



# MTH 309T LINEAR ALGEBRA

## EXAM 1

October 3, 2019

Name:

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UB Person Number:

5	0	2	6	2	8	0	3
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1      2      3      4      5      6      7      TOTAL      GRADE

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TOTAL

GRADE



1. (20 points) Consider the following vectors in  $\mathbb{R}^3$ :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of  $b$  such that  $w \in \text{Span}(v_1, v_2, v_3)$ .

b) Is the set  $\{v_1, v_2, v_3\}$  linearly independent? Justify your answer.

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \xrightarrow{-2} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -5 & -2 & 4+b \end{bmatrix} \xrightarrow{\cdot 5} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 8 & 14+b \end{bmatrix} \quad \begin{matrix} 8=14+b \\ b=-6 \end{matrix}$$

$$a) \boxed{b = -6}$$

$$b) \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 8 \end{bmatrix} \xrightarrow{\cdot \frac{1}{8}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot -2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot -1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot 1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) The set is linearly independent b/c  
the homogeneous vector equation  $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$   
will only have one solution.



2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute  $A^{-1}$ .

$$\begin{aligned} & \left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\cdot -1} \left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\cdot -2} \left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \\ & \xrightarrow{\cdot -2} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \xrightarrow{\cdot -1} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array} \right] \end{aligned}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$



3. (10 points) Let  $A$  be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix  $C$  such that  $A^T C = B$  (where  $A^T$  is the transpose of  $A$ ).

$$(A^T)^{-1} = (A^{-1})^T \quad C = (A^{-1})^T B$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$(A^T)^{-1} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+4+6 \\ 3-4-6 \\ -1+4+3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix}$$

$$(A^T)^{-1} \cdot \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -4+5+4 \\ 6-5-4 \\ -2+5+2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$$

$$(A^T)^{-1} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+4+2 \\ 9-4+2 \\ -3+4+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$



4. (20 points) Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix} \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a) Find the standard matrix of  $T$ .

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

b) Find all vectors  $u$  satisfying  $T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ .

$$a) \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2(1) \\ 1(1) \\ -3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$$b) Tu = Au = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array} \right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1}} \left[ \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 8 \\ 0 & -1 & -4 \end{array} \right] \rightarrow$$

$$\rightarrow \left[ \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & -1 & -4 \\ 0 & 3 & 8 \end{array} \right] \xrightarrow{R_2 \cdot (-1)} \left[ \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 3 & 8 \end{array} \right] \xrightarrow{R_3 - 3R_2} \left[ \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{array} \right]$$

no solutions  
for  $u$



5. (20 points) For each matrix  $A$  given below determine if the matrix transformation  $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T_A(v) = Av$  is one-to-one or not. If  $T_A$  is not one-to-one, find two vectors  $v_1$  and  $v_2$  such that  $T_A(v_1) = T_A(v_2)$ .

a)  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{.-3}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{.-2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

a) is one-to-one

b)  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{.-3}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{.-2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{.-1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

b) not one-to-one

$$T\left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}\right)$$

$\uparrow \quad \quad \uparrow$   
 $v_1 \quad \quad v_2$

$$\begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_3 &= x_3 \\ \text{Nul}(A) \end{aligned}$$





6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If  $u, v, w$  are vectors in  $\mathbb{R}^3$  such that  $w + u \in \text{Span}(u, v)$  then  $w \in \text{Span}(u, v)$ .

**True**

Since  $u \in \text{span}(u, v)$

for  $u+w \in \text{span}(u, v)$   $w$  would have to be some combination of  $cu + dv$  where  $c$  &  $d$  are some constants.

Therefore  $w \in \text{Span}(u, v)$

b) If  $u, v, w$  are vectors in  $\mathbb{R}^3$  such that the set  $\{u, v, w\}$  is linearly independent then the set  $\{u, v\}$  must be linearly independent. \*Linearly dependent - homogenous vector equation has more than 1 solution

**False** let  $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

~~$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$  is linearly independent~~

~~$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$  is linearly dependent~~

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  is linearly independent

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$  is linearly dependent

**True** For a set of  $p$  vectors (2 in this case) of  $\mathbb{R}^n$  ( $\mathbb{R}^3$  in this case) can only be linearly dependent if  $p > n$ .  
 $2 \not> 3$  so the set  $\{u, v\}$  would be linearly independent.



7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If  $A$  is a  $2 \times 2$  matrix and  $u, v$  are vectors in  $\mathbb{R}^2$  such that  $Au, Av$  are linearly dependent then  $u, v$  also must be linearly dependent.

False Counterexample:  $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$   $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$   $v = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$Au$  &  $Av$  both  $= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  which is linearly dependent

but  $u$  and  $v$  are not linearly dependent

b) If  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear transformation and  $u, v, w \in \mathbb{R}^2$  are vectors such that  $u$  is in  $\text{Span}(v, w)$  then  $T(u)$  must be in  $\text{Span}(T(v), T(w))$ .

True If  $u$  is in  $\text{Span}(v, w)$  it is a combination of  $cv + dw$  where  $c$  and  $d$  are some constants

$$\text{So } T(u) = T(cv + dw) = T(cv) + T(dw) = cT(v) + dT(w)$$

Which means that  $T(u)$  is a combination of constant values of  $T(v)$  &  $T(w)$  which is what it means to be in  $\text{Span}(T(v), T(w))$ .