5 | Closed Sets, Interior, Closure, Boundary

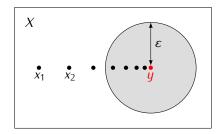
5.1 Definition. Let X be a topological space. A set $A \subseteq X$ is a *closed set* if the set $X \setminus A$ is open.

5.5 Proposition. Let X be a topological space.

- 1) The sets X, \varnothing are closed.
- 2) If $A_i \subseteq X$ is a closed set for $i \in I$ then $\bigcap_{i \in I} A_i$ is closed.
- 3) If A_1 , A_2 are closed sets then the set $A_1 \cup A_2$ is closed.

5.7 Definition. Let (X, ϱ) be a metric space, and let $\{x_n\}$ be a sequence of points in X. We say that $\{x_n\}$ converges to a point $y \in X$ if for every $\varepsilon > 0$ there exists N > 0 such that $\varrho(y, x_n) < \varepsilon$ for all n > N. We write: $x_n \to y$.

Equivalently: $x_n \to y$ if for every $\varepsilon > 0$ there exists N > 0 such that $x_n \in B(y, \varepsilon)$ for all n > N.



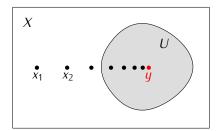
5.8 Proposition. Let (X, ϱ) be a metric space and let $A \subseteq X$. The following conditions are equivalent:

- 1) The set A is closed in X.
- 2) If $\{x_n\} \subseteq A$ and $x_n \to y$ then $y \in A$.

Proof. Exercise.

5.10 Definition. Let X be a topological space and $y \in X$. If $U \subseteq X$ is an open set such that $y \in U$ then we say that U is an *open neighborhood of* y.

5.11 Definition. Let X be a topological space. A sequence $\{x_n\} \subseteq X$ converges to $y \in X$ if for every open neighborhood U of y there exists N > 0 such that $x_n \in U$ for n > N.



5.12 Note. In general topological spaces a sequence may converge to many points at the same time.

5.13 Proposition. Let (X, ϱ) be a metric space and let $\{x_n\}$ be a sequence in X. If $x_n \to y$ and $x_n \to z$ for some $y, z \in X$ then y = z.

Proof. Exercise.

5.14 Proposition.	Let X	be a	topological	space	and le	<i>t A</i> ⊆	X	be a	closed	set.	If $\{x_n\}$	$\subseteq A$	ana
$x_n \to y$ then $y \in A$	4.												

Proof. Exercise. □

5.16 Example. Let $X = \mathbb{R}$ with the following topology:

$$\mathfrak{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \setminus S) \text{ for some countable set } S \subseteq \mathbb{R}\}$$

Closed sets in X are the whole space $\mathbb R$ and all countable subsets of $\mathbb R$. If $\{x_n\}\subseteq X$ is a sequence then $x_n\to y$ if and only if there exists N>0 such that $x_n=y$ for all n>N (exercise). It follows that if A is any (closed or not) subset of X, $\{x_n\}\subseteq A$, and $x_n\to y$ then $y\in A$.

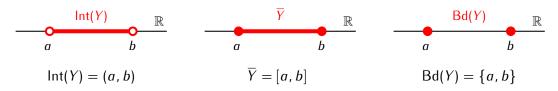
5.17 Definition. Let X be a topological space and let $Y \subseteq X$.

- The *interior of* Y is the set $Int(Y) := \bigcup \{U \mid U \subseteq Y \text{ and } U \text{ is open in } X\}.$
- The closure of Y is the set $\overline{Y} := \bigcap \{A \mid Y \subseteq A \text{ and } A \text{ is closed in } X\}.$
- The boundary of Y is the set $Bd(Y) := \overline{Y} \cap (\overline{X \setminus Y})$.

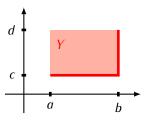
5.18 Example. Consider the set Y = (a, b] in \mathbb{R} :



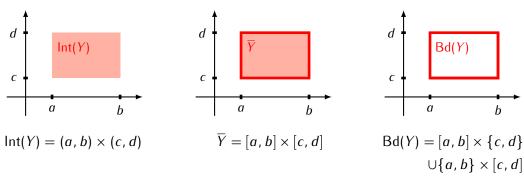
We have:



5.19 Example. Consider the set $Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid a < x_1 \le b, c \le x_2 < d\}$ in \mathbb{R}^2 :



We have:



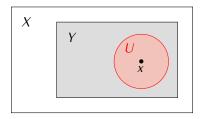
5.20 Proposition. Let X be a topological space and let $Y \subseteq X$.

- 1) The set Int(Y) is open in X. It is the biggest open set contained in Y: if U is open and $U \subseteq Y$ then $U \subseteq Int(Y)$.
- 2) The set \overline{Y} is closed in X. It is the smallest closed set that contains Y: if A is closed and $Y \subseteq A$ then $\overline{Y} \subseteq A$.

Proof. Exercise.

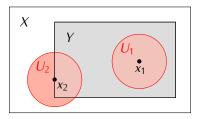
5.21 Proposition. Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:

- 1) $x \in Int(Y)$
- 2) There exists an open neighborhood U of x such that $U \subseteq Y$.



5.22 Proposition. Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:

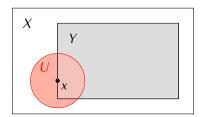
- 1) $x \in \overline{Y}$
- 2) For every open neighborhood U of x we have $U \cap Y \neq \emptyset$.



Proof. Exercise.

5.23 Proposition. Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:

- 1) $x \in Bd(Y)$
- 2) For every open neighborhood U of x we have $U \cap Y \neq \emptyset$ and $U \cap (X \setminus Y) \neq \emptyset$.



5.24 Definition. Let X be a topological space. A set $Y \subseteq X$ is *dense in* X if $\overline{Y} = X$.

5.25 Proposition. Let X be a topological space and let $Y \subseteq X$. The following conditions are equivalent:

- 1) Y is dense in X
- 2) If $U \subseteq X$ is an open set and $U \neq \emptyset$ then $U \cap Y \neq \emptyset$.

5.26 Example. The set of rational numbers $\mathbb Q$ is dense in $\mathbb R.$