3 | Open Sets

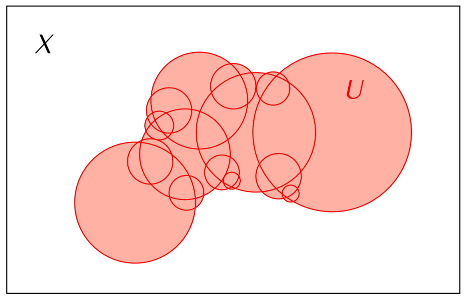
3.1 Definition. Let ϱ_1 and ϱ_2 be two metrics on the same set X. We say that the metrics ϱ_1 and ϱ_2 are *equivalent* if for every $x \in X$ and for every $r > 0$ there exist $s_1, s_2 > 0$ such that $B_{\varrho_1}(x, s_1) \subseteq B_{\varrho_2}(x, r)$ and $B_{\varrho_2}(x, s_2) \subseteq B_{\varrho_1}(x, r)$.

3.2 Proposition. *Let ϱ_1, ϱ_2 be equivalent metrics on a set X, and let μ_1, μ_2 be equivalent metrics on a set Y. A function $f\colon X \to Y$ is continuous with respect to the metrics ϱ_1 and μ_1 if and only if it is continuous with respect to the metrics ϱ_2 and μ_2.*

3.3 Example. The Euclidean metric d, the orthogonal metric ϱ_{ort} and the maximum metric ϱ_{max} are equivalent metrics on $\mathbb{R}^n$ (exercise).

3.4 Example. The following metrics on $\mathbb{R}^2$ are not equivalent to one another: the Euclidean metric d, the hub metric ϱ_h, and the discrete metric ϱ_{disc} (exercise).

3.5 Definition. Let (X, ϱ) be a metric space. A subset $U \subseteq X$ is an *open set* if U is a union of (perhaps infinitely many) open balls in X: $U = \bigcup_{i \in I} B(x_i, r_i)$.



3.6 Proposition. *Let (X, ϱ) be a metric space and let $U \subseteq X$. The following conditions are equivalent:*

1) *The set U is open.*
2) *For every $x \in U$ there exists $r_x > 0$ such that $B(x, r_x) \subseteq U$.*

Proof. Exercise. □

3.7 Proposition. *Let X be a set and let ϱ_1, ϱ_2 be two metrics on X. The following conditions are equivalent:*

1) *The metrics ϱ_1 and ϱ_2 are equivalent.*
2) *A set $U \subseteq X$ is open with respect to the metric ϱ_1 if and only if it is open with respect to the metric ϱ_2.*

3.8 Proposition. *Let (X, ϱ) be a metric space.*

1) *The sets X and $\varnothing$ are open sets.*
2) *If U_i is an open set for $i \in I$ then the set $\bigcup_{i \in I} U_i$ is open.*
3) *If U_1, U_2 are open sets then the set $U_1 \cap U_2$ is open.*

Proof. Exercise. □

3.10 Proposition. *Let* (X, ϱ), (Y, μ) *be metric spaces and let* $f\colon X \to Y$ *be a function. The following conditions are equivalent:*

1) *The function* f *is continuous.*
2) *For every open set* $U \subseteq Y$ *the set* $f^{-1}(U)$ *is open in* X.

3.11 Lemma. *Let* (X, ϱ), (Y, μ) *be metric spaces and let* $f\colon X \to Y$ *be a continuous function. If* $B := B(y_0, r)$ *is an open ball in* Y *then the set* $f^{-1}(B)$ *is open in* X.

Proof. Exercise. $\square$

3.12 Definition. Let X be a set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X satisfying the following conditions:

1) $X, \varnothing \in \mathcal{T}$;
2) If $U_i \in \mathcal{T}$ for $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$;
3) If $U_1, U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called *open sets*.

A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

3.13 Definition. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f\colon X \to Y$ is *continuous* if for every $U \in \mathcal{T}_Y$ we have $f^{-1}(U) \in \mathcal{T}_X$.

3.14 Example. If (X, ϱ) is a metric space then X is a topological space with the topology

$$\mathcal{T} = \{U \subseteq X \mid U \text{ is a union of open balls}\}$$

We say that the topology $\mathcal{T}$ is *induced by the metric* ϱ.

3.16 Example. Let X be an arbitrary set and let

$$\mathcal{T} = \{\text{all subsets of } X\}$$

The topology $\mathcal{T}$ is called the *discrete topology* on X. If X is equipped with this topology then we say that it is a *discrete topological space*.

3.17 Example. Let X be an arbitrary set and let

$$\mathcal{T} = \{X, \varnothing\}$$

The topology $\mathcal{T}$ is called the *antidiscrete topology* on X.

3.18 Example. Let $X = \mathbb{R}$ and let

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \smallsetminus S) \text{ for some finite set } S \subseteq \mathbb{R}\}$$

The topology $\mathcal{T}$ is called the *Zariski topology* on $\mathbb{R}$.

3.19 Definition. A topological space $(X, \mathcal{T})$ is *metrizable* if there exists a metric ϱ on X such that $\mathcal{T}$ is the topology induced by ϱ.

3.20 Lemma. *If $(X, \mathcal{T})$ is a metrizable topological space and $x, y \in X$ are points such that $x \neq y$ then there exists an open set $U \subseteq X$ such that $x \in U$ and $y \notin U$.*

Proof. Exercise. $\square$

3.21 Proposition. *If X is a set containing more than one point then the antidiscrete topology on X is not metrizable.*