

1. Consider the following vectors in \mathbb{R}^4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V .

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V .

$$v_1 \cdot v_2 = (1)(2) + (0)(1) + (-1)(-1) + 0$$

a.) Use Gram-Schmidt: $\langle w_1, w_1 \rangle = 3$

$$w_1 = v_1 \rightarrow \langle w_1, v_2 \rangle = (1)(2) + 0 + 1 + 0 = 3$$

$$w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = w_2$$

$$w_3 = v_3 - \frac{\langle w_1, v_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_3 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 - 2w_1 + w_2 = \begin{bmatrix} 2 \\ -2 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\langle w_1, v_3 \rangle = 2 + 0 + 1 + 3 = 6$$

$$\langle w_2, v_3 \rangle = 2 - 2 + 0 - 3 = -3$$

$$\langle w_3, w_2 \rangle = 3$$

$$\text{b.) } \text{proj}_V u = \frac{\langle u, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle u, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \frac{\langle u, w_3 \rangle}{\langle w_3, w_3 \rangle} w_3$$

$$= w_1 + w_2 + w_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \text{proj}_V u$$

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \checkmark$$

$$\langle u, w_1 \rangle = 3 + 0 - 3 + 3 = 3$$

$$\langle u, w_2 \rangle = 3 + 1 + 0 - 3 = 1$$

$$\langle u, w_3 \rangle = 3 - 3 + 3 + 0 = 3$$

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2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0), (-1, 2), (2, 1)$.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

~~$$A^T A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 3 & -1 & 5 \end{bmatrix}$$~~

~~$$A^T A x = A^T b$$~~

$$b = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad 0 + (-2) + 2$$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

~~$$A^T b = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$~~

?

~~$$R3 \xrightarrow{\frac{1}{2}} \left(\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & 2 & -1 & 2 \\ 3 & -1 & 5 & 1 \end{array} \right) \xrightarrow{\frac{1}{2}R2} \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & -1 & -4 & 1 \end{array} \right) \xrightarrow{R3 + R1} \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & -9/2 & 2 \end{array} \right) \xrightarrow{-\frac{2}{9}R3} \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 1 & \frac{2}{9} \end{array} \right)$$~~

$$\xrightarrow{\frac{1}{3}R3} \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 1 & \frac{2}{9} \end{array} \right) \xrightarrow{\frac{1}{2}R2} \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & \frac{7}{9} \\ 0 & 0 & 1 & \frac{4}{9} \end{array} \right) \xrightarrow{R1 + R2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{7}{9} \\ 0 & 0 & 1 & \frac{4}{9} \end{array} \right) \quad x_1 = \frac{3}{2}, \quad x_2 = \frac{7}{9}, \quad x_3 = \frac{4}{9}$$

$$\frac{9}{10} \quad \frac{2}{9} \quad \frac{3}{2}$$

$$\frac{2}{3} \neq \frac{1}{2}$$

$$(\frac{3}{2}, 0) \quad (\frac{7}{9}, 2) \quad (-\frac{4}{9}, 1)$$

$$a = \frac{1 - 0}{-\frac{8}{18} - \frac{36}{18}} = -\frac{1}{44}$$

$$b = 0$$

$$\frac{0.4}{1580} = \frac{1}{3900}$$

$$f(x) = 0.4x + 0$$

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$$\frac{1}{18} \quad \frac{3}{18} \quad \frac{8}{18}$$

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3. Consider the following matrix A :

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A .

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

b) $\lambda = -1$

c) $\lambda = -2$

$$\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{vmatrix} = \det \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{vmatrix} \rightarrow \text{Picks-Minor.}$$

$$\det A = (1) \det \begin{bmatrix} 1 & 2 \\ 2 & 2-\lambda \end{bmatrix} (-1^3) + (1-\lambda) \det \begin{bmatrix} -\lambda & 2 \\ 4 & 2-\lambda \end{bmatrix} (-1^4) + 0$$

$$= (2-f-4)(1)(-1) + (1-f)(f^2 - 2f + 8) = f^3 + 2f^2 + 8f + f^2 - 2f - 8$$

$$= (-2-h)(-1) + (-h^3 + 2h^2 + 8h + h^2 - 2h - 8)$$

$$= (h+2) + (-h^3 + 3h^2 + 6h - 8) = -h^3 + 3h^2 + 7h - 6 = 0$$

Check: $\int_0^1 f(x) dx = 0$

check: $f = -1$

check $f = -2$

$$\begin{aligned} -O + O + O - 6 &= 0 \\ -6 &= 0 \times \end{aligned}$$

$$8 + 12 + (-14) - 6 = 0$$

$0 = 0 \checkmark$

$$1+3-7-6=0$$
$$-9=0$$

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Only $\lambda = -2$ is an eigen val for this one.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

$$\text{Null}(A - \lambda_1 I) = \left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ -2 & 8 & 4 & 0 \\ 2 & -8 & -4 & 0 \end{array} \right] \xrightarrow{\text{R2} - \text{R1}, \text{R3} + \text{R1}} \left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{R1} \rightarrow \frac{1}{2} \text{R1}} \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] = \begin{cases} x_1 = 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{cases}$$

$$x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Null}(A - \lambda_2 I) = \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array} \right] \xrightarrow{\text{R2} + \text{R1}, \text{R3} + \text{R1}} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array} \right] \xrightarrow{\text{R1} + \text{R2}, \text{R3} + 2\text{R2}} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \end{array} \right] \xrightarrow{\text{R3} + 4\text{R2}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x_1 = x_3 \\ x_2 = -x_3 \\ x_3 \end{cases} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$A = PDP^{-1}$$

~~$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$~~

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$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ . Assumption: false.

b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \quad z = w - w = 0$$

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix. Assump: false.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a.) $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ ~~$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} = (\lambda-2)(\lambda-2)$~~

$\text{Null}(A - 2I) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{x_1=0} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{Row 1} - \text{Row 2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{x_1=0} \begin{bmatrix} 0 & 0 \\ x_2 & 1 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$2v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

+1 a.) this is false since for a given Matrix ~~there~~

~~there~~ exists $P = [v_1 \dots v_n]$ such that

$v_1 \dots v_n$ are eigenvectors, and $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

where the eigenvalues are found on diagonal

The Eq $A^K = P D P^{-1}$ doesn't involve
a change in P at all so why should

$2v$ correspond to $2R$.

[Don Back]

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The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

- a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V .
 b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V .

von Schmidt

$$w_1 = v_1 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1} \right) v_2 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1} \right) v_3$$

$$w_2 = v_2 - \left(\frac{w_2 \cdot v_1}{w_2 \cdot w_2} \right) v_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2} \right) v_3$$

$$w_3 = v_3 - \left(\frac{w_3 \cdot v_1}{w_3 \cdot w_3} \right) v_1 - \left(\frac{w_3 \cdot v_2}{w_3 \cdot w_3} \right) v_2$$

$$w_1 \cdot v_2 = 1(2) + 0(1) + (-1)(-1) + 1(0) = 2 + 0 + 1 + 0 = 3$$

$$w_1 \cdot w_1 = 1^2 + 0^2 + (-1)^2 + (1)^2 = 3$$

$$w_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$w_2 \cdot v_3 = 1(2) + 1(-2) + 0(-1) + (-1)(3) \\ = 2 - 2 + 0 - 3 = -3$$

$$w_2 \cdot w_2 = 1^2 + 1^2 + 0^2 + (-1)^2 = 3$$

$$w_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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$$\text{proj}_V u = \left(\frac{u \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2} \right) w_2 + \left(\frac{u \cdot w_3}{w_3 \cdot w_3} \right) w_3 = \frac{3}{3} w_1 + \frac{3}{3} w_2 + \frac{3}{3} w_3 = w_1 + w_2 + w_3$$

$$w_1 = 3 + 0 - 3 + 3 = 3$$

$$w_2 = 3 + 3 + 0 - 3 = 3$$

$$w_3 = 6 - 3 + 0 + 3 = 6$$

$$w_1 + w_2 + w_3 = 6 + 6 = 12$$

$$\text{proj}_V u = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0), (-1, 2), (2, 1)$.

$$\underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_b \quad (A^T A)x = A^T b$$

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & -1+1+2 \\ 1-1+2 & 1+1+1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-2+2 \\ 0+2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array} \right] \xrightarrow{R_1-3R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & -7 & -9 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 1 & \frac{9}{7} \end{array} \right] \xrightarrow{-2R_2+R_1 \rightarrow R_1}$$

$$\left[\begin{array}{cc|c} 6 & 0 & -\frac{18}{7} \\ 0 & 1 & \frac{9}{7} \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -\frac{3}{7} \\ 0 & 1 & \frac{9}{7} \end{array} \right] \rightarrow \begin{aligned} a &= -\frac{3}{7} \\ b &= \frac{9}{7} \end{aligned}$$

$$\boxed{f(x) = -\frac{3}{7}x + \frac{9}{7}} \quad \checkmark$$

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where $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is eigenvector corresponding to λ

3. Consider the following matrix A:

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

$$\begin{aligned} v_2 + v_3 &= \lambda v_1 \\ v_1 + v_2 &= \lambda v_2 \\ 4v_1 + 2v_2 + 2v_3 &= \lambda v_3 \end{aligned}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ No

b) $\lambda = -1$ No

c) $\lambda = -2$ Yes

$$-\lambda I = \det \begin{pmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{pmatrix} = 0 = -1 \begin{pmatrix} 2-\lambda & 4 \end{pmatrix} + (1-\lambda) \begin{pmatrix} -2\lambda + \lambda^2 - 8 \end{pmatrix} = 0$$

$$2+\lambda + (-2\lambda + \lambda^2 - 8 + 2\lambda^2 - \lambda^3 + 8\lambda) = 0$$

$$-\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0$$

$$\lambda^3 - 3\lambda^2 - 7\lambda + 6 = 0 \quad \checkmark$$

a) $0^3 - 3(0)^2 - 7(0) + 6 = 0 \quad 6 \neq 0 \rightarrow \text{Not an eigenvalue}$

b) $(-1)^3 - 3(-1)^2 - 7(-1) + 6 = 0 \quad \checkmark$

$-1 - 3 + 7 + 6 = 0 \quad \checkmark$

$9 \neq 0 \rightarrow \text{Not an eigenvalue}$

c) $(-2)^3 - 3(-2)^2 - 7(-2) + 6 = 0 \quad \checkmark \quad 20/20$

$-8 - 12 + 14 + 6 = 0$

$0 = 0 \quad \checkmark \rightarrow \text{IS an eigenvalue}$

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

For $\lambda = 3$, $A - 3I = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$Nul(A - 3I) = \text{Span} \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ = eigenspace of A corresponding to λ_1

$$A - 5I = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -2 \\ -2 & 4 & 2 \\ -1 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & -2 & -2 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Nul(A - 5I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad P = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

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5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ .

$\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$ b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) False, just because there is an eigenvalue λ does not mean there is an eigenvalue 2λ . There is no guarantee $2\lambda = \text{val}$ is another solution to $|A - I(\text{val})| = 0$. Ex) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda_1 = 1$ $\lambda_2 = -1$

b)

$$\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} \xrightarrow{\frac{W_1 V_{11} + W_2 V_{21}}{\sqrt{V_{11}^2 + V_{21}^2}}} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} -W_1 \\ -W_2 \end{bmatrix} \quad \text{say } W_1 = W_2 = 1$$

$$\begin{bmatrix} W_1 + W_2 \\ W_1 + W_2 \end{bmatrix} + \begin{bmatrix} W_1 + W_2 \\ W_1 + W_2 \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

True, the projection of a vector on some space can be visualized as the "shadow" of that vector on that space, \therefore any projection of w_1, w_2 somewhere else should carry the same sign, unless they both are zero

) False, $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $A = A^T \checkmark$ Symmetric \checkmark $A^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i) False $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

↑ does not have 2 linearly independent eigenvectors

1. Consider the following vectors in \mathbb{R}^4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $B = \{v_1, v_2, v_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

a) Find an orthogonal basis $D = \{w_1, w_2, w_3\}$ of the subspace V .

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V .

Row reduction of matrix A (with annotations):

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 3 \\ -1 & -1 & -1 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{R_3 + R_4} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 3 \\ -1 & 0 & -2 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{R_4 - 2R_1} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{bmatrix} \xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

orthogonal basis D is the basis of $\text{col}(A)$

Calculation of dot products:

$$W_1 \cdot V_2 = 1 \cdot 2 + 0 \cdot 1 + -1 \cdot -1 + 1 \cdot 3 = 3$$

$$W_1 \cdot W_1 = 1 \cdot 1 + 0 \cdot 0 + 1 \cdot -1 + 1 \cdot 1 = 3$$

$$W_2 \cdot V_3 = 1 \cdot 2 + 1 \cdot -1 + 0 \cdot -1 + 1 \cdot 3 = 3$$

$$W_2 \cdot W_2 = 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 + -1 \cdot -1 = 3$$

a) $W_1 = V_1$

$$W_2 = V_2 - \left(\frac{W_1 \cdot V_2}{W_1 \cdot W_1} \right) W_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$W_3 = V_3 - \left(\frac{W_1 \cdot V_3}{W_1 \cdot W_1} \right) W_1 - \left(\frac{W_2 \cdot V_3}{W_2 \cdot W_2} \right) W_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{-3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$D = \boxed{\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}}$

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part b
on back

~~*~~ ~~*~~

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0), (-1, 2), (2, 1)$.

$$W_1 = \langle v_1 \rangle$$

$$W_2 = v_2$$

$$\frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} W_1$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$W_1 = \langle v_1 \rangle$$

$$W_1 = \sqrt{\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}} W_1 = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$W_2 = \langle v_2 \rangle - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} W_1$$

~~f(x) = 30x - 2~~

I'm sorry,
I don't know
how to do this

$$A^T \cdot A x = A^T b$$

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\det A^T A = \boxed{30} 6 \cdot 5 = 30$$

$$\det A^T = (-1 \cdot 1 - 2 \cdot 0) + (1 \cdot 1 - 2 \cdot 0) + (1 \cdot 2 - 0) = -2$$

$$f(x) = 30x - 2$$

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3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$$\lambda = 0$$

a) $\text{Null}(A - \lambda_1 I)$ $\rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_1]{\frac{1}{2}R_3} \begin{bmatrix} -1 & 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 2 \end{bmatrix} \xrightarrow[R_1 + R_2]{R_1, R_2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$
 $\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow[\frac{1}{3}R_3]{R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_2, R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{no basis}$

0 is not an eigen value of A

b) $\lambda = -1$ $A + I I \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & -6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \underline{-1 \text{ is not an eigen value of } A}$

c) $\lambda = -2$ $A + 2I \rightarrow \begin{bmatrix} 2 & 0 & 2 \\ 1 & 3 & 0 \\ 4 & 2 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{R_3 \rightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}$
 $\xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[-1R_3]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \underline{-2 \text{ is not an eigen value of } A}$

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4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

$$\text{basis of } (A - 3I) = \left[\begin{array}{ccc} -3 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{array} \right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc} -1 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{array} \right] \xrightarrow{R_3 + R_1} \left[\begin{array}{ccc} -1 & 8 & 4 \\ -2 & 8 & 4 \\ 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{ccc} -1 & 8 & 4 \\ 0 & 0 & 0 \\ 2 & -8 & -4 \end{array} \right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc} -1 & 8 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \begin{aligned} x_1 - 4x_2 - 2x_3 &= 0 \rightarrow x_1 = 4x_2 + 2x_3 \\ 0x_1 + 0x_2 &= x_3 \\ 0x_1 + 0x_2 &= x_3 \end{aligned}$$

$$\rightarrow x = x_3 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \rightarrow \text{basis} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{basis of } (A - 5I) = \left[\begin{array}{ccc} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{array} \right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{ccc} -1 & 2 & 1 \\ -2 & 3 & 2 \\ 2 & -8 & -6 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & -8 & -6 \end{array} \right] \xrightarrow{R_3 + R_1} \left[\begin{array}{ccc} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -3 & -5 \end{array} \right] \xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{-1R_1} \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \begin{aligned} x_1 + x_3 &= 0 \quad x_1 = -x_3 \\ x_2 + x_3 &= 0 \quad x_2 = -x_3 \\ 0x_1 + 0x_2 &= x_3 \quad x_3 = x_3 \end{aligned}$$

$$\rightarrow x = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \text{basis} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

so $P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ & $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

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$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{row } 1 - 2 \cdot \text{row } 2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{x_1 = x_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{\text{row } 2 + \text{row } 1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{x_2 = 0} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ .

b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) ~~false, because multiplying~~ $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ $\lambda_1 = 1$ then $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
 but $\lambda_2 = 2$ then $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $x_1 = 0$ no v_2

b) True, because ~~for~~ for the proj to equal the original vector,
 +3 the vector must already fall on the projected plan
 with $0=0$ as the only vector that could possibly be
 equal to its negative proj

c) ~~false~~, because $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ is a square, symmetric, & orthogonal
 +1 matrix but $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is not the Identity matrix

D) True, because the resulting matrix will still be
 +5 symmetrical which means that it will also be
 orthogonally diagonalizable

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} // w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1} \right) w_1$$

$$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1} \right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2} \right) w_2$$

1. Consider the following vectors in \mathbb{R}^4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} // w_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

The set $B = \{v_1, v_2, v_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

a) Find an orthogonal basis $D = \{w_1, w_2, w_3\}$ of the subspace V .

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V .

$$w_1 = v_1 //, \quad w_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{3}{3} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \checkmark$$

$$w_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \left(\frac{6}{3} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{-3}{3} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \\ -1 & 0 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\text{proj}_V u = \left(\frac{u \cdot v_1}{v_1 \cdot v_1} \right) v_1 + \left(\frac{u \cdot v_2}{v_2 \cdot v_2} \right) v_2 + \left(\frac{u \cdot v_3}{v_3 \cdot v_3} \right) v_3$$

$$= \left(\frac{3}{3} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{3}{3} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \left(\frac{0}{18} \right) \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$\text{proj}_V u = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} //$$

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

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2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$A \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array} \right] \xrightarrow{-2/6 R_1 + R_2} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 7/3 & 3 \end{array} \right] \xrightarrow{3/7 R_2} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 1 & 9/7 \end{array} \right] \xrightarrow{-2R_2 + R_1} \left[\begin{array}{cc|c} 6 & 0 & 18/7 \\ 0 & 1 & 9/7 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 6 & 0 & 18/7 \\ 0 & 1 & 9/7 \end{array} \right] \xrightarrow{1/6 R_1} \left[\begin{array}{cc|c} 1 & 0 & 3/7 \\ 0 & 1 & 9/7 \end{array} \right] \quad \boxed{\text{Crossed out}}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3/7 \\ 9/7 \end{bmatrix}$$

$$f(x) = -3/7x + 9/7$$

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3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$

b) $\lambda = -1$

c) $\lambda = -2$

$$\left(\begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix} \right) = 0$$

$$\begin{aligned} -\lambda((1-\lambda)(2-\lambda)) - 1(2-\lambda) + 2[2-4(1-\lambda)] &= 0 \\ -\lambda^3 + 3\lambda^2 - 2\lambda - 2 + \lambda + 4 - 8 + 8\lambda &= 0 \\ -\lambda^3 + 3\lambda^2 + 7\lambda - 6 &= 0 \end{aligned}$$

i) $\lambda = 0$

$$-(0)^3 + 3(0)^2 + 7(0) - 6 \neq 0$$

$-6 \neq 0 \rightarrow$ Not an eigenvalue of A

$$-(-1)^3 + 3(-1)^2 + 7(-1) - 6 \neq 0$$

$1 + 3 - 7 - 6 \neq 0 \rightarrow$ not an eigenvalue of A

$$-(-2)^3 + 3(-2)^2 + 7(-2) - 6 = 0$$

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$$8 + 12 - 14 - 6 = 0$$

$0 = 0 \rightarrow \lambda = -2$ is an eigenvalue of A

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

$\lambda_1 = 3$ $\begin{bmatrix} 1-3 & 8 & 4 \\ -2 & 11-3 & 4 \\ 2 & -8 & -1-3 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 8 & 4 & | & 0 \\ 2 & 8 & 4 & | & 0 \\ 2 & -8 & -4 & | & 0 \end{bmatrix} \xrightarrow{\substack{R_1+R_2 \rightarrow R_2 \\ R_1+R_3}} \begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} 0 & -4 & -2 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 = 4x_2 + 2x_3 \Rightarrow \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} x_3 \Rightarrow \text{NUL}(A - \lambda_1 I_n) = \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$x_2 = \text{free}$

$x_3 = \text{free}$

$\lambda_2 = 5$ $\begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -4 & 8 & 4 & | & 0 \\ -2 & 6 & 4 & | & 0 \\ 2 & -8 & -6 & | & 0 \end{bmatrix} \xrightarrow{\substack{-\frac{1}{2}R_1 + R_2 \rightarrow R_2 \\ \frac{1}{2}R_1 + R_3 \rightarrow R_3}} \begin{bmatrix} 0 & 2 & 2 & | & 0 \\ 0 & -4 & -4 & | & 0 \end{bmatrix} \xrightarrow{2R_2 + R_3 \rightarrow R_3} \begin{bmatrix} -4 & 8 & 4 & | & 0 \\ 0 & 2 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{-4R_2 + R_1 \rightarrow R_1} \begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 0 & 2 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 & -4 & | & 0 \\ 0 & 2 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 = \text{free} \end{array} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3 \Rightarrow \text{NUL}(A - \lambda_2 I_n) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} //$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [v] = \lambda v$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c & d \\ a & c \end{bmatrix}$$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ .

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

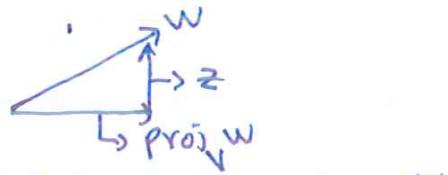
d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) False The eigenvalue is the root of the characteristic polynomial which means its the solution that makes the equation $p(\lambda) = 0$ true. An $n \times n$ matrix can have ~~no~~ more than n roots so 2λ would not be the eigenvalue.

b) True projection of a vector w on subspace V is unique ~~to every vector~~ such that $z = w - \text{proj}_V w \Rightarrow w = z + \text{proj}_V w$

where z is an orthogonal vector to V

\rightarrow This means the only way $\text{proj}_V w = -w$ is if vector w is a zero vector.



\rightarrow If w is an element of vectorspace of V then $\text{proj}_V w = w$ which is different from $\text{proj}_V w = -w$

c) False If A is symmetric then it has a orthogonal eigenvector

A can then be expressed as $A = Q D Q^T$

$A^2 = Q D^2 Q^T$ not identity matrix

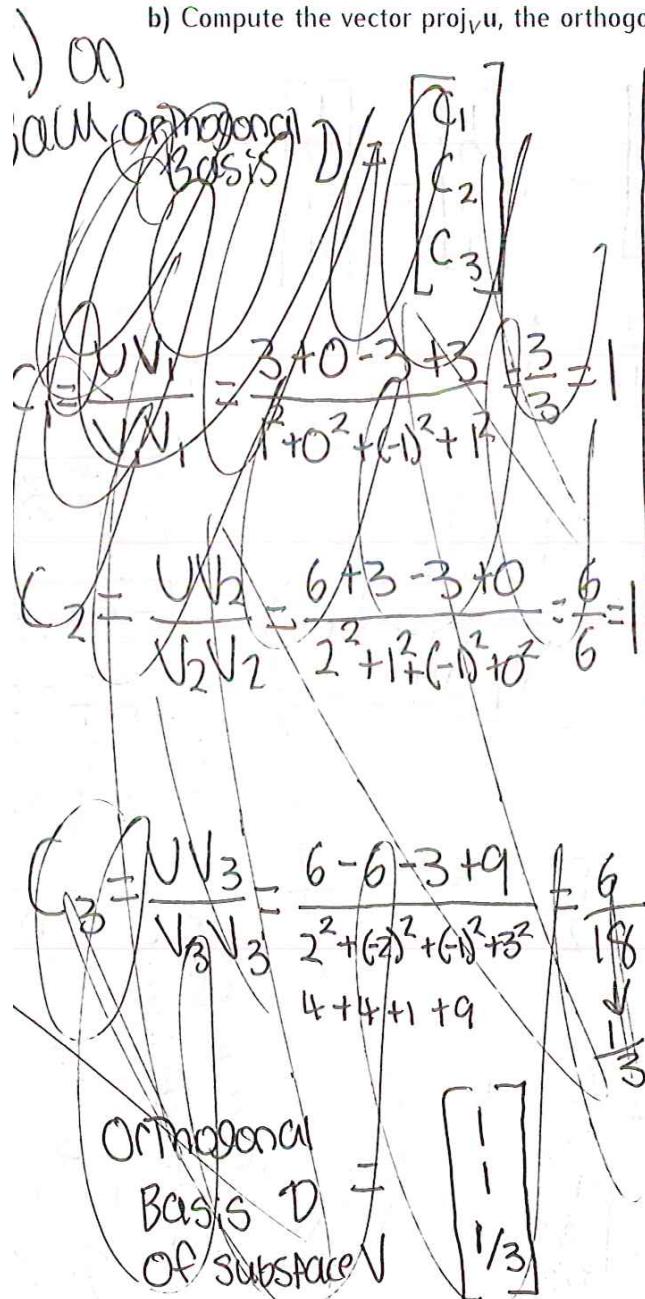
1. Consider the following vectors in \mathbb{R}^4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $B = \{v_1, v_2, v_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

a) Find an orthogonal basis $D = \{w_1, w_2, w_3\}$ of the subspace V .

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V .



$$\text{proj}_V u = \frac{u \cdot v_1}{\|v_1\|^2} v_1 + \frac{u \cdot v_2}{\|v_2\|^2} v_2 + \frac{u \cdot v_3}{\|v_3\|^2} v_3$$

$$u \cdot v_1 = 3+0-3+3 = 3$$

$$\|v_1\|^2 = 1^2+0^2+(-1)^2+1^2 = 3$$

$$u \cdot v_2 = 6+3-3+0 = 6$$

$$\|v_2\|^2 = 2^2+1^2+(-1)^2+0^2 = 6$$

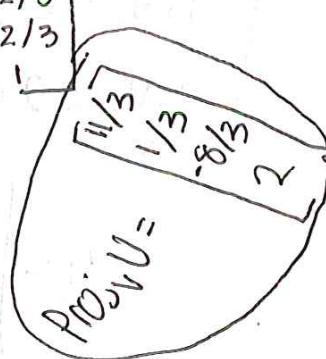
$$u \cdot v_3 = 6-6-3+9 = 6$$

$$\|v_3\|^2 = 2^2+(-2)^2+(-1)^2+3^2 = 18$$

$$\text{proj}_V u = \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{6} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{6}{18} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

$$\text{proj}_V u = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

$$\text{proj}_V u = \begin{bmatrix} 11/3 \\ 1/3 \\ -8/3 \\ 2 \end{bmatrix}$$



2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$(A \text{ and } B)$$

$$A^T A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad A \quad B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \xrightarrow{\cdot \frac{1}{6}}$$

$$f(x) = \cancel{-3x+9}$$

18/20

$$\begin{bmatrix} 1 & 1/3 \\ 2 & 3 \end{bmatrix} + (-2)R_1$$

$$a = \cancel{17/3} - 3$$

$$b = 9$$

$$\begin{bmatrix} 1 & 1/3 \\ 0 & 1/3 \end{bmatrix} + (-1)R_2 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \cdot 3 \rightarrow \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 9 \end{bmatrix}$$

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ $0+0+0-6=0$ X	b) $\lambda = -1$ $1+3-7-6=0$ $-a=0$ X	c) $\lambda = -2$ $8+12-14-6=0$ $20-20=0$ $0=0$ $\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0$ $\frac{8+12}{\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0}$ ✓
---	--	---

20

~~✗~~ ~~✗~~ ~~✗~~
 $\lambda = -2$ is the
only eigenvalue
of A

$$\begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}$$

20/20

$$(-\lambda)((1-\lambda)(2-\lambda)-0) - 1(1(2-\lambda)-0) + 2(2-4(1-\lambda))$$

$$(-\lambda)(2-3\lambda+\lambda^2) - (2-\lambda) + 2(2-4+4\lambda)$$

~~$\lambda^3 + 3\lambda^2 + 7\lambda - 6$~~

$$-\lambda^3 + 3\lambda^2 + 7\lambda - 6$$

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

$$\lambda = 3$$

$$A - 3I:$$

$$\begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} + (-1)R_1 + R_3$$



$$\begin{bmatrix} -2 & 8 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$x_1 = 4x_2 + 2x_3$$

$$\lambda = 3: \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

also eigenvectors $[v_1, v_2, v_3]$

$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

with eigenvalues
in D corresponding
to v_1, v_2, v_3 respectively

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \end{aligned}$$

$$\lambda = 5: \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = 5: A - 5I:$$

$$\begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \xrightarrow{-\frac{1}{4}}$$



$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \xrightarrow{+2R_1} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 2 & -8 & -6 \end{bmatrix} \xrightarrow{+(-2)R_1}$$



$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & -4 & -4 \end{bmatrix} \xrightarrow{+2R_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{2/2}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{+2R_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_3$$

$$x_2 = -x_3$$

$$\lambda = 5: \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Ans

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ .

b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) ~~(False)~~ \checkmark

\checkmark false. The eigenvalue would stay the same, as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ would be eigenvectors but linearly independent since $\lambda \rightarrow V = 2V \neq 2\lambda$

b) ~~(True)~~ \checkmark

\checkmark true; The Identity matrix, which is both A and A^2 in this case, is the only matrix that could be both symmetric and orthogonal

b) True; This has to be true because this would be the only way proj_V would equal $-w$. No other combo would produce this besides the trivial solution

d) ~~(True)~~ \checkmark false; This is not always the case. When matrices are added, properties are not always preserved, such as diagonalizability:

$$6. \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 2 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix}$$

diagonalizable diagonalizable not diagonalizable

1. Consider the following vectors in \mathbb{R}^4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V .

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V .

$$\begin{aligned} X_1 &= V_1, \quad X_2 = V_2 - \frac{V_2 \cdot V_1}{V_1 \cdot V_1} V_1, \quad X_3 = V_3 - \left(\frac{V_3 \cdot V_1}{V_1 \cdot V_1} V_1 + \frac{V_3 \cdot V_2}{V_2 \cdot V_2} V_2 \right) \\ V_1 &= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad V_2 \cdot V_1 = 2+0+1+0=3, \quad V_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad V_3 \cdot V_1 = 2+0+1+3=6 \\ V_1 \cdot V_1 &= 1+0+1+1=3, \quad V_1 \cdot V_1 = 3, \quad \frac{V_3 \cdot V_1}{V_1 \cdot V_1} = 2 \\ \frac{V_2 \cdot V_1}{V_1 \cdot V_1} &= \frac{3}{3}=1, \quad \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} \\ X_2 &= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ X_3 &= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \\ 3/2 \\ 0 \end{bmatrix} \\ X_3 &= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \\ 3/2 \\ 0 \end{bmatrix} \end{aligned}$$

$$\text{proj}_V u = \left(\frac{u \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2} \right) w_2 + \left(\frac{u \cdot w_3}{w_3 \cdot w_3} \right) w_3$$

$$w_1 = 3+0+3+3=3 \quad \text{Proj}_V u = w_1 + w_2 + \frac{9}{13} w_3 = w_1 + w_2 + \frac{18}{13} w_3$$

$$w_1 = 1+0+1+1=3$$

$$w_2 = 3+3+0-3=3$$

$$w_2 = 1+1+0+1=3$$

$$w_3 = -3+\frac{9}{2}+\frac{9}{2}+3=9$$

$$w_3 = 1+0.25+0.25+1=\frac{13}{2}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -18/13 \\ 87/13 \\ 27/13 \\ 18/13 \end{bmatrix} = \begin{bmatrix} 0-18/13 \\ 1+27/13 \\ -1-57/13 \\ 18/13 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 40/13 \\ -14/13 \\ 18/13 \end{bmatrix}$$

14/20

$$\text{Proj}_V u = \begin{bmatrix} 8/13 \\ 40/13 \\ -14/13 \\ 18/13 \end{bmatrix}$$

$\frac{27}{13}$

26

1.3
4.5
5
0
5

What's 6.  A₁ A₂ A₃

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Least Square

$$\text{Least Square} \quad (A^T A)^{-1} (A^T b) = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$(A^T A)^{-1} (A^T b) = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{2}{1-0} = \cancel{1} \cancel{\sqrt{3+1}}^2 = \cancel{2}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\begin{array}{l} \text{1} - \frac{1}{2} \\ \text{5} - 2 \\ \hline \end{array} \quad \left(\begin{array}{c} 2 \\ -2 \end{array} \right) \quad \sqrt{2^2 + 2^2} = \sqrt{8} \quad \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 2 & 1 \end{array} \right] \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] \stackrel{?}{=} \begin{array}{l} 18+10 \\ 0+2\sqrt{10}+\sqrt{2} \end{array}$$

$$\frac{1}{z^2} = -3 \sqrt{a+1} = \sqrt{10}$$

$$\frac{2}{1} - \frac{1}{0} = \frac{1}{1} \sqrt{-1} \sqrt{2}$$

$$50 \mid \Gamma_8 + \sqrt{2}$$

$$P(x) = \frac{18}{5} - \frac{\sqrt{10}}{5} + \frac{2\sqrt{2}}{5}x + \frac{2\sqrt{10}}{5} + \frac{\sqrt{2}}{5}$$

3. Consider the following matrix A :

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A .

- a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix} = -\lambda[(-\lambda)(2-\lambda)] - 1[0 - \lambda(2-\lambda)] + 2[0 + 4 - 4\lambda]$$

The roots are $\lambda = 0, 3, 5/4$
 therefore $\lambda = 0$ is an eigenvalue while $\lambda = -1$ and $\lambda = -2$ are not

$$1 - 2\lambda - \lambda + \lambda^2 - \lambda[\lambda^2 - 3\lambda + 1] - \lambda^3 + 3\lambda^2 - \lambda - 2 + 4 + 8 - 8\lambda (-\lambda^3 + 3\lambda^2 - 8\lambda + 10)$$

$$\lambda(-\lambda + 3) 2(4\lambda + 5) - 4\lambda = -\frac{5}{4} =$$

$$\lambda = 0 \quad \lambda = 3 \quad \lambda = 5/4$$

$$-\lambda^3 + 3\lambda^2 - \lambda - 2 + 4 + 8 - 16\lambda$$

$$-\lambda^3 + 3\lambda^2 - 16\lambda + 14$$

$$\lambda^2(-\lambda + 3)^2 - 6$$

$$\lambda = 0 \quad \lambda = 3$$

14/20

$$\lambda = 0 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

$$\lambda = -1 = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\lambda = -2 = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 3 & 0 \\ 4 & 2 & 4 \end{bmatrix}$$

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

$$\lambda_1 = 3 \quad \begin{bmatrix} 1-3 & 8 & 4 \\ -2 & 11-3 & 4 \\ 2 & -8 & -1-3 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 0 & -8 & -4 \end{bmatrix} \quad -2x_1 + 8x_2 + 4x_3 = 0 \quad \begin{aligned} x_1 &= -\frac{8}{2}x_2 - \frac{4}{2}x_3 \\ x_1 &= -4x_2 - 2x_3 \end{aligned}$$

$$V_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 5 \quad \begin{bmatrix} 1-5 & 8 & 4 \\ -2 & 11-5 & 4 \\ 2 & -8 & -1-5 \end{bmatrix} = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \quad \begin{aligned} -4x_1 + 8x_2 + 4x_3 &= 0 & x_1 &= +2x_2 + x_3 \\ -2x_1 + 6x_2 + 4x_3 &= 0 & -4x_2 - 2x_3 + 6x_2 + 4x_3 &= 2x_2 + 2x_3 & x_2 &= x_3 \\ 2x_1 - 8x_2 - 6x_3 &= 0 & x_1 &= 4x_3 \end{aligned}$$

$$V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = 4 - 2 + 0 = 2 \quad D^{-1} = \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$D = P^{-1}AP =$$

$$D = P^{-1} \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{array}{r} 1+8+0 \\ -8+11 \\ 8-8 \end{array} \quad \begin{array}{r} 2+1 \\ -4+4 \\ 4-1 \end{array} \quad \begin{array}{r} 9-4 \\ -2+11+4 \end{array}$$

$$D = P^{-1} \boxed{\begin{bmatrix} 9 & 6 & 5 \\ 3 & 0 & 5 \\ 0 & 3 & -5 \end{bmatrix}}$$

$$\text{Q10: } \text{Or if } \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \text{ if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^+ = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ .

b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

$$A^T = A \quad B^T B = I$$

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ then $\begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$ gives $\lambda=1, 2$ for $\lambda=1$ Eigenvector $= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
 $\therefore 2(\lambda)=2 \quad \begin{bmatrix} 1-2 & 1 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ \therefore False

True, the formula for $\text{Proj}_V w = \frac{(w \cdot v)}{(v \cdot v)} v$ if w zero vector $(w \cdot v) = 0$
 $\therefore (w \cdot v) = 0$ thus being true.

True because the matrix is symmetric $A^T = A$, and orthogonal $A^T \cdot A = I$. A^2 will be equivalent to $A \cdot A^T$ as $A = A^T$ therefore equating to the identity matrix.

True
~~True~~, only Symmetric 2×2 matrices and a Symmetric Matrix plus another symmetric matrix will be Symmetric.

5

1) ~~False~~, if $V = -W$ then $\text{Proj}_V W = \left(\frac{-w^2}{-w^2} \right) - w = 1(-w) = -w$
 therefore W can be a non-zero vector

354x976

$$400 \cdot 480 = 476 + 13$$

1. Consider the following vectors in \mathbb{R}^4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\frac{1}{18}, -\frac{1}{18}, -\frac{1}{18}, \frac{3}{18}$$

The set $B = \{v_1, v_2, v_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

- a) Find an orthogonal basis $D = \{w_1, w_2, w_3\}$ of the subspace V .
 b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V .

Using Gram-Schmidt Process we computed.

Rough. $4+4+1+9$
 $2+0+1+3$
 $\frac{5}{18} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \begin{pmatrix} 10/18 \\ -10/18 \\ -5/18 \\ 5/18 \end{pmatrix}$

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{w_1 \cdot v_2}{v_2 \cdot v_2} \cdot v_2$$

$$= \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} - \frac{2+0+1+0}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

8/20

✓ $w_3 = v_3 - \frac{w_1 \cdot v_3}{v_3 \cdot v_3} v_3 - \frac{w_2 \cdot v_3}{v_3 \cdot v_3} v_2$

$$D = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, ? \right\}$$

$$\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{2+0+0-3}{4+4+1+9} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{18} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$- \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \Rightarrow 2+0+0-3$$

$$= \begin{bmatrix} 19/18 \\ -19/18 \\ -9/18 \\ 19/16 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 10/18 \\ -19/18 \\ -9/18 \\ 13/16 \end{bmatrix} + \begin{bmatrix} 1/18 \\ -1/18 \\ -1/18 \\ 1/16 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 10/18 - 1/18 + 1/16 \\ -19/18 - 1/18 - 1/18 + 1/16 \\ -9/18 - 1/18 - 1/18 + 1/16 \\ 13/16 \end{bmatrix} = \begin{bmatrix} 10/18 - 9/18 + 13/16 \\ -19/18 - 1/18 - 1/18 + 1/16 \\ -9/18 - 1/18 - 1/18 + 1/16 \\ 13/16 \end{bmatrix}$$

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & 2 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & 2 \\ 0 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$x_1 + \frac{1}{2}x_3 = 0 \\ 4x_2 + x_3 = 0 \\ x_3 = 0$$

$\frac{1}{2}x_2$

$$0 = a + b$$

$$2 = -a + b$$

$$2 = a + b$$

$2x_3 = 3x_2$

?
 $a + b = 0$
 $-a + b = 2$
 $a + b = 2$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Let $b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$$b = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

3. Consider the following matrix A :

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A .

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$$\begin{aligned} 0 \det(A - \lambda I) &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} -\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix} \\ &= (-\lambda)(1-\lambda)(2-\lambda) \end{aligned}$$

a) Yes as $-\lambda_1 = 0$ $\lambda = 0$

b) No $\lambda_2 = 1$

c) No $\lambda_3 = 2$

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4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

$$\begin{array}{l} \left[\begin{array}{ccc} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & 3 \end{array} \right] \\ \Rightarrow \left[\begin{array}{ccc} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 0 & -1 & 5 \end{array} \right] \\ \Rightarrow \left[\begin{array}{ccc} -2 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & -1 & 5 \end{array} \right] \end{array}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

$$\lambda_1 = 3 \quad \begin{bmatrix} 1-\lambda & 8 & 4 \\ -2 & 11-\lambda & 4 \\ 2 & -8 & -1-\lambda \end{bmatrix} \quad 18/20$$

$$= \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -8 & -4 \end{bmatrix}$$

$$2x_1 - 8x_2 + 4x_3 = 0 \quad \text{or} \quad x_1 - 4x_2 + 2x_3 = 0$$

$$x_1 = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 5 \quad \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \sim \begin{bmatrix} -4 & 8 & 4 \\ 0 & -2 & -2 \\ 2 & -8 & -6 \end{bmatrix} \sim \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 4 & 8 & 4 \\ -2 & 6 & 4 \\ 0 & -6 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & 3 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 & 0 \\ -2 & 6 & 4 \\ 0 & -6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_2 = x_3, \quad \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{P}_1 = \begin{bmatrix} 4/\sqrt{17} \\ 1/\sqrt{17} \\ 5/0 \end{bmatrix} \quad \text{P}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \quad \text{P}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$\xrightarrow{\text{Reg B}}$

$$\begin{array}{l} (1-\lambda)(1-\lambda) \\ , -\lambda, \lambda=1 \\ 0, -\lambda \\ \lambda=0 \end{array}$$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ .

b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

$\frac{a}{+2}$ False, consider $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \lambda = 2$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \quad ?$$



$\frac{b}{+3}$ True, because $\text{proj}_V w$ is 0 if w is orthogonal to V .
and projection cannot reverse a direction.)?
The only case where $\text{proj}_V w = -w$ would be if $w=0$ because $w=0$

c) True, $A^T = A^{-1}$ (orthogonal) $A = A^T$ (symmetric)

$\cancel{A \cdot A^T = I} \Rightarrow A^T \cdot A = I$ (orthogonal)
 $(A^T)^T \cdot A = I$ since symmetric $A^2 = I$

d) $\frac{+5}{+5}$ True, since both A & B are $n \times n$ matrices, which are

orthogonally diagonalizable, which means they have to be symmetric if $A = \begin{bmatrix} x & m \\ m & y \end{bmatrix}$ & $B = \begin{bmatrix} p & r \\ r & q \end{bmatrix}$

$$6 \quad \text{then } A+B = \begin{bmatrix} x+p & m+r \\ m+r & y+q \end{bmatrix}$$

$A+B$ is symmetric & $n \times n$
so orthogonally diagonalizable.

1. Consider the following vectors in \mathbb{R}^4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V .

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V .

a) Gram-Schmidt Process

$$\begin{aligned} w_1 &= v_1 & v_2 &= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} & w_1 \cdot v_2 &= 2 + 0 + 1 + 0 = 3 \\ w_2 &= v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1} \right) w_1 & = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{3}{3} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} & = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & w_1 \cdot w_1 &= 1 + 0 + 1 + 1 = 3 \\ w_3 &= v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1} \right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2} \right) w_2 & = v_3 &= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} & w_2 \cdot v_3 &= 2 + (-2) + 0 + (-3) = -3 \\ & & & & w_2 \cdot w_2 &= 1 + 0 + 1 + 1 = 3 \\ & & & & w_1 \cdot v_3 &= 2 + 1 + 3 = 6 \end{aligned}$$

$$\begin{array}{c} \downarrow \\ \begin{bmatrix} 2 & -1 & +1 & 2 \\ -2 & 0 & +1 & -1 \\ -1 & -1 & +0 & 0 \\ 3 & -1 & +1 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} \downarrow \\ \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \left(\frac{6}{3} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{-3}{3} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \\ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \end{array} \quad \begin{array}{c} \downarrow \\ \mathcal{D} = \left[\begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \right] = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{array} \right] \end{array}$$

$$\begin{array}{l} w_1 \cdot w_2 = 0 \\ w_2 \cdot w_3 = 0 \\ w_1 \cdot w_3 = 0 \end{array}$$

b) $\text{proj}_V u$

$$\begin{aligned} w_1 &= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} & w_2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & w_3 &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} & w_1 \cdot w_1 &= 3 & u \cdot w_1 &= 3 \\ w_2 \cdot w_2 &= 3 & u \cdot w_2 &= 3 & w_2 \cdot w_3 &= 3 & u \cdot w_2 &= 3 & u \cdot w_3 &= 3 \\ w_3 \cdot w_3 &= 3 & u \cdot w_3 &= 3 & & & & & & \end{aligned}$$

$$\text{Proj}_V u = \left(\frac{u \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2} \right) w_2 + \left(\frac{u \cdot w_3}{w_3 \cdot w_3} \right) w_3$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} \left[\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right] + \left(\frac{3}{3} \right) \left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right] + \left(\frac{3}{3} \right) \left[\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right]$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Proj}_V u = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 2_1 \\ 2_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & -1+1+2 \\ -1-1+2 & 1+1+1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \\ \hline 6 & 2 & 0 \end{array} \right] \xrightarrow{\text{R}_2 - 3\text{R}_1} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 1 & 3 \\ \hline 6 & 2 & 0 \end{array} \right] \xrightarrow{\text{R}_3 - \frac{1}{2}\text{R}_1} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{R}_1 \rightarrow \frac{1}{6}\text{R}_1} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{R}_1 \rightarrow 3\text{R}_1} \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 0 \end{array} \right]$$

$$\begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} z_1 = -3/17$$

$$\begin{vmatrix} 1 & 3 \\ 9 & 7 \end{vmatrix} z_2 = 9/17$$

$$9/3 - 2/3 = \frac{7}{7} - \frac{2}{7} = \frac{5}{7}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array} \right] \xrightarrow{\frac{1}{3}} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 1 & 1 \end{array} \right] \xrightarrow{\cdot \frac{1}{6}} \left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} \end{array} \right] \xrightarrow{\cdot 3} \left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 1 & 1 \end{array} \right] \xrightarrow{\cdot \frac{1}{3}} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & n & \\ 1 & 0 & -3/7 \\ 0 & 1 & 9/7 \end{array} \right]$$

$$-\frac{3}{7}X + \frac{9}{7}$$

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$$\det(A - \lambda I_n)$$

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

$$\det \begin{pmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{pmatrix}$$

$$\begin{array}{|ccc|} \hline -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \\ \hline \end{array}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$

$$\left. \begin{array}{l} \text{No} \\ -\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0 \\ 0 + 0 + 0 - 6 \neq 0 \\ -(-1)^3 + 3(-1)^2 + 7(-1) - 6 = 0 \\ 1 + 3 - 7 - 6 \neq 0 \\ -(-2)^3 + 3(-2)^2 + 7(-2) - 6 = 0 \\ 8 + 12 + (-14) - 6 \neq 0 \\ 20 + 20 \\ 0 = 0 \end{array} \right\} \text{No}$$

b) $\lambda = -1$

$$\left. \begin{array}{l} -\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0 \\ 0 + 0 + 0 - 6 \neq 0 \\ -(-1)^3 + 3(-1)^2 + 7(-1) - 6 = 0 \\ 1 + 3 - 7 - 6 \neq 0 \\ -(-2)^3 + 3(-2)^2 + 7(-2) - 6 = 0 \\ 8 + 12 + (-14) - 6 \neq 0 \\ 20 + 20 \\ 0 = 0 \end{array} \right\} \text{No}$$

c) $\lambda = -2$

yes ✓

$$\begin{aligned} & (-\lambda + 1)(-\lambda + 2 - \lambda) + 0 + 4 \\ & (2 + 4 + 1 - \lambda) - 0 - (2 - \lambda) \\ & \cancel{\lambda} \cdot \cancel{1} \cdot \cancel{2} \\ & \cancel{8} \cdot \cancel{1} \cdot \cancel{-\lambda} \\ & (1 - \lambda)(2 - \lambda) \\ & \lambda^2 - \lambda - 2\lambda + 2 \\ & -\lambda (\lambda^2 - 3\lambda + 2) \\ & -\lambda^3 + 3\lambda^2 + 2\lambda + 4 - (8 + 8\lambda) \\ & -\lambda^3 + 3\lambda^2 + 16\lambda - 4 - (2 - \lambda) \\ & -\lambda^3 + 3\lambda^2 + 17\lambda - 6 = 0 \end{aligned}$$

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4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

$$\lambda_1 = 3, [A - 3I | 0]$$

$$\left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ 2 & 8 & 4 & 0 \\ 2 & -8 & -4 & 0 \end{array} \right] \xrightarrow{-1}, \left[\begin{array}{ccc|c} 0 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\frac{1}{8}} \left[\begin{array}{ccc|c} 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{aligned} x_1 &= +4x_2 + 2x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

$$\lambda_2 = 5, [A - 5I | 0]$$

$$\left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array} \right] \xrightarrow{\frac{1}{2}} \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array} \right] \xrightarrow{\frac{1}{4}} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array} \right] \xrightarrow{1,2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{1,2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned} \quad x_3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 6 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ .

x5 ✓ False, while you can multiply eigenvectors to scale them they still correspond to the original eigenvalue, λ .

b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

x1 True, if it is impossible for the projection to return $-w$?

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

x1 True, property of orthonormal vectors $\cdot v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$?

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A+B$ is also orthogonally diagonalizable.

x5 d) True, If A and B are orthogonally diagonalizable then they have to be symmetric Matrices

$A+B$ will still be symmetric as adding B top of diagonal will cancel out with B 's bottom of diagonal.

Since $(A+B)$ is symmetric it is orthogonally diagonalizable

a)

$$\det \begin{bmatrix} 1 & 0 & 0 \\ 3-\lambda & 2 & 0 \\ 1 & 0 & \lambda \end{bmatrix}$$

$$(3-\lambda)(8-\lambda) - 2$$

$$24 - 11\lambda + \lambda^2$$

$$-(6-3) (\lambda-6)(\lambda-3)=0$$

$$\lambda=6 \quad \lambda=3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -\frac{2}{2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{0}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 4/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}, \quad \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 4/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

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$$1/9 + 4/9 + 4/9 = 1 \quad 2/9 + 2/9 + -4/9 = 0 \quad 2/9 - 4/9 + 4/9 = 0$$

$$2/9 + 2/9 - 4/9 = 0 \quad 4/9 + 1/9 + 4/9 = 1 \quad 4/9 - 4/9 - 4/9 = 0$$

$$4/9 - 4/9 + 4/9 = 0 \quad 4/9 - 2/9 - 2/9 = 0 \quad 4/9 + 4/9 + 1/9 = 1$$

1. Consider the following vectors in \mathbb{R}^4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\frac{0+0+(-3)}{2+(-2)+0+(-3)}$$

$$\frac{1+1+0+1}{1+1+0+1}$$

$$(-1) \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The set $B = \{v_1, v_2, v_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

a) Find an orthogonal basis $D = \{w_1, w_2, w_3\}$ of the subspace V .

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V .

$$a) w_1 = v_1, \quad w_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1} \right) w_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1 \cdot 2 + 0 \cdot 1 + (-1) \cdot (-1) + 1 \cdot 0}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1} \right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2} \right) w_2$$

$$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{1 \cdot 2 + 0 \cdot (-2) + (-1) \cdot (-1) + 1 \cdot 3}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1 \cdot 2 + 0 \cdot (-2) + (-1) \cdot (-1) + 1 \cdot 3}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

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$$b) \text{proj}_V u = \left(\frac{u \cdot v_1}{v_1 \cdot v_1} \right) v_1 + \left(\frac{u \cdot v_2}{v_2 \cdot v_2} \right) v_2 + \left(\frac{u \cdot v_3}{v_3 \cdot v_3} \right) v_3$$

$$\frac{6}{3} + \frac{-1}{3}$$

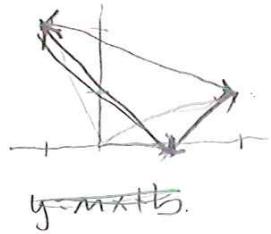
$$\textcircled{1} \left(\frac{3+0+(-3)+3}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{4+1+1+0} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{6}{4+4+1+9} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right)$$

$$\frac{9}{3} + \frac{2}{3}$$

$$\left[\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 1/3 \\ -7/3 \\ 2 \end{bmatrix} \right]$$

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$$



$$1+(-1)+2=2+1 \quad 0+2+1=y=ax+b$$

$$1+(-1)+2=2+1 \quad 0=a(1)+b$$

$$2=a(-1)+b$$

$$1=a(2)+b$$

$$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b$$

$$Ax=b$$

$$\checkmark A^T A x = A^T b$$

$$A^T A x = A^T b$$

what's A ? what's b ?

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \text{A/20}$$

$$2+1+2 \quad 5x=-1$$

$$0-2+1 \quad x=-\frac{1}{5}$$

$$f(x) = -\frac{1}{5}x + \frac{1}{5}$$

$$(8\lambda + \dots)(\lambda - \dots)$$

$$Ax = \lambda I x$$

$$-8 + 4\lambda - 16\lambda + 8\lambda^2$$

$$8\lambda^2 - 12\lambda - 8$$

3. Consider the following matrix A:

$$1/2(\lambda - 0) \begin{bmatrix} -2 & -4\lambda \\ 2 & -4 - 4\lambda \end{bmatrix}$$

$$(2-\lambda)\left(\frac{1}{2} + \lambda\right)^{-8}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 6 \\ 1 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$

b) $\lambda = -1$

c) $\lambda = -2$

$$\begin{array}{c|cc|c} 0 & 1 & 2 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 4 & 2 & 2 & 0 \end{array} \xrightarrow{-4} \begin{array}{c|cc|c} 1 & 1 & 2 & 0 \\ \hline 1 & 2 & 0 & 0 \\ 4 & 2 & 3 & 0 \end{array} \xrightarrow{-1} \begin{array}{c|cc|c} 1 & 1 & 2 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 4 & 2 & 3 & 0 \end{array} \xrightarrow{-4} \begin{array}{c|cc|c} 1 & 1 & 2 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 0 & 2 & 3 & 0 \end{array}$$

$$\begin{array}{c|cc|c} 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array} \xrightarrow{-2} \begin{array}{c|cc|c} 1 & 1 & 2 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}$$

$$\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{array} \xrightarrow{-1} \begin{array}{c|cc|c} 1 & 1 & 2 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \xrightarrow{-1} \begin{array}{c|cc|c} 1 & 0 & 2 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}$$

$$\begin{bmatrix} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{bmatrix}$$

NO because $A \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

eigenvector can't
be \emptyset

$$x_2 - 2x_3 = 0 \quad x_2 = 2x_3$$

$$x_3 = 0 \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

NO because $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
can't be \emptyset

4

$$\begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{-2} Ax = 2\lambda$$

$$0 \quad 1 \quad 0 \quad 0 \quad Ax = 2\lambda$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{-2} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \quad Ax = \lambda x$$

yes free

$$x_1 + x_2 + 2x_3 = 0 \quad x_1 = x_2 = 2x_3$$

$$1 \quad 0 \quad -2 \quad 0$$

$$0 \quad 1 \quad 0 \quad 0$$

$$0 \quad 0 \quad -2 \quad 0$$

$$\boxed{\text{No}} \quad \boxed{14/20}$$

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

$$A - \lambda_2 = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right. \xrightarrow{\text{Row operations}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -8 & -4 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & -8 & -8 & 0 \\ 0 & -2 & -2 & 0 \\ 2 & -8 & -6 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -4 & -3 & 0 \end{array} \right) \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{\text{Row operations}} \begin{array}{l} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_1 = x_3 \\ x_2 = -x_3 \end{array}$$

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A - \lambda_1 = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -8 & -4 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ .

*5 False, The eigenvector $2v$ still corresponds to the eigenvalue λ . Consider (A) below.

b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

*5 True, see (B) below.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

*5 True by definition to determine if a sym. Matrix is orthogonal, you compute $A^T A$ and if you get identity matrix its orthogonal. Because $A^T = A$

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

*5 By definition a matrix is orthogonally diagonalizable if it is symmetric and every symmetric matrix is diagonalizable, so the answer is true because any 2×2 symmetric matrix for example:

$$\begin{bmatrix} A & C \\ C & B \end{bmatrix} + \begin{bmatrix} D & F \\ F & E \end{bmatrix} = \begin{bmatrix} A+D & C+F \\ C+E & B+E \end{bmatrix}$$

added to another symmetric matrix is still symmetric. TRUE.

D) $\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow{\lambda^2 - 5\lambda + 6} \lambda = 2, 3 \quad \begin{bmatrix} a & 0 \\ 1 & 2 \end{bmatrix} \xrightarrow{x_1 = 2x_2} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \leftarrow \text{eigenvector}$
 but $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ is still an eigenvector
 that corresponds to λ , not 2λ . FALSE


 In order for $\text{proj}_V w$ to be w , we would have to have $w_{\perp} = -w_{\perp}$. That isn't possible unless w is the zero vector because $(w - \text{proj}_V w)$ must be in W^\perp and not in W . Unless its the zero vector. TRUE.

1. Consider the following vectors in \mathbb{R}^4 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of \mathbb{R}^4 .

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V .

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of \mathbf{u} on V .

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{(1 \cdot 2 + 0 \cdot 1 + -1 \cdot -1 + 1 \cdot 0)}{(1 \cdot 1 + 0 \cdot 0 + -1 \cdot -1 + 1 \cdot 1)} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \checkmark$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2} \right) \mathbf{w}_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{(2 \cdot 2 + 0 \cdot -2 + -1 \cdot -1 + 3 \cdot 3)}{(2 \cdot 2 + 0 \cdot 0 + -1 \cdot -1 + 3 \cdot 3)} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{(2 \cdot 1 + 0 \cdot -2 + -1 \cdot -1 + 3 \cdot 0)}{(1 \cdot 1 + 0 \cdot 0 + -1 \cdot -1 + 1 \cdot 1)} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \checkmark$$

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\} \checkmark$$

$$\text{proj}_V \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 + \left(\frac{\mathbf{u} \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \right) \mathbf{w}_2 + \left(\frac{\mathbf{u} \cdot \mathbf{w}_3}{\mathbf{w}_3 \cdot \mathbf{w}_3} \right) \mathbf{w}_3 = \frac{(3+0-3+3)}{(1+0+1+1)} \mathbf{w}_1 + \frac{(3+3+0-3)}{(1+1+0+1)} \mathbf{w}_2 + \frac{(3-3+3+0)}{(1+1+1+0)} \mathbf{w}_3$$

$$= \mathbf{w}_1 + 3\mathbf{w}_2 + \mathbf{w}_3 = \begin{bmatrix} 1+3+1 \\ 0+3-1 \\ -1+0+1 \\ 1+3+0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 4 \end{bmatrix}$$

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2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & -1+2 \\ 1-1+2 & 1+1+1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-2+2 \\ 0+2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$(A^T A) \vec{x} = (A^T b)$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & \frac{7}{3} & 3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -\frac{3}{7} \\ 0 & 1 & \frac{9}{7} \end{array} \right]$$

$$\boxed{f(x) = -\frac{3}{7}x + \frac{9}{7}} \quad \checkmark$$

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3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$

b) $\lambda = -1$

c) $\lambda = -2$

$$\begin{aligned} A - 0I &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 4 & 2 & 2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \rightarrow X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{not an eigenvalue} \end{aligned}$$

$$\begin{aligned} A + I &= \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & | & 0 \\ 1 & 2 & 0 & | & 0 \\ 4 & 2 & 3 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & -2 & -1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 4 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & -9 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \rightarrow X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{an eigenvalue} \end{aligned}$$

$$\begin{aligned} A + 2I &= \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 0 \\ 4 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 2 & | & 0 \\ 1 & 3 & 0 & | & 0 \\ 4 & 2 & 5 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & | & 0 \\ 2 & 1 & 2 & | & 0 \\ 4 & 2 & 5 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & | & 0 \\ 0 & -8 & 2 & | & 0 \\ 0 & -10 & 5 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \rightarrow X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{an eigenvalue} \end{aligned}$$

$$\begin{aligned} -\lambda &\quad 1 & 2 \\ 1 &\quad -\lambda & 0 \\ 4 &\quad 2 & -2-\lambda \end{aligned} \rightarrow (-\lambda)(1-\lambda)(2-\lambda) + 0 + 2 - 8(1-\lambda) - 0 - (2-\lambda)$$

$$= (-\lambda + \lambda^2)(2-\lambda) + 2 - 8 + 8\lambda - 2 + \lambda$$

$$= (-2\lambda + 3\lambda^2 - \lambda^3) - 8 + 9\lambda$$

$$= -\lambda^3 + 3\lambda^2 + 7\lambda - 8 = 0$$

$$(0)^3 + 3(0)^2 + 7(0) - 8 = -8 \neq 0 \leftarrow \boxed{\text{not an eigenvalue}}$$

$$(-1)^3 + 3(-1)^2 + 7(-1) - 8 = -11 \neq 0 \leftarrow \boxed{\text{not an eigenvalue}}$$

$$(-2)^3 + 3(-2)^2 + 7(-2) - 8 = -2 \neq 0 \leftarrow \boxed{\text{not an eigenvalue}}$$

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4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1} .

$$\begin{aligned} 4 - 3I &= \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -8 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \quad x_2 = 1, x_3 = 1 \\ x_2 = 1, x_3 = 1 & \quad x_2 = -1, x_3 = 1 \\ I - 5I &= \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -8 & -6 \\ -2 & 6 & 4 \\ -4 & 8 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & -3 \\ 0 & -2 & -2 \\ 0 & -8 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} -x_3 \\ -x_2 \\ x_1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad x_3 = 1 \\ x_3 = 1 & \quad x_2 = 1 \\ P &= \begin{bmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{lin. dependent} \\ D &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \checkmark \quad 15/20 \end{aligned}$$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then $2v$ is an eigenvector of A corresponding to the eigenvalue 2λ .

b) If V is a subspace of \mathbb{R}^2 and w is a vector such that $\text{proj}_V w = -w$ then w must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

~~False.~~ Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 4 = -\lambda + \lambda^2 - 4 = 0 \in (\lambda-2)(\lambda+1)$

$\lambda=2 \quad \lambda=-1$

For $\lambda=1 \rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} x=0 \rightarrow x=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda=-2 \rightarrow \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} x=0 \rightarrow x=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq 2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$)

~~True.~~ For any u such that $\text{proj}_V u = v$, $u \cdot v \geq 0$. $\|w\| \cdot \|w\| \leq 0$ so the
only way for this to be true is if $w=0$.

~~True.~~ If A is orthogonal and symmetric then it is of the form PD^2P^T . Since P is an orthogonal basis for A , PD^2P^T is going to be the identity matrix.

~~False.~~ Let $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Both A and B are orthogonal and diagonalizable, let $w = A + B \rightarrow \begin{bmatrix} 1+\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -2/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1+\sqrt{2} \end{bmatrix}$. Though W is symmetrical, the columns of W aren't orthogonal to each other so W isn't orthogonally diagonalizable.