19 | Quotient Spaces

19.1 Definition. Let X be a set. An *equivalence relation on X* is a binary relation $\sim$ satisfying three properties:

1) $x \sim x$ for all $x \in X$ (reflexivity)
2) if $x \sim y$ then $y \sim x$ (symmetry)
3) if $x \sim y$ and $y \sim z$ then $x \sim z$ (transitivity)

19.4 Definition. Let X we a set with an equivalence relation $\sim$ and let $x \in X$. The *equivalence class* of x is the subset $[x] \subseteq X$ consisting of all elements that are in the relation with x:

$$[x] = \{y \in X \mid x \sim y\}$$

19.7 Proposition. *Let X be a set with an equivalence relation $\sim$, and let $x, y \in X$.*

1) If $x \sim y$ then $[x] = [y]$.
2) If $x \not\sim y$ then $[x] \cap [y] = \varnothing$.

19.9 Definition. Let X be a set with an equivalence relation $\sim$. The *quotient set* of X is the set $X/\sim$ whose elements are all distinct equivalence classes of $\sim$. The function

$$\pi\colon X \to X/\sim$$

given by $\pi(x) = [x]$ is called the *quotient map*.

19.11 Definition. Let X be a topological space and let $\sim$ be an equivalence relation on X. The *quotient topology* on the set $X/\sim$ is the topology where a set $U \subseteq X/\sim$ is open if the set $\pi^{-1}(U)$ is open in X. The set $X/\sim$ with this topology is called the *quotient space* of X taken with respect to the relation $\sim$.

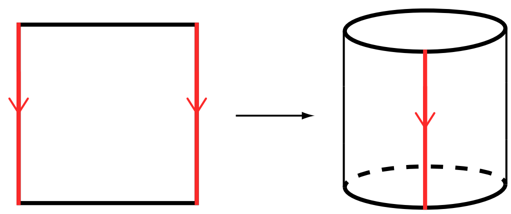
19.12 Proposition. *Let X be a topological space and let $\sim$ be an equivalence relation on X. A set $A \subseteq X/\sim$ is closed if and only the set $\pi^{-1}(A)$ is closed in X.*

Proof. Exercise. $\square$

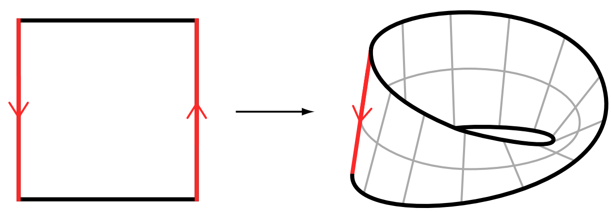
19.13 Proposition. *Let X, Y be a topological spaces and let $\sim$ be an equivalence relation on X. A function $f\colon X/{\sim} \to Y$ is continuous if and only if the function $f\pi\colon X \to Y$ is continuous.*

Proof. Exercise. $\square$

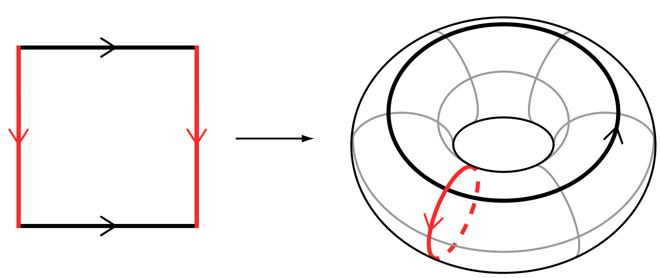
19.17 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation defined as in Example 19.2: $(0, t) \sim (1, t)$ for all $t \in [0, 1]$. Using arguments similar as in Example 19.15 we can show that the quotient space is homeomorphic to the cylinder $S^1 \times [0, 1]$:



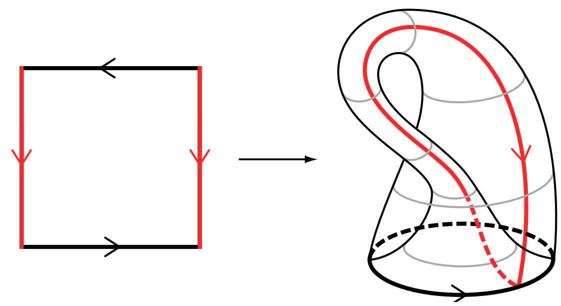
19.18 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, 1 - t)$ for all $t \in [0, 1]$. The space obtained as a quotient space is called the *Möbius band*:



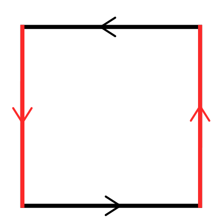
19.19 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, t)$ for all $t \in [0, 1]$ and $(s, 0) \sim (s, 1)$ for all $s \in [0, 1]$. Using arguments similar to these given in Example 19.15 one can show that the quotient space in this case is homeomorphic to the torus:



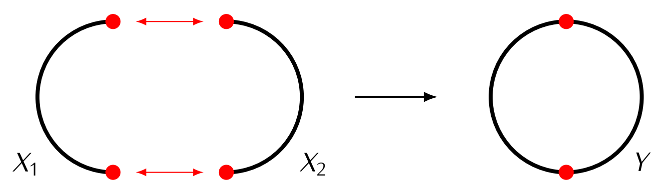
19.20 Example. Take the square $[0,1]\times[0,1]$ with the equivalence relation given by $(0,t)\sim(1,t)$ for all $t\in[0,1]$ and $(s,0)\sim(1-s,1)$ for all $s\in[0,1]$. The resulting quotient space is called the *Klein bottle*. One can show that the Klein bottle is a two dimensional manifold.



19.21 Example. Following the scheme of the last two examples we can consider the square $[0,1]\times[0,1]$ with the equivalence relation given by $(0,t)\sim(1,1-t)$ and $(s,0)\sim(1-s,1)$ for all $s,t\in[0,1]$:



Disjoint unions



19.25 Proposition. *For any family of continuous functions $\{f_i \colon X_i \to Y\}_{i\in I}$, there exists a unique continuous function $f \colon \bigsqcup_{i\in I} X_i \to Y$ such that $k_j f = f_j$ for each $j \in I$.*

Proof. Exercise. $\square$