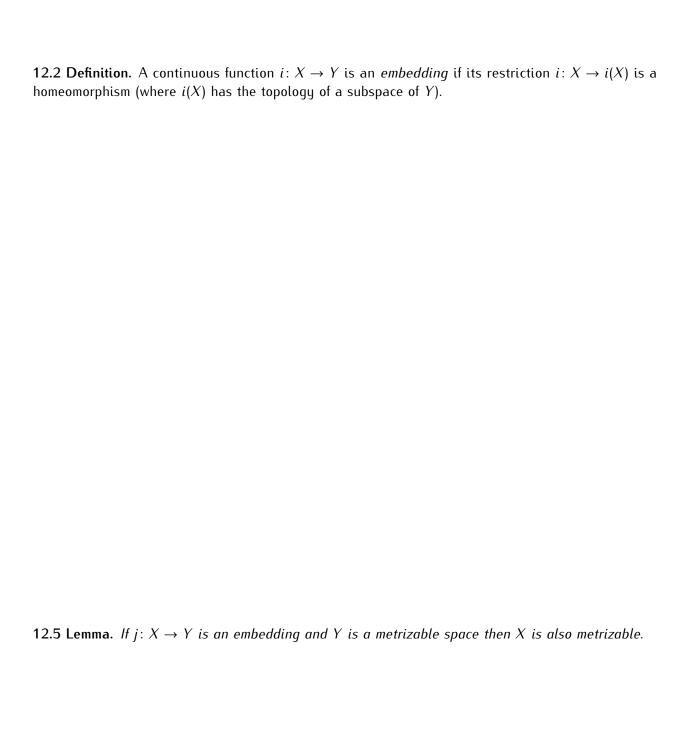
12 Urysohn Metrization Theorem

12.1 Urysohn Metrization Theorem. Every second countable normal space is metrizable.



12.6 Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The *product topology* on $\prod_{i\in I} X_i$ is the topology generated by the basis

 $\mathcal{B} = \left\{ \prod_{i \in I} U_i \mid U_i \text{ is open in } X_i \text{ and } U_i \neq X_i \text{ for finitely many indices } i \text{ only} \right\}$

12.8 Proposition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and for $j\in I$ let

$$p_j \colon \prod_{i \in I} X_i \to X_j$$

be the projection onto the j-th factor: $p_j((x_i)_{i \in I}) = x_j$. Then:

- 1) for any $j \in I$ the function p_j is continuous.
- 2) A function $f: Y \to \prod_{i \in I} X_i$ is continuous if and only if the composition $p_j f: Y \to X_j$ is continuous for all $j \in I$

Proof. Exercise.

12.10 Proposition. If X_1, \ldots, X_n are metrizable spaces then $X_1 \times \cdots \times X_n$ is also a metrizable space.

12.11 Proposition. If $\{X_i\}_{i=1}^{\infty}$ is an infinite countable family of metrizable spaces then $\prod_{i=1}^{\infty} X_i$ is also a metrizable space.

12.12 Example. The *Hilbert cube* is the topological space $[0,1]^{\aleph_0}$ obtained as the infinite countable product of the closed interval [0,1]:

$$[0,1]^{\aleph_0} = \prod_{i=1}^{\infty} [0,1]$$

Elements of $[0,1]^{\aleph_0}$ are infinite sequences $(t_i)=(t_1,t_2,\ldots)$ where $t_i\in[0,1]$ for $i=1,2,\ldots$ The Hilbert cube is a metric space with a metric ϱ given by

$$\varrho((t_i), (s_i)) = \sum_{i=1}^{\infty} \frac{1}{2^i} |t_i - s_i|$$

12.13 Definition. Let X be a topological space and let $\{f_i\}_{i\in I}$ be a family of continuous functions $f_i\colon X\to [0,1]$. We say that the family $\{f_i\}_{i\in I}$ separates points from closed sets if for any point $x_0\in X$ and any closed set $A\subseteq X$ such that $x_0\notin A$ there is a function $f_j\in \{f_i\}_{i\in I}$ such that $f_j(x_0)>0$ and $f_j|_A=0$.

12.14 Embedding Lemma. Let X be a T_1 -space. If $\{f_i \colon X \to [0,1]\}_{i \in I}$ is a family that separates points from closed sets then the map

$$f_{\infty} \colon X \to \prod_{i \in I} [0, 1]$$

given by $f_{\infty}(x) = (f_i(x))_{i \in I}$ is an embedding.

Proof of Theorem 12.1.

| 12.16 Proposition. Every second countable regular space is normal. |
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| Proof. Exercise. |
| 12.17 Urysohn Metrization Theorem (v.2). Every second countable regular space is metrizable. |
| 12.18 Definition. Let X be a topological space. A collection $\mathcal{U} = \{U_i\}_{i \in I}$ of open sets in X is <i>locally finite</i> if each point $x \in X$ has an open neighborhood V_X such that $V_X \cap U_i \neq \emptyset$ for finitely many $i \in \mathcal{U}$ only. |
| A collection $\mathcal U$ is countably locally finite if it can be decomposed into a countable union $\mathcal U=\bigcup_{n=1}^\infty \mathcal U_n$ where each collection $\mathcal U_n$ is locally finite. |
| 12.19 Nagata-Smirnov Metrization Theorem. Let X be a topological space. The following conditions are equivalent: 1) X is metrizable. |
| 2) X is regular and it has a basis which is countably locally finite. |