22 | Mapping Spaces

22.1 Definition. Let X, Y be topological spaces. By $\mathrm{Map}(X, Y)$ we will denote the set of all continuous functions $f\colon X \to Y$.

Our main goal in this chapter is to show how the set $\mathrm{Map}(X, Y)$ can be given the structure of a topological space. Most of the constructions of new topological spaces from existing spaces that we have already encountered were motivated by the choice of continuous functions from or into the new space that we wanted to have. For example, the product topology is defined in such way, that a map $f\colon Y \to \prod_{i\in I} X_i$ is continuous if and only if its compositions with all projection maps $p_i f\colon Y \to X_i$ are continuous (12.8). Similarly, the quotient topology on a space $X/\!\sim$ is defined so that a function $f\colon X/\!\sim \to Y$ is continuous if and only if its composition with the quotient map $f\pi\colon X \to Y$ is continuous (19.13). The choice of topology on $\mathrm{Map}(X, Y)$ will be based on similar considerations.

Denote by $\mathrm{Func}(X, Y)$ the set of all functions (continuous or not) $X \to Y$. Any function $F\colon Z \times X \to Y$ defines a function $F_*\colon Z \to \mathrm{Func}(X, Y)$, where for $z \in Z$ the function $F_*(z)\colon X \to Y$ is given by $F_*(z)(x) = F(z, x)$. Conversely, any function $F_*\colon Z \to \mathrm{Func}(X, Y)$ defines a function $F\colon Z \times X \to Y$ given by $F(z, x) = F_*(z)(x)$. For any spaces X, Y, Z the assignment $F \mapsto F_*$ gives a bijective correspondence:

$$\begin{pmatrix} \text{functions} \\ Z \times X \to Y \end{pmatrix} \cong \begin{pmatrix} \text{functions} \\ Z \to \mathrm{Func}(X, Y) \end{pmatrix}$$

If $F\colon Z \times X \to Y$ is a continuous function, then for any $z \in Z$ the function $F_*(z)\colon X \to Y$ is continuous. This shows that in this case we get a well defined function

$$F_*\colon Z \to \mathrm{Map}(X, Y)$$

With this in mind, it is reasonable to attempt to define a topology on $\mathrm{Map}(X, Y)$ in such way, that for any function $F\colon Z \times X \to Y$ the induced function $F_*\colon Z \to \mathrm{Map}(X, Y)$ is continuous if any only if F is continuous. This motivates the following definition:

22.2 Definition. Let X, Y be a topological spaces, and let $\mathcal{T}$ be a topology on $\mathrm{Map}(X, Y)$.

1) We will say that the topology $\mathcal{T}$ is *lower admissible* if for any continuous function $F\colon Z \times X \to Y$ the function $F_*\colon Z \to \mathrm{Map}(X, Y)$ is continuous.
2) We will say that the topology $\mathcal{T}$ is *upper admissible* if for any function $F\colon Z \times X \to Y$ if the function $F_*\colon Z \to \mathrm{Map}(X, Y)$ is continuous then F is continuous.
3) We will say that the topology $\mathcal{T}$ is *admissible* if it is both lower and upper admissible.

The definition of upper admissible topology can be reformulated using the notion of the evaluation map:

22.3 Definition. Let X, Y be topological spaces. The *evaluation map* is the function

$$\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$$

given by $\mathrm{ev}((f, x)) = f(x)$.

Notice that $\mathrm{ev}_*\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Y)$ is the identity function. We have:

22.4 Lemma. *Let X, Y be topological spaces, and let $\mathcal{T}$ be a topology on* $\mathrm{Map}(X, Y)$*. The following conditions are equivalent:*

1) The topology $\mathcal{T}$ is upper admissible.
2) The evaluation map $\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$ *is continuous.*

Proof. 1) $\Rightarrow$ 2) For any choice of topology on $\mathrm{Map}(X, Y)$ the identity function $\mathrm{id}_{\mathrm{Map}(X,Y)}\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Y)$ is continuous. Since by assumption $\mathcal{T}$ is upper admissible and $\mathrm{ev}_* = \mathrm{id}_{\mathrm{Map}(X,Y)}$ this implies that ev is continuous.

2) $\Rightarrow$ 1) Assume that ev in continuos, and let $F\colon Z \times X \to Y$ be a function such that F_* is continuous. Then $F_* \times \mathrm{id}_X\colon Z \times X \to \mathrm{Map}(X, Y) \times X$ is a continuous function. Since $F = \mathrm{ev} \circ (F_* \times \mathrm{id}_X)$ it follows that F is continuous. $\square$

22.5 Example. Let X, Y be topological spaces. If we consider $\mathrm{Map}(X, Y)$ with the antidiscrete topology then every function $Z \to \mathrm{Map}(X, Y)$ is continuous. Therefore the antidiscrete topology on $\mathrm{Map}(X, Y)$ is lower admissible.

On the other hand, consider $\mathrm{Map}(X, Y)$ with the discrete topology. We will show that this topology is upper admissible. By Lemma 22.4 it suffices to verify that the evaluation map $\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$

is continuous, i.e. that for any open set $U \subseteq Y$ the set $\mathrm{ev}^{-1}(U)$ is open in $\mathrm{Map}(X, Y) \times X$. Notice that

$$\begin{aligned}\mathrm{ev}^{-1}(U) &= \{(f, x) \in \mathrm{Map}(X, Y) \times X \mid f(x) \in U\} \\ &= \{(f, x) \in \mathrm{Map}(X, Y) \times X \mid x \in f^{-1}(U)\} \\ &= \bigcup_{f \in \mathrm{Map}(X,Y)} \{f\} \times f^{-1}(U)\end{aligned}$$

For any $f \in \mathrm{Map}(X, Y)$ the set $f^{-1}(U)$ is open in X, and since the topology on $\mathrm{Map}(X, Y)$ is discrete the set $\{f\}$ is open in $\mathrm{Map}(X, Y)$. It follows that $\mathrm{ev}^{-1}(U)$ is open in $\mathrm{Map}(X, Y) \times X$.

22.6 Proposition. *Let X, Y be topological spaces.*

1) *If $\mathcal{U}, \mathcal{U}'$ are two topologies on $\mathrm{Map}(X, Y)$ such that $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{U}$ is upper admissible, then $\mathcal{U}'$ also is upper admissible.*
2) *If $\mathcal{L}, \mathcal{L}'$ are two topologies on $\mathrm{Map}(X, Y)$ such that $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{L}$ is lower admissible, then $\mathcal{L}'$ also is lower admissible.*
3) *If $\mathcal{U}$, $\mathcal{L}$ are two topologies on $\mathrm{Map}(X, Y)$ such that $\mathcal{U}$ is upper admissible and $\mathcal{L}$ is lower admissible then $\mathcal{L} \subseteq \mathcal{U}$.*

Proof. Proofs of 1) and 2) are straightforward. To prove part 3), denote by $\mathrm{Map}(X, Y)_{\mathcal{U}}$ and $\mathrm{Map}(X, Y)_{\mathcal{L}}$ the set $\mathrm{Map}(X, Y)$ equipped with the topology, respectively, $\mathcal{U}$ and $\mathcal{L}$. Since $\mathcal{U}$ is upper admissible the evaluation map $\mathrm{ev}\colon \mathrm{Map}(X, Y)_{\mathcal{U}} \times X \to Y$ is continuous. Since $\mathcal{L}$ is lower admissible we get that $\mathrm{id}_{\mathrm{Map}(X,Y)} = \mathrm{ev}_*\colon \mathrm{Map}(X, Y)_{\mathcal{U}} \to \mathrm{Map}(X, Y)_{\mathcal{L}}$ is continuous. Therefore any set U open in $\mathrm{Map}(X, Y)_{\mathcal{L}}$ is also open in $\mathrm{Map}(X, Y)_{\mathcal{U}}$, and so $\mathcal{L} \subseteq \mathcal{U}$. □

22.7 Corollary. *Given spaces X and Y, if there exists an admissible topology on $\mathrm{Map}(X, Y)$ then such topology is unique.*

Proof. This follows directly from Proposition 22.6. □

The next proposition shows that in general an admissible topology on $\mathrm{Map}(X, Y)$ may not exist:

22.8 Proposition. *Let X be completely regular space. If there exist an admissible topology on $\mathrm{Map}(X, \mathbb{R})$ then X is locally compact.*

22.9 Example. Since the space $\mathbb{Q}$ of rational numbers is completely regular but not locally compact (Exercise 18.4), there is no admissible topology on $\mathrm{Map}(\mathbb{Q}, \mathbb{R})$.

The proof Proposition 22.8 will depend on the following fact:

22.10 Definition. Let X, Y be topological spaces. For sets $A \subseteq X$ and $B \subseteq Y$ denote

$$P(A, B) = \{f \in \mathrm{Map}(X, Y) \mid f(A) \subseteq B\}$$

22.11 Lemma. *Let X, Y topological spaces, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Let $\mathcal{T}$ be a topology on* $\mathrm{Map}(X, Y)$ *with subbasis given by all sets $P(A, V)$ where $A \subseteq X$ is a closed set such that $A \subseteq U_i$ for some $i \in I$, and $V \subseteq Y$ is an open set. If X is a regular space then $\mathcal{T}$ upper admissible.*

Proof. Exercise. □

Proof of Proposition 22.8. Let $\mathcal{A}$ be an admissible topology on $\mathrm{Map}(X, \mathbb{R})$, and let $\mathrm{Map}(X, \mathbb{R})_{\mathcal{A}}$ denote $\mathrm{Map}(X, \mathbb{R})$ taken with this topology. Take $x_0 \in X$. We need to show that there exists an open set $V \subseteq X$ such that $x_0 \in V$ and $\overline{V}$ is compact.

Let $f\colon X \to \mathbb{R}$ be a constant function given by $f(x) = 0$ for all $x \in X$. Then $\mathrm{ev}((f, x_0)) \in (-1, 1)$. Since $(-1, 1)$ is open in $\mathbb{R}$ and the function $\mathrm{ev}\colon \mathrm{Map}(X, \mathbb{R})_{\mathcal{A}} \times X \to \mathbb{R}$ is continuous, there exist open sets $W \subseteq \mathrm{Map}(X, \mathbb{R})_{\mathcal{A}}$ and $V \subseteq X$ such that $f \in W$, $x_0 \in V$, and $\mathrm{ev}(W \times V) \subseteq (-1, 1)$. We will prove that $\overline{V}$ is compact. It will suffice to show that if $\mathcal{U}$ is an open cover of X then $\overline{V} \subseteq U_{i_1} \cup \ldots \cup U_{i_n}$ for some $i_1, \ldots, i_n \in I$ (Exercise 14.3). Let $\mathcal{U} = \{U_i\}_{\in I}$ be such an open cover, and let $\mathcal{T}$ be a topology on $\mathrm{Map}(X, \mathbb{R})$ with subbasis consisting of all sets $P(A, Z)$ where $A \subseteq X$ is a closed, $A \subseteq U_i$ for some $i \in I$, and $Z \subseteq \mathbb{R}$ is an open set. By Lemma 22.11 $\mathcal{T}$ is upper admissible. Since $\mathcal{A}$ is lower admissible, by Proposition 22.6 we obtain that $\mathcal{A} \subseteq \mathcal{T}$. This implies that there exist elements $P(A_1, Z_1), \ldots P(A_n, Z_n)$ of the subbasis of $\mathcal{T}$ such that $f \in \bigcap_{k=1}^n P(A_k, Z_k) \subseteq W$. Notice that since $f(A_k) = 0$ for all k, we must have $0 \in \bigcap_{k=1}^n Z_k$. Assume that there exists a point $y \in V \smallsetminus \bigcup_{k=1}^n A_k$. Since the set $\bigcup_{k=1}^n A_k$ is closed in X and the space X is completely regular, this would give a continuous function $g\colon X \to \mathbb{R}$ such that $g(\bigcup_{k=1}^n A_k) = 0$ (and so $g \in \bigcap_{k=1}^n P(A_k, Z_k) \subseteq W$) and $g(y) = 1$. This is however impossible, since by the choice of W and V we have $h(v) \in (-1, 1)$ for every $h \in W$ and $v \in V$. Therefore $V \subseteq \bigcup_{k=1}^n A_k$, and since $\bigcup_{k=1}^n A_k$ is closed, also $\overline{V} \subseteq \bigcup_{k=1}^n A_k$. By assumption for each $k = 1, \ldots, n$ there exists $U_{i_k} \in \mathcal{U}$ such that $A_k \subseteq U_{i_k}$. This gives $\overline{V} \subseteq \bigcup_{k=1}^n U_{i_k}$.

□

In view of Proposition 22.8 a natural question is whether an admissible topology on $\mathrm{Map}(X, Y)$ exists when X is a completely regular, locally compact space. The condition that X is completely regular can replaced by the condition that X is Hausdorff, since every locally compact Hausdorff space is completely regular (18.20). Our next goal is to show that under these assumptions on X the set $\mathrm{Map}(X, Y)$ has an admissible topology, and that this topology can be described as follows:

22.12 Definition. Let X, Y be topological spaces. The *compact-open* topology on $\mathrm{Map}(X, Y)$ is the topology defined by the subbasis consisting of all sets $P(A, U)$ where $A \subseteq X$ is compact and $U \subseteq Y$ is an open set.

22.13 Theorem. *For any spaces X, Y the compact-open topology on* $\mathrm{Map}(X, Y)$ *is lower admissible.*

Proof. Consider $\mathrm{Map}(X, Y)$ as a space with compact-open topology, and let $F\colon Z \times X \to Y$ be a continuous function. We need to show that if $F_*\colon Z \to \mathrm{Map}(X, Y)$ is continuous. By Proposition 4.14 it is enough to show that for any compact set $A \subseteq Y$ and an open set $U \subseteq Y$ the set $F_*^{-1}(P(A, U))$ is open in Z. It will suffice to check that for any $z_0 \in F_*^{-1}(P(A, U))$ there exists an open neighborhood $V \subseteq Z$ such that $V \subseteq F_*^{-1}(P(A, U))$. Notice that

$$\begin{aligned} F_*^{-1}(P(A, U)) &= \{z \in Z \mid F(\{z\} \times A) \subseteq U\} \\ &= \{z \in Z \mid \{z\} \times A \subseteq F^{-1}(U)\} \end{aligned}$$

In particular, since $z_0 \in F_*^{-1}(P(A, U))$ we have $\{z_0\} \times A \subseteq F^{-1}(U)$. The set $F^{-1}(U)$ is open in $Z \times Y$, so $F^{-1}(U) = \bigcup_{i \in I}(V_i \times W_i)$ for some open sets $V_i \in Z$ and $W_i \in X$. Since $\{z_0\} \times A \cong A$ is compact, there exist $i_1, \dots i_n \in I$ such that $\{z_0\} \times A \subseteq \bigcup_{k=1}^n (V_{i_k} \times W_{i_k})$. Take $V = \bigcap_{k=1}^n V_{i_k}$. Then $V \times A \subseteq \bigcup_{k=1}^n V_{i_k} \times W_{i_k} \subseteq F^{-1}(U)$, and so $V \subseteq F_*^{-1}(P(A, U))$. □

22.14 Theorem. *Let X, Y be topological spaces. If X is locally compact Hausdorff space then the compact-open topology on* $\mathrm{Map}(X, Y)$ *is upper admisible.*

Proof. Let $\mathcal{C}$ denote the compact-open topology on $\mathrm{Map}(X, Y)$. Let $\mathcal{U} = \{U_i\}$ be an open cover of X such that $\overline{U}_i$ is compact for each $i \in I$. Such open cover exists by the assumption that X is locally compact. Let $\mathcal{T}$ be the topology on $\mathrm{Map}(X, Y)$ with subbasis consisting of all sets $P(A, V)$ where $A \subseteq X$ is a closed, $A \subseteq U_i$ for some $i \in I$, and $V \subseteq Y$ is an open. Notice that by Proposition 14.13 for any such $P(A, V)$ the set A is compact, since $A \subseteq \overline{U}_i$ for some $i \in I$, and $\overline{U}_i$ is compact. Therefore $P(A, V) \in \mathcal{C}$, and so $\mathcal{T} \subseteq \mathcal{C}$.

By Lemma 22.11 the topology $\mathcal{T}$ is upper admissible. Since by Theorem 22.13 $\mathcal{C}$ is lower admissible using Proposition 22.6 we obtain that $\mathcal{C} \subseteq \mathcal{T}$. This shows that $\mathcal{C} = \mathcal{T}$, and so $\mathcal{C}$ is upper admissible. □

22.15 Corollary. *If X is a locally compact Hausdorff space and Y is any space then the compact-open topology on* $\mathrm{Map}(X, Y)$ *is admissible.*

Proof. Follows from Theorem 22.13 and Theorem 22.14 . □

22.16 Note. Let X, Y, Z be topological spaces. By Corollary 22.15 if X is locally compact Hausdorff and $\mathrm{Map}(X, Y)$ is taken with the compact-open topology then the map

$$\Psi\colon \mathrm{Map}(Z \times X, Y) \to \mathrm{Map}(Z, \mathrm{Map}(X, Y))$$

given by $\Psi(F) = F_*$ is a well defined bijection. One can show that if in addition Z is a Hausdorff space, and both $\mathrm{Map}(Z \times X, Y)$ and $\mathrm{Map}(Z, \mathrm{Map}(X, Y))$ are considered as topological spaces with compact-open topology, then Ψ is a homeomorphism.

In some cases the compact open-topology on $\mathrm{Map}(X, Y)$ can be described more explicitly. Let X be a topological space and let S be a set. Recall (1.18) that the Cartesian product $\prod_{s \in S} X$ is formally defined as the set of all functions $S \to X$. We have:

22.17 Proposition. *Let X be a topological space, and let S be a set considered as a discrete topological space. There exists a homeomorphism*

$$\mathrm{Map}(S, X) \cong \prod_{s \in S} X$$

where $\mathrm{Map}(S, X)$ *is taken with the compact-open topology, and* $\prod_{s \in S} X$ *with the product topology.*

Proof. Exercise. □

22.18 Note. In the special case where $S = \{*\}$ is a set consisting of a single point we obtain a homeomorphism $\mathrm{Map}(\{*\}, X) \cong X$.

Next, let X be a topological space and (Y, ϱ) be a metric space. If $f, g \colon X \to Y$ are continuous function then the function $\Phi_{f,g} \colon X \to \mathbb{R}$ given by $\Phi_{f,g}(x) = \varrho(f(x), g(x))$ is continuous (exercise). If X is a compact space then by Exercise 14.6 this function attains its maximum value at some point $x_0 \in X$. We have:

22.19 Proposition. *Let X be a compact Hausdorff space, and let (Y, ϱ) be a metric space. For $f, g \in \mathrm{Map}(X, Y)$ define*

$$d(f, g) = \max\{\varrho(f(x), g(x)) \mid x \in X\}$$

Then d is a metric on $\mathrm{Map}(X, Y)$*. Moreover, in the topology induced by this metric is the compact-open topology.*

Proof. Exercise. □

We conclude this chapter with a result that says that compact-open topology behaves well with respect to composition of functions:

22.20 Theorem. *Let X, Y, Z be topological spaces. Let*

$$\Phi \colon \mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

be a function given by $\Phi(f, g) = g \circ f$. If Y is a locally compact Hausdorff space, and all mapping spaces are equipped with the compact-open topology then Φ is continuous.

The proof will use the following fact:

22.21 Lemma. *Let X be a locally compact Hausdorff space, and let $A, W \subseteq X$ be sets such that A is compact, W is open, and $A \subseteq W$. Then there exists an open set $V \subseteq X$ such that $A \subseteq V$, $\overline{V} \subseteq W$, and $\overline{V}$ is compact.*

Proof. Exercise. □

Proof of Theorem 22.20. Let $A \subseteq X$ be a compact set, $U \subseteq Z$ be an open set, and let $(f, g) \in \Phi^{-1}(P(A, U))$. It will suffice to show that (f, g) has an open neighborhood contained in $\Phi^{-1}(P(A, U))$. Since $g \circ f(A) \subseteq U$, thus $f(A) \subseteq g^{-1}(U)$. By (14.9) the set $f(A)$ is compact, so using Lemma 22.21 we obtain that there exists an open set $V \subseteq Y$ such that $f(A) \subseteq V$, $\overline{V} \subseteq g^{-1}(U)$, and $\overline{V}$ is compact. It remains to notice that the set $P(A, V) \times P(\overline{V}, U)$ is an open neighborhood of (f, g) in $\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z)$, and $P(A, V) \times P(\overline{V}, U) \subseteq \Phi^{-1}(P(A, U))$. □

Exercises to Chapter 22

E22.1 Exercise. Prove Proposition 22.8.

E22.2 Exercise. Prove Proposition 22.17.

E22.3 Exercise. Prove Proposition 22.19.

E22.4 Exercise. Prove Proposition 22.21.

E22.5 Exercise. Let X, Y be topological spaces, and let $A \subseteq X$, $B \subseteq Y$ be closed sets. Show that in the compact-open topology on $\mathrm{Map}(X, Y)$ the set $P(A, B)$ is closed.

E22.6 Exercise. Let X, Y, Z be topological spaces, and let $f\colon X \to Y$ be a continuous function.

a) Define a function $f_*\colon \mathrm{Map}(Z, X) \to \mathrm{Map}(Z, Y)$ by $f_*(g) = f \circ g$. Show that f_* is continuous.

b) Define a function $f^*\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ by $f^*(g) = g \circ f$. Show that f^* is continuous.

All mapping spaces are considered with the compact-open topology.

E22.7 Exercise. Let X, Y_i, $i \in I$ be topological spaces. Show that there is a homeomorphism:

$$\mathrm{Map}(X, \textstyle\prod_{i\in I} Y_i) \simeq \prod_{i\in I} \mathrm{Map}(X, Y_i)$$

All mapping spaces are taken with the compact-open topology.