

# 11 | Tietze Extension Theorem

The Urysohn Lemma, which we proved in the last chapter, shows that every normal space  $X$  is equipped with an ample supply of continuous functions  $X \rightarrow [0, 1]$ : any two closed, disjoint sets in  $X$  give one such function. However, an inconvenient constraint is that these functions are of very special type: they map one closed set to 0, and the other one to 1.

It is easy to modify the Urysohn Lemma to expand this collection of functions a bit:

**11.1 Generalized Urysohn Lemma.** *Let  $X$  be a normal space and let  $A, B \subseteq X$  be closed sets such that  $A \cap B = \emptyset$ . For any  $a, b \in \mathbb{R}$ ,  $a < b$  there exists a continuous function  $f: X \rightarrow [a, b]$  such that  $A \subseteq f^{-1}(\{a\})$  and  $B \subseteq f^{-1}(\{b\})$ .*

*Proof.* By the Urysohn Lemma 10.1 we can find a function  $g: X \rightarrow [0, 1]$  such that  $g(A) = \{0\}$  and  $g(B) = \{1\}$ . Take  $f = h \circ g$ , where  $h: [0, 1] \rightarrow [a, b]$  is any continuous function such that  $h(0) = a$  and  $h(1) = b$ .  $\square$

The collection of functions described by Lemma 11.1 is still very narrow: these functions are constant when restricted to either set  $A$  or  $B$ . The main result of this chapter is to show that such restriction is not necessary; any function defined on a closed subset of a normal space gives a function defined on the whole space:

**11.2 Tietze Extension Theorem (v.1).** *Let  $X$  be a normal space, let  $A \subseteq X$  be a closed subspace, and let  $f: A \rightarrow [a, b]$  be a continuous function for some  $[a, b] \subseteq \mathbb{R}$ . There exists a continuous function  $\tilde{f}: X \rightarrow [a, b]$  such that  $\tilde{f}|_A = f$ .*

The main idea of the proof is to use the Urysohn Lemma 10.1 to construct functions  $\tilde{f}_n: X \rightarrow [a, b]$  for  $n = 1, 2, \dots$  such that as  $n$  increases  $\tilde{f}_n|_A$  gives ever closer approximations of  $f$ . Then we take  $\tilde{f}$  to be

the limit of the sequence  $\{\tilde{f}_n\}$ . We start by looking at sequences of functions and their convergence.

**11.3 Definition.** Let  $X, Y$  be a topological spaces and let  $\{f_n: X \rightarrow Y\}$  be a sequence of functions. We say that the sequence  $\{f_n\}$  *converges pointwise* to a function  $f: X \rightarrow Y$  if for each  $x \in X$  the sequence  $\{f_n(x)\} \subseteq Y$  converges to the point  $f(x)$ .

**11.4 Note.** If  $\{f_n: X \rightarrow Y\}$  is a sequence of continuous functions that converges pointwise to  $f: X \rightarrow Y$  then  $f$  need not be continuous. For example, let  $f_n: [0, 1] \rightarrow \mathbb{R}$  be the function given by  $f_n(x) = x^n$ . Notice that  $f_n(x) \rightarrow 0$  for all  $x \in [0, 1)$  and that  $f_n(1) \rightarrow 1$ . Thus the sequence  $\{f_n\}$  converges pointwise to the function  $f: [0, 1] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 0 & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases}$$

The functions  $f_n$  are continuous but  $f$  is not.

**11.5 Definition.** Let  $X$  be a topological space, let  $(Y, \varrho)$  be a metric space, and let  $\{f_n: X \rightarrow Y\}$  be a sequence of functions. We say that the sequence  $\{f_n\}$  *converges uniformly* to a function  $f: X \rightarrow Y$  if for every  $\varepsilon > 0$  there exists  $N > 0$  such that

$$\varrho(f(x), f_n(x)) < \varepsilon$$

for all  $x \in X$  and for all  $n > N$ .

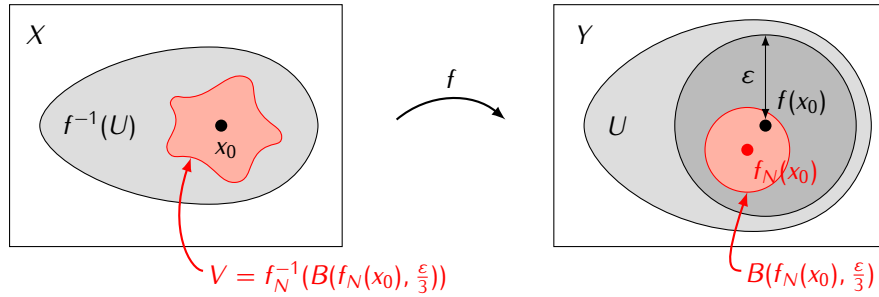
**11.6 Note.** If a sequence  $\{f_n\}$  converges uniformly to  $f$  then it also converges pointwise to  $f$ , but the converse is not true in general.

**11.7 Proposition.** Let  $X$  be a topological space and let  $(Y, \varrho)$  be a metric space. Assume that  $\{f_n: X \rightarrow Y\}$  is a sequence of functions that converges uniformly to  $f: X \rightarrow Y$ . If all functions  $f_n$  are continuous then  $f$  is also a continuous function.

*Proof.* Let  $U \subseteq Y$  be an open set. We need to show that the set  $f^{-1}(U) \subseteq X$  is open. It suffices to check that each point  $x_0 \in f^{-1}(U)$  has an open neighborhood  $V$  such that  $V \subseteq f^{-1}(U)$ . Since  $U$  is an open set there exists  $\varepsilon > 0$  such  $B(f(x_0), \varepsilon) \subseteq U$ . Choose  $N > 0$  such that  $\varrho(f(x), f_N(x)) < \frac{\varepsilon}{3}$  for all  $x \in X$ , and take  $V = f_N^{-1}(B(f_N(x_0), \frac{\varepsilon}{3}))$ . Since  $f_N$  is a continuous function the set  $V$  is an open neighborhood of  $x_0$  in  $X$ . It remains to show that  $V \subseteq f^{-1}(U)$ . For  $x \in V$  we have:

$$\varrho(f(x), f(x_0)) \leq \varrho(f(x), f_N(x)) + \varrho(f_N(x), f_N(x_0)) + \varrho(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

This means that  $f(x) \in B(f(x_0), \varepsilon) \subseteq U$ , and so  $x \in f^{-1}(U)$ .



□

**11.8 Lemma.** Let  $X$  be a normal space,  $A \subseteq X$  be a closed set, and let  $f: A \rightarrow \mathbb{R}$  be a continuous function such that for some  $C > 0$  we have  $|f(x)| \leq C$  for all  $x \in A$ . There exists a continuous function  $g: X \rightarrow \mathbb{R}$  such that  $|g(x)| \leq \frac{1}{3}C$  for all  $x \in X$  and  $|f(x) - g(x)| \leq \frac{2}{3}C$  for all  $x \in A$ .

*Proof.* Define  $Y := f^{-1}([-C, -\frac{1}{3}C])$ ,  $Z := f^{-1}([\frac{1}{3}C, C])$ . Since  $f: A \rightarrow \mathbb{R}$  is a continuous function these sets are closed in  $A$ , but since  $A$  is closed in  $X$  the sets  $Y$  and  $Z$  are also closed in  $X$ . Since  $Y \cap Z = \emptyset$  by the Generalized Urysohn Lemma 11.1 there exists a continuous function  $g: X \rightarrow [-\frac{C}{3}, \frac{C}{3}]$  such that  $h(x) = -\frac{C}{3}$  for all  $x \in Y$  and  $h(x) = \frac{C}{3}$  for all  $x \in Z$ . It is straightforward to check that  $|f(x) - g(x)| \leq \frac{2}{3}C$  for all  $x \in A$ . □

*Proof of Theorem 11.2.* Since  $f$  takes values in an interval  $[a, b]$ , we can find a number  $C > 0$  such that  $|f(x)| \leq C$  for all  $x \in A$ . For  $n = 1, 2, \dots$  we will construct continuous functions  $g_n: X \rightarrow \mathbb{R}$  such that

- (i)  $|g_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \cdot C$  for all  $x \in X$ ;
- (ii)  $|f(x) - \sum_{i=1}^n g_i(x)| \leq \left(\frac{2}{3}\right)^n \cdot C$  for all  $x \in A$ .

We argue by induction. Existence of  $g_1$  follows directly from Lemma 11.8. Assume that for some  $n \geq 1$  we already have functions  $g_1, \dots, g_n$  satisfying (i) and (ii). Apply Lemma 11.8 to the function  $f - \sum_{k=1}^n g_k$ . We can take  $g_{n+1} := g$  where  $g$  is the function given by the lemma.

Let  $\tilde{f}_n := \sum_{k=1}^n g_k$  and let  $\tilde{f}_\infty := \sum_{k=1}^\infty g_k$ . Using condition (i) we obtain that the sequence  $\{\tilde{f}_n\}$  converges uniformly to  $\tilde{f}$  (exercise). Since each of the functions  $\tilde{f}_n$  is continuous, by Proposition 11.7 we obtain that  $\tilde{f}_\infty$  is a continuous function. Also, using (ii) we obtain that  $\tilde{f}_\infty(x) = f(x)$  for all  $x \in A$  (exercise).

The only remaining issue is that the function  $\tilde{f}_\infty$  takes its values in  $\mathbb{R}$ , and not in the interval  $[a, b]$ . However, it is not difficult to modify it to obtain a continuous function  $\tilde{f}: X \rightarrow [a, b]$  such that  $\tilde{f}(x) = \tilde{f}_\infty(x)$  for all  $x \in A$  (exercise). □

Here is another useful reformulation of Tietze Extension Theorem:

**11.9 Tietze Extension Theorem (v.2).** *Let  $X$  be a normal space, let  $A \subseteq X$  be a closed subspace, and let  $f: A \rightarrow \mathbb{R}$  be a continuous function. There exists a continuous function  $\tilde{f}: X \rightarrow \mathbb{R}$  such that  $\tilde{f}|_A = f$ .*

*Proof.* It is enough to show that for any continuous function  $g: A \rightarrow (-1, 1)$  we can find a continuous function  $\bar{g}: X \rightarrow (-1, 1)$  such that  $\bar{g}|_A = g$ . Indeed, if this holds then given a function  $f: A \rightarrow \mathbb{R}$  let  $g = hf$  where  $h: \mathbb{R} \rightarrow (-1, 1)$  is an arbitrary homeomorphism. Then we can take  $\tilde{f} = h^{-1}\bar{g}$ .

Assume then that  $g: A \rightarrow (-1, 1)$  is a continuous function. By Theorem 11.2 there is a function  $g_1: X \rightarrow [-1, 1]$  such that  $g_1|_A = g$ . Let  $B := g_1^{-1}(\{-1, 1\})$ . The set  $B$  is closed in  $X$  and  $A \cap B = \emptyset$  since  $g_1(A) = g(A) \subseteq (-1, 1)$ . By Urysohn Lemma 10.1 there is a continuous function  $k: X \rightarrow [0, 1]$  such that  $B \subseteq k^{-1}(\{0\})$  and  $A \subseteq k^{-1}(\{1\})$ . Let  $\bar{g}(x) := k(x) \cdot g_1(x)$ . We have:

- 1) if  $g_1(x) \in (-1, 1)$  then  $\bar{g}(x) \in (-1, 1)$
- 2) if  $g_1(x) \in \{-1, 1\}$  then  $x \in B$  so  $\bar{g}(x) = 0 \cdot g_1(x) = 0$

It follows that  $\bar{g}: X \rightarrow (-1, 1)$ . Also,  $\bar{g}$  is a continuous function since  $k$  and  $g_1$  are continuous. Finally, if  $x \in A$  then  $\bar{g}(x) = 1 \cdot g_1(x) = g(x)$ , so  $\bar{g}|_A = g$ .  $\square$

Tietze Extension Theorem holds for functions defined on normal spaces. It turns out the function extension property is actually equivalent to the notion of normality of a space:

**11.10 Theorem.** *Let  $X$  be a space satisfying  $T_1$ . The following conditions are equivalent:*

- 1)  $X$  is a normal space.
- 2) For any closed sets  $A, B \subseteq X$  such that  $A \cap B = \emptyset$  there is a continuous function  $f: X \rightarrow [0, 1]$  such that  $A \subseteq f^{-1}(\{0\})$  and  $B \subseteq f^{-1}(\{1\})$ .
- 3) If  $A \subseteq X$  is a closed set then any continuous function  $f: A \rightarrow \mathbb{R}$  can be extended to a continuous function  $\tilde{f}: X \rightarrow \mathbb{R}$ .

*Proof.* The implication 1)  $\Rightarrow$  2) is the Urysohn Lemma 10.1 and 2)  $\Rightarrow$  1) is Proposition 9.15. The implication 1)  $\Rightarrow$  3) is the Tietze Extension Theorem 11.9. The proof of implication 3)  $\Rightarrow$  1) is an exercise.  $\square$

## Exercises to Chapter 11

**E11.1 Exercise.** The goal of this exercise is to fill a gap in the proof of Theorem 11.2. For a topological space  $X$  and  $A \subseteq X$  let  $f: A \rightarrow [a, b]$  and  $\tilde{f}: X \rightarrow \mathbb{R}$  be continuous functions satisfying  $\tilde{f}(x) = f(x)$  for all  $x \in A$ . Show that there exists a continuous function  $\tilde{f}': X \rightarrow [a, b]$  such that  $\tilde{f}'(x) = f(x)$  for all  $x \in A$ .

**E11.2 Exercise.** Prove implication 3)  $\Rightarrow$  1) of Theorem 11.10.

**E11.3 Exercise.** Let  $X$  be a normal space, let  $A \subseteq X$  be a closed subspace, and let  $f: A \rightarrow \mathbb{R}$  be a continuous function.

a) Assume that  $g: X \rightarrow \mathbb{R}$  is a continuous function such that  $f(x) \leq g(x)$  for all  $x \in A$ . Show that there exists a continuous function  $F: X \rightarrow \mathbb{R}$  satisfying  $F|_A = f$  and  $F(x) \leq g(x)$  for all  $x \in X$ .

b) Assume that  $g, h: X \rightarrow \mathbb{R}$  are a continuous function such that  $h(x) \leq f(x) \leq g(x)$  for all  $x \in A$  and  $h(x) \leq g(x)$  for all  $x \in X$ . Show that there exists a continuous function  $F': X \rightarrow \mathbb{R}$  satisfying  $F'|_A = f$  and  $h(x) \leq F'(x) \leq g(x)$  for all  $x \in X$ .

**E11.4 Exercise.** Recall that if  $X$  is a topological space then a subspace  $Y \subseteq X$  is called a retract of  $X$  if there exists a continuous function  $r: X \rightarrow Y$  such that  $r(x) = x$  for all  $x \in Y$ . Let  $X$  be a normal space and let  $Y \subseteq X$  be a closed subspace of  $X$  such that  $Y \cong \mathbb{R}$ . Show that  $Y$  is a retract of  $X$ .

**E11.5 Exercise.** Let  $X$  be topological space. Recall from Exercise 10.3 that a set  $A \subseteq X$  is a  $G_\delta$ -set if there exists a countable family of open sets  $U_1, U_2, \dots$  such that  $A = \bigcap_{n=1}^{\infty} U_n$ .

a) Show that if  $X$  is a normal space and  $A \subseteq X$  is a closed  $G_\delta$ -set then there exists a continuous function  $f: X \rightarrow [0, 1]$  such that  $A = f^{-1}(\{0\})$ .

b) Show that if  $X$  is a normal space and  $A, B \subseteq X$  are closed  $G_\delta$ -sets such that  $A \cap B = \emptyset$  then there exists a continuous function  $f: X \rightarrow [0, 1]$  such that  $A = f^{-1}(\{0\})$  and  $B = f^{-1}(\{1\})$ .