17 | Tychonoff Theorem

We have seen already that a product of finitely many compact spaces is compact (15.6) . The main goal here is to show that the same is true for arbitrary products of compact spaces:

17.1 Tychonoff Theorem. *If $\{X_s\}_{s\in S}$ is a family of topological spaces and X_s is compact for each $s \in S$ then the product space $\prod_{s\in S} X_s$ is compact.*

The proof of Theorem 17.1 involves two main ideas. The first is reformulation of compactness in terms of closed sets.

17.2 Definition. Let $\mathcal{A}$ be a family of subsets of a space X. The family $\mathcal{A}$ is *centered* if for any finite number of sets $A_1, \dots, A_n \in \mathcal{A}$ we have $A_1 \cap \dots \cap A_n \neq \varnothing$

17.3 Example. If $\mathcal{A} = \{A_i\}_{i\in I}$ is a family of subsets of X such that $\bigcap_{i\in I} A_i \neq \varnothing$ then $\mathcal{A}$ is centered.

17.4 Example. Let $X = \mathbb{R}$. For $n = 1, 2, \dots$ define $A_n = [n, +\infty)$. Then the family $\{A_n\}_{n\in\mathbb{Z}}$ is centered even though $\bigcap_{n=1}^{\infty} A_n = \varnothing$.

17.5 Lemma. *Let X be a topological space. The following conditions are equivalent:*

1) *The space X is compact.*
2) *For any centered family $\mathcal{A}$ of closed subsets of X we have $\bigcap_{A\in\mathcal{A}} A \neq \varnothing$.*

Proof. 2) $\Rightarrow$ 1) Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. We need to show that $\mathcal{U}$ has a finite subcover. For $i \in I$ define $A_i := X \smallsetminus U_i$. This gives a family $\mathcal{A} = \{A_i\}_{i\in I}$ of closed sets in X. We have:

$$\bigcap_{i\in I} A_i = \bigcap_{i\in I} (X \smallsetminus U_i) = X \smallsetminus \bigcup_{i\in I} U_i = X \smallsetminus X = \varnothing$$

This implies that $\mathcal{A}$ is not a centered family, so there exist sets $A_{i_1}, \dots, A_{i_n} \in \mathcal{A}$ such that $A_{i_1} \cap \dots \cap A_{i_n} =$

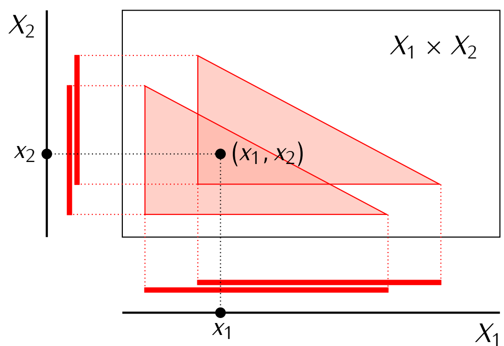
$\varnothing$. This gives:

$$\varnothing = A_{i_1} \cap \dots \cap A_{i_n} = (X \smallsetminus U_{i_1}) \cap \dots \cap (X \smallsetminus U_{i_n}) = X \smallsetminus (U_{i_1} \cup \dots \cup U_{i_n})$$

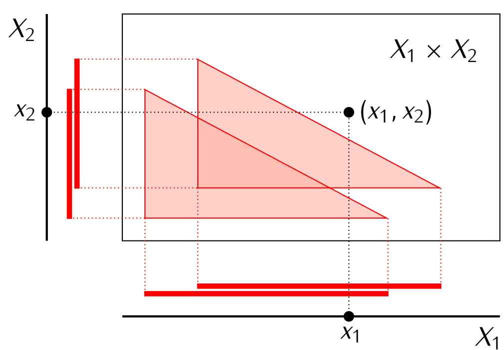
Therefore $X = U_{i_1} \cup \dots \cup U_{i_n}$, and so $\{U_{i_1}, \dots, U_{i_n}\}$ is a finite subcover of $\mathcal{U}$.

1) $\Rightarrow$ 2) Follows from a similar argument. $\square$

Having Lemma 17.5 at our disposal we can try to prove the Theorem 17.1 in the following way. Given a centered family $\mathcal{A}$ of subsets of $\prod_{s\in S} X_s$ we need to show that $\bigcap_{A\in\mathcal{A}} A \neq \varnothing$. Let $p_{s_0}\colon \prod_{s\in S} X_s \to X_{s_0}$ be the projection onto the s_0-th factor. For each $s \in S$ the family $\{\,\overline{p_s(A)}\,\}_{A\in\mathcal{A}}$ is a centered family of closed subsets of X_s. Since X_s is compact we can find $x_s \in X_s$ such that $x_s \in \bigcap_{A\in\mathcal{A}} \overline{p_s(A)}$. If we can show that the point $(x_s)_{s\in S} \in \prod_{s\in S} X_s$ is in $\bigcap_{A\in\mathcal{A}} A$ then we are done.



The problem with this approach is that in general not every choice of points $x_s \in \bigcap_{A\in\mathcal{A}} \overline{p_s(A)}$ will give a point $(x_s)_{s\in S}$ that belongs to $\bigcap_{A\in\mathcal{A}} A$:



This brings in the second main idea of the proof of Tychonoff Theorem, which (modulo a few technical details) can be outlined as follows. We will start with an arbitrary centered family $\mathcal{A}$ of closed subsets of $\prod_{s\in S} X_s$, but then we will replace it by a certain family $\mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}$. This inclusion will give $\bigcap_{M\in\mathcal{M}} M \subseteq \bigcap_{A\in\mathcal{A}} A$, so it will be enough to show that $\bigcap_{M\in\mathcal{M}} M \neq \varnothing$. The advantage of working with the family $\mathcal{M}$ will be that for any choice of points $x_s \in \bigcap_{M\in\mathcal{M}} \overline{p_s(M)}$ the point $(x_s)_{s\in S}$ will belong to $\bigcap_{M\in\mathcal{M}} M$, which will let us avoid the issues indicated above.

The main difficulty is to show that for a given centered family $\mathcal{A}$ we can find a family $\mathcal{M}$ that has the above propreties. This will be accomplished using Zorn's Lemma. This lemma is a very useful result in set theory that appears in proofs of many theorems in various areas of mathematics. Here is a concise introduction to Zorn's Lemma:

17.6 Definition. A *partially ordered set* (or *poset*) is a set S equipped with a binary relation $\leq$ satisfying

(i) $x \leq x$ for all $x \in S$ (reflexivity)
(ii) if $x \leq y$ and $y \leq x$ then $y = x$ (antisymmetry)
(iii) if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).

17.7 Definition. A *linearly ordered set* is a poset $(S, \leq)$ such that for any $x, y \in S$ we have either $x \leq y$ or $y \leq x$.

17.8 Example. If A is a set and S is the set of all subsets of A then S is a poset with ordering given by inclusion of subsets.

17.9 Definition. If $(S, \leq)$ is a poset then an element $x \in S$ is a *maximal element* of S if we have $x \leq y$ only for $y = x$.

17.10 Example. If S is the set of all subsets of a set A ordered by inclusion then S has only one maximal element: the whole set A.

If we take S' to be the set of all *proper* subsets of a A then S' has many maximal elements: for every $a \in A$ the set $A - \{a\}$ is a maximal element of S'.

17.11 Example. In general a poset does not need to have any maximal elements. For example, take the set of integers $\mathbb{Z}$ with the usual ordering $\leq$. The set $\mathbb{Z}$ does not have any maximal elements since for every number $n \in \mathbb{Z}$ we can find a larger number (e.g. $n + 1$).

17.12 Note. If $(S, \leq)$ is a poset and $T \subseteq S$ then T is also a poset with ordering inherited from S.

17.13 Definition. Let $(S, \leq)$ is a poset and let $T \subseteq S$. An *upper bound of* T is an element $x \in S$ such that $y \leq x$ for all $y \in T$.

17.14 Definition. If $(S, \leq)$ is a poset. A *chain* in S is a subset $T \subseteq S$ such that T is linearly ordered.

17.15 Zorn's Lemma. *If $(S, \leq)$ is a non-empty poset such that every chain in S has an upper bound* *in S then S contains a maximal element.*

Proof. See any book on set theory. $\square$

We are finally ready for the proof of the Tychonoff Theorem:

Proof of Theorem 17.1. Let $X = \prod_{s \in S} X_s$ where X_s is a compact space for each $s \in S$. Let $\mathcal{A}$ be a centered family of closed subsets of X. We will show that there exists $x = (x_s)_{s \in S} \in X$ such that $x \in \bigcap_{A \in \mathcal{A}} A$. Let T denote the set consisting of all centered families $\mathcal{F}$ of (not necessarily closed) subsets of X such that $\mathcal{A} \subseteq \mathcal{F}$. The set T is partially ordered by the inclusion.

Claim. Every chain in T has an upper bound.

Indeed, if $\{\mathcal{F}_j\}_{j \in J}$ is a chain in T then take $\mathcal{F} = \bigcup_{j \in J} \mathcal{F}_j$. Since $\mathcal{F}$ is a centered family (exercise) and $\mathcal{F}_j \subseteq \mathcal{F}$ for all $j \in J$ thus $\mathcal{F}$ is an upper bound of $\{\mathcal{F}_j\}_{j \in J}$.

By Zorn's Lemma 17.15 we obtain that the set T contains a maximal element $\mathcal{M}$. We will show that there exists $x \in X$ such that

$$x \in \bigcap_{M \in \mathcal{M}} \overline{M}$$

Since $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{A}$ consists of closed sets we have $\bigcap_{M \in \mathcal{M}} \overline{M} \subseteq \bigcap_{A \in \mathcal{A}} A$. Therefore it will follow that $x \in \bigcap_{A \in \mathcal{A}} A$, and thus $\bigcap_{A \in \mathcal{A}} A \neq \varnothing$.

Construction of the element x proceeds as follows. For $s \in S$ let $p_s \colon X \to X_s$ by the projection onto the s-th coordinate. For each $s \in S$ the family $\{\overline{p_s(M)}\}_{M \in \mathcal{M}}$ is a centered family of closed subsets of X_s, so by compactness of X_s there is $x_s \in X_s$ such that $x_s \in \bigcap_{M \in \mathcal{M}} \overline{p_s(M)}$. We set $x = (x_s)_{s \in S}$.

In order to see that $x \in \bigcap_{M \in \mathcal{M}} \overline{M}$ notice that $\mathcal{M}$ has the following property:

$$\text{if } B \subseteq X \text{ and } B \cap M \neq \varnothing \text{ for all } M \in \mathcal{M} \text{ then } B \in \mathcal{M} \tag{$*$}$$

Indeed, if $\mathcal{M}' = \mathcal{M} \cup \{B\}$ then $\mathcal{M}' \in T$ (exercise) and $\mathcal{M} \subseteq \mathcal{M}'$, so by the maximality of $\mathcal{M}$ we must have $\mathcal{M} = \mathcal{M}'$.

For $s \in S$ let $U_s \subseteq X_s$ be an open neighborhood of x_s. Since $x_s \in \overline{p_s(M)}$ for all $M \in \mathcal{M}$, thus $U_s \cap p_s(M) \neq \varnothing$ for all $M \in \mathcal{M}$. Equivalently, $p_s^{-1}(U_s) \cap M \neq \varnothing$ for all $M \in \mathcal{M}$. By property ($*$) we obtain that $p^{-1}(U_s) \in \mathcal{M}$ for all $s \in S$. Since $\mathcal{M}$ is a centered family we obtain

$$p^{-1}(U_{s_1}) \cap \cdots \cap p^{-1}(U_{s_n}) \cap M \neq \varnothing \quad \text{for all } M \in \mathcal{M} \tag{$**$}$$

Recall that by (12.9) the sets of the form $p^{-1}(U_{s_1}) \cap \cdots \cap p^{-1}(U_{s_n})$. are precisely the open neighborhoods of x that belong to the basis of the product topology on X, and thus any open neighborhood of X contains a neighborhood of this type. Therefore using ($**$) we obtain that if $M \in \mathcal{M}$ then for any open neighborhood U of x we have $M \cap U \neq \varnothing$. This means that for every $M \in \mathcal{M}$ we have $x \in \overline{M}$, and thus $x \in \bigcap_{M \in \mathcal{M}} \overline{M}$.

$\square$

17.16 Proposition. *If X_i is a Hausdorff space for each $i \in I$ then the product space $\prod_{i\in I} X_i$ is also Hausdorff.*

Proof. Exercise. $\square$

17.17 Corollary. *If X_i is a compact Hausdorff space space for each $i \in I$ then the product space $\prod_{i\in I} X_i$ is also compact Hausdorff.*

Proof. Follows from Tychonoff Theorem 17.1 and Proposition 17.16. $\square$

Exercises to Chapter 17

E17.1 Exercise. This problem does not involve topology, it is an exercise in using Zorn's Lemma 17.15. A subset $H \subseteq \mathbb{R}$ is a *subgroup* of $\mathbb{R}$ if it satisfies three conditions:

1) $0 \in H$
2) if $x \in H$ then $-x \in H$
3) if $x, y \in H$ then $x + y \in H$

For example, the set of integers $\mathbb{Z}$ and the set of rational numbers $\mathbb{Q}$ are subgroups of $\mathbb{R}$. Show that for any real number $r \neq 0$ there exists a subgroup $H \subseteq \mathbb{R}$ such that $r \notin H$, but $r \in H'$ for any subgroup H' such that $H \subseteq H'$ and $H \neq H'$.

E17.2 Exercise. This is another exercise on Zorn's Lemma. Recall (1.24) that any binary relation on a set S is formally defined as a subset $R \subseteq S \times S$. We say that R is a *partial order relation* if S equipped with this relation is a partially ordered set (17.6). In the subset notation this mean that R satisfies the following conditions:

(i) $(x, x) \in R$ for all $x \in S$
(ii) if $(x, y) \in R$ and $(y, x) \in S$ then $x = y$
(iii) if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

A partial order relation R is a *linear order relation* if X equipped with this relation becomes a linearly ordered set (17.7) . Explicitly, this mean that R satisfies conditions (i) - (iii), and that for any $x, y \in S$ either $(x, y) \in R$ or $(y, x) \in R$.

If R, R' are binary relations on S then we will say that R' *extends* R if $R \subseteq R'$.

a) Show that if R is a partial order relation on S and $x_0, y_0 \in S$ are elements such that $(x_0, y_0) \notin R$ and $(y_0, x_0) \notin R$ then R can be extended to a partial order relation R' such that $(x_0, y_0) \in R'$.

b) Show that if R is a partial order relation on a set S then R can be extended to a linear order relation $\overline{R}$ on S.

E17.3 Exercise. The goal of this exercise is to complete two details in the proof of the Tychonoff Theorem 17.1.

a) For $j \in J$ let $\mathcal{F}_j$ be a centered family of subsets of a space X. Show that if the set $\{\mathcal{F}_j\}_{j\in J}$ is linearly ordered with respect to inclusion then $\mathcal{F} = \bigcup_{j\in J} \mathcal{F}_j$ is a centered family.

b) Let T denote the collection of all centered families of subsets of X. Consider T with ordering given by inclusion. Let $\mathcal{M}$ be a maximal element in T, and let $A \subseteq X$ be s set such that $A \cap M \neq \varnothing$ for all $M \in \mathcal{M}$. Show that the family $\mathcal{M}' = \mathcal{M} \cup \{A\}$ is centered.

E17.4 Exercise. Prove Proposition 17.16.

E17.5 Exercise. The *Cantor set* is the subspace C of the real line defined as follows. Take $A_0 = [0, 1]$. The set A_1 is then obtained by removing the open middle third subinterval of A_0:

$$A_1 = [0, 1] \smallsetminus (\tfrac{1}{3}, \tfrac{2}{3}) = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$

Next, A_2 is obtained from A_2 by removing open middle third subinterval out of each connected component of A_2. Explicitly:

$$A_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$

Inductively we construct A_{n+1} from A_n by removing the middle third open subintervals from all connected components of A_n. Then we define $C = \bigcap_{n=0}^{\infty} A_n$.

Show that the Cantor set is homeomorphic to the space $\prod_{n=1}^{\infty} D$ where D is the discrete space with two elements $D = \{0, 1\}$.