MTH 427/527
Practice Final Exam

MTH 427 Instructions. Solve any **six** problems. If you attempt more problems, the best six solutions will be used for the final score.

MTH 527 Instructions. Solve any **seven** problems. If you attempt more problems, the best seven solutions will be used for the final score.

1. Let X be a Hausdorff space, and let $f\colon X \to Y$ be a continuous function which is onto. Assume that f has the property that for any open set $U \subseteq X$ we have $f(U) \cap \overline{f(X \smallsetminus U)} = \varnothing$. Show that f is a homeomorphism.

2. Let X be a topological space, and let $\Delta \subseteq X \times X$ be the set defined by

$$\Delta = \{(x, x) \in X \times X \mid x \in X\}$$

Show that if Δ is closed in $X \times X$ then X is a Hausdorff space.

3. Let X be a non-empty connected, compact space, and let $f\colon X \to \mathbb{R}$ be a continuous function. Show that $f(X) = [a, b]$ for some $a, b \in \mathbb{R}$

4. Let $f\colon X \to Y$ be a continuous function such that for any open set $U \subseteq X$ the set $f(U)$ is open in Y. Assume $f(X)$ is connected, and that for each point $y \in Y$ the set $f^{-1}(y)$ is connected. Show that X is connected.

5. Let X be a connected space. Assume that $A \subseteq X$ is a closed set such that the boundary $\mathrm{Bd}(A)$ is connected. Show that A is connected.

6. Recall that a space X is locally compact if every point $x \in X$ has an open neighborhood $V \subseteq X$ such that $\overline{V}$ is compact. Show that if M is a topological manifold (without boundary) then M is locally compact.

7. Recall that a continuous function $f\colon X \to Y$ is proper if for each compact set $K \subseteq Y$ the set $f^{-1}(K)$ is compact. Let (X, ϱ), (Y, μ) be metric spaces and let $f\colon X \to Y$ be a continuous proper function. Show that for any closed set $A \subseteq X$ the set $f(A) \subseteq Y$ is closed.

8. Let X be a topological space, and let $A \subseteq X$. Let X/A denote the quotient space of X defined by the equivalence relation which identifies all points of A: $a \sim a'$ for all $a, a' \in A$.

a) Show that if the quotient space X/A is a Hausdorff space, then A is closed in X.

b) Show that if X is a regular (T_3) space and $A \subseteq X$ is a closed set, then X/A is a Hausdorff space.