MTH-309T-F19-EX1-037-P00

MTH 309T LINEAR ALGEBRA
EXAM 1
October 3, 2019

Name:

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UB Person Number:

5	0	2	6	4	1	5	8

Instructions:

- Textbooks, calculators and any other electronic devices are not permitted. You may use one sheet of notes.
- For full credit solve each problem fully, showing all relevant work.

1	2	3	4	5	6	7	TOTAL	GRADE

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1	2	3	4	5	6	7	TOTAL	GRADE

1. **(20 points)** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix}$$

a) Find all values of b such that $w \in \mathrm{Span}(v_1, v_2, v_3)$.

b) Is the set $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & b \end{array}\right] R3 \cdot R2 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & 0 & 2b \end{array}\right] R3 - 3 \cdot R2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -6 & 2b-6 \end{array}\right] \cdot -\frac{1}{6} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -\frac{1}{3}b+1 \end{array}\right] R1 + R2$$

$3(-\frac{1}{3}b+1)$

$-b+3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -\frac{1}{3}b+1 \end{array}\right] -3 \cdot R3 \quad \left[\begin{array}{cccc} 1 & 0 & 0 & -b+3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -\frac{1}{3}b+1 \end{array}\right] R2 - 2 \cdot R3$$

$$\begin{array}{cccc} 1 & 0 & 0 & -b+3 \\ 0 & 1 & 0 & \frac{2}{3}b \\ 0 & 0 & 1 & -\frac{1}{3}b \end{array}$$

$x_1 = -b+3$

$x_2 = \frac{2}{3}b$

$x_3 = -\frac{1}{3}b$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -3 & 0 & 0 \end{array}\right] R3 + 3 \cdot R2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

-2

$x_1 = x_2 - x_3$

$x_2 = -3x_3$

$x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

It is not linearly independent bc there are infinite solutions

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. **(10 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] R2 \cdot \tfrac{1}{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] R3 \cdot -2 \cdot R2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] + \tfrac{1}{2}R3$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] R1 + R2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \begin{matrix} R1- \\ 2R3 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \quad A^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

3. **(10 points)** Let A be the same matrix as in Problem 2, and let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find a matrix C such that $A^T C = B$ (where A^T is the transpose of A).

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \cdot C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 2 & 0 & 1 & 0 \end{array}\right] R2 + 2R3 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \end{array}\right] +R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \end{array}\right] R3 - R2 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 \end{array}\right] \cdot -\frac{1}{2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right] - 4 \cdot R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right] - R2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 0 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right]$$

$$\frac{1}{2}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$C = \begin{bmatrix} -1 & 4 & 0 \\ -4 & 16 & 0 \\ -3 & 4 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -4 & 0 \\ -2 & 4 & 0 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

$$3\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + -4\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix}$$

$$-2\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + 0 = \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix}$$

0 1 2 3 4 5 6 7 8 9 10

4. **(20 points)** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

a) Find the standard matrix of T.

b) Find all vectors **u** satisfying $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 10 \\ -2 \end{bmatrix}$.

$e1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad T(e_2) = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Standard matrix $\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 0 & -3 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 10 \\ 0 & -3 & -2 \end{array}\right] \xrightarrow{R2,\ -\frac{1}{3}R3} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 20/3 \\ 0 & -3 & -2 \end{array}\right] R3+3R2$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 20/3 \\ 0 & 0 & 18 \end{array}\right] R1+2R2 \quad \left[\begin{array}{cc|c} 1 & 0 & 10/3 \\ 0 & 1 & 20/3 \\ 0 & 0 & 18 \end{array}\right]$$

$$\boxed{u = \begin{bmatrix} 10/3 \\ 20/3 \\ 18 \end{bmatrix}}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

5. (20 points) For each matrix A given below determine if the matrix transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to one or not. If T_A is not one-to-one, find two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$.

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \cdot \tfrac{1}{2} \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] R_3 \cdot R_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right] R_3 - 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ a) not one to one

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \cdot \tfrac{1}{2} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] R3-4 \cdot R2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & -2 & 0 \end{array}\right] R3 \cdot R2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \cdot -\tfrac{1}{4} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$ b) one to one

✓

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

6. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{w} + \mathbf{u} \in \text{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.

True, V_0 tn is in the span

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ must be linearly independent.

false, $\{u, v\}$ could have a differente solution with nothing trivial

0 1 2 3 4 5 6 7 8 9 10

7. (10 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A is a 2×2 matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ such that $A\mathbf{u}$, $A\mathbf{v}$ are linearly dependent then $\mathbf{u}, \mathbf{v}$ also must be linearly dependent.

~~false~~ false, u, v can be linearly independent before being multiplied by A.

b) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are vectors such that $\mathbf{u}$ is in Span(v, w) then $T(\mathbf{u})$ must be in Span(T(v), T(w)).

True, T(u) is in span (T(v), T(w)) if u is in span(v, w)

0 1 2 3 4 5 6 7 8 9 10