MTH-309T-F19-EX1-048-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	2	6	6	3	6

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
20	4	10	10	15	1	2	2	10	70	B-

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer.

$c_1V_1 + c_2V_2 + c_3V_3 = w$

a) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} V_1 + \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} V_2 + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{②+①} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{①(-2)+③} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & b \end{array}\right] \xrightarrow{②(3)+③} \left[\begin{array}{cccc} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & \underbrace{0}_{\text{free variable}} & b+6 \end{array}\right]$$

$\boxed{b = -6}$ ✓

b) $x_1 = -3x_3$

$x_2 = 2 - 2x_3$

$x_3 = \text{free}$

The set $\{V_1, V_2, V_3\}$ is not linearly independent. This is because x_3 is a free variable, therefore the set has infinite solutions (and is linearly dependent.) ✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1+3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1 \cdot 2} \left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{1+3(-1)} \left[\begin{array}{ccc|ccc} \cancel{1} & \cancel{2} & \cancel{1} & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1+(-1)3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)1+2\,2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2+3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 0 \\ 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2+(-1)1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{3+(-1)2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{2 \cdot \frac{1}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow{3 \cdot (-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

?

$$\boxed{A^{-1} = \begin{bmatrix} -2 & \cancel{2} & \cancel{0} \\ 1 & \cancel{-\frac{1}{2}} & \cancel{\frac{1}{2}} \\ 2 & \cancel{-1} & \cancel{0} \end{bmatrix}}$$

This is a very complicated (and ultimately incorrect) way of performing row reduction.

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☒ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$A^T C = B$

$C = (A^{-1})^T B$ ✓

Based on problem 2, $A^{-1} = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 & 0 \end{bmatrix}$ ← Not quite, but ok.

$$(A^{-1})^T = \begin{bmatrix} -2 & 1 & 2 \\ 2 & -\frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$C = (A^{-1})^T B = \begin{bmatrix} -2 & 1 & 2 \\ 2 & -\frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 4 & 6 \\ 4 & -\frac{5}{2} & -2 \\ 0 & 2 & 0 \end{bmatrix}}$$

ok, given the incorrect A^{-1}

0 1 2 3 4 5 6 7 8 9 10 ✗

4. (20 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T(u) = T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 - 2a_2 \\ a_1 + a_2 \\ a_1 - 3a_2 \end{bmatrix}$ ✓ ?

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -2 & -8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & 0 & -9 \end{array}\right]$

?

→ ~~No vector satisfies T(u).~~

This is the standard matrix of T.

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

a)
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 2 & 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

✓

one-to-one

b)
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc c|c} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \to$$

✓

Not one to one.

$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$T_A(v_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $T_A(v_1) \neq T_A(v_2)$

$T_A(v_2) = \begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True ← why?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

~~False~~ ← why?

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

True because Au, Av are a linear combination of A. !?

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True because of the matrix property $c \cdot T = c(T)$?

0 1 2 3 4 5 6 7 8 9 10