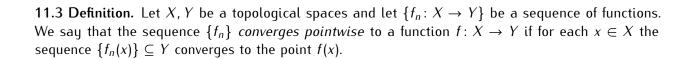
11 Tietze Extension Theorem

11.1 Generalized Urysohn Lemma. Let X be a normal space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \emptyset$. For any $a, b \in \mathbb{R}$, a < b there exists a continuous function $f: X \to [a, b]$ such that $A \subseteq f^{-1}(\{a\})$ and $B \subseteq f^{-1}(\{b\})$.

11.2 Tietze Extension Theorem (v.1). Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f: A \to [a,b]$ be a continuous function for some $[a,b] \subseteq \mathbb{R}$. There exits a continuous function $\bar{f}: X \to [a,b]$ such that $\bar{f}|_A = f$.



11.5 Definition. Let X be a topological space, let (Y,ϱ) be a metric space, and let $\{f_n\colon X\to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ converges uniformly to a function $f\colon X\to Y$ if for every $\varepsilon>0$ there exists N>0 such that

$$\varrho(f(x),f_n(x))<\varepsilon$$

for all $x \in X$ and for all n > N.

11.6 Note. If a sequence $\{f_n\}$ converges uniformly to f then it also converges pointwise to f, but the converse is not true in general.

11.7 Proposition. Let X be a topological space and let (Y, ϱ) be a metric space. Assume that $\{f_n \colon X \to Y\}$ is a sequence of functions that converges uniformly to $f \colon X \to Y$. If all functions f_n are continuous then f is also a continuous function.

11.8 Lemma. Let X be a normal space, $A \subseteq X$ be a closed set, and let $f: A \to \mathbb{R}$ be a continuous function such that for some C > 0 we have $|f(x)| \le C$ for all $x \in A$. There exists a continuous function $g: X \to \mathbb{R}$ such that $|g(x)| \le \frac{1}{3}C$ for all $x \in X$ and $|f(x) - g(x)| \le \frac{2}{3}C$ for all $x \in A$.

Proof of Theorem 11.2.

11.9 Tietze Extension Theorem (v.2). Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f: A \to \mathbb{R}$ be a continuous function. There exits a continuous function $\bar{f}: X \to \mathbb{R}$ such that $\bar{f}|_A = f$.

- **11.10 Theorem.** Let X be a space satisfying T_1 . The following conditions are equivalent:
 - 1) X is a normal space.
 - 2) For any closed sets $A, B \subseteq X$ such that $A \cap B = \emptyset$ there is a continuous function $f: X \to [0, 1]$ such that such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$.
 - 3) If $A \subseteq X$ is a closed set then any continuous function $f: A \to \mathbb{R}$ can be extended to a continuous function $\bar{f}: X \to \mathbb{R}$.