MTH-309T-F19-EX1-004-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	2	0	6	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
20	10	10	19	20	5	3	--	--	87	A-

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$[\, v_1 \quad v_2 \quad v_3 \,] = \vec{w}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b-6 \end{array}\right]$$

$b - 6 = 0$

$\boxed{\text{if } b = 6 \text{, then } \vec{w} \in \text{span}(\vec{v_1}, \vec{v_2}, \vec{v_3})}$ ✓

$X_1 - X_2 + X_3 = -2$

X_2

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↑ X_3 is free

to be linearly independent, every column after row reduction must be a pivot column. Therefore, the set is not linearly independent. ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1/2 & 1 & -1 & 1/2 \end{array}\right]$$

$$\xrightarrow{2R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow[-2R_3+R_1]{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$ ✓

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$(A^T)^{-1} = (A^{-1})^T \qquad AA^{-1} = I$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$x_1 + y_1 = 1$

$-x_1 + 2z_1 = 4$

$2x_1 + y_1 - z_1 = 3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$(A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

Simpler:

$(A^T)^{-1} = (A^{-1})^T$,

and A^{-1} was computed in problem 2.

Then $C = (A^{-1})^T B$

$z_1 = 6$

$y_1 = 5 - 2(6)$

$y_1 = -7$

$x_1 = 1 - -7$

$x_1 = 8$

$$C_1 = \begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -1 & 0 & 2 & 5 \\ 2 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

check

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix} = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{matrix} \checkmark$$

$z_2 = 5$

$y_2 = 7 - 10$

$y_2 = -3$

$x_2 = 2 - -3$

$x_2 = 5$

$$C_2 = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 2 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -1 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$z_3 = 2$

$y_3 = 7 - 4 = 3$

$x_3 = 3 - 3 = 0$

$$C_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

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4. (20 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$Tx = Ax$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{matrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$. $= A\vec{u}$ $\quad \vec{u}$ is 2×1 $= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$ ✓

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}\vec{u} = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$ ✓ $\quad \boxed{\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$ ✓

$u_1 - u_2 = 1$

$u_2 = 3$

$u_1 = 4$

$$\begin{bmatrix} \cancel{4} \\ 3 \end{bmatrix}$$ 7

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

One to one if pivot position in every column.

a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to$

$\boxed{\text{A is one to one}}$ ✓

b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Not a pivot column

A is not 1-1. ✓

$T_A(v_1) = A \cdot v_1$

$A \cdot v_1 = A \cdot v_2$

let $x_3 = 2$

$x_3 = 2$

$x_2 = -2(2) = -4$

$x_1 = 0 - -4 = 4$

$\begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = v_1$ ✓ $v_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = x_3$ $x_3 = 1$

$x_2 + 2x_3 = 0$ $x_2 = 0 - 2$ $= -2$

$x_1 + x_2 = 0$ $x_1 = 0 - -2$ $= 2$

let $x_3 = 1$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True. lin. combinations of vectors in a span are still within a span and vice versa ?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True ✓. In the set of 3, they are all lin. ind. from each other, meaning if one vector was taken away, the remaining two would still be lin. ind.

why?

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~true~~. A matrix is a linear transform, so anything that is linear dependent that is linearly transformed stays linearly dependent.

True, but it is not what this problem states.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

true. linear transforms will not change is something is within a span ← why?

$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \overset{u}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

☐ 0 ☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10