12 | Urysohn Metrization Theorem

In this chapter we return to the problem of determining which topological spaces are metrizable i.e. can be equipped with a metric which is compatible with their topology. We have seen already that any metrizable space must be normal, but that not every normal space is metrizable. We will show, however, that if a normal space space satisfies one extra condition then it is metrizable. Recall that a space X is second countable if it has a countable basis. We have:

12.1 Urysohn Metrization Theorem. *Every second countable normal space is metrizable.*

The main idea of the proof is to show that any space as in the theorem can be identified with a subspace of some metric space. To make this more precise we need the following:

12.2 Definition. A continuous function $i\colon X \to Y$ is an *embedding* if its restriction $i\colon X \to i(X)$ is a homeomorphism (where $i(X)$ has the topology of a subspace of Y).

12.3 Example. The function $i\colon (0,1) \to \mathbb{R}$ given by $i(x) = x$ is an embedding. The function $j\colon (0,1) \to \mathbb{R}$ given by $j(x) = 2x$ is another embedding of the interval $(0,1)$ into $\mathbb{R}$.

12.4 Note. 1) If $j\colon X \to Y$ is an embedding then j must be 1-1.

2) Not every continuous 1-1 function is an embedding. For example, take $\mathbb{N} = \{0, 1, 2, \dots\}$ with the discrete topology, and let $f\colon \mathbb{N} \to \mathbb{R}$ be given

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} & \text{if } n > 0 \end{cases}$$

The function f is continuous and it is 1-1, but it is not an embedding since $f\colon \mathbb{N} \to f(\mathbb{N})$ is not a homeomorphism.

12.5 Lemma. *If $j\colon X \to Y$ is an embedding and Y is a metrizable space then X is also metrizable.*

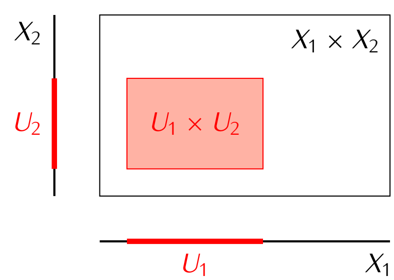
Proof. Let μ be a metric on Y. Define a metric ϱ on X by $\varrho(x_1, x_2) = \mu(j(x_1), j(x_2))$. It is easy to check that the topology on X is induced by the metric ϱ (exercise). $\square$

Let now X be a space as in Theorem 12.1. In order to show that X is metrizable it will be enough to construct an embedding $j\colon X \to Y$ where Y is metrizable. The space Y will be obtained as a product of topological spaces:

12.6 Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The *product topology* on $\prod_{i\in I} X_i$ is the topology generated by the basis

$$\mathcal{B} = \left\{\prod_{i\in I} U_i \mid U_i \text{ is open in } X_i \text{ and } U_i \neq X_i \text{ for finitely many indices } i \text{ only}\right\}$$

12.7 Note. 1) If X_1, X_2 are topological spaces then the product topology on $X_1 \times X_2$ is the topology induced by the basis $\mathcal{B} = \{U_1 \times U_2 \mid U_1 \text{ is open in } X_1,\ U_2 \text{ is open in } X_2\}$.



2) In general if $X_1, \dots, X_n$ are topological spaces then the product topology on $X_1 \times \dots \times X_n$ is the topology generated by the basis $\mathcal{B} = \{U_1 \times \dots \times U_n \mid U_i \text{ is open in } X_i\}$.

3) If $\{X_i\}_{i=1}^{\infty}$ is an infinitely countable family of topological spaces then the basis of the product topology on $\prod_{i=1}^{\infty} X_i$ consists of all sets of the form

$$U_1 \times \dots \times U_n \times X_{n+1} \times X_{n+2} \times X_{n+3} \times \dots$$

where $n \geq 0$ and $U_i \subseteq X_i$ is an open set for $i = 1, \dots, n$.

12.8 Proposition. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces and for $j \in I$ let*

$$p_j\colon \prod_{i\in I} X_i \to X_j$$

be the projection onto the j-th factor: $p_j((x_i)_{i\in I}) = x_j$. Then:

1) *for any $j \in I$ the function p_j is continuous.*
2) *A function $f\colon Y \to \prod_{i\in I} X_i$ is continuous if and only if the composition $p_j f\colon Y \to X_j$ is continuous for all $j \in I$*

Proof. Exercise. □

12.9 Note. Notice that the basis $\mathcal{B}$ given in Definition 12.6 consists of all sets of the form

$$p_{i_1}^{-1}(U_1) \cap \cdots \cap p_{i_n}^{-1}(U_{i_n})$$

where $i_1, \dots, i_n \in I$ and $U_{i_1} \subseteq X_{i_1}, \dots, U_{i_n} \subseteq X_{i_n}$ are open sets.

12.10 Proposition. *If $\{X_i\}_{i=1}^{\infty}$ is a countable family of metrizable spaces then $\prod_{i=1}^{\infty} X_i$ is also a metrizable space.*

Proof. Let ϱ_i be a metric on X_i. We can assume that for any $x, x' \in X_i$ we have $\varrho_i(x, x') \leq 1$. Indeed, if ϱ_i does not have this property then we can replace it by the metric ϱ_i' given by:

$$\varrho_i'(x, x') = \begin{cases} \varrho_i(x, x') & \text{if } \varrho_i(x, x') \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

The metrics ϱ_i and ϱ_i' are equivalent (exercise), and so they define the same topology on the space X_i.

Given metrics ϱ_i on X_i satisfying the above condition define a metric ϱ_∞ on $\prod_{i=1}^{\infty} X_i$ by:

$$\varrho_\infty((x_i), (x_i')) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varrho_i(x_i, x_i')$$

The topology induced by the metric ϱ_∞ on $\prod_{i=1}^{\infty} X_i$ is the product topology (exercise). □

12.11 Example. The *Hilbert cube* is the topological space $[0, 1]^{\aleph_0}$ obtained as the infinite countable product of the closed interval $[0, 1]$:

$$[0, 1]^{\aleph_0} = \prod_{i=1}^{\infty} [0, 1]$$

Elements of $[0, 1]^{\aleph_0}$ are infinite sequences $(t_i) = (t_1, t_2, \dots)$ where $t_i \in [0, 1]$ for $i = 1, 2, \dots$ The Hilbert cube is a metric space with a metric ϱ given by

$$\varrho((t_i), (s_i)) = \sum_{i=1}^{\infty} \frac{1}{2^i} |t_i - s_i|$$

Theorem 12.1 is a consequence of the following fact:

12.12 Theorem. *If X is a second countable normal space then there exists an embedding $j\colon X \to [0,1]^{\aleph_0}$.*

Theorem 12.12 will follow in turn from a more general result on embeddings of topological spaces:

12.13 Definition. Let X be a topological space and let $\{f_i\}_{i\in I}$ be a family of continuous functions $f_i\colon X \to [0,1]$. We say that the family $\{f_i\}_{i\in I}$ *separates points from closed sets* if for any point $x_0 \in X$ and any closed set $A \subseteq X$ such that $x_0 \notin A$ there is a function $f_j \in \{f_i\}_{i\in I}$ such that $f_j(x_0) > 0$ and $f_j|_A = 0$.

12.14 Embedding Lemma. *Let X be a T_1-space. If $\{f_i\colon X \to [0,1]\}_{i\in I}$ is a family that separates points from closed sets then the map*

$$f_\infty\colon X \to \prod_{i\in I} [0,1]$$

given by $f_\infty(x) = (f_i(x))_{i\in I}$ is an embedding.

12.15 Note. If the family $\{f_i\}_{i\in I}$ in Lemma 12.14 is infinitely countable then f_∞ is an embedding of X into the Hilbert cube $[0,1]^{\aleph_0}$.

We will show first that Theorem 12.12 follows from Lemma 12.14, and then we will prove the lemma.

Proof of Theorem 12.12. Let $\mathcal{B} = \{V_i\}_{i=1}^{\infty}$ be a countable basis of X, and let S the set given by

$$S := \{(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \overline{V}_i \subseteq V_j\}$$

If $(i,j) \in S$ then the sets $\overline{V}_i$ and $X \smallsetminus V_j$ are closed and disjoint, so by the Urysohn Lemma 10.1 there is a continuous function $f_{ij}\colon X \to [0,1]$ such that

$$f_{ij}(x) = \begin{cases} 1 & \text{if } x \in \overline{V_i} \\ 0 & \text{if } x \in X \smallsetminus V_j \end{cases}$$

We will show that the family $\{f_{ij}\}_{(i,j)\in S}$ separates points from closed sets. Take $x_0 \in X$ and let $A \subseteq X$ be an closed set such that $x_0 \notin A$. Since $\mathcal{B} = \{V_i\}_{i=1}^{\infty}$ is a basis of X there is $V_j \in \mathcal{B}$ such that $x_0 \in V_j$ and $V_j \subseteq X \smallsetminus A$. Using Lemma 10.3 we also obtain that there exists $V_i \in \mathcal{B}$ such that $x_0 \in V_i$ and $\overline{V}_i \subseteq V_j$. We have $f_{ij}(x_0) = 1$. Also, since $A \subseteq X \smallsetminus V_j$ we have $f_{ij}|_A = 0$.

By the Embedding Lemma 12.14 the family $\{f_{ij}\}_{(i,j)\in S}$ defines an embedding

$$f_\infty\colon X \to \prod_{(i,j)\in S} [0,1]$$

The set S is countable. If it is infinite then $\prod_{(i,j)\in S}[0,1] \cong [0,1]^{\aleph_0}$. If S is finite then $\prod_{(i,j)\in S}[0,1] \cong [0,1]^N$ for some $N \geq 0$ and $[0,1]^N$ can be identified with a subspace of $[0,1]^{\aleph_0}$. $\square$

Proof of Theorem 12.1. Follows from Theorem 12.12, Lemma 12.5, and the fact that the Hilbert cube is a metric space (12.11). □

It remains to prove Lemma 12.14:

Proof of Lemma 12.14. We need to show that the function f_∞ satisfies the following conditions:

1) f_∞ is continuous;
2) f_∞ is 1-1;
3) $f_\infty\colon X \to f_\infty(X)$ is a homeomorphism.

1) Let $p_j\colon \prod_{i\in I}[0,1] \to [0,1]$ be the projection onto the j-th coordinate. Since $p_j f_\infty = f_j$, thus $p_j f_\infty$ is a continuous function for all $j \in I$. Therefore by Proposition 12.8 the function f_∞ is continuous.

2) Let $x, y \in X$, $x \neq y$. Since X is a T_1-space the set $\{y\}$ is closed in X. Therefore there is a function $f_j \in \{f_i\}_{i\in I}$ such that $f_j(x) > 0$ and $f_j(y) = 0$. In particular $f_j(x) \neq f_j(y)$. Since $f_j = p_j f_\infty$ this gives $p_j f_\infty(x) \neq p_j f_\infty(y)$. Therefore $f_\infty(x) \neq f_\infty(y)$.

3) Let $U \subseteq X$ be an open set. We need to prove that the set $f_\infty(U)$ is open in $f(X)$. It will suffice to show that for any $x_0 \in U$ there is a set V open in $f_\infty(X)$ such that $f_\infty(x_0) \in V$ and $V \subseteq f_\infty(U)$.

Given $x_0 \in U$ let $f_j \in \{f_i\}_{i\in I}$ be a function such that $f_j(x_0) > 0$ and $f_j|_{X\smallsetminus U} = 0$. Let $p_j\colon \prod_{i\in I}[0,1] \to [0,1]$ be the projection onto the j-th coordinate. Define

$$V := f_\infty(X) \cap p_j^{-1}(\,(0,1]\,)$$

The set V is open in $f_\infty(X)$ since $p_j^{-1}(\,(0,1]\,)$ is open in $\prod_{i\in I}[0,1]$. Notice that

$$\begin{aligned} V &= \{f_\infty(x) \mid \ x \in X \text{ and } p_j f_\infty(x) > 0\} \\ &= \{f_\infty(x) \mid \ x \in X \text{ and } f_j(x) > 0\} \end{aligned}$$

Since $f_j(x_0) > 0$ we have $f_\infty(x_0) \in V$. Finally, since $f_j(x) = 0$ for all $x \in X \smallsetminus U$ we get that $V \subseteq f_\infty(U)$. □

One can show that the following holds:

12.16 Proposition. *Every second countable regular space is normal.*

Proof. Exercise. □

As a consequence Theorem 12.1 can be reformulated as follows:

12.17 Urysohn Metrization Theorem (v.2). *Every second countable regular space is metrizable.*

While every metrizable space is normal (and regular) such spaces do not need to be second countable. For example, any discrete space X is metrizable, but if X consists of uncountably many points it does not have a countable basis (Exercise 4.10). This means that the converse of the Urysohn Metrization Theorem does not hold. However, this theorem can be generalized to give conditions that are both sufficient and necessary for metrizability of a space. We finish this chapter by giving the statement of such result without proof.

12.18 Definition. Let X be a topological space. A collection $\mathcal{U} = \{U_i\}_{i\in I}$ of open sets in X is *locally finite* if each point $x \in X$ has an open neighborhood V_x such that $V_x \cap U_i \neq \varnothing$ for finitely many $i \in I$ only.

A collection $\mathcal{U}$ is *countably locally finite* if it can be decomposed into a countable union $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ where each collection $\mathcal{U}_n$ is locally finite.

12.19 Nagata-Smirnov Metrization Theorem. *Let X be a topological space. The following conditions are equivalent:*

1) *X is metrizable.*
2) *X is regular and it has a basis which is countably locally finite.*

Exercises to Chapter 12

E12.1 Exercise. Show that the product topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ is the same as the topology induced by the Euclidean metric.

E12.2 Exercise. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The *box topology* on $\prod_{i\in I} X_i$ is the topology generated by the basis

$$\mathcal{B} = \left\{\prod_{i\in I} U_i \mid U_i \text{ is open in } X_i\right\}$$

Notice that for products of finitely many spaces the box topology is the same as the product topology, but that it differs if we take infinite products.

Let $X = \prod_{n=1}^{\infty}[0, 1]$ be the product of countably many copies of the interval $[0, 1]$. Consider X as a topological space with the box topology. Show that the map $f\colon [0, 1] \to X$ given by $f(t) = (t, t, t, \dots)$ is not continuous.

E12.3 Exercise. Prove Proposition 12.8

E12.4 Exercise. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and for $i \in I$ let A_i be a closed set in X_i. Show that the set $\prod_{i\in I} A_i$ is closed in the product topology on $\prod_{i\in I} X_i$.

E12.5 Exercise. Let X and Y be non-empty topological spaces. Show that the space $X \times Y$ is connected if and only if X and Y are connected.

E12.6 Exercise. Assume that X, Y are spaces such that $\mathbb{R} \cong X \times Y$. Show that either X or Y is consists of only one point.

E12.7 Exercise. Let X, Y be topological spaces. For a (not necessarily continuous) function $f\colon X \to Y$ the *graph* of f is the subspace $\Gamma(f)$ of $X \times Y$ given by

$$\Gamma(f) = \{(x, f(x)) \in X \times Y \mid x \in X\}$$

Show that if Y is a Hausdorff space and $f\colon X \to Y$ is a continuous function then $\Gamma(f)$ is closed in $X \times Y$.

E12.8 Exercise. Let X_1, X_2 be topological spaces, and for $i = 1, 2$ let $p_i\colon X_1 \times X_2 \to X_i$ be the projection map.

a) Show that if a set $U \subseteq X_1 \times X_2$ is open in $X_1 \times X_2$ then $p_i(U)$ is open in X_i.

b) Is it true that if $A \subseteq X_1 \times X_2$ is a closed set then $p_i(A)$ must be closed is X_i ? Justify your answer.

E12.9 Exercise. The goal of this exercise is to complete the proof of Proposition 12.10. For $i = 1, 2, \ldots$ let (X_i, ϱ_i) be a metric space such that $\varrho_i(x, x') \leq 1$ for all $x, x' \in X_i$. Let ϱ_∞ be a metric the Cartesian product $\prod_{i=1}^{\infty} X_i$ given by

$$\varrho_\infty((x_i), (x_i')) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varrho_i(x_i, x_i')$$

Show that the topology defined by ϱ_∞ is the same as the product topology.

E12.10 Exercise. The goal of this exercise is to give a proof of Proposition 12.16. Let X be a second countable regular space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$.

a) Show that there exist countable families of open sets $\{U_1, U_2, \ldots\}$ and $\{V_1, V_2, \ldots\}$ such that

(i) $A \subseteq \bigcup_{i=1}^{\infty} U_i$ and $B \subseteq \bigcup_{i=1}^{\infty} V_i$

(ii) for all $i \geq 1$ we have $\overline{U}_i \cap B = \varnothing$ and $\overline{V}_i \cap A = \varnothing$

b) For $n \geq 1$ define

$$U_n' := U_n \smallsetminus \bigcup_{i=1}^{n} \overline{V_i} \quad \text{and} \quad V_n' := V_n \smallsetminus \bigcup_{i=1}^{n} \overline{U}_i$$

Let $U' = \bigcup_{n=1}^{\infty} U_n'$ and $V' = \bigcup_{n=1}^{\infty} V_n'$. Show that U' and V' are open sets, that $A \subseteq U'$ and $B \subseteq V'$, and that $U' \cap V' = \varnothing$.

E12.11 Exercise. Let $\mathbb{R}_{disc}$ denote the real line with the discrete topology and let $X = \prod_{n=1}^{\infty} \mathbb{R}_{disc}$.

a) Show that X is not second countable.

b) By Proposition 12.10 we know that X is a metrizable space. Verify this fact without using Proposition 12.10, but using instead only topological properties of X and the Nagata-Smirnov Metrization Theorem 12.19.