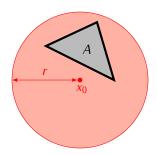
## 15 | Heine-Borel Theorem

**15.1 Definition.** Let  $(X, \varrho)$  be a metric space. A set  $A \subseteq X$  is *bounded* if there exists an open ball  $B(x_0, r) \subseteq X$  such that  $A \subseteq B(x_0, r)$ .



**15.2 Proposition.** Let  $(X, \varrho)$  be a metric space and let  $A \subseteq X$ . The following conditions are equivalent:

- 1) A is bounded.
- 2) For each  $x \in X$  there exists  $r_x > 0$  such that  $A \subseteq B(x, r_x)$ .
- 3) There exists R > 0 such that  $\varrho(x_1, x_2) < R$  for all  $x_1, x_2 \in A$ .

*Proof.* Exercise.

**15.3 Heine-Borel Theorem.** A set  $A \subseteq \mathbb{R}^n$  is compact if and only if A is closed and bounded.

**15.5 Theorem.** If X, Y are compact spaces then the space  $X \times Y$  is also compact.

**15.6 Corollary.** If  $X_1, \ldots, X_n$  are compact spaces spaces then the space  $X_1 \times \cdots \times X_n$  is compact.

**15.7 Corollary.** For  $i=1,\ldots,n$  let  $[a_i,b_i]\subseteq\mathbb{R}$  be a closed interval. The closed box

$$[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$$

is compact.

*Proof of Theorem 15.3.* ( $\Rightarrow$ ) Exercise.

(⇐)