1 | Some Set Theory

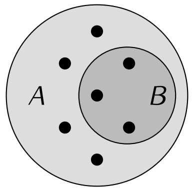
Sets.

Frequently used sets:

$\varnothing =$ the empty set (i.e. the set that contains no elements)
$\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ the set of integers
$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ the set of positive integers
$\mathbb{Q} =$ the set of rational numbers
$\mathbb{R} =$ the set of real numbers

We will write $x \in A$ to denote that x is an element of the set A and $y \notin A$ to indicate that y is not an element of A.

1.1 Definition. A set B is a *subset* of a set A if every element of B is in A. In such case we write $B \subseteq A$.



A set B is a *proper subset* of A if $B \subseteq A$ and $B \neq A$.

1.3 Example. Here are some often used subsets of $\mathbb{R}$:

1) an open interval:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

2) a closed interval:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

3) a half open interval:

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

1.4 Definition. The *union* of sets A and B is the set $A \cup B$ that consists of all elements that belong to either A or B:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The *intersection* of sets A and B is the set $A \cap B$ that consists of all elements that belong to both A and B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

1.7 Definition. If $A \cap B = \varnothing$ then we say that A and B are *disjoint sets*.

Definition 1.4 can be extended to unions and intersections of arbitrary families of sets. If $\{A_i\}_{\in I}$ is a family of sets then

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$
$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

1.10 Definition. The *difference* of sets A and B is the set $A \smallsetminus B$ consisting of the elements of A that do not belong to B:

$$A \smallsetminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

1.12 Definition. If $A \subseteq B$ then the set $B \smallsetminus A$ is called the *complement* of A in B.

1.13 Properties of the algebra of sets.

Distributivity:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

De Morgan's Laws:

$$A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$$
$$A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$$

1.14 Definition. The *Cartesian product* of sets A, B is the set consisting of all ordered pairs of elements of A and B:

$$A \times B = \{(a, b) \mid a \in A, \quad b \in B\}$$

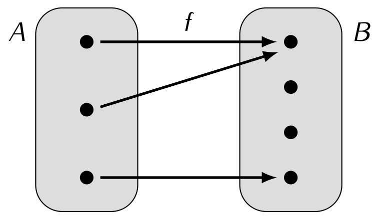
1.16 Notation. Given a set A by A^n we will denote the n-fold Cartesian product of A:

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

1.18 Infinite products.

1.27 Definition. Let A, B be sets

1) A function $f\colon A \to B$ is *1-1* if $f(x) = f(x')$ only if $x = x'$.



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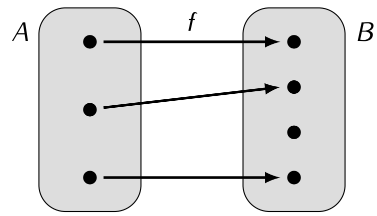

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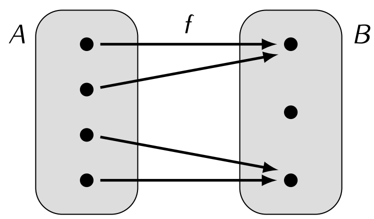
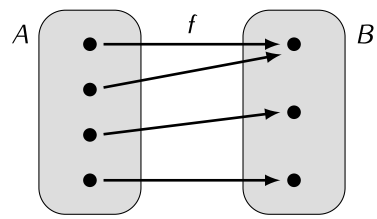
2) A function $f\colon A \to B$ is *onto* if for every $y \in B$ there is $x \in A$ such that $f(x) = y$

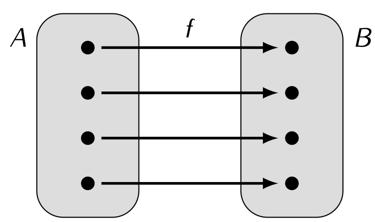


not onto



onto

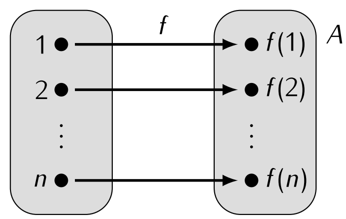
3) A function $f\colon A \to B$ is a *bijection* if f is both 1-1 and onto.



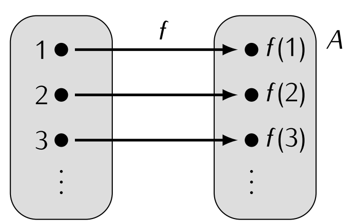
bijection

1.29 Definition. Sets A, B *have the same cardinality* if there exists a bijection $f\colon A \to B$. In such case we write $|A| = |B|$.

1.30 Definition. A set A is *finite* if either $A = \varnothing$ or A has the same cardinality as the set $\{1, \dots, n\}$ for some $n \geq 1$.



1.31 Definition. A set A is *infinitely countable* if it is has the same cardinality as the set $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$



1.32 Definition. A set A is *countable* if it is either finite or infinitely countable.

1.33 Example. The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is countable since we have a bijection $f\colon \mathbb{Z}^+ \to \mathbb{N}$ given by $f(k) = k - 1$.

1.34 Example. The set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is countable since we have a bijection $f\colon \mathbb{Z}^+ \to \mathbb{Z}$ given by

$$f(k) = \begin{cases} k/2 & \text{if } k \text{ is even} \\ (1-k)/2 & \text{if } k \text{ is odd} \end{cases}$$

In other words:

$$f(1) = 0,\ \ f(2) = 1,\ \ f(3) = -1,\ \ f(4) = 2,\ \ f(5) = -2,\ \ f(6) = 3,\ \ \dots$$

1.35 Example. The set of rational numbers $\mathbb{Q}$ is countable. A bijection $f\colon \mathbb{Z}^+ \to \mathbb{Q}$ can be constructed as follows:

$$
\begin{array}{cccccccc}
0/1 & 1/1 & -1/1 & 2/1 & -2/1 & 3/1 & -3/1 & \cdots \\
0/2 & 1/2 & -1/2 & 2/2 & -2/2 & 3/2 & -3/2 & \cdots \\
0/3 & 1/3 & -1/3 & 2/3 & -2/3 & 3/3 & -3/3 & \cdots \\
0/4 & 1/4 & -1/4 & 2/4 & -2/4 & 3/4 & -3/4 & \cdots \\
0/5 & 1/5 & -1/5 & 2/5 & -2/5 & 3/5 & -3/5 & \cdots \\
0/6 & 1/6 & -1/6 & 2/6 & -2/6 & 3/6 & -3/6 & \cdots \\
0/7 & 1/7 & -1/7 & 2/7 & -2/7 & 3/7 & -3/7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
\qquad
\begin{array}{rcl}
0/1 & = & f(1) \\
0/2 = 0/1 & = & f(1) \\
1/1 & = & f(2) \\
-1/1 & = & f(3) \\
1/2 & = & f(4) \\
0/3 = 0/1 & = & f(1) \\
0/4 = 0/1 & = & f(1) \\
1/3 & = & f(5) \\
-1/2 & = & f(6) \\
2/1 & = & f(7) \\
\cdots & & \cdots
\end{array}
$$

1.36 Theorem. *1) If A is a countable set and $B \subseteq A$ then B is countable.*
2) If $\{A_1, A_2, \ldots\}$ is a collection of countably many countable sets then the set $\bigcup_{i=1}^{\infty} A_i$ is countable.
3) If $\{A_1, A_2, \ldots, A_n\}$ is a collection of finitely many countable sets then the set $A_1 \times \cdots \times A_n$ is countable.

1.37 Example. The set of all real numbers in the interval (0, 1) is not countable.

$$\begin{aligned}
f(1) &= 0.31415\ldots\\
f(2) &= 0.12345\ldots\\
f(3) &= 0.75149\ldots\\
f(4) &= 0.00032\ldots\\
f(5) &= 0.11111\ldots\\
\cdots & \quad \cdots\ \cdots\ \cdots
\end{aligned}$$

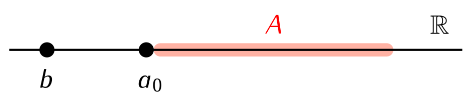
1.38 Example. The function $f\colon (0,1) \to \mathbb{R}$ given by $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ is a bijection. It follows that $|\mathbb{R}| = |(0,1)|$, and so the set $\mathbb{R}$ is not countable.

Infima and Suprema.

1.40 Definition. Let $A \subseteq \mathbb{R}$. The set A is *bounded below* if there exists a number b such that $b \leq x$ for all $x \in A$. The set A is *bounded above* if there exists a number c such that $x \leq c$ for all $x \in A$. The set A is *bounded* if it is both bounded below and bounded above.

1.41 Definition. Let $A \subseteq \mathbb{R}$. If the set A is bounded below then the *greatest lower bound* of A (or *infimum* of A) is a number $a_0 \in \mathbb{R}$ such that:

1) $a_0 \leq x$ for all $x \in A$
2) if $b \leq x$ for all $x \in A$ then $b \leq a_0$



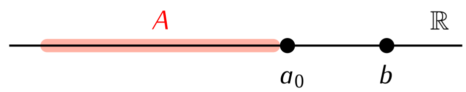
We write: $a_0 = \inf A$.

If the set A is not bounded below then we set $\inf A := -\infty$.

1.43 Theorem. *For any non-empty bounded below subset $A \subseteq \mathbb{R}$ the number* $\inf A$ *exists.*

1.44 Definition. Let $A \subseteq \mathbb{R}$. If the set A is bounded above then the *least upper bound* of A (or *supremum* of A) is a number $a_0 \in \mathbb{R}$ such that:

1) $x \leq a_0$ for all $x \in A$
2) if $x \leq b$ for all $x \in A$ then $a_0 \leq b$



We write: $a_0 = \sup A$.

If the set A is not bounded above then we set $\sup A := +\infty$.

1.46 Theorem. *For any non-empty bounded above subset $A \subseteq \mathbb{R}$ the number $\sup A$ exists.*