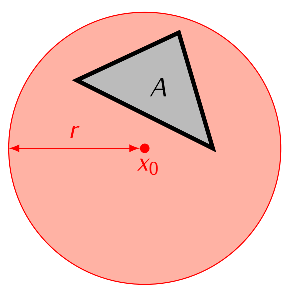
# 15 | Heine-Borel Theorem

**15.1 Definition.** Let $(X, \varrho)$ be a metric space. A set $A \subseteq X$ is *bounded* if there exists an open ball $B(x_0, r) \subseteq X$ such that $A \subseteq B(x_0, r)$.




**15.2 Proposition.** *Let $(X, \varrho)$ be a metric space and let $A \subseteq X$. The following conditions are equivalent:*

1) *$A$ is bounded.*
2) *For each $x \in X$ there exists $r_x > 0$ such that $A \subseteq B(x, r_x)$.*
3) *There exists $R > 0$ such that $\varrho(x_1, x_2) < R$ for all $x_1, x_2 \in A$.*

*Proof.* Exercise. □

**15.3 Heine-Borel Theorem.** *A set $A \subseteq \mathbb{R}^n$ is compact if and only if $A$ is closed and bounded.*

**15.5 Theorem.** *If $X$, $Y$ are compact spaces then the space $X \times Y$ is also compact.*

**15.6 Corollary.** *If $X_1, \dots, X_n$ are compact spaces spaces then the space $X_1 \times \dots \times X_n$ is compact.*

**15.7 Corollary.** *For $i = 1, \dots, n$ let $[a_i, b_i] \subseteq \mathbb{R}$ be a closed interval. The closed box*

$$[a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$$

*is compact.*

*Proof of Theorem 15.3.* ($\Rightarrow$) Exercise.

($\Leftarrow$)

$\square$