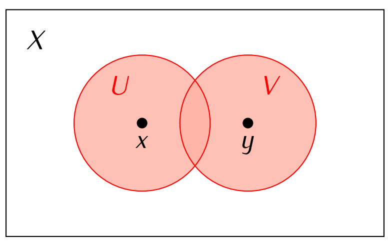
9 | Separation Axioms

9.1 Definition. A topological space X *satisfies the axiom* T_1 if for every points $x, y \in X$ such that $x \neq y$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

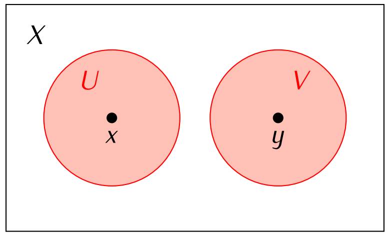


9.3 Proposition. *Let X be a topological space. The following conditions are equivalent:*

1) *X satisfies T_1.*
2) *For every point $x \in X$ the set $\{x\} \subseteq X$ is closed.*

Proof. Exercise. □

9.4 Definition. A topological space X *satisfies the axiom* T_2 if for any points $x, y \in X$ such that $x \neq y$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.



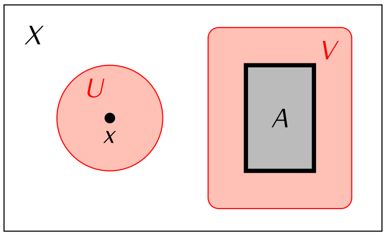
A space that satisfies the axiom T_2 is called a *Hausdorff space*.

9.6 Note. If X satisfies T_2 then it satisfies T_1.

9.8 Proposition. *Let X be a Hausdorff space and let $\{x_n\}$ be a sequence in X. If $x_n \to y$ and $x_n \to z$ for some $y, z \in X$ then $y = z$.*

Proof. Exercise. □

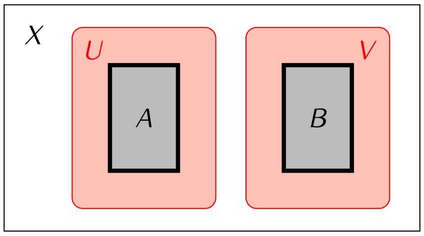
9.9 Definition. A topological space X *satisfies the axiom* T_3 if X satisfies T_1 and if for each point $x \in X$ and each closed set $A \subseteq X$ such that $x \notin A$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $A \subseteq V$, and $U \cap V = \varnothing$.



A space that satisfies the axiom T_3 is called a *regular space*.

9.10 Note. Since in spaces satisfying T_1 sets consisting of a single point are closed (9.3) it follows that if a space satisfies T_3 then it satisfies T_2.

9.12 Definition. A topological space X *satisfies the axiom* T_4 if X satisfies T_1 and if for any closed sets $A, B \subseteq X$ such that $A \cap B = \varnothing$ there exist open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$.



A space that satisfies the axiom T_4 is called a *normal space*.

9.13 Note. If X satisfies T_4 then it satisfies T_3.

9.14 Theorem. *Every metric space is normal.*

9.15 Proposition. *Let X be a topological space satisfying T_1. If for any pair of closed sets $A, B \subseteq X$ satisfying $A \cap B = \varnothing$ there exists a continuous function $f\colon X \to [0, 1]$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$ then X is a normal space.*

Proof. Exercise. $\square$

9.16 Definition. Let (X, ϱ) be a metric space. The *distance between a point $x \in X$ and a set $A \subseteq X$* is the number

$$\varrho(x, A) := \inf\{\varrho(x, a) \mid a \in A\}$$

9.17 Lemma. *If (X, ϱ) is a metric space and $A \subseteq X$ is a closed set then $\varrho(x, A) = 0$ if and only if $x \in A$.*

Proof. Exercise. $\square$

9.18 Lemma. *Let (X, ϱ) be a metric space and $A \subseteq X$. The function $\varphi\colon X \to \mathbb{R}$ given by*

$$\varphi(x) = \varrho(x, A)$$

is continuous.

9.19 Corollary. *If (X, ϱ) is a metric space and $A, B \subseteq X$ are closed sets such that $A \cap B = \varnothing$ then there exists a continuous function $f \colon X \to [0,1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.*

9.20 Note. The results described above can be summarized by the following picture: