

# 7 | Connectedness

**7.1** Let  $[a, b] \subseteq \mathbb{R}$  be a closed interval and let  $(a, b) \subseteq \mathbb{R}$  be an open interval. We would like to show that  $[a, b]$  and  $(a, b)$  are non-homeomorphic topological spaces. The idea of a proof of this fact is as follows. Assume that there exists a homeomorphism

$$f: [a, b] \rightarrow (a, b)$$

Recall that by Proposition 6.10 for any  $Y \subseteq [a, b]$  the function  $f|_Y: Y \rightarrow f(Y)$  also would be a homeomorphism. If we take  $Y = [a, b] \setminus \{a\} = (a, b]$  then

$$f(Y) = f([a, b] \setminus \{a\}) = (a, b) \setminus \{f(a)\}$$

Intuitively the spaces  $Y$  and  $f(Y)$  are different in an essential way since  $Y$  comes in one piece while  $f(Y)$  is split into two pieces by removal of the point  $f(a)$ :



For this reason we can expect that the spaces  $Y$  and  $f(Y)$  are not homeomorphic, and that, as a consequence,  $[a, b]$  and  $(a, b)$  are not homeomorphic as well.

In order to make this intuitive argument into a rigorous proof we need to define precisely what it means that a topological space is “in one piece” and then show that this feature is preserved by homeomorphisms. The property of being “in one piece” is captured by the definition of a connected space:

**7.2 Definition.** A topological space  $X$  is *connected* if for any two open sets  $U, V \subseteq X$  such that  $U \cup V = X$  and  $U, V \neq \emptyset$  we have  $U \cap V \neq \emptyset$ .

**7.3 Definition.** If  $X$  is a topological space and  $U, V \subseteq X$  are non-empty open sets such that  $U \cap V = \emptyset$  and  $U \cup V = X$  then we say that  $\{U, V\}$  is a *separation* of  $X$ .

Thus, a space  $X$  is connected if there does not exist a separation of  $X$ .

**7.4 Example.** For  $a < b$  take  $(a, b) \subseteq \mathbb{R}$  and let  $c \in (a, b)$ . The space  $X = (a, b) \setminus \{c\}$  is not connected. Indeed, the sets  $U = (a, c)$  and  $V = (c, b)$  form a separation of  $X$ .

**7.5 Proposition.** Let  $a < b$ . The intervals  $(a, b)$ ,  $[a, b]$ ,  $(a, b]$ , and  $[a, b)$  are connected topological spaces.

*Proof.* Assume first that  $[a, b]$  is a closed interval and that  $U, V \subseteq [a, b]$  are open sets such that  $a \in U$ ,  $b \in V$ , and  $U \cup V = [a, b]$ . We will show that  $U \cap V \neq \emptyset$ . Let  $x_0 = \inf V$ . There are two possibilities: either  $x_0 \notin U$  or  $x_0 \in U$ . In the first case  $x_0 \in V$  and  $x_0 > a$ . Since  $V$  is an open set there exists  $\varepsilon > 0$  such that  $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq V$ . This implies that there is  $x \in V$  such that  $x < x_0$  which is impossible by the definition of  $x_0$ .

Thus the only possible option is  $x_0 \in U$ . Since  $U$  is an open set there exists  $\varepsilon' > 0$  such that  $[x_0, x_0 + \varepsilon') \subseteq U$ . On the other hand, by the definition of  $x_0$  we have  $[x_0, x_0 + \varepsilon') \cap V \neq \emptyset$ . Therefore  $U \cap V \neq \emptyset$ .

Assume now that  $I$  is an interval (either closed, open, or half-open) and that  $U, V \subseteq I$  are non-empty open sets such that  $U \cup V = I$ . We will show that  $U \cap V \neq \emptyset$ . Let  $c, d \in I$  be points such that  $c \in U$  and  $d \in V$ . We can assume that  $c < d$ . Take  $U' = U \cap [c, d]$  and  $V' = V \cap [c, d]$ . The sets  $U', V'$  are open in  $[c, d]$ ,  $c \in U'$ ,  $d \in V'$ , and  $U' \cup V' = [c, d]$ . By the observation above we have  $U' \cap V' \neq \emptyset$ , and so  $U \cap V \neq \emptyset$ .

□

One can show that intervals are in fact the only subspaces of  $\mathbb{R}$  that are connected:

**7.6 Proposition.** If  $X$  is a connected subspace of  $\mathbb{R}$  then  $X$  is an interval (either open, closed, or half-closed, finite or infinite).

*Proof.* Exercise.

□

**7.7** Going back to the argument outlined in 7.1, by Proposition 7.5 we get that the space  $Y = (a, b]$  is connected, and the space  $f(Y) = (a, b) \setminus f(a)$  is not connected by Example 7.4. We still need to show however that a connected space cannot be homeomorphic to one that is not connected. In fact a stronger statement is true:

**7.8 Proposition.** *Let  $f: X \rightarrow Y$  be a continuous function. If  $f$  is onto and the space  $X$  is connected then  $Y$  is also connected.*

*Proof.* Assume that  $Y$  is not connected and let  $U, V \subseteq Y$  be a separation of  $Y$ . Then the sets  $f^{-1}(U), f^{-1}(V)$  form a separation of  $X$  which contradicts the assumption that  $X$  is connected.  $\square$

**7.9 Corollary.** *If  $f: X \rightarrow Y$  is a continuous function and  $X$  is a connected space then  $f(X)$  is connected.*

*Proof.* By restricting the range of  $f$  we obtain a function  $f: X \rightarrow f(X)$  which is continuous and onto, and so it we can apply Proposition 7.8.  $\square$

A very useful consequence of Corollary 7.9 is the following fact:

**7.10 Intermediate Value Theorem.** *Let  $X$  be a connected topological space and let  $f: X \rightarrow \mathbb{R}$  be a continuous function. If  $a < b$  are points in  $\mathbb{R}$  such that  $a = f(x)$  and  $b = f(y)$  for some  $x, y \in X$  then for each  $c \in [a, b]$  there exists  $z \in X$  such that  $c = f(z)$ .*

*Proof.* By Corollary 7.9 the set  $f(X)$  is connected, and so by Proposition 7.6  $f(X)$  is an interval. It follows that for any  $a, b \in f(X)$  we have  $[a, b] \subseteq f(X)$ .  $\square$

Since every homeomorphism  $f: X \rightarrow Y$  is onto directly from Corollary 7.9 we get:

**7.11 Corollary.** *If  $X \cong Y$  and  $X$  is a connected space then  $Y$  is also connected.*

**7.12 Corollary.** *The space  $\mathbb{R}$  is connected.*

*Proof.* This follows from Corollary 7.11 and Proposition 7.5 since  $\mathbb{R} \cong (a, b)$  for any  $a < b$ .  $\square$

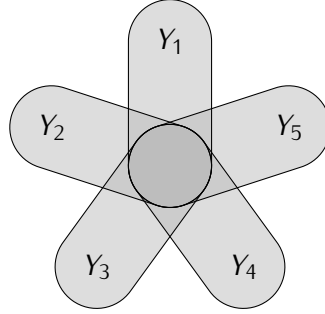
**7.13 Note.** A *topological invariant* is a property of topological spaces such that if a space  $X$  has this property and  $X \cong Y$  then  $Y$  also has this property. By Corollary 7.11 connectedness is a topological invariant.

**7.14 Proposition.** *Let  $X$  be a topological space. The following conditions are equivalent :*

- 1)  $X$  is connected
- 2) For any closed sets  $A, B \subseteq X$  such that  $A, B \neq X$  and  $A \cap B = \emptyset$  we have  $A \cup B \neq X$ .
- 3) If  $A \subseteq X$  is a set that is both open and closed then either  $A = X$  or  $A = \emptyset$ .
- 4) If  $D = \{0, 1\}$  is a space with the discrete topology then any continuous function  $f: X \rightarrow D$  is a constant function.

*Proof.* Exercise.  $\square$

**7.15 Proposition.** Let  $X$  be a topological space and for  $i \in I$  let  $Y_i$  be a subspace of  $X$ . Assume that  $\bigcup_{i \in I} Y_i = X$  and  $\bigcap_{i \in I} Y_i \neq \emptyset$ . If  $Y_i$  is connected for each  $i \in I$  then  $X$  is also connected.



*Proof.* Let  $D = \{0, 1\}$  be a space with the discrete topology and let  $f: X \rightarrow D$  be a continuous function. By Proposition 7.14 it is enough to show that  $f$  is a constant function. Let  $x_0 \in \bigcap_{i \in I} Y_i$ . We can assume that  $f(x_0) = 0$ . For any  $i \in I$  the function  $f|_{Y_i}: Y_i \rightarrow D$  is constant since  $Y_i$  is connected. Since  $x_0 \in Y_i$  and  $f(x_0) = 0$  we get that  $f(x) = 0$  for all  $x \in Y_i$ . Since this applies to all subspaces  $Y_i$  we obtain that  $f(x) = 0$  for all  $x \in \bigcup_{i \in I} Y_i = X$ .  $\square$

**7.16 Corollary.** The space  $\mathbb{R}^n$  is connected for all  $n \geq 1$ .

*Proof.* For  $0 \neq x \in \mathbb{R}^n$  let  $L_x \subseteq \mathbb{R}^n$  be the line passing through  $x$  and the origin:

$$L_x = \{tx \in \mathbb{R}^n \mid t \in \mathbb{R}\}$$

For every  $x \in \mathbb{R}^n$  consider the continuous function  $f_x: \mathbb{R} \rightarrow \mathbb{R}^n$  given by  $f_x(t) = tx$ . Since  $\mathbb{R}$  is connected and  $f_x(\mathbb{R}) = L_x$  it follows that  $L_x$  is connected. We have  $\mathbb{R}^n = \bigcup_{x \in \mathbb{R}^n} L_x$  and  $\bigcap_{x \in \mathbb{R}^n} L_x = \{0\}$ . Therefore by Proposition 7.15 the space  $\mathbb{R}^n$  is connected.  $\square$

**7.17 Definition.** Let  $X$  be a topological space. A *connected component* of  $X$  is a subspace  $Y \subseteq X$  such that

- 1)  $Y$  is connected
- 2) if  $Y \subseteq Z \subseteq X$  and  $Z$  is connected then  $Y = Z$ .

**7.18 Proposition.** Let  $X$  be a topological space.

- 1) For every point  $x_0 \in X$  there exist a connected component  $Y \subseteq X$  such that  $x_0 \in Y$ .
- 2) If  $Y, Y'$  are connected components of  $X$  then either  $Y \cap Y' = \emptyset$  or  $Y = Y'$ .

*Proof.* 1) Given a point  $x_0 \in X$  let  $\{C_i\}_{i \in I}$  be the collection of all subspaces of  $X$  such that  $x_0 \in C_i$  and  $C_i$  is connected. Define  $Y := \bigcup_{i \in I} C_i$ . We have  $x_0 \in Y$ . Also, since  $x_0 \in \bigcap_{i \in I} C_i$  by Proposition

7.15 we obtain that  $Y$  is connected. If  $Y \subseteq Z \subseteq X$  and  $Z$  is connected then  $Z = C_{i_0}$  for some  $i_0 \in I$ , and so  $Z = Y$ . Therefore  $Y$  is a connected component of  $X$ .

2) Let  $Y, Y'$  be two connected components of  $X$ . Assume that  $Y \cap Y' \neq \emptyset$ . By Proposition 7.15 we get then that  $Y \cup Y'$  is connected. Since  $Y \subseteq Y \cup Y'$  we must have  $Y = Y \cup Y'$ . By the same argument we obtain that  $Y' = Y \cup Y'$ . Therefore  $Y = Y'$

□

**7.19 Corollary.** *Let  $X$  be a topological space. If  $Z \subseteq X$  is a connected subspace then there exists a connected component  $Y \subseteq X$  such that  $Z \subseteq Y$ .*

*Proof.* Exercise.

□

**7.20 Corollary.** *Let  $f: X \rightarrow Y$  be a continuous function. If  $X$  is a connected space then there exists a connected component  $Z \subseteq Y$  such that  $f(X) \subseteq Z$ .*

*Proof.* Exercise.

□

## Exercises to Chapter 7

**E7.1 Exercise.** Let  $X$  be a topological space and let  $Y \subseteq X$  be a subspace. Show that if  $Y$  is a connected space and  $Y$  is dense in  $X$  then  $X$  is connected.

**E7.2 Exercise.** Prove Proposition 7.6.

**E7.3 Exercise.** Show that the sphere  $S^n$  is connected for all  $n \geq 1$ .

**E7.4 Exercise.** Let  $a < b$ . Show that the closed interval  $[a, b] \subseteq \mathbb{R}$  is not homeomorphic to the half-closed interval  $(a, b]$ .

**E7.5 Exercise.** A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is *strictly increasing* if for all  $x, y \in \mathbb{R}$  such that  $x > y$  we have  $f(x) > f(y)$ , and is *strictly decreasing* if for all  $x, y \in \mathbb{R}$  such that  $x > y$  we have  $f(x) < f(y)$ . Show that if  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a continuous 1-1 function then  $f$  is either strictly increasing or strictly decreasing.

**E7.6 Exercise.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that  $f(x) \cdot f(f(x)) = 1$  for all  $x \in \mathbb{R}$  and that  $f(10) = 9$ . Find the value of  $f(5)$ . Justify your answer.

**E7.7 Exercise.** Let  $f: S^n \rightarrow \mathbb{R}$  be a continuous function. Show that there exists a point  $x \in S^n$  such that  $f(x) = f(-x)$ . Here if  $x = (x_1, \dots, x_n) \in S^n$  then  $-x = (-x_1, \dots, -x_n)$ .

**E7.8 Exercise.** Let  $a < b$ . Show that there does not exist a continuous bijection  $f: (a, b) \rightarrow [a, b]$ . Remember that a continuous bijection need not be a homeomorphism since the inverse function may be

not continuous (see 6.11).

**E7.9 Exercise.** Prove Proposition 7.14.

**E7.10 Exercise.** Let  $X$  be a topological space. Show that the following conditions are equivalent:

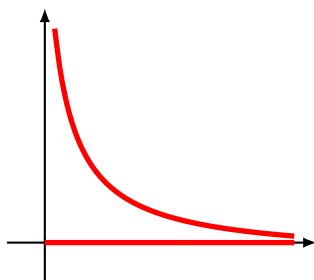
- 1)  $X$  is connected
- 2) if  $A \subseteq X$  is any set such that  $A \neq X$  and  $A \neq \emptyset$  then  $\text{Bd}(A) \neq \emptyset$ .

**E7.11 Exercise.** Let  $X$  be a topological space. Show that every connected component of  $X$  is closed in  $X$ .

**E7.12 Exercise.** Let  $(X, \varrho)$  be a metric space. Assume for some  $x_0 \in X$  and  $r > 0$  the open ball  $B(x_0, r)$  consists of countably many points. Show that  $X$  is not connected.

**E7.13 Exercise.** Let  $X$  be the subspace of  $\mathbb{R}^2$  consisting of the positive  $x$ -axis and of the graph of the function  $f(x) = \frac{1}{x}$  for  $x > 0$ :

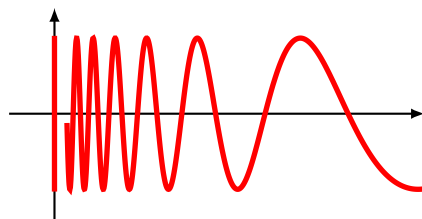
$$X := \{(x, 0) \in \mathbb{R}^2 \mid x \geq 0\} \cup \{(x, \frac{1}{x}) \in \mathbb{R}^2 \mid x > 0\}$$



Show that  $X$  is not connected.

**E7.14 Exercise.** The *topologist's sine curve* is the subspace  $Y$  of  $\mathbb{R}^2$  that consists of a segment of the  $y$ -axis and of the graph of the function  $f(x) = \sin(\frac{1}{x})$ :

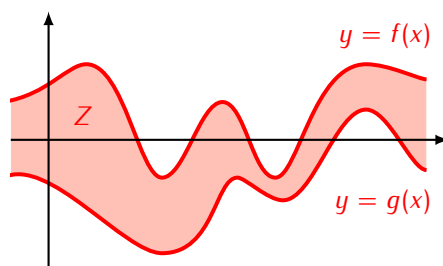
$$Y := \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\} \cup \{(x, \sin(\frac{1}{x})) \in \mathbb{R}^2 \mid x > 0\}$$



Show that  $Y$  is connected.

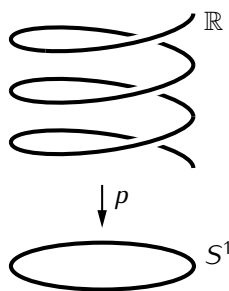
**E7.15 Exercise.** Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions such that  $g(x) < f(x)$  for all  $x \in \mathbb{R}$ . Let  $Z$  be the subspace of  $\mathbb{R}^2$  given by

$$Z = \{(x, y) \mid g(x) \leq y \leq f(x)\}$$



Show that  $Z$  is connected.

**E7.16 Exercise.** Consider the unit circle  $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ . Let  $p: \mathbb{R} \rightarrow S^1$  denote the function given by  $p(t) = (\sin 2\pi t, \cos 2\pi t)$ . Geometrically speaking this function wraps  $\mathbb{R}$  infinitely many times around the circle:



Show that there does not exist a continuous function  $g: S^1 \rightarrow \mathbb{R}$  such that  $pg = \text{id}_{S^1}$ .

**E7.17 Exercise.** A space  $X$  is *totally disconnected* if every connected component of  $X$  consists of a single point. Obviously every discrete topological space is totally disconnected. Consider the set of rational numbers  $\mathbb{Q}$  as a subspace of  $\mathbb{R}$ . Show that  $\mathbb{Q}$  is totally disconnected. Note that by Exercise 6.1  $\mathbb{Q}$  is not a discrete space.

**E7.18 Exercise.** Show that metrizability is a topological invariant. That is, if  $X$  and  $Y$  are homeomorphic spaces, and  $X$  metrizable then so is  $Y$ .