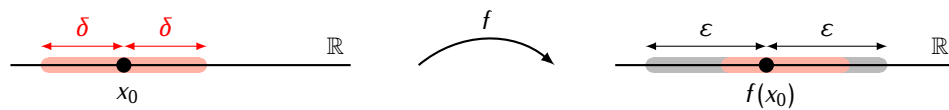
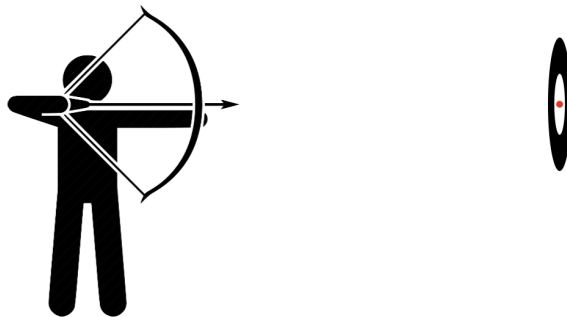


## 2 | Metric Spaces

Recall that a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is *continuous at a point*  $x_0 \in \mathbb{R}$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $|x_0 - x| < \delta$  then  $|f(x_0) - f(x)| < \varepsilon$ :



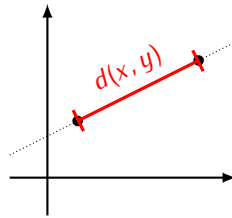
A function is *continuous* if it is continuous at every point  $x_0 \in \mathbb{R}$ .



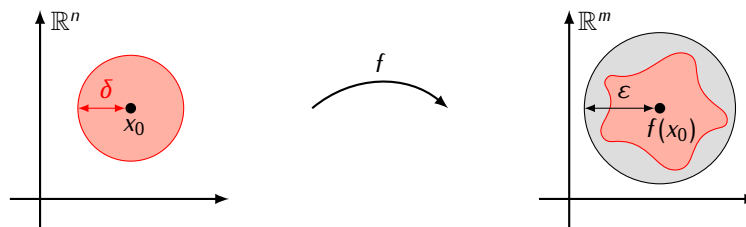
Continuity of functions of several variables  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is defined in a similar way. Recall that  $\mathbb{R}^n := \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$ . If  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  are two points in  $\mathbb{R}^n$  then the distance between  $x$  and  $y$  is given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

The number  $d(x, y)$  is the length of the straight line segment joining the points  $x$  and  $y$ :

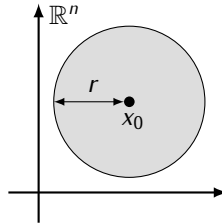


**2.1 Definition.** A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is *continuous at*  $x_0 \in \mathbb{R}^n$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $d(x_0, x) < \delta$  then  $d(f(x_0), f(x)) < \varepsilon$ .

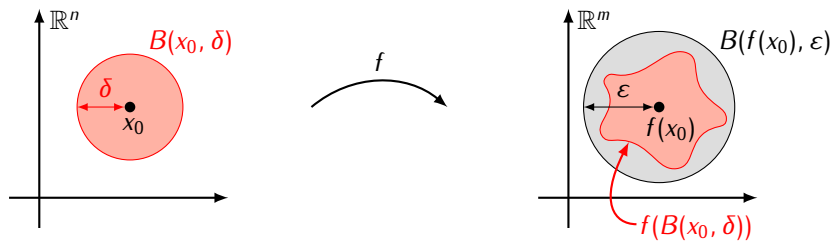


**2.2 Definition.** Let  $x_0 \in \mathbb{R}^n$  and let  $r > 0$ . An *open ball* with radius  $r$  and with center at  $x_0$  is the set

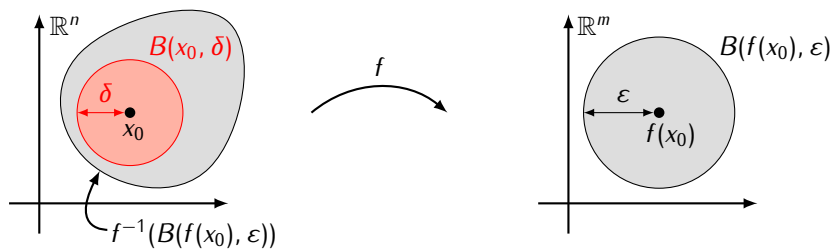
$$B(x_0, r) = \{x \in \mathbb{R}^n \mid d(x_0, x) < r\}$$



Using this terminology we can say that a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous at  $x_0$  if for each  $\varepsilon > 0$  there is a  $\delta > 0$  such  $f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$ :



Here is one more way of rephrasing the definition of continuity:  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous at  $x_0$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$ :



**2.3 Definition.** A *metric space* is a pair  $(X, \varrho)$  where  $X$  is a set and  $\varrho$  is a function

$$\varrho: X \times X \rightarrow \mathbb{R}$$

that satisfies the following conditions:

- 1)  $\varrho(x, y) \geq 0$  and  $\varrho(x, y) = 0$  if and only if  $x = y$ ;
- 2)  $\varrho(x, y) = \varrho(y, x)$ ;
- 3) for any  $x, y, z \in X$  we have  $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$ .

The function  $\varrho$  is called a *metric* on the set  $X$ . For  $x, y \in X$  the number  $\varrho(x, y)$  is called the *distance* between  $x$  and  $y$ .

**2.4 Definition.** Let  $(X, \varrho)$  and  $(Y, \mu)$  be metric spaces. A function  $f: X \rightarrow Y$  is *continuous at*  $x_0 \in X$  if for each  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $\varrho(x_0, x) < \delta$  then  $\mu(f(x_0), f(x)) < \epsilon$ .

A function  $f: X \rightarrow Y$  is *continuous* if it is continuous at every point  $x_0 \in X$ .

**2.5 Definition.** Let  $(X, \varrho)$  be a metric space. For  $x_0 \in X$  and let  $r > 0$  the *open ball* with radius  $r$  and with center at  $x_0$  is the set

$$B_\varrho(x_0, r) = \{x \in X \mid \varrho(x_0, x) < r\}$$

Notice that a function  $f: X \rightarrow Y$  between metric spaces  $(X, \varrho)$  and  $(Y, \mu)$  is continuous at  $x_0 \in X$  if and only if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $B_\varrho(x_0, \delta) \subseteq f^{-1}(B_\mu(f(x_0), \varepsilon))$ .

**2.6 Example.** Let  $X = \mathbb{R}^n$ . For  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  define:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

The metric  $d$  is called the *Euclidean metric* on  $\mathbb{R}^n$ .

**2.7 Example.** Let  $X = \mathbb{R}^n$ . For  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  define:

$$q_{ort}(x, y) = |x_1 - y_1| + \dots + |x_n - y_n|$$

The metric  $q_{ort}$  is called the *orthogonal metric* on  $\mathbb{R}^n$ .

**2.8 Example.** Let  $X = \mathbb{R}^n$ . For  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  define:

$$\varrho_{\max}(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

The metric  $\varrho_{\max}$  is called the *maximum metric* on  $\mathbb{R}^n$ .



**2.9 Example.** Let  $X = \mathbb{R}^n$ . For  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  define  $q_h(x, y)$  as follows. If  $x = y$  then  $q_h(x, y) = 0$ . If  $x \neq y$  then

$$q_h(x, y) = \sqrt{x_1^2 + \dots + x_n^2} + \sqrt{y_1^2 + \dots + y_n^2}$$

The metric  $q_h$  is called the *hub metric* on  $\mathbb{R}^n$ .

**2.10 Example.** Let  $X$  be any set. Define a metric  $\varrho_{disc}$  on  $X$  by

$$\varrho_{disc}(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The metric  $\varrho_{disc}$  is called the *discrete metric* on  $X$ .

**2.11 Example.** If  $(X, \varrho)$  is a metric space and  $A \subseteq X$  then  $A$  is a metric space with the metric induced from  $X$ .