20 | Simplicial Complexes

20.1 Definition. A *simplicial complex* $K = (V, S)$ consists of a set V together with a set S of finite, non-empty subsets of V such that the following conditions are satisfied:

1) For each $v \in V$ the set $\{v\}$ is in S.
2) If $\sigma \in S$ and $\varnothing \neq \tau \subseteq \sigma$ then $\tau \in S$.

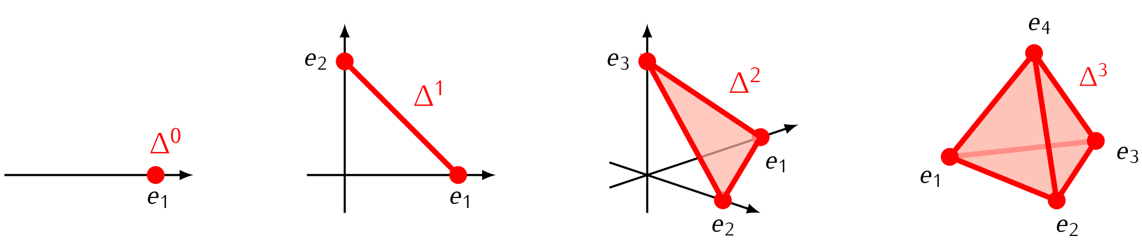
20.2 Notation. If $K = (V, S)$ is a simplicial complex then:

- Elements of V are called *vertices* of K.
- Elements of S are called *simplices* of K.
- If a simplex $\sigma \in S$ consists of $n + 1$ elements then we say that σ is an *n-simplex*.
- If $\sigma \in S$ and $\tau \subseteq \sigma$ then we say that τ is a *face* of σ. If $\tau \neq \sigma$ then τ is a *proper face* of σ. The inclusion $j^{\sigma}_{\tau} \colon \tau \to \sigma$ is called a *face map*.
- We say that K is a simplicial complex of dimension n if K has n-simplices, but it does not have m-simplices for $m > n$. We write: $\dim K = n$. If K has simplices in all dimensions then $\dim K = \infty$.
- We say that K is a finite simplicial complex if K consists of finitely many simplices.

20.6 Definition. If $K = (V, S)$ is a simplicial complex, then a *subcomplex* of K is a simplicial complex $L = (V', S')$ such that $V' \subseteq V$ and $S' \subseteq S$. In such case we write $L \subseteq K$.

20.8 Definition. Let $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots,$ $e_{n+1} = (0, 0, 0, \dots, 1)$ be the standard basis vectors in $\mathbb{R}^{n+1}$. The *standard geometric n-simplex* is a subspace $\Delta^n \subseteq \mathbb{R}^{n+1}$ given by

$$\Delta^n = \left\{ \sum_{i=1}^{n+1} t_i e_i \in \mathbb{R}^{n+1} \mid t_i \in [0, 1],\ \textstyle\sum_{i=0}^{n} t_i = 1 \right\}$$



20.9 Definition. Let A be a finite set. The *geometric A-simplex* is a metric space (Δ^A, ϱ), such that elements of Δ^A are formal sums $\sum_{a\in A} t_a a$ where $t_a \in [0,1]$ for each $a \in A$, and $\sum_{a\in A} t_a = 1$. If $x = \sum_{a\in A} t_a a$ and $y = \sum_{a\in A} t'_a a$ then

$$\varrho(x,y) = \sqrt{\textstyle\sum_{a\in A}(t_a - t'_a)^2}$$

20.10 Proposition. *If A is a set consisting of $n+1$ elements then Δ^A is homeomorphic to the standard n-simplex Δ^n.*

Proof. Exercise. $\square$

20.11 Definition. Let K be a simplicial complex. The *geometric realization* of K is the topological space $|K|$ defined by:

$$|K| = \coprod_{\sigma \in K} \Delta^{\sigma} \Big/ \sim$$

where the equivalence relation $\sim$ is given by $x \sim \Delta(j^{\sigma}_{\tau})(x)$ for each face map $j^{\sigma}_{\tau} \colon \tau \to \sigma$ and $x \in \Delta^{\tau}$.

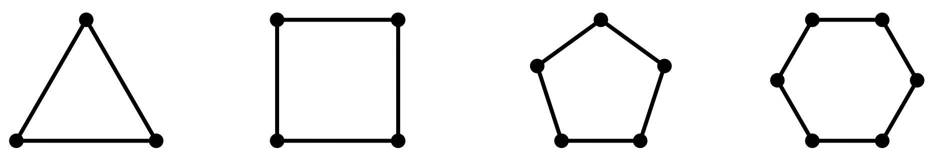
20.13 Proposition. *If L is a subcomplex of a simplicial complex K, then $|L|$ is a closed subspace of $|K|$.*

Proof. Exercise. $\square$

20.14 Definition. Let K be a finite simplicial complex. For $n = 0, 1, 2, \dots$ let $s_n(K)$ denote the number of n-simplices of K. The *Euler characteristic* of K is the integer

$$\chi(K) = \sum_{n=0}^{\infty} (-1)^n s_n(K)$$

20.15 Theorem. *If K, L are finite simplicial complexes such that $|K|$ is homeomorphic to $|L|$ then $\chi(K) = \chi(L)$.*



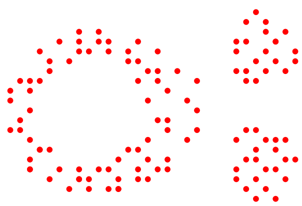
$|K_3|$ $|K_4|$ $|K_5|$ $|K_6|$

20.17 Definition. If X is a topological space such that $X \cong |K|$ for some finite simplicial complex K then we define the Euler characteristic $\chi(X)$ of X as the Euler characteristic $\chi(K)$ of K.

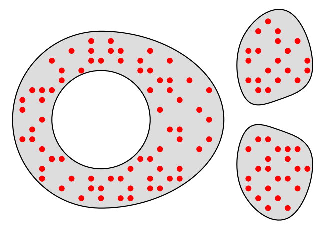
20.18 Proposition. *The Euler characteristic is a topological invariant: if X, Y are spaces such that $X \cong Y$ and $\chi(X)$ is defined, then $\chi(Y)$ is defined and $\chi(Y) = \chi(X)$.*

20.19 Example. We will use the Euler characteristic to show that the 2-dimensional sphere S^2 is not homeomorphic to the torus $T = S^1 \times S^1$.

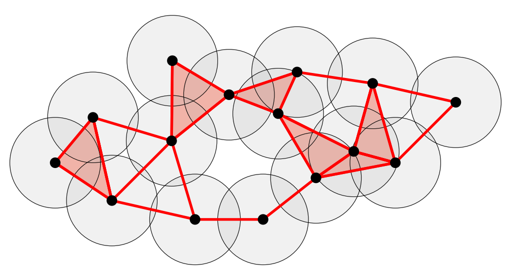
Topological data analysis.



a set of data points



data points and the hypothetical underlying space X



20.21 Theorem. *If K is a simplicial complex then the geometric realization $|K|$ is a normal space.*

20.22 Definition. The *n-skeleton* of a simplicial complex K is a subcomplex $K^{(n)} \subseteq K$ given as follows:

- vertices of $K^{(n)}$ are the same as vertices of K;
- m-simplices of $K^{(n)}$ are the same as m-simplices of K for any $m \leq n$;
- $K^{(n)}$ has no m-simplices for $m > n$.

20.23 Proposition. *Let K be a simplicial complex, and let X be a topological space. A function $f\colon |K| \to X$ is continuous if and only if $f|_{|K^{(n)}|}\colon |K^{(n)}| \to X$ is continuous for each $n = 0, 1, \ldots$.*

Proof. Exercise. □

20.24 Lemma. *Let K be a simplicial complex, and let $f_n\colon |K^{(n)}| \to X$ be a continuous function. Assume that for each $\sigma \in S_{n+1}$ we have a continuous function $f_\sigma\colon |\overline{\sigma}| \to X$ such that $f_\sigma|_{|\partial\sigma|} = f_n|_{|\partial\sigma|}$. Then f_n extends to a function $f_{n+1}\colon |K^{(n+1)}| \to X$ such that $f_{n+1}|_{|\overline{\sigma}|} = f_\sigma$.*

Proof. Exercise. □

Proof of Theorem 20.21.

□