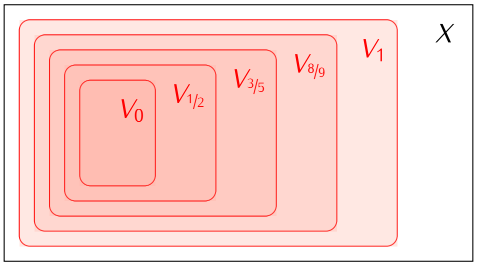
10 | Urysohn Lemma

10.1 Urysohn Lemma. *Let X be a normal space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$. There exists a continuous function $f\colon X \to [0,1]$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$.*

10.2 Lemma. *Let X be a topological space. Assume that for each $r \in [0,1] \cap \mathbb{Q}$ we are given an open set $V_r \subseteq X$ such that $\overline{V}_r \subseteq V_{r'}$ if $r < r'$. There exists a continuous function $f\colon X \to [0,1]$ such that if $x \in V_r$ then $f(x) \leq r$ and if $x \notin V_1$ then $f(x) = 1$.*

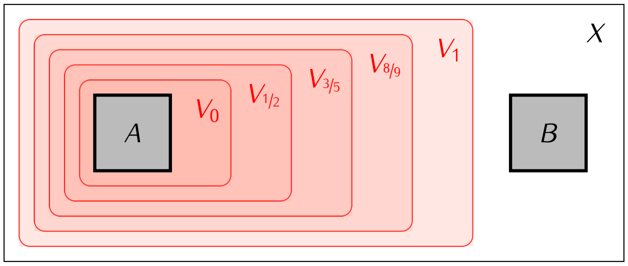


10.3 Lemma. *Let X be a normal space, let $A \subseteq X$ be a closed set and let $U \subseteq X$ be an open set such that $A \subseteq U$. There exists an open set V such that $A \subseteq V$ and $\overline{V} \subseteq U$.*

Proof. Exercise. $\square$

Proof of Urysohn Lemma 10.1. We will show that for each $r \in [0,1] \cap \mathbb{Q}$ there exists an open set $V_r \subseteq X$ such that

1) $A \subseteq V_0$
2) $B \subseteq X \smallsetminus V_1$
3) if $r < r'$ then $\overline{V}_r \subseteq V_{r'}$.



$\square$

10.4 Definition. A topological space X *satisfies the axiom* $T_{3^1/_2}$ if X satisfies T_1 and if for each point $x \in X$ and each closed set $A \subseteq X$ such that $x \notin A$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f|_A = 0$.

A space that satisfies the axiom $T_{3^1/_2}$ is called a *completely regular space* or a *Tychonoff space*.