

# 12 | Urysohn Metrization Theorem

12.1 Urysohn Metrization Theorem. *Every second countable normal space is metrizable.*

**12.2 Definition.** A continuous function  $i: X \rightarrow Y$  is an *embedding* if its restriction  $i: X \rightarrow i(X)$  is a homeomorphism (where  $i(X)$  has the topology of a subspace of  $Y$ ).

**12.5 Lemma.** If  $j: X \rightarrow Y$  is an embedding and  $Y$  is a metrizable space then  $X$  is also metrizable.

**12.6 Definition.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. The *product topology* on  $\prod_{i \in I} X_i$  is the topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i \mid U_i \text{ is open in } X_i \text{ and } U_i \neq X_i \text{ for finitely many indices } i \text{ only} \right\}$$

**12.8 Proposition.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces and for  $j \in I$  let*

$$p_j: \prod_{i \in I} X_i \rightarrow X_j$$

*be the projection onto the  $j$ -th factor:  $p_j((x_i)_{i \in I}) = x_j$ . Then:*

- 1) for any  $j \in I$  the function  $p_j$  is continuous.*
- 2) A function  $f: Y \rightarrow \prod_{i \in I} X_i$  is continuous if and only if the composition  $p_j f: Y \rightarrow X_j$  is continuous for all  $j \in I$*

*Proof.* Exercise.

□

**12.10 Proposition.** *If  $X_1, \dots, X_n$  are metrizable spaces then  $X_1 \times \dots \times X_n$  is also a metrizable space.*

**12.11 Proposition.** *If  $\{X_i\}_{i=1}^{\infty}$  is an infinite countable family of metrizable spaces then  $\prod_{i=1}^{\infty} X_i$  is also a metrizable space.*

**12.12 Example.** The *Hilbert cube* is the topological space  $[0, 1]^{\aleph_0}$  obtained as the infinite countable product of the closed interval  $[0, 1]$ :

$$[0, 1]^{\aleph_0} = \prod_{i=1}^{\infty} [0, 1]$$

Elements of  $[0, 1]^{\aleph_0}$  are infinite sequences  $(t_i) = (t_1, t_2, \dots)$  where  $t_i \in [0, 1]$  for  $i = 1, 2, \dots$ . The Hilbert cube is a metric space with a metric  $\varrho$  given by

$$\varrho((t_i), (s_i)) = \sum_{i=1}^{\infty} \frac{1}{2^i} |t_i - s_i|$$

**12.13 Definition.** Let  $X$  be a topological space and let  $\{f_i\}_{i \in I}$  be a family of continuous functions  $f_i: X \rightarrow [0, 1]$ . We say that the family  $\{f_i\}_{i \in I}$  *separates points from closed sets* if for any point  $x_0 \in X$  and any closed set  $A \subseteq X$  such that  $x_0 \notin A$  there is a function  $f_j \in \{f_i\}_{i \in I}$  such that  $f_j(x_0) > 0$  and  $f_j|_A = 0$ .

**12.14 Embedding Lemma.** *Let  $X$  be a  $T_1$ -space. If  $\{f_i: X \rightarrow [0, 1]\}_{i \in I}$  is a family that separates points from closed sets then the map*

$$f_\infty: X \rightarrow \prod_{i \in I} [0, 1]$$

*given by  $f_\infty(x) = (f_i(x))_{i \in I}$  is an embedding.*

*Proof of Theorem 12.1.*

□



**12.16 Proposition.** *Every second countable regular space is normal.*

*Proof.* Exercise. □

**12.17 Urysohn Metrization Theorem (v.2).** *Every second countable regular space is metrizable.*

**12.18 Definition.** Let  $X$  be a topological space. A collection  $\mathcal{U} = \{U_i\}_{i \in I}$  of open sets in  $X$  is *locally finite* if each point  $x \in X$  has an open neighborhood  $V_x$  such that  $V_x \cap U_i \neq \emptyset$  for finitely many  $i \in I$  only.

A collection  $\mathcal{U}$  is *countably locally finite* if it can be decomposed into a countable union  $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$  where each collection  $\mathcal{U}_n$  is locally finite.

**12.19 Nagata-Smirnov Metrization Theorem.** *Let  $X$  be a topological space. The following conditions are equivalent:*

- 1)  $X$  is metrizable.
- 2)  $X$  is regular and it has a basis which is countably locally finite.