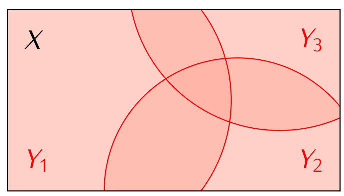
14 | Compact Spaces

14.1 Definition. Let X be a topological space. A *cover* of X is a collection $\mathcal{Y} = \{Y_i\}_{i\in I}$ of subsets of X such that $\bigcup_{i\in I} Y_i = X$.



If the sets Y_i are open in X for all $i \in I$ then $\mathcal{Y}$ is an *open cover* of X. If $\mathcal{Y}$ consists of finitely many sets then $\mathcal{Y}$ is a *finite cover* of X.

14.2 Definition. Let $\mathcal{Y} = \{Y_i\}_{i\in I}$ be a cover of X. A *subcover* of $\mathcal{Y}$ is cover $\mathcal{Y}'$ of X such that every element of $\mathcal{Y}'$ is in $\mathcal{Y}$.

14.4 Definition. A space X is *compact* if every open cover of X contains a finite subcover.

14.8 Proposition. *Let $f\colon X \to Y$ be a continuous function. If X is compact and f is onto then Y is compact.*

Proof. Exercise. $\square$

14.9 Corollary. *Let $f\colon X \to Y$ be a continuous function. If $A \subseteq X$ is compact then $f(A) \subseteq Y$ is compact.*

14.10 Corollary. *Let X, Y be topological spaces. If X is compact and $Y \cong X$ then Y is compact.*

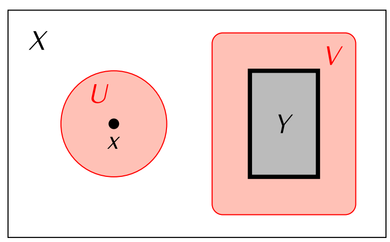
14.12 Proposition. *For any $a < b$ the closed interval $[a, b] \subseteq \mathbb{R}$ is compact.*

14.13 Proposition. *Let X be a compact space. If Y is a closed subspace of X then Y is compact.*

Proof. Exercise. $\square$

14.14 Proposition. *Let X be a Hausdorff space and let $Y \subseteq X$. If Y is compact then it is closed in X.*

14.15 Lemma. *Let X be a Hausdorff space, let $Y \subseteq X$ be a compact subspace, and let $x \in X \smallsetminus Y$. There exists open sets $U, V \subseteq X$ such that $x \in U$, $Y \subseteq V$ and $U \cap V = \varnothing$.*



14.16 Corollary. *Let X be a compact Hausdorff space. A subspace $Y \subseteq X$ is compact if and only if Y is closed in X.*

Proof. Let $A \subseteq X$ be a closed set. By Proposition 14.13 A is a compact space and thus by Corollary 14.9 $f(A)$ is a compact subspace of Y. Since Y is a Hausdorff space, using Proposition 14.14 we obtain that $f(A)$ is closed in Y. □

14.18 Proposition. *Let $f\colon X \to Y$ be a continuous bijection. If X is a compact space and Y is a Hausdorff space then f is a homeomorphism.*

14.19 Theorem. *If X is a compact Hausdorff space then X is normal.*