12 | Urysohn Metrization Theorem

12.1 Urysohn Metrization Theorem. *Every second countable normal space is metrizable.*

12.2 Definition. A continuous function $i\colon X \to Y$ is an *embedding* if its restriction $i\colon X \to i(X)$ is a homeomorphism (where $i(X)$ has the topology of a subspace of Y).

12.5 Lemma. *If $j\colon X \to Y$ is an embedding and Y is a metrizable space then X is also metrizable.*

12.6 Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The *product topology* on $\prod_{i\in I} X_i$ is the topology generated by the basis

$$\mathcal{B} = \left\{\prod_{i\in I} U_i \mid U_i \text{ is open in } X_i \text{ and } U_i \neq X_i \text{ for finitely many indices } i \text{ only}\right\}$$

12.8 Proposition. *Let* $\{X_i\}_{i\in I}$ *be a family of topological spaces and for* $j \in I$ *let*

$$p_j \colon \prod_{i\in I} X_i \to X_j$$

be the projection onto the j*-th factor:* $p_j((x_i)_{i\in I}) = x_j$*. Then:*

1) *for any* $j \in I$ *the function* p_j *is continuous.*
2) *A function* $f \colon Y \to \prod_{i\in I} X_i$ *is continuous if and only if the composition* $p_j f \colon Y \to X_j$ *is continuous for all* $j \in I$

Proof. Exercise. □

12.10 Proposition. *If $\{X_i\}_{i=1}^{\infty}$ is a countable family of metrizable spaces then $\prod_{i=1}^{\infty} X_i$ is also a metrizable space.*

12.11 Example. The *Hilbert cube* is the topological space $[0,1]^{\aleph_0}$ obtained as the infinite countable product of the closed interval $[0,1]$:

$$[0,1]^{\aleph_0} = \prod_{i=1}^{\infty} [0,1]$$

Elements of $[0,1]^{\aleph_0}$ are infinite sequences $(t_i) = (t_1, t_2, \dots)$ where $t_i \in [0,1]$ for $i = 1, 2, \dots$ The Hilbert cube is a metric space with a metric ϱ given by

$$\varrho((t_i), (s_i)) = \sum_{i=1}^{\infty} \frac{1}{2^i} |t_i - s_i|$$

12.12 Theorem. *If X is a second countable normal space then there exists an embedding $j: X \to [0,1]^{\aleph_0}$.*

12.13 Definition. Let X be a topological space and let $\{f_i\}_{i\in I}$ be a family of continuous functions $f_i: X \to [0,1]$. We say that the family $\{f_i\}_{i\in I}$ *separates points from closed sets* if for any point $x_0 \in X$ and any closed set $A \subseteq X$ such that $x_0 \notin A$ there is a function $f_j \in \{f_i\}_{i\in I}$ such that $f_j(x_0) > 0$ and $f_j|_A = 0$.

12.14 Embedding Lemma. *Let X be a T_1-space. If $\{f_i \colon X \to [0,1]\}_{i \in I}$ is a family that separates points from closed sets then the map*

$$f_\infty \colon X \to \prod_{i \in I} [0,1]$$

given by $f_\infty(x) = (f_i(x))_{i \in I}$ is an embedding.

Proof of Theorem 12.12.

□

12.16 Proposition. *Every second countable regular space is normal.*

Proof. Exercise. □

12.17 Urysohn Metrization Theorem (v.2). *Every second countable regular space is metrizable.*

12.18 Definition. Let X be a topological space. A collection $\mathcal{U} = \{U_i\}_{i \in I}$ of open sets in X is *locally finite* if each point $x \in X$ has an open neighborhood V_x such that $V_x \cap U_i \neq \varnothing$ for finitely many $i \in I$ only.

A collection $\mathcal{U}$ is *countably locally finite* if it can be decomposed into a countable union $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ where each collection $\mathcal{U}_n$ is locally finite.

12.19 Nagata-Smirnov Metrization Theorem. *Let X be a topological space. The following conditions are equivalent:*

1) *X is metrizable.*
2) *X is regular and it has a basis which is countably locally finite.*