21 | Embeddings of Manifolds

21.1 Definition. Let X be a topological space and let $f: X \to \mathbb{R}$ be a continuous function. The *support* of f is the closure of the subset of X consisting of points with non-zero values:

$$supp(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$$

- **21.2 Definition.** Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. A partition of unity subordinate to \mathcal{U} is a family of continuous functions $\{\lambda_i \colon X \to [0,1]\}_{i \in I}$ such that
 - (i) $supp(\lambda_i) \subseteq U_i$ for each $i \in I$;
 - (ii) each point $x \in X$ has an open neighborhood U_x such that $U_x \cap \text{supp}(\lambda_i) \neq \emptyset$ for finitely many $i \in I$ only;
- (iii) for each $x \in X$ we have $\sum_{i \in I} \lambda_i(x) = 1$.

- **21.4 Lemma.** Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let $\{\lambda_i\}_{i \in I}$ be a partition of unity subordinate to \mathcal{U} .
 - 1) Let $i \in I$ and let $f_i \colon U_i \to \mathbb{R}^n$ be a continuous function. Then the function $\tilde{f}_i \colon X \to \mathbb{R}^n$ given by

$$\tilde{f}_i(x) = \begin{cases} \lambda_i(x)f_i(x) & \text{for } x \in U_i \\ 0 & \text{for } x \in X \setminus U_i \end{cases}$$

is continuous.

2) Assume that for each $i \in I$ we have a continuous function $f_i \colon U_i \to \mathbb{R}^n$, and let $\tilde{f}_i \colon X \to \mathbb{R}^n$ be the function defined as above. Then the function $\tilde{f} \colon X \to \mathbb{R}^n$ given by

$$\tilde{f}(x) = \sum_{i \in I} \tilde{f}_i(x)$$

is continuous.

Proof. Exercise. □

21.5 Proposition. Let X be a normal space. For any finite open cover $\{U_1, \ldots, U_n\}$ of X there exists a partition of unity subordinate to this cover.
21.6 Finite Shrinking Lemma. Let X be a normal space and let $\{U_1, \ldots, U_n\}$ be a finite open cover of X . There exists an open cover $\{V_1, \ldots, V_n\}$ of X such that $\overline{V}_i \subseteq U_i$ for each $i \geq 1$.
21.7 Shrinking Lemma. Let X be a normal space and let $\{U_i\}_{i\in I}$ be a open cover of X such that each point of X belongs to finitely many sets U_i only. There exists an open cover $\{V_i\}_{i\in I}$ of X such that $\overline{V}_i\subseteq U_i$ for all $i\in I$.
Proof. Exercise.

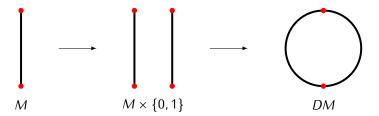
Proof of Proposition 21.5.
1.8 Corollary. If X is a compact Hausdorff space then for every open cover $\mathcal U$ of X there exists an artition of unity subordinate to $\mathcal U$.

21.9 Theorem. If M is a compact manifold without boundary then for some $N \geq 0$ there exists an embedding $j \colon M \to \mathbb{R}^N$.

21.11 Definition. Let M be a manifold with boundary ∂M . The double of M is the topological space

$$DM = M \times \{0, 1\}/\sim$$

where $\{0,1\}$ is the discrete space with two points and \sim is the equivalence relation on $M \times \{0,1\}$ given by $(x,0) \sim (x,1)$ for all $x \in \partial M$.



21.12 Proposition. If M is an n-dimensional manifold with boundary then DM is an n-dimensional manifold without boundary. Moreover, if M is compact then so is DM.

Proof. Exercise.

21.13 Corollary. If M is a compact manifold with boundary then for some N > 0 there exists an embedding $M \to \mathbb{R}^N$.