20 | Simplicial Complexes

The straightforward way of constructing a topological space is to describe its sets of points and the collection of open sets. Many constructions in geometry and topology, however, take a less direct approach. They specify a collection of simple topological spaces that serve as building blocks. Then, in order to construct more complex spaces, one just needs to give a recipe how these building blocks should be put together. Simplicial complexes introduced in this chapter are one example of such framework, but there are several others: cubical complexes, simplicial sets, CW-complexes etc. These constructions differ in the selection of spaces which serve as the building blocks, how “recipes” for assembling these building blocks look like, and how the building process is conducted.

20.1 Definition. A *simplicial complex* $K = (V, S)$ consists of a set V together with a set S of finite, non-empty subsets of V such that the following conditions are satisfied:

1) For each $v \in V$ the set $\{v\}$ is in S.
2) If $\sigma \in S$ and $\varnothing \neq \tau \subseteq \sigma$ then $\tau \in S$.

20.2 Notation. If $K = (V, S)$ is a simplicial complex then:

- Elements of V are called *vertices* of K.
- Elements of S are called *simplices* of K.
- If a simplex $\sigma \in S$ consists of $n + 1$ elements then we say that σ is an *n-simplex*.
- If $\sigma \in S$ and $\tau \subseteq \sigma$ then we say that τ is a *face* of σ. If $\tau \neq \sigma$ then τ is a *proper face* of σ. The inclusion $j^{\sigma}_{\tau} \colon \tau \to \sigma$ is called a *face map*.
- We say that K is a simplicial complex of dimension n if K has n-simplices, but it does not have m-simplices for $m > n$. We write: $\dim K = n$. If K has simplices in all dimensions then $\dim K = \infty$.
- We say that K is a finite simplicial complex if K consists of finitely many simplices.

We will write $\sigma \in K$ to denote that σ is a simplex of K.

20.3 Example. If Γ is a graph then then we can define a simplicial complex $K(\Gamma)$ whose vertices are vertices of Γ, and whose 1-simplices are sets $\{v_1, v_2\}$ such that there is an edge of Γ that joins v_1 with v_2. The complex $K(\Gamma)$ has no simplices in dimensions greater than 1.

20.4 Example. Let X be a set, and let $\mathcal{A} = \{A_i\}_{i\in I}$ be a collection of subsets of X. The *nerve* of $\mathcal{A}$ is the simplicial complex $N(\mathcal{A})$ whose vertices are elements of $\mathcal{A}$. A set $\{A_{i_0}, \dots, A_{i_n}\}$ is a simplex in $N(\mathcal{A})$ if $A_{i_1} \cap \dots \cap A_{i_n} = \varnothing$.

20.5 Example. As a special case of the construction of a nerve (20.4), let (X, ϱ) be a metric space, and for a fixed $\varepsilon > 0$ let $\mathcal{B}_\varepsilon$ be the collection of all open ball in X of radius ε:

$$\mathcal{B}_\varepsilon = \{B(x, \varepsilon) \mid x \in X\}$$

The nerve $N(\mathcal{B}_\varepsilon)$ is called the *Čech complex* of X and it is denoted $\check{C}(X, \varepsilon)$.

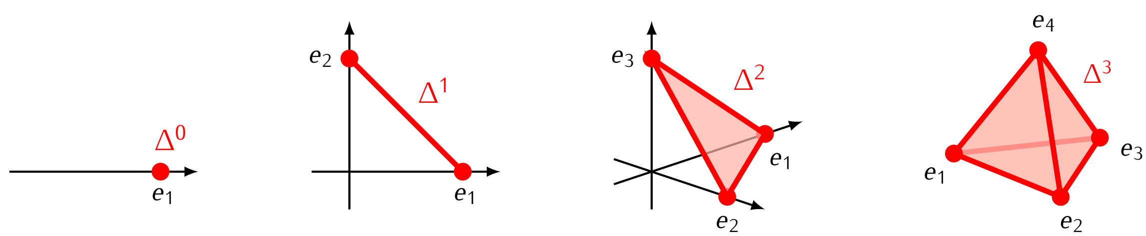
20.6 Definition. If $K = (V, S)$ is a simplicial complex, then a *subcomplex* of K is a simplicial complex $L = (V', S')$ such that $V' \subseteq V$ and $S' \subseteq S$. In such case we write $L \subseteq K$.

20.7 Example. A simplex σ of a simplicial complex K defines a subcomplex $\overline{\sigma} \subseteq K$ consisting of all faces of σ and a subcomplex $\partial\sigma \subseteq K$ consisting of all proper faces of σ.

We will show that every simplicial complex K defines a certain topological space $|K|$. This space is assembled from building blocks given by geometric simplices. The structure of K provides a blueprint how the assembly process should be performed.

20.8 Definition. Let $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots,$ $e_{n+1} = (0, 0, 0, \dots, 1)$ be the standard basis vectors in $\mathbb{R}^{n+1}$. The *standard geometric n-simplex* is a subspace $\Delta^n \subseteq \mathbb{R}^{n+1}$ given by

$$\Delta^n = \left\{ \sum_{i=1}^{n+1} t_i e_i \in \mathbb{R}^{n+1} \mid t_i \in [0, 1],\ \textstyle\sum_{i=0}^{n} t_i = 1 \right\}$$



By the above definition vertices of the standard geometric simplex Δ^n correspond to the vectors $e_1, \dots, e_{n+1}$. It will be useful to modify this construction, so that vertices of a geometric simplex can be indexed by elements of any given finite set:

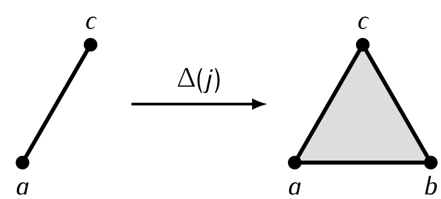
20.9 Definition. Let A be a finite set. The *geometric A-simplex* is a metric space (Δ^A, ϱ), such that elements of Δ^A are formal sums $\sum_{a\in A} t_a a$ where $t_a \in [0,1]$ for each $a \in A$, and $\sum_{a \in A} t_a = 1$. If $x = \sum_{a\in A} t_a a$ and $y = \sum_{a \in A} t'_a a$ then

$$\varrho(x, y) = \sqrt{\textstyle\sum_{a\in A}(t_a - t'_a)^2}$$

20.10 Proposition. *If A is a set consisting of $n+1$ elements then Δ^A is homeomorphic to the standard n-simplex Δ^n.*

Proof. Exercise. □

If A is a finite set and $B \subseteq A$, then the simplex Δ^B is a subspace of Δ^A and the inclusion map $j\colon B \to A$ induces a continuous inclusion $\Delta(j)\colon \Delta^B \to \Delta^A$. We will say that Δ^B is a *face* of Δ^A. If $B \neq A$ then it is a *proper face*.



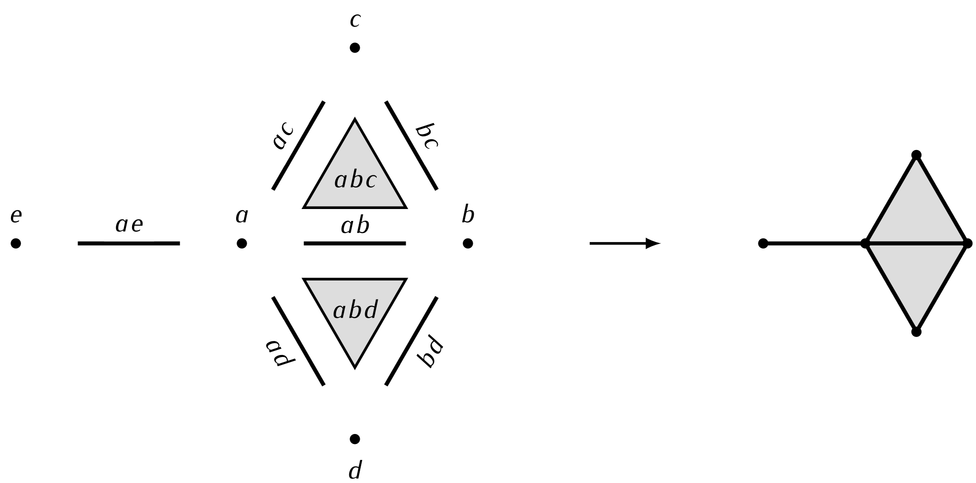
In particular, if σ is a simplex of a simplicial complex K, and τ is a face of σ then the face map $j^\sigma_\tau\colon \tau \to \sigma$ defines a map $\Delta(j^\sigma_\tau)\colon \Delta^\tau \to \Delta^\sigma$.

20.11 Definition. Let K be a simplicial complex. The *geometric realization* of K is the topological space $|K|$ defined by:

$$|K| = \bigsqcup_{\sigma\in K} \Delta^\sigma \Big/ \sim$$

where the equivalence relation $\sim$ is given by $x \sim \Delta(j^\sigma_\tau)(x)$ for each face map $j^\sigma_\tau\colon \tau \to \sigma$ and $x \in \Delta^\tau$.

20.12 Example. Let K be a simplicial complex with vertices $\{a, b, c, d, e\}$ and simplices $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, d\}$, $\{b, d\}$, $\{a, e\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$. The picture on the left shows the disjoint union $\bigsqcup_{\sigma\in K} \Delta^\sigma$, and the picture on the right shows the geometric realization $|K|$:



The following fact will be useful later on:

20.13 Proposition. *If L is a subcomplex of a simplicial complex K, then $|L|$ is a closed subspace of $|K|$.*

Proof. Exercise. □

Simplicial complexes and their geometric realizations are very useful tools, because they are a source of algebraic invariants of topological spaces. We will illustrate this using an example of one such invariant, the Euler characteristic.

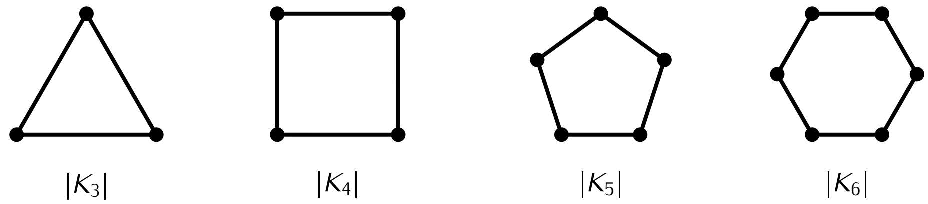
20.14 Definition. Let K be a finite simplicial complex. For $n = 0, 1, 2, \dots$ let $s_n(K)$ denote the number of n-simplices of K. The *Euler characteristic* of K is the integer

$$\chi(K) = \sum_{n=0}^{\infty} (-1)^n s_n(K)$$

The following fact is one of the most important properties of the Euler characteristic. We will omit its proof.

20.15 Theorem. *If K, L are finite simplicial complexes such that $|K|$ is homeomorphic to $|L|$ then $\chi(K) = \chi(L)$.*

20.16 Example. For $n \geq 3$ let K_n be a simplicial complex of dimension 1 whose set of vertices is $\{a_1, \dots, a_n\}$, and whose 1-simplices are sets $\{a_i, a_{i+1}\}$ for $1 \leq i \leq n-1$ and $\{a_n, a_1\}$. Notice that each space $|K_n|$ is homeomorphic to the circle S^1:



Since in each complex K_n the number of 0-simplices is the same as the number of 1-simplices, we have $\chi(K_n) = 0$ for any $n \geq 3$.

We can extend the notion of the Euler characteristic to the realm of topological spaces as follows:

20.17 Definition. If X is a topological space such that $X \cong |K|$ for some finite simplicial complex K then we define the Euler characteristic $\chi(X)$ of X as the Euler characteristic $\chi(K)$ of K.

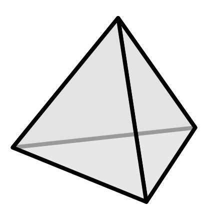
Notice that by Theorem 20.15 the Euler characteristic of a space X is a well defined. Indeed, if $X \cong |K|$ and $X \cong |L|$ for some finite simplicial complexes K and L then $|K| \cong |L|$, and so $\chi(K) = \chi(L)$. Therefore the value of $\chi(X)$ does not depend on the choice of a simplicial complex K such that $X \cong |K|$. Moreover, we have:

20.18 Proposition. *The Euler characteristic is a topological invariant: if X, Y are spaces such that $X \cong Y$ and $\chi(X)$ is defined, then $\chi(Y)$ is defined and $\chi(Y) = \chi(X)$.*

Proof. If $\chi(X)$ is defined then $X \cong |K|$ for some finite simplicial complex K. Then $Y \cong X \cong |K|$, and so $\chi(Y) = \chi(K) = \chi(X)$. □

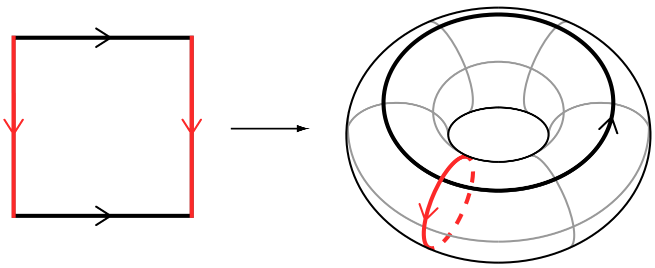
20.19 Example. We will use the Euler characteristic to show that the 2-dimensional sphere S^2 is not homeomorphic to the torus $T = S^1 \times S^1$.

The sphere S^2 is homeomorphic to the surface of the tetrahedron:

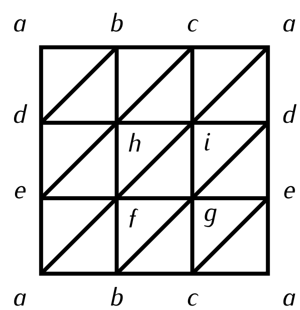


This surface can be obtained as a geometric realization of a simplicial complex which has 4 0-simplices, 6 1-simplices and 4 2-simplices. This gives: $\chi(S^2) = 4 - 6 + 4 = 2$.

Next, recall that the torus T can be constructed by identifying edges of a square:

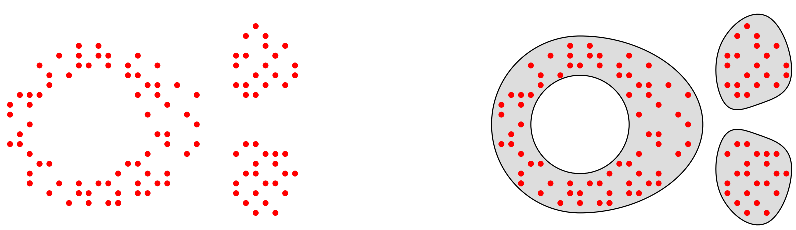


Subdividing the square we can represent the torus as a geometric realization of a simplicial complex:



This simplicial complex has 9 0-simplices, 27 1-simplices, and 18 2-simplices. This gives $\chi(T) = 9 - 27 + 18 = 0$. Since $\chi(T) \neq \chi(S^2)$ by Proposition 20.15 we obtain that $T \not\cong S^2$.

20.20 Topological Data Analysis. While simplicial complexes have been known and studied for a long time, more recently they appeared in the field of topological data analysis (TDA), which uses topological methods to study sets of data. The premise of TDA is that a set of data can be considered as a sample of points taken from some underlying, unknown topological space X, and that properties of that underlying space (its number of connected components, the Euler characteristic etc.) provide an insight into the data.

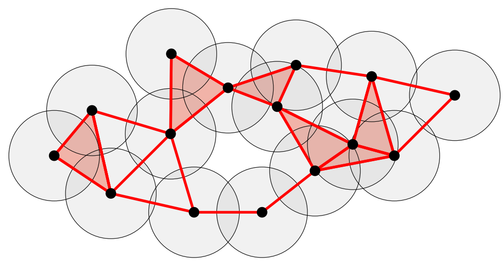


a set of data points

data points and the hypothetical underlying space X

Let S be a set of data points. In order to study the space X underlying it, we attempt to reconstruct this space from the set S. For this purpose we assume that the set S is a metric space, i.e. we can measure distances between data points. For example, if each data point consists of medical data of a patient in the numerical form (age, weight, blood pressure, level of blood sugar etc.), then it can be represented as a vector in $\mathbb{R}^n$ for some n, and the Euclidean metric or some variant of it may be suitable to measure distances between data points in the set S.

Since S is a metric space, for $\varepsilon > 0$ we can consider the Čech complex $\check{C}(S, \varepsilon)$ of S (20.5). Recall that $\check{C}(S, \varepsilon)$ is a simplicial complex whose vertices are points of S, and whose simplices are finite sets $\sigma \subseteq S$ such that $\bigcap_{x \in S} B(x, \varepsilon) \neq \varnothing$. Let X_ε be the geometric realization of $\check{C}(S, \varepsilon)$. We can view this space as an approximation of the space X.



Naturally, properties of of the space X_ε depend not just on the data set S, but also on the choice of ε. In topological data analysis one considers how the space X_ε evolves as ε increases. Properties of X_ε which persists for a wide range of values of ε are considered to be meaningful, reflecting properties of the space X. Properties which are observed only for a short span of values of ε are disregarded.

We finish this chapter with a proof of the following fact:

20.21 Theorem. *If K is a simplicial complex then the geometric realization $|K|$ is a normal space.*

Out goal here is twofold. First, Theorem 20.21 describes a useful property of simplicial complexes. Second, its proof will illustrate how one can work with simplicial complexes and their realizations.

The proof of Theorem 20.21 will use Theorem 11.10, which says that a space X is normal if it is a T_1 space and if each continuous function $f\colon A \to \mathbb{R}$ defined on a closed subspace $A \subseteq X$ can be extended to a continuous function $\bar{f}\colon X \to \mathbb{R}$. In order to verify that this condition holds for the space $|K|$, we need to understand better how to construct continuous functions from $|K|$. The following notion will be useful:

20.22 Definition. The *n-skeleton* of a simplicial complex K is a subcomplex $K^{(n)} \subseteq K$ given as follows:

- vertices of $K^{(n)}$ are the same as vertices of K;
- m-simplices of $K^{(n)}$ are the same as m-simplices of K for any $m \leq n$;

– $K^{(n)}$ has no m-simplices for $m > n$.

Since $K^{(n)}$ is a subcomplex of K, by Proposition 20.13 the geometric realization $|K^{(n)}|$ is a closed subspace of $|K|$. We have:

20.23 Proposition. *Let K be a simplicial complex, and let X be a topological space. A function $f\colon |K| \to X$ is continuous if and only if $f|_{|K^{(n)}|}\colon |K^{(n)}| \to X$ is continuous for each $n = 0, 1, \dots$.*

Proof. Exercise. □

Since $|K^{(n)}| \subseteq |K^{(n+1)}|$ for each $n \geq 0$, continuous functions $|K| \to X$ can be constructed inductively: we start with a continuous function $f_0\colon |K^{(0)}| \to X$, we extend it to a continuous function $f_1\colon |K^{(1)}| \to X$ and so on. The resulting functions $f_n\colon |K^{(n)}| \to X$ define a function $f\colon |K| \to X$ such that $f|_{|K^{(n)}|} = f_n$. By Proposition 20.23 the function f is continuous.

The inductive step of extending a function $f_n\colon |K^{(n)}| \to X$ to a function $f_{n+1}\colon |K^{(n+1)}| \to X$ can be described as follows. Recall (20.7) that each simplex $\sigma \in K$ defines subcomplexes $\overline{\sigma}$ and $\partial\sigma$ of K, which consist, respectively, of all faces and all proper faces of σ. In this way we obtain subspaces $|\overline{\sigma}|$ and $|\partial\sigma|$ of $|K|$. Let S_{n+1} denote the set of $(n+1)$-simplices of K. We have:

$$|K^{(n+1)}| = |K^{(n)}| \cup \bigcup_{\sigma \in S_{n+1}} |\overline{\sigma}|$$

Also, if $\sigma \in S_n$ then $|\overline{\sigma}| \cap |K^{(n)}| = |\partial\sigma|$.

20.24 Lemma. *Let K be a simplicial complex, and let $f_n\colon |K^{(n)}| \to X$ be a continuous function. Assume that for each $\sigma \in S_{n+1}$ we have a continuous function $f_\sigma\colon |\overline{\sigma}| \to X$ such that $f_\sigma|_{|\partial\sigma|} = f_n|_{|\partial\sigma|}$. Then f_n extends to a function $f_{n+1}\colon |K^{(n+1)}| \to X$ such that $f_{n+1}|_{|\overline{\sigma}|} = f_\sigma$.*

Proof. Exercise. □

Proof of Theorem 20.21. The proof that $|K|$ is a T_1 space is left as an exercise. By Theorem 11.10 in order to verify that $|K|$ is normal we need to show that if $A \subseteq |K|$ is a closed set, and $f\colon A \to \mathbb{R}$ is a continuous function, then f can be extended to a continuous function $\bar{f}\colon |K| \to \mathbb{R}$. We will construct the function $\bar{f}$ by induction on skeleta of $|K|$. The 0-th skeleton $|K^{(0)}|$ is a discrete space whose points are vertices of K. For $v \in |K^{(0)}|$ define

$$\bar{f}_0(v) = \begin{cases} f(v) & \text{if } v \in A \\ 0 & \text{otherwise} \end{cases}$$

This gives a continuous function $\bar{f}_0\colon |K^{(0)}| \to \mathbb{R}$ such that $f|_{A\cap|K^{(0)}|} = \bar{f}_0|_{A\cap|K^{(0)}|}$.

Next, assume for each $m \leq n$ we have defined a continuous function $\bar{f}_m\colon |K^{(m)}| \to \mathbb{R}$ satisfying $f|_{A\cap|K^{(m)}|} = \bar{f}_m|_{A\cap|K^{(m)}|}$ and $f_m = f_n|_{|K^{(m)}|}$. Let σ be an $(n+1)$-simplex in K. Since $A \cap |\bar{\sigma}|$ and $|\partial\sigma|$ are closed sets in $|\bar{\sigma}|$, by the Closed Pasting Lemma 6.7 the function $f_\sigma\colon (A\cap|\bar{\sigma}|)\cup|\partial\sigma| \to \mathbb{R}$ given by

$$f_\sigma(x) = \begin{cases} f(x) & \text{if } x \in A \cap |\bar{\sigma}| \\ \bar{f}_n(x) & \text{if } x \in |\partial\sigma| \end{cases}$$

is continuous. The space $|\bar{\sigma}|$ is homeomorphic to Δ^{n+1}, thus it is a normal space. Therefore, by the Tietze Extension Theorem 11.9 the function f_σ can be extended to a continuous function $\bar{f}_\sigma\colon |\bar{\sigma}| \to \mathbb{R}$. By Lemma 20.24 the functions $\bar{f}_\sigma$ taken for all $\sigma \in S_n$, define a continuous function $\bar{f}_{n+1}\colon |K^{(n+1)}| \to \mathbb{R}$ which extends $\bar{f}_n$. Moreover, $\bar{f}_{n+1}|_{A\cap|K^{(n+1)}|} = f|_{A\cap|K^{(n+1)}|}$.

It remains to notice that by Proposition 20.23 the functions $\bar{f}_n$ taken for $n \geq 0$, define a continuous function $\bar{f}\colon |K| \to \mathbb{R}$ such that $\bar{f}|_A = f$. ☐

Exercises to Chapter 20

E20.1 Exercise. Prove Proposition 20.13.

E20.2 Exercise. Let K be a simplicial complex. Show that a set $A \subseteq |K|$ is closed in $|K|$ if and only if $A \subseteq |K^{(n)}|$ is closed in $|K^{(n)}|$ for each $n \geq 0$

E20.3 Exercise. Prove Proposition 20.23.

E20.4 Exercise. Prove Lemma 20.24.

E20.5 Exercise. Let K be a simplicial complex. Show that the space $|K|$ is a compact if and only if K is finite (i.e. it consists of finitely many sets). Hint: Consider two cases: $\dim K = \infty$ and $\dim K < \infty$. In the first case take $S = \{x_0, x_1, \dots\} \subseteq |K|$ to be a set such that $x_n \in |K^{(n)}| \smallsetminus |K^{(n-1)}|$ for each $n \geq 0$. Show that S is a discrete subspace of $|K|$.

E20.6 Exercise. The goal of this exercise is to show that the geometric realization of a simplicial complex need not be a metrizable space.

Let K be a simplicial complex given as follows. Vertices of K are indexed by natural numbers $x_0, x_1, x_2, \dots$. Simplices of K are sets $\sigma_n = \{x_0, x_n\}$ for $n > 0$ and singletons $\{x_n\}$ for $n \geq 0$. Show that the space $|K|$ is not metrizable. Hint: assume that ϱ is a metric on $|K|$, and let $B(x_0, r) \subseteq |K|$ the an open ball with the center at the vertex x_0 taken with respect to this metric. Show that there exists an open set $U \subseteq |K|$ such that $x_0 \in U$ but that for any $r > 0$ the ball $B(x_0, r)$ is not contained in U.

E20.7 Exercise. Compute the Euler characteristic of the the cylinder $X = S^1 \times [0, 1]$.

E20.8 Exercise. In Example 20.19 we observed that the sphere S^2 is homeomorphic to the surface of the tetrahedron, i.e the union of proper faces of the standard geometric 2-simplex Δ^3. Similarly, for any $n \geq 0$ the n-dimensional sphere S^n is homeomorphic to the union of proper faces of the standard $(n+1)$-simplex Δ^{n+1}. Use this to compute the Euler characteristic $\chi(S^n)$.

E20.9 Exercise. The goal of this exercise is to show that the Euler characteristic is essentially the only topological invariant that can be obtained by counting simplices in a finite simplicial complex. Let K be a finite simplicial complex. Given a sequence of integers $\{c_n\} = \{c_0, c_1, c_2, \dots\}$ define

$$\psi_{\{c_n\}}(K) = \sum_{n=0}^{\infty} c_n s_n(K)$$

where $s_n(K)$ is the number of n-simplices of K. Assume that $\psi_{\{c_n\}}$ is a topological invariant, that is if K, L are finite simplicial complexes such that $|K| \cong |L|$ then $\psi_{\{c_n\}}(K) = \psi_{\{c_n\}}(L)$. Show that there exists $N \in \mathbb{Z}$ such that $\psi_{\{c_n\}}(K) = N \cdot \chi(K)$ for every finite simplicial complex K (or equivalently: $c_n = (-1)^n N$ for all $n \geq 0$).