7 | Connectedness

7.2 Definition. A topological space X is *connected* if for any two open sets $U, V \subseteq X$ such that $U \cup V = X$ and $U, V \neq \varnothing$ we have $U \cap V \neq \varnothing$.

7.3 Definition. If X is a topological space and $U, V \subseteq X$ are non-empty open sets such that $U \cap V = \varnothing$ and $U \cup V = X$ then we say that $\{U, V\}$ is a *separation* of X.

7.5 Proposition. *Let $a < b$. The intervals (a, b), $[a, b]$, $(a, b]$, and $[a, b)$ are connected topological spaces.*

7.6 Proposition. *If X is a connected subspace of $\mathbb{R}$ then X is an interval (either open, closed, or half-closed, finite or infinite).*

Proof. Exercise. $\square$

7.8 Proposition. *Let $f\colon X \to Y$ be a continuous function. If f is onto and the space X is connected then Y is also connected.*

7.9 Corollary. *If $f\colon X \to Y$ is a continuous function and X is a connected space then $f(X)$ is connected.*

7.10 Intermediate Value Theorem. *Let X be a connected topological space and let $f\colon X \to \mathbb{R}$ be a continuous function. If $a < b$ are points in $\mathbb{R}$ such that $a = f(x)$ and $b = f(y)$ for some $x, y \in X$ then for each $c \in [a, b]$ there exists $z \in X$ such that $c = f(z)$.*

7.11 Corollary. *If $X \cong Y$ and X is a connected space then Y is also connected.*

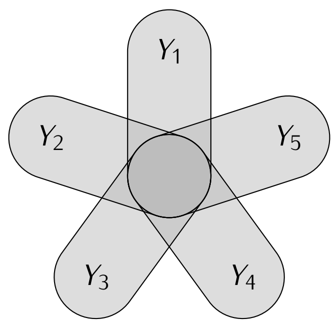
7.13 Note. A *topological invariant* is a property of topological spaces such that if a space X has this property and $X \cong Y$ then Y also has this property. By Corollary 7.11 connectedness is a topological invariant.

7.14 Proposition. *Let X be a topological space. The following conditions are equivalent :*

1) *X is connected*
2) *For any closed sets $A, B \subseteq X$ such that $A, B \neq X$ and $A \cap B = \varnothing$ we have $A \cup B \neq X$.*
3) *If $A \subseteq X$ is a set that is both open and closed then either $A = X$ or $A = \varnothing$.*
4) *If $D = \{0, 1\}$ is a space with the discrete topology then any continuous function $f\colon X \to D$ is a constant function.*

Proof. Exercise. □

7.15 Proposition. *Let X be a topological space and for $i \in I$ let Y_i be a subspace of X. Assume that $\bigcup_{i\in I} Y_i = X$ and $\bigcap_{i\in I} Y_i \neq \varnothing$. If Y_i is connected for each $i \in I$ then X is also connected.*



7.16 Corollary. *The space $\mathbb{R}^n$ is connected for all $n \geq 1$.*

7.17 Definition. Let X be a topological space. A *connected component* of X is a subspace $Y \subseteq X$ such that

1) Y is connected
2) if $Y \subseteq Z \subseteq X$ and Z is connected then $Y = Z$.

7.18 Proposition. *Let X be a topological space.*

1) For every point $x_0 \in X$ there exist a connected component $Y \subseteq X$ such that $x_0 \in Y$.
2) If Y, Y' are connected components of X then either $Y \cap Y' = \varnothing$ or $Y = Y'$.

7.19 Corollary. *Let X be a topological space. If $Z \subseteq X$ is a connected subspace then there exists a connected component $Y \subseteq X$ such that $Z \subseteq Y$.*

Proof. Exercise. $\square$

7.20 Corollary. *Let $f\colon X \to Y$ be a continuous function. If X is a connected space then there exists a connected component $Z \subseteq Y$ such that $f(X) \subseteq Z$.*

Proof. Exercise. $\square$