22 | Mapping Spaces

22.1 Definition. Let X, Y be topological spaces. By $\mathrm{Map}(X, Y)$ we will denote the set of all continuous functions $f\colon X \to Y$.

22.2 Definition. Let X, Y be a topological spaces, and let $\mathcal{T}$ be a topology on $\mathrm{Map}(X, Y)$.

1) We will say that the topology $\mathcal{T}$ is *lower admissible* if for any continuous function $F\colon Z \times X \to Y$ the function $F_*\colon Z \to \mathrm{Map}(X, Y)$ is continuous.
2) We will say that the topology $\mathcal{T}$ is *upper admissible* if for any function $F\colon Z \times X \to Y$ if the function $F_*\colon Z \to \mathrm{Map}(X, Y)$ is continuous then F is continuous.
3) We will say that the topology $\mathcal{T}$ is *admissible* if it is both lower and upper admissible.

22.3 Definition. Let X, Y be topological spaces. The *evaluation map* is the function

$$\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$$

given by $\mathrm{ev}((f, x)) = f(x)$.

22.4 Lemma. *Let X, Y be topological spaces, and let $\mathcal{T}$ be a topology on* $\mathrm{Map}(X, Y)$*. The following conditions are equivalent:*

1) *The topology $\mathcal{T}$ is upper admissible.*
2) *The evaluation map* $\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$ *is continuous.*

22.6 Proposition. *Let X, Y be topological spaces.*

1) *If $\mathcal{U}, \mathcal{U}'$ are two topologies on $\mathrm{Map}(X, Y)$ such that $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{U}$ is upper admissible, then $\mathcal{U}'$ also is upper admissible.*
2) *If $\mathcal{L}, \mathcal{L}'$ are two topologies on $\mathrm{Map}(X, Y)$ such that $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{L}$ is lower admissible, then $\mathcal{L}'$ also is lower admissible.*
3) *If $\mathcal{U}$, $\mathcal{L}$ are two topologies on $\mathrm{Map}(X, Y)$ such that $\mathcal{U}$ is upper admissible and $\mathcal{L}$ is lower admissible then $\mathcal{L} \subseteq \mathcal{U}$.*

22.7 Corollary. *Given spaces X and Y, if there exists an admissible topology on $\mathrm{Map}(X, Y)$ then such topology is unique.*

22.8 Proposition. *Let X be completely regular space. If there exist an admissible topology on* $\mathrm{Map}(X, \mathbb{R})$ *then X is locally compact.*

22.10 Definition. Let X, Y be topological spaces. For sets $A \subseteq X$ and $B \subseteq Y$ denote

$$P(A, B) = \{f \in \mathrm{Map}(X, Y) \mid f(A) \subseteq B\}$$

22.11 Lemma. *Let X, Y topological spaces, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Let $\mathcal{T}$ be a topology on* $\mathrm{Map}(X, Y)$ *with subbasis given by all sets $P(A, V)$ where $A \subseteq X$ is a closed set such that $A \subseteq U_i$ for some $i \in I$, and $V \subseteq Y$ is an open set. If X is a regular space then $\mathcal{T}$ upper admissible.*

Proof. Exercise. $\square$

Proof of Proposition 22.8.

□

22.12 Definition. Let X, Y be topological spaces. The *compact-open* topology on $\mathrm{Map}(X, Y)$ is the topology defined by the subbasis consisting of all sets $P(A, U)$ where $A \subseteq X$ is compact and $U \subseteq Y$ is an open set.

22.13 Theorem. *For any spaces X, Y the compact-open topology on $\mathrm{Map}(X, Y)$ is lower admissible.*

22.14 Theorem. *Let X, Y be topological spaces. If X is locally compact Hausdorff space then the compact-open topology on* $\mathrm{Map}(X, Y)$ *is upper admisible.*

22.15 Corollary. *If X is a locally compact Hausdorff space and Y is any space then the compact-open topology on* $\mathrm{Map}(X, Y)$ *is admissible.*

22.17 Proposition. *Let X be a topological space, and let S be a set considered as a discrete topological space. There exists a homeomorphism*

$$\mathrm{Map}(S, X) \cong \prod_{s\in S} X$$

where $\mathrm{Map}(S, X)$ *is taken with the compact-open topology, and* $\prod_{s\in S} X$ *with the product topology.*

Proof. Exercise. □

22.19 Proposition. *Let X be a compact Hausdorff space, and let (Y, ϱ) be a metric space. For $f, g \in \mathrm{Map}(X, Y)$ define*

$$d(f, g) = \max\{\varrho(f(x), g(x)) \mid x \in X\}$$

Then d is a metric on $\mathrm{Map}(X, Y)$*. Moreover, in the topology induced by this metric is the compact-open topology.*

Proof. Exercise. □

22.20 Theorem. *Let X, Y, Z be topological spaces. Let*

$$\Phi\colon \mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

be a function given by $\Phi(f, g) = g \circ f$. If Y is a locally compact Hausdorff space, and all mapping spaces are equipped with the compact-open topology then Φ is continuous.

22.21 Lemma. *Let X be a locally compact Hausdorff space, and let $A, W \subseteq X$ be sets such that A is compact, W is open, and $A \subseteq W$. Then there exists an open set $V \subseteq X$ such that $A \subseteq V$, $\overline{V} \subseteq W$, and $\overline{V}$ is compact.*

Proof. Exercise. □

Proof of Theorem 22.20.

□