Quantum Symmetries of Graphs and Nonlocal Games

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What is this talk about?

Some intriguing connections that have recently emerged between

- **"Pure" Mathematics** – noncommutative algebra, representation theory, operator algebras, combinatorics, ...
- **Quantum Information Theory (QIT)** – nonlocal games, interactive proof systems, quantum entanglement, complexity theory, ...

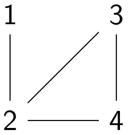
E.g., the Connes Embedding Problem in operator algebras, capacity problems for noisy quantum channels, ...
Today I will illustrate these connection by considering graphs, their (quantum) symmetries, and some related nonlocal games of interest in QIT.

Graphs

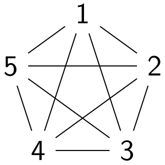
A (finite, simple) graph is a pair $X = (V, E)$ where

- V is a finite set (the vertices of X) and
- $E \subset V \times V$ (the edges of X) satisfying $(x, x) \notin E$ for all $x \in V$ and $(x, y) \in E \iff (y, x) \in E$.

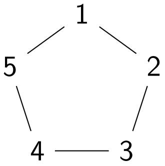
$X =$



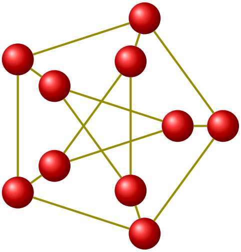
$K_5 =$



$C_5 =$



$P =$



Symmetries of Graphs

Let $X = (V(X), E(X))$ and $Y = (E(Y), V(Y))$ be graphs.

- An graph isomorphism from X to Y is a **bijection** $\varphi : V(X) \to V(Y)$ that preserves edges:

$$(x, y) \in E(X) \iff (\varphi(x), \varphi(y)) \in E(Y).$$

- If $X = Y$, then graph isomorphisms $X \to X$ form a group under composition, called the symmetry group $G(X)$ of X.

Examples: (From previous slide)

$$G(X) = S_2,\ G(K_5) = S_5,\ G(C_5) = D_5,\ G(P) = S_5.$$

$$\mathrm{Z} = \quad \begin{array}{ccc} 1 & — & 3 \\ | & / & | \\ 2 & — & 4 \\ | & & | \\ 5 & & 6 \end{array} \implies \mathrm{G(Z)} = \{\mathrm{e}\} \text{ (i.e., } Z \text{ is rigid)}$$

$G(X)$ as a Matrix Group

- Given a graph $X = (V, E)$ with $|V| = n$ vertices, its symmetry group $G(X)$ is a subgroup of the symmetric group S_n.
- Therefore automorphisms of X can be identified with permutation matrices

  $$\sigma = [\sigma_{xy}]_{x,y \in V} \in M_n(\mathbb{C})$$

  that respect the edge structure of X. That is,

  $$\boxed{\sigma A_X = A_X \sigma,}$$

  where $A_X = [A_X(x, y)]_{x,y \in V} = [\chi_E(x, y)]_{x,y \in V}$ is the adjacency matrix of X.
- So
  $$G(X) = \{\sigma \in S_n \mid A_X \sigma = \sigma A_X\} \subset M_n(\mathbb{C}).$$
- In general, $G(X)$ is difficult to compute!

Quantum Symmetries of Graphs

Let's make life more complicated and generalize the notion of symmetries of graphs, using ideas from noncommutative algebra

Definition

Let $X = (V, E, A_X)$ be a graph with $n = |V|$. Let B be a unital associative $\mathbb{C}$-algebra. A matrix $p = [p_{xy}]_{x,y \in V} \in M_n(B)$ is called a quantum automorphism of X if its entries satisfy the relations

$$(1) \quad p_{xy}p_{xz} = \delta_{\{y=z\}}p_{xy}, \quad p_{xy}p_{zy} = \delta_{\{x=z\}}p_{xy}$$

$$(2) \quad \sum_x p_{xy} = 1, \quad \sum_y p_{xy} = 1$$

$$(3) \quad pA_X = A_X p.$$

- Think of a quantum automorphisms $p \in M_n(B)$ of X as "B-valued permutation matrices". We regard these as quantum symmetries of the graph X.
- **Do they form a group?** No.

The Quantum Symmetry Group of a Graph

Although quantum automorphisms of X don't form a group, they can be assembled into a quantum group, denoted by $G^+(X)$.

Definition (Banica, Bichon)

Let $\mathcal{C}(G^+(X))$ be the universal unital algebra with **generators** $\{q_{xy}\}_{x,y\in V}$ and **relations** making $q = [q_{xy}]$ a quantum automorphism of X. $\in M_n(\mathcal{C}(G^+(X)))$

- $\mathcal{C}(G^+(X))$ captures "all" quantum symmetries of X via its representation theory: Quantum automorphisms $p = [p_{xy}] \in M_n(B)$ of X correspond to **representations**

$$\pi : \mathcal{C}(G^+(X)) \to B; \qquad \pi(q_{xy}) = p_{xy}.$$

- In particular, the **algebra of coordinate functions** $\mathcal{C}(G(X))$ on the symmetry group $G(X)$ is the **abelianization** of $\mathcal{C}(G^+(X))$:

$$q_{xy} \mapsto \text{ the } (x,y) \text{ coordinate function on } G(X) \subset S_n \subset M_n.$$

The Quantum Symmetry Group of a Graph

$$\mathcal{C}(G^+(X)) = \left\langle q_{xy},\ x, y \in V \;\middle|\; q = [q_{xy}] \text{ is a quantum aut. of } X \right\rangle.$$

$$\mathcal{C}(G(X)) = \mathcal{C}(G^+(X))/\langle \text{commutators} \rangle$$

- We say that a graph X has quantum symmetry if $\mathcal{C}(G^+(X))$ is non-abelian. I.e., if $\mathcal{C}(G^+(X)) \neq \mathcal{C}(G(X))$.
- If $\mathcal{C}(G^+(X))$ is non-abelian, it is not an algebra of functions on a group. But it still has many "group-like" properties.
- $\mathcal{C}(G^+(X))$ is actually a Hopf-$*$-algebra with structure maps:

$$\Delta(q_{xy}) = \sum_z q_{xz} \otimes q_{zy},\ S(q_{xy}) = q_{yx},\ \epsilon(q_{xy}) = \delta_{x,y},\ q_{xy}^* = q_{xy}.$$

- Get a **quantum group** $G^+(X) = (\mathcal{C}(G^+(X)), \Delta, S, \epsilon)$ – the quantum symmetry group of X.

Graphs with Quantum Symmetry

Do some graphs actually admit quantum symmetries?

- Yes! Take $X = K_n$. Know $G(K_n) = S_n$.
- $\mathcal{C}(G^+(K_n))$ has generators q_{xy} with relations $q_{xy}q_{xz} = \delta_{y,z}q_{xy}$, $q_{yx}q_{zx} = \delta_{y,z}q_{yx}$, $\sum_x q_{xy} = 1 = \sum_y q_{xy}$
- We write $G^+(K_n) = S_n^+$. (Wang's Q. Permutation gp '98)
- Take $n = 4$, a, b self-adjoint idempotents in a $*$-algebra B. Put

$$p = \begin{pmatrix} a & 1-a & 0 & 0 \\ 1-a & a & 0 & 0 \\ 0 & 0 & b & 1-b \\ 0 & 0 & 1-b & b \end{pmatrix} \quad (4 \times 4 \text{ q. permutation})$$

- Get a quotient map $\mathcal{C}(S_4^+) \to \mathbb{C}\langle a, b\rangle \subseteq B$; $q_{xy} \mapsto p_{xy}$.
- In particular, we have $\mathcal{C}(S_4^+) \to \mathbb{C}[\mathbb{Z}_2 \star \mathbb{Z}_2]$.
- In general $\mathcal{C}(G^+(X))$ can be ∞-dimensional and very noncommutative!

Why study $G^+(X)$?

- Determining $G^+(X)$ for a given graph X is an interesting (and difficult) algebraic problem.
- $G^+(X)$ defines new classes quantum algebraic invariants for graphs (quantum vertex transitivity, quantum rigidity, etc...)
- $G^+(X)$ is known for most small graphs ($|V| \leq 11$) and certain families of graphs [Banica-Bichon, Ren, Schmidt].
- For **operator algebraists**, $\mathcal{C}(G^+(X))$ can be viewed as a quantum version of the a group algebra $\mathbb{C}\Gamma$ for some discrete group Γ.
- In fact, one can complete $\mathcal{C}(G^+(X))$ into a Hilbert spaace $L^2(G^+(X))$, and form the quantum group von Neumann algebra

$$\underbrace{L^\infty(G^+(X)) = \mathcal{C}(G^+(X))''}_{\text{an analogue of the group von Neumann algebra } \mathcal{L}(\Gamma)=\mathbb{C}\Gamma''} \subset B(L^2(G^+(X))).$$

Some operator algebraic results

For complete graphs, we a have a fairly good handle on the operator algebraic structure of $G^+(K_n) = S_n^+$:

Theorem (B-Chirvasitu-Freslon)

For each n, $\mathcal{C}(S_n^+)$ is a residually finite-dimensional (RFD) $*$-algebra. Moreover, $L^\infty(S_n^+)$ is a full II_1-factor with the Connes Embedding Property (CEP).

- Informally: a von Neumann algebra M has the CEP if it can be "well-approximated" by matrices. (Not all von Neumann algebras have the CEP! **[Ji-Natarajan-Vidick-Wright-Yuen]**

Question: What about general $G^+(X)$? How big/complicated can we make $\mathcal{C}(G^+(X)), L^\infty(G^+(X))$?

Theorem (B.-Chirvasitu-Roberson)

Let Γ be a finitely generated, finitely presented group. Then there exists a finite graph X and a Hopf $*$-algebra embedding

$$\mathbb{C}\Gamma \hookrightarrow \mathcal{C}(G^+(X)).$$

- The preceding theorems say that you can make $G^+(X)$ very complicated from an algebraic/operator algebraic perspective.
- Most interestingly, these results arise from studying the quantum groups $G^+(X)$ from the perspectives of quantum information theory and representation theory.

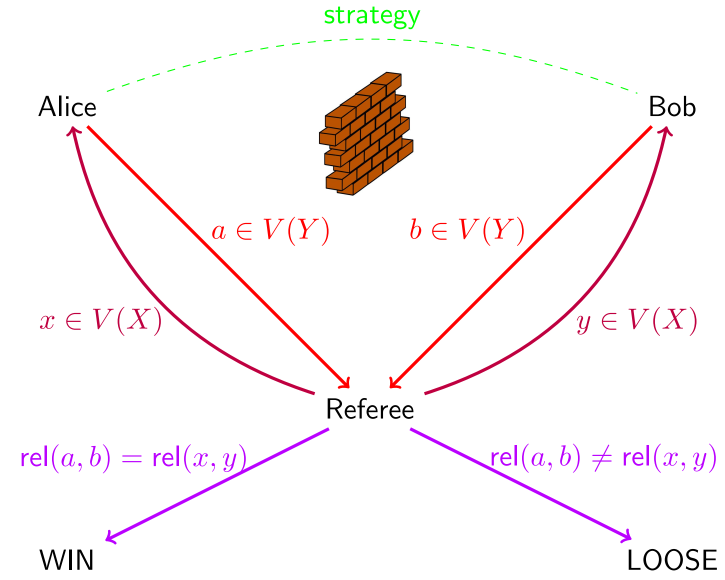
Quantum Symmetries via Quantum Information

- Suppose your friends, Alice and Bob, hand you two graphs X, Y, and claim that they are isomorphic. You'd like to verify their claim.
- One way to accomplish this is to play a **nonlocal game**, called the Graph Isomorphism Game, $\mathsf{Iso}(X, Y)$.
  1. You are the referee, Alice and Bob are the players, and they cooperate to win each round of the game.
  2. During each round, the referee randomly samples a pair of vertices $(x, y) \in V(X) \times V(X)$, and sends x to Alice and y to Bob. Alice and Bob then return vertices $a \in V(Y)$ and $b \in V(Y)$, respectively.
  3. Alice and Bob win the round if $\mathsf{rel}_X(x, y) = \mathsf{rel}_Y(a, b)$, where

$$\mathsf{rel}_X(x,y) = \begin{cases} 1, & (x,y) \in E(X) \\ 0, & x = y \\ -1, & (x,y) \notin E(X) \end{cases}.$$

  4. **Nonlocality Assumption**: During play, Alice and Bob are spatially separated and can't communicate with each other!

Schematic for a round of $\mathsf{Iso}(X, Y)$



Winning Strategies for Iso(X, Y)

- As the referee, you observe Alice and Bob's behavior via their joint input-output correlations:

$$p(a, b|x, y) = \text{“}\mathbb{P}(\text{Alice and Bob return } (a, b) \text{ given } (x, y))\text{”}$$

- You'd be confident that X and Y are isomorphic if Alice and Bob win every round of the game with probability 1:

$$p(a, b|x, y) = 0 \quad \text{whenever} \quad \mathrm{rel}_X(x, y) \neq \mathrm{rel}_Y(a, b).$$

- Such a correlation $[p(a, b|x, y)]_{a,b,x,y}$ is called a winning strategy for Iso(X, Y).
- **Example**: Alice and Bob can fix an isomorphism $\varphi : X \to Y$, and agree to use φ each round to supply answers to the referee:

$$p(a, b|x, y) = \delta_{a,\varphi(x)}\delta_{b,\varphi(y)}.$$

  This is a deterministic winning strategy.

Winning Strategies for Iso(X, Y)

- **Example**: More generally, before each round, Alice and Bob could jointly sample an isomorphism $\varphi : X \to Y$ according to some probability distribution $\mathbb{P}$:

$$p(a, b|x, y) = \sum_{\varphi} \mathbb{P}(\varphi)\delta_{a,\varphi(x)}\delta_{b,\varphi(y)} = \mathbb{P}(\varphi|\varphi(x) = a,\ \varphi(y) = b).$$

- The above strategies are often referred to as "classical strategies."
- **Questions**: What is the most general form of a winning strategy for Iso(X, Y)? Does a winning strategy for Iso(X, Y) *really* imply that the graphs are isomorphic?

Using Quantum Mechanics to Win

- Our physical world is described by the laws of quantum mechanics!
- This means that Alice and Bob can play the game $\mathrm{Iso}(X, Y)$ by performing measurements on quantum systems.
- Mathematically, a quantum system is given by a Hilbert space H. The state of the quantum system H is described by a unit vector $|\psi\rangle \in H$. Measurements of the state $|\psi\rangle \in H$ are described by operators on H: A (projective) measurement with n outcomes is given by a family of self-adjoint projections

$$(P_a)_{a=1}^n \subset B(H) \quad \text{such that} \quad \sum_a P_a = 1.$$

The probability of measuring the state $|\psi\rangle$ with outcome a is given by

$$p(a) = \langle\psi|P_a|\psi\rangle.$$

Quantum Strategies for Iso(X, Y)

In the graph isomorphism game Iso(X, Y), Alice and Bob can devise a quantum strategy as follows.

- Before each round, prepare a joint quantum system $H = H_A \otimes H_B$ in state $|\psi\rangle \in H_A \otimes H_B$. ‘
- Since Alice and Bob are spatially separated and cannot communicate, Alice's equipment can only perform local measurements on her half of the system H_A. Similarly Bob's measurements act on H_B.
- A quantum strategy for Iso(X, Y) is given by $(H_A \otimes H_B, \psi)$ and a choice of local measurement systems

  $$(P_{xa})_{a \in V(Y)} \subset B(H_A), \quad (Q_{yb})_{b \in V(Y)} \subset B(H_B) \quad \forall x, y \in V(X),$$

  with joint measurement outcomes given by

  $$p(a, b|x, y) = \langle\psi|P_{xa} \otimes Q_{yb}|\psi\rangle.$$

Quantum Strategies for Iso(X, Y)

- Physical experiments (aka "Bell tests") tell us that it is possible for Alice and Bob to better correlate their outputs if they perform measurements on a shared entangled quantum state $|\psi\rangle \in H_A \otimes H_B$.
- $|\psi\rangle \in H_A \otimes H_B$ is entangled if it cannot be written as a product vector $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$.

  Example: Bell state/EPR pair: $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle\big)$.

Theorem (Atserias et al '16)

There exist pairs of non-isomorphic graphs X, Y for which there exists a winning quantum strategy $(H, |\psi\rangle, (P_{xa}), (Q_{yb}))$ for Iso(X, Y).

- The pair of graphs X, Y above are called quantum isomorphic, and we write $X \cong_q Y$.

Quantum Pseudo-Telepathy

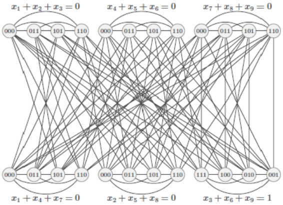
- In the above scenario, Alice and Bob (without communicating with each other!) can use quantum mechanics to trick you into believing that a pair of graphs is isomorphic when it is not.
- This is an instance of a "pseudo-telepathy" game: A nonlocal game with a winning quantum strategy, but no winning classical strategy.
- The fact that Alice and Bob can fool us is actually good news! In quantum computation pseudo-telepathy games provide physical models in which can experimentally verify the advantages of certain quantum algorithms over classical ones.
- These games are also useful for certifying entanglement (self-testing quantum states).

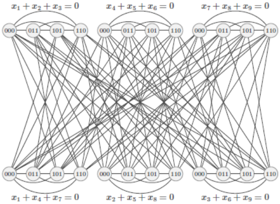
An Example: The Magic Square Graphs

- Consider the Mermin-Perez Magic Square:

x_1	x_2	x_3	0
x_4	x_5	x_6	0
x_7	x_8	x_9	0
0	0	1	

- It's impossible to globally fill this table with $x_i \in \{0, 1\}$ so that row sums and column sums are as shown. But this linear binary constraint system has "matrix solutions".
- Can build a pair of graphs X, Y from this data satisfying $X \cong_q Y$ but $X \ncong Y$:





Quantum Isomorphisms and Representation Theory

- Currently, the above construction using linear binary constraint systems is the only effective way to find pseudo-telepathic pairs of graphs X, Y.
- **Important Problem**: How to find more examples?
- One approach is to try and relate quantum isomorphisms $X \cong_q Y$ to the underlying **algebraic quantum symmetries** of the graphs X and Y. I.e., the quantum groups $G^+(X)$ and $G^+(Y)$ introduced earlier.
- In fact, there turns out to be a deep connection between the existence of quantum isomorphisms $X \cong_q Y$ and the (finite dimensional unitary) representation theory of the quantum groups $G^+(X)$ and $G^+(Y)$.

The tensor category of unitary representations of $G^+(X)$

Just as in the case for classical groups (e.g., $G(X)$), we can study the tensor category of finite-dimensional unitary representations of the quantum group $G^+(X)$, denoted by $\mathrm{Rep}(G^+(X))$.

Quantum Isomorphisms and Representation Theory

Let X, Y be graphs with adjacency matrices A_X, A_Y.

Theorem (B.,Chirvastu,Eifler,Harris,Paulsen,Su,Wasilewski)

The following conditions are equivalent.

1. $X \cong_q Y$
2. *The representation categories Rep$(G^+(X))$ and Rep$(G^+(Y))$ are "the same".*

More precisely, in (2) we mean that there is a **unitary monoidal equivalence** of tensor categories $F : \mathrm{Rep}(G^+(X)) \to \mathrm{Rep}(G^+(Y))$ such that

$$F(\mathbb{C}^{|V(X)|}) = \mathbb{C}^{|V(Y)|},\ F(m_X) = m_Y, F(u_X) = u_Y,\ \&\ F(A_X) = A_Y.$$

Is this categorical characterization of $X \cong_q Y$ useful? Yes!!!

Some Applications

- This shows that understanding quantum isomorphisms $X \cong_q Y$ amounts to understanding the structure of $G^+(X)$.
- The above theorem + some deep work of Slofstra work on linear BCS games $\implies$ our generic group algebra embeddings

$$\mathbb{C}\Gamma \hookrightarrow \mathcal{C}(G^+(X)).$$

- The representation categories $\mathrm{Rep}(G^+(X))$ admit a nice diagrammatic description via **simple planar graphs**. This can be exploited to planar analogue of Lovász' criterion for graph isomorphism.

A Really Beautiful Combinatorial Application

Theorem (Mančinska-Roberson '19)

Given two graphs X, Y, TFAE:

1. $X \cong_q Y$
2. *For every planar graph K,*

$$|\mathit{Hom}(K, X)| = |\mathit{Hom}(K, Y)|$$

where $\mathit{Hom}(K, Z)$ is the set of graph homomorphisms $K \to Z$.

THANKS!