MTH-309T-F19-EX1-065-P00

MTH 309T LINEAR ALGEBRA
EXAM 1
October 3, 2019

Name:

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UB Person Number:

5	0	2	2	4	3	9	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$2v_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} \overset{b=-6}{\Rightarrow} w = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$ $\boxed{b=-6}$

$v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $-3v_1 = \begin{bmatrix} -3 \\ 0 \\ -6 \end{bmatrix}$

$-2v_1 = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \xrightarrow{(-2)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_1 = R_2 + R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & -2 \end{bmatrix} \xrightarrow{R_3 = R_2 + R_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

yes, linearly independent because every col. is a pivot column

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-2)} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{R_1 = R_1 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -1 & -1 \\ 2 & -2 & -1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -1 & -1 \\ 2 & -2 & -1 \end{bmatrix} \qquad (A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T C = B \qquad \underline{(A^T)^{-1} = (A^{-1})^T}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$(A^T)^{-1} A^T C = (A^T)^{-1} B$$

$$C = (A^T)^{-1} B = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2+4+6 & -4+5+4 & -6+4+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ 1-4-3 & 2-5-2 & 3-4-1 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & +3 \\ 6 & +5 & +2 \end{bmatrix}}$$

Check $A^T C = B$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & +3 \\ 6 & +5 & 2 \end{bmatrix} = \begin{bmatrix} 8-7+0 & 5-3+0 & 0+3+0 \\ -8+0+12 & -5+0+10 & 4 \\ 16-7-6 & 10-3-5 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 4 & 12 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{-2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{array}\right]$ $\Big\}$ $x_1 = 7$, $x_2 = 3$

$T\left(\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7-6 \\ 7+3 \\ 7-9 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{array}\right] \xrightarrow{-3} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{-1}$

$A = \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{array}\right]$

yes one to one

pivot position in every col.

b) $A = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{array}\right] \xrightarrow{-3} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array}\right]$

yes pivot position in every col.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$W \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad u \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{W} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{u} \in \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

false

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True, if u, v, w are all lin. independent, u, v must be linearly independent

$$\underbrace{u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{all 3 lin. ind.}}$$

$\underbrace{\hphantom{u = [\,] \quad v = [\,]}}_{}$

u, v $\Rightarrow$ lin. ind.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True, since multipying by the same matrix give linear dependance

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True since every matrix transformation is a linear transformation

0 1 2 3 4 5 6 7 8 9 10