MTH-309T-F19-EX1-028-P00

# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

**Name:**

Brendan DesRosiers

**UB Person Number:**

| 5 | 0 | 1 | 8 | 6 | 5 | 8 | 4 |
|---|---|---|---|---|---|---|---|

**Instructions:**

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | PIAZZA | HILL | TOTAL | GRADE |
|---|---|---|---|---|---|---|---|---|---|---|
| 13 | 9 | 10 | 15 | 20 | 6 | 3 | 2 | -- | 78 | B |

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

**a)** Find all values of $b$ such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

**b)** Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\mathrm{Span}(v_1, v_2, v_3) = \left\{ \begin{matrix} \text{the set of all} \\ \text{vectors } c_1v_1 + c_2v_2 + c_3v_3 \end{matrix} \right\}$ ✓

$x_1v_1 + x_2v_2 + x_3v_3 = w$

$R_3 = -2R_1 + R_3$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

$R_3 = +R_2 + R_3$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & \cancel{b} \end{array}\right]$ $b+4$

$R_1 = R_2 + R_1$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b \end{array}\right]$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & \cancel{b} \end{array}\right] \quad b+6$$

Apply reduction to the whole matrix, not just the first three columns!

$\therefore$ b must equal $\cancel{0}$ $-6$

$$\begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 + 2 \\ x_3 = x_3 \end{cases}$$

b) the set $\{v_1, v_2, v_3\}$ is not linearly independent because every column of the matrix is not a pivot column ✓

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☒ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

**2. (10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute $A^{-1}$.

$$R_2 = R_1 - R_2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$R_2 = -R_2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$R_3 = -2R_1 + R_3 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$R_1 = R_2 + R_1 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$R_2 = R_3 + R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$R_1 = -R_3 + R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \Longrightarrow A^{-1} = \begin{bmatrix} -2 & 3 & \overset{-1}{\cancel{0}} \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☒ 9 ☐ 10

3. **(10 points)** Let $A$ be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix $C$ such that $A^T C = B$ (where $A^T$ is the transpose of $A$).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$A^T C = B$

Matrix Algebra Property: $(A^T)^{-1} = (A^{-1})^T$

$C = (A^T)^{-1} \cdot B$

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$C = \underset{3\times 3}{\begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}} \cdot \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}}$$

defined with

$C = 3\times 3$

$$C = \begin{bmatrix} -2(1)+1(4)+2(3) & -2(2)+1(5)+2(2) & -2(3)+1(4)+2(1) \\ 3(1)+(-1)(4)+(-2)(3) & 3(2)+(-1)(5)+(-2)(2) & 3(3)+(-1)(4)+(-2)(2) \\ 0(1)+1(4)+1(3) & 0(2)+1(5)+1(2) & 0(3)+1(4)+1(1) \end{bmatrix}$$

$$C = \begin{bmatrix} -2+4+6 & -4+5+4 & -6+4+2 \\ 3-4-6 & 6-5-4 & 9-4-4 \\ 4+3 & 5+2 & 4+1 \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 1 \\ 7 & 7 & 5 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

**a)** Find the standard matrix of $T$.

**b)** Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a)

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$A = [\, T(e_1) \quad T(e_2) \,]$

$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b)

$R_2 = -R_1 + R_2$
$R_3 = -R_1 + R_3$
$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$ ✓

$\downarrow$

$R_3 = -R_3$
$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{array}\right]$ ← This is not reduced yet

$\cancel{u = \begin{bmatrix} 2x_2 + 1 \\ -3x_2 + 9 \\ -x_2 + 3 \end{bmatrix}}$

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☒ | ☐ | ☐ | ☐ | ☐ | ☐ |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |

5. **(20 points)** For each matrix $A$ given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If $T_A$ is not one-to-one, find two vectors $\mathrm{v}_1$ and $\mathrm{v}_2$ such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$  b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$R_3 = -3R_1 + R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right]$

$\downarrow$

$R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$

$\downarrow$

$R_3 = -2R_2 + R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$

$\downarrow$

$R_3 = -\frac{1}{4}R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$T_A$ is one-to-one because there is a pivot position in every column

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$R_3 = -3R_1 + R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$

$R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$R_3 = -2R_2 + R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$

$R_1 = -R_2 + R_1$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$T_A$ is not one-to-one because only columns 1 and 2 have pivot positions

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x_1 = 2x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases}$

• For $V_1$, $x_3 = 1$

$V_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

• For $V_2$, $x_3 = 2$

$V_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w}+\mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad w = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

TRUE ✓ because if $w+u$ is in Span$(u,v)$ then $w$ will also be in Span $(u,v)$ because they are linear combinations of one another ?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$u = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad v = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad w = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

• Since the set is linearly independent the matrix would have to be

$\begin{bmatrix} 1 & b_1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & b_3 & 1 \end{bmatrix}$

in order for every column to have a pivot column

set $\{u, v\}$

$\begin{bmatrix} 1 & b_1 \\ a_2 & 1 \\ a_3 & b_3 \end{bmatrix}$

this set is still linearly independent so the statement is TRUE ✓

0 1 2 3 4 5 6 7 8 9 10

**7. (10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

**a)** If $A$ is a $2\times 2$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$$A = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}, \quad u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad v = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Au and Av are LD

No matter what the vectors u, v in $\mathbb{R}^2$ are they will have a pivot column so statement is FALSE ✓ ?

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**b)** If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

TRUE ✓ because the properties of linear transformation tells us that if u is in the Span(v, w) then T(u) has to be in the Span(T(v), T(w))

why?

0 1 2 3 4 5 6 7 8 9 10