MTH 309Y LINEAR ALGEBRA

EXAM 3

December 11, 2018

Name: John Braganza

Person Number: 502-333-12

- Textbooks and electronic devices (calculators, cellphones etc.) are not permitted.
- You may use one sheet of notes.
- For full credit explain your answers fully, showing all work.
- Each problem is worth 20 points.

1	20
2	10
3	20
4	14
5	9
Total:	73/100

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

$\cancel{v_1 \cdot v_2 = (1)(2) + (0)(1) + (-1)(-1) + 0}$

a.) Use Gram-Smit:

$\langle w_1, w_1 \rangle = 3$

$w_1 = v_1$

$\langle w_1, v_2 \rangle = (1)(2) + 0 + 1 + 0 = 3$

$$w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - 1w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = w_2$$

$$w_3 = v_3 - \frac{\langle w_1, v_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_3 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 - 2w_1 + w_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\langle w_1, v_3 \rangle = 2 + 0 + 1 + 3 = 6$

$\langle w_2, v_3 \rangle = 2 - 2 + 0 - 3 = -3$

$\langle w_2, w_2 \rangle = 3$

$$D = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$ ✓

$\langle u, w_1 \rangle = 3 + 0 - \cancel{3} + \cancel{3} = 3$

$\langle u, w_2 \rangle = 3 + \cancel{3} + 0 - \cancel{3} = 3$

$\langle u, w_3 \rangle = \cancel{3} - \cancel{3} + 3 + 0 = 3$

$$\text{b.)} \quad \text{proj}_V u = \frac{\langle u, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle u, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \frac{\langle u, w_3 \rangle}{\langle w_3, w_3 \rangle} w_3$$

$$= w_1 + w_2 + w_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \text{proj}_V u$$ ✓

20/20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1,0)$, $(-1,2)$, $(2,1)$.

$-1+1 \quad -2$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \qquad \cancel{A^TA = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 3 & -1 & 5 \end{bmatrix}}$$

$3\times 2 \qquad 2\times 3$

$$A^TAx = A^Tb \qquad b = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$0 + (-2) + 2$

$2\times 3 \quad 3\times 1$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 3 & -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad A^Tb = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\frac{1}{2}\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & 2 & -1 & 2 \\ 3 & -1 & 5 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & -1 & -4 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & -1 & -4 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & -9/2 & 2 \end{array}\right] \frac{2}{9}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & -1 & 4/9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 7/9 \\ 0 & 0 & -1 & 4/9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 7/9 \\ 0 & 0 & +1 & -4/9 \end{array}\right]$$

$x_1 = 3/2$
$x_2 = 7/9$
$x_3 = -4/9$

$\frac{4}{10} \quad \frac{2}{5} \quad \frac{3}{2}$

$(3/2, 0) \quad (7/9, 2) \quad (-4/9, 1)$

$$a = \frac{1-0}{-\frac{8}{18} - \frac{36}{18}} = \frac{1}{-\frac{44}{18}} \qquad b = 0$$

0.4
$18 \overline{)80}$
-72
8

$$f(x) = 0.\bar{4}x + 0$$

10/20

$1\frac{8}{18}$

$18 \times 4 = 72$

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ **b)** $\lambda = -1$ **c)** $\lambda = -2$

$$\det(A - \lambda I) = \det\begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix} = \det\begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}$$ → Pick this row.

$$\det = (1)\det\begin{bmatrix} 1 & 2 \\ 2 & 2-\lambda \end{bmatrix}(-1^3) + (1-\lambda)\det\begin{bmatrix} -\lambda & 2 \\ 4 & 2-\lambda \end{bmatrix}(-1^4) + 0$$

$$= (2-\lambda-4)(1)(-1) + (1-\lambda)(\lambda^2 - 2\lambda - 8)$$

$-\lambda^3 + 2\lambda^2 + 8\lambda + \lambda^2 - 2\lambda - 8$

$-\lambda^3 + 3\lambda^2 + 6\lambda - 8$

$$= (-2-\lambda)(-1) + (-\lambda^3 + 2\lambda^2 + 8\lambda + \lambda^2 - 2\lambda - 8)$$

$$= (\lambda+2) + (-\lambda^3 + 3\lambda^2 + 6\lambda - 8) = -\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0$$ ✓

Check: $\lambda = 0$

$-0+0+0-6=0$

$-6=0$ x

check: $\lambda = -1$

$1+3-7-6=0$

$-9=0$

check $\lambda = -2$

$8+12+(-14)-6=0$

$0=0$ ✓

20/20

Only $\lambda = -2$ is an eigen val for this one.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$$Null(A-\lambda_1 I) = \left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ -2 & 8 & 4 & 0 \\ 2 & -8 & -4 & 0 \end{array}\right] \rightarrow -\frac{1}{2}\left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} x_1 = 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$Null(A-\lambda_2 I) = -\frac{1}{4}\left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ +2 & -8 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \tfrac{1}{2} \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \cancel{\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}}$$

14/20

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ. Assumption: false.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\quad z = w - w = 0$

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix. Assump: false.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A+B$ is also orthogonally diagonalizable.

a.) $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ $\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4$

$(\lambda - 2)(\lambda - 2)$

$\lambda_1 = 2 \quad \lambda_2 = 2$

$\text{Null}(A - 2I) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \end{matrix} \xrightarrow{\times 1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} x_1 = 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$2v = [$

+1 a.) this is false since for a given Matrix $P = [v_1 \dots v_n]$ where $v_1 \dots v_n$ are eigenvectors. and $D = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ where the eigenvalues are found on diagonal

The Eq $A^k = PD^kP^{-1}$ doesn't involve a change in P at all so why should $2v$ correspond to 2λ.

?

b.) if $\text{proj}_V w = -w$

+3 then $\longrightarrow$ where z is orthogonal to V space

$z = w - \text{proj}_V w$

$z = w + w$

? $z = 2w$ by definition z cannot be in V unless $w = 0$

$\therefore$ True.

c.) True ~~if A is symmetric and orthogonal~~

+3 ~~then~~ consider

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\rightarrow$ it is symmetric because the 0's match up

$\rightarrow$ ~~Orthogonal because dot Product=0~~

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \rightarrow \therefore$ orthogonal

because $1^2 = 1$

In order to have a Orthogonally symmetric Matrix the entries on the diagonal Must be 1 $\therefore A^2$ will result in Identity

Don Back

MTH 309Y LINEAR ALGEBRA

EXAM 3

December 11, 2018

Name: Adam Bonisteel

Person Number: 50143375

- Textbooks and electronic devices (calculators, cellphones etc.) are not permitted.
- You may use one sheet of notes.
- For full credit explain your answers fully, showing all work.
- Each problem is worth 20 points.

1	16
2	20
3	20
4	18
5	8
Total:	82/100

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

Gram-Schmidt

$$w_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 \qquad w_3 = v_3 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2$$

$$w_1 \cdot v_2 = 1(2) + 0(1) + (-1)(-1) + 1(0) = 2 + 0 + 1 + 0 = 3$$

$$w_1 \cdot w_1 = 1^2 + 0^2 + (-1)^2 + (1)^2 = 3$$

$$w_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$w_2 \cdot v_3 = 1(2) + 1(-2) + 0(-1) + (-1)(3)$$
$$= 2 - 2 + 0 - 3 = -3$$

$$w_2 \cdot w_2 = 1^2 + 1^2 + 0^2 - (-1)^2 = 3$$

$$w_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{\cancel{3}\ 6}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \cancel{2}\ 1 \\ -1 \\ \cancel{0}\ 1 \\ \cancel{1}\ 0 \end{bmatrix}$$

$$\boxed{\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \cancel{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}} \right\}}$$

16/20

$$\text{proj}_V u = \left(\frac{u \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2}\right) w_2 + \left(\frac{u \cdot w_3}{w_3 \cdot w_3}\right) w_3 = \frac{3}{3} w_1 + \frac{3}{3} w_2 + \frac{3}{3} w_3 = w_1 + w_2 + w_3$$

$$u \cdot w_1 = 3 + 0 - 3 + 3 = 3$$

$$u \cdot w_2 = 3 + 3 + 0 - 3 = 3$$

$$u \cdot w_3 = 6 - 3 + 0 + 3 = 6$$

$$w_3 \cdot w_3 = 4 + 1 + 0 + 1 = 6$$

$$\text{proj}_V u = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ -1 \\ 1 \end{bmatrix}}$$

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{b}$$

$$(A^T A)x = A^T b$$

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & 1-1+2 \\ 1-1+2 & 1+1+1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-2+2 \\ 0+2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \xrightarrow{R_1 - 3R_2 \to R_2} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & -7 & -9 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 1 & 9/7 \end{array}\right] \xrightarrow{-2R_2 + R_1 \to R_1}$$

$$\left[\begin{array}{cc|c} 6 & 0 & -18/7 \\ 0 & 1 & 9/7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 9/7 \end{array}\right] \to \begin{array}{l} a = -3/7 \\ b = 9/7 \end{array}$$

$$\boxed{f(x) = \frac{-3}{7}x + \frac{9}{7}}$$ ✓

20/20

where $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is eigenvector corresponding to λ

3. Consider the following matrix A:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \qquad A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}.$$

$v_2 + v_3 = \lambda v_1$

$v_1 + v_2 = \lambda v_2$

$4v_1 + 2v_2 + 2v_3 = \lambda v_3$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ NO b) $\lambda = -1$ NO c) $\lambda = -2$ Yes

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}\right) = 0 = -1\left[2-\lambda-4\right] + (1-\lambda)\left[-2\lambda+\lambda^2-8\right] - 0$$

$$2+\lambda + (-2\lambda+\lambda^2-8+2\lambda^2-\lambda^3+8\lambda) = 0$$

$$-\lambda^3+3\lambda^2+7\lambda-6=0$$

$$\lambda^3-3\lambda^2-7\lambda+6=0 \quad ✓$$

a) $0^3-3(0)^2-7(0)+6=0 \qquad 6 \neq 0 \rightarrow$ Not an eigenvalue

b) $(-1)^3-3(-1)^2-7(-1)+6=0$ ✓

$-1-3+7+6=0$ ✓

$9 \neq 0 \rightarrow$ Not an eigenvalue

c) $(-2)^3-3(-2)^2-7(-2)+6=0$ ✓

20/20

$-8-12+14+6=0$

$0=0$ ✓ $\rightarrow$ IS an eigenvalue

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

For $\lambda = 3$, $A - \lambda I = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \to \begin{bmatrix} -1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\text{Nul}(A - 3I) = \text{Span}\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ = eigenspace of A corresponding to λ_1

$A - 5I = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -2 \\ -2 & 4 & 2 \\ -1 & 4 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 & 2 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{Nul}(A - 5I) = \text{Span}\left\{ \begin{bmatrix} \cancel{0}\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$\boxed{D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad P = \begin{bmatrix} 4 & 2 & \cancel{0}\ -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}}$$

18/20

$$\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) False, just because there is an eigenvalue λ does not mean there is an eigenvalue 2λ. There is no guarantee $2\lambda = \text{val}$ is another solution to $|A - I(\text{val})| = 0$ Ex) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda_1 = 1$, $\lambda_2 = -1$

+5

b) $\cancel{\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} \frac{w_1 v_{11} + w_2 v_{21}}{v_{11}^2 + v_{21}^2} + \frac{w_1 v_{12} + w_2 v_{22}}{v_{12}^2 + v_{22}^2} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -w_1 \\ -w_2 \end{bmatrix}}$ Say $w_1 = w_2 = 1$

$\cancel{\frac{v_{11} + v_{21}}{v_{11}^2 + v_{21}^2}}$

$\cancel{\begin{bmatrix} w_1 + w_2 \\ w_1 + w_2 \end{bmatrix} + \begin{bmatrix} w_1 + w_2 \\ w_1 + w_2 \end{bmatrix}}$ $W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

+2

True, the projection of a vector on some space can be visualized as the "shadow" of that vector on that space, $\therefore$ any projection of w_1, w_2 somewhere else should carry the same sign, only they both are zero

?

c) ~~False~~, $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $A = A^T$ ✓ Symmetric ✓ $A^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

d) ~~False~~ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

+1

does not have 2 linearly independent eigenvectors

MTH 309Y LINEAR ALGEBRA

EXAM 3

December 11, 2018

Name: Tyler Betts

Person Number: 50213643

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1	10
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5	14
Total:	68/100

1. Consider the following vectors in $\mathbb{R}^4$:

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The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

$$V = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow[R_4 - R_1]{R_3 + R_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 - R_3 \\ R_4 + 2R_3}]{R_1 - 2R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 - 2R_4]{R_2 + R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$

orthogonal basis D is the basis of Col(A)

R =

a) $w_1 = v_1$

$$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$w_1 \cdot v_2 = 1 \cdot 2 + 0 \cdot 1 + -1 \cdot -1 + 1 \cdot 0 = 3$

$w_1 \cdot w_1 = 1 \cdot 1 + 0 \cdot 0 + -1 \cdot -1 + 1 \cdot 1 = 3$

$w_2 \cdot v_3 = 1 \cdot 2 + 1 \cdot -2 + 0 \cdot -1 + -1 \cdot 3 = -3$

$w_2 \cdot w_2 = 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 + -1 \cdot -1 = 3$

$$w_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{-3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{D = \begin{bmatrix} 1 & 1 & \cancel{2}\ 1 \\ 0 & 1 & -1 \\ -1 & 0 & \cancel{0}\ 1 \\ 1 & -1 & \cancel{1}\ 0 \end{bmatrix}}$$

2

10/20

part b on back →

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$\cancel{W_1 = \langle v_1 \rangle}$

$\cancel{W_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1}$

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$W_1 = \langle v_1^1 \rangle$

$W_2 = \langle v_2^2 \rangle - \frac{\langle w_1 \cdot v_2 \rangle}{\langle w_1 \cdot w_1 \rangle} w_1$

$w_1 = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

I'm sorry,
I don't know
how to do this

$A^T \cdot A x = A^T b$

$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$

$\det A^T A = 6 \cdot 5 = 30$

$\det A^T = (-1 \cdot 1 - 2 \cdot 2) + (1 \cdot 1 - 2 \cdot 0) + (1 \cdot 2 - 0) = -2$

$\boxed{f(x) = 30x - 2}$

8/20

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$\lambda = 0$

a) $Null(A - \lambda_1 I) \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 + R_3]{R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ no basis

0 is not an eigen value of A

---

b) $\lambda = -1$ $A + 1I \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & -6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ -1 is not an eigen value of A

---

c) $\lambda = -2$ $A + 2I \rightarrow \begin{bmatrix} 2 & 0 & 2 \\ 1 & 3 & 0 \\ 4 & 2 & 4 \end{bmatrix} \xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}$

$\xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[R_1 - R_3]{-1R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ -2 is ~~not~~ an eigen value of A

16/20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

basis of $(A-3I) = \begin{bmatrix} -2 & 8 & 4 \\ -2 & \cancel{8} & 4 \\ 2 & -8 & -4 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2,\ \frac{1}{2}R_3]{\frac{1}{2}R_1} \begin{bmatrix} -1 & 4 & 2 \\ -1 & 4 & 2 \\ 1 & -4 & -2 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} -1 & 4 & 2 \\ -1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_2-R_1} \begin{bmatrix} -1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_1} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x_1 - 4x_2 - 2x_3 = 0 \rightarrow x_1 = 4x_2 + 2x_3$

$x_2 = x_2$

$x_3 = x_3$

$\rightarrow x = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \rightarrow \text{basis} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

basis of $(A-5I) = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2,\ \frac{1}{2}R_3]{\frac{1}{4}R_1} \begin{bmatrix} -1 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -4 & -3 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\xrightarrow[R_1-2R_2]{R_3-R_2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 = x_3 \end{matrix} \rightarrow \begin{matrix} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{matrix}$

$\rightarrow x = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \rightarrow \text{basis} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

so $\boxed{P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \ \& \ D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$

20/20

$\begin{bmatrix}0&1\\0&-1\end{bmatrix} \to \begin{bmatrix}0&1\\0&0\end{bmatrix} \Rightarrow x_2=0$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) ~~true,~~ false ~~because multiplying~~ $A = \begin{bmatrix}2&1\\0&1\end{bmatrix}$ $\lambda_1 = 1$ then $\begin{bmatrix}1&1\\0&0\end{bmatrix} \to v_1 = \begin{bmatrix}1\\1\end{bmatrix}$

but

+5 so false

$\lambda_2 = 2$ then $\begin{bmatrix}0&1\\0&-1\end{bmatrix} \to \begin{bmatrix}0&1\\0&0\end{bmatrix}$ $x_2 = 0$, $0=0$ no v_2

b) True, because for the proj to equal the original vector

+3 the vector must already fall on the projected plan with $0 = -0$ as the only vector that could possibly be equal to it's negative proj

c) ~~false~~, because $A = \begin{bmatrix}1&0&1\\0&1&0\\1&0&-1\end{bmatrix}$ is a square, symmetric, & orthogonal

+1 matrix but $A^2 = \begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$ which is not the Identity matrix

D) True, because the resulting matrix will still be

+5 symmetrical which means that it will also be orthogonally diagonizable

MTH 309Y LINEAR ALGEBRA

EXAM 3

December 11, 2018

Name: Yonatan Belay

Person Number: 50161483

- Textbooks and electronic devices (calculators, cellphones etc.) are not permitted.
- You may use one sheet of notes.
- For full credit explain your answers fully, showing all work.
- Each problem is worth 20 points.

1	16
2	20
3	20
4	20
5	17
Total:	93/100

$w_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ // $W_2 = V_2 - \left(\frac{W_1 \cdot V_2}{W_1 \cdot W_1}\right) W_1$

$W_3 = V_3 - \left(\frac{W_1 \cdot V_3}{W_1 \cdot W_1}\right) W_1 - \left(\frac{W_2 \cdot V_3}{W_2 \cdot W_2}\right) W_2$

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3\}$ of the subspace V.

b) Compute the vector $\mathrm{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

a) $w_1 = v_1$ //, $w_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{3}{3}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ //

$W_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{6}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{-3}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \cancel{\begin{bmatrix} 0 \\ -3 \\ 0 \\ 3 \end{bmatrix}}$

$$D = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \\ -1 & 0 & 0 \\ 1 & -1 & 3 \end{bmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \end{array}$$

b) $\mathrm{Proj}_V U = \left(\frac{U \cdot V_1}{V_1 \cdot V_1}\right) V_1 + \left(\frac{U \cdot V_2}{V_2 \cdot V_2}\right) V_2 + \left(\frac{U \cdot V_3}{V_3 \cdot V_3}\right) V_3$

$= \left(\frac{3}{3}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{3}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \cancel{\left(\frac{0}{18}\right)\begin{bmatrix} 0 \\ -3 \\ 0 \\ 3 \end{bmatrix}}$

$\mathrm{Proj}_V U = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ //

16/20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1,0)$, $(-1,2)$, $(2,1)$.

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix} = \overset{b}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}$$

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \rightarrow -\tfrac{2}{6}R_1 + R_2 \rightarrow \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 7/3 & 3 \end{array}\right] \tfrac{3}{7}R_2 \rightarrow \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 1 & 9/7 \end{array}\right] -2R_2 + R_1 \rightarrow$$

$$\left[\begin{array}{cc|c} 6 & 0 & -18/7 \\ 0 & 1 & 9/7 \end{array}\right] \tfrac{1}{6}R_1 \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 9/7 \end{array}\right]$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3/7 \\ 9/7 \end{bmatrix}$$

$f(x) = -3/7\, x + 9/7$

20/20

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$2-4+4\lambda$
$4-8+8\lambda$

$$\det\left(\begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}\right) = 0$$

$$-\lambda((1-\lambda)(2-\lambda)) - 1(2-\lambda) + 2[2-4(1-\lambda)] = 0$$

$$-\lambda^3 + 3\lambda^2 - 2\lambda - 2 + \lambda + 4 - 8 + 8\lambda = 0$$

$$-\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0$$

a) $\lambda = 0$

$-(0)^3 + 3(0)^2 + 7(0) - 6 \neq 0$

$-6 \neq 0 \rightarrow$ Not an eigenvalue of A

b) $-(-1)^3 + 3(-1)^2 + 7(-1) - 6 \neq 0$

$1 + 3 - 7 - 6 \neq 0 \rightarrow$ not an eigenvalue of A

c) $-(-2)^3 + 3(-2)^2 + 7(-2) - 6 = 0$

20/20

$8 + 12 - 14 - 6 = 0$

$0 = 0 \rightarrow \lambda = -2$ is an eigenvalue of A

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

for $\lambda_1 = 3$ $\begin{bmatrix} 1-3 & 8 & 4 \\ -2 & 11-3 & 4 \\ 2 & -8 & -1-3 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ -2 & 8 & 4 & 0 \\ 2 & -8 & -4 & 0 \end{array}\right] \xrightarrow{\substack{-R_1+R_2\to R_2 \\ R_1+R_3}} \left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] -\frac{1}{2}R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 4x_2 + 2x_3$
$x_2 = \text{free}$
$x_3 = \text{free}$

$\Rightarrow \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} x_3 \Rightarrow \text{Nul}(A - \lambda I_n) = \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

for $\lambda_2 = 5$ $\begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \xrightarrow{\substack{-\frac{1}{2}R_1+R_2\to R_2 \\ \frac{1}{2}R_1+R_3\to R_3}} \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \xrightarrow{2R_2+R_3\to R_3} \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] -4R_2+R_1\to R$

$\left[\begin{array}{ccc|c} -4 & 0 & -4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\substack{-\frac{1}{4}R_1 \\ \frac{1}{2}R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $x_1 = -x_3$, $x_2 = -x_3$, $x_3 = \text{free}$ $= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3 \Rightarrow \text{Nul}(A - \lambda I_n) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

20/20

$A = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}[v] = \lambda v$

$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} c & d \\ d & c \end{bmatrix}$

$\begin{bmatrix} a+c & b+d \\ b+d & a+c \end{bmatrix}$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then 2v is an eigenvector of A corresponding to the eigenvalue 2λ.

$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

b) If V is a subspace of $\mathbb{R}^2$ and w is a vector such that $\text{proj}_V \, \text{w} = -\text{w}$ then w must be the zero vector.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

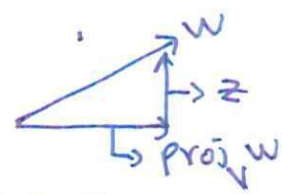
a) False The eigenvalue is the root of the characteristic polynomial which means its the solution that makes the equation $P(\lambda) = 0$ True an $n \times n$ matric can have no more than n roots so 2λ would not be the eigen value.

+5

b) True projection of a vector w on subspace V is unique ~~to each vector being projected~~ such that $z = w - \text{proj}_V w \Rightarrow w = z + \text{proj}_V w$ where z is an orthogonal vector to V

+5

-> This means the only way $\text{proj}_V w = -w$ is if vector w is a zero vector.



-> If w is an element of vectorspace of V then $\text{proj}_V w = w$ which is different from $\text{proj}_V w = -w$

c) False

+2

If A is symmetric then it has n orthogonal eigenvector A can then be expressed as $A = QDQ^T$

$A^2 = QD^2Q^T$ not identity matrix

MTH 309Y LINEAR ALGEBRA

EXAM 3

December 11, 2018

Name: Thomas Bassett

Person Number: 50190029

- Textbooks and electronic devices (calculators, cellphones etc.) are not permitted.
- You may use one sheet of notes.
- For full credit explain your answers fully, showing all work.
- Each problem is worth 20 points.

20	1	6
15	2	18
20	3	20
20	4	20
	5	8
	Total:	72/100

14?

2

1. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V.

6/20

a) on

Basis orthogonal Basis $D = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$

$C_1 = \frac{UV_1}{V_1V_1} = \frac{3+0-3+3}{1^2+0^2+(-1)^2+1^2} = \frac{3}{3} = 1$

$C_2 = \frac{UV_2}{V_2V_2} = \frac{6+3-3+0}{2^2+1^2+(-1)^2+0^2} = \frac{6}{6} = 1$

$C_3 = \frac{UV_3}{V_3V_3} = \frac{6-6-3+9}{2^2+(-2)^2+(-1)^2+3^2} = \frac{6}{18} = \frac{1}{3}$

$4+4+1+9$

Orthogonal Basis D of subspace V $= \begin{bmatrix} 1 \\ 1 \\ 1/3 \end{bmatrix}$

$\text{Proj}_V U = \frac{UV_1}{V_1V_1}V_1 + \frac{UV_2}{V_2V_2}V_2 + \frac{UV_3}{V_3V_3}V_3$

$UV_1 = 3+0-3+3 = 3$

$V_1V_1 = 1^2+0^2+(-1)^2+1^2 = 3$

$UV_2 = 6+3-3+0 = 6$

$V_2V_2 = 2^2+1^2+(-1)^2+0^2 = 6$

$UV_3 = 6-6-3+9 = 6$

$V_3V_3 = 2^2+(-2)^2+(-1)^2+3^2 = 18$

$\text{Proj}_V U = \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{6}\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{6}{18}\begin{bmatrix} 2 \\ -2 \\ -2 \\ 3 \end{bmatrix}$

$\text{Proj}_V U = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \\ 1 \end{bmatrix}$

$\text{Proj}_V U = \begin{bmatrix} 11/3 \\ 1/3 \\ -8/3 \\ 2 \end{bmatrix}$

$\text{Proj}_V U = \begin{bmatrix} 11/3 \\ 1/3 \\ -8/3 \\ 2 \end{bmatrix}$

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1,0)$, $(-1,2)$, $(2,1)$.

$$A\begin{bmatrix}a\\b\end{bmatrix} =$$

$$\underbrace{\begin{bmatrix}1 & 1\\-1 & 1\\2 & 1\end{bmatrix}}_{A}\begin{bmatrix}a\\b\end{bmatrix} = \underbrace{\begin{bmatrix}0\\2\\1\end{bmatrix}}_{B}$$

$$A^TA = \begin{bmatrix}1 & -1 & 2\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\\2 & 1\end{bmatrix} \to \begin{bmatrix}6 & 2\\2 & 3\end{bmatrix}$$

$$A^TB = \begin{bmatrix}1 & -1 & 2\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}\cancel{4}\ 0\\3\end{bmatrix}$$

$$\begin{bmatrix}6 & 2 & \cancel{4}\ 0\\2 & 3 & 3\end{bmatrix}\cdot 1/6$$

$$\downarrow$$

$$\begin{bmatrix}1 & 1/3 & 0\\2 & 3 & 3\end{bmatrix} + (-2)R_1$$

$$\downarrow$$

$$\begin{bmatrix}1 & 1/3 & 0\\0 & 1/3 & 3\end{bmatrix} + (-1)R_2 \to \begin{bmatrix}1 & 0 & -3\\0 & 1/3 & 3\end{bmatrix}\cdot 3 \Rightarrow \begin{array}{c}\begin{matrix}a & b & \end{matrix}\\\begin{bmatrix}1 & 0 & -3\\0 & 1 & 9\end{bmatrix}\end{array}$$

$$\boxed{f(x) = -3x + 9}$$

18/20

$a = -3$

$b = 9$

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$

$\cancel{0+0+0-6 \neq 0}$

$0+0+0-6=0$

X

b) $\lambda = -1$

$1+3-7-6=0$

$-9=0$

X

c) $\lambda = -2$

$8+12-14-6=0$

$20-20=0$

$0=0$

✓

$-\lambda^3+3\lambda^2+7\lambda-6=0$ ✓

$8+12$

20

$$\begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}$$

20/20

$\lambda = -2$ is the only eigenvalue of A

$(-\lambda)((1-\lambda)(2-\lambda)-0) - 1(1(2-\lambda)-0) + 2(2-4(1-\lambda))$

$(-\lambda)(2-3\lambda+\lambda^2) - (2-\lambda) + 2(2-4+4\lambda)$

$\cancel{-2\lambda}+3\lambda^2 - \lambda^3 - 2 + \cancel{\lambda} + 4 - 8 + 8\lambda$

$-\lambda^3+3\lambda^2+7\lambda-6$

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$\lambda = 3$

$A - 3I$:

$$\begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \begin{matrix} \\ +(-1)R_1 \\ +R_1 \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} -2 & 8 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot -\tfrac{1}{2} \rightarrow \begin{bmatrix} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$x_1 = 4x_2 + 2x_3$$

$$\lambda = 3: \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\lambda = 5$ $A - 5I$:

$$\begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \cdot -\tfrac{1}{4}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \begin{matrix} \\ +2R_1 \\ +(-2)R_1 \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & -4 & -4 \end{bmatrix} \begin{matrix} \\ \\ +2R_2 \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \tfrac{1}{2} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} +2R_2 \\ \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

20/20

Note:

$x_1 = -x_3$

$x_2 = -x_3$

$$\lambda = 5: \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

also equal to $[v_1, v_2, v_3]$

$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

With eigenvalues in D corresponding to v_1, v_2, v_3 respectively

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then 2v is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and w is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then w must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a) +5

if $\lambda = 2$ false: The
and x_2 in $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ eigen value would
is a free root, ? stay the same, as
both $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ v and 2v wouldn't
could be eigenvectors be linearly independent
meaning $\lambda \to v = 2v \not\Rightarrow 2\lambda$

b) True; This has to
+1 be true because this would
be the only way proj_V would
equal $-w$. No other combo
? would produce this besides
the trivial solution

c)

+1 true; The Identity matrix,
which is both A and A^2
in this case, is the only
matrix that could
be both symmetric and
orthogonal

d) ~~false~~; This is not
always the case. When
+1 matrixies are added, properties
are not always preserved, such
as diagonizability:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 2 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix}$$

diagonizable diagonizable not diagonizable

MTH 309Y LINEAR ALGEBRA

EXAM 3

December 11, 2018

Name: George Bainbridge

Person Number: 50178478

- Textbooks and electronic devices (calculators, cellphones etc.) are not permitted.
- You may use one sheet of notes.
- For full credit explain your answers fully, showing all work.
- Each problem is worth 20 points.

1	14
2	4
3	14
4	12
5	15
Total:	59/100

1. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V u$, the orthogonal projection of u on V.

A) $X_1 = V_1 \quad X_2 = V_2 - \frac{V_2 \cdot V_1}{V_1 \cdot V_1} \cdot V_1 \quad X_3 = V_3 - \left(\frac{V_3 \cdot V_1}{V_1 \cdot V_1}\right) V_1 - \left(\frac{V_3 \cdot V_2}{V_2 \cdot V_2}\right) V_2$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$V_2 \cdot V_1 = 2+0+1+0 = 3$

$V_1 \cdot V_1 = 1+0+1+1 = 3$

$\frac{V_2 \cdot V_1}{V_1 \cdot V_1} = \frac{3}{3} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$\begin{matrix} 2 \\ 1 \\ -1 \\ 0 \end{matrix} - \begin{matrix} 1 \\ 0 \\ -1 \\ 1 \end{matrix} = \begin{matrix} 1 \\ 1 \\ 0 \\ -1 \end{matrix}$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad x_2 = \begin{matrix} 2 \\ 1 \\ -1 \\ 0 \end{matrix} - 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$V_3 \cdot V_1 = 2+0+1+3 = 6$

$V_1 \cdot V_1 = 3 \quad \frac{V_3 \cdot V_1}{V_1 \cdot V_1} = 2$

$V_3 \cdot V_2 = 4 - 2 + 1 + 0 = 3 \quad \frac{V_3 \cdot V_2}{V_2 \cdot V_2} = \frac{1}{2}$

$V_2 \cdot V_2 = 4+1+1 = 6$

$x_3 = \begin{matrix} 2 \\ -2 \\ -1 \\ 3 \end{matrix} - 2 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

$x_3 = \begin{matrix} 2 \\ 2 \\ -1 \\ 3 \end{matrix} - \begin{matrix} 2 \\ 0 \\ -2 \\ 2 \end{matrix} - \begin{matrix} 1 \\ 1/2 \\ -1/2 \\ 0 \end{matrix} = \begin{bmatrix} -1 \\ 3/2 \\ 3/2 \\ 1 \end{bmatrix}$

$$\boxed{\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3/2 \\ 3/2 \\ 1 \end{bmatrix} \right\}}$$

B)

$\text{Proj}_V U = \left(\frac{U \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{U \cdot w_2}{w_2 \cdot w_2}\right) w_2 + \left(\frac{U \cdot w_3}{w_3 \cdot w_3}\right) w_3$

$U \cdot w_1 = 3+0-3+3 = 3$

$w_1 \cdot w_1 = 1+0+1+1 = 3$

$U \cdot w_2 = 3+3+0-3 = 3$

$w_2 \cdot w_2 = 1+1+0+1 = 3$

$U \cdot w_3 = -3 + 9/2 + 9/2 + 3 = 9$

$w_3 \cdot w_3 = 1 + 2.25 + 2.25 + 1 = 13/2$

$\text{Proj}_U V = w_1 + w_2 + \frac{9}{13/2} w_3 = w_1 + w_2 + \frac{18}{13} w_3$

$= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -18/13 \\ 27/13 \\ 27/13 \\ 18/13 \end{bmatrix} = \begin{matrix} 2 - \frac{18}{13} = \\ 1+27/13 \\ 1-27/13 \\ \frac{18}{13} \end{matrix} \begin{pmatrix} 8/13 \\ 40/13 \\ -14/13 \\ 18/13 \end{pmatrix}$

14/20

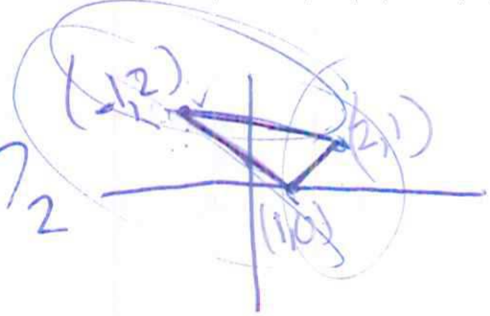
$1.5 \times 1.5 = 75 + 150 = 2.25$ $\quad 1.5 \times 3 = 4.5$

$$\boxed{\text{Proj}_U V = \begin{bmatrix} 8/13 \\ 40/13 \\ -14/13 \\ 18/13 \end{bmatrix}}$$

$18 \times 3 = 54 \quad 26 \quad \frac{27}{13}$

What's b? A_1 A_2 A_3

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1,0)$, $(-1,2)$, $(2,1)$.



~~$A = \begin{matrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{matrix}$ $B = \begin{matrix} 1 \\ \sqrt{5} \\ \sqrt{5} \end{matrix}$ $A^T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ $A^T B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{5} \\ \sqrt{5} \end{bmatrix} =$~~

Least Square

$(A^T A)^{-1} (A^T b) = \begin{bmatrix} x \\ x \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$ ~~$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$~~ $\quad A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} = 1+1+4$

?

$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

4/20

~~$\begin{matrix} 2 & 1 \\ 1 & 0 \end{matrix} = 1 \quad \sqrt{1^2+1}^2 = 8$~~

$\begin{matrix} 1 & -1 \\ 0 & 2 \end{matrix}$ $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ $\sqrt{2^2+2^2} = \sqrt{8}$ $\quad \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{8} \\ \sqrt{10} \\ \sqrt{2} \end{bmatrix} \stackrel{?}{=} \begin{matrix} \sqrt{8} + \sqrt{10} + 2\sqrt{2} \\ 0 + 2\sqrt{10} + \sqrt{2} \end{matrix}$

$\begin{matrix} -1 & 2 \\ 2 & 1 \end{matrix} = \begin{matrix} -3 \\ 1 \end{matrix}$ $\sqrt{9+1} = \sqrt{10}$ $\quad \left.\begin{matrix} 5 & 0 \\ 0 & 5 \end{matrix}\right| \begin{matrix} \sqrt{8} + \sqrt{10} + 2\sqrt{2} \\ 2\sqrt{10} + \sqrt{2} \end{matrix}$

$\begin{matrix} 2 & 1 \\ 1 & 0 \end{matrix} = \begin{matrix} 1 \\ 1 \end{matrix}$ $\sqrt{1+1}$ $\sqrt{2}$ $\quad \boxed{f(x) = \cancel{\frac{\sqrt{8}}{5} + \frac{\sqrt{10}}{5} + \frac{2\sqrt{2}}{5} x + \frac{2\sqrt{10}}{5} + \frac{\sqrt{2}}{5}}}$

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix} = -\lambda\left[(1-\lambda)(2-\lambda)\right] - 1\left[(2-\lambda)\right] + 2\left[2(+4-4\lambda)\right]$$

$1-2\lambda-\lambda+\lambda^2$

$-\lambda\left[\lambda^2-3\lambda+1\right]$

$-\lambda^3+3\lambda^2-\lambda-2+\lambda+4+8-8\lambda$

$(-\lambda^3+3\lambda^2-8\lambda+10)$

$\lambda^2(-\lambda+3)\ 2(4\lambda+5)$ $-4\lambda = \frac{-5}{-4} =$

~~$\lambda=0\ \lambda=3\ \lambda=5/4$~~

$-\lambda^3+3\lambda^2-\lambda+2+\lambda+4+16-16\lambda$

$-\lambda^3+3\lambda^2 \cancel{-16\lambda+14}$

$\lambda^2(-\lambda+3)$ +7 -6

$\lambda=0\ \lambda=3$

The roots are $\lambda=0$, ~~3, 5/4~~ therefore $\lambda=0$ is an eigenvalue while $\lambda=-1$ and $\lambda=-2$ are not

$\lambda=0=$
$$\begin{matrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{matrix}$$

$\lambda=-1=$
$$\begin{matrix} -1 & 1 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 3 \end{matrix}$$

$\lambda=-2=$
$$\begin{matrix} -2 & 1 & 2 \\ 1 & 3 & 0 \\ 4 & 2 & 4 \end{matrix}$$

14/20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$\lambda_1 = 3 = \begin{bmatrix} 1-3 & 8 & 4 \\ -2 & 11-3 & 4 \\ 2 & -8 & -1-3 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix}$ $-2x_1 + 8x_2 + 4x_3 = 0$ $\frac{-2x_1}{-2} = \frac{-8x_2}{-2} - \frac{4}{2}x_3$

$x_1 = 4x_2 + 2x_3$

$v_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $v_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 5 \begin{bmatrix} 1-5 & 8 & 4 \\ -2 & 11-5 & 4 \\ 2 & -8 & -1-5 \end{bmatrix} = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix}$ $-4x_1 + 8x_2 + 4x_3 = 0$ $x_1 = +2x_2 + x_3$

$-2x_1 + 6x_2 + 4x_3 = 0$ $-4x_2 - 2x_3 + 6x_2 + 4x_3$ $2x_2 + 2x_3$ $x_2 = -x_3$

$2x_1 - 8x_2 - 6x_3 = 0$ $v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = 4 - 2 + 0 = 2$ $D^{-1} = \frac{1}{2}\begin{bmatrix} & 0 & -1 \\ 0 & & \end{bmatrix}$

12/20

$D = P^{-1}AP =$

$D = P^{-1}\begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}\begin{bmatrix} 4 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $1+8+0$ $2+4$ $9-4$ $\cancel{\ }$ $-2+11+4$

$-8+11$ $-4+4$

$8-8$ $4-1$

$$\boxed{D = P^{-1}\begin{bmatrix} 9 & 6 & 5 \\ 3 & 0 & 5 \\ 0 & 3 & -5 \end{bmatrix}}$$

$\begin{matrix}0 & 1 & 0 & 1\\ 1 & 1 & 1 & 1\end{matrix}$ $0+1$ $0+1$ $\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}^{+}$ $A\begin{Bmatrix}a & b\\ c & a\end{Bmatrix}$ $A^{+} = \begin{matrix}a & c\\ b & d\end{matrix}$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

$A^T = A$ $\quad B^+B = I$

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

A) Let $A = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$ then $\begin{bmatrix}1-\lambda & 1\\ 0 & 2-\lambda\end{bmatrix}$ gives $\lambda = 1, 2$ for $\lambda = 1$ Eigenvector $= \begin{bmatrix}0\\ 1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$

for $2(1) = 2$ $\begin{bmatrix}1-2 & 1\\ 0 & 2-2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}$ Eigenvector $= \begin{bmatrix}1\\ 1\end{bmatrix}$ and $2 \cdot \begin{bmatrix}0\\ 1\end{bmatrix} \neq \begin{bmatrix}1\\ 1\end{bmatrix}$ $\boxed{\therefore \text{False}}$

+5

B.) ~~True, the formula for Proj$_V$w $= \left(\frac{w \cdot v}{v \cdot v}\right)v$ if w = zero vector $\left(\frac{w \cdot v}{v \cdot v}\right) = 0$ $\therefore \left(\frac{w \cdot v}{v \cdot v}\right)v$ with w = 0 thus being true.~~

c) True because the matrix is symmetric $A^T = A$, and orthogonal $A^+ \cdot A = I$. A^2 will be equivalent to $A \cdot A^+$ as $A = A^T$ there for equating to the identity matrix.

+5

D) True ~~False~~, only symetric 2×2 matrices and a symmetric matrix plus another symetric matrix will be symmetric.

+5

B.) ~~False~~, if $V = -w$ then $\text{Proj}_V W = \left(\frac{-w^2}{-w^2}\right) - w = 1(-w) = -w$ therefore w can be a non-zero vector

MTH 309Y LINEAR ALGEBRA

EXAM 3

December 11, 2018

Name: Aditya Kishan Ankaraboyana

Person Number: 50118925

- Textbooks and electronic devices (calculators, cellphones etc.) are not permitted.
- You may use one sheet of notes.
- For full credit explain your answers fully, showing all work.
- Each problem is worth 20 points.

1	8
2	1
3	4
4	16
5	15
Total:	44/100

480 − 476 + 13

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$\frac{1}{9}$ $-\frac{1}{9}$ $-\frac{1}{18}$ $\frac{3}{18}$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

Rough. $4+4+1+9$

$2+0+1+3$

$\frac{5}{18}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 10/18 \\ -10/18 \\ -5/18 \\ 5/6 \end{bmatrix}$

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

Using Gram-Smidth Process we computeD.

$W_1 = V_1$

$W_2 = V_2 - \frac{W_1 \cdot V_2}{V_2 \cdot V_2} \cdot V_2$

$= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

8/20

$= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{2+0+1+0}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \cancel{0}\ 1 \\ 0 \\ -1 \end{bmatrix}$

$\mathcal{D} = \left\{ \cancel{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} ? \end{bmatrix} \right\}$

$W_3 = V_3 - \frac{W_2 \cdot V_3}{\cancel{V_3 \cdot V_3}} \cancel{V_3} - \frac{W_2 \cdot V_3}{V_3 \cdot V_3} V_2$

$2+0+0-3$

$\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}}{18} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} \Rightarrow$

$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{2+0+0-3}{4+4+1+9} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{18}\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1/9 \\ -1/9 \\ -1/18 \\ 1/6 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 19/9 \\ -19/9 \\ -9/18 \\ 19/6 \end{bmatrix}^2 - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 10/9 \\ -19/9 \\ -9/18 \\ \ldots \end{bmatrix}$

$\frac{10}{9} - \frac{9}{8} + \frac{13}{6}$

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1,0)$, $(-1,2)$, $(2,1)$.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & 2 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & 2 \\ 0 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \sim$$

$x_1 + \frac{5}{2}x_3 = 0$

$4x_2 + x_3 = 0$

~~$x_3 = 0$~~

1/20

$0 = a + b$

$2 = -a + b$

$2 = a + b$

?

$$\boxed{\begin{aligned} a + b &= 0 \\ -a + b &= 2 \\ a + b &= 2 \end{aligned}}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$2\times3 \quad 3\times2$

$3\times2 \quad 2\times3$

Let $b = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

$$\hat{b} = \frac{\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

$$\det(A - \lambda I) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}$$

$$= \cancel{(-\lambda)(1-\lambda)(2-\lambda)}$$

a) Yes as $-\lambda_1 = 0$ $\lambda = 0$

b) ~~No~~ $\lambda_2 = 1$

c) ~~No~~ $\lambda_3 = 2.$

4/20

$$\begin{bmatrix} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 4 & 2 \\ -1 & 3 & 2 \\ 0 & -1 & 5 \end{bmatrix}$$

$\Rightarrow$

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$\lambda = 3$ $\begin{bmatrix} 1-\lambda & 8 & 4 \\ -2 & 11-\lambda & 4 \\ 2 & -8 & -1-\lambda \end{bmatrix}$

16/20

$$= \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -8 & -4 \end{bmatrix}$$

$2x_1 - 8x_2 + 4x_3 = 0$ or $x_1 - 4x_2 + 2x_3 = 0$

$x_1 = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} +2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 5$ $\begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \sim \cancel{\begin{bmatrix} -4 & 8 & 4 \\ 0 & -2 & -2 \\ 2 & -8 & -6 \end{bmatrix}} \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 0 & -6 & -6 \end{bmatrix}$

$\begin{bmatrix} 2 & 4 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & -3 \end{bmatrix}$ $\sim \begin{bmatrix} -2 & 2 & 0 \\ -2 & 6 & 4 \\ 0 & -6 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & - \\ -3 & 0 & \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_2 = x_3.$
$x_1 = -x_3$
$x_1 = x_3.$

$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $P_1 = \begin{bmatrix} 4/\sqrt{17} \\ 1/\sqrt{17} \\ 0 \end{bmatrix}$ $P_2 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$ $P_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$\begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\begin{bmatrix}1&0\\1&3\end{bmatrix}$ $(1-\lambda)(1-\lambda)$ $\lambda=1$ $\lambda=0$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\text{proj}_V\,\mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A+B$ is also orthogonally diagonalizable.

$\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&3\\3&2\end{bmatrix}$

a False, consider $\begin{bmatrix}2&3\\1&5\end{bmatrix}$ $\lambda=2$.

+2

?

b True, because $\text{proj}_V w$ is 0 if w is orthogonal to V. and projection cannot reverse a direction. ?
The only case where $\text{proj}_V w = -w$ would be if $w=0$ because $-w=0$

+3

c) True, $A^T = A^{-1}$ (orthogonal) $A = A^T$ (symmetric)

+5

~~$A\cdot A^{-1} = I$~~ $A^T\cdot A = I$ (orthogonal)

$(A)\cdot A = I$ since symmetric $A^2 = I$

d) True, since both A & B are $n\times n$ matrices, which are orthogonally diagonizable, which means they have to be symmetric if $A=\begin{bmatrix}x&m\\m&y\end{bmatrix}$ & $B=\begin{bmatrix}p&r\\r&q\end{bmatrix}$

+5

then $A+B = \begin{bmatrix}x+p & m+r\\ m+r & y+q\end{bmatrix}$

$A+B$ is symmetric & $n\times n$
so orthogonally diagonizable.

MTH 309Y LINEAR ALGEBRA
EXAM 3
December 11, 2018

Name: Taha Alam

Person Number: 50190425

- Textbooks and electronic devices (calculators, cellphones etc.) are not permitted.
- You may use one sheet of notes.
- For full credit explain your answers fully, showing all work.
- Each problem is worth 20 points.

1	20
2	20
3	20
4	20
5	12
Total:	92/100

1. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{w_1, w_2, w_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

a) Gram-Schmidt Process

$w_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ $w_1 \cdot v_2 = 2+0+1+0 = 3$

$w_1 \cdot w_1 = 1+0+1+1 = 3$

$w_1 = v_1$

$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{3}{3}\right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \leftarrow w_2$

$v_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$ $w_2 \cdot v_3 = 2 + (-2) + 0 + (-3) = -3$

$w_2 \cdot w_2 = 1+0+1+1 = 3$

$w_1 \cdot v_3 = 2+1+3 = 6$

$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 =$

$2-1+1 \quad 2$
$-2-0+1 \quad -1$
$-1--1+0 \quad 0$
$3-1+-1 \quad 1$

$\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \left(\frac{6}{3}\right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{-3}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} =$

$\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \cancel{-} - \left(\frac{3}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \leftarrow w_3$

w_1, w_2, w_3

$\mathcal{D} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ ✓

$w_1 \cdot w_2 = 0$
$w_2 \cdot w_3 = 0$ ✓
$w_1 \cdot w_3 = 0$

b) $\text{proj}_V u$

$w_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ $w_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ $w_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ $u = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$

$w_1 \cdot w_1 = 3$ $\quad u \cdot w_1 = 3$
$w_2 \cdot w_2 = 3$ $\quad u \cdot w_2 = 3$
$w_3 \cdot w_3 = 3$ $\quad u \cdot w_3 = 3$

$\text{proj}_V u = \left(\frac{u \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2}\right) w_2 + \left(\frac{u \cdot w_3}{w_3 \cdot w_3}\right) w_3$

$\frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{3}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \left(\frac{3}{3}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\text{proj}_V u = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

20/20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{b}$$

$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & 1-1+2 \\ 1-1+2 & 1+1+1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$-7x = -3$

$x = \frac{3}{7}$

$\frac{4}{7} \cdot -9 + \frac{-27}{7} + \frac{21}{7} \frac{-6}{7}$

$A^TAx = A^Tb$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \updownarrow \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 6 & 2 & 0 \end{array}\right] -3 \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 0 & -7 & -9 \end{array}\right] + \frac{3}{7} \rightarrow \left[\begin{array}{cc|c} 2 & 0 & \frac{-6}{7} \\ 0 & -7 & -9 \end{array}\right] \cdot -\frac{1}{7} \left[\begin{array}{cc|c} 2 & 0 & \frac{-6}{7} \\ 0 & 1 & 9/7 \end{array}\right] \cdot \frac{1}{2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{-3}{7} \\ 0 & 1 & 9/7 \end{array}\right] \quad z_1 = -3/7 \quad z_2 = 9/7$$

$9/3 - 2/3$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] -\frac{1}{3} \quad \left[\begin{array}{cc|c} 6 & 6/3 & 0 \\ 0 & 7/3 & 3 \end{array}\right] \cdot \frac{1}{6} = \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 7/3 & 3 \end{array}\right] \cdot 3/7 = \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 1 & 9/7 \end{array}\right] -\frac{1}{3}$$

$$\begin{array}{cc} z_1 & z_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 9/7 \end{array}\right] \quad z_1 = -3/7 \quad z_2 = 9/7$$

$$-\frac{3}{7}x + \frac{9}{7}$$

20/20

$3 \quad 3\lambda$

$\det(A - \lambda I_n)$

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

$\det\begin{pmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{pmatrix}$

$\begin{matrix} -\lambda & 1 & 2 & -\lambda & 1 \\ 1 & 1-\lambda & 0 & 1 & 1-\lambda \\ 4 & 2 & 2-\lambda & 4 & 2 \end{matrix}$

For each value of λ given below determine if it is an eigenvalue of A. $(-\lambda \cdot 1-\lambda \cdot 2-\lambda) + 0 + 4 -$

$(2 \cdot 4 \cdot 1-\lambda) - 0 - (2-\lambda)$

a) $\lambda = 0$ NO

b) $\lambda = -1$ NO

c) $\lambda = -2$ Yes

$8 \cdot 1-\lambda$

$8 - 8\lambda$

$(1-\lambda)(2-\lambda)$

$\lambda^2 - \lambda - 2\lambda + 2$

$-\lambda\ (\lambda^2 - 3\lambda + 2)$

$-\lambda^3 + 3\lambda^2 - 2\lambda + 4 - (8 - 8\lambda)$

$-\lambda^3 + 3\lambda^2 + 6\lambda - 4 - (2-\lambda)$

$-\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0$

$-\lambda^3 + 3\lambda^2 + 7\lambda - 6 = 0$

$0 + 0 + 0 - 6 \neq 0$

$-(-1)^3 + 3(-1)^2 + 7(-1) - 6 = 0$

$1 + 3 - 7 - 6 \neq 0$

$-(-2)^3 + 3(-2)^2 + 7(-2) - 6 = 0$

$8 + (12) + (-14) - 6$

$20 \quad -20$

$0 = 0$

20/20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$\lambda_1 = 3$ $[A - 3I \mid 0]$

$$\left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ -2 & 8 & 4 & 0 \\ 2 & -8 & -4 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} -2 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \div \tfrac{1}{2} \quad \begin{array}{c} x_1 \; x_2 \; x_3 \\ \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = +4x_2 + 2x_3$
$x_2 = x_2$
$x_3 = x_3$

$$x^2 \begin{bmatrix} +4 \\ 1 \\ 0 \end{bmatrix} + x^3 \begin{bmatrix} +2 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = 5$ $[A - 5I \mid 0]$

$$\left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \begin{array}{l} -\tfrac{1}{2} \\ \tfrac{1}{2} \end{array} \quad \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \div -\tfrac{1}{4} \quad \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] 2 \quad \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot \tfrac{1}{2} \quad \begin{array}{c} x_1 \; x_2 \; x_3 \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = -x_3$
$x_2 = -x_3$
$x_3 = x_3$

$$x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

20/20

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

+5 ✓ False, while you can multiply eigenvectors to scale them they still correspone to the original eigenvalue, λ.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector. True, it is impossible for the projection to return $-w$ on a non-zero vector

+1 2

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix. True, propertiey of orthonormal vectors $v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ +1 ?.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

+5

d) True, If A and B are orthognally diagonalizable then they have to be symmetric Matrices

A + B will still be symmetric as adding B top of diagonal will cancel out with B's bottom of diagonal.

Since $(A+B)$ is symetric it is orthogonally diagonalizable

a)

$$\det\left(\begin{bmatrix} 3-\lambda & 2 \\ 1 & 8-\lambda \end{bmatrix}\right)$$

$(3-\lambda)(8-\lambda) - 2$

$24 - 11\lambda + \lambda^2$

$-8 \; -3$

$(\lambda - 8)(\lambda - 3) = 0$

$\lambda = 8 \quad \lambda = 3$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \frac{-2}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{0}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

c. $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$, $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

$1/9 + 4/9 + 4/9 = 1$ $\quad 2/9 + 2/9 + -4/9 = 0$ $\quad 2/9 - 4/9 + 2/9 = 0$

$\in$ $\quad 2/9 + 2/9 - 4/9 = 0$ $\quad 4/9 + 1/9 + 4/9 = 1$ $\quad 4/9 - 2/9 - 2/9 = 0$

$2/9 - 4/9 \; 2/9 = 0$ $\quad 4/9 - 2/9 - 2/9 = 0$ $\quad 4/9 + 4/9 + 1/9 = 1$

MTH 309Y LINEAR ALGEBRA
EXAM 3
December 11, 2018

Name: David Adkins

Person Number: 50208978

- Textbooks and electronic devices (calculators, cellphones etc.) are not permitted.
- You may use one sheet of notes.
- For full credit explain your answers fully, showing all work.
- Each problem is worth 20 points.

1	12
2	4
3	14
4	20
5	20
Total:	70/100

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

a) $w_1 = v_1 \quad \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{w_1}$

$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1$

$\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{2+0+1+0}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ (3, 3)

$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{w_2}$

$w_3 = \overset{v_3}{\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}} - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - w_2 \frac{w_2 \cdot v_3}{w_2 \cdot w_2}$

$\frac{0+0+-3}{2+(-2)+0+(-3)}$

$1+1+0+1$

$(-1)\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$\frac{2+0+1+3}{3} \quad \frac{6}{3} = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 2 \end{bmatrix}$

$w_3 = \begin{bmatrix} \cancel{2} \\ -2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} \cancel{2} \\ \cancel{0} \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} \cancel{-1} \\ \cancel{-1} \\ 0 \\ 1 \end{bmatrix}$

$w_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$$\boxed{\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$ ✓

12/20

b) $\text{proj}_V u = \left(\frac{u \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{u \cdot v_2}{v_2 \cdot v_2}\right) v_2 + \left(\frac{u \cdot v_3}{v_3 \cdot v_3}\right) v_3$

$-\frac{6}{3} + -\frac{1}{3}$

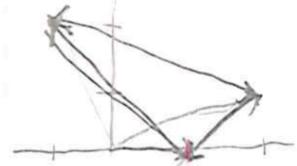
$1 \left(\frac{3+0+-3+3}{1+0+1+1}\right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{6}\, 1\, \frac{6+3+(-3)+0}{4+1+1+0} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{6}{18}\, \frac{1}{3}\, \frac{6+(-6)+(-3)+9}{4+4+1+9} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$ $\begin{matrix} 2/3 \\ -2/3 \\ -1/3 \\ 1 \end{matrix}$

$\frac{9}{3} + \frac{2}{3}$

$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 11/3 \\ 1/3 \\ -7/3 \\ 2 \end{bmatrix}}$

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ \\ \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$



~~y = mx + b.~~

$f(x) = ax + b$

$y = ax + b$

$Ax = b$ $\quad = \quad A^T A x = A^T b$

$0 = a(1) + b$

$2 = a(-1) + b$

$1 = a(2) + b$

$$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

?

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \quad ?$$

$A^T A x = A^T b$

what's A? what's b?

4/20

$$A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$2 + 1 + 2$

$0 - 2 + 1$

$5x = -1$

$x = -\frac{1}{5}$

$$\cancel{\boxed{f(x) = -\frac{1}{5}x + \frac{1}{5}}}$$

$-8+4\lambda-16\lambda+8\lambda^2$

$8\lambda^2-12\lambda-8$

$Ax=\lambda I x$

3. Consider the following matrix A:

$1(2-\lambda-0)[(2-4-4\lambda)2]$

$$A=\begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix}$$

$(2-\lambda)(\frac{-1}{2}+\lambda)-8$

$\begin{bmatrix}1&2\\1&4\end{bmatrix}-\begin{bmatrix}2&0\\0&2\end{bmatrix}$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda=0$ $\boxed{NO}$ b) $\lambda=-1$ $\boxed{NO}$ c) $\lambda=-2$ $\boxed{NO}$

$\left(\begin{array}{ccc|c} 0&1&2&0\\1&1&0&0\\4&2&2&0\end{array}\right)$

$\left[\begin{array}{ccc|c} 1&1&2&0\\1&2&0&0\\4&2&3&0\end{array}\right]$

$\left[\begin{array}{ccc|c} -2&1&2&0\\1&-1&0&0\\4&2&0&0\end{array}\right]$

$\lambda^2-5\lambda+4+2$

$\lambda^2-5\lambda+6$

$(\lambda-3)(\lambda-2)$

$\lambda=2,3$

$\left[\begin{array}{ccc|c} 1&1&0&0\\0&1&2&0\\0&-2&2&0\end{array}\right]$

$\left[\begin{array}{ccc|c} 1&1&2&0\\0&1&-2&0\\0&-2&-5&0\end{array}\right]$

$\begin{matrix}0&1&-2&0\\1&-1&0&0\\0&1&0&0\end{matrix}$ $=\lambda A$

$Ax=2x$

$\left[\begin{array}{ccc|c} 1&0&2&0\\0&1&2&0\\0&0&4&0\end{array}\right]$

$\begin{matrix}1&1&2&0\\0&1&-2&0\\0&0&1&0\end{matrix}$

$\begin{matrix}1&0&-2&0\\0&1&-2&0\\0&1&0&0\end{matrix}$

$Ax=\lambda x$

$\begin{matrix}-2&2&0\\1&1&0\end{matrix}$

$x_1=0$
$x_2=0$
$x_3=0$

yes free

$x_1+x_2+2x_3=0$ $x_1=x_2+2x_3$

$x_2-2x_3=0$ $x_2=2x_3$

$x_3=0$ $\begin{bmatrix}0\\0\\0\end{bmatrix}$

NO because $A\begin{bmatrix}0\\0\\0\end{bmatrix}=2\begin{bmatrix}0\\0\\0\end{bmatrix}$ eigenvector can't be $\emptyset$

NO because $\begin{bmatrix}0\\0\\0\end{bmatrix}$ can't be $\emptyset$,

?

$\begin{matrix}1&0&-2&0\\0&1&0&0\\0&0&-12&0\end{matrix}$

$\boxed{No}$

14/20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$$A-\lambda_1 = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \cdot 1$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -8 & -4 & 0 \end{array} \cdot \tfrac{1}{2}$$

$$A-\lambda_2 = \left[\begin{array}{ccc|c} -4 & 8 & 4 & 0 \\ -2 & 6 & 4 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \cdot 2$$

$$\begin{array}{ccc} 1 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & -8 & -8 & 0 \\ 0 & -2 & -2 & 0 \\ 2 & -8 & -6 & 0 \end{array}\right] \begin{array}{l} \cdot -4 \\ \cdot -\tfrac{1}{2} \\ \cdot \tfrac{1}{2} \end{array}$$

$$x_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -4 & -3 & 0 \end{array}\right] \cdot 4$$

$$\boxed{P = \begin{bmatrix} -1 & 4 & 2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$

$x_2 + x_3 = 0$

$x_1 = -x_3$

$x_2 = -x_3$

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = x_1$$

20/20

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and v is an eigenvector of A corresponding to an eigenvalue λ then 2v is an eigenvector of A corresponding to the eigenvalue 2λ.

+5 False. The eigenvector 2v still corresponds to the eigenvalue λ. Consider (A) below.

b) If V is a subspace of $\mathbb{R}^2$ and w is a vector such that $\text{proj}_V\, \mathbf{w} = -\mathbf{w}$ then w must be the zero vector.

True, see (B) below.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

+5 True by definition to determine if a sym. Matrix is othogonal, you compute A^TA and if you get identity matrix its orthogonal. Because $A^T = A$ then $A^2 = A^TA$ therefore A will be identity matrix

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

+5 By definition a matrix is orthogonally diagonizable is λ is symmetric and every symmetric matrix is diagonizable, so the answer is true because any 2×2 symmetric matrix

for example: $\begin{bmatrix} A & C \\ C & B \end{bmatrix} + \begin{bmatrix} D & F \\ F & E \end{bmatrix} = \begin{bmatrix} A+D & C+F \\ C+F & B+E \end{bmatrix}$

added to another symmetric matrix is still symmetric. TRUE.

(A) $\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ $\lambda^2 - 5\lambda + 6$

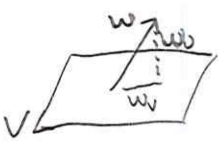
$(\lambda-2)(\lambda-3)$

$\lambda = 2, 3$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow x_1 = -2x_2$

$\rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix} \leftarrow$ eigenvector

but $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ is still an eigenvector that corresponds to λ, not 2λ. FALSE

(B) 

In order for $\text{proj}_V w$ to be $-w$ we would have to have $w_V = -w$. That isn't

+5 possible unless w is the zero vector because $(w - \text{proj}_V w)$ must be in $W^\perp$ and not in W. Unless its the zero vector. TRUE.

MTH 309Y LINEAR ALGEBRA

EXAM 3

December 11, 2018

Name: Jerin Abraham

Person Number: 50184971

- Textbooks and electronic devices (calculators, cellphones etc.) are not permitted.
- You may use one sheet of notes.
- For full credit explain your answers fully, showing all work.
- Each problem is worth 20 points.

1	19
2	20
3	19
4	15
5	9
Total:	82/100

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\text{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ on V.

a) $w_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{(1\cdot 2 + 0\cdot 1 + -1\cdot -1 + 1\cdot 0)}{(1\cdot 1 + 0\cdot 0 + -1\cdot -1 + 1\cdot 1)} w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - \frac{(2+0+1+3)}{(1+0+1+1)}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{(2-2+0-3)}{(1+1+0+1)}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\mathcal{D} = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}}$$

b) $$\text{proj}_V u = \left(\frac{u \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2}\right) w_2 + \left(\frac{u \cdot w_3}{w_3 \cdot w_3}\right) w_3 = \frac{(3+0-3+3)}{(1+0+1+1)} w_1 + \frac{(3+3+0\cancel{+3})^{-3}}{(1+1+0+1)} w_2 + \frac{(3-3+3+0)}{(1+1+1+0)} w_3$$

$$= w_1 + \cancel{3}w_2 + w_3 = \begin{bmatrix} 1+3+1 \\ 0+3-1 \\ -1+0+1 \\ 1+3+0 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 2 \\ 0 \\ 4 \end{bmatrix}}$$

19/20

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\cancel{\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \\ 0 & 0 & 1 \end{array}\right]}$$

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & 1-1+2 \\ 1-1+2 & 1+1+1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-2+2 \\ 0+2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$(A^TA)\hat{x} = (A^Tb)$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 2 & 3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 7/3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 9/7 \end{array}\right]$$

$$\boxed{f(x) = -\frac{3}{7}x + \frac{9}{7}}$$ ✓

20/20

3. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

For each value of λ given below determine if it is an eigenvalue of A.

a) $\lambda = 0$ b) $\lambda = -1$ c) $\lambda = -2$

a) $A - 0I = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 4 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow$ not an eigenvalue

b) $A + I = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 3 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & -5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow$ not an eigenvalue

c) $A + 2I = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 0 \\ 4 & 2 & 5 \end{bmatrix} \to \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 3 & 0 & 0 \\ 4 & 2 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & -10 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 6/5 & 0 \\ 0 & 1 & -2/5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{bmatrix} -\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 4 & 2 & 2-\lambda \end{bmatrix} \to (-\lambda)(1-\lambda)(2-\lambda) + 0 + 2 - 8(1-\lambda) - 0 - (2-\lambda)$

$= (-\lambda + \lambda^2)(2-\lambda) + 2 - 8 + 8\lambda - 2 + \lambda$

$= (-2\lambda + 3\lambda^2 - \lambda^3) - 8 + 9\lambda$

$= -\lambda^3 + 3\lambda^2 + 7\lambda - \cancel{8}\ 6 = 0$

a) $-(0)^3 + 3(0)^2 + 7(0) - 8 = -8 \neq 0 \leftarrow \boxed{\text{not an eigenvalue}}$

b) $-(-1)^3 + 3(-1)^2 + 7(-1) - 8 = -11 \neq 0 \leftarrow \boxed{\text{not an eigenvalue}}$

c) $-(-2)^3 + 3(-2)^2 + 7(-2) - 8 = -2 \neq 0 \leftarrow \boxed{\text{~~not~~ an eigenvalue}}$

19/20

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 4 \\ -2 & 11 & 4 \\ 2 & -8 & -1 \end{bmatrix}$$

Knowing that eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$ diagonalize this matrix; that is, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Note: you do not need to compute P^{-1}.

$$A - 3I = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 8 & 4 \\ 2 & -8 & -4 \end{bmatrix} \to \left[\begin{array}{ccc|c} 2 & -8 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} \to \overset{x_2=1,\, x_3=-1}{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}, \overset{x_2=-1,\, x_3=1}{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}}$$

$$A - 5I = \begin{bmatrix} -4 & 8 & 4 \\ -2 & 6 & 4 \\ 2 & -8 & -6 \end{bmatrix} \to \left[\begin{array}{ccc|c} 2 & -8 & -6 & 0 \\ -2 & 6 & 4 & 0 \\ -4 & 8 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -4 & -3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} \to \overset{x_3=1}{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$$

$$\boxed{P = \begin{bmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$$

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5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ then $2\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 2λ.

b) If V is a subspace of $\mathbb{R}^2$ and $\mathbf{w}$ is a vector such that $\text{proj}_V \mathbf{w} = -\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

c) If A is a square matrix which is both symmetric and orthogonal then A^2 is the identity matrix.

d) If A and B are 2×2 matrices which are both orthogonally diagonalizable, then the matrix $A + B$ is also orthogonally diagonalizable.

a. False. let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 4 = -\lambda + \lambda^2 - 2 = 0 \leftarrow (\lambda-2)(\lambda+1)$

+3

$\lambda = 2 \quad \lambda = -1$

For $\lambda = -1 \rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} x = 0 \rightarrow x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda = -2 \rightarrow \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} x = 0 \rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq 2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

why?

b. True. For any u such that $\text{proj}_V u = v$, $u \cdot v \geq 0$. $\therefore w \cdot -w \leq 0$ so the only way for this to be true is if $w = 0$

+4

c. True. If A is orthogonal and symmetric then it is of the form PDP^T. $A^2 = PD^2P^T$. Since P is an orthogonal basis for A, PD^2P^T is going to be the identity matrix

+1

?

d. ~~False.~~ let $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Both A and B are orthogonaly diagonalizable, let $W = A + B \rightarrow \begin{bmatrix} 1+1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -2/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1+1/\sqrt{2} \end{bmatrix}$. Though W is symmetrical, the columns of W arent orthogonal to each other so W isn't orthogonally diagonalizable.

+1