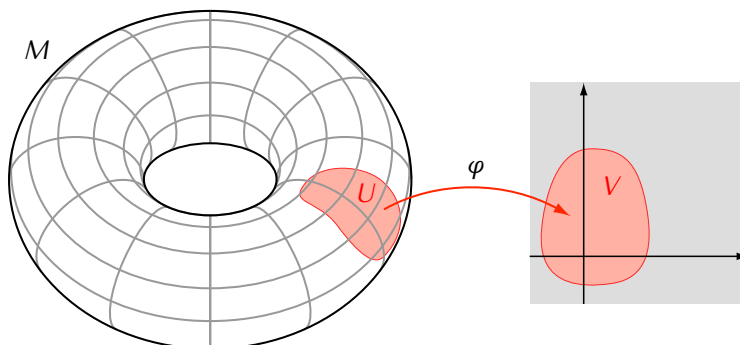


# 13 | Metrization of Manifolds

**13.1 Definition.** A *topological manifold of dimension  $n$*  is a topological space  $M$  which is a Hausdorff, second countable, and such that every point of  $M$  has an open neighborhood homeomorphic to an open subset of  $\mathbb{R}^n$  (we say that  $M$  is *locally homeomorphic to  $\mathbb{R}^n$* ).

**13.2 Note.** Let  $M$  be a manifold of dimension  $n$ . If  $U \subseteq M$  is an open set and  $\varphi: U \rightarrow V$  is a homeomorphism of  $U$  with some open set  $V \subseteq \mathbb{R}^n$  then we say that  $U$  is a *coordinate neighborhood* and  $\varphi$  is a *coordinate chart* on  $M$ .



**13.3 Lemma.** *If  $M$  is an  $n$ -dimensional manifold then:*

- 1) for any point  $x \in M$  there exists a coordinate chart  $\varphi: U \rightarrow V$  such that  $x \in U$ ,  $V$  is an open ball  $V = B(y, r)$ , and  $\varphi(x) = y$ ;*
- 2) for any point  $x \in M$  there exists a coordinate chart  $\psi: U \rightarrow V$  such that  $x \in U$ ,  $V = \mathbb{R}^n$ , and  $\psi(x) = 0$ .*

*Proof.* Exercise. □

**13.4 Example.** A space  $M$  is a manifold of dimension 0 if and only if  $M$  is a countable (finite or infinite) discrete space.

**13.5 Example.** If  $U$  is an open set in  $\mathbb{R}^n$  then  $U$  is an  $n$ -dimensional manifold. The identity map  $\text{id}: U \rightarrow U$  is then a coordinate chart defined on the whole manifold  $U$ . In particular  $\mathbb{R}^n$  is an  $n$ -dimensional manifold.

**13.6 Example.** The  $n$ -dimensional sphere

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

is an  $n$ -dimensional manifold.

**13.7 Proposition.** *If  $M$  is an  $m$ -dimensional manifold and  $N$  is an  $n$ -dimensional manifold then  $M \times N$  is an  $m + n$ -dimensional manifold.*

*Proof.* Exercise. □

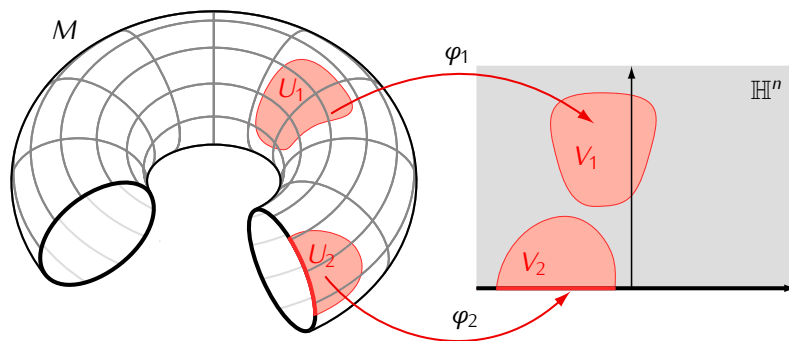
**13.9 Note.** There exist topological spaces that are locally homeomorphic to  $\mathbb{R}^n$ , but do not satisfy the other conditions of the definition of a manifold (13.1).

**13.10 Invariance of Dimension Theorem.** *If  $M$  is a non-empty topological space such that  $M$  is a manifold of dimension  $m$  and  $M$  is also a manifold of dimension  $n$  then  $m = n$ .*

**13.11 Definition.** A *topological  $n$ -dimensional manifold with boundary* is a topological space  $M$  which is a Hausdorff, second countable, and such that every point of  $M$  has an open neighborhood homeomorphic to an open subset of  $\mathbb{H}^n$ .

**13.12** Let  $\partial\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{H}^n \mid x_n = 0\}$ . If  $M$  is an  $n$ -dimensional manifold with boundary,  $\varphi: U \rightarrow V$  is a coordinate chart, and  $x \in U$  then there are two possibilities:

- 1)  $\varphi(x) \in \partial\mathbb{H}^n$
- 2)  $\varphi(x) \notin \partial\mathbb{H}^n$



In the first case we say that the point  $x$  is a *boundary point* of  $M$ , and in the the second case that  $x$  is an *interior point* of  $M$ .

**13.13 Theorem.** Let  $M$  be an  $n$ -dimensional manifold with boundary, let  $x_0 \in M$  and let  $\varphi: U \rightarrow V$  be a local coordinate chart such that  $x_0 \in U$ . If  $\varphi(x_0) \in \partial\mathbb{H}^n$  then for any other local coordinate chart  $\psi: U' \rightarrow V'$  such that  $x_0 \in U'$  we have  $\psi(x_0) \in \partial\mathbb{H}^n$ .

**13.14 Definition.** Let  $M$  be a manifold with boundary. The subspace of  $M$  consisting of all boundary points of  $M$  is called *the boundary of  $M$*  and it is denoted by  $\partial M$ .

**13.15 Example.** The space  $\mathbb{H}^n$  is trivially an  $n$ -dimensional manifold with boundary.

**13.16 Example.** For any  $n$  the closed  $n$ -dimensional ball

$$\overline{B}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$$

is an  $n$ -dimensional manifold with boundary (exercise). In this case we have  $\partial \overline{B}^n = S^{n-1}$ .

**13.17 Example.** If  $M$  is a manifold (without boundary) then we can consider it as a manifold with boundary, where  $\partial M = \emptyset$ .

**13.19 Proposition.** *If  $M$  is an  $n$ -dimensional manifold with boundary then:*

- 1)  $M \setminus \partial M$  is an open subset of  $M$  and it is an  $n$ -dimensional manifold (without boundary);
- 2)  $\partial M$  is a closed subset of  $M$  and it is an  $(n - 1)$ -dimensional manifold (without boundary).

*Proof.* Exercise. □

**13.20 Theorem.** *Every topological manifold (with or without boundary) is metrizable.*

**13.21 Lemma.** *Let  $M$  be an  $n$ -dimensional topological manifold, and let  $\varphi: U \rightarrow V$  be a coordinate chart on  $M$ . If  $\bar{B}(x, r)$  is a closed ball in  $\mathbb{R}^n$  such that  $\bar{B}(x, r) \subseteq V$  then the set  $\varphi^{-1}(\bar{B}(x, r))$  is closed in  $M$ .*

