MTH-309T-F19-EX1-079-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

Charles Bakleh

UB Person Number:

5	0	2	7	0	3	8	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
20	10	10	18	8	6	6	2	10	88	A-

1. (20 points) Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

$b_1v_1 + b_2v_2 + b_3v_3 = w$

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) Augmented matrix

$$-2\cdot R_1 \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right]$$

for $w \in \text{Span}(v_1, v_2, v_3)$

$b+6=0$

$b=-6$ ✓

$$1\cdot R_2 \to \left[\begin{array}{cccc} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & b+4 \end{array}\right]$$

$$1\cdot \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$

b)

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & b+6 \end{array}\right]$$

reduce row echolont form

null space

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free variable

b) $\{v_1, v_2, v_3\}$ is not linearly independent because it's reduce row echolon form has a free variable so the null space has more then one trivial solution (it has infinitly many solutions), meaning that it is linearly dependent, (linearly independent vectors would have only one trivial solution to the null space which would be the 0 vector. ✓

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad -1\cdot$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \quad -2\cdot$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \quad 1\cdot$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \quad -2\cdot$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right] \quad 1\cdot$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

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0	1	2	3	4	5	6	7	8	9	10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad (A^T)^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 1 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 1 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$-1\cdot\left[\begin{array}{ccc|ccc} -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$2\cdot\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$A^T C = B$$

$$C = (A^T)^{-1} B$$

$$C = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2+4+6 & -4+5+4 & -6+4+2 \\ 3-4-6 & 6-5-4 & 9-4-2 \\ -1+4+3 & -2+5+2 & -3+4+1 \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 5 & 0 \\ -7 & -3 & 3 \\ 6 & 5 & 2 \end{bmatrix}$$ ✓

Simpler:

$C = (A^T)^{-1} \cdot B = (A^{-1})^T \cdot B$

Then use A^{-1} from problem 2.

0	1	2	3	4	5	6	7	8	9	10
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4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - 2(0) \\ 1 + 0 \\ 1 - 3 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{matrix} 0-2 \\ 0+1 \\ 0-3 \end{matrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

$3 \times 2 \quad 2 \times 1$

$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ A_5 & A_6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ A_5 & A_6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$A_1 = 1$ $\quad A_2 = -2$

$A_3 = 1$ $\quad A_4 = 1$

$A_5 = 1$ $\quad A_6 = -3$

standard matrix of T is $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$

b) $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right]$ -1

$\begin{bmatrix} 0 & -3 & -9 \\ 1 & 1 & 10 \\ 1 & 3 & -2 \end{bmatrix}$ 1

$-\frac{1}{3}$ $\begin{bmatrix} 0 & -3 & -9 \\ 1 & 1 & 10 \\ 1 & 0 & -11 \end{bmatrix}$

-1 $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 10 \\ 1 & 0 & -11 \end{bmatrix}$

-1 $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 7 \\ 1 & 0 & -11 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 18 \\ 1 & 0 & -11 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 3 \\ 0 & 0 & 13 \end{bmatrix}$

no ~~solutions~~

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. **(20 points)** For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ $\cdot -3$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \cdot 1/2$$

b) ?

$$-1 \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot 1/2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot -2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot -1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

✓

Solution is one to one as every row ~~and~~ column has a pivot ~~pos~~ position or leading one

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

True: because is $w + u$ is in Span (U, V) then $U - V = W$ which is also in Span (U, V)

?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True: is $\{U, V, W\}$ are lineorly independent than $\{U, V\}$ is lineorly independent because $c_1U + c_2V + c_3W = 0$ would have only one trivial solution, meaning that $c_1U + c_2V + c_3W$ or $c_1U + c_3W$ or $c_1U + c_2V$ or $c_2V + c_3W$, have no combination to sum to 0 other than multiplying by 0, if they did than $\{u, v, w\}$ would have a nother solution to $c_1U + c_2V + c_3W = 0$ which would make $\{U, V, W\}$ linearly dependent. Since $c_1U + c_2V$ have no combination to 0 other than 0 they are lineorly independent

0	1	2	3	4	5	6	7	8	9	10
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7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

False: ✓ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$

This matrix will not give a counterexample.

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span$(\mathbf{v}, \mathbf{w})$ then $T(\mathbf{u})$ must be in Span$(T(\mathbf{v}), T(\mathbf{w}))$.

True: if u is in the span (v, w) then a combination $c_1 v + c_2 w = u$, so $T(c_1 v + c_2 w) = T(u)$ as $T(c_1 v + c_2 w)$ can be split into $c_1 T(v) + c_2 T(w) = T(u)$ which means that there is a linear combination $T(v)$ and $T(w)$ that equal $T(u)$

✓

0 1 2 3 4 5 6 7 8 9 10