MTH-309T-F19-EX1-055-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

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Name:

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UB Person Number:

5	0	2	6	2	8	0	3

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
12	10	10	19	20	3	9	2	10	94	A

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{bmatrix} \cdot -2 \rightarrow \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & \cancel{3}\ -1 & -2 & 4+b \end{bmatrix} \cdot 5 \rightarrow \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & \cancel{8}\ 0 & \cancel{14+b}\ b+6 \end{bmatrix}$$

Correct answer for wrong reasons

$b+6=0$

~~$8 = 14+b$~~

a) $\boxed{b=-6}$

b) $$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \cancel{8} \end{bmatrix} \cdot \tfrac{1}{8} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot -2 \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot -1 \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot 1$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) The set is linearly independent b/c the homogenous vector equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$ will only have one solution.

Correct conclusion, but it comes from incorrect computations.

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☒	☐	☐	☐	☐	☐	☐	☐	☐
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot -1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot -2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \begin{matrix} \cdot -2 \\ \cdot 1 \end{matrix}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}}$$ ✓

0 1 2 3 4 5 6 7 8 9 10

3. (10 points) Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$(A^T)^{-1} = (A^{-1})^T \qquad C = (A^{-1})^T B$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad (A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$(A^T)^{-1} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+4+6 \\ 3-4-6 \\ -1+4+3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 6 \end{bmatrix} \qquad (A^T)^{-1} \cdot \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -4+5+4 \\ 6-5-4 \\ -2+5+2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$$

$$(A^T)^{-1} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+4+2 \\ 9-4+2 \\ -3+4+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10

4. (20 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

a) Find the standard matrix of T.

$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

b) Find all vectors u satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2(1) \\ 1 \\ -3(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\boxed{a) \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$ ✓

b) $Tu = Au = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \cdot -1 \; \cdot -1 \rightarrow$ ✓ $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & \cancel{8}\ 9 \\ 0 & -1 & \cancel{-4}\ -3 \end{array}\right] \rightarrow$

$\rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & -1 & -4 \\ 0 & 3 & 8 \end{array}\right] \cdot -1 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 3 & 8 \end{array}\right] \cdot -3 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{array}\right]$

$\boxed{\text{no solutions for } u}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

* one-to-one
pivot pos in ever col

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ $\cdot -3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \cdot 2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$

a) is one-to-one

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ $\cdot -3$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \cdot -2 \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot -1 \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - 2x_3 = 0$
$x_2 + 2x_3 = 0$
$x_3 = x_3$
Nul(A)

b) not one-to-one

$\boxed{T\left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}\right)}$
v_1 v_2

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

$\boxed{\text{True}}$ ✓ Since $u \in \text{span}(u, v)$
for $u+w \in \text{span}(u,v)$ w would have to
be some combination of $cu + dv$ where c & d are
some constants.
Therefore $w \in \text{Span}(u+v)$

Why?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent. *Linearly dependent - homogenous vector equation has more than 1 solution

$\boxed{\text{~~False~~}}$ let $u = \begin{bmatrix}1\\0\\0\end{bmatrix}, v = \begin{bmatrix}0\\1\\0\end{bmatrix}, w = \begin{bmatrix}0\\0\\1\end{bmatrix}$

~~$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is linearly independent~~

~~$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ is linearly dependent~~

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is linearly independent

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is linearly ~~dependent~~ independent

$\boxed{\text{True}}$ ~~For a set of p vectors (2 in this case) of $\mathbb{R}^n$ ($\mathbb{R}^3$ in this case) can only be linearly dependent if $p > n$. $2 \not> 3$ so the set $\{u, v\}$ would be linearly independent.~~

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}, A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\boxed{\text{False}}$ ✓ Counterexample: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $v = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

Au & Av both $= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is linearly dependent

but u and v are not linearly dependent

These vectors are actually dependent since $v = 4 \cdot u$. Try e.g. $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ instead.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

$\boxed{\text{True}}$ If u is in Span(v, w) it is a combination of $cv + dw$ where c and d are some constants

So $T(u) = T(cv + dw) = T(cv) + T(dw) = cT(v) + dT(w)$

Which means that $T(u)$ is a combination of constant values of $T(v)$ & $T(w)$ which is what it means to be in Span$(T(v), T(w))$.

✓

0 1 2 3 4 5 6 7 8 9 10