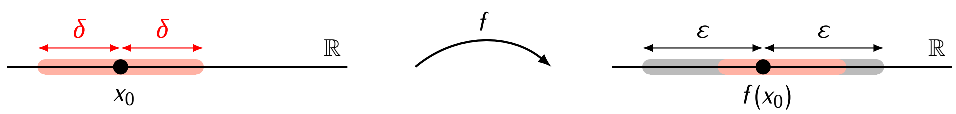
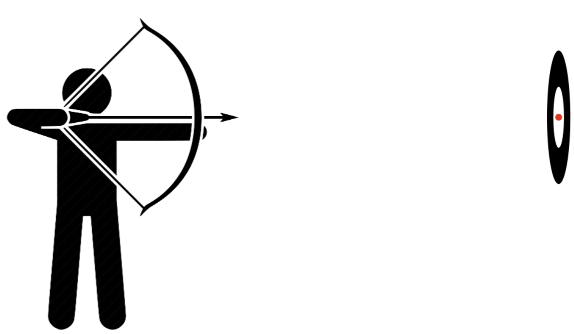
2 | Metric Spaces

Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $x_0 \in \mathbb{R}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x_0 - x| < \delta$ then $|f(x_0) - f(x)| < \varepsilon$:



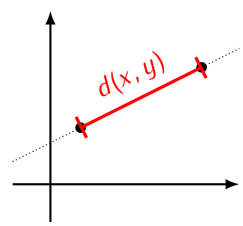
A function is *continuous* if it is continuous at every point $x_0 \in \mathbb{R}$.



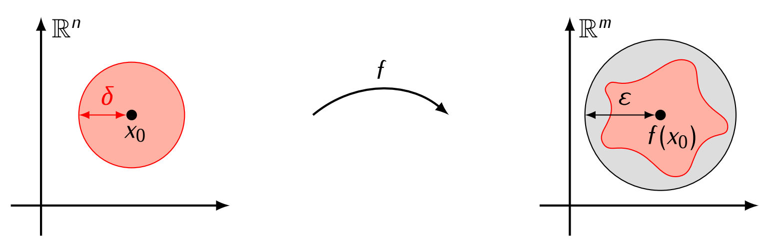
Continuity of functions of several variables $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is defined in a similar way. Recall that $\mathbb{R}^n := \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are two points in $\mathbb{R}^n$ then the distance between x and y is given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

The number $d(x, y)$ is the length of the straight line segment joining the points x and y:

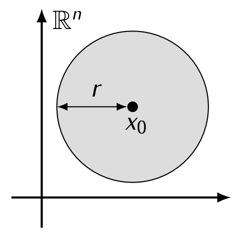


2.1 Definition. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *continuous at* $x_0 \in \mathbb{R}^n$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x_0, x) < \delta$ then $d(f(x_0), f(x)) < \varepsilon$.

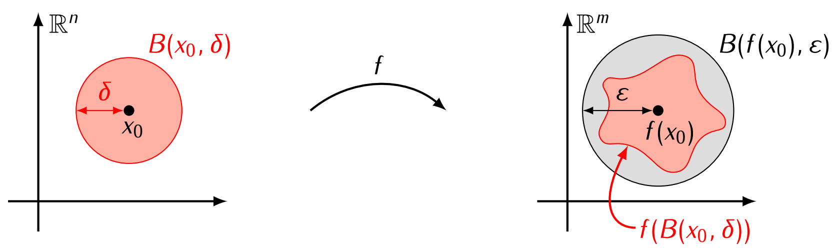


2.2 Definition. Let $x_0 \in \mathbb{R}^n$ and let $r > 0$. An *open ball* with radius r and with center at x_0 is the set

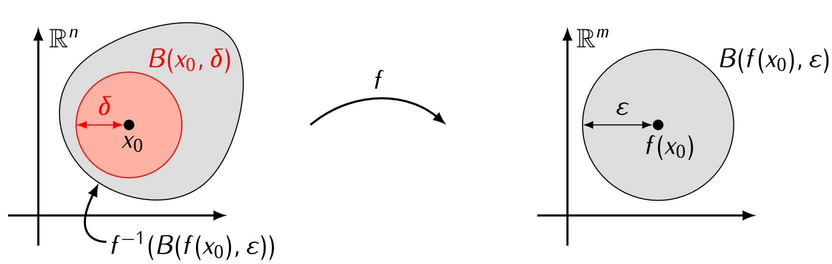
$$B(x_0, r) = \{x \in \mathbb{R}^n \mid d(x_0, x) < r\}$$



Using this terminology we can say that a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x_0 if for each $\varepsilon > 0$ there is a $\delta > 0$ such $f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$:



Here is one more way of rephrasing the definition of continuity: $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$:



2.3 Definition. A *metric space* is a pair (X, ϱ) where X is a set and ϱ is a function

$$\varrho\colon X \times X \to \mathbb{R}$$

that satisfies the following conditions:

1) $\varrho(x, y) \geq 0$ and $\varrho(x, y) = 0$ if and only if $x = y$;
2) $\varrho(x, y) = \varrho(y, x)$;
3) for any $x, y, z \in X$ we have $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$.

The function ϱ is called a *metric* on the set X. For $x, y \in X$ the number $\varrho(x, y)$ is called the *distance* between x and y.

2.4 Definition. Let (X, ϱ) and (Y, μ) be metric spaces. A function $f\colon X \to Y$ is *continuous at* $x_0 \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varrho(x_0, x) < \delta$ then $\mu(f(x_0), f(x)) < \epsilon$.

A function $f\colon X \to Y$ is *continuous* if it is continuous at every point $x_0 \in X$.

2.5 Definition. Let (X, ϱ) be a metric space. For $x_0 \in X$ and let $r > 0$ the *open ball* with radius r and with center at x_0 is the set

$$B_\varrho(x_0, r) = \{x \in X \mid \varrho(x_0, x) < r\}$$

Notice that a function $f\colon X \to Y$ between metric spaces (X, ϱ) and (Y, μ) is continuous at $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $B_\varrho(x_0, \delta) \subseteq f^{-1}(B_\mu(f(x_0), \varepsilon))$.

2.6 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

The metric d is called the *Euclidean metric* on $\mathbb{R}^n$.

2.7 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define:

$$\varrho_{ort}(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$$

The metric ϱ_{ort} is called the *orthogonal metric* on $\mathbb{R}^n$.

2.8 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define:

$$\varrho_{max}(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

The metric ϱ_{max} is called the *maximum metric* on $\mathbb{R}^n$.

2.9 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define $\varrho_h(x, y)$ as follows. If $x = y$ then $\varrho_h(x, y) = 0$. If $x \neq y$ then

$$\varrho_h(x, y) = \sqrt{x_1^2 + \cdots + x_n^2} + \sqrt{y_1^2 + \cdots + y_n^2}$$

The metric ϱ_h is called the *hub metric* on $\mathbb{R}^n$.

2.10 Example. Let X be any set. Define a metric ϱ_{disc} on X by

$$\varrho_{disc}(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The metric ϱ_{disc} is called the *discrete metric* on X.

2.11 Example. If (X, ϱ) is a metric space and $A \subseteq X$ then A is a metric space with the metric induced from X.