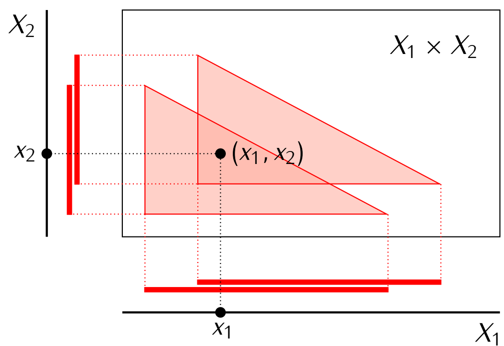
17 | Tychonoff Theorem

17.1 Tychonoff Theorem. *If $\{X_s\}_{s\in S}$ is a family of topological spaces and X_s is compact for each $s \in S$ then the product space $\prod_{s\in S} X_s$ is compact.*

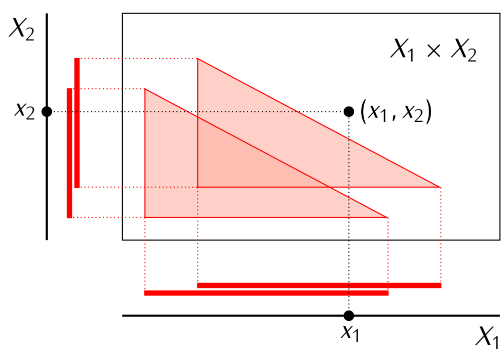
17.2 Definition. Let $\mathcal{A}$ be a family of subsets of a space X. The family $\mathcal{A}$ is *centered* if for any finite number of sets $A_1, \dots, A_n \in \mathcal{A}$ we have $A_1 \cap \dots \cap A_n \neq \varnothing$

17.5 Lemma. *Let X be a topological space. The following conditions are equivalent:*

1) *The space X is compact.*
2) *For any centered family $\mathcal{A}$ of closed subsets of X we have $\bigcap_{A\in\mathcal{A}} A \neq \varnothing$.*



X_2
$X_1 \times X_2$
x_2
(x_1, x_2)
x_1
X_1



X_2
$X_1 \times X_2$
x_2
(x_1, x_2)
x_1
X_1

17.6 Definition. A *partially ordered set* (or *poset*) is a set S equipped with a binary relation $\leq$ satisfying

(i) $x \leq x$ for all $x \in S$ (reflexivity)
(ii) if $x \leq y$ and $y \leq x$ then $y = x$ (antisymmetry)
(iii) if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).

17.8 Definition. A *linearly ordered set* is a poset $(S, \leq)$ such that for any $x, y \in S$ we have either $x \leq y$ or $y \leq x$.

17.9 Definition. If $(S, \leq)$ is a poset then an element $x \in S$ is a *maximal element* of S if we have $x \leq y$ only for $y = x$.

17.13 Definition. Let $(S, \leq)$ is a poset and let $T \subseteq S$. An *upper bound of* T is an element $x \in S$ such that $y \leq x$ for all $y \in T$.

17.14 Definition. If $(S, \leq)$ is a poset. A *chain* in S is a subset $T \subseteq S$ such that T is linearly ordered.

17.15 Zorn's Lemma. *If $(S, \leq)$ is a non-empty poset such that every chain in S has an upper bound in S then S contains a maximal element.*

Proof. See any book on set theory. $\square$

Proof of Theorem 17.1.

□

17.16 Proposition. *If X_i is a Hausdorff space for each $i \in I$ then the product space $\prod_{i \in I} X_i$ is also Hausdorff.*

Proof. Exercise. $\square$

17.17 Corollary. *If X_i is a compact Hausdorff space space for each $i \in I$ then the product space $\prod_{i \in I} X_i$ is also compact Hausdorff.*