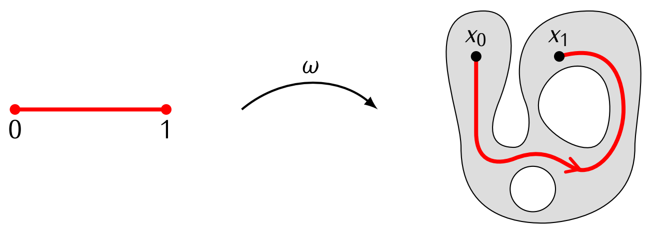
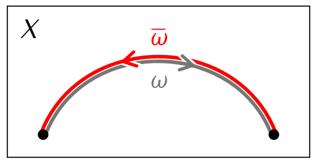
8 | Path Connectedness

8.1 Definition. Let X be a topological space. A *path* in X is a continuous function $\omega\colon [0,1] \to X$. If $\omega(0) = x_0$ and $\omega(1) = x_1$ then we say that ω joins x_0 with x_1.

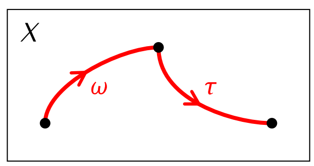


8.2 Definition. 1) If $\omega\colon [0,1] \to X$ is a path in X then the *inverse* of ω is the path $\overline{\omega}$ given by $\overline{\omega}(t) = \omega(1-t)$ for $t \in [0,1]$.



2) If $\omega.\tau\colon [0,1] \to X$ are paths such that $\omega(1) = \tau(0)$ then the *concatenation* of ω and τ if the path $\omega * \tau$ given by

$$(\omega * \tau)(t) = \begin{cases} \omega(2t) & \text{for } t \in [0, 1/2] \\ \tau(2t-1) & \text{for } t \in [1/2, 1] \end{cases}$$

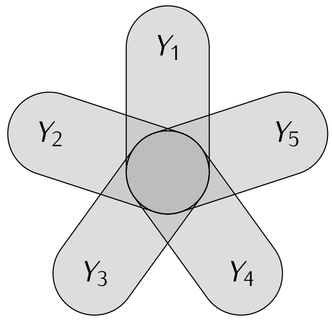


8.3 Definition. A space X is *path connected* if for every $x_0, x_1 \in X$ there is a path joining x_0 with x_1.

8.5 Proposition. *Every path connected space is connected.*

Proof. Exercise. $\square$

8.7 Proposition. *Let X be a topological space and for $i \in I$ let Y_i be a subspace of X. Assume that $\bigcup_{i\in I} Y_i = X$ and $\bigcap_{i\in I} Y_i \neq \varnothing$. If Y_i is path connected for each $i \in I$ then X is also path connected.*



8.8 Definition. Let X be a topological space. A *path connected component* of X is a subspace $Y \subseteq X$ such that

1) Y is path connected
2) if $Y \subseteq Z \subseteq X$ and Z is path connected then $Y = Z$.

8.9 Proposition. *Let X be a topological space.*

1) For every point $x_0 \in X$ there exist a path connected component $Y \subseteq X$ such that $x_0 \in Y$.
2) If Y, Y' are path connected components of X then either $Y \cap Y' = \varnothing$ or $Y = Y'$.

Proof. Similar to the proof of Proposition 7.18. □

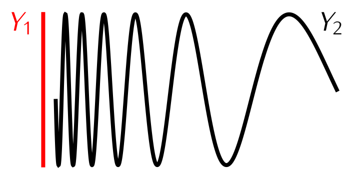
8.10 Proposition. *Let $x_0 \in X$ The path connected component $Y \subseteq X$ that contains x_0 is given by:*

$$Y = \{x \in X \mid \textit{there exists a path joining } x \textit{ with } x_0\}$$

Proof. Exercise. □

8.11 Example. Let Y be the topologist's sine curve. The space Y has only one connected component (since Y is connected). On the other hand it has two path connected components:

$$Y_1 = \{(0, y) \mid -1 \leq y \leq 1\} \quad \text{and} \quad Y_2 = \{(x, \sin(\tfrac{1}{x})) \mid x > 0\}$$



8.12 Definition. Let X be a topological space.

1) X is *locally connected* if for any $x \in X$ and any open neighborhood U of x there is an open neighborhood V of x such that $V \subseteq U$ and V is connected.

2) X is *locally path connected* if for any $x \in X$ and any open neighborhood U of x there is an open neighborhood V of x such that $V \subseteq U$ and V is path connected.

8.15 Proposition. *If X is locally path connected then it is locally connected.*

Proof. Exercise. □

8.16 Proposition. *If a space X is locally connected then connected components of X are open in X.*

Proof. Exercise. □

8.17 Proposition. *If a space X is locally path connected then path connected components of X are open in X.*

Proof. Exercise. □

8.18 Proposition. *If X is a connected and locally path connected space then X is path connected.*