1. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace $V$ of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace $V$.

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of $u$ on $V$.

$$V = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow[R_4-R_1]{R_3+R_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4+2R_3]{R_1-2R_3,\ R_2-R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1-2R_4]{R_2+R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

→ orthogonal basis D is the basis of col(A)

D =

a) $w_1 = v_1$

$w_1 \cdot v_2 = 1\cdot 2 + 0\cdot 1 + -1\cdot -1 + 1\cdot 0 = 3$

$w_1 \cdot w_1 = 1\cdot 1 + 0\cdot 0 + -1\cdot -1 + 1\cdot 1 = 3$

$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$w_2 \cdot v_3 = 1\cdot 2 + 1\cdot -2 + 0\cdot -1 + -1\cdot 3 = -3$

$w_2 \cdot w_2 = 1\cdot 1 + 1\cdot 1 + 0\cdot 0 + -1\cdot -1 = 3$

$w_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{-3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$w_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{D = \begin{bmatrix} 1 & 1 & \cancel{2}\ 1 \\ 0 & 1 & -1 \\ -1 & 0 & \cancel{0}\ 1 \\ 1 & -1 & \cancel{1}\ 0 \end{bmatrix}}$$

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part b on back →