MTH-309T-F19-EX1-016-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	2	0	5	2	0

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	--	--	--	--	--	--	1	nan
1	2	3	4	5	6	7	TOTAL	GRADE

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a).
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{-2} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{bmatrix} \xrightarrow{+} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{bmatrix}$$

When $b = -6$, it has infinite solutions.

$x_1 + 3x_3 = 0$

$x_2 + 2x_3 = 2$

b)
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{\times -2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & -2 \end{bmatrix} \xrightarrow{+} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -5 & -2 \end{bmatrix} \times 5$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \times -\tfrac{1}{2} \leftarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} +$$

It's linearly independent.

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 2 & -1 \end{bmatrix} 2 \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix} \times -2$$

$$\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times -1 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ A is inverstible.

$$C_1 : \begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \cdot -2 \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} -1 \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$C_2 : \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \to 2 \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & -1 \\ 2 & 0 & 3 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ \cancel{2} & \cancel{0} & \cancel{3} & \cancel{0} \end{bmatrix} - \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ \cancel{1} & \cancel{0} & \cancel{2} & \cancel{-1} \\ 0 & 0 & 1 & 0 \end{bmatrix} -1 \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] -1$$

C_2

$$C_3 \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} 2 \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} \times -2$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} +1 \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \times -2$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

0 1 2 3 4 5 6 7 8 9 10

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3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^T C = B$$

$$A^T \cdot A^{T^{-1}} C = B A^{T^{-1}}$$

$$C = B A^{T^{-1}}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \qquad \therefore \ A^T \text{ inverse} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \overset{-1}{-1\times1} + \overset{4}{1\times4} + 0\times3 & \overset{-2}{-1\times2} + \overset{5}{1\times5} + 0\times2 & -1\times3 + 1\times4 + 0\times1 \\ \overset{-1}{-1\times1} + 0\times4 + 2\times3 & -1\times2 + 0\times5 + 2\times2 & -1\times3 + 0\times4 + -1\times1 \\ \overset{2}{2\times1} + 1\times4 + -1\times3 & 2\times2 + 1\times5 + -1\times2 & 2\times3 + 1\times4 + -1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 1 \\ 5 & 2 & -4 \\ 3 & 7 & 9 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a). $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -1 & -3 \end{bmatrix}_{x-1} \to \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 21 \\ 1 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix}_{x-1}$

$-4 \quad -7 \quad 0 \quad x - \frac{1}{2}$

$\begin{matrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{matrix} \quad x-2$

$\begin{matrix} 0 & +1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{matrix} \quad x-1$

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{matrix} \quad -1$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{bmatrix} \times\tfrac{1}{2} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 3 & 4 & 4 & 3 & 4 & 2 \end{bmatrix} \times -3$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{bmatrix} \times -1$$

It's not one to one.

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 2 \end{bmatrix} \times -2$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \times\tfrac{1}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} -1$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$V_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$x_1 = -2$

$x_2 + x_5 = 2$

$x_3 = 0$

$x_4 = \text{free}$

$x_5 = \text{free}$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

F

$$u\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad w = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

linearly independent $\rightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

F

linearly independent mean have exactly one solution.

if $\{u, v, w\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\{u, v\} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\leftarrow$ have infinite solution.

$\therefore$ It's False.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

T

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in $\text{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\text{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

T

0 1 2 3 4 5 6 7 8 9 10