + v_1 - 2u_2 - 2v_2 \\ u_1 + v_1 + u_2 + v_2 \\ u_1 + v_1 - 3u_2 - 3v_2 \end{bmatrix} =$$

$$T(v) = \begin{bmatrix} v_1 - 2v_2 \\ v_1 + v_2 \\ v_1 - 3v_2 \end{bmatrix} \qquad = \begin{bmatrix} (u_1+v_1) - 2(u_2+v_2) \\ (u_1+v_1) + (u_2+v_2) \\ (u_1+v_1) - 3(u_2+v_2) \end{bmatrix}$$

standard matrix

$$A\left(T(e_1)\ T(e_2)\right) =$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

standard matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

b)

$$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} \Rightarrow A: \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 10 \\ 1 & 3 & 0 & -2 \end{array}\right] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 3 \\ -17 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 10 \\ 1 & 3 & 0 & -2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 3 & 0 & 9 \\ 0 & 5 & 0 & -2 \end{array}\right] = \left[\begin{array}{cccc} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 5 & 0 & -2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & -17 \end{array}\right]$$

0 1 2 3 4 5 6 7 8 **9** 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} (-3) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \cdot 1/2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} (-1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \cdot (-1/4) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} (-2) \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (-1) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A is one-to one
because every column has a pivot position and the matrix is a homogeneous equation

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} (-3) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \cdot 1/2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} (-1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot 1/2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} -2 \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A is one-to one
because every column has a pivot position and the matrix is a homogeneous equation.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

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6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}$

True

$V_1 + 0V_2 = 1$
$0V_1 + V_2 = 2$ ✓
$0V_1 + 0V_2 = 0$

$u + w = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$V_1 + 0V_2 = 4$
$0V_1 + V_2 = 8$
$0V_1 + 0V_2 = 0$

$V_1 + 0V_2 = 4$
$0V_1 + V_2 = 6$ ✓
$0V_1 + 0V_2 = 0$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True because in order for a set to be linearly independent they must have 1 solution (Homogeneous equation) so that means that u, v, w all only have 1 solution which ~~means~~ means that the set $\{u, v\}$ is linearly independent

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $u = [\ ]$, $v = [\ ]$

True

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span(T(v), T(w)).

This is true

0 1 2 3 4 5 6 7 8 9 10