MTH-309T-F19-EX1-049-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	9	9	7	8	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	TOTAL	GRADE
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1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

$b \neq 0$

$b > 0$

$b < 0$

b) b is linearly independent as the reduced form has exactly one solution when A = 0

$$\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & 0 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix} \qquad \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix} \qquad \begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}$$

$$A^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array} \rightarrow \begin{array}{cccccc} 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}$$

0 1 2 3 4 5 6 7 8 9 10

$B \cdot \frac{1}{A^T}$

$B \cdot A^{T^{-1}}$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A^T C = B \to C = \frac{B}{A^T}$ $\quad C = B \cdot (A^T)^{-1}$

$C = B \cdot (A^{-1})^T$

$A^{-1} = \begin{matrix} 2 & 3 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 1 \end{matrix}$

$(A^{-1})^T = \begin{bmatrix} 2 & -1 & 2 \\ 3 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 2 \\ 3 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$$\begin{array}{c|ccc} & c_1 & c_2 & c_3 \\ \hline c_1 & 11 & -6 & 9 \\ c_2 & 27 & -13 & 22 \\ c_3 & 13 & -6 & 11 \end{array}$$

$c_{11} = \overset{-2}{1 \cdot 2} + \overset{6}{2 \cdot 3} + \overset{3}{1 \cdot 3}$

$c_{12} - \overset{-1}{1 \cdot 1} + \overset{-2}{2 \cdot -1} + \overset{-3}{3 \cdot -1}$

$c_{13} = \overset{2}{2 \cdot 1} + \overset{4}{2 \cdot 2} + \overset{3}{3 \cdot 1}$

$c_{21} = \overset{8}{4 \cdot 2} + \overset{15}{5 \cdot 3} + \overset{4}{4 \cdot 1}$

$c_{22} = \overset{-4}{4 \cdot -1} + \overset{-5}{5 \cdot -1} + \overset{-4}{4 \cdot -1}$

$c_{23} = \overset{8}{4 \cdot 2} + \overset{10}{5 \cdot 2} + \overset{4}{4 \cdot 1}$

$c_{31} = \overset{6}{3 \cdot 2} + \overset{6}{3 \times 2} + \overset{1}{1 \cdot 1} = 13$

$c_{32} = -3 + -2 + -1 = -6$

$c_{33} = 6 + 4 + 1 = 11$

$C = \begin{bmatrix} 11 & -6 & 9 \\ 27 & -13 & 22 \\ 13 & -6 & 11 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$A = [T(e_1), T(e_2)]$

where $e_1 = \begin{matrix} 1 \\ 0 \end{matrix}$ $e_2 = \begin{matrix} 0 \\ 1 \end{matrix}$

a)

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$

b)

$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & -1 & 4 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & -1 & 4 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{matrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a is

One to one

$\begin{matrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & -1 & 2 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & -1 & 2 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 8 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{matrix}$

$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{matrix}$

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{matrix}$

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

b is

one to one

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

False take $w = \begin{matrix}1\\0\\0\end{matrix}$ $u = \begin{matrix}1\\1\\0\end{matrix}$ $v = \begin{matrix}2\\1\\0\end{matrix}$

$w + u = \begin{matrix}2\\1\\0\end{matrix} \in \text{Span}(u, v)$, but

$w = \begin{matrix}1\\0\\0\end{matrix} \notin \text{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

Let $x_1 V = 3$
$x_2 V = 3$
$x_3 w = -6$

$x_1 V + x_2 V + x_3 W = 0$

$3 + 3 + -6 = 0$

$x_1 U + x_2 V = 0$

$x_1 U = x_2 V$, so $x_1 U$ is a scalar multiple of $x_2 V$

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Au, AV are depedent

but u, V are indepedent

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in $\mathrm{Span}(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in $\mathrm{Span}(T(\mathbf{v}), T(\mathbf{w}))$.

False

$u = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(u) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(v) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ $T(w) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

$u \in \mathrm{Span}(v, w)$

$T(u) \notin \mathrm{Span}(T(v), T(w))$

0 1 2 3 4 5 6 7 8 9 10