MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	5	0	9	7	2
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\text{SPAN}(V_1\ V_2\ V_3) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \rightarrow 2.R1 - R3 = R3 \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow R1 + R2 = R1 \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow R2 - R3 = R3$$

$[2.1-2]$ $[2.-1-(-3)]$ $2.1-0$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

CONTINUED HERE

A AFTER ROW REDUCTION

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} V_1 + 3V_3 = -2 \quad / \cdot (-2) \\ V_2 + 2V_3 = 2 \quad / \cdot (3) \Rightarrow \end{matrix} \quad \begin{matrix} -2V_1 - 6V_3 = 4 \\ 3V_2 + 6V_3 = 6 \end{matrix} \Rightarrow -2V_1 + 3V_2 = 10$$

$$\Downarrow$$

$$2V_1 - 3V_2 = -10$$

$$2V_1 = 3V_2 - 10$$

$$\boxed{V_1 = \frac{3}{2}V_2 - 5}$$

CONTINUED HERE

$V_1 = \frac{3}{2}V_2 - 5$

FOR $V_1 + 3V_3 = -2$

$\frac{3}{2}V_2 - 5 + 3V_3 = -2$

$\frac{3}{2}V_2 + 3V_3 = 3$

$\frac{1}{2}V_2 + V_3 = 1$

$\boxed{V_2 = 2 - 2V_3}$

$V_1 = \frac{3}{2} \cdot 2(1 - V_3) - 5$

$V_1 = 3 - 3V_3 - 5$

$\boxed{V_1 = -3V_3 - 2}$

$V_1 \rightarrow -3V_3 - 2$

$V_2 \rightarrow 2 - 2V_3$

$V_3 \rightarrow V_3$

- b HAS INFINITE MANY OF VALUES
- SET OF $\{V_1, V_2, V_3\}$ IS LINEARLY DEPENDENT SINCE V_3 IS A FREE VARIABLE.

2. **(10 points)** Consider the following matrix:

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \end{array}$$

Compute A^{-1}.

$$A^{-1} = \frac{1}{|A|} \cdot A^{T}$$

$$|A| = 1.[0,(-1) - 2.1] - (-1)[(1)(-1) - 1,0] + 2[1.2 - 0,0]$$

$$= -2 - 1 + 4 = 1$$

DETERMINENT A $\Rightarrow |A| = 1$

$$A^{T} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

TO VERIFY $A . A^{-1} = I$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+1+2 & 1+0+2 & 0-2-2 \\ 1+0+2 & 1+0+1 & 0+0-1 \\ 0-2-2 & 0+0-1 & 0+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & -4 \\ 3 & 2 & -1 \\ -4 & -1 & 5 \end{bmatrix} \xrightarrow[\text{(ON SCRAP PAPER)}]{\text{ROW REDUCTION}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$A^T . C = \begin{bmatrix} x_1 + y_1 + 0 & x_2 + y_2 + 0 & x_3 + y_3 + 0 \\ -x_1 + 0 + 2z_1 & -x_2 + 0 + 2z_2 & -x_3 + 0 + 2z_3 \\ 2x_1 + y_1 - z_1 & 2x_2 + y_2 - z_2 & 2x_3 + y_3 - z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$x_1 + y_1 = 1 \rightarrow y_1 = 1 - x_1$

$-x_1 + 2z_1 = 4$

$2x_1 + y_1 - z_1 = 3 \rightarrow 2x_1 + 1 - x_1 - z_1 = 3$

$z_1 = 6$, $x_1 = 8$, $y_1 = -7$ $\Leftarrow \begin{cases} x_1 - z_1 = 2 \\ -x_1 + 2z_1 = 4 \end{cases}$

$x_2 + y_2 = 2 \rightarrow y_2 = 1 - x_2$

$-x_2 + 2z_2 = 5$

$2x_2 + y_2 - z_2 = 2$

$\begin{rcases} x_2 - z_2 = 1 \\ -x_2 + 2z_2 = 5 \end{rcases}$ $z_2 = 6$, $x_2 = 7$, $y_2 = -6$

$x_3 + y_3 = 3 \rightarrow y_3 = 3 - x_3$

$-x_3 + 2z_3 = 4$

$2x_3 + y_3 - z_3 = 1$

$\begin{rcases} x_3 - z_3 = -2 \\ -x_3 + 2z_3 = 4 \end{rcases}$ $z_3 = 2$, $x_3 = 0$, $y_3 = 3$

$$\left.\begin{matrix} x_1 = 8 & x_2 = 7 & x_3 = 0 \\ y_1 = -7 & y_2 = -6 & y_3 = 3 \\ z_1 = 6 & z_2 = 6 & z_3 = 2 \end{matrix}\right\} \quad C = \begin{bmatrix} 8 & 7 & 0 \\ -7 & -6 & 3 \\ 6 & 6 & 2 \end{bmatrix}$$

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$T(v) = T \cdot v$

$v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad T(v) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} \checkmark \qquad T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

$T(u) = T \cdot u \qquad T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

$\underset{3\times 2}{\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}} \cdot u = \underset{3\times 1}{\begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}}$

$2\times 1 \rightarrow u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 - 2u_2 \\ u_1 + u_2 \\ u_1 - 3u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$u_1 - 2u_2 = 1$
$u_1 + u_2 = 10$
$u_1 - 3u_2 = -2$

$\begin{cases} u_1 - 2u_2 = 1 \\ -u_1 - u_2 = -10 \end{cases}$

$-3u_2 = -9$
$\boxed{u_2 = 3}$

$u_1 + 3 = 10 \quad \boxed{u_1 = 7}$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

SCRAP PAPER

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow[\text{REDUCTION}]{\text{ROW}} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$

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5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

SINCE $\mathbb{R}^3 \to \mathbb{R}^3$

$$V = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$T_A(V) = A \cdot V$

$$A \cdot V = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + y_1 + 0 & x_2 + y_2 + 0 & x_3 + y_3 + 0 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 4z_1 & 3x_2 + 4y_2 + 4z_2 & 3x_3 + 4y_3 + 4z_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 4z_1 & 3x_2 + 4y_2 + 4z_2 & 3x_3 + 4y_3 + 4z_3 \end{bmatrix}$$

$$A \cdot V = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + y_1 + 0 & x_2 + y_2 + 0 & x_3 + y_3 + 0 \\ 2y_1 + 4z_1 & 2y_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 2z_1 & 3x_2 + 4y_2 + 2z_2 & 3x_3 + 4y_3 + 2z_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ 2y_1 + 4z_1 & 2x_2 + 4z_2 & 2y_3 + 4z_3 \\ 3x_1 + 4y_1 + 2z_1 & 3x_2 + 4y_2 + 2z_2 & 3x_3 + 4y_3 + 2z_3 \end{bmatrix}$$

FOR $T_A(V_1) = T_A(V_2)$

$\to 3x_{31} + 4y_{31} + 4z_{31} = 3x_{32} \; 4y_{32} + 2z_{32}$

$4z_{31} = 2z_{32}$

$2z_{31} = 1z_{32}$ $\quad \dfrac{z_{31}}{z_{32}} = \dfrac{1}{2}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 12 & 16 & 16 \\ 15 & 22 & 24 \end{bmatrix}$$

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

TRUE → IF u, v, w ARE VECTORS IN $\mathbb{R}^3$

THEN SPAN (u v w) = { SET OF ALL LINEAR COMBINATIONS }

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

FALSE → ONLY IF $u . x_1 + v . x_2 + w . x_3 = 0$ → LINEARLY INDEPENDENT

$u x_1 + v x_2 = 0$

$\{u, v\}$ LINEARLY INDEPENDENT → THIS EQUATION DOES NOT GURANTEED $u x_1 + v x_2 = 0$

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

TRUE $\longrightarrow$ IF A IS 2X2 MATRIX THEN NUL(A) IS SPAN OF SOME VECTORS IN $\mathbb{R}^2$.

NUL(A) = (SET OF SOLUTIONS OF $A.V = 0$)

↳ LINEARLY INDEPENDENT

IF $Au \neq 0$
$AV \neq 0$ THEN u, v LINEARLY DEPENDENT.

$Au \neq 0$
$AV \neq 0$ FOR DEPENDENCE

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span(T(v), T(w)).

TRUE