11 | Tietze Extension Theorem

11.1 Generalized Urysohn Lemma. *Let X be a normal space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$. For any $a, b \in \mathbb{R}$, $a < b$ there exists a continuous function $f\colon X \to [a, b]$ such that $A \subseteq f^{-1}(\{a\})$ and $B \subseteq f^{-1}(\{b\})$.*

11.2 Tietze Extension Theorem (v.1). *Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f\colon A \to [a, b]$ be a continuous function for some $[a, b] \subseteq \mathbb{R}$. There exits a continuous function $\bar{f}\colon X \to [a, b]$ such that $\bar{f}|_A = f$.*

11.3 Definition. Let X, Y be a topological spaces and let $\{f_n \colon X \to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ *converges pointwise* to a function $f \colon X \to Y$ if for each $x \in X$ the sequence $\{f_n(x)\} \subseteq Y$ converges to the point $f(x)$.

11.5 Definition. Let X be a topological space, let (Y, ϱ) be a metric space, and let $\{f_n \colon X \to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ *converges uniformly* to a function $f \colon X \to Y$ if for every $\varepsilon > 0$ there exists $N > 0$ such that

$$\varrho(f(x), f_n(x)) < \varepsilon$$

for all $x \in X$ and for all $n > N$.

11.6 Note. If a sequence $\{f_n\}$ converges uniformly to f then it also converges pointwise to f, but the converse is not true in general.

11.7 Proposition. *Let X be a topological space and let (Y, ϱ) be a metric space. Assume that $\{f_n \colon X \to Y\}$ is a sequence of functions that converges uniformly to $f \colon X \to Y$. If all functions f_n are continuous then f is also a continuous function.*

11.8 Lemma. *Let X be a normal space, $A \subseteq X$ be a closed set, and let $f\colon A \to \mathbb{R}$ be a continuous function such that for some $C > 0$ we have $|f(x)| \leq C$ for all $x \in A$. There exists a continuous function $g\colon X \to \mathbb{R}$ such that $|g(x)| \leq \frac{1}{3}C$ for all $x \in X$ and $|f(x) - g(x)| \leq \frac{2}{3}C$ for all $x \in A$.*

Proof of Theorem 11.2.

□

11.9 Tietze Extension Theorem (v.2). *Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f\colon A \to \mathbb{R}$ be a continuous function. There exits a continuous function $\bar{f}\colon X \to \mathbb{R}$ such that $\bar{f}|_A = f$.*

11.10 Theorem. *Let X be a space satisfying T_1. The following conditions are equivalent:*

1) *X is a normal space.*
2) *For any closed sets $A, B \subseteq X$ such that $A \cap B = \varnothing$ there is a continuous function $f\colon X \to [0,1]$ such that such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$.*
3) *If $A \subseteq X$ is a closed set then any continuous function $f\colon A \to \mathbb{R}$ can be extended to a continuous function $\bar{f}\colon X \to \mathbb{R}$.*