11 | Tietze Extension Theorem

The Urysohn Lemma, which we proved in the last chapter, shows that every normal space X is equipped with an ample supply of continuous functions $X \to [0,1]$: any two closed, disjoint sets in X give one such function. However, an inconvenient constraint is that these functions are of very special type: they map one closed set to 0, and the other one to 1.

It is easy to modify the Urysohn Lemma to expand this collection of functions a bit:

11.1 Generalized Urysohn Lemma. *Let X be a normal space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$. For any $a, b \in \mathbb{R}$, $a < b$ there exists a continuous function $f\colon X \to [a,b]$ such that $A \subseteq f^{-1}(\{a\})$ and $B \subseteq f^{-1}(\{b\})$.*

Proof. By the Urysohn Lemma 10.1 we can find a function $g\colon X \to [0,1]$ such that $g(A) = \{0\}$ and $g(B) = \{1\}$. Take $f = h \circ f$, where $h\colon [0,1] \to [a,b]$ is any continuous function such that $h(0) = a$ and $h(1) = b$. $\square$

The collection of functions described by Lemma 11.1 is still very narrow: these functions are constant when restricted to either set A or B. The main result of this chapter is to show that such restriction is not necessary; any function defined on a closed subset of a normal space gives a function defined on the whole space:

11.2 Tietze Extension Theorem (v.1). *Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f\colon A \to [a,b]$ be a continuous function for some $[a,b] \subseteq \mathbb{R}$. There exits a continuous function $\bar{f}\colon X \to [a,b]$ such that $\bar{f}|_A = f$.*

The main idea of the proof is to use the Urysohn Lemma 10.1 to construct functions $\bar{f}_n\colon X \to [a,b]$ for $n = 1, 2, \dots$ such that as n increases $\bar{f}_n|_A$ gives ever closer approximations of f. Then we take $\bar{f}$ to be the limit of the sequence $\{\bar{f}_n\}$. We start by looking at sequences of functions and their convergence.

11.3 Definition. Let X, Y be a topological spaces and let $\{f_n\colon X \to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ *converges pointwise* to a function $f\colon X \to Y$ if for each $x \in X$ the sequence $\{f_n(x)\} \subseteq Y$ converges to the point $f(x)$.

11.4 Note. If $\{f_n\colon X \to Y\}$ is a sequence of continuous functions that converges pointwise to $f\colon X \to Y$ then f need not be continuous. For example, let $f_n\colon [0,1] \to \mathbb{R}$ be the function given by $f_n(x) = x^n$. Notice that $f_n(x) \to 0$ for all $x \in [0,1)$ and that $f_n(1) \to 1$. Thus the sequence $\{f_n\}$ converges pointwise to the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases}$$

The functions f_n are continuous but f is not.

11.5 Definition. Let X be a topological space, let (Y, ϱ) be a metric space, and let $\{f_n\colon X \to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ *converges uniformly* to a function $f\colon X \to Y$ if for every $\varepsilon > 0$ there exists $N > 0$ such that

$$\varrho(f(x), f_n(x)) < \varepsilon$$

for all $x \in X$ and for all $n > N$.

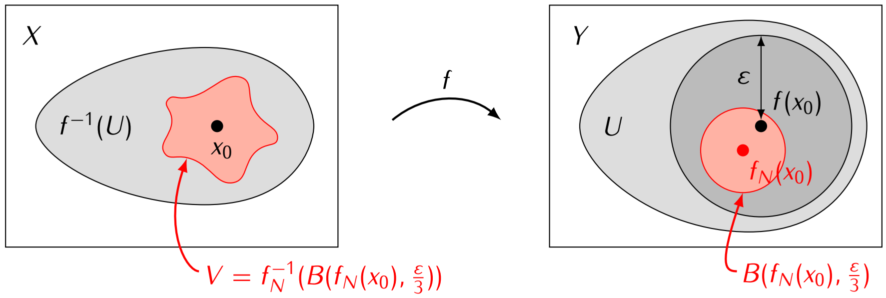
11.6 Note. If a sequence $\{f_n\}$ converges uniformly to f then it also converges pointwise to f, but the converse is not true in general.

11.7 Proposition. *Let X be a topological space and let (Y, ϱ) be a metric space. Assume that $\{f_n\colon X \to Y\}$ is a sequence of functions that converges uniformly to $f\colon X \to Y$. If all functions f_n are continuous then f is also a continuous function.*

Proof. Let $U \subseteq Y$ be an open set. We need to show that the set $f^{-1}(U) \subseteq X$ is open. If suffices to check that each point $x_0 \in f^{-1}(U)$ has an open neighborhood V such that $V \subseteq f^{-1}(U)$. Since U is an open set there exists $\varepsilon > 0$ such $B(f(x_0), \varepsilon) \subseteq U$. Choose $N > 0$ such that $\varrho(f(x), f_N(x)) < \frac{\varepsilon}{3}$ for all $x \in X$, and take $V = f_N^{-1}(B(f_N(x_0), \frac{\varepsilon}{3}))$. Since f_N is a continuous function the set V is an open neighborhood of x_0 in X. It remains to show that $V \subseteq f^{-1}(U)$. For $x \in V$ we have:

$$\varrho(f(x), f(x_0)) \leq \varrho(f(x), f_N(x)) + \varrho(f_N(x), f_N(x_0)) + \varrho(f_N(x_0), f(x_0)) < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon$$

This means that $f(x) \in B(f(x_0), \varepsilon) \subseteq U$, and so $x \in f^{-1}(U)$.



□

11.8 Lemma. *Let X be a normal space, $A \subseteq X$ be a closed set, and let $f\colon A \to \mathbb{R}$ be a continuous function such that for some $C > 0$ we have $|f(x)| \leq C$ for all $x \in A$. There exists a continuous function $g\colon X \to \mathbb{R}$ such that $|g(x)| \leq \frac{1}{3}C$ for all $x \in X$ and $|f(x) - g(x)| \leq \frac{2}{3}C$ for all $x \in A$.*

Proof. Define $Y := f^{-1}([-C, -\frac{1}{3}C])$, $Z := f^{-1}([\frac{1}{3}C, C])$. Since $f\colon A \to \mathbb{R}$ is a continuous function these sets are closed in A, but since A is closed in X the sets Y and Z are also closed in X. Since $Y \cap Z = \varnothing$ by the Generalized Urysohn Lemma 11.1 there exists a continuous function $g\colon X \to [-\frac{C}{3}, \frac{C}{3}]$ such that $h(x) = -\frac{C}{3}$ for all $x \in Y$ and $h(x) = \frac{C}{3}$ for all $x \in Z$. It is straightforward to check that $|f(x) - g(x)| \leq \frac{2}{3}C$ for all $x \in A$. □

Proof of Theorem 11.2. Since f takes values in an interval $[a, b]$, we can find a number $C > 0$ such that $|f(x)| \leq C$ for all $x \in A$. For $n = 1, 2, \ldots$ we will construct continuous functions $g_n\colon X \to \mathbb{R}$ such that

(i) $|g_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \cdot C$ for all $x \in X$;
(ii) $\left|f(x) - \sum_{i=1}^{n} g_i(x)\right| \leq \left(\frac{2}{3}\right)^{n} \cdot C$ for all $x \in A$.

We argue by induction. Existence of g_1 follows directly from Lemma 11.8. Assume that for some $n \geq 1$ we already have functions $g_1, \ldots, g_n$ satisfying (i) and (ii). Apply Lemma 11.8 to the function $f - \sum_{k=1}^{n} g_k$. We can take $g_{n+1} := g$ where g is the function given by the lemma.

Let $\bar{f}_n := \sum_{k=1}^{n} g_k$ and let $\bar{f}_\infty := \sum_{k=1}^{\infty} g_k$. Using condition (i) we obtain that the sequence $\{\bar{f}_n\}$ converges uniformly to $\bar{f}$ (exercise). Since each of the functions $\bar{f}_n$ is continuous, by Proposition 11.7 we obtain that $\bar{f}_\infty$ is a continuous function. Also, using (ii) be obtain that $\bar{f}_\infty(x) = f(x)$ for all $x \in A$ (exercise).

The only remaining issue is that the function $\bar{f}_\infty$ takes its values in $\mathbb{R}$, and not in the interval $[a, b]$. However, it is not difficult to modify it to obtain a continuous function $\bar{f}\colon X \to [a, b]$ such that $\bar{f}(x) = \bar{f}_\infty(x)$ for all $x \in A$ (exercise). □

Here is another useful reformulation of Tietze Extension Theorem:

11.9 Tietze Extension Theorem (v.2). *Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f: A \to \mathbb{R}$ be a continuous function. There exits a continuous function $\bar{f}: X \to \mathbb{R}$ such that $\bar{f}|_A = f$.*

Proof. It is enough to show that for any continuous function $g: A \to (-1, 1)$ we can find a continuous function $\bar{g}: X \to (-1, 1)$ such that $\bar{g}|_A = g$. Indeed, if this holds then given a function $f: A \to \mathbb{R}$ let $g = hf$ where $h: \mathbb{R} \to (-1, 1)$ is an arbitrary homeomorphism. Then we can take $\bar{f} = h^{-1}\bar{g}$.

Assume then that $g: A \to (-1, 1)$ is a continuous function. By Theorem 11.2 there is a function $g_1: X \to [-1, 1]$ such that $g_1|_A = g$. Let $B := g_1^{-1}(\{-1, 1\})$. The set B is closed in X and $A \cap B = \varnothing$ since $g_1(A) = g(A) \subseteq (-1, 1)$. By Urysohn Lemma 10.1 there is a continuous function $k: X \to [0, 1]$ such that $B \subseteq k^{-1}(\{0\})$ and $A \subseteq k^{-1}(\{1\})$. Let $\bar{g}(x) := k(x) \cdot g_1(x)$. We have:

1) if $g_1(x) \in (-1, 1)$ then $\bar{g}(x) \in (-1, 1)$
2) if $g_1(x) \in \{-1, 1\}$ then $x \in B$ so $\bar{g}(x) = 0 \cdot g_1(x) = 0$

It follows that $\bar{g}: X \to (-1, 1)$. Also, $\bar{g}$ is a continuous function since k and g_1 are continuous. Finally, if $x \in A$ then $\bar{g}(x) = 1 \cdot g_1(x) = g(x)$, so $\bar{g}|_A = g$. □

Tietze Extension Theorem holds for functions defined on normal spaces. It turns out the function extension property is actually equivalent to the notion of normality of a space:

11.10 Theorem. *Let X be a space satisfying T_1. The following conditions are equivalent:*

1) *X is a normal space.*
2) *For any closed sets $A, B \subseteq X$ such that $A \cap B = \varnothing$ there is a continuous function $f: X \to [0, 1]$ such that such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$.*
3) *If $A \subseteq X$ is a closed set then any continuous function $f: A \to \mathbb{R}$ can be extended to a continuous function $\bar{f}: X \to \mathbb{R}$.*

Proof. The implication 1) ⇒ 2) is the Urysohn Lemma 10.1 and 2) ⇒ 1) is Proposition 9.15. The implication 1) ⇒ 3) is the Tietze Extension Theorem 11.9. The proof of implication 3) ⇒ 1) is an exercise.

□

Exercises to Chapter 11

E11.1 Exercise. The goal of this exercise is to fill a gap in the proof of Theorem 11.2. For a topological space X and $A \subseteq X$ let $f: A \to [a, b]$ and $\bar{f}: X \to \mathbb{R}$ be continuous functions satisfying $\bar{f}(x) = f(x)$ for all $x \in A$. Show that there exists a continuous function $\bar{f}': X \to [a, b]$ such that $\bar{f}'(x) = f(x)$ for all $x \in A$.

E11.2 Exercise. Prove implication 3) $\Rightarrow$ 1) of Theorem 11.10.

E11.3 Exercise. Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f\colon A \to \mathbb{R}$ be a continuous function.

a) Assume that $g\colon X \to \mathbb{R}$ is a continuous function such that $f(x) \leq g(x)$ for all $x \in A$. Show that there exists a continuous function $F\colon X \to \mathbb{R}$ satisfying $F|_A = f$ and $F(x) \leq g(x)$ for all $x \in X$.

b) Assume that $g, h\colon X \to \mathbb{R}$ are a continuous function such that $h(x) \leq f(x) \leq g(x)$ for all $x \in A$ and $h(x) \leq g(x)$ for all $x \in X$. Show that there exists a continuous function $F'\colon X \to \mathbb{R}$ satisfying $F'|_A = f$ and $h(x) \leq F'(x) \leq g(x)$ for all $x \in X$.

E11.4 Exercise. Recall that if X is a topological space then a subspace $Y \subseteq X$ is a called a retract of X if there exists a continuous function $r\colon X \to Y$ such that $r(x) = x$ for all $x \in Y$. Let X be a normal space and let $Y \subseteq X$ be a closed subspace of X such that $Y \cong \mathbb{R}$. Show that Y is a retract of X.

E11.5 Exercise. Let X be topological space. Recall from Exercise 10.3 that a set $A \subseteq X$ is a G_δ-set if there exists a countable family of open sets $U_1, U_2, \dots$ such that $A = \bigcap_{n=1}^{\infty} U_n$.

a) Show that if X is a normal space and $A \subseteq X$ is a closed G_δ-set then there exists a continuous function $f\colon X \to [0, 1]$ such that $A = f^{-1}(\{0\})$.

b) Show that if X is a normal space and $A, B \subseteq X$ are closed G_δ-sets such that $A \cap B = \varnothing$ then there exists a continuous function $f\colon X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.

E11.6 Exercise. Assume that we have a sequence of spaces

$$X_1 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

Let $X_\infty = \bigcup_{n=1}^{\infty}$. Define a topology on X_∞ in such way that a set $U \subseteq X_\infty$ is open in X_∞ if and only if the set $U \cap X_n$ is open in X_n for each $n = 1, 2, \dots$

a) Show that a function $f\colon X_\infty \to Y$ is continuous if and only if its restriction $f|_{X_n}\colon X_n \to Y$ is continuous for each $n = 1, 2, \dots$.

b) Assume that for each n the space X_n is normal, and that X_n is closed in X_{n+1}. Show that X_∞ is normal. (Hint: Use Proposition 9.15).