MTH-309T-F19-EX1-076-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	1	0	2	1	0

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	TOTAL	GRADE
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1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $C_1V_1 + C_2V_2 + C_3V_3 = W$

$C_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + C_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + C_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$

$C_1 - C_2 + C_3 = -2$
$0 \quad C_2 + 2C_3 = 2$
$2C_1 + C_2 + \quad = b$

$b = 2C_1 + C_2$

$C_1 = -3$ $\quad C_1 = -\frac{3}{2}$
$C_2 = 0$ $\quad C_2 = 1$
$C_3 = 1$ $\quad C_3 = \frac{1}{2}$

$b = -5, -2$

b) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$

No, there must be a pivot column in every column and have only one solution

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = -R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \; R_1 = R_2 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \; R_3 = -2R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \; R_1 = -R_3 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \; R_2 = R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$c_1 \; c_2 \; c_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 2 & 1 & -1 & 3 & 2 & 1 \end{array}\right] \xrightarrow{R_3 = 2R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 4 & 5 & 4 \\ 0 & 1 & 3 & 11 & 12 & 9 \end{array}\right] \quad R_2 = R_1 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 1 & 3 & 11 & 12 & 9 \end{array}\right] R_3 = R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 & 7 & 7 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] (-2)$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] (-1)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 5 & 0 \\ 0 & 1 & 0 & -7 & -3 & 3 \\ 0 & 0 & 1 & 6 & 5 & 2 \end{array}\right] \qquad C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$ V

$T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} u = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$

$u_1 - 2u_2 = 1$
$u_1 + u_2 = 10$
$u_1 - 3u_2 = -2$

$u = \begin{bmatrix} \ \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

NUl(A) = {0}

pivot position in every column

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \cdot\tfrac{1}{2} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \xrightarrow{R_3 = -3R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] R_3 = -R_2 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] R_2 = R_2 - R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 = R_1 - R_2$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\therefore$ one-to-one

b) $A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \cdot\tfrac{1}{2} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] R_3 = -3R_1 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \cdot(-1) \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

not one-to-one

$$\begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} V_1 = T_A(V_1)$$

$V_1 + V_2 = 0$

$V_2 + 2V_3 = 0$

$\therefore$ when $V_2 = 2$

$V_1 = -2$; $V_3 = -1$

$V_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$

$V_2 = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}$

when $V_2 = 4$

$V_1 = -4$ $V_2 = -2$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True, because $w + v = w'$ so

w has to be in $\text{Span}(u, v)$

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

TRUE,

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

because $x_1u + x_2v + x_3w = 0$ has only one solution, $x_1u + x_2v$ will only have one solution as well.

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

True because

$A(u+v) = Au + Av$

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then T(u) must be in Span(T(v), T(w)).

False, T(u) could not be in Span (T(V), T(w))

0 1 2 3 4 5 6 7 8 9 10