MTH-309T-F19-EX1-078-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

Ricky Chen

UB Person Number:

5	0	2	3	1	1	6	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

16	10	10	16	13	6	3	--	10	81	B
1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE

MTH-309T-F19-EX1-078-P01

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & 0 & b+4 \end{array}\right] +$$

$$\left[\begin{array}{cccc} 1 & -1 & -2 & -2 \\ 0 & 0 & 2 & b+6 \\ 0 & -1 & 0 & b+4 \end{array}\right] + \xrightarrow{R_1 = R_1 - R_3} \left[\begin{array}{cccc} 1 & 0 & -2 & -b-6 \\ 0 & 0 & 2 & b+6 \\ 0 & -1 & 0 & b+4 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & 0 & -2 & -b-6 \\ 0 & -1 & 0 & b+4 \\ 0 & 0 & \cancel{2}\ 0 & b+6 \end{array}\right] \xrightarrow{R_1 = R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & b+4 \\ 0 & 0 & \cancel{2}\ 0 & b+6 \end{array}\right] \begin{array}{l} \\ \cdot(-1) \\ \cdot(\frac{1}{2}) \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -b-4 \\ 0 & 0 & 1 & \frac{b+6}{2} \end{array}\right]$$

a) ~~All values of b are $c_1(2) + c_2(-3) + c_3(0)$ that $w \in \mathrm{Span}(v_1, v_2, v_3)$~~

b) Yes, the set $\{v_1, v_2, v_3\}$ is linearly independent because after row reduction every column is a pivot column.

Only $b = -6$

Correct conclusion, but it comes from an incorrect row reduction.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$ ← this is the answer

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 + 2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right]$$

$\downarrow R_1 = R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 2 & -3 & 1 \end{array}\right]$$

$\downarrow R_2 = R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ -1 & 0 & 0 & 2 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ -1 & 0 & 0 & 2 & -3 & 1 \end{array}\right] \begin{array}{l} \cdot(-1) \\ \\ \cdot(-1) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ 1 & 0 & 0 & -2 & 3 & -1 \end{array}\right]$$

1. $R_3 \leftrightarrow R_1$
2. $R_2 \leftrightarrow R_3$

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒
0	1	2	3	4	5	6	7	8	9	10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A^T = [$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$(A^{-1})^T$

$$A^T C = B \qquad A^T \qquad A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ \cancel{2} & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\cancel{(A^T)^{-1} A^T} C = (A^T)^{-1} B \checkmark$$

$$C = (A^{-1})^T B \checkmark \qquad (A^{-1})^T = \begin{bmatrix} -2 & \cancel{2} & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \checkmark$$

$$3 \times 3 \qquad 3 \times 3 \quad = 3 \times 3$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = 1\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+8+6=12 \\ 3-4-6=-7 \\ -1+4+3=6 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+6+4=10 \\ 6-5-4=-3 \\ -2+5+2=5 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = 3\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} +2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+8+2=4 \\ 9-4-2=3 \\ -3+4+1=2 \end{bmatrix}$$

$$C = \begin{bmatrix} \cancel{12} & \cancel{10} & \cancel{4} \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10 ☒

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(u) = \begin{bmatrix} a_1 - 2a_2 \\ a_1 + a_2 \\ a_1 - 3a_2 \end{bmatrix}$ $T(v) = \begin{bmatrix} b_1 - 2b_2 \\ b_1 + b_2 \\ b_1 - 3b_2 \end{bmatrix}$

$T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ standard matrix of $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) $R_3 = R_3 - R_1$ $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$

$+2R_2$ $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & -3 \end{array}\right]$

$\downarrow$

$\begin{bmatrix} 3 & 0 & 21 \\ 1 & 1 & 10 \\ 0 & 1 & -3 \end{bmatrix} * \left(\frac{1}{3}\right) \to \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 1 & 1 & 10 \\ 0 & 1 & -3 \end{array}\right]$

+3

$\downarrow$

$\underbrace{\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 1 & -3 \end{bmatrix}}$

$x_1 = 7$
$x_2 = x_2$

all vectors u satisfying $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ are: $\cancel{\begin{bmatrix} 7 \\ x_2 \end{bmatrix}}$ ~~where x_2 is free~~

This would mean that there are no solutions

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $-(3R_1)\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} 2- \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & -1 & 0 \end{bmatrix} \cdot(\tfrac{1}{2})$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} +$

$\cdot(\tfrac{1}{2})\cdot$ $(-1)\cdot$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \leftarrow + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \leftarrow$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yes A is one-to-one ✓

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} -2R_2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 0 & -6 \end{bmatrix} \begin{matrix} \cdot\frac{1}{2} \\ \cdot -\frac{1}{3} \end{matrix} \to + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} -(\cdot 2) \leftarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & \cancel{1} & \cancel{3} \end{bmatrix} -$

0 0

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ~~yes, A is one to one.~~

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐	☐	☐

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

True, because since $u+w$ is in the span (u,v) then it must be true that $w \in \text{Span}(u,v)$ because the sum of the two vectors $(u+w)$ are in the span (u,v).

?

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

True

~~False~~, for example

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then the set $\{u,v\}$ $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ is also linearly independent because it has only one solution and pivot col in every column. If u, v, w are independent then it has pivot col. in every column that means with two vectors it still holds true.

0 1 2 3 4 5 6 7 8 9 10

7. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False, ✓

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

u and v could still be independent and be multiplied with A to make it linearly dependant. ← example ?

This matrix preserves independence

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

Yes, true ✓ because if u is in the span of (v, w) then the transformation could just be the scalar of u vector and it would still let T(u) be in the span of T(v), T(w) ?

0 1 2 3 4 5 6 7 8 9 10