

6 | Continuous Functions

Let X, Y be topological spaces. Recall that a function $f: X \rightarrow Y$ is continuous if for every open set $U \subseteq Y$ the set $f^{-1}(U) \subseteq X$ is open. In this chapter we study some properties of continuous functions. We also introduce the notion of a *homeomorphism* that plays a central role in topology: from the topological perspective interesting properties of spaces are the properties that are preserved by homeomorphisms.

6.1 Proposition. *Let X, Y be topological spaces. A function $f: X \rightarrow Y$ is continuous if and only if for every closed set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is closed.*

Proof. Assume that $f: X \rightarrow Y$ is a continuous function and let $A \subseteq Y$ be a closed set. We have

$$f^{-1}(A) = X \setminus f^{-1}(Y \setminus A)$$

The set $Y \setminus A$ is open in Y so by continuity of f the set $f^{-1}(Y \setminus A) \subseteq X$ is open in X . It follows that $f^{-1}(A)$ is closed in X .

Conversely, assume that $f: X \rightarrow Y$ is a function such that for every closed set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is closed. Let $U \subseteq Y$ be an open set. We have

$$f^{-1}(U) = X \setminus f^{-1}(Y \setminus U)$$

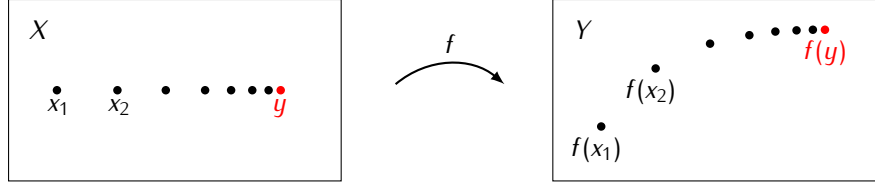
The set $Y \setminus U$ is closed in Y so by assumption the set $f^{-1}(Y \setminus U)$ is closed in X . It follows that $f^{-1}(U)$ is open in X . Therefore f is a continuous function. \square

For metric spaces continuous functions are precisely the functions that preserve convergence of sequences:

6.2 Proposition. *Let (X, ρ) be a metric space, let Y be a topological space, and let $f: X \rightarrow Y$ be a function. The following conditions are equivalent:*

- 1) f is continuous.

2) For any sequence $\{x_n\} \subseteq X$ if $x_n \rightarrow y$ for some $y \in X$ then $f(x_n) \rightarrow f(y)$.



Proof. 1) \Rightarrow 2) Exercise.

2) \Rightarrow 1) Let $A \subseteq Y$ be a closed set. We will show that the set $f^{-1}(A)$ is closed in X . By Proposition 5.8 it suffices to show that if $\{x_n\} \subseteq f^{-1}(A)$ is a sequence and $x_n \rightarrow x$ then $x \in f^{-1}(A)$.

If $x_n \rightarrow x$ then by assumption we have $f(x_n) \rightarrow f(x)$. Since $\{f(x_n)\} \subseteq A$ and A is a closed set, thus by Proposition 5.8 we obtain that $f(x) \in A$, and so $x \in f^{-1}(A)$. \square

The implication 1) \Rightarrow 2) in Proposition 6.2 holds for maps between general topological spaces:

6.3 Proposition. Let $f: X \rightarrow Y$ be a continuous function of topological spaces. If $\{x_n\} \subseteq X$ is a sequence and $x_n \rightarrow x$ for some $x \in X$ then $f(x_n) \rightarrow f(x)$.

Proof. Exercise. \square

6.4 Example. We will show that the implication 2) \Rightarrow 1) in Proposition 6.2 is not true if X is a general topological space. Let X be the space defined in Example 5.16: $X = \mathbb{R}$ with the topology

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \emptyset \text{ or } U = (\mathbb{R} \setminus S) \text{ for some countable set } S \subseteq \mathbb{R}\}$$

Recall that if $\{x_n\}$ is a sequence in X then $x_n \rightarrow x$ if and only if there exists $N > 0$ such that $x_n = x$ for all $n > N$. Let $f: X \rightarrow X$ be a function given by

$$f(x) = \begin{cases} 0 & \text{if } x \in (0, 1) \\ 1 & \text{if } x \notin (0, 1) \end{cases}$$

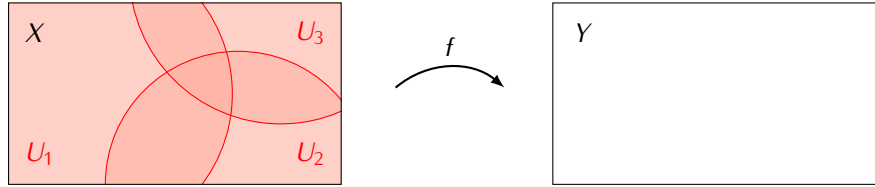
This function is not continuous since the set $\{0\}$ is closed in X and the set $(0, 1) = f^{-1}(\{0\})$ is not closed in X . On the other hand let $\{x_n\} \subseteq X$ be a sequence and let $x_n \rightarrow x$. There is $N > 0$ such that $x_n = x$ for $n > N$, so $f(x_n) = f(x)$ for all $n > N$ and so $f(x_n) \rightarrow f(x)$.

6.5 Proposition. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are continuous functions then the function $gf: X \rightarrow Z$ is also continuous.

Proof. Exercise. \square

Frequently functions $f: X \rightarrow Y$ are constructed by gluing together several functions defined on subspaces of X . The next two facts are useful for verifying that functions obtained in this way are continuous.

6.6 Open Pasting Lemma. *Let X, Y be topological spaces and let $\{U_i\}_{i \in I}$ be a family of open sets in X such that $\bigcup_{i \in I} U_i = X$. Assume that for $i \in I$ we have a continuous function $f_i: U_i \rightarrow Y$ such that $f_i(x) = f_j(x)$ if $x \in U_i \cap U_j$. Then the function $f: X \rightarrow Y$ given by $f(x) = f_i(x)$ for $x \in U_i$ is continuous.*



Proof. Let $V \subseteq Y$ be an open set. We will show that the set $f^{-1}(V) \subseteq X$ is open. Since $\bigcup_{i \in I} U_i = X$ we have

$$f^{-1}(V) = \bigcup_{i \in I} f^{-1}(V) \cap U_i = \bigcup_{i \in I} f_i^{-1}(V)$$

Since $f_i: U_i \rightarrow Y$ is a continuous function the set $f_i^{-1}(V)$ is open in U_i . Also, since U_i is open in X by Exercise 5.8 we obtain that the set $f_i^{-1}(V)$ is open in X . Thus $f^{-1}(V)$ is an open set. \square

6.7 Closed Pasting Lemma. *Let X, Y be topological spaces and let $A_1, \dots, A_n \subseteq X$ be a finite family of closed sets such that $\bigcup_{i=1}^n A_i = X$. Assume that for $i = 1, 2, \dots, n$ we have a continuous function $f_i: A_i \rightarrow Y$ such that $f_i(x) = f_j(x)$ if $x \in A_i \cap A_j$. Then the function $f: X \rightarrow Y$ given by $f(x) = f_i(x)$ for $x \in A_i$ is continuous.*

Proof. Exercise. \square

6.8 Example. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the absolute value function, $f(x) = |x|$. On the set $A_1 = (-\infty, 0]$ this function is given by $f|_{A_1}(x) = -x$, and on $A_2 = [0, +\infty)$ it is given by $f|_{A_2}(x) = x$. Since both $f|_{A_1}$ and $f|_{A_2}$ are continuous functions and A_1, A_2 are closed sets in \mathbb{R} by the Closed Pasting Lemma 6.7 we obtain that $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

6.9 Definition. A *homeomorphism* is a continuous function $f: X \rightarrow Y$ such that f is a bijection and the inverse function $f^{-1}: Y \rightarrow X$ is continuous.

6.10 Proposition. 1) *For any topological space the identify function $\text{id}_X: X \rightarrow X$ given by $\text{id}_X(x) = x$ is a homeomorphism.*

2) *If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are homeomorphisms then the function $gf: X \rightarrow Z$ is also a homeomorphism.*

3) If $f: X \rightarrow Y$ is a homeomorphism then the inverse function $f^{-1}: Y \rightarrow X$ is also a homeomorphism.

4) If $f: X \rightarrow Y$ is a homeomorphism and $Z \subseteq X$ then the function $f|_Z: Z \rightarrow f(Z)$ is also a homeomorphism.

Proof. Exercise. □

6.11 Note. If $f: X \rightarrow Y$ is a continuous bijection then f need not be a homeomorphism since the inverse function f^{-1} may be not continuous. For example, let $X = \{x_1, x_2\}$ be a space with the discrete topology and let $Y = \{y_1, y_2\}$ be a space with the antidiscrete topology. Let $f: X \rightarrow Y$ be given by $f(x_i) = y_i$. The function f is continuous but f^{-1} is not continuous since the set $\{x_1\}$ is open in X , but the set $(f^{-1})^{-1}(\{x_1\}) = \{y_1\}$ is not open in Y .

6.12 Proposition. Let $f: X \rightarrow Y$ be a continuous bijection. The following conditions are equivalent:

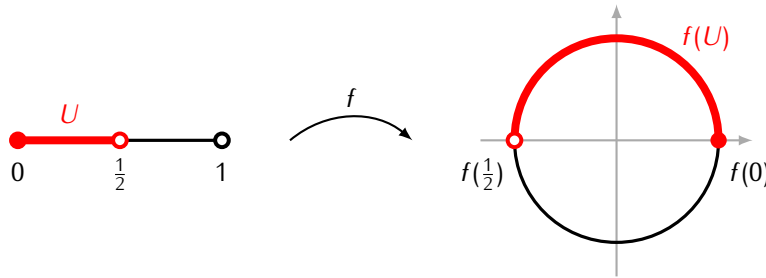
- (i) The function f is a homeomorphism.
- (ii) For each open set $U \subseteq X$ the set $f(U) \subseteq Y$ is open.
- (iii) For each closed set $A \subseteq X$ the set $f(A) \subseteq Y$ is closed.

Proof. Exercise. □

6.13 Example. Recall that S^1 denotes the unit circle:

$$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

The function $f: [0, 1) \rightarrow S^1$ given by $f(x) = (\cos 2\pi x, \sin 2\pi x)$ is a continuous bijection, but it is not a homeomorphism since the set $U = [0, \frac{1}{2})$ is open in $[0, 1)$, but $f(U)$ is not open in S^1 .



6.14 Definition. We say that topological spaces X, Y are *homeomorphic* if there exists a homeomorphism $f: X \rightarrow Y$. In such case we write: $X \cong Y$.

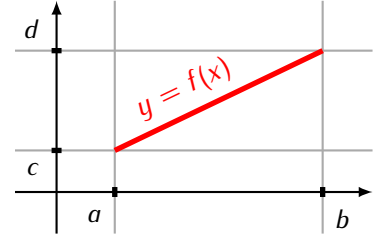
6.15 Note. Notice that if $X \cong Y$ and $Y \cong Z$ then $X \cong Z$.

6.16 Example. For any $a < b$ and $c < d$ the open intervals $(a, b), (c, d) \subseteq \mathbb{R}$ are homeomorphic. To see this take e.g. the function $f: (a, b) \rightarrow (c, d)$ defined by

$$f(x) = \left(\frac{c-d}{a-b} \right) x + \left(\frac{ad-bc}{a-b} \right)$$

This function is a continuous bijection. Its inverse function $f^{-1}: (c, d) \rightarrow (a, b)$ is given by

$$f^{-1}(x) = \left(\frac{a-b}{c-d} \right) x + \left(\frac{cb-da}{c-d} \right)$$



so it is also continuous. By the same argument for any $a < b$ and $c < d$ the closed intervals $[a, b], [c, d] \subseteq \mathbb{R}$ are homeomorphic.

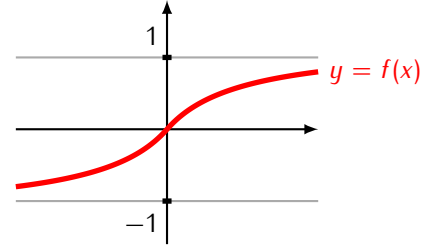
6.17 Note. In Chapter 7 we will show that an open interval (a, b) is not homeomorphic to a closed interval $[c, d]$.

6.18 Example. We will show that for any $a < b$ the open interval (a, b) is homeomorphic to \mathbb{R} . Since $(a, b) \cong (-1, 1)$ it will be enough to check that $\mathbb{R} \cong (-1, 1)$. Take the function $f: \mathbb{R} \rightarrow (-1, 1)$ given by

$$f(x) = \frac{x}{1 + |x|}$$

This function is a continuous bijection with the inverse function $f^{-1}: (-1, 1) \rightarrow \mathbb{R}$ is given by

$$f^{-1}(x) = \frac{x}{1 - |x|}$$



Since f^{-1} is continuous we obtain that f is a homeomorphism.

6.19 Note. If spaces X and Y are homeomorphic then usually there are many homeomorphisms $X \rightarrow Y$. For example, the function $g: (-1, 1) \rightarrow \mathbb{R}$ given by

$$g(x) = \tan\left(\frac{\pi}{2}x\right)$$

is another homeomorphism between the spaces $(-1, 1)$ and \mathbb{R} .

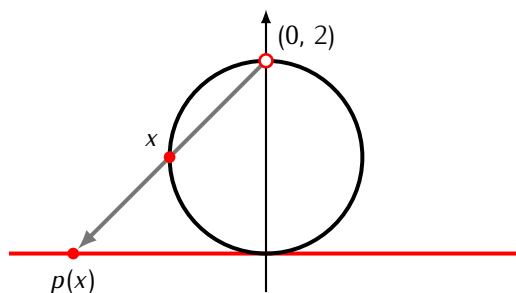
6.20 Example. We will show that for any point $x_0 \in S^1$ there is a homeomorphism $S^1 \setminus \{x_0\} \cong \mathbb{R}$. Denote by $S^1_{(0,1)} \subseteq \mathbb{R}^2$ the circle of radius 1 with the center at the point $(0, 1) \in \mathbb{R}^2$:

$$S^1_{(0,1)} := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 1)^2 = 1\}$$

It is easy to check that for $x_0 \in S^1$ the space $S^1 \setminus \{x_0\}$ is homeomorphic to the space $X = S^1_{(0,1)} \setminus \{(0, 2)\}$. Likewise, it is easy to check that \mathbb{R} is homeomorphic to the subspace $Y \subseteq \mathbb{R}^2$ that consists of all points of the x -axis:

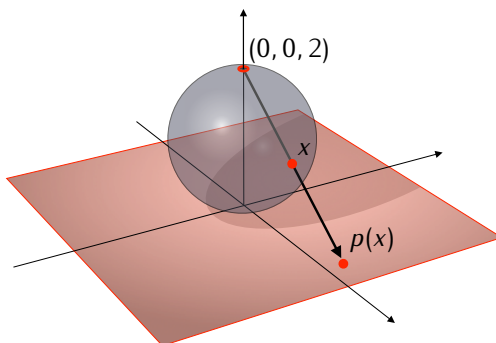
$$Y := \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\}$$

It is then enough to show that $X \cong Y$. A homeomorphism $p: X \rightarrow Y$ can be constructed as follows. For any point $x \in X$ there is a unique line in \mathbb{R}^2 that passes through x and through the point $(0, 2) \in \mathbb{R}^2$. We define $p(x)$ to be the point of intersection of this line with the x -axis:



The function p is called the *stereographic projection*.

In a similar way we can construct a stereographic projection in any dimension $n \geq 1$ that gives a homeomorphism between the space $S^n \setminus \{x_0\}$ (i.e. the n -dimensional sphere with one point deleted) and the space \mathbb{R}^n :



Exercises to Chapter 6

E6.1 Exercise. Consider the set of rational numbers \mathbb{Q} as a subspace of \mathbb{R} . Show that \mathbb{Q} is not homeomorphic to a space with the discrete topology.

E6.2 Exercise. Prove Proposition 6.3.

E6.3 Exercise. Prove Proposition 6.5.

E6.4 Exercise. Prove Lemma 6.7.

E6.5 Exercise. Prove Proposition 6.12.

E6.6 Exercise. Let X be a topological space and let $f, g: X \rightarrow \mathbb{R}$ be continuous functions.

a) Show that the set

$$A = \{x \in X \mid f(x) \geq g(x)\}$$

is closed in X .

b) Let $h_{\max}, h_{\min}: X \rightarrow \mathbb{R}$ be functions given by $h_{\max}(x) = \max\{f(x), g(x)\}$ and $h_{\min}(x) = \min\{f(x), g(x)\}$. Show that h_{\max} and h_{\min} are continuous functions.

E6.7 Exercise. Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions such that $f(x) > g(x)$ for all $x \in \mathbb{R}$. Define subspaces X, Y of \mathbb{R}^2 as follows.

$$X := \{(x, y) \in \mathbb{R}^2 \mid g(x) \leq y \leq f(x)\} \quad Y := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$$

Show that $X \cong Y$.

E6.8 Exercise. Let $x_0 = (0, 0) \in \mathbb{R}^2$ and let $\bar{B}(x_0, 1) \subseteq \mathbb{R}^2$ be a closed ball defined by the Euclidean metric d :

$$\bar{B}(x_0, 1) = \{x \in \mathbb{R}^2 \mid d(x, x_0) \leq 1\}$$

Define subspaces $X, Y \subseteq \mathbb{R}^2$ as follows:

$$X := \mathbb{R}^2 \setminus \{x_0\} \quad Y := \mathbb{R}^2 \setminus \bar{B}(x_0, 1)$$

Show that $X \cong Y$.

E6.9 Exercise. Let (X, ρ) be a metric space. A subspace $Y \subseteq X$ is a *retract* of X if there exists a continuous function $r: X \rightarrow Y$ such that $r(x) = x$ for all $x \in Y$. Show that if $Y \subseteq X$ is a retract of X then Y is a closed in X .