

## 6 | Continuous Functions

Let  $X, Y$  be topological spaces. Recall that a function  $f: X \rightarrow Y$  is continuous if for every open set  $U \subseteq Y$  the set  $f^{-1}(U) \subseteq X$  is open. In this chapter we study some properties of continuous functions. We also introduce the notion of a *homeomorphism* that plays a central role in topology: from the topological perspective interesting properties of spaces are the properties that are preserved by homeomorphisms.

**6.1 Proposition.** *Let  $X, Y$  be topological spaces. A function  $f: X \rightarrow Y$  is continuous if and only if for every closed set  $A \subseteq Y$  the set  $f^{-1}(A) \subseteq X$  is closed.*

*Proof.* Assume that  $f: X \rightarrow Y$  is a continuous function and let  $A \subseteq Y$  be a closed set. We have

$$f^{-1}(A) = X \setminus f^{-1}(Y \setminus A)$$

The set  $Y \setminus A$  is open in  $Y$  so by continuity of  $f$  the set  $f^{-1}(Y \setminus A) \subseteq X$  is open in  $X$ . It follows that  $f^{-1}(A)$  is closed in  $X$ .

Conversely, assume that  $f: X \rightarrow Y$  is a function such that for every closed set  $A \subseteq Y$  the set  $f^{-1}(A) \subseteq X$  is closed. Let  $U \subseteq Y$  be an open set. We have

$$f^{-1}(U) = X \setminus f^{-1}(Y \setminus U)$$

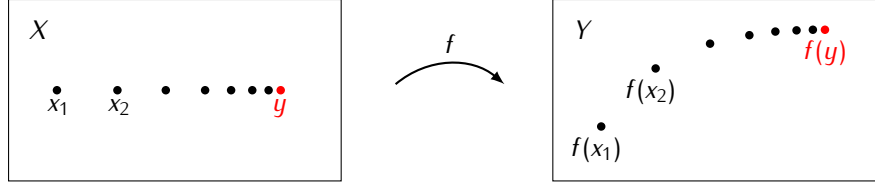
The set  $Y \setminus U$  is closed in  $Y$  so by assumption the set  $f^{-1}(Y \setminus U)$  is closed in  $X$ . It follows that  $f^{-1}(U)$  is open in  $X$ . Therefore  $f$  is a continuous function.  $\square$

For metric spaces continuous functions are precisely the functions that preserve convergence of sequences:

**6.2 Proposition.** *Let  $(X, \rho)$  be a metric space, let  $Y$  be a topological space, and let  $f: X \rightarrow Y$  be a function. The following conditions are equivalent:*

- 1)  $f$  is continuous.

2) For any sequence  $\{x_n\} \subseteq X$  if  $x_n \rightarrow y$  for some  $y \in X$  then  $f(x_n) \rightarrow f(y)$ .



*Proof.* 1)  $\Rightarrow$  2) Exercise.

2)  $\Rightarrow$  1) Let  $A \subseteq Y$  be a closed set. We will show that the set  $f^{-1}(A)$  is closed in  $X$ . By Proposition 5.8 it suffices to show that if  $\{x_n\} \subseteq f^{-1}(A)$  is a sequence and  $x_n \rightarrow x$  then  $x \in f^{-1}(A)$ .

If  $x_n \rightarrow x$  then by assumption we have  $f(x_n) \rightarrow f(x)$ . Since  $\{f(x_n)\} \subseteq A$  and  $A$  is a closed set, thus by Proposition 5.8 we obtain that  $f(x) \in A$ , and so  $x \in f^{-1}(A)$ .  $\square$

The implication 1)  $\Rightarrow$  2) in Proposition 6.2 holds for maps between general topological spaces:

**6.3 Proposition.** Let  $f: X \rightarrow Y$  be a continuous function of topological spaces. If  $\{x_n\} \subseteq X$  is a sequence and  $x_n \rightarrow x$  for some  $x \in X$  then  $f(x_n) \rightarrow f(x)$ .

*Proof.* Exercise.  $\square$

**6.4 Example.** We will show that the implication 2)  $\Rightarrow$  1) in Proposition 6.2 is not true if  $X$  is a general topological space. Let  $X$  be the space defined in Example 5.16:  $X = \mathbb{R}$  with the topology

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \emptyset \text{ or } U = (\mathbb{R} \setminus S) \text{ for some countable set } S \subseteq \mathbb{R}\}$$

Recall that if  $\{x_n\}$  is a sequence in  $X$  then  $x_n \rightarrow x$  if and only if there exists  $N > 0$  such that  $x_n = x$  for all  $n > N$ . Let  $f: X \rightarrow X$  be a function given by

$$f(x) = \begin{cases} 0 & \text{if } x \in (0, 1) \\ 1 & \text{if } x \notin (0, 1) \end{cases}$$

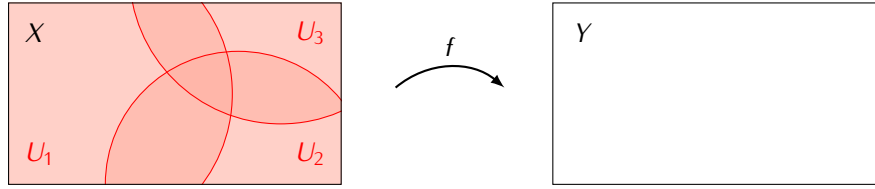
This function is not continuous since the set  $\{0\}$  is closed in  $X$  and the set  $(0, 1) = f^{-1}(\{0\})$  is not closed in  $X$ . On the other hand let  $\{x_n\} \subseteq X$  be a sequence and let  $x_n \rightarrow x$ . There is  $N > 0$  such that  $x_n = x$  for  $n > N$ , so  $f(x_n) = f(x)$  for all  $n > N$  and so  $f(x_n) \rightarrow f(x)$ .

**6.5 Proposition.** If  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are continuous functions then the function  $gf: X \rightarrow Z$  is also continuous.

*Proof.* Exercise.  $\square$

Frequently functions  $f: X \rightarrow Y$  are constructed by gluing together several functions defined on subspaces of  $X$ . The next two facts are useful for verifying that functions obtained in this way are continuous.

**6.6 Open Pasting Lemma.** *Let  $X, Y$  be topological spaces and let  $\{U_i\}_{i \in I}$  be a family of open sets in  $X$  such that  $\bigcup_{i \in I} U_i = X$ . Assume that for  $i \in I$  we have a continuous function  $f_i: U_i \rightarrow Y$  such that  $f_i(x) = f_j(x)$  if  $x \in U_i \cap U_j$ . Then the function  $f: X \rightarrow Y$  given by  $f(x) = f_i(x)$  for  $x \in U_i$  is continuous.*



*Proof.* Let  $V \subseteq Y$  be an open set. We will show that the set  $f^{-1}(V) \subseteq X$  is open. Since  $\bigcup_{i \in I} U_i = X$  we have

$$f^{-1}(V) = \bigcup_{i \in I} f^{-1}(V) \cap U_i = \bigcup_{i \in I} f_i^{-1}(V)$$

Since  $f_i: U_i \rightarrow Y$  is a continuous function the set  $f_i^{-1}(V)$  is open in  $U_i$ . Also, since  $U_i$  is open in  $X$  by Exercise 5.8 we obtain that the set  $f_i^{-1}(V)$  is open in  $X$ . Thus  $f^{-1}(V)$  is an open set.  $\square$

**6.7 Closed Pasting Lemma.** *Let  $X, Y$  be topological spaces and let  $A_1, \dots, A_n \subseteq X$  be a finite family of closed sets such that  $\bigcup_{i=1}^n A_i = X$ . Assume that for  $i = 1, 2, \dots, n$  we have a continuous function  $f_i: A_i \rightarrow Y$  such that  $f_i(x) = f_j(x)$  if  $x \in A_i \cap A_j$ . Then the function  $f: X \rightarrow Y$  given by  $f(x) = f_i(x)$  for  $x \in A_i$  is continuous.*

*Proof.* Exercise.  $\square$

**6.8 Example.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the absolute value function,  $f(x) = |x|$ . On the set  $A_1 = (-\infty, 0]$  this function is given by  $f|_{A_1}(x) = -x$ , and on  $A_2 = [0, +\infty)$  it is given by  $f|_{A_2}(x) = x$ . Since both  $f|_{A_1}$  and  $f|_{A_2}$  are continuous functions and  $A_1, A_2$  are closed sets in  $\mathbb{R}$  by the Closed Pasting Lemma 6.7 we obtain that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

**6.9 Definition.** A *homeomorphism* is a continuous function  $f: X \rightarrow Y$  such that  $f$  is a bijection and the inverse function  $f^{-1}: Y \rightarrow X$  is continuous.

**6.10 Proposition.** 1) *For any topological space the identify function  $\text{id}_X: X \rightarrow X$  given by  $\text{id}_X(x) = x$  is a homeomorphism.*

2) *If  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are homeomorphisms then the function  $gf: X \rightarrow Z$  is also a homeomorphism.*

3) If  $f: X \rightarrow Y$  is a homeomorphism then the inverse function  $f^{-1}: Y \rightarrow X$  is also a homeomorphism.

4) If  $f: X \rightarrow Y$  is a homeomorphism and  $Z \subseteq X$  then the function  $f|_Z: Z \rightarrow f(Z)$  is also a homeomorphism.

*Proof.* Exercise. □

**6.11 Note.** If  $f: X \rightarrow Y$  is a continuous bijection then  $f$  need not be a homeomorphism since the inverse function  $f^{-1}$  may be not continuous. For example, let  $X = \{x_1, x_2\}$  be a space with the discrete topology and let  $Y = \{y_1, y_2\}$  be a space with the antidiscrete topology. Let  $f: X \rightarrow Y$  be given by  $f(x_i) = y_i$ . The function  $f$  is continuous but  $f^{-1}$  is not continuous since the set  $\{x_1\}$  is open in  $X$ , but the set  $(f^{-1})^{-1}(\{x_1\}) = \{y_1\}$  is not open in  $Y$ .

**6.12 Proposition.** Let  $f: X \rightarrow Y$  be a continuous bijection. The following conditions are equivalent:

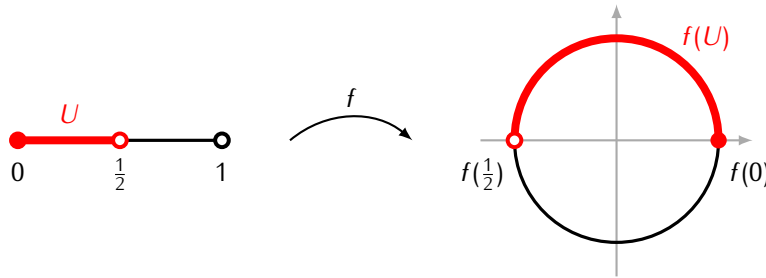
- (i) The function  $f$  is a homeomorphism.
- (ii) For each open set  $U \subseteq X$  the set  $f(U) \subseteq Y$  is open.
- (iii) For each closed set  $A \subseteq X$  the set  $f(A) \subseteq Y$  is closed.

*Proof.* Exercise. □

**6.13 Example.** Recall that  $S^1$  denotes the unit circle:

$$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

The function  $f: [0, 1) \rightarrow S^1$  given by  $f(x) = (\cos 2\pi x, \sin 2\pi x)$  is a continuous bijection, but it is not a homeomorphism since the set  $U = [0, \frac{1}{2})$  is open in  $[0, 1)$ , but  $f(U)$  is not open in  $S^1$ .



**6.14 Definition.** We say that topological spaces  $X, Y$  are *homeomorphic* if there exists a homeomorphism  $f: X \rightarrow Y$ . In such case we write:  $X \cong Y$ .

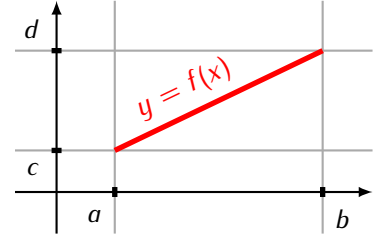
**6.15 Note.** Notice that if  $X \cong Y$  and  $Y \cong Z$  then  $X \cong Z$ .

**6.16 Example.** For any  $a < b$  and  $c < d$  the open intervals  $(a, b), (c, d) \subseteq \mathbb{R}$  are homeomorphic. To see this take e.g. the function  $f: (a, b) \rightarrow (c, d)$  defined by

$$f(x) = \left( \frac{c-d}{a-b} \right) x + \left( \frac{ad-bc}{a-b} \right)$$

This function is a continuous bijection. Its inverse function  $f^{-1}: (c, d) \rightarrow (a, b)$  is given by

$$f^{-1}(x) = \left( \frac{a-b}{c-d} \right) x + \left( \frac{cb-da}{c-d} \right)$$



so it is also continuous. By the same argument for any  $a < b$  and  $c < d$  the closed intervals  $[a, b], [c, d] \subseteq \mathbb{R}$  are homeomorphic.

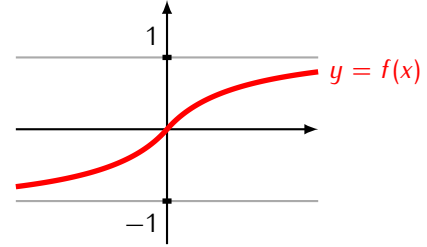
**6.17 Note.** In Chapter 7 we will show that an open interval  $(a, b)$  is not homeomorphic to a closed interval  $[c, d]$ .

**6.18 Example.** We will show that for any  $a < b$  the open interval  $(a, b)$  is homeomorphic to  $\mathbb{R}$ . Since  $(a, b) \cong (-1, 1)$  it will be enough to check that  $\mathbb{R} \cong (-1, 1)$ . Take the function  $f: \mathbb{R} \rightarrow (-1, 1)$  given by

$$f(x) = \frac{x}{1 + |x|}$$

This function is a continuous bijection with the inverse function  $f^{-1}: (-1, 1) \rightarrow \mathbb{R}$  is given by

$$f^{-1}(x) = \frac{x}{1 - |x|}$$



Since  $f^{-1}$  is continuous we obtain that  $f$  is a homeomorphism.

**6.19 Note.** If spaces  $X$  and  $Y$  are homeomorphic then usually there are many homeomorphisms  $X \rightarrow Y$ . For example, the function  $g: (-1, 1) \rightarrow \mathbb{R}$  given by

$$g(x) = \tan\left(\frac{\pi}{2}x\right)$$

is another homeomorphism between the spaces  $(-1, 1)$  and  $\mathbb{R}$ .

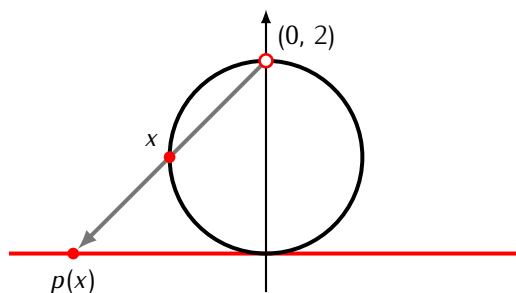
**6.20 Example.** We will show that for any point  $x_0 \in S^1$  there is a homeomorphism  $S^1 \setminus \{x_0\} \cong \mathbb{R}$ . Denote by  $S^1_{(0,1)} \subseteq \mathbb{R}^2$  the circle of radius 1 with the center at the point  $(0, 1) \in \mathbb{R}^2$ :

$$S^1_{(0,1)} := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 1)^2 = 1\}$$

It is easy to check that for  $x_0 \in S^1$  the space  $S^1 \setminus \{x_0\}$  is homeomorphic to the space  $X = S^1_{(0,1)} \setminus \{(0, 2)\}$ . Likewise, it is easy to check that  $\mathbb{R}$  is homeomorphic to the subspace  $Y \subseteq \mathbb{R}^2$  that consists of all points of the  $x$ -axis:

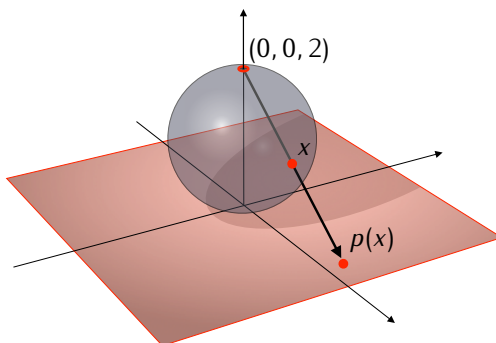
$$Y := \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\}$$

It is then enough to show that  $X \cong Y$ . A homeomorphism  $p: X \rightarrow Y$  can be constructed as follows. For any point  $x \in X$  there is a unique line in  $\mathbb{R}^2$  that passes through  $x$  and through the point  $(0, 2) \in \mathbb{R}^2$ . We define  $p(x)$  to be the point of intersection of this line with the  $x$ -axis:



The function  $p$  is called the *stereographic projection*.

In a similar way we can construct a stereographic projection in any dimension  $n \geq 1$  that gives a homeomorphism between the space  $S^n \setminus \{x_0\}$  (i.e. the  $n$ -dimensional sphere with one point deleted) and the space  $\mathbb{R}^n$ :



### Exercises to Chapter 6

**E6.1 Exercise.** Consider the set of rational numbers  $\mathbb{Q}$  as a subspace of  $\mathbb{R}$ . Show that  $\mathbb{Q}$  is not homeomorphic to a space with the discrete topology.

**E6.2 Exercise.** Prove Proposition 6.3.

**E6.3 Exercise.** Prove Proposition 6.5.

**E6.4 Exercise.** Prove Lemma 6.7.

**E6.5 Exercise.** Prove Proposition 6.12.

**E6.6 Exercise.** Let  $X$  be a topological space and let  $f, g: X \rightarrow \mathbb{R}$  be continuous functions.

a) Show that the set

$$A = \{x \in X \mid f(x) \geq g(x)\}$$

is closed in  $X$ .

b) Let  $h: X \rightarrow \mathbb{R}$  be a function given by  $h(x) = \max\{f(x), g(x)\}$ . Show that  $h$  is continuous.

**E6.7 Exercise.** Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions such that  $f(x) > g(x)$  for all  $x \in \mathbb{R}$ . Define subspaces  $X, Y$  of  $\mathbb{R}^2$  as follows.

$$X := \{(x, y) \in \mathbb{R}^2 \mid g(x) \leq y \leq f(x)\} \quad Y := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$$

Show that  $X \cong Y$ .

**E6.8 Exercise.** Let  $x_0 = (0, 0) \in \mathbb{R}^2$  and let  $\bar{B}(x_0, 1) \subseteq \mathbb{R}^2$  be a closed ball defined by the Euclidean metric  $d$ :

$$\bar{B}(x_0, 1) = \{x \in \mathbb{R}^2 \mid d(x, x_0) \leq 1\}$$

Define subspaces  $X, Y \subseteq \mathbb{R}^2$  as follows:

$$X := \mathbb{R}^2 \setminus \{x_0\} \quad Y := \mathbb{R}^2 \setminus \bar{B}(x_0, 1)$$

Show that  $X \cong Y$ .

**E6.9 Exercise.** Let  $(X, \varrho)$  be a metric space. A subspace  $Y \subseteq X$  is a *retract* of  $X$  if there exists a continuous function  $r: X \rightarrow Y$  such that  $r(x) = x$  for all  $x \in Y$ . Show that if  $Y \subseteq X$  is a retract of  $X$  then  $Y$  is a closed in  $X$ .