MTH-309T-F19-EX1-021-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	3	6	7	8	7

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

20	9	10	18	16	7	10	2	10	101	A
1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

(a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$$

$$\frac{1}{2}\times \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 2 & -3 & 0 & b \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\begin{array}{c} -1\times \\ b \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 1 & -\frac{3}{2} & 0 & \frac{b}{2} \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -\frac{1}{2} & -1 & \frac{b}{2}+2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$x-2 \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -\frac{1}{2} & -1 & \frac{b}{2}+2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$x-1 \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & -2(\frac{b}{2}+2) = -b-4 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & -b-4 \\ 0 & 0 & 0 & b+6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & -b-4 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$

Since $b+6$ is in last colum $b+6$ has to be equal to 0 which $b=-6$ to make zero otherwise $b+6$ will be 1. which is undefind. So $\boxed{b=-6}$

✓

(b)

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & -\frac{1}{2} & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = -x_2 + x_3 \\ x_2 = x_3 \\ x_3 = x_3 \end{array}$$

No the set is not linear Independent Some Scalar mutiple of v_1 add to Scalar multiple of v_2 will product v_3.

✓

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{-1}\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ \cancel{4} & -2 & 1 \end{bmatrix}$$

$A - D + 2G = 1$
$A + G = 0$
$2D - G = 0$

$-D + G = 1$
$2D - G = 0$

$-2D + 2G = 2$
$2D - G = 0$

$G = 2$
$A = -2$
$D = 1$

$B - E + 2H = 0$
$B + H = 1$
$2E - H = 0$

$2B - 2E + 4H = 0$
$2E - H = 0$

$2B + 3H = 0$
$2B + 2H = 2$

$-H = 2$
$H = -2$
$B = 3$
$E = -1$

$C - F + 2I = 0$
$C + I = 0$
$2F - I = 1$

$2C - 2F + 4I = 0$
$2F - I = 1$

$2C + 3I = 1$
$2C + 2I = 0$

$I = 1$
$C = -1$
$F = 1$

← Not the best way, but it (mostly) worked.

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad A^T \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix} \checkmark$$

$A+D=1$
$-A+2G=4$
$2A+D-G=3$

$2A+D-G=3$
$A+D=1$

$A-G=2$
$-A+2G=4$

$G=6$
$A=8$

$D=-7$

$B+E=2$
$-B+2H=5$
$2B+E-H=2$

$2B+E-H=2$
$B+E=2$

$B-H=0$
$-B+2H=5$

$H=5$
$B=5$
$E=-3$

$C+F=3$
$-C+2I=4$
$2C+F-I=1$

$2C+F-I=1$
$C+F=3$

$C-I=-2$
$-C+2I=4$

$I=2$

$-C+4=4$

$-C=0$
$C=0$
$F=3$

Simpler:

$$C = (A^T)^{-1} \cdot B$$
$$= (A^{-1})^T \cdot B$$

and A^{-1} was computed in problem 2

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors u satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-2\cdot 0 \\ 1+0 \\ 1-3\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-2\cdot 1 \\ 0+1 \\ 0-3\cdot 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

Standard matrix of T is equal

$T\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$ ✓

b)

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \times -1$ $\quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \times 2$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \times -1$ $\quad \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \times \frac{1}{3}$ $\quad x_1 = 7$

$x_2 = 3$

$x_3 = x_3$

There is no x_3

✓

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \times -1$ $\quad$ So when u equal $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$ will satisfaing $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] \times -1$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

☒ 18

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(v) = Av$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(v_1) = T_A(v_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \times \frac{1}{2}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \times -3$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \times -1$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \times \frac{1}{2}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \times -2$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \times -1$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Since every column of matrix A has pivot position TA is one to one ✓

$Nul(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \times \frac{1}{2}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \times -3$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \times -1$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + x_2 = 0$
$x_2 + 2x_3 = 0$
$x_3 = x_3$

$x_1 = -x_2$
$x_2 = -2x_3$
$x_3 = x_3$

$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} x_3$

x_2 is not a free variable

$Nul(A) = \left\{ \cancel{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}} \right\}$ $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

Since not all column of matrix A has pivot column TA is not one to one.

Since $Nu(A)$ in part a is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$
and $Nu(A)$ in part b is $\left\{ \cancel{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}} \right\}$ $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

and, $TA(v_1) = TA(v_2)$
v_1 has to equal v_2
which v_1 and v_2 are both $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
because each Null space contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

v_1, v_2 should be different vectors, but since it was not explicitly stated I will accept it

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w}+\mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

~~false~~

$w+u = x_1u + x_2v$

$w+u-u = x_1u + x_2v - u$

$w = x_1u + x_2v - u$

since $w+u \in \mathrm{span}(u, v)$

so $w+u = x_1u + x_2v$ ✓

and $w = x_1u + x_2v - u$ ✓

does not equal to $w = x_1u + x_2v$

which w is not linear combination of $x_1u + x_2v$

which w not in $\mathrm{Span}(u, v)$

$w = (x_1 - 1)u + x_2v$

so $w \in \mathrm{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

$$\begin{array}{ccc} u & v & w \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

True ✓

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

because u v have pivot position at every column

$\Downarrow$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This could be explained more clearly...

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

false

A $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ u $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ v $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Au \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Av \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

u and v are not linearly dependent
no scalar multiple of u could product v

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True

Yes because linear transformation has some algebra properties as Nector addiction with $u = x_1 v + x_2 w$

$T(u) = x_1 T(v) + x_2 T(w)$

A $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ u $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ v $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ w $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10