MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	3	8	7	0	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
14	10	10	20	14	10	6	2	--	86	B+

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\cancel{V_1 + V_3 = w \mid b = 2}$

$\cancel{-V_1 + V_3 = V}$

$2V_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}, b = -6$ $\boxed{b=-6}$ ✓

$-3V_1 + V_3 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}, b = -6$

b) The set $\{V_1, V_2, V_3\}$ ~~is linearly independent~~, because the vectors cannot linearly combine to cancel one another

cannot cancel all terms

$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$

cannot cancel all terms

no scaler multiple

They can. E.g. $3v_1 + 2v_2 - v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

MTH-309T-F19-EX1-074-P02

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$[A \mid I] \to [I \mid A^{-1}]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A=\begin{bmatrix}1&-1&2\\1&0&1\\0&2&-1\end{bmatrix}\qquad B=\begin{bmatrix}1&2&3\\4&5&4\\3&2&1\end{bmatrix}$$

Find a matrix C such that $A^TC=B$ (where A^T is the transpose of A).

$$A^T=\begin{bmatrix}1&1&0\\-1&0&2\\2&1&-1\end{bmatrix}$$

$$A^TC=B$$

$$C=(A^T)^{-1}B \qquad (A^T)^{-1}=(A^{-1})^T=\begin{bmatrix}-2&3&-1\\1&-1&1\\2&-2&1\end{bmatrix}^T=\begin{bmatrix}-2&1&2\\3&-1&-2\\-1&1&1\end{bmatrix}$$ ✓

$$C=\begin{bmatrix}-2&1&2\\3&-1&-2\\-1&1&1\end{bmatrix}\cdot\begin{bmatrix}1&2&3\\4&5&4\\3&2&1\end{bmatrix}=\begin{bmatrix}(-2+4+6)&\overset{-4+5+4}{(\cancel{\quad})}&(-6+4+2)\\(3-4-6)&(6-5-4)&(9-4-2)\\(-1+4+3)&(-2+5+2)&(-3+4+1)\end{bmatrix}$$

$$\boxed{C=\begin{bmatrix}8&5&0\\-7&-3&3\\6&5&2\end{bmatrix}}$$ ✓

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\boxed{T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ ✓

b) $1 = x_1 - 2x_2$

$10 = x_1 + x_2$

$-2 = x_1 - 3x_2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 7$

$x_2 = 3$

$\boxed{u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}}$ ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) One to one -> pivot in every column

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ A is one-to-one ✓

b) $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ not one-to-one ✓

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 + 4x_3 \\ 3x_1 + 4x_2 + 2x_3 \end{bmatrix}$$

=

$x_2 = x_3$

$\begin{bmatrix} 1 & 1 \\ 0 & 6 \\ 3 & 6 \end{bmatrix}$ $x_1 = 1$ $x_2 = 1$ $\begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$

$x_1 = 2$ $x_2 = 1$ $\begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix}$

no such vectors exist, Since $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ are linearly independent

Since T_A is not one-to-one these vectors must exist.
Try e.g. $v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True

For any equation $w + u = a_1 u + a_2 v$, $w = a_1 u - u + a_2 v = cu + a_2 v$

$w \in \mathrm{Span}(u, v)$

✓

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True

If $\{u, v\}$ were dependent, then $a_1 u + a_2 v + 0w = 0$ would prove $\{u, v, w\}$ to be dependent

✓

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False

$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ $V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$AV_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $AV_2 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

✓

V_1, V_2 are independent

AV_1, AV_2 are dependent

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

$u = a_1 V + a_2 w$ True ✓

~~$T(u) = a_1 T(v) + a_2 T(w)$~~

u, $a_1 V + a_2 w$ are equivolent vectors ← actually equal (by your assumtion)

$\therefore$

Their transformations are the same | so ?

0 1 2 3 4 5 6 7 8 9 10