

# 18 | Compactification

We have seen that compact Hausdorff spaces have several interesting properties that make this class of spaces especially important in topology. If we are working with a space  $X$  which is not compact we can ask if  $X$  can be embedded into some compact Hausdorff space  $Y$ . If such embedding exists we can identify  $X$  with a subspace of  $Y$ , and some arguments that work for compact Hausdorff spaces will still apply to  $X$ . This approach leads to the notion of a *compactification* of a space. Our goal in this chapter is to determine which spaces have compactifications. We will also show that compactifications of a given space  $X$  can be ordered, and we will look for the largest and smallest compactifications of  $X$ .

**18.1 Proposition.** *Let  $X$  be a topological space. If there exists an embedding  $j: X \rightarrow Y$  such that  $Y$  is a compact Hausdorff space then there exists an embedding  $j_1: X \rightarrow Z$  such that  $Z$  is compact Hausdorff and  $\overline{j_1(X)} = Z$ .*

*Proof.* Assume that we have an embedding  $j: X \rightarrow Y$  where  $Y$  is a compact Hausdorff space. Let  $\overline{j(X)}$  be the closure of  $j(X)$  in  $Y$ . The space  $\overline{j(X)}$  is compact (by Proposition 14.13) and Hausdorff, so we can take  $Z = \overline{j(X)}$  and define  $j_1: X \rightarrow Z$  by  $j_1(x) = j(x)$  for all  $x \in X$ .  $\square$

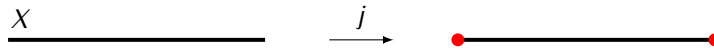
**18.2 Definition.** A space  $Z$  is a *compactification* of  $X$  if  $Z$  is compact Hausdorff and there exists an embedding  $j: X \rightarrow Z$  such that  $\overline{j(X)} = Z$ .

**18.3 Corollary.** *Let  $X$  be a topological space. The following conditions are equivalent:*

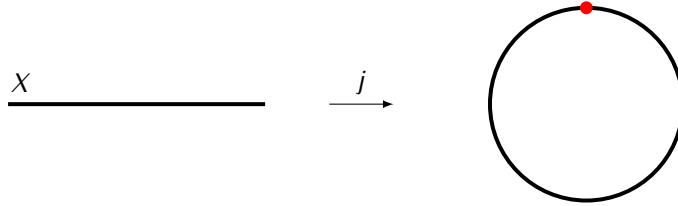
- 1) *There exists a compactification of  $X$ .*
- 2) *There exists an embedding  $j: X \rightarrow Y$  where  $Y$  is a compact Hausdorff space.*

*Proof.* Follows from Proposition 18.1.  $\square$

**18.4 Example.** The closed interval  $[-1, 1]$  is a compactification of the open interval  $(-1, 1)$ . with the embedding  $j: (-1, 1) \rightarrow [-1, 1]$  is given by  $j(t) = t$  for  $t \in (-1, 1)$ .



**18.5 Example.** The unit circle  $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$  is another compactification of the interval  $(-1, 1)$ . The embedding  $j: (-1, 1) \rightarrow S^1$  is given by  $j(t) = (\sin \pi t, -\cos \pi t)$ .

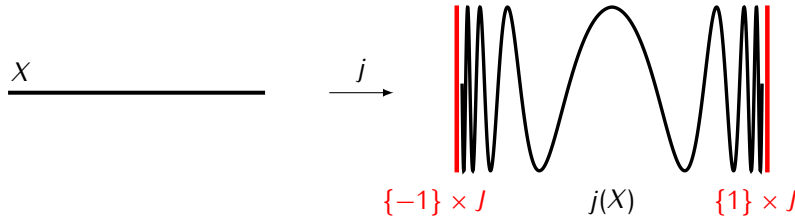


**18.6 Example.** A more complex compactification of the space  $X = (-1, 1)$  can be obtained as follows. Let  $J = [-1, 1]$ . Consider the function  $j: X \rightarrow J \times J$  given by

$$j(t) = \left( t, \cos \left( \frac{|t|}{1-|t|} \right) \right)$$

The map  $j$  is an embedding, and so  $\overline{j(X)} \subseteq J \times J$  is a compactification of  $X$ . We have:

$$\overline{j(X)} = \{-1\} \times J \cup j(X) \cup \{1\} \times J$$



**18.7 Theorem.** A space  $X$  has a compactification if and only if  $X$  is completely regular (i.e. it is a  $T_{3\frac{1}{2}}$ -space).

*Proof.* ( $\Rightarrow$ ) Assume that  $X$  has a compactification. Let  $j: X \rightarrow Y$  be an embedding where  $Y$  is a compact Hausdorff space. By Theorem 14.19 the space  $Y$  is normal, so it is completely regular. Since subspaces of completely regular spaces are completely regular (exercise) we obtain that  $j(X) \subseteq Y$  is completely regular. Finally, since  $j(X) \cong X$  we get that  $X$  is completely regular.

( $\Leftarrow$ ) Assume that  $X$  is completely regular. We need to show that there exists an embedding  $j: X \rightarrow Y$  where  $Y$  is a compact Hausdorff space. Let  $C(X)$  denote the set of all continuous functions  $f: X \rightarrow [0, 1]$ .

Complete regularity of  $X$  implies that  $C(X)$  is a family of functions that separates points from closed sets in  $X$  (12.13). Consider the map

$$j_X: X \rightarrow \prod_{f \in C(X)} [0, 1]$$

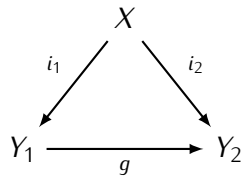
given by  $j_X(x) = (f(x))_{f \in C(X)}$ . By the Embedding Lemma 12.14 we obtain that this map is an embedding. It remains to notice that by Corollary 17.17 the space  $\prod_{f \in C(X)} [0, 1]$  is compact Hausdorff.  $\square$

**18.8 Note.** In the part  $(\Rightarrow)$  of the proof of Theorem 18.7 we used the fact that subspaces of completely regular spaces are completely regular. An analogous property does not hold for normal spaces: a subspace of a normal space need not be normal. For this reason it is not true that a space that has a compactification must be a normal space.

**18.9 Definition.** Let  $X$  be a completely regular space and let  $j_X: X \rightarrow \prod_{f \in C(X)} [0, 1]$  be the embedding defined in the proof of Theorem 18.7 and let  $\beta(X)$  be the closure of  $j_X(X)$  in  $\prod_{f \in C(X)} [0, 1]$ . The compactification  $j_X: X \rightarrow \beta(X)$  is called the *Čech-Stone compactification* of  $X$ .

The Čech-Stone compactification is the largest compactification of a space  $X$  in the following sense:

**18.10 Definition.** Let  $X$  be a space and let  $i_1: X \rightarrow Y_1$ ,  $i_2: X \rightarrow Y_2$  be compactifications of  $X$ . We will write  $Y_1 \geq Y_2$  if there exists a continuous function  $g: Y_1 \rightarrow Y_2$  such that  $i_2 = gi_1$ :



**18.11 Proposition.** Let  $i_1: X \rightarrow Y_1$ ,  $i_2: X \rightarrow Y_2$  be compactifications of a space  $X$ .

- 1) If  $Y_1 \geq Y_2$  then there exists only one map  $g: Y_1 \rightarrow Y_2$  satisfying  $i_2 = gi_1$ . Moreover  $g$  is onto.
- 2)  $Y_1 \geq Y_2$  and  $Y_2 \geq Y_1$  if and only if the map  $g: Y_1 \rightarrow Y_2$  is a homeomorphism.

*Proof.* Exercise.  $\square$

**18.12 Theorem.** Let  $X$  be a completely regular space and let  $j_X: X \rightarrow \beta(X)$  be the Čech-Stone compactification of  $X$ . For any compactification  $i: X \rightarrow Y$  of  $X$  we have  $\beta(X) \geq Y$ .

The proof Theorem 18.12 will use the following fact:

**18.13 Lemma.** *If  $f: X_1 \rightarrow X_2$  is a continuous map of compact Hausdorff spaces then  $f(\overline{A}) = \overline{f(A)}$  for any  $A \subseteq X_1$ .*

*Proof.* Exercise. □

*Proof of Theorem 18.12.* Let  $i: X \rightarrow Y$  be a compactification of  $X$ . We need to show that there exists a map  $g: \beta(X) \rightarrow Y$  such that the following diagram commutes:

$$\begin{array}{ccc} & X & \\ j_X \swarrow & & \searrow i \\ \beta(X) & \xrightarrow{g} & Y \end{array}$$

Let  $C(X)$ ,  $C(Y)$  denote the sets of all continuous functions  $X \rightarrow [0, 1]$  and  $Y \rightarrow [0, 1]$  respectively. Consider the continuous functions  $j_X: X \rightarrow \prod_{f \in C(X)} [0, 1]$  and  $j_Y: Y \rightarrow \prod_{f' \in C(Y)} [0, 1]$  defined as in the proof of Theorem 18.7. Notice that we have a continuous function

$$i_*: \prod_{f \in C(X)} [0, 1] \rightarrow \prod_{f' \in C(Y)} [0, 1]$$

given by  $i_*((t_f)_{f \in C(X)}) = (s_{f'})_{f' \in C(Y)}$  where  $s_{f'} = t_{if'}$ . Moreover, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ j_X \downarrow & & \downarrow j_Y \\ \prod_{f \in C(X)} [0, 1] & \xrightarrow{i_*} & \prod_{f' \in C(Y)} [0, 1] \end{array}$$

We have:

$$i_*(\beta(X)) = i_*(\overline{j_X(X)}) = \overline{i_* j_X(X)} = \overline{j_Y i(X)} = j_Y(\overline{i(X)}) = j_Y(Y)$$

Here the first equality comes from the definition of  $\beta(X)$ , the second from Lemma 18.13, the third from commutativity of the diagram above, the fourth again from Lemma 18.13, and the last from the assumption that  $i: X \rightarrow Y$  is a compactification. Since the map  $j_Y: Y \rightarrow \prod_{f' \in C(Y)} [0, 1]$  is embedding the map  $j_Y: Y \rightarrow j_Y(Y)$  is a homeomorphism. We can take  $g = j_Y^{-1} i_*: \beta(X) \rightarrow Y$ . □

Motivated by the fact that Čech-Stone compactification is the largest compactification of a space  $X$  one can ask if the smallest compactification of  $X$  also exists. If  $X$  is a non-compact space then we need to add at least one point to  $X$  to compactify it. If adding only one point suffices then it gives an obvious candidate for the smallest compactification:

**18.14 Definition.** A space  $Z$  is a *one-point compactification* of a space  $X$  if  $Z$  is a compactification of  $X$  with embedding  $j: X \rightarrow Z$  such that the set  $Z \setminus j(X)$  consists of only one point.

**18.15 Example.** The unit circle  $S^1$  is a one-point compactification of the open interval  $(0, 1)$ .

**18.16 Proposition.** If a space  $X$  has a one-point compactification  $j: X \rightarrow Z$  then this compactification is unique up to homeomorphism. That is, if  $j': X \rightarrow Z'$  is another one-point compactification of  $X$  then there exists a homeomorphism  $h: Z \rightarrow Z'$  such that  $j' = hj$ .

*Proof.* Exercise. □

Our next goal is to determine which spaces admit a one-point compactification.

**18.17 Definition.** A topological space  $X$  is *locally compact* if every point  $x \in X$  has an open neighborhood  $U_x \subseteq X$  such that the closure  $\overline{U_x}$  is compact.

**18.18 Note.** 1) If  $X$  is a compact space then  $X$  is locally compact since for any  $x \in X$  we can take  $U_x = X$ .

2) The real line  $\mathbb{R}$  is not compact but it is locally compact. For  $x \in \mathbb{R}$  we can take  $U_x = (x - 1, x + 1)$ , and then  $\overline{U_x} = [x - 1, x + 1]$  is compact. Similarly, for each  $n \geq 0$  the space  $\mathbb{R}^n$  is a non-compact but locally compact.

3) The set  $\mathbb{Q}$  of rational numbers, considered as a subspace of the real line, is not locally compact (exercise).

**18.19 Theorem.** Let  $X$  be a non-compact topological space. The following conditions are equivalent:

- 1) The space  $X$  is locally compact and Hausdorff.
- 2) There exists a one-point compactification of  $X$ .

*Proof.* 1)  $\Rightarrow$  2) Assume that  $X$  is locally compact and Hausdorff. We define a space  $X^+$  as follows. Points of  $X^+$  are points of  $X$  and one extra point that we will denote by  $\infty$ :

$$X^+ := X \cup \{\infty\}$$

A set  $U \subseteq X^+$  is open if either of the following conditions holds:

- (i)  $U \subseteq X$  and  $U$  is open in  $X$
- (ii)  $U = \{\infty\} \cup (X \setminus K)$  where  $K \subseteq X$  is a compact set.

The collection of subsets of  $X^+$  defined in this way is a topology on  $X^+$  (exercise). One can check that the function  $j: X \rightarrow X^+$  given by  $j(x) = x$  is continuous and that it is an embedding (exercise). Moreover, since  $X$  is not compact for every open neighborhood  $U$  of  $\infty$  we have  $U \cap X \neq \emptyset$ , so  $\overline{j(X)} = X^+$ .

To see that  $X^+$  is a compact space assume that  $\mathcal{U} = \{U_i\}_{i \in I}$  is an open cover of  $X^+$ . Let  $U_{i_0} \in \mathcal{U}$  be a set such that  $\infty \in U_{i_0}$ . By the definition of the topology on  $X^+$  we have  $X^+ \setminus U_{i_0} = K$  where  $K \subseteq X$  is a compact set. Compactness of  $K$  gives that

$$K \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

for some  $U_{i_1}, \dots, U_{i_n} \in \mathcal{U}$ . It follows that  $\{U_{i_0}, U_{i_1}, \dots, U_{i_n}\}$  is a finite cover of  $X^+$ .

It remains to check that  $X^+$  is a Hausdorff space (exercise).

2)  $\Rightarrow$  1) Let  $j: X \rightarrow Z$  be a one-point compactification of  $X$ . Since  $X \cong j(X)$  it suffices to show that the space  $j(X)$  is locally compact and Hausdorff. We will denote by  $\infty$  the unique point in  $Z \setminus j(X)$ .

Since  $Z$  is a Hausdorff space and subspaces of a Hausdorff space are Hausdorff we get that  $j(X)$  is a Hausdorff space.

Next, we will show that  $j(X)$  is locally compact. Let  $x \in j(X)$ . Since  $Z$  is Hausdorff there are sets  $U, V \subseteq Z$  open in  $Z$  such that  $x \in U$ ,  $\infty \in V$ , and  $U \cap V = \emptyset$ . Since  $\infty \notin U$  the set  $U$  is open in  $X$ . Let  $\overline{U}$  denote the closure of  $U$  in  $X$ . We will show that  $\overline{U}$  is a compact set. Notice that we have

$$\overline{U} \subseteq Z \setminus V \subseteq Z$$

Since  $Z \setminus V$  is closed in the compact space  $Z$  thus it is compact by Proposition 14.13. Also, since  $\overline{U}$  is a closed subset of  $Z \setminus V$ , thus  $\overline{U}$  is compact by the same result.  $\square$

**18.20 Corollary.** *If  $X$  is a locally compact Hausdorff space then  $X$  is completely regular.*

*Proof.* Follows from Theorem 18.7 and Theorem 18.19.  $\square$

**18.21 Corollary.** *Let  $X$  be a topological space. The following conditions are equivalent:*

- 1) *The space  $X$  is locally compact and Hausdorff.*
- 2) *There exists an embedding  $i: X \rightarrow Y$  where  $Y$  is compact Hausdorff space and  $i(X)$  is an open set in  $Y$ .*

*Proof.* 1)  $\Rightarrow$  2) If  $X$  is compact then we can take  $i$  to be the identity map  $\text{id}_X: X \rightarrow X$ . If  $X$  is not compact take the one-point compactification  $j: X \rightarrow X^+$ . By the definition of topology on  $X^+$  the set  $j(X)$  is open in  $X^+$ .

2)  $\Rightarrow$  1) exercise.  $\square$

The next proposition says that one-point compactification, when it exists, is the smallest compactification of a space in the sense of Definition 18.10:

**18.22 Proposition.** *Let  $X$  be a non-compact, locally compact space and let  $j: X \rightarrow X^+$  be the one-point compactification of  $X$ . For every compactification  $i: X \rightarrow Y$  of  $X$  we have  $Y \geq X^+$ .*

*Proof.* Exercise. □

One can also show that if a space  $X$  is not locally compact (and so it does not have a one-point compactification) then no compactification of  $X$  has the property of being the smallest (see Exercise 18.15).

### Exercises to Chapter 18

**E18.1 Exercise.** Show that a subspace of a completely regular space is completely regular (this will complete the proof of Theorem 18.7).

**E18.2 Exercise.** Prove Proposition 18.11.

**E18.3 Exercise.** Prove Lemma 18.13.

**E18.4 Exercise.** Consider the set  $\mathbb{Q}$  of rational numbers with the subspace topology of the real line. Show that  $\mathbb{Q}$  is not locally compact.

**E18.5 Exercise.** Let  $X$  be a locally compact Hausdorff space, let  $x_0 \in X$ , and let  $U \subseteq X$  be an open neighborhood of  $x_0$ . Show that there exists an open neighborhood  $W$  of  $x_0$  such that  $\overline{W} \subseteq U$  and  $\overline{W}$  is compact.

**E18.6 Exercise.** Prove Proposition 18.16.

**E18.7 Exercise.** The goal of this exercise is to fill one of the gaps left in the proof of Theorem 18.19. Let  $X$  be a locally compact Hausdorff space and let  $X^+ = X \cup \{\infty\}$  be the space defined in part 1)  $\Rightarrow$  2) of the proof of (18.19). Show that  $X^+$  is a Hausdorff space.

**E18.8 Exercise.** Prove the implication 2)  $\Rightarrow$  1) of Corollary 18.21.

**E18.9 Exercise.** A continuous function  $f: X \rightarrow Y$  is *proper* if for every compact set  $A \subseteq Y$  the set  $f^{-1}(A) \subseteq X$  is compact. Let  $X, Y$  be locally compact, Hausdorff spaces and let  $X^+, Y^+$  be their one-point compactifications. Let  $f: X \rightarrow Y$  be a continuous function. Show that the following conditions are equivalent:

- 1) The function  $f$  is proper.
- 2) The function  $f^+: X^+ \rightarrow Y^+$  given by  $f^+(x) = f(x)$  for  $x \in X$  and  $f^+(\infty) = \infty$  is continuous.

**E18.10 Exercise.** Let  $(X, \varrho), (Y, \mu)$  be metric spaces and let  $f: X \rightarrow Y$  be a continuous function. Show that the following conditions are equivalent:

- 1)  $f$  is proper (Exercise 18.9)
- 2) If  $\{x_n\} \subseteq X$  is a sequence such that  $\{f(x_n)\} \subseteq Y$  converges then  $\{x_n\} \subseteq X$  has a convergent subsequence.

**E18.11 Exercise.** Let  $X, Y$  be locally compact Hausdorff spaces, and let  $j: X \rightarrow Y$  be an embedding such that  $j(X)$  is an open in  $Y$ . Define  $j^\# : Y^+ \rightarrow X^+$  as follows:

$$j^\#(y) = \begin{cases} j^{-1}(y) & \text{if } y \in j(X) \\ \infty & \text{otherwise} \end{cases}$$

Show that  $j^\#$  is a continuous function.

**E18.12 Exercise.** Let  $X, Y$  be locally compact, Hausdorff spaces and let  $X^+, Y^+$  be their one-point compactifications. Let  $f: X^+ \rightarrow Y^+$  be a continuous function such  $f(\infty) = \infty$ . Show that there exists an open set  $U \subseteq X$  such  $f = g^+ j^\#$  where  $j: U \rightarrow X$  is the inclusion map,  $g = f|_U: U \rightarrow Y$  is a proper map,  $j^\#: X^+ \rightarrow U^+$  is obtained from  $j$  as in Exercise 18.11, and  $g^+: U^+ \rightarrow Y^+$  obtained from  $g$  as in Exercise 18.9.

**E18.13 Exercise.** Let  $X$  be topological space and let  $j: X \rightarrow Y$  be a compactification of  $X$ . Show that if  $X$  is locally compact the set  $j(X)$  is open in  $Y$ .

**E18.14 Exercise.** Prove Proposition 18.22.

**E18.15 Exercise.** The goal of this exercise is to show that the smallest compactification of a non-compact space  $X$  exists only if  $X$  has a one-point compactification (i.e. if  $X$  is a locally compact space).

Let  $X$  be a completely regular non-compact space. Assume that there exists a compactification  $j: X \rightarrow Y$  of  $X$  such that for any other compactification  $i: X \rightarrow Z$  we have  $Z \geq Y$ . Show that  $Y$  is a one-point compactification of  $X$ . As a consequence  $X$  must be locally compact. (Hint: Assume that  $Y$  is not a one-point compactification of  $X$  and let  $y_1, y_2 \in Y \setminus j(X)$ . Show that the space  $W = Y \setminus \{y_1, y_2\}$  has a one-point compactification  $k: W \rightarrow W^+$  and that  $kj: X \rightarrow W^+$  is a compactification of  $X$ . Show that it is not true that  $W^+ \geq Y$ ).