1. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V.

$\cancel{v_1 \cdot v_2 = (1)(2) + (0)(1) + (-1)(-1) + 0}$

a.) Use Gram-Smit: $\langle w_1, w_1 \rangle = 3$

$\langle w_1, v_2 \rangle = (1)(2) + 0 + 1 + 0 = 3$

$w_1 = v_1$

$$w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - 1w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = w_2$$

$$w_3 = v_3 - \frac{\langle w_1, v_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_3 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 - 2w_1 + w_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\langle w_1, v_3 \rangle = 2 + 0 + 1 + 3 = 6$

$\langle w_2, v_3 \rangle = 2 - 2 + 0 - 3 = -3$

$\langle w_2, w_2 \rangle = 3$

$$D = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$ ✓

b.) $$\text{proj}_V u = \frac{\langle u, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle u, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \frac{\langle u, w_3 \rangle}{\langle w_3, w_3 \rangle} w_3$$

$\langle u, w_1 \rangle = 3 + 0 - \cancel{3} + \cancel{3} = 3$

$\langle u, w_2 \rangle = 3 + \cancel{3} + 0 - \cancel{3} = 3$

$\langle u, w_3 \rangle = \cancel{3} - \cancel{3} + 3 + 0 = 3$

$$= w_1 + w_2 + w_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \text{proj}_V u$$ ✓

20/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$ 3×2 $\quad A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ 2×3

$-1+1 \quad -2$

$\cancel{A^TA = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 3 & -1 & 5 \end{bmatrix}}$

$A^TAx = A^Tb$ ✓

$b = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ $\quad 0 + (-2) + 2$

$2\times 3 \quad 3\times 1$

$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ ✓

$\frac{1}{2}$, $x{-3}$: $\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & 2 & -1 & 2 \\ 3 & -1 & 5 & 1 \end{array}\right] \rightarrow \frac{1}{2}\left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & -1 & -4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & -1 & -4 & 1 \end{array}\right] x1 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & -9/2 & 2 \end{array}\right] \frac{2}{9}$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & -1 & 4/9 \end{array}\right] \frac{1}{2} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 7/9 \\ 0 & 0 & -1 & 4/9 \end{array}\right] -3/2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 7/9 \\ 0 & 0 & +1 & 4/9 \end{array}\right]$

$x_1 = 3/2$
$x_2 = 7/9$
$x_3 = -4/9$

$\frac{4}{10} \quad \frac{2}{5} \quad \frac{3}{2}$

$(3/2, 0) \quad (7/9, 2) \quad (-4/9, 1)$

$\frac{4}{15}$

$a = \dfrac{1 - 0}{\frac{-8}{18} - \frac{36}{18}} = \dfrac{1}{-\frac{44}{18}}$

$b = 0$

0.4
$\overline{)80}$
-72
8

$f(x) = 0.4x + 0$

10/20

$1\frac{8}{18}$

$18 \times 4 = 72$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$$\det(A - \lambda I) = \det\begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix} = \det\begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}$$ → Pick this row.

$$\det = (1)\det\begin{bmatrix} 1 & 2 \\ 2 & 2-\lambda \end{bmatrix}(-1^3) + (1-\lambda)\det\begin{bmatrix} -\lambda & 2 \\ 4 & 2-\lambda \end{bmatrix}(-1^4) + 0$$

$$= (2-\lambda-4)(1)(-1) + (1-\lambda)(\lambda^2 - 2\lambda - 8)$$

$-\lambda^3 + 2\lambda^2 + 8\lambda + \lambda^2 - 2\lambda - 8$

$-\lambda^3 + 3\lambda^2 + 6\lambda - 8$

$$= (-2-\lambda)(-1) + (-\lambda^3 + 2\lambda^2 + 8\lambda + \lambda^2 - 2\lambda - 8)$$

$$= (\lambda+2) + (-\lambda^3 + 3\lambda^2 + 6\lambda - 8) = -\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0$$ ✓

Check: $\lambda = 0$

$-0 + 0 + 0 - 6 = 0$

$-6 = 0$ ×

check: $\lambda = -1$

$1 + 3 - 7 - 6 = 0$

$-9 = 0$

Check $\lambda = -2$

$8 + 12 + (-14) - 6 = 0$

$0 = 0$ ✓

20/20

Only $\lambda = -2$ is an eigen val for this one.

0 1 2 ☑3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$$Null(A - \lambda_1 I) = \left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ -2 & 8 & 4 & 0 \\ 2 & -8 & -4 & 0 \end{array}\right] \rightarrow -\frac{1}{2}\left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} x_1 = 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_2\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$Null(A - \lambda_2 I) = -\frac{1}{4}\left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ +2 & -8 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \tfrac{1}{2} \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \cancel{\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}}$$

14/20

0 1 2 3 ☑4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then 2v is an eigenvector of A corresponding to the eigenvalue 2λ. Assumption: false.

b) If V is a subspace of $\mathbb{R}^2$ and w is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then w must be the zero vector.

$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $z = w - w = 0$

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix. Assump: false.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

~~a.) $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ $\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4$~~

~~$(\lambda - 2)(\lambda - 2)$~~

~~$\lambda_1 = 2 \quad \lambda_2 = 2$~~

~~$\text{Null}(A - 2I) = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} x_1 = 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$~~

~~$2v = [$~~

+1 a.) this is false since for a given Matrix ~~given ...~~ $P = [v_1 \dots v_n]$ where $v_1 \dots v_n$ are eigenvectors, and $D = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ where the eigenvalues are found on diagonal. The Eq $A^k = PD^kP^{-1}$ doesn't involve a change in P at all so why should 2v correspond to 2λ.

?

b.) if $\text{proj}_V w = -w$

+3 then where z is orthogonal to V_{space}

$z = w - \text{proj}_V w$

$z = w + w$

?

$z = 2w$ by definition z cannot be in V unless $w = 0$

$\therefore$ True.

True

c.) ~~if A is symmetric and orthogonal~~

+3 ~~then~~ consider

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ → it is symmetric because the 0's match up

→ ~~Orthogonal because dot Product = 0~~

→ $\therefore$ orthogonal

$6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ Identity

because $1^2 = 1$

In order to have a Orthogonally symmetric Matrix the entries on the diagonal Must be 1 $\therefore$ A^2 will result in Identity

$\boxed{\text{Do on Back}}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\mathrm{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

am-Schmidt

$$w_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 \qquad w_3 = v_3 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2$$

$$w_1 \cdot v_2 = 1(2) + 0(1) + (-1)(-1) + 1(0) = 2 + 0 + 1 + 0 = 3$$

$$w_1 \cdot w_1 = 1^2 + 0^2 + (-1)^2 + (1)^2 = 3$$

$$w_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$w_2 \cdot v_3 = 1(2) + 1(-2) + 0(-1) + (-1)(3)$$
$$= 2 - 2 + 0 - 3 = -3$$
$$w_2 \cdot w_2 = 1^2 + 1^2 + 0^2 + (-1)^2 = 3$$

$$w_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{\cancel{3}\ 6}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{-3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \cancel{2}\ 1 \\ -1 \\ \cancel{0}\ 1 \\ \cancel{1}\ 0 \end{bmatrix}$$

$$\boxed{\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \cancel{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}} \right\}}$$

16/20

$$\mathrm{proj}_V\, u = \left(\frac{u \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2}\right) w_2 + \left(\frac{u \cdot w_3}{w_3 \cdot w_3}\right) w_3 = \frac{3}{3} w_1 + \frac{3}{3} w_2 + \frac{3}{3} w_3 = w_1 + w_2 + w_3$$

$$w_1 = 3 + 0 - 3 + 3 = 3$$
$$w_2 = 3 + 3 + 0 - 3 = 3$$
$$w_3 = 6 - 3 + 0 + 3 = 6$$
$$w_3 = 4 + 1 + 0 + 1 = 6$$

$$\mathrm{proj}_V\, u = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ -1 \\ 1 \end{bmatrix}}$$

						✔														
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{b} \qquad (A^T A)x = A^T b$$

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & 1-1+2 \\ 1-1+2 & 1+1+1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-2+2 \\ 0+2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \xrightarrow{R_1 - 3R_2 \to R_2} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & -7 & -9 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 1 & 9/7 \end{array}\right] \xrightarrow{-2R_2 + R_1 \to R_1}$$

$$\left[\begin{array}{cc|c} 6 & 0 & -18/7 \\ 0 & 1 & 9/7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 9/7 \end{array}\right] \to \begin{array}{l} a = -3/7 \\ b = 9/7 \end{array}$$

$$\boxed{f(x) = \frac{-3}{7}x + \frac{9}{7}}$$

✓

20/20

0 1 2 3 4 5 6 ☑7 8 9 10 11 12 13 14 15 16 17 18 19 20

where $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is eigenvector corresponding to λ

3. Consider the following matrix A:

$+ \quad - \quad +$
$- \quad + \quad -$
$+ \quad - \quad +$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}.$$

$v_2 + v_3 = \lambda v_1$
$v_1 + v_2 = \lambda v_2$
$4v_1 + 2v_2 + 2v_3 = \lambda v_3$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ NO **b)** $\lambda = -1$ NO **c)** $\lambda = -2$ Yes

$$-\lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}\right) = 0 = -1\left[2-\lambda-4\right] + (1-\lambda)\left[-2\lambda+\lambda^2-8\right] - 0$$

$$2+\lambda+(-2\lambda+\lambda^2-8+2\lambda^2-\lambda^3+8\lambda) = 0$$

$$-\lambda^3+3\lambda^2+7\lambda-6=0$$

$$\lambda^3-3\lambda^2-7\lambda+6=0 \quad \checkmark$$

a) $0^3-3(0)^2-7(0)+6=0$ $6 \neq 0 \rightarrow$ Not an eigenvalue

b) $(-1)^3-3(-1)^2-7(-1)+6=0$ $\checkmark$

$-1-3+7+6=0$ $\checkmark$

$9 \neq 0 \rightarrow$ Not an eigenvalue

20/20

c) $(-2)^3-3(-2)^2-7(-2)+6=0$ $\checkmark$

$-8-12+14+6=0$

$0=0 \checkmark \rightarrow$ IS an eigenvalue

;

0 1 2 3 4 5 6 7 ✔8 9 10 11 12 13 14 15 16 17 18 19 20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

For $\lambda = 3$, $A - \lambda I = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \to \begin{bmatrix} -1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$Nul(A-3I) = \text{Span}\left\{\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$ = eigenspace of A corresponding to λ_1

$A - 5I = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -2 \\ -2 & 4 & 2 \\ -1 & 4 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 & 2 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$Nul(A-5I) = \text{Span}\left\{\begin{bmatrix} \cancel{0}\ -1 \\ -1 \\ 1 \end{bmatrix}\right\}$

$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $P = \begin{bmatrix} 4 & 2 & \cancel{0}\ -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

18/20

0 1 2 3 4 5 6 7 8 ✔9 10 11 12 13 14 15 16 17 18 19 20

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

v_{12} v_{22}

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\operatorname{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) False, just because there is an eigenvalue λ does not mean there is an eigenvalue 2λ. There is no guarantee $2\lambda = \text{val}$ is another solution to $|A - I(\text{val})| = 0$ Ex) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda_1 = 1$, $\lambda_2 = -1$

+5

b) ~~$\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} \dfrac{w_1 v_{11} + w_2 v_{21}}{v_{11}^2 + v_{21}^2} + \dfrac{w_1 v_{12} + w_2 v_{22}}{v_{12}^2 + v_{22}^2} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -w_1 \\ -w_2 \end{bmatrix}$~~ Say $w_1 = w_2 = 1$ ~~$\dfrac{v_{11} + v_{21}}{v_{11}^2 + v_{21}^2}$~~

+2

~~$\begin{bmatrix} w_1 + w_2 \\ w_1 + w_2 \end{bmatrix} + \begin{bmatrix} w_1 + w_2 \\ w_1 + w_2 \end{bmatrix}$~~ $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

True, the projection of a vector on some space can be visualized as the "shadow" of that vector on that space, $\therefore$ any projection of w_1, w_2 somewhere else should carry the same sign, unless they both are zero

?

c) False, $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $A = A^T$ ✓ Symmetric ✓ $A^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

d) False $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

+1

does not have 2 linearly independent eigenvectors

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☑	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

$$V = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow[R_4-R_1]{R_3+R_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2-R_3 \\ R_4+2R_3}]{R_1-2R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1-2R_4]{R_2+R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$

orthogonal basis D is the basis of col(A)

D =

a) $W_1 = V_1$

$W_1 \cdot V_2 = 1\cdot 2 + 0\cdot 1 + -1\cdot -1 + 1\cdot 0 = 3$

$W_1 \cdot W_1 = 1\cdot 1 + 0\cdot 0 + -1\cdot -1 + 1\cdot 1 = 3$

$W_2 = V_2 - \left(\frac{W_1\cdot V_2}{W_1 \cdot W_1}\right) W_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$W_2 \cdot V_3 = 1\cdot 2 + 1\cdot -2 + 0\cdot -1 + -1\cdot 3 = -3$

$W_2 \cdot W_2 = 1\cdot 1 + 1\cdot 1 + 0\cdot 0 + -1\cdot -1 = 3$

$W_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$W_3 = V_3 - \left(\frac{W_1\cdot V_3}{W_1 \cdot W_1}\right) W_1 - \left(\frac{W_2\cdot V_3}{W_2 \cdot W_2}\right) W_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{-3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$W_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{D = \begin{bmatrix} 1 & 1 & \cancel{2}\ 1 \\ 0 & 1 & -1 \\ -1 & 0 & \cancel{0}\ 1 \\ 1 & -1 & \cancel{1}\ 0 \end{bmatrix}}$$

10/20

part b on back →

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1,0)$, $(-1,2)$, $(2,1)$.

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$W_1 = \langle v_1^2 \rangle$

$W_2 = \langle v_2^2 \rangle - \dfrac{\langle w_1 \cdot v_2 \rangle}{\langle w_1 \cdot w_1 \rangle} w_1$

$w_1 = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

I'm sorry, I don't know how to do this

$A^T \cdot A x = A^T b$

$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$$

$\det A^T A = 6 \cdot 5 = 30$

$\det A^T = (-1 \cdot 1 - 2 \cdot 2) + (1 \cdot 1 - 2 \cdot 0) + (1 \cdot 2 - 0) = -2$

$$\boxed{f(x) = 30x - 2}$$

8/20

0 1 2 3 4 5 6 7 8 9 10 11 12 ✔ 13 14 15 16 17 18 19 20

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$\lambda = 0$

a) $\text{Null}(A - \lambda_i I) \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 + R_3]{R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ no basis

0 is not an eigen value of A

---

b) $\lambda = -1$ $A + 1I \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & -6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ -1 is not an eigen value of A

---

c) $\lambda = -2$ $A + 2I \rightarrow \begin{bmatrix} 2 & 0 & 2 \\ 1 & 3 & 0 \\ 4 & 2 & 4 \end{bmatrix} \xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}$

$\xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[R_1 - R_3]{-1R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow$ -2 is ~~not~~ an eigen value of A

16/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 ✔ 14 15 16 17 18 19 20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

basis of $(A-3I) = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2,\ \frac{1}{2}R_3]{\frac{1}{2}R_1} \begin{bmatrix} -1 & 4 & 2 \\ -1 & 4 & 2 \\ 1 & -4 & -2 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} -1 & 4 & 2 \\ -1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_2-R_1} \begin{bmatrix} -1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_1} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x_1 - 4x_2 - 2x_3 = 0 \rightarrow x_1 = 4x_2 + 2x_3$

$x_2 = x_2$

$x_3 = x_3$

$\rightarrow x = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \rightarrow$ basis $= \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

basis of $(A-5I) = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2,\ \frac{1}{2}R_3]{\frac{1}{4}R_1} \begin{bmatrix} -1 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -4 & -3 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\xrightarrow[R_1-2R_2]{R_3-R_2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 = x_3 \end{matrix} \rightarrow \begin{matrix} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{matrix}$

$\rightarrow x = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow$ basis $= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

so $\boxed{P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \text{ \& } D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$

20/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ✔ 15 16 17 18 19 20

1 2

$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ ~~xxo~~ $\begin{matrix} x_1 = x_2 \\ \lambda_2 = x_2 \end{matrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 = 0$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) false ~~true~~, ~~because multiplying~~ $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ $\lambda_1 = 1$ then $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

but

+5 so false $\lambda_2 = 2$ then $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{matrix} x_2 = 0 \\ 0 = 0 \end{matrix}$ no V_2

---

b) True, because for the proj to equal the original vector,

+3 the vector must already fall on the projected plan with $0 = -0$ as the only vector that could possibly be equal to it's negative proj

---

c) ~~false~~, because $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ is a square, symmetric, + orthogonal

+1 matrix but $A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ which is not the Identity matrix

---

D) True, because the resulting matrix will still be

+5 symmetrical which means that it will also be orthogonally diagonizable

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ✔ 16 17 18 19 20

$w_1 = \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$ //

$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1$

$w_3 = v_3 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2$

1. Consider the following vectors in $\mathbb{R}^4$:

$1+0+1+1$ $\quad 2+0+1+1+0$ $\quad 3$

$$v_1 = \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \quad v_2 = \begin{bmatrix}2\\1\\-1\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}2\\-2\\-1\\3\end{bmatrix}, \quad u = \begin{bmatrix}3\\3\\3\\3\end{bmatrix}$$

w_2 $\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}$ $\begin{bmatrix}1\\0\\-1\end{bmatrix}$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

$\begin{bmatrix}1\\-2\\0\\2\end{bmatrix} + \begin{bmatrix}1\\1\\0\\-1\end{bmatrix}$

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V.

a) $w_1 = v_1$ //, $w_2 = \begin{bmatrix}2\\1\\-1\\0\end{bmatrix} - \left(\frac{3}{3}\right)\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\-1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\\-1\end{bmatrix}$ // ✓

$w_3 = \begin{bmatrix}2\\-2\\-1\\3\end{bmatrix} - \frac{6}{3}\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} - \left(\frac{-3}{3}\right)\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} = \begin{bmatrix}2\\-2\\-1\\3\end{bmatrix} - \begin{bmatrix}2\\1\\-1\\0\end{bmatrix} = \cancel{\begin{bmatrix}0\\-3\\0\\3\end{bmatrix}}$

$$D = \begin{array}{c} \begin{bmatrix}1 & 1 & 0\\0 & 1 & -3\\-1 & 0 & 0\\1 & -1 & 3\end{bmatrix} \\ v_1 \quad v_2 \quad v_3 \end{array}$$

$9+9$

b) $\text{proj}_V u = \left(\frac{u \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{u \cdot v_2}{v_2 \cdot v_2}\right) v_2 + \left(\frac{u \cdot v_3}{v_3 \cdot v_3}\right) v_3$

$= \left(\frac{3}{3}\right)\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} + \left(\frac{3}{3}\right)\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \cancel{\left(\frac{0}{18}\right)\begin{bmatrix}0\\-3\\0\\3\end{bmatrix}}$

$\text{proj}_V u = \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\0\\-1\end{bmatrix} = \begin{bmatrix}2\\1\\-1\\0\end{bmatrix}$ //

16/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ✔16 17 18 19 20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1,0)$, $(-1,2)$, $(2,1)$.

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix} = \overset{b}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}$$

$$A^TA = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \xrightarrow{-2/6 R_1 + R_2} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 7/3 & 3 \end{array}\right] \xrightarrow{3/7 R_2} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 1 & 9/7 \end{array}\right] \xrightarrow{-2R_2 + R_1}$$

$$\left[\begin{array}{cc|c} 6 & 0 & -18/7 \\ 0 & 1 & 9/7 \end{array}\right] \xrightarrow{\frac{1}{6}R_1} \left[\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 9/7 \end{array}\right]$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3/7 \\ 9/7 \end{bmatrix}$$

$f(x) = -3/7 x + 9/7$

20/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 ✔ 18 19 20

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$2-4+4\lambda$
$4-8+8\lambda$

$$\left(\begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix} \right) = 0$$

$-\lambda((1-\lambda)(2-\lambda)) - 1(2-\lambda) + 2[2-4(1-\lambda)] = 0$

$-\lambda^3 + 3\lambda^2 - 2\lambda - 2 + \lambda + 4 - 8 + 8\lambda = 0$ ✓

$-\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0$

) $\lambda = 0$

$-(0)^3 + 3(0)^2 + 7(0) - 6 \neq 0$

$-6 \neq 0 \rightarrow$ Not an eigenvalue of A

$-(-1)^3 + 3(-1)^2 + 7(-1) - 6 \neq 0$

$1 + 3 - 7 - 6 \neq 0$ $\rightarrow$ not an eigenvalue of A

) $-(-2)^3 + 3(-2)^2 + 7(-2) - 6 = 0$

20/20

$8 + 12 - 14 - 6 = 0$

$0 = 0 \rightarrow \lambda = -2$ is an eigenvalue of A

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 ✔ 19 20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

r $\lambda_1 = 3$ $\begin{bmatrix} 1-3 & 8 & 4 \\ -2 & 11-3 & 4 \\ 2 & -8 & -1-3 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 0 \\ 2 & 8 & 4 & 0 \\ 2 & -8 & -4 & 0 \end{array}\right] \xrightarrow{\substack{-R_1+R_2 \to R_2 \\ R_1+R_3}} \left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 4x_2 + 2x_3$
$x_2 =$ free
$x_3 =$ free

$\Rightarrow \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} x_3 \Rightarrow \mathrm{Nul}(A - \lambda I_n) = \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

r $\lambda_2 = 5$ $\begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \xrightarrow{\substack{-\frac{1}{2}R_1 + R_2 \to R_2 \\ \frac{1}{2}R_1 + R_3 \to R_3}} \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \xrightarrow{2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-4R_2 + R_1 \to R_1}$

$\left[\begin{array}{ccc|c} -4 & 0 & -4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\substack{-\frac{1}{4}R_1 \\ \frac{1}{2}R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $x_1 = -x_3$, $x_2 = -x_3$, $x_3 =$ free $= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3 \Rightarrow \mathrm{Nul}(A - \lambda I_n) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

20/20

$A = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}[v] = \lambda v$

$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} c & d \\ d & c \end{bmatrix}$

$\begin{bmatrix} a+c & b+d \\ b+d & a+c \end{bmatrix}$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then 2v is an eigenvector of A corresponding to the eigenvalue 2λ.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$

b) If V is a subspace of $\mathbb{R}^2$ and w is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then w must be the zero vector.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) False The eigenvalue is the root of the characteristic polynomial which means its the solution that makes the equation $p(\lambda) = 0$ true

5

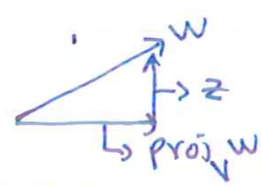
an $n \times n$ matric can have ~~only~~ no more than n roots so 2λ would not be the eigen value.

b) True projection of a vector w on subspace V is unique ~~to each vector~~ ~~being projected~~ such that $z = w - \text{proj}_V w \Rightarrow w = z + \text{proj}_V w$

+5

where z is an orthogonal vector to V

→ This means the only way $\text{proj}_V w = -w$ is if vector w is a zero vector.



→ If w is an element of vectorspace of V then $\text{proj}_V w = w$ which is different from $\text{proj}_V w = -w$

c) False

+2

If A is symmetric then it has n orthogonal eigenvector

A can then be expressed as $A = QDQ^T$

$A^2 = QD^2Q^T$ not identity matrix

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ✔

1. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V.

6/20

a) On basis Orthogonal Basis $D = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$c_1 = \frac{UV_1}{VV_1} = \frac{3+0-3+3}{1^2+0^2+(-1)^2+1^2} = \frac{3}{3} = 1$$

$$c_2 = \frac{UV_2}{V_2V_2} = \frac{6+3-3+0}{2^2+1^2+(-1)^2+0^2} = \frac{6}{6} = 1$$

$$c_3 = \frac{UV_3}{V_3V_3} = \frac{6-6-3+9}{2^2+(-2)^2+(-1)^2+3^2} = \frac{6}{18}$$

$$4+4+1+9 \qquad \frac{1}{3}$$

Orthogonal Basis D of subspace V $= \begin{bmatrix} 1 \\ 1 \\ 1/3 \end{bmatrix}$

$$\text{Proj}_V U = \frac{UV_1}{V_1V_1}V_1 + \frac{UV_2}{V_2V_2}V_2 + \frac{UV_3}{V_3V_3}V_3$$

$UV_1 = 3+0-3+3 = 3$

$V_1V_1 = 1^2+0^2+(-1)^2+1^2 = 3$

$UV_2 = 6+3-3+0 = 6$

$V_2V_2 = 2^2+1^2+(-1)^2+0^2 = 6$

$UV_3 = 6-6-3+9 = 6$

$V_3V_3 = 2^2+(-2)^2+(-1)^2+3^2 = 18$

$$\text{Proj}_V U = \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{6}\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{6}{18}\begin{bmatrix} 2 \\ -2 \\ -2 \\ 3 \end{bmatrix}$$

$$\text{Proj}_V U = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \\ 1 \end{bmatrix}$$

$$\text{Proj}_V U = \begin{bmatrix} 11/3 \\ 1/3 \\ -8/3 \\ 2 \end{bmatrix}$$

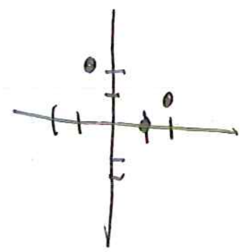
$$\text{Proj}_V U = \begin{bmatrix} 11/3 \\ 1/3 \\ -8/3 \\ 2 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$A\begin{bmatrix}a\\b\end{bmatrix} =$

$$\underbrace{\begin{bmatrix}1 & 1\\-1 & 1\\2 & 1\end{bmatrix}}_{A}\begin{bmatrix}a\\b\end{bmatrix} = \underbrace{\begin{bmatrix}0\\2\\1\end{bmatrix}}_{B}$$



$$A^TA = \begin{bmatrix}1 & -1 & 2\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\\2 & 1\end{bmatrix} \rightarrow \begin{bmatrix}6 & 2\\2 & 3\end{bmatrix}$$

$$A^TB = \begin{bmatrix}1 & -1 & 2\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}\cancel{4}\ 0\\3\end{bmatrix}$$

$$\left[\begin{array}{cc|c}6 & 2 & \cancel{4}\ 0\\2 & 3 & 3\end{array}\right]\cdot 1/6$$

$\Downarrow$

$$\left[\begin{array}{cc|c}1 & 1/3 & 4/6\\2 & 3 & 3\end{array}\right] + (-2)R_1$$

$\downarrow$

$$\left[\begin{array}{cc|c}1 & 1/3 & -2/3\\0 & 1/3 & 3\end{array}\right] + (-1)R_2 \rightarrow \left[\begin{array}{cc|c}1 & 0 & -3\\0 & 1/3 & 3\end{array}\right]\cdot 3 \rightarrow \begin{array}{ccc}a & b & \\ \hline 1 & 0 & -3\\0 & 1 & 9\end{array}$$

$a = -3$

$b = 9$

$$\boxed{f(x) = -3x + 9}$$

18/20

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	✔	☐

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$

$0+0+0-6 \neq 0$

X

$0+0+0-6=0$

X

b) $\lambda = -1$

$1+3-7-6=0$

$-9=0$

X

c) $\lambda = -2$

$8+12-14-6=0$

$20-20=0$

$0=0$

✓

$-\lambda^3+3\lambda^2+7\lambda-6=0$ ✓

$8+12$ 20

$$\begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}$$

20/20

$\underline{\lambda=-2}$ is the only eigenvalue of A

$(-\lambda)((1-\lambda)(2-\lambda)-0)-1(1(2-\lambda)-0)+2(2-4(1-\lambda))$

$(-\lambda)(2-3\lambda+\lambda^2)-(2-\lambda)+2(2-4+4\lambda)$

$\cancel{-2\lambda}+3\lambda^2 \cancel{-\lambda^3} -2+\cancel{\lambda}+4-8+8\lambda$

$-\lambda^3+3\lambda^2+7\lambda-6$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 ✔ 19 20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$\lambda = 3$

$A - 3I$:

$$\begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \begin{matrix} \\ +(-1)R_1 \\ +R_1 \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} -2 & 8 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot -\tfrac{1}{2} \rightarrow \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$x_1 = 4x_2 + 2x_3$

$\lambda = 3$: $\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 5$ $A - 5I$:

$$\begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \cdot -\tfrac{1}{4}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \begin{matrix} \\ +2R_1 \\ +(-2)R_1 \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & -4 & -4 \end{bmatrix} \begin{matrix} \\ \\ +2R_2 \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \tfrac{1}{2} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} +2R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

20/20

$x_1 = -x_3$

$x_2 = -x_3$

$\lambda = 5$: $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Note:

also equal to $[v_1, v_2, v_3]$

$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

With eigenvalues in D corresponding to v_1, v_2, v_3 respectively

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then 2v is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and **w** is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then **w** must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) false: The eigenvalue would stay the same, as V and 2V wouldn't be linearly independent

+5

f $\lambda = 2$ d X_2 in $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ s a free root, ?

m $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ould be eigenvectors

aining $\lambda \to V = 2V \not\to 2\lambda$

b) True; This has to be true because This would be The only way Proj V would equal -W. No other combo would produce this besides The trivial Solution

+1 ?

c) True; The Identity matrix, which is both A and A^2 in This case, is The only matrix that could be both symmetric and orthogonal

+1

d) ~~false~~; This is not always The case. When matrixies are added, Properties are not always preserved, such as diagonizability:

+1

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 2 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix}$$

diagonizable diagonizable not diagonizable

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 ✔ 17 18 19 20

1. Consider the following vectors in $\mathbb{R}^4$:

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The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

$X_1 = V_1 \quad X_2 = V_2 - \frac{V_2 \cdot V_1}{V_1 \cdot V_1} \cdot V_1 \quad X_3 = V_3 - \left(\frac{V_3 \cdot V_1}{V_1 \cdot V_1}\right) V_1 - \left(\frac{V_3 \cdot V_2}{V_2 \cdot V_2}\right) V_2$

$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$V_2 \cdot V_1 = 2+0+1+0 = 3$

$V_1 \cdot V_1 = 1+0+1+1 = 3$

$\frac{V_2 \cdot V_1}{V_1 \cdot V_1} = \frac{3}{3} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$\begin{matrix} 2 \\ 1 \\ -1 \\ 0 \end{matrix} - \begin{matrix} 1 \\ 0 \\ -1 \\ 1 \end{matrix} = \begin{matrix} 1 \\ 1 \\ 0 \\ -1 \end{matrix}$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$X_2 = \begin{matrix} 2 \\ 1 \\ -1 \\ 0 \end{matrix} - 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$V_3 \cdot V_1 = 2+0+1+3 = 6$

$V_1 \cdot V_1 = 3 \quad \frac{V_3 \cdot V_1}{V_1 \cdot V_1} = 2$

$V_3 \cdot V_2 = 4+-2+1+0 = 3 \quad \frac{V_3 \cdot V_2}{V_2 \cdot V_2} = \frac{1}{2}$

$V_2 \cdot V_2 = 4+1+1 = 6$

$X_3 = \begin{matrix} 2 \\ -2 \\ -1 \\ 3 \end{matrix} - 2 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

$X_3 = \begin{matrix} 2 \\ -2 \\ -1 \\ 3 \end{matrix} - \begin{matrix} 2 \\ 0 \\ -2 \\ 2 \end{matrix} - \begin{matrix} 1 \\ 1/2 \\ -1/2 \\ 0 \end{matrix} = \begin{bmatrix} -1 \\ 3/2 \\ 3/2 \\ 1 \end{bmatrix}$

$\boxed{D = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3/2 \\ 3/2 \\ 1 \end{bmatrix} \right.}$

$\text{Proj}_V U = \left(\frac{U \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{U \cdot w_2}{w_2 \cdot w_2}\right) w_2 + \left(\frac{U \cdot w_3}{w_3 \cdot w_3}\right) w_3$

$U \cdot w_1 = 3+0+3+3 = 3$

$w_1 \cdot w_1 = 1+0+1+1 = 3$

$U \cdot w_2 = 3+3+0-3 = 3$

$w_2 \cdot w_2 = 1+1+0+1 = 3$

$U \cdot w_3 = -3 + 9/2 + 9/2 + 3 = 9$

$w_3 \cdot w_3 = 1 + 2.25 + 2.25 + 1 = 13/2$

$\text{Proj}_U V = w_1 + w_2 + \frac{9}{13/2} w_3 = w_1 + w_2 + \frac{18}{13} w_3$

$= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -18/13 \\ 27/13 \\ 27/13 \\ 18/13 \end{bmatrix} = \begin{matrix} 2 - \frac{18}{13} \\ 1 + 27/13 \\ 1 - 27/13 \\ \frac{18}{13} \end{matrix} = \begin{bmatrix} 8/13 \\ 40/13 \\ -14/13 \\ \frac{18}{13} \end{bmatrix}$

$\boxed{\text{Proj}_U V = \begin{bmatrix} 8/13 \\ 40/13 \\ -14/13 \\ 18/13 \end{bmatrix}}$

14/20

$1.5 \times 3 = 4.5$

$18 \times 3 = 54$

$\frac{27}{13}$

26

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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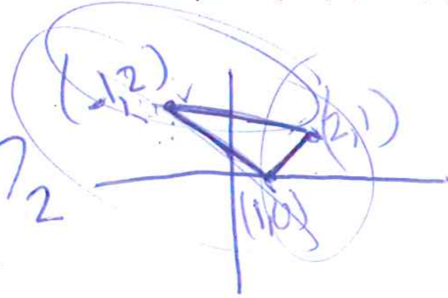
What's b? A_1 A_2 A_3

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1,0)$, $(-1,2)$, $(2,1)$.

$A = \begin{matrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{matrix}$ $B = \begin{matrix} 1 \\ \sqrt{5} \\ \sqrt{5} \end{matrix}$



$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ $A^T B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{5} \\ \sqrt{5} \end{bmatrix} = \quad 2$

Least Square

$(A^TA)^{-1}(A^Tb) = \begin{bmatrix} x \\ x \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad 1 \quad -1$

$A^TA = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} = 1+1+4$

?

$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

4/20

$\begin{matrix} 2 & 1 \\ 1 & 0 \end{matrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sqrt{1^2+1}^2 = 8$

$\begin{matrix} 1 & -1 \\ 0 & 2 \end{matrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \sqrt{2^2+2^2} = \sqrt{8}$

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{8} \\ \sqrt{10} \\ \sqrt{2} \end{bmatrix} ? = \begin{matrix} \sqrt{8} + \sqrt{10} + 2\sqrt{2} \\ 0 + 2\sqrt{10} + \sqrt{2} \end{matrix}$

$\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} = \begin{matrix} -3 \\ 1 \end{matrix} \sqrt{9+1} = \sqrt{10}$

$\left.\begin{matrix} 5 & 0 \\ 0 & 5 \end{matrix}\right| \begin{matrix} \sqrt{8} + \sqrt{10} + 2\sqrt{2} \\ 2\sqrt{10} + \sqrt{2} \end{matrix}$

$\begin{matrix} 2 & 1 \\ 1 & 0 \end{matrix} = \begin{matrix} 1 \\ 1 \end{matrix} \sqrt{1+1} \sqrt{2}$

$\boxed{f(x) = \frac{\sqrt{8}}{5} - \frac{\sqrt{10}}{5} + \frac{2\sqrt{2}}{5} x + \frac{2\sqrt{10}}{5} + \frac{\sqrt{2}}{5}}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 ✔14 15 16 17 18 19 20

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ **b)** $\lambda = -1$ **c)** $\lambda = -2$

$$\begin{bmatrix} \cdot & 1 & 2 \\ \cdot & 1 & 0 \\ \cdot & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix} = -\lambda[(1-\lambda)(2-\lambda)] - 1[(2-\lambda)(0)] + 2[2(+4-4\lambda)]$$

$1-2\lambda-\lambda+\lambda^2$

$-\lambda[\lambda^2-3\lambda+1]$

$-\lambda^3+3\lambda^2\cancel{-\lambda}-2\cancel{+\lambda}+4+8-8\lambda$

$(-\lambda^3+3\lambda^2-8\lambda+10)$ $-4\lambda = -5$ $\frac{-5}{-4}$

$\lambda^2(-\lambda+3)$ $2(-4\lambda+5)$

~~$\lambda=0$ $\lambda=3$ $\lambda=5/4$~~

$-\lambda^3+3\lambda^2-\lambda-2+\lambda+4+16-16\lambda$

$-\lambda^3+3\lambda^2\cancel{-16\lambda+14}$

$+7$ -6

$\lambda^2(-\lambda+3)$

$\lambda=0$ $\lambda=3$

The roots are $\lambda=$ ~~0, 3, 5/4~~ therefore $\lambda=0$ is an eigenvalue while $\lambda=-1$ and $\lambda=-2$ are not

$\lambda=0=$ $\begin{matrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{matrix}$

$\lambda=-1=$ $\begin{matrix} -1 & 1 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 3 \end{matrix}$

$\lambda=-2=$ $\begin{matrix} -2 & 1 & 2 \\ 1 & 3 & 0 \\ 4 & 2 & 4 \end{matrix}$

14/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 ✔ 14 15 16 17 18 19 20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$3 = \begin{bmatrix} 1-3 & 8 & 4 \\ -2 & 11-3 & 4 \\ 2 & -8 & -1-3 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix}$ $-2x_1 + 8x_2 + 4x_3 = 0$ $\frac{-2X_1}{-2} = \frac{-8}{-2}x_2 - \frac{4}{2}x_3$

$x_1 = 4x_2 + 2x_3$

$v_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $v_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(1) (1) (-1)

$5 \begin{bmatrix} 1-5 & 8 & 4 \\ -2 & 11-5 & 4 \\ 2 & -8 & -1-5 \end{bmatrix} = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix}$ $-4x_1 + 8x_2 + 4x_3 = 0$ $-2x_1 + 6x_2 + 4x_3 = 0$ $2x_1 - 8x_2 - 6x_3 = 0$

$x_1 = +2x_2 + x_3$

$-4x_2 - 2x_3 + 6x_2 + 4x_3$ $2x_2 + 2x_3$ $x_2 = -x_3$

$v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = 4 - 2 + 0 = 2$ $D^{-1} = \frac{1}{2}\begin{bmatrix} & 0^{-1} \\ 0 & \end{bmatrix}$

12/20

$D = P^{-1}AP =$

$D = P^{-1}\begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}\begin{bmatrix} 4 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$1+8+0$ $2+4$ $9-4$
$-8+11$ $-4+4$ $\cancel{-2+5}$ $-2+11+4$
$8-8$ $4-1$

$$\boxed{D = P^{-1}\begin{bmatrix} 9 & 6 & 5 \\ 3 & 0 & 5 \\ 0 & 3 & -5 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10 11 ✔12 13 14 15 16 17 18 19 20

$\begin{matrix}0 & 1 & 0 & 1\\ 1 & 1 & 1 & 1\end{matrix}$ O+1 O+1 $\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$ $A=\begin{bmatrix}a & b\\ c & a\end{bmatrix}$ $A^t = \begin{matrix}a & c\\ b & d\end{matrix}$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then 2v is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and **w** is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then **w** must be the zero vector.

$A^T = A$ $\quad B^T B = I$

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

et $A=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$ then $\begin{bmatrix}1-\lambda & 1\\ 0 & 2-\lambda\end{bmatrix}$ gives $\lambda = 1, 2$ for $\lambda = 1$ Eigenvector $= \begin{bmatrix}0\\ 1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$

$2(\lambda) = 2$ $\begin{bmatrix}1-2 & 1\\ 0 & 2-2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}$ Eigenvector $= \begin{bmatrix}1\\ 1\end{bmatrix}$ and $2 \cdot \begin{bmatrix}0\\ 1\end{bmatrix} \neq \begin{bmatrix}1\\ 1\end{bmatrix}$ $\boxed{\therefore \text{False}}$

+5

~~True, the formula for $\text{Proj}_V w = \left(\frac{w \cdot v}{v \cdot v}\right) v$ if w = zero vector $\left(\frac{w \cdot v}{v \cdot v}\right) = 0$~~

~~$\therefore \left(\frac{w \cdot v}{v \cdot v}\right) v$ will $= 0$ thus being true.~~

True because the matrix is symmetric $A^t = A$, and orthogonal $A^t \cdot A = I$. A^2 will be equivalent to $A \cdot A^t$ as $A = A^T$ there for equating to the identity matrix.

True
~~False~~, only symetric 2×2 matrices and a symmetric matrix plus another symetric matrix will be symmetric.

5

) ~~False~~, if $V = -w$ then $\text{Proj}_V W = \left(\frac{-w^2}{-w^2}\right) \cdot -w = 1(-w) = -w$

therefore w can be a non-zero vector

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

$354 \times 9 = 476$

$480 - 476 + 13$

$\frac{1}{9}$

$-\frac{1}{9}$

$-\frac{1}{18}$

$\frac{3}{18}$

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\mathrm{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

Rough.

$4+4+1+9$

$2+0+1+3$

$$\frac{5}{18}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 10/18 \\ -10/18 \\ -5/18 \\ 5/6 \end{bmatrix}$$

Using Gram-Smidth Process we computed.

$W_1 = V_1$

$$W_2 = V_2 - \frac{W_1 \cdot V_2}{V_2 \cdot V_2} \cdot V_2$$

$$= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

8/20

$$= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{2+0+1+0}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{\cancel{3}}{\cancel{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \cancel{0}\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathcal{D} = \left\{ \cancel{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ \cancel{0}\ 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} ? \end{bmatrix} \right\}$$

$$W_3 = V_3 - \frac{W_2 \cdot V_3}{\cancel{V_3 \cdot V_3}} \cancel{V_3} - \frac{W_1 \cdot V_3}{V_3 \cdot V_3} V_3$$

$2+0+0-3$

$$\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}}{18} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \Rightarrow$$

$$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{2+0+0-3}{4+4+1+9} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{18}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1/9 \\ -1/9 \\ -1/18 \\ 1/6 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 19/9 \\ -19/9 \\ -9/18 \\ 19/6 \end{bmatrix}^{2} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 10/9 \\ -19/9 \\ -9/18 \\ 13/6 \end{bmatrix}$$

$\frac{10}{9} - \frac{9}{8} + \frac{13}{6}$

0 1 2 3 4 5 6 7 8 9 10 ✔ 11 12 13 14 15 16 17 18 19 20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1,0)$, $(-1,2)$, $(2,1)$.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & 2 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & 2 \\ 0 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \sim \Big($$

$x_1 + \frac{5}{2}x_3 = 0$

$4x_2 + x_3 = 0$

~~$x_3 = 0$~~

1/20

$0 = a + b$

$2 = -a + b$

$2 = a + b$

$$\boxed{\begin{aligned} a + b &= 0 \\ -a + b &= 2 \\ a + b &= 2 \end{aligned}}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

?

$2\times3 \cdot 3\times2$

3×2 2×3

Let $b = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

$$\hat{b} = \frac{\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
☐	☐	☐	☐	☐	☐	☐	☐	☐	☑	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$$\det(A - \lambda I) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}$$

$$= \cancel{(-\lambda)(1-\lambda)(2-\lambda)}$$

a) Yes as $-\lambda_1 = 0$ $\lambda = 0$

b) No $\lambda_2 = 1$

c) No $\lambda_3 = 2.$

4/20

0 1 2 3 4 5 6 7 ✔ 8 9 10 11 12 13 14 15 16 17 18 19 20

$$\begin{bmatrix} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 0 & -1 & 5 \end{bmatrix}$$

$\Rightarrow$

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$\lambda = 3$ $\begin{bmatrix} 1-\lambda & 8 & 4 \\ -2 & 11-\lambda & 4 \\ 2 & -8 & -1-\lambda \end{bmatrix}$

16/20

$$= \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -8 & -4 \end{bmatrix}$$

$2x_1 - 8x_2 + 4x_3 = 0$ or $x_1 - 4x_2 + 2x_3 = 0$

$$x_1 = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 5$ $\begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \sim \cancel{\begin{bmatrix} -4 & 8 & 4 \\ 0 & -2 & -2 \\ 2 & -8 & -6 \end{bmatrix}} \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 0 & -6 & -6 \end{bmatrix}$

$\begin{bmatrix} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & -3 \end{bmatrix}$ $\sim \begin{bmatrix} -2 & 2 & 0 \\ -2 & 6 & 4 \\ 0 & -6 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ -3 & 0 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -3 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_2 = x_3$. $x_1 = -x_3$ $x_1 = x_3$. $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $P_1 = \begin{bmatrix} 4/\sqrt{17} \\ 1/\sqrt{17} \\ 0 \end{bmatrix}$ 5 $P_2 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$ $P_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

0 1 2 3 4 5 6 ☑7 8 9 10 11 12 13 14 15 16 17 18 19 20

$\begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\begin{bmatrix}1&0\\1&3\end{bmatrix}$ $(1-\lambda)(1-\lambda)$, $\lambda=1$

$\begin{matrix}1-\lambda & 1\\ 0 & -\lambda\end{matrix}$ $\lambda=0$.

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\operatorname{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A+B$ is also orthogonally diagonalizable.

$\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&3\\3&2\end{bmatrix}$ [

a False, consider $\begin{bmatrix}2&3\\1&5\end{bmatrix}$ $\lambda=2$.

+2 ?

b True, because $\operatorname{proj}_V w$ is 0 if w is orthogonal to V. and projection cannot reverse a direction. ?
The only case when $\operatorname{proj}_V w = -w$ would be if $w=0$ because $-w=0$

+3

c) True, $A^T = A^{-1}$ (orthogonal) $A = A^T$ (symmetric)

+5

~~$A\cdot A^{-1} = I \Rightarrow$~~ $A^T\cdot A = I$ (orthogonal)

$(A)\cdot A = I$ since symmetric $A^2 = I$

d) True, since both A & B are $n\times n$ matrices, which are orthogonally diagonizable, which means they have to be symmetric if $A = \begin{bmatrix}x&m\\m&y\end{bmatrix}$ & $B = \begin{bmatrix}p&r\\r&q\end{bmatrix}$

+5

then $A+B = \begin{bmatrix}x+p & m+r\\ m+r & y+q\end{bmatrix}$

$A+B$ is symmetric & $n\times n$ so orthogonally diagonizable.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathrm{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3\}$ of the subspace V.

b) Compute the vector $\mathrm{proj}_V \mathrm{u}$, the orthogonal projection of u on V.

a) Gram-Schmidt Process

$w_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ $w_1 \cdot v_2 = 2 + 0 + 1 + 0 = 3$

$w_1 \cdot w_1 = 1 + 0 + 1 + 1 = 3$

$w_1 = v_1$

$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{3}{3}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \leftarrow w_2$

$v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$ $w_2 \cdot v_3 = 2 + (-2) + 0 + (-3) = -3$

$w_2 \cdot w_2 = 1 + 0 + 1 + 1 = 3$

$w_1 \cdot v_3 = 2 + 1 + 3 = 6$

$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 =$

$2 - 1 + 1 \quad 2$
$-2 - 0 + 1 \quad -1$
$-1 - -1 + 0 \quad 0$
$3 - 1 + -1 \quad 1$

$\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \left(\frac{6}{3}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{-3}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} =$

$\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \cancel{-} - \left(\frac{3}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \leftarrow w_3$

w_1, w_2, w_3

$\mathcal{D} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ ✓

$w_1 \cdot w_2 = 0$
$w_2 \cdot w_3 = 0$ ✓
$w_1 \cdot w_3 = 0$

b) $\mathrm{proj}_V U$

$w_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ $w_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ $w_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ $U = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$

$w_1 \cdot w_1 = 3$ $\quad U \cdot w_1 = 3$
$w_2 \cdot w_2 = 3$ $\quad U \cdot w_2 = 3$
$w_3 \cdot w_3 = 3$ $\quad U \cdot w_3 = 3$

$\mathrm{Proj}_V U = \left(\frac{U \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{U \cdot w_2}{w_2 \cdot w_2}\right) w_2 + \left(\frac{U \cdot w_3}{w_3 \cdot w_3}\right) w_3$

$\frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{3}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \left(\frac{3}{3}\right)\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\mathrm{Proj}_V U = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

20/20

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	✔
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$7x = -3 \quad \frac{}{/7} \quad \frac{}{/7} \qquad x = \frac{3}{7}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underset{b}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}$$

$$A^TA = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & 1-1+2 \\ 1-1+2 & 1+1+1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\frac{4}{7} \cdot -9 + \frac{-27}{7} + \frac{21}{7} \frac{-6}{7} \qquad A^TAx = A^Tb$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \updownarrow \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 6 & 2 & 0 \end{array}\right] \cdot -3 \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 0 & -7 & -9 \end{array}\right] \cdot 3/7 \rightarrow \left[\begin{array}{cc|c} 2 & 0 & \frac{-6}{7} \\ 0 & -7 & -9 \end{array}\right] \cdot \frac{1}{7} \quad \left[\begin{array}{cc|c} 2 & 0 & \frac{-6}{7} \\ 0 & 1 & 9/7 \end{array}\right] \cdot \frac{1}{2}$$

$$\left[\begin{array}{c} \frac{-3}{7} \\ 9/7 \end{array}\right] \quad z_1 = -3/7 \quad z_2 = 9/7$$

$$9/3 - 2/3$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \cdot -\frac{1}{3} \quad \left[\begin{array}{cc|c} 6 & 6/3 & 0 \\ 0 & 7/3 & 3 \end{array}\right] \cdot \frac{1}{6} = \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 7/3 & 3 \end{array}\right] \cdot 3/7 = \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 1 & 9/7 \end{array}\right] \cdot -\frac{1}{3}$$

$$\begin{array}{cc} z_1 & z_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 9/7 \end{array}\right] \qquad z_1 = -3/7 \quad z_2 = 9/7$$

$$-\frac{3}{7}x + \frac{9}{7}$$

20/20

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	✔

$\lambda,\ 3\lambda$

$\det(A - \lambda I_n)$

3. Consider the following matrix A:

$$\det\begin{pmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{pmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

$$\begin{matrix} -\lambda & 1 & 2 & -\lambda & 1 \\ 1 & 1-\lambda & 0 & 1 & 1-\lambda \\ 4 & 2 & 2-\lambda & 4 & 2 \end{matrix}$$

For each value of λ given below determine if it is an eigenvalue of A. $(-\lambda \cdot 1-\lambda \cdot 2-\lambda) + 0 + 4 \cdot$

$(2 \cdot 4 \cdot 1-\lambda) - 0 - (2-\lambda)$

$8 \cdot 1-\lambda$

$8 - 8\lambda$

$(1-\lambda)(2-\lambda)$

$\lambda^2 - \lambda \cdot 2\lambda + 2$

$-\lambda\ (\lambda^2 - 3\lambda + 2)$

$-\lambda^3 + 3\lambda^2 + 2\lambda + 4 - (8 + 8\lambda)$

$-\lambda^3 + 3\lambda^2 + 16\lambda - 4 - (2 - \lambda)$

$-\lambda^3 + 3\lambda^2 + 17\lambda - 6 = 0$

a) $\lambda = 0$ NO

b) $\lambda = -1$ NO

c) $\lambda = -2$ Yes ✓

$-\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0$ ✓

$0 + 0 + 0 - 6 \neq 0$ ✓

$-(-1)^3 + 3(-1)^2 + 7(-1) - 6 = 0$

$1 + 3 - 7 - 6 \neq 0$ ✓

$-(-2)^3 + 3(-2)^2 + 7(-2) - 6 = 0$

$8 + (12) + (-14) - 6$ ✓

$20 \quad -20$

$0 = 0$ ✓

20/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ✔

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$\lambda_1 = 3$ $[A - 3I \mid 0]$

$$\left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ -2 & 8 & 4 & 0 \\ 2 & -8 & -4 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \div 1/2 \quad \begin{array}{c} x_1 \; x_2 \; x_3 \\ \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = +4x_2 + 2x_3$
$x_2 = x_2$
$x_3 = x_3$

$$x^2 \begin{bmatrix} +4 \\ 1 \\ 0 \end{bmatrix} + x^3 \begin{bmatrix} +2 \\ 6 \\ 1 \end{bmatrix}$$

$\lambda_2 = 5$ $[A - 5I \mid 0]$

$$\left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \div -\tfrac{1}{4} \quad \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot \tfrac{1}{2} \quad \begin{array}{c} x_1 \; x_2 \; x_3 \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = -x_3$
$x_2 = -x_3$
$x_3 = x_3$

$$x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

20/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ✔

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then 2v is an eigenvector of A corresponding to the eigenvalue 2λ.

+5 ✓ False, while you can multiply eigenvectors to scale them they still correspond to the original eigenvalue, λ.

b) If V is a subspace of $\mathbb{R}^2$ and w is a vector such that $\text{proj}_V\, w = -w$ then w must be the zero vector.

+1 True, it is impossible for the projection to return $-w$ on a non-zero vector 2

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

+1 True, propertiey of orthonormal vectors $v_i \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$?.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

+5

d) True, If A and B are orthognally diagoralizable then they have to be symmetric Matrices

A+B will still be symmetric as adding B top of diagonal will cancel out with B's bottom of diagonal.

Since (A+B) is symetric it is orthogonally diagonilizable

a)

$\det\left(\begin{bmatrix} 3-\lambda & 2 \\ 1 & 8-\lambda \end{bmatrix}\right)$

$(3-\lambda)(8-\lambda) - 2$

$24 - 11\lambda + \lambda^2$

$-8 \quad -3$

$(\lambda - 8)(\lambda - 3) = 0$

$\lambda = 8 \quad \lambda = 3$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad -\frac{2}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{0}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

$1/9 + 4/9 + 4/9 = 1$ $\quad$ $2/9 + 2/9 + -4/9 = 0$

$2/9 - 4/9 + 2/9 = 0$

$2/9 + 2/9 - 4/9 = 0$ $\quad$ $4/9 + 1/9 + 4/9 = 1$

$4/9 - 2/9 - 2/9 = 0$

$2/9 - 4/9 \; 2/9 = 0$ $\quad$ $4/9 - 2/9 - 2/9 = 0$

$4/9 + 4/9 + 1/9 = 1$

0 1 2 3 4 5 6 7 8 9 10 11 12 ✔ 13 14 15 16 17 18 19 20

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$0+0+{-3}$

$$\frac{2+(-2)+0+(-3)}{1+1+0+1}$$

$$(-1)\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

a) $w_1 = v_1 \quad \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{w_1}$

$$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 \qquad \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{w_2}$$

$$\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{2+0+1+0}{1+0+1+1}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1}\right) w_1 - w_2\left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right)$$

$$\frac{2+0+1+3}{3} \quad \frac{6}{3} = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 2 \end{bmatrix}$$

$$\boxed{\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

✓

$$w_3 = \begin{bmatrix} \cancel{2} \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} \cancel{2} \\ \cancel{0} \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} \cancel{-1} \\ \cancel{-1} \\ 0 \\ 1 \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

12/20

b) $$\text{proj}_V u = \left(\frac{u \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{u \cdot v_2}{v_2 \cdot v_2}\right) v_2 + \left(\frac{u \cdot v_3}{v_3 \cdot v_3}\right) v_3$$

$-\frac{6}{3} + {-\frac{1}{3}}$

$$1 \left(\frac{3+0+-3+3}{1+0+1+1}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{6}\;1\;\frac{6+3+(-3)+0}{4+1+1+0}\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{6}{18}\;\frac{1}{3}\;\frac{6+(-6)+(-3)+9}{4+4+1+9}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \quad \begin{matrix} 2/3 \\ -2/3 \\ -1/3 \\ 1 \end{matrix}$$

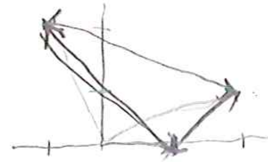
$\frac{9}{3} + \frac{2}{3}$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 11/3 \\ 1/3 \\ -7/3 \\ 2 \end{bmatrix}}$$

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☑ 12 ☐ 13 ☐ 14 ☐ 15 ☐ 16 ☐ 17 ☐ 18 ☐ 19 ☐ 20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$



~~y = mx + b~~

$f(x) = ax + b$

$y = ax + b$

$Ax = b$ $\quad$ ✓ $\quad A^T A x = A^T b$

$0 = a(1) + b$

$2 = a(-1) + b$

$1 = a(2) + b$

$$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

?

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b$$?

$A^T A x = A^T b$

what's A? what's b?

4/20

$A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$2 + 1 + 2$

$0 - 2 + 1$

$5x = -1$

$x = -\frac{1}{5}$

$\boxed{f(x) = -\frac{1}{5}x + \frac{1}{5}}$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
☐	☐	☐	☐	✔	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

$(8\lambda + \quad)(\lambda - \quad)$

$Ax = \lambda I x$

$-8 + 4\lambda - 16\lambda + 8\lambda^2$

$8\lambda^2 - 12\lambda - 8$

3. Consider the following matrix A:

$1(2-\lambda-0)[(2-4-4\lambda)2]$

$$A = \begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$(2-\lambda)(-\frac{1}{2} + \lambda) - 8$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ $\boxed{NO}$ **b)** $\lambda = -1$ $\boxed{NO}$ **c)** $\lambda = -2$ $\boxed{NO}$

$\begin{bmatrix} -2 & 2 \\ 1 & \end{bmatrix}$ $\begin{bmatrix} 1-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}$

$\lambda^2 - 5\lambda + 4 + 2$

$\lambda^2 - 5\lambda + 6$

$(\lambda-3)(\lambda-2)$

$\lambda = 2, 3$

$$\left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 2 & 2 & 0 \end{array}\right) \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & 2 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & -5 & 0 \end{array}\right] \quad \begin{array}{cccc} 0 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$$

$= \lambda A$

$Ax = 2x$

$$\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \quad \begin{array}{cccc} 1 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \quad \begin{array}{cccc} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \end{array}$$

$Ax = \lambda x$

$\begin{matrix} -2 & 2 & 0 \\ 1 & 1 & 0 \end{matrix}$

$\begin{matrix} x_1 = \\ x_2 = \\ x_3 = \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + x_2 + 2x_3 = 0 \quad x_1 = x_2 + 2x_3$

yes free

$x_2 - 2x_3 = 0 \quad x_2 = 2x_3$

NO because $A\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ eigenvector can't be $\emptyset$

?

$x_3 = 0$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{matrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -12 & 0 \end{matrix}$

NO because $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ can't be $\emptyset$,

$\boxed{No}$

14/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 ✔14 15 16 17 18 19 20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A - \lambda_1 = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \cdot 1$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -8 & -4 & 0 \end{array}\right] \cdot \tfrac{1}{2}$$

$$\begin{matrix} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$x_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$A - \lambda_2 = \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \cdot 2, \cdot 1$$

$$\left[\begin{array}{ccc|c} 0 & -8 & -8 & 0 \\ 0 & -2 & -2 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \begin{matrix} \cdot -4 \\ \cdot -\tfrac{1}{2} \\ \cdot \tfrac{1}{2} \end{matrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -4 & -3 & 0 \end{array}\right] \cdot 4$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$

$x_2 + x_3 = 0$

$x_1 = -x_3$

$x_2 = -x_3$

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = x_1$$

$$\boxed{P = \begin{bmatrix} -1 & 4 & 2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

20/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ✔

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then 2v is an eigenvector of A corresponding to the eigenvalue 2λ.

+5 False, The eigenvector 2v still corresponds to the eigenvalue λ. Consider (A) below.

b) If V is a subspace of $\mathbb{R}^2$ and **w** is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then **w** must be the zero vector.

True, see (B) below.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

+5 True by definition to determine if a sym. Matrix is othogonal, you compute A^TA and if you get identity matrix its orthogonal. Because $A^T = A$ then $A^2 = A^TA$ therefore it will be identity matrix

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

+5 By definition a matrix is orthogonally diagonizable is A is symmetric and every symmetric matrix is diagonizable, so the answer is true because any 2×2 symmetric matrix

for example: $\begin{bmatrix} A & C \\ C & B \end{bmatrix} + \begin{bmatrix} D & F \\ F & E \end{bmatrix} = \begin{bmatrix} A+D & C+F \\ C+F & B+E \end{bmatrix}$

added to another symmetric matrix is still symmetric. TRUE.

(A) $\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ $\lambda^2 - 5\lambda + 6$, $(\lambda-2)(\lambda-3)$, $\lambda = 2, 3$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \to x_1 = -2x_2 \to \begin{bmatrix} 2 \\ -1 \end{bmatrix} \leftarrow$ eigenvector

but $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ is still an eigenvector that corresponds to λ, not 2λ. FALSE

In order for $\text{proj}_V w$ to be $-w$ we would have to have $w_V = -w$. That isn't possible unless w is the zero vector because $(w - \text{proj}_V w)$ must be in $W^{\perp}$ and not in W. Unless its the zero vector. TRUE.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\mathrm{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

$$w_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{(1\cdot 2 + 0\cdot 1 + -1\cdot -1 + 1\cdot 0)}{(1\cdot 1 + 0\cdot 0 + -1\cdot -1 + 1\cdot 1)} w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \checkmark$$

$$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{(2+0+1+3)}{(1+0+1+1)}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{(2-2+0-3)}{(1+1+0+1)}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

$$\boxed{\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}} \checkmark$$

$$\mathrm{proj}_V u = \left(\frac{u \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2}\right) w_2 + \left(\frac{u \cdot w_3}{w_3 \cdot w_3}\right) w_3 = \frac{(3+0-3+3)}{(1+0+1+1)} w_1 + \frac{(3+3+0\overset{-3}{+3})}{(1+1+0+1)} w_2 + \frac{(3-3+3+0)}{(1+1+1+0)} w_3$$

$$= w_1 + \cancel{3} w_2 + w_3 = \begin{bmatrix} 1+3+1 \\ 0+3-1 \\ -1+0+1 \\ 1+3+0 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 2 \\ 0 \\ 4 \end{bmatrix}}$$

19/20

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2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & 1-1+2 \\ 1-1+2 & 1+1+1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-2+2 \\ 0+2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$(A^TA)\hat{x} = (A^Tb)$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 7/3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 9/7 \end{array}\right]$$

$$\boxed{f(x) = -\frac{3}{7}x + \frac{9}{7}}$$ ✓

20/20

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14 ☐ 15 ☐ 16 ☐ 17 ☐ 18 ☐ 19 ☑ 20

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ **b)** $\lambda = -1$ **c)** $\lambda = -2$

$$A - 0I = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 4 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{not an eigenvalue}$$

$$A + I = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 3 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & -5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{not an eigenvalue}$$

$$A + 2I = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 0 \\ 4 & 2 & 5 \end{bmatrix} \to \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 3 & 0 & 0 \\ 4 & 2 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & -10 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 6/5 & 0 \\ 0 & 1 & -2/5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{vmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{vmatrix} \to (-\lambda)(1-\lambda)(2-\lambda) + 0 + 2 - 8(1-\lambda) - 0 - (2-\lambda)$$

$$= (-\lambda + \lambda^2)(2-\lambda) + 2 - 8 + 8\lambda - 2 + \lambda$$

$$= (-2\lambda + 3\lambda^2 - \lambda^3) - 8 + 9\lambda$$

$$= -\lambda^3 + 3\lambda^2 + 7\lambda \cancel{-8}\ 6 = 0$$

19/20

$-(0)^3 + 3(0)^2 + 7(0) - 8 = -8 \neq 0 \leftarrow \boxed{\text{not an eigenvalue}}$

$-(-1)^3 + 3(-1)^2 + 7(-1) - 8 = -11 \neq 0 \leftarrow \boxed{\text{not an eigenvalue}}$

$-(-2)^3 + 3(-2)^2 + 7(-2) - 8 = -2 \neq 0 \leftarrow \boxed{\cancel{\text{not}}\ \text{an eigenvalue}}$

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14 ☐ 15 ☐ 16 ☐ 17 ☐ 18 ☑ 19 ☐ 20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$$A-3I = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \to \left[\begin{array}{ccc|c} 2 & -8 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} \to \underset{x_2=1,\, x_3=-1}{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}, \underset{x_2=-1,\, x_3=1}{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}}$$

$$A-5I = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \to \left[\begin{array}{ccc|c} 2 & -8 & -6 & 0 \\ -2 & 6 & 4 & 0 \\ -4 & 8 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -4 & -3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} \to \underset{x_3=1}{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$$

lin. dependent

$$\boxed{P = \begin{bmatrix} 2 & \cancel{-2} & -1 \\ 1 & \cancel{-1} & -1 \\ -1 & \cancel{1} & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$$ ✓

15/20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ☑15 16 17 18 19 20

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\mathrm{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

False. let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 4 = -\lambda + \lambda^2 - 2 = 0 \leftarrow (\lambda-2)(\lambda+1)$ $\lambda = 2$ $\lambda = -1$

+3

For $\lambda = -1 \rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} x = 0 \rightarrow x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda = -2 \rightarrow \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} x = 0 \rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq 2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

True. For any u such that $\mathrm{proj}_V u = v$, $u \cdot v \geq 0$. ! $w \cdot -w \leq 0$ so the only way for this to be true is if $w = 0$ why?

+4

True. If A is orthogonal and symmetric then it is of the form PDP^T. $A^2 = PD^2P^T$. Since P is an orthogonal basis for A, PD^2P^T is going to be the identity matrix

~~False~~. let $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Both A and B are orthogonaly diagonalizable, let $W = A + B \rightarrow \begin{bmatrix} 1+1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -2\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1+1/\sqrt{2} \end{bmatrix}$. Though W is symmetrical, the columns of W arent orthogonal to each other so W isn't orthogonally diagonalizable.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
										✔	✔	✔								