MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

SM Hoque

UB Person Number:

5	0	2	2	8	1	6	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
18	10	4	15	8	4	3	2	--	64	C

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a)
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right] R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$
$x_1 \quad x_2 \quad x_3$

$x_1 - x_2 + x_3 = -2$
$x_2 + 2x_3 = 2$
$0 = b+6$
$b = -6$ ✓

b)
$c_1v_1 + c_2v_2 + c_3v_3 = 0$

$$c_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1 - c_2 + c_3 = 0 \rightarrow c_1 = 0$
$c_2 + 2c_3 = 0 \rightarrow c_2 = 0$
$c_3 = 0$

~~Yes~~ because all c_n values are equal to 0.

These matrices should be the same!

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] R_2 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] R_3 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] R_2 + R_1 : R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] R_1 - R_3 : R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] R_2 : R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$ ✓

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^{-1} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & -1 \\ 2 & -2 & 1 \end{bmatrix} \qquad (A^T)^{-1} = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$C = (A^T)^{-1} \cdot B$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 2 & 3 \\ -3 & -1 & -2 & 4 & 5 & 4 \\ 1 & 1 & 1 & 3 & 2 & 1 \end{array}\right] \quad R_2 - R_1 : R_1$$

?

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ -3 & -1 & -2 & 4 & 5 & 4 \\ 1 & 1 & 1 & 3 & 2 & 1 \end{array}\right] \quad 3R_1 + R_2 : R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & -1 & -2 & -11 & -16 & -17 \\ 1 & 1 & 1 & 3 & 2 & 1 \end{array}\right] \quad \begin{matrix} R_3 : R_3 - R_1 \\ R_2 : -R_2 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & +1 & +2 & +11 & +16 & +17 \\ 0 & 1 & 1 & 8 & 9 & 8 \end{array}\right] \quad R_3 : R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & 1 & 2 & 11 & 16 & 17 \\ 0 & 0 & +1 & +3 & +7 & +9 \end{array}\right] \quad \begin{matrix} R_3 = -R_3 \\ R_2 = R_2 - 2R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -7 \\ 0 & 1 & 0 & 5 & 2 & -1 \\ 0 & 0 & 1 & 3 & 7 & 9 \end{array}\right]$$

$$C = \begin{bmatrix} -5 & -7 & -7 \\ 5 & 2 & -1 \\ 3 & 7 & 9 \end{bmatrix}$$

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☒ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10

4. **(20 points)** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right] \in \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{array}{c} \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \\ x_1 \quad x_2 \end{array}$$ ✓

$$u = \begin{bmatrix} 7 \\ -5 \\ 16 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & \cancel{-5}\ 3 \\ 0 & 0 & \cancel{16}\ 0 \end{array}\right]$$

free variable) ?

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$$T(u) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$$

$$T(u) = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$$ ✓ $R_2 : R_2 - R_1$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right] \quad R_3 : R_3 + R_2$$

$$\left[\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & 0 & 7 \end{array}\right]\right. \quad R_3 : R_3 - R_1$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 6 \end{array}\right] \quad R_2 : R_2 - R_3$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -5 \\ 0 & 2 & 6 \end{array}\right]$$

$R_1 : R_1 + R_3$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -5 \\ 0 & 2 & 6 \end{array}\right]$$

$R_3 : R_3 - 2R_2$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathrm{v}) = A\mathrm{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors v_1 and v_2 such that $T_A(\mathrm{v}_1) = T_A(\mathrm{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ ~~Not~~ one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ ~~one-to-one~~

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ $R_3 : R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$ $R_2 : R_2 - R_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ $R_1 : -R_2 + R_1$

$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

pivot in each column

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $R_3 : R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ $R_1 : R_1 - R_3$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ $R_2 : R_2 - R_3$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

no pivot

$\underline{v_1, v_2}$?

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

~~False~~

$w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

w + u is a linear combination so

just w itself may not be in span

w + u is not a lin. combination of u and v in this case.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True ✓

?

L. I. depends on the values $c_1, c_2, \ldots c_n$

Set $\{u, v, w\} \in \mathbb{R}^3$

$c_1 = c_2 = c_3 = 0$

Set $\{u, v\} \in \mathbb{R}^2$: Set gets dropped by one dimension

$c_1 = c_2 = 0$

Solution remains the same

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False ✓

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

$Au = 0$

$Av = 0$

?

u & v need to be set to 0 and checked to see if a solution exists. i.e. $c_1, c_2, \ldots c_n$ all equal 0.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True ✓

If the Span(v, w) contains u then the span of (T(v), T(w)) contains T(u)

why?

0 1 2 3 4 5 6 7 8 9 10