



# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

Name:

Kristiyan Aleksandrov

UB Person Number:

5	0	2	3	9	2	0	9
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

11

8

7

13

13

2

1

2

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57

C-

1

2

3

4

5

6

7

PIAZZA

HILL

TOTAL

GRADE



1. (20 points) Consider the following vectors in  $\mathbb{R}^3$ :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of  $b$  such that  $w \in \text{Span}(v_1, v_2, v_3)$ .

b) Is the set  $\{v_1, v_2, v_3\}$  linearly independent? Justify your answer.

$$\cancel{-3 \leq b \leq 2}$$

$$b = -6$$

$$\begin{array}{c} v_1 \ v_2 \ v_3 \\ \left[ \begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array} \right] \xrightarrow{+(-2)} \left[ \begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b-2 \end{array} \right] \xrightarrow{+} \\ \begin{array}{c} x_1 \ x_2 \ x_3 \ b \\ \left[ \begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b \end{array} \right] \end{array} \end{array}$$

$b=0$

$$b) \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array} \right] \xrightarrow{\cdot -2} \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array} \right] \xrightarrow{+} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\cdot -1}$$

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= 0 \\ x_3 &= x_3 \end{aligned}$$

$$\begin{array}{c} x_1 \ x_2 \ x_3 \\ \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

$x_3 = \text{free}$

So infinite solutions

Which means that it is linearly dependent. ✓



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute  $A^{-1}$ .

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 - R_1} \left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 - 2R_2} \left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \xrightarrow{R_1 + R_2}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \xrightarrow{R_1 - R_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \xrightarrow{R_2 + R_3}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

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$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$B \cdot (A^T)^{-1} = C$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

$$C = (A^T)^{-1} \cdot B$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{bmatrix}$$

$$K \rightarrow \left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{ccccccc} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right]$$

$$(A^T)^{-1} = \begin{bmatrix} -2 & 5 & -1 \\ 1 & -1 & 1 \\ 2 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 4 \\ 5 & 5 & 5 \\ -2 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -7 & 7 \\ 8 & 9 & 8 \\ 5 & 7 & 7 \end{bmatrix}$$

$$(A^T)^{-1} \cdot B \quad \checkmark$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -5 & -7 & -7 \\ 8 & 9 & 8 \\ 5 & 7 & 7 \end{bmatrix} =$$



4. (20 points) Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of  $T$ .

b) Find all vectors  $u$  satisfying  $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ .

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \checkmark$$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$\left( \begin{array}{cc|c} c_1 & c_2 & \\ 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array} \right) \xrightarrow{(-1)} \left( \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -2 \end{array} \right) \xrightarrow{\div 3} \left( \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \end{array} \right)$$

$$\left( \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & \times \end{array} \right) + \left( \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \end{array} \right) \leftarrow \left( \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -2 \end{array} \right) \div 3$$

?



5. (20 points) For each matrix  $A$  given below determine if the matrix transformation  $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T_A(v) = Av$  is one-to-one or not. If  $T_A$  is not one-to-one, find two vectors  $v_1$  and  $v_2$  such that  $T_A(v_1) = T_A(v_2)$ .

a)  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\downarrow \cdot -3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\div 2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\div 2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot -1}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot 2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot -2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A$  is  
one to one

b)  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\downarrow$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Also ~~one to one~~

Pivot Position in every column

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot -1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot -2}$$





6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If  $u, v, w$  are vectors in  $\mathbb{R}^3$  such that  $w + u \in \text{Span}(u, v)$  then  $w \in \text{Span}(u, v)$ .

True <sup>✓</sup> because  $u$  is not needed for  $w$  to be in the span. If it was given that all 3 were in  $\mathbb{R}^3$ . ) ?

b) If  $u, v, w$  are vectors in  $\mathbb{R}^3$  such that the set  $\{u, v, w\}$  is linearly independent then the set  $\{u, v\}$  must be linearly independent.

? ~~False~~ it would depend on what vectors  $u$  and  $v$  are because the homogeneous solution may be different when using only the 2 vectors out of the 3.



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If  $A$  is a  $2 \times 2$  matrix and  $u, v$  are vectors in  $\mathbb{R}^2$  such that  $Au, Av$  are linearly dependent then  $u, v$  also must be linearly dependent.

~~True~~

b) If  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear transformation and  $u, v, w \in \mathbb{R}^2$  are vectors such that  $u$  is in  $\text{Span}(v, w)$  then  $T(u)$  must be in  $\text{Span}(T(v), T(w))$ .

~~True~~ ← why?