MTH-309T-F19-EX1-054-P00

MTH 309T LINEAR ALGEBRA

EXAM 1

October 3, 2019

Name:

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UB Person Number:

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Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

1	2	3	4	5	6	7	PIAZZA	HILL	TOTAL	GRADE
12	9	5	18	20	5	4	2	--	75	B-

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \text{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 2 & -3 & 0 & b \end{array}\right] \xrightarrow{\cdot -1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & b \end{array}\right] \xrightarrow{\cdot 3}$

$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6+b \end{array}\right]$ $\therefore$ for ~~all values of b greater than -5~~, $w \in \text{Span}(v_1, v_2, v_3)$

We must have $b+6=0$, i.e. $b=-6$

b) The $\{v_1, v_2, v_3\}$ is ~~linearly independent~~ because no vector in that set is a multiple of another.

This does not suffice to verify that three vectors are lin. independent.

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

Identity Matrix

$$[\, A \mid I \,] \xrightarrow{\cdot -1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\cdot -1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1\ \cancel{0} & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdot -2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \cancel{1} & 1 & 0 \\ 0 & 1 & -1 & \cancel{0} & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\cdot -1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{\cdot 1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

Identity Matrix

$\therefore$ A is invertible and

$$A^{-1} = \begin{bmatrix} \cancel{1} & 3 & -1 \\ \cancel{0} & -1 & 1 \\ \cancel{0} & -2 & 1 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$3 \times 3 \quad 3 \times 3 \quad 3 \times 3$

$$C = B \cdot \frac{1}{A^T} = B \cdot A^{T^{-1}}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A^{T^{-1}} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 2 & 6 & 2 & 2 & 0 \\ 0 & 0 & 6 & -3 & 3 & 3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 5 & -1 & -3 \\ 0 & 0 & 6 & -3 & 3 & 3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 6 & 0 & -6 & -6 & 0 & 6 \\ 0 & 2 & 0 & 5 & -1 & -3 \\ 0 & 0 & 6 & -3 & 3 & 3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 6 & 0 & 0 & -3 & -3 & 3 \\ 0 & 2 & 0 & 5 & -1 & -3 \\ 0 & 0 & 6 & -3 & 3 & 3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 5/2 & -\frac{1}{2} & -3/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

Identity Matrix $\qquad A^{T^{-1}}$

$$C = B \cdot A^{T^{-1}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 5/2 & -\frac{1}{2} & -3/2 \\ -\frac{1}{2} & \frac{1}{2} & 1/2 \end{bmatrix}$$

$3 \times 3 \quad 3 \times 3$

$$\boxed{C = \begin{bmatrix} -\frac{1}{2} & 5 & -3/2 \\ -2 & -5/2 & 2 \\ 3/2 & -3 & 1/2 \end{bmatrix}}$$

Simpler: $(A^T)^{-1} = (A^{-1})^T$

Then use problem 2.

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors $\mathbf{u}$ satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

a) $A = [\,T(e_1), T(e_2)\,]$ $\quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad e_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(e_1) = \begin{bmatrix} 1 - 2(0) \\ 1 + 0 \\ 1 - 3(0) \end{bmatrix} = \begin{bmatrix} 1-0 \\ 1+0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 0 - 2(1) \\ 0 + 1 \\ 0 - 3(1) \end{bmatrix} = \begin{bmatrix} 0-2 \\ 0+1 \\ 0-3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}}$$ ✓

b)
$$\cdot -1 \quad \begin{array}{c} \begin{array}{cc|c} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 1 & -3 & -2 \end{array}\right] \end{array} \checkmark \rightarrow \cdot -1 \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 1 & -3 & -2 \end{array}\right]$$

$$\rightarrow \begin{array}{c} \cdot \tfrac{1}{3} \\ \cdot -1 \end{array} \begin{array}{c} \begin{array}{cc|c} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 9 \\ 0 & -1 & -3 \end{array}\right] \end{array} \rightarrow \cdot -1 \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cc|c} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right] \end{array}$$

$$\boxed{\begin{aligned} x_1 &= 2x_2 + 1 \\ x_2 &= \cancel{x_2}\ 3 \end{aligned}} \Rightarrow \begin{aligned} x_1 &= 7 \\ x_2 &= 3 \end{aligned}$$

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\cdot\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

$\cdot -1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$\cdot -1 \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

$\cdot 1 \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\cdot \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

∴ A has a pivot position in every column so it is one-to-one ✓

b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$\cdot\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\cdot -1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\cdot -1 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$
$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 2x_3$
$x_2 = -2x_3$
$x_3 = x_3$

vector form

$\begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} \to x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

∴ A is not one-to-one since there is not a pivot position in every column, so

$v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ & $V_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$ ✓

s.t. $T_A(v_1) = T_A(v_2)$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. **(10 points)** For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

False ✓ w+v can be in the span of u,v but w ∈ has no correlation with the span of u,v since w is added to u.

?

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

True ✓ since $\{u, v, w\}$ in $\mathbb{R}^3$ are all vectors with leading ones and in row reduced e form to be linearly independent, also to be linearly independent u, v, w cannot be multiples of each other so $\{u, v\}$ must be linearly independent because they are not multiples of each other as proven by $\{u, v, w\}$ linear independence ✓

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $Au = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$ $Av = \begin{bmatrix} 10 \\ 22 \end{bmatrix}$

2×1 2×1

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 2×2 $v = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $Au = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$ $Av = \begin{bmatrix} 13 \\ 24 \end{bmatrix}$

$Au = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $Av = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ← indep.

$Au = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Av = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ← dependent

~~True~~

If Au and Av are linearly dependent the u, v must be linearly dependent since the scalar multiple multiplied to either Au or Av to get the other is the same scalar multiple to get c·u to v or c·v to u

b) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span($\mathbf{v}, \mathbf{w}$) then $T(\mathbf{u})$ must be in Span($T(\mathbf{v}), T(\mathbf{w})$).

True ✓ since T(u) the u is a vector $\mathbb{R}^2$ and v, w is a vector in $\mathbb{R}^2$ ∴ T(u) must be in the span of T(v), T(w).

?

0	1	2	3	4	5	6	7	8	9	10
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