5 | Closed Sets, Interior, Closure, Boundary

5.1 Definition. Let X be a topological space. A set $A \subseteq X$ is a *closed set* if the set $X \smallsetminus A$ is open.

5.2 Example. A closed interval $[a, b] \subseteq \mathbb{R}$ is a closed set since the set $\mathbb{R} \smallsetminus [a, b] = (-\infty, a) \cup (b, +\infty)$ is open in $\mathbb{R}$.

5.3 Example. Let $\mathcal{T}_{Za}$ be the Zariski topology on $\mathbb{R}$. Recall that $U \in \mathcal{T}_{Za}$ if either $U = \varnothing$ or $U = \mathbb{R} \smallsetminus S$ where $S \subset \mathbb{R}$ is a finite set. As a consequence closed sets in the Zariski topology are the whole space $\mathbb{R}$ and all finite subsets of $\mathbb{R}$.

5.4 Example. If X is a topological space with the discrete topology then every subset $A \subseteq X$ is closed in X since every set $X \smallsetminus A$ is open in X.

5.5 Proposition. *Let X be a topological space.*

1) *The sets X, $\varnothing$ are closed.*
2) *If $A_i \subseteq X$ is a closed set for $i \in I$ then $\bigcap_{i \in I} A_i$ is closed.*
3) *If A_1, A_2 are closed sets then the set $A_1 \cup A_2$ is closed.*

Proof. 1) The set X is closed since the set $X \smallsetminus X = \varnothing$ is open. Similarly, the set $\varnothing$ is closed since the set $X \smallsetminus \varnothing = X$ is open.

2) We need to show that the set $X \smallsetminus \bigcap_{i \in I} A_i$ is open. By De Morgan's Laws (1.13) we have:

$$X \smallsetminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \smallsetminus A_i)$$

By assumption the sets A_i are closed, so the sets $X \smallsetminus A_i$ are open. Since any union of open sets is open we get that $X \smallsetminus \bigcap_{i \in I} A_i$ is an open set.

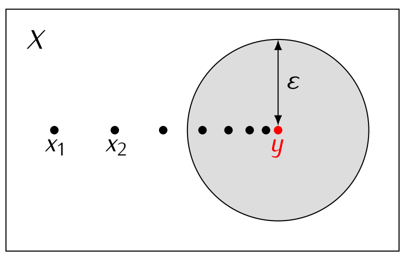
3) Exercise. □

5.6 Note. By induction we obtain that if $\{A_1, \dots, A_n\}$ is a finite collection of closed sets then the set $A_1 \cup \dots \cup A_n$ is closed. It is not true though that an infinite union of closed sets must be closed. For example, the sets $A_n = [\frac{1}{n}, 1]$ are closed in $\mathbb{R}$, but the set $\bigcup_{n=1}^{\infty} A_n = (0, 1]$ is not closed.

In metric spaces closed sets can be characterized using the notion of convergence of sequences:

5.7 Definition. Let (X, ϱ) be a metric space, and let $\{x_n\}$ be a sequence of points in X. We say that $\{x_n\}$ *converges* to a point $y \in X$ if for every $\varepsilon > 0$ there exists $N > 0$ such that $\varrho(y, x_n) < \varepsilon$ for all $n > N$. We write: $x_n \to y$.

Equivalently: $x_n \to y$ if for every $\varepsilon > 0$ there exists $N > 0$ such that $x_n \in B(y, \varepsilon)$ for all $n > N$.



5.8 Proposition. *Let (X, ϱ) be a metric space and let $A \subseteq X$. The following conditions are equivalent:*

1) *The set A is closed in X.*
2) *If $\{x_n\} \subseteq A$ and $x_n \to y$ then $y \in A$.*

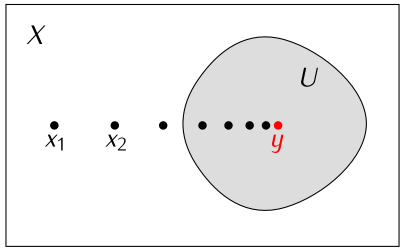
Proof. Exercise. □

5.9 Example. Take $\mathbb{R}$ with the Euclidean metric, and let $A = (0, 1]$. Let $x_n = \frac{1}{n}$. Then $\{x_n\} \subseteq A$, but $x_n \to 0 \notin A$. This shows that A is not a closed set in $\mathbb{R}$.

The notion of convergence of a sequence can be extended from metric spaces to general topological spaces by replacing open balls with center at a point y with open neighborhoods of y:

5.10 Definition. Let X be a topological space and $y \in X$. If $U \subseteq X$ is an open set such that $y \in U$ then we say that U is an *open neighborhood of y*.

5.11 Definition. Let X be a topological space. A sequence $\{x_n\} \subseteq X$ *converges* to $y \in X$ if for every open neighborhood U of y there exists $N > 0$ such that $x_n \in U$ for $n > N$.



5.12 Note. In general topological spaces a sequence may converge to many points at the same time. For example let $(X, \mathcal{T})$ be a space with the antidiscrete topology $\mathcal{T} = \{X, \varnothing\}$. Any sequence $\{x_n\} \subseteq X$ converges to any point $y \in X$ since the only open neighborhood of y is whole space X, and $x_n \in X$ for all n. The next proposition says that such situation cannot happen in metric spaces:

5.13 Proposition. *Let (X, ϱ) be a metric space and let $\{x_n\}$ be a sequence in X. If $x_n \to y$ and $x_n \to z$ for some $y, z \in X$ then $y = z$.*

Proof. Exercise. □

5.14 Proposition. *Let X be a topological space and let $A \subseteq X$ be a closed set. If $\{x_n\} \subseteq A$ and $x_n \to y$ then $y \in A$.*

Proof. Exercise. □

5.15 Note. For a general topological space X the converse of Proposition 5.14 is not true. That is, assume that $A \subseteq X$ is a set with the property that if $\{x_n\} \subseteq A$ and $x_n \to y$ then $y \in A$. The next example shows that this does not imply that the set A must be closed in X.

5.16 Example. Let $X = \mathbb{R}$ with the following topology:

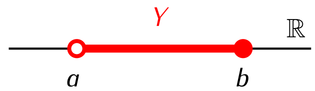
$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \smallsetminus S) \text{ for some countable set } S \subseteq \mathbb{R}\}$$

Closed sets in X are the whole space $\mathbb{R}$ and all countable subsets of $\mathbb{R}$. If $\{x_n\} \subseteq X$ is a sequence then $x_n \to y$ if and only if there exists $N > 0$ such that $x_n = y$ for all $n > N$ (exercise). It follows that if A is any (closed or not) subset of X, $\{x_n\} \subseteq A$, and $x_n \to y$ then $y \in A$.

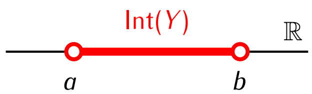
5.17 Definition. Let X be a topological space and let $Y \subseteq X$.

- The *interior of Y* is the set $\operatorname{Int}(Y) := \bigcup \{U \mid U \subseteq Y \text{ and } U \text{ is open in } X\}$.
- The *closure of Y* is the set $\overline{Y} := \bigcap \{A \mid Y \subseteq A \text{ and } A \text{ is closed in } X\}$.
- The *boundary of Y* is the set $\operatorname{Bd}(Y) := \overline{Y} \cap \overline{(X \smallsetminus Y)}$.

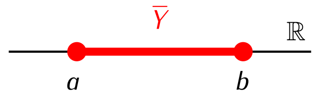
5.18 Example. Consider the set $Y = (a, b]$ in $\mathbb{R}$:



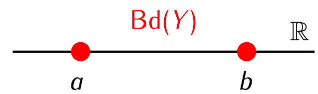
We have:



$\mathrm{Int}(Y) = (a, b)$

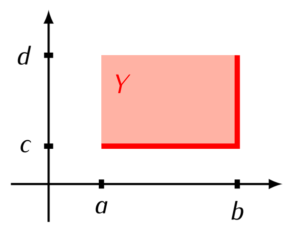


$\overline{Y} = [a, b]$

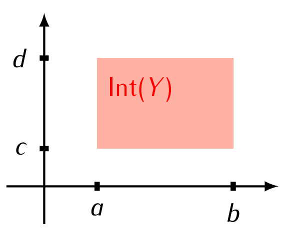


$\mathrm{Bd}(Y) = \{a, b\}$

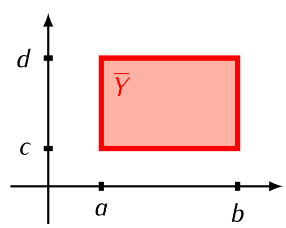
5.19 Example. Consider the set $Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid a < x_1 \leq b,\ c \leq x_2 < d\}$ in $\mathbb{R}^2$:



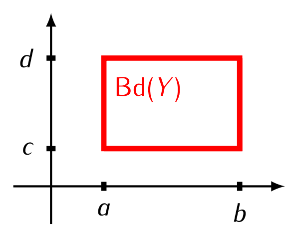
We have:



$\mathrm{Int}(Y) = (a, b) \times (c, d)$



$\overline{Y} = [a, b] \times [c, d]$



$$\mathrm{Bd}(Y) = [a, b] \times \{c, d\} \\ \cup \{a, b\} \times [c, d]$$

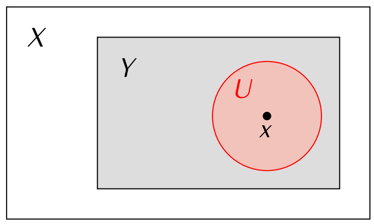
5.20 Proposition. *Let X be a topological space and let $Y \subseteq X$.*

1) *The set $\mathrm{Int}(Y)$ is open in X. It is the biggest open set contained in Y: if U is open and $U \subseteq Y$ then $U \subseteq \mathrm{Int}(Y)$.*
2) *The set $\overline{Y}$ is closed in X. It is the smallest closed set that contains Y: if A is closed and $Y \subseteq A$ then $\overline{Y} \subseteq A$.*

Proof. Exercise. □

5.21 Proposition. *Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:*

1) *$x \in \operatorname{Int}(Y)$*
2) *There exists an open neighborhood U of x such that $U \subseteq Y$.*

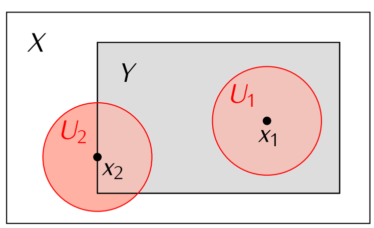


Proof. 1) $\Rightarrow$ 2) Assume that $x \in \operatorname{Int}(Y)$. Since $\operatorname{Int}(Y)$ is an open set and $\operatorname{Int}(Y) \subseteq Y$ we can take $U = \operatorname{Int}(Y)$.

2) $\Rightarrow$ 1) Assume that $x \in U$ for some open set U such that $U \subseteq Y$. Since $\operatorname{Int}(Y)$ is the union of all open sets contained in Y thus $U \subseteq \operatorname{Int}(Y)$ and so $x \in \operatorname{Int}(Y)$. □

5.22 Proposition. *Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:*

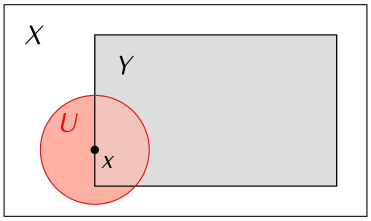
1) *$x \in \overline{Y}$*
2) *For every open neighborhood U of x we have $U \cap Y \neq \varnothing$.*



Proof. Exercise. □

5.23 Proposition. *Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:*

1) *$x \in \operatorname{Bd}(Y)$*
2) *For every open neighborhood U of x we have $U \cap Y \neq \varnothing$ and $U \cap (X \smallsetminus Y) \neq \varnothing$.*



Proof. This follows from the definition of $\mathrm{Bd}(Y)$ and Proposition 5.22. □

5.24 Definition. Let X be a topological space. A set $Y \subseteq X$ is *dense in* X if $\overline{Y} = X$.

5.25 Proposition. *Let X be a topological space and let $Y \subseteq X$. The following conditions are equivalent:*

1) *Y is dense in X*
2) *If $U \subseteq X$ is an open set and $U \neq \varnothing$ then $U \cap Y \neq \varnothing$.*

Proof. This follows directly from Proposition 5.22. □

5.26 Example. The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercises to Chapter 5

E5.1 Exercise. Prove Proposition 5.8

E5.2 Exercise. Prove Proposition 5.13

E5.3 Exercise. Let (X, ϱ) be a metric space. A sequence $\{x_n\}$ is called a *Cauchy sequence* if for any $N > 0$ there exists $\varepsilon > 0$ such that if $n, m > N$ then $\varrho(x_m, x_n) < \varepsilon$. Show that if $\{x_n\}$ is a sequence in X that converges to some point $x_0 \in X$ then $\{x_n\}$ is a Cauchy sequence.

E5.4 Exercise. Prove Proposition 5.14

E5.5 Exercise. Let X be the topological space defined in Example 5.16 and let $\{x_n\}$ be a sequence in X. Show that $x_n \to y$ for some $y \in X$ iff there exists $N > 0$ such that $x_n = y$ for all $n > N$.

E5.6 Exercise. Prove Proposition 5.22

E5.7 Exercise. Let X be a topological space and let Y be a subspace of X. Show that a set $A \subseteq Y$ is closed in Y if and only if there exists a set B closed in X such that $Y \cap B = A$.

E5.8 Exercise. Let X be a topological space and let $Y \subseteq X$ be a subspace.

a) Assume that Y is open in X. Show that if $U \subseteq Y$ is open in Y then U is open in X.

b) Assume that Y is closed in X. Show that if $A \subseteq Y$ is closed in Y then A is closed in X.

E5.9 Exercise. Let (X, ϱ) be a metric space. The *closed ball* with center at a point $x_0 \in X$ and radius $r > 0$ is the set

$$\overline{B}(x_0, r) = \{x \in X \mid \varrho(x_0, x) \leq r\}$$

a) Show that for any $x_0 \in X$ and any $r > 0$ the closed ball $\overline{B}(x_0, r)$ is a closed set.

b) Consider $\mathbb{R}^n$ with the Euclidean metric d. Show that for any $x_0 \in \mathbb{R}^n$ and any $r > 0$ the closed ball $\overline{B}(x_0, r)$ is the closure of the open ball $B(x_0, r)$ (i.e. $\overline{B}(x_0, r) = \overline{B(x_0, r)}$).

c) Give an example showing that in a general metric space (X, ϱ) the closed ball $\overline{B}(x_0, r)$ need not be the closure of the open ball $B(x_0, r)$.

E5.10 Exercise. Consider the following subset of $\mathbb{R}$:

$$Y = \left\{ -\frac{1}{n} \;\middle|\; n \in \mathbb{Z},\ n \geq 1 \right\}$$

Describe $\mathrm{Int}(Y)$, $\overline{Y}$, and $\mathrm{Bd}(Y)$ in the following topological spaces:

a) $\mathbb{R}$ with the Euclidean topology.
b) $\mathbb{R}$ with the Zariski topology.
c) $\mathbb{R}$ with the arrow topology.
d) $\mathbb{R}$ with the discrete topology.
e) $\mathbb{R}$ with the antidiscrete topology.
f) $\mathbb{R}$ with the topology defined in Example 5.16.

E5.11 Exercise. Let (X, ϱ) be a metric space. We say that a set $Y \subseteq X$ is *bounded* if there exists an open ball $B(x, r) \subseteq X$ such that $Y \subseteq B(x, r)$. Show that if Y is a bounded set then $\overline{Y}$ is also bounded.

E5.12 Exercise. Let X be a topological space and let $Y_1, Y_2 \subseteq X$.

a) Show $\overline{Y}_1 \cup \overline{Y}_2 = \overline{Y_1 \cup Y_2}$
b) Is it true always true that $\overline{Y}_1 \cap \overline{Y}_2 = \overline{Y_1 \cap Y_2}$? Justify your answer.

E5.13 Exercise. Let X be a topological space and let $Y \subseteq X$ be a dense subset of X. Show that if $f, g\colon X \to \mathbb{R}$ are continuous functions such that $f(x) = g(x)$ for all $x \in Y$ then $f(x) = g(x)$ for all $x \in X$.

E5.14 Exercise. Let X be a topological space, and let $A, B \subseteq X$. Show that if $\overline{B} \subseteq \mathrm{Int}(A)$ then $X = \mathrm{Int}(X \smallsetminus B) \cup \mathrm{Int}(A)$.

E5.15 Exercise. Let $\mathbb{R}_{Ar}$ denote the set of real numbers with the arrow topology (4.8). The goal of this exercise is to show that this space is not metrizable.

a) Recall that a space X is second countable if it has a countable basis. We say that a space X is *separable* if there is a set $Y \subseteq X$ such that Y is countable and dense in X. Show that if X is a metrizable space then X is separable if and only if X is second countable.

b) Show that $\mathbb{R}_{Ar}$ is a separable space.

Since by Exercise 4.11 $\mathbb{R}_{Ar}$ is not second countable this implies that $\mathbb{R}_{Ar}$ is not metrizable.