MTH-309T-F19-EX1-014-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	1	8	4	9	9	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
8	9	2	20	20	3	2	--	10	70	B-

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) as $w \in$ span (V_1, V_2, V_3) $\quad \therefore V_1 + V_2 + V_3 = W$

$$-4 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

$\therefore b = (\cancel{-8})$ $\therefore -6$ (why?)

b) $$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ ✓

① $c_1 - c_2 + c_3 = 0$ $\quad \therefore c_1 = c_2 - c_3$

② $c_2 + 2c_3 = 0$ $\quad \therefore c_3 = -\frac{c_2}{2}$ $\quad \therefore c_3 = 0$ $[\because c_2 = 0$

③ $2c_1 - 3c_2 = 0$ $\quad \therefore c_1 = \frac{3c_2}{2}$ $\quad \therefore c_1 = 0$ $[\because c_1 = 0$

?

Use row reduction!

$\therefore c_2 - c_3 = \frac{3c_2}{2}$

$\therefore 2c_2 - 2c_3 = 3c_2$ $\quad \therefore 2c_2 + 2\left(\frac{c_2}{2}\right) = 3c_2$ $\quad \therefore 3c_2 = 3c_2$ $\quad \therefore c_2 = 0$

as, $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so, we can say the set $\{V_1, V_2, V_3\}$ is ~~linearly independent~~.

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☒ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14 ☐ 15 ☐ 16 ☐ 17 ☐ 18 ☐ 19 ☐ 20

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$-1-2(-1)$
$=-1+2$
$=1$
$0-2(-1)$

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_1} = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_3} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad \therefore A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & c_4 & c_7 \\ c_2 & c_5 & c_8 \\ c_3 & c_6 & c_9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

✓

0 1 2 3 4 5 6 7 8 9 10

$\frac{9}{9}$ 3

4. (20 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) Standard matrix of T :- $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b) as linear transformation,

$Au = T(u)$

$$\therefore \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

① $u_1 - 2u_2 = 1$

② $u_1 + u_2 = 10$

③ $u_1 - 3u_2 = -2$

$\therefore u_1 = 1 + 2u_2$ $\quad \therefore u_1 = 1 + 2(3) = 1 + 6 = 7$

$\therefore 1 + 2u_2 + u_2 = 10$ $[\because u_1 = 1 + 2u_2]$ $\therefore 3u_2 = 9$ $\therefore u_2 = 3$

$\therefore u_1 - 3(3) = -2$ $[\because u_2 = 3]$

$\therefore u_1 - 9 = -2$ $\quad \therefore u_1 = 7$

$$\therefore u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 - R_3 - 3R_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ a is one-to one. but b is not one-to-one ✓

Now, $T_A(V_1) = T_A(V_2)$

$\therefore T_A(V_1 - V_2) = 0$

$$\therefore \underset{V_1}{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}} - \underset{V_2}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \underbrace{\underset{V_1}{\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}} - \underset{V_2}{\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_1 and v_2 are supposed to be different. Since it was not explicitly stated I will accept it though.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

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6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If u, v, w are vectors in $\mathbb{R}^3$ such that $w + u \in \text{Span}(u, v)$ then $w \in \text{Span}(u, v)$.

False. because, $w+u = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \in$ span (u, v).

~~not~~ necessary,

$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \in$ span (u, v).

b) If u, v, w are vectors in $\mathbb{R}^3$ such that the set $\{u, v, w\}$ is linearly independent then the set $\{u, v\}$ must be linearly independent.

False.

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

the set $\{u, v, w\}$ is linearly independent. ~~be~~ cause, $x_1 u_1 + x_2 v_2 + x_3 w_3 = 0$ has only one, trival solution. on the other, $\{u, v, w\}$ is linearly dependent because $x_1 u + x_2 v + x_3 w = 0$ has non trival solution

This is the same set!

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False. ← why ?

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

~~False.~~

True. ← why ?

0 1 2 3 4 5 6 7 8 9 10