1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

$X_1 = V_1 \quad X_2 = V_2 - \frac{V_2 \cdot V_1}{V_1 \cdot V_1} \cdot V_1 \quad X_3 = V_3 - \left(\frac{V_3 \cdot V_1}{V_1 \cdot V_1}\right)V_1 - \left(\frac{V_3 \cdot V_2}{V_2 \cdot V_2}\right)V_2$

$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$V_2 \cdot V_1 = 2+0+1+0 = 3$

$V_1 \cdot V_1 = 1+0+1+1 = 3$

$\frac{V_2 \cdot V_1}{V_1 \cdot V_1} = \frac{3}{3} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$V_2 - \begin{matrix} 1 \\ 0 \\ -1 \\ 1 \end{matrix}$

$\begin{matrix} 2 \\ 1 \\ -1 \\ 0 \end{matrix} - \begin{matrix} 1 \\ 0 \\ -1 \\ 1 \end{matrix} = \begin{matrix} 1 \\ 1 \\ 0 \\ -1 \end{matrix}$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$X_2 = \begin{matrix} 2 \\ 1 \\ -1 \\ 0 \end{matrix} - 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$V_3 \cdot V_1 = 2+0+1+3 = 6$

$V_1 \cdot V_1 = 3 \quad \frac{V_3 \cdot V_1}{V_1 \cdot V_1} = 2$

$V_3 \cdot V_2 = 4 + -2 + 1 + 0 = 3 \quad \frac{V_3 \cdot V_2}{V_2 \cdot V_2} = \frac{1}{2}$

$V_2 \cdot V_2 = 4+1+1 = 6$

$\cancel{X_3 = \begin{matrix} 2 \\ -2 \\ -1 \\ 3 \end{matrix} - 2 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}}$

$\cancel{X_3 = \begin{matrix} 2 \\ -2 \\ -1 \\ 3 \end{matrix} - \begin{matrix} 2 \\ 0 \\ -2 \\ 2 \end{matrix} - \begin{matrix} 1 \\ 1/2 \\ -1/2 \\ 0 \end{matrix} = \begin{bmatrix} -1 \\ 3/2 \\ 3/2 \\ 1 \end{bmatrix}}$

$$\boxed{\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3/2 \\ 3/2 \\ 1 \end{bmatrix} \right\}}$$

$\text{Proj}_V U = \left(\frac{U \cdot W_1}{W_1 \cdot W_1}\right) W_1 + \left(\frac{U \cdot W_2}{W_2 \cdot W_2}\right) W_2 + \left(\frac{U \cdot W_3}{W_3 \cdot W_3}\right) W_3$

$U \cdot W_1 = 3+0-3+3 = 3$

$W_1 \cdot W_1 = 1+0+1+1 = 3$

$U \cdot W_2 = 3+3+0-3 = 3$

$W_2 \cdot W_2 = 1+1+0+1 = 3$

$U \cdot W_3 = -3 + 9/2 + 9/2 + 3 = 9$

$W_3 \cdot W_3 = 1 + 2.25 + 2.25 + 1 = 13/2$

$\text{Proj}_U V = W_1 + W_2 + \frac{9}{13/2} W_3 = W_1 + W_2 + \frac{18}{13} W_3$

$= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -18/13 \\ 27/13 \\ 27/13 \\ 18/13 \end{bmatrix}$

$2 - \frac{18}{13} = \frac{8}{13}$

$1 + 27/13 = 40/13$

$-1 - 27/13 = -14/13$

$\frac{18}{13}$

$\begin{bmatrix} 8/13 \\ 40/13 \\ -14/13 \\ \frac{18}{13} \end{bmatrix}$

$$\boxed{\text{Proj}_U V = \begin{bmatrix} 8/13 \\ 40/13 \\ -14/13 \\ 18/13 \end{bmatrix}}$$

$1.5 \times 3 = 4.5$

$18 \times 3 = 54$

$\frac{27}{13}$

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14/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20