

18 | Compactification

We have seen that compact Hausdorff spaces have several interesting properties that make this class of spaces especially important in topology. If we are working with a space X which is not compact we can ask if X can be embedded into some compact Hausdorff space Y . If such embedding exists we can identify X with a subspace of Y , and some arguments that work for compact Hausdorff spaces will still apply to X . This approach leads to the notion of a *compactification* of a space. Our goal in this chapter is to determine which spaces have compactifications. We will also show that compactifications of a given space X can be ordered, and we will look for the largest and smallest compactifications of X .

18.1 Proposition. *Let X be a topological space. If there exists an embedding $j: X \rightarrow Y$ such that Y is a compact Hausdorff space then there exists an embedding $j_1: X \rightarrow Z$ such that Z is compact Hausdorff and $\overline{j_1(X)} = Z$.*

Proof. Assume that we have an embedding $j: X \rightarrow Y$ where Y is a compact Hausdorff space. Let $\overline{j(X)}$ be the closure of $j(X)$ in Y . The space $\overline{j(X)}$ is compact (by Proposition 14.13) and Hausdorff, so we can take $Z = \overline{j(X)}$ and define $j_1: X \rightarrow Z$ by $j_1(x) = j(x)$ for all $x \in X$. \square

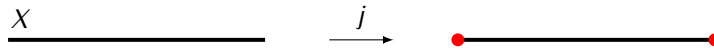
18.2 Definition. A space Z is a *compactification* of X if Z is compact Hausdorff and there exists an embedding $j: X \rightarrow Z$ such that $\overline{j(X)} = Z$.

18.3 Corollary. *Let X be a topological space. The following conditions are equivalent:*

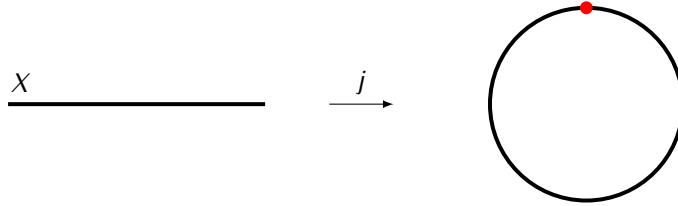
- 1) *There exists a compactification of X .*
- 2) *There exists an embedding $j: X \rightarrow Y$ where Y is a compact Hausdorff space.*

Proof. Follows from Proposition 18.1. \square

18.4 Example. The closed interval $[-1, 1]$ is a compactification of the open interval $(-1, 1)$. with the embedding $j: (-1, 1) \rightarrow [-1, 1]$ is given by $j(t) = t$ for $t \in (-1, 1)$.



18.5 Example. The unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ is another compactification of the interval $(-1, 1)$. The embedding $j: (-1, 1) \rightarrow S^1$ is given by $j(t) = (\sin \pi t, -\cos \pi t)$.

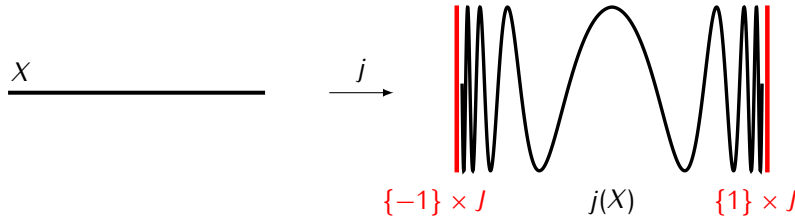


18.6 Example. A more complex compactification of the space $X = (-1, 1)$ can be obtained as follows. Let $J = [-1, 1]$. Consider the function $j: X \rightarrow J \times J$ given by

$$j(t) = \left(t, \cos \left(\frac{|t|}{1-|t|} \right) \right)$$

The map j is an embedding, and so $\overline{j(X)} \subseteq J \times J$ is a compactification of X . We have:

$$\overline{j(X)} = \{-1\} \times J \cup j(X) \cup \{1\} \times J$$



18.7 Theorem. A space X has a compactification if and only if X is completely regular (i.e. it is a $T_{3\frac{1}{2}}$ -space).

Proof. (\Rightarrow) Assume that X has a compactification. Let $j: X \rightarrow Y$ be an embedding where Y is a compact Hausdorff space. By Theorem 14.19 the space Y is normal, so it is completely regular. Since subspaces of completely regular spaces are completely regular (exercise) we obtain that $j(X) \subseteq Y$ is completely regular. Finally, since $j(X) \cong X$ we get that X is completely regular.

(\Leftarrow) Assume that X is completely regular. We need to show that there exists an embedding $j: X \rightarrow Y$ where Y is a compact Hausdorff space. Let $C(X)$ denote the set of all continuous functions $f: X \rightarrow [0, 1]$.

Complete regularity of X implies that $C(X)$ is a family of functions that separates points from closed sets in X (12.13). Consider the map

$$j_X: X \rightarrow \prod_{f \in C(X)} [0, 1]$$

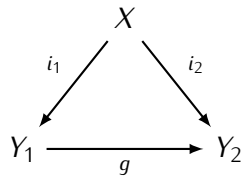
given by $j_X(x) = (f(x))_{f \in C(X)}$. By the Embedding Lemma 12.14 we obtain that this map is an embedding. It remains to notice that by Corollary 17.17 the space $\prod_{f \in C(X)} [0, 1]$ is compact Hausdorff. \square

18.8 Note. In the part (\Rightarrow) of the proof of Theorem 18.7 we used the fact that subspaces of completely regular spaces are completely regular. An analogous property does not hold for normal spaces: a subspace of a normal space need not be normal. For this reason it is not true that a space that has a compactification must be a normal space.

18.9 Definition. Let X be a completely regular space and let $j_X: X \rightarrow \prod_{f \in C(X)} [0, 1]$ be the embedding defined in the proof of Theorem 18.7 and let $\beta(X)$ be the closure of $j_X(X)$ in $\prod_{f \in C(X)} [0, 1]$. The compactification $j_X: X \rightarrow \beta(X)$ is called the *Čech-Stone compactification* of X .

The Čech-Stone compactification is the largest compactification of a space X in the following sense:

18.10 Definition. Let X be a space and let $i_1: X \rightarrow Y_1$, $i_2: X \rightarrow Y_2$ be compactifications of X . We will write $Y_1 \geq Y_2$ if there exists a continuous function $g: Y_1 \rightarrow Y_2$ such that $i_2 = gi_1$:



18.11 Proposition. Let $i_1: X \rightarrow Y_1$, $i_2: X \rightarrow Y_2$ be compactifications of a space X .

- 1) If $Y_1 \geq Y_2$ then there exists only one map $g: Y_1 \rightarrow Y_2$ satisfying $i_2 = gi_1$. Moreover g is onto.
- 2) $Y_1 \geq Y_2$ and $Y_2 \geq Y_1$ if and only if the map $g: Y_1 \rightarrow Y_2$ is a homeomorphism.

Proof. Exercise. \square

18.12 Theorem. Let X be a completely regular space and let $j_X: X \rightarrow \beta(X)$ be the Čech-Stone compactification of X . For any compactification $i: X \rightarrow Y$ of X we have $\beta(X) \geq Y$.

The proof Theorem 18.12 will use the following fact:

18.13 Lemma. *If $f: X_1 \rightarrow X_2$ is a continuous map of compact Hausdorff spaces then $f(\overline{A}) = \overline{f(A)}$ for any $A \subseteq X_1$.*

Proof. Exercise. □

Proof of Theorem 18.12. Let $i: X \rightarrow Y$ be a compactification of X . We need to show that there exists a map $g: \beta(X) \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} & X & \\ j_X \swarrow & & \searrow i \\ \beta(X) & \xrightarrow{g} & Y \end{array}$$

Let $C(X)$, $C(Y)$ denote the sets of all continuous functions $X \rightarrow [0, 1]$ and $Y \rightarrow [0, 1]$ respectively. Consider the continuous functions $j_X: X \rightarrow \prod_{f \in C(X)} [0, 1]$ and $j_Y: Y \rightarrow \prod_{f' \in C(Y)} [0, 1]$ defined as in the proof of Theorem 18.7. Notice that we have a continuous function

$$i_*: \prod_{f \in C(X)} [0, 1] \rightarrow \prod_{f' \in C(Y)} [0, 1]$$

given by $i_*((t_f)_{f \in C(X)}) = (s_{f'})_{f' \in C(Y)}$ where $s_{f'} = t_{if'}$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ j_X \downarrow & & \downarrow j_Y \\ \prod_{f \in C(X)} [0, 1] & \xrightarrow{i_*} & \prod_{f' \in C(Y)} [0, 1] \end{array}$$

We have:

$$i_*(\beta(X)) = i_*(\overline{j_X(X)}) = \overline{i_* j_X(X)} = \overline{j_Y i(X)} = j_Y(\overline{i(X)}) = j_Y(Y)$$

Here the first equality comes from the definition of $\beta(X)$, the second from Lemma 18.13, the third from commutativity of the diagram above, the fourth again from Lemma 18.13, and the last from the assumption that $i: X \rightarrow Y$ is a compactification. Since the map $j_Y: Y \rightarrow \prod_{f' \in C(Y)} [0, 1]$ is embedding the map $j_Y: Y \rightarrow j_Y(Y)$ is a homeomorphism. We can take $g = j_Y^{-1} i_*: \beta(X) \rightarrow Y$.

□

Motivated by the fact that Čech-Stone compactification is the largest compactification of a space X one can ask if the smallest compactification of X also exists. If X is a non-compact space then we need to add at least one point to X to compactify it. If adding only one point suffices then it gives an obvious candidate for the smallest compactification:

18.14 Definition. A space Z is a *one-point compactification* of a space X if Z is a compactification of X with embedding $j: X \rightarrow Z$ such that the set $Z \setminus j(X)$ consists of only one point.

18.15 Example. The unit circle S^1 is a one-point compactification of the open interval $(0, 1)$.

18.16 Proposition. If a space X has a one-point compactification $j: X \rightarrow Z$ then this compactification is unique up to homeomorphism. That is, if $j': X \rightarrow Z'$ is another one-point compactification of X then there exists a homeomorphism $h: Z \rightarrow Z'$ such that $j' = hj$.

Proof. Exercise. □

Our next goal is to determine which spaces admit a one-point compactification.

18.17 Definition. A topological space X is *locally compact* if every point $x \in X$ has an open neighborhood $U_x \subseteq X$ such that the closure $\overline{U_x}$ is compact.

18.18 Note. 1) If X is a compact space then X is locally compact since for any $x \in X$ we can take $U_x = X$.

2) The real line \mathbb{R} is not compact but it is locally compact. For $x \in \mathbb{R}$ we can take $U_x = (x - 1, x + 1)$, and then $\overline{U_x} = [x - 1, x + 1]$ is compact. Similarly, for each $n \geq 0$ the space \mathbb{R}^n is a non-compact but locally compact.

3) The set \mathbb{Q} of rational numbers, considered as a subspace of the real line, is not locally compact (exercise).

18.19 Theorem. Let X be a non-compact topological space. The following conditions are equivalent:

- 1) The space X is locally compact and Hausdorff.
- 2) There exists a one-point compactification of X .

Proof. 1) \Rightarrow 2) Assume that X is locally compact and Hausdorff. We define a space X^+ as follows. Points of X^+ are points of X and one extra point that we will denote by ∞ :

$$X^+ := X \cup \{\infty\}$$

A set $U \subseteq X^+$ is open if either of the following conditions holds:

- (i) $U \subseteq X$ and U is open in X
- (ii) $U = \{\infty\} \cup (X \setminus K)$ where $K \subseteq X$ is a compact set.

The collection of subsets of X^+ defined in this way is a topology on X^+ (exercise). One can check that the function $j: X \rightarrow X^+$ given by $j(x) = x$ is continuous and that it is an embedding (exercise). Moreover, since X is not compact for every open neighborhood U of ∞ we have $U \cap X \neq \emptyset$, so $\overline{j(X)} = X^+$.

To see that X^+ is a compact space assume that $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X^+ . Let $U_{i_0} \in \mathcal{U}$ be a set such that $\infty \in U_{i_0}$. By the definition of the topology on X^+ we have $X^+ \setminus U_{i_0} = K$ where $K \subseteq X$ is a compact set. Compactness of K gives that

$$K \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

for some $U_{i_1}, \dots, U_{i_n} \in \mathcal{U}$. It follows that $\{U_{i_0}, U_{i_1}, \dots, U_{i_n}\}$ is a finite cover of X^+ .

It remains to check that X^+ is a Hausdorff space (exercise).

2) \Rightarrow 1) Let $j: X \rightarrow Z$ be a one-point compactification of X . Since $X \cong j(X)$ it suffices to show that the space $j(X)$ is locally compact and Hausdorff. We will denote by ∞ the unique point in $Z \setminus j(X)$.

Since Z is a Hausdorff space and subspaces of a Hausdorff space are Hausdorff we get that $j(X)$ is a Hausdorff space.

Next, we will show that $j(X)$ is locally compact. Let $x \in j(X)$. Since Z is Hausdorff there are sets $U, V \subseteq Z$ open in Z such that $x \in U$, $\infty \in V$, and $U \cap V = \emptyset$. Since $\infty \notin U$ the set U is open in X . Let \overline{U} denote the closure of U in X . We will show that \overline{U} is a compact set. Notice that we have

$$\overline{U} \subseteq Z \setminus V \subseteq Z$$

Since $Z \setminus V$ is closed in the compact space Z thus it is compact by Proposition 14.13. Also, since \overline{U} is a closed subset of $Z \setminus V$, thus \overline{U} is compact by the same result. \square

18.20 Corollary. *If X is a locally compact Hausdorff space then X is completely regular.*

Proof. Follows from Theorem 18.7 and Theorem 18.19. \square

18.21 Corollary. *Let X be a topological space. The following conditions are equivalent:*

- 1) *The space X is locally compact and Hausdorff.*
- 2) *There exists an embedding $i: X \rightarrow Y$ where Y is compact Hausdorff space and $i(X)$ is an open set in Y .*

Proof. 1) \Rightarrow 2) If X is compact then we can take i to be the identity map $\text{id}_X: X \rightarrow X$. If X is not compact take the one-point compactification $j: X \rightarrow X^+$. By the definition of topology on X^+ the set $j(X)$ is open in X^+ .

2) \Rightarrow 1) exercise. \square

The next proposition says that one-point compactification, when it exists, is the smallest compactification of a space in the sense of Definition 18.10:

18.22 Proposition. *Let X be a non-compact, locally compact space and let $j: X \rightarrow X^+$ be the one-point compactification of X . For every compactification $i: X \rightarrow Y$ of X we have $Y \geq X^+$.*

Proof. Exercise. □

One can also show that if a space X is not locally compact (and so it does not have a one-point compactification) then no compactification of X has the property of being the smallest (see Exercise 18.15).

Exercises to Chapter 18

E18.1 Exercise. Show that a subspace of a completely regular space is completely regular (this will complete the proof of Theorem 18.7).

E18.2 Exercise. Prove Proposition 18.11.

E18.3 Exercise. Prove Lemma 18.13.

E18.4 Exercise. Consider the set \mathbb{Q} of rational numbers with the subspace topology of the real line. Show that \mathbb{Q} is not locally compact.

E18.5 Exercise. Let X be a locally compact Hausdorff space, let $x_0 \in X$, and let $U \subseteq X$ be an open neighborhood of x_0 . Show that there exists an open neighborhood W of x_0 such that $\overline{W} \subseteq U$ and \overline{W} is compact.

E18.6 Exercise. Prove Proposition 18.16.

E18.7 Exercise. The goal of this exercise is to fill one of the gaps left in the proof of Theorem 18.19. Let X be a locally compact Hausdorff space and let $X^+ = X \cup \{\infty\}$ be the space defined in part 1) \Rightarrow 2) of the proof of (18.19). Show that X^+ is a Hausdorff space.

E18.8 Exercise. Prove the implication 2) \Rightarrow 1) of Corollary 18.21.

E18.9 Exercise. A continuous function $f: X \rightarrow Y$ is *proper* if for every compact set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is compact. Let X, Y be locally compact, Hausdorff spaces and let X^+, Y^+ be their one-point compactifications. Let $f: X \rightarrow Y$ be a continuous function. Show that the following conditions are equivalent:

- 1) The function f is proper.
- 2) The function $f^+: X^+ \rightarrow Y^+$ given by $f^+(x) = f(x)$ for $x \in X$ and $f^+(\infty) = \infty$ is continuous.

E18.10 Exercise. Let $(X, \varrho), (Y, \mu)$ be metric spaces and let $f: X \rightarrow Y$ be a continuous function. Show that the following conditions are equivalent:

- 1) f is proper (Exercise 18.9)
- 2) If $\{x_n\} \subseteq X$ is a sequence such that $\{f(x_n)\} \subseteq Y$ converges then $\{x_n\} \subseteq X$ has a convergent subsequence.

E18.11 Exercise. Let X, Y be locally compact Hausdorff spaces, and let $j: X \rightarrow Y$ be an embedding such that $j(X)$ is an open in Y . Define $j^\# : Y^+ \rightarrow X^+$ as follows:

$$j^\#(y) = \begin{cases} j^{-1}(y) & \text{if } y \in j(X) \\ \infty & \text{otherwise} \end{cases}$$

Show that $j^\#$ is a continuous function.

E18.12 Exercise. Let X, Y be locally compact, Hausdorff spaces and let X^+, Y^+ be their one-point compactifications. Let $f: X^+ \rightarrow Y^+$ be a continuous function such $f(\infty) = \infty$. Show that there exists an open set $U \subseteq X$ such $f = g^+ j^\#$ where $j: U \rightarrow X$ is the inclusion map, $g = f|_U: U \rightarrow Y$ is a proper map, $j^\#: X^+ \rightarrow U^+$ is obtained from j as in Exercise 18.11, and $g^+: U^+ \rightarrow Y^+$ obtained from g as in Exercise 18.9.

E18.13 Exercise. Let X be topological space and let $j: X \rightarrow Y$ be a compactification of X . Show that if X is locally compact the set $j(X)$ is open in Y .

E18.14 Exercise. Prove Proposition 18.22.

E18.15 Exercise. The goal of this exercise is to show that the smallest compactification of a non-compact space X exists only if X has a one-point compactification (i.e. if X is a locally compact space).

Let X be a completely regular non-compact space. Assume that there exists a compactification $j: X \rightarrow Y$ of X such that for any other compactification $i: X \rightarrow Z$ we have $Z \geq Y$. Show that Y is a one-point compactification of X . As a consequence X must be locally compact. (Hint: Assume that Y is not a one-point compactification of X and let $y_1, y_2 \in Y \setminus j(X)$. Show that the space $W = Y \setminus \{y_1, y_2\}$ has a one-point compactification $k: W \rightarrow W^+$ and that $kj: X \rightarrow W^+$ is a compactification of X . Show that it is not true that $W^+ \geq Y$).