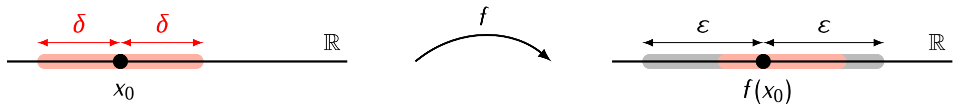
2 | Metric Spaces

Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $x_0 \in \mathbb{R}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x_0 - x| < \delta$ then $|f(x_0) - f(x)| < \varepsilon$:

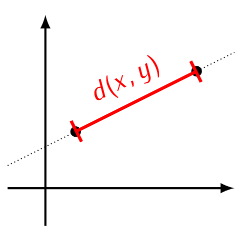


A function is *continuous* if it is continuous at every point $x_0 \in \mathbb{R}$.

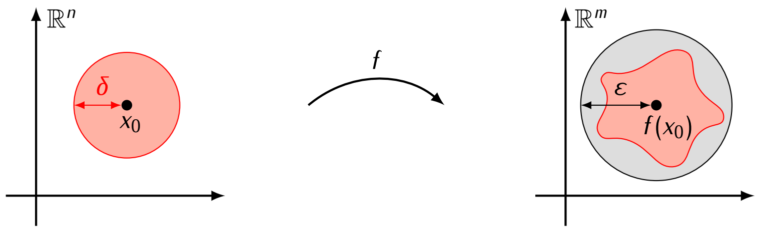
Continuity of functions of several variables $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is defined in a similar way. Recall that $\mathbb{R}^n := \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are two points in $\mathbb{R}^n$ then the distance between x and y is given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

The number $d(x, y)$ is the length of the straight line segment joining the points x and y:



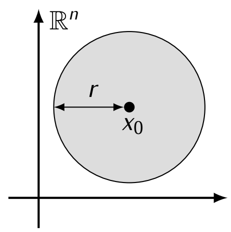
2.1 Definition. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *continuous at* $x_0 \in \mathbb{R}^n$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x_0, x) < \delta$ then $d(f(x_0), f(x)) < \varepsilon$.



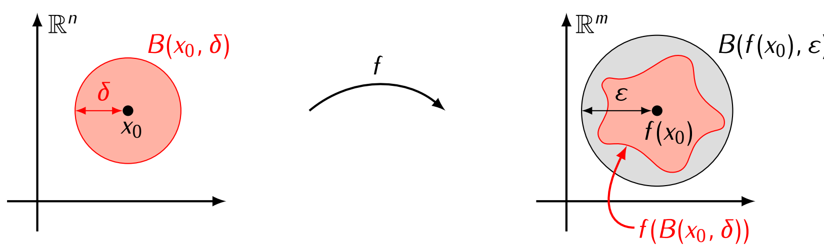
The above picture motivates the following, more geometric reformulation of continuity:

2.2 Definition. Let $x_0 \in \mathbb{R}^n$ and let $r > 0$. An *open ball* with radius r and with center at x_0 is the set

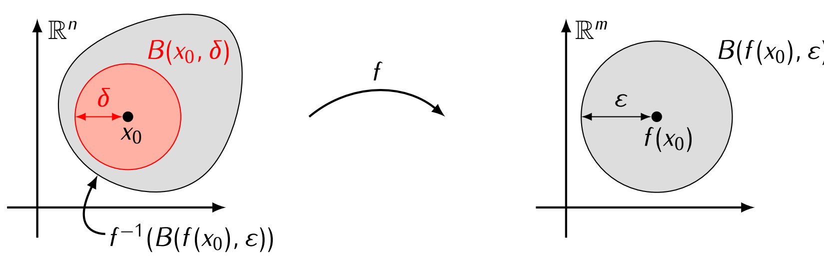
$$B(x_0, r) = \{x \in \mathbb{R}^n \mid d(x_0, x) < r\}$$



Using this terminology we can say that a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x_0 if for each $\varepsilon > 0$ there is a $\delta > 0$ such $f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$:



Here is one more way of rephrasing the definition of continuity: $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$:



Notice that in order to define continuity of functions $\mathbb{R}^n \to \mathbb{R}^m$ we used only the fact the for any two points in $\mathbb{R}^n$ or $\mathbb{R}^m$ we can compute the distance between these points. This suggests that we could define similarly what is means that a function $f\colon X \to Y$ is continuous where X and Y are any sets, provided that we have some way of measuring distances between points in these sets. This observation leads to the notion of a metric space:

2.3 Definition. A *metric space* is a pair (X, ϱ) where X is a set and ϱ is a function

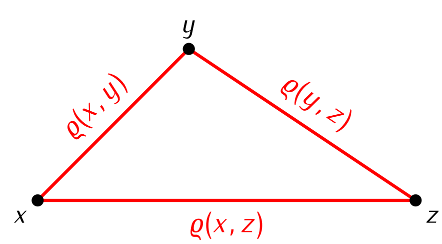
$$\varrho\colon X \times X \to \mathbb{R}$$

that satisfies the following conditions:

1) $\varrho(x, y) \geq 0$ and $\varrho(x, y) = 0$ if and only if $x = y$;
2) $\varrho(x, y) = \varrho(y, x)$;
3) for any $x, y, z \in X$ we have $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$.

The function ϱ is called a *metric* on the set X. For $x, y \in X$ the number $\varrho(x, y)$ is called the *distance* between x and y.

The first condition in Definition 2.3 says that distances between points of X are non-negative, and that the only point located within the distance zero from a point x is the point x itself. The second condition says that the distance from x to y is the same as the distance from y to x. The third condition is called the *triangle inequality*. It says that the distance between points x and z measured directly will never be bigger than the number we obtain by taking the distance from x to some intermediary point y and adding it to the distance between y and z:



We define continuity of functions between metric spaces the same way as for functions between $\mathbb{R}^n$ and $\mathbb{R}^m$:

2.4 Definition. Let (X, ϱ) and (Y, μ) be metric spaces. A function $f\colon X \to Y$ is *continuous at* $x_0 \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varrho(x_0, x) < \delta$ then $\mu(f(x_0), f(x)) < \epsilon$.

A function $f\colon X \to Y$ is *continuous* if it is continuous at every point $x_0 \in X$.

We can reformulate this definition in terms of open balls:

2.5 Definition. Let (X, ϱ) be a metric space. For $x_0 \in X$ and let $r > 0$ the *open ball* with radius r and with center at x_0 is the set

$$B_\varrho(x_0, r) = \{x \in X \mid \varrho(x_0, x) < r\}$$

We will often write $B(x_0, r)$ instead of $B_\varrho(x_0, r)$ when it will be clear from the context which metric is being used.

Notice that a function $f\colon X \to Y$ between metric spaces (X, ϱ) and (Y, μ) is continuous at $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $B_\varrho(x_0, \delta) \subseteq f^{-1}(B_\mu(f(x_0), \varepsilon))$.

Here are some examples of metric spaces:

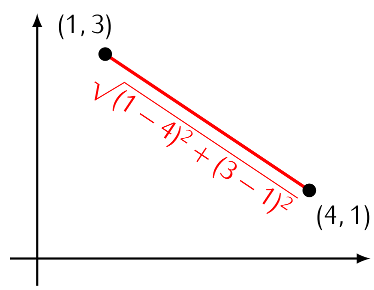
2.6 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

The metric d is called the *Euclidean metric* on $\mathbb{R}^n$.

For example, if $x = (1, 3)$ and $y = (4, 1)$ are points in $\mathbb{R}^2$ then

$$d(x, y) = \sqrt{(1 - 4)^2 + (3 - 1)^2} = \sqrt{13}$$



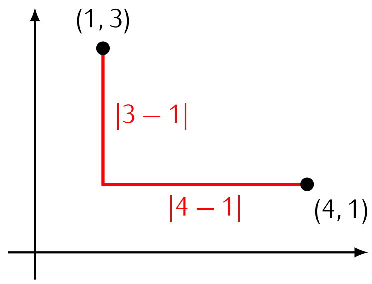
2.7 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define:

$$\varrho_{ort}(x, y) = |x_1 - y_1| + \dots + |x_n - y_n|$$

The metric ϱ_{ort} is called the *orthogonal metric* on $\mathbb{R}^n$.

For example, if $x = (1, 3)$ and $y = (4, 1)$ are points in $\mathbb{R}^2$ then

$$\varrho_{ort}(x, y) = |1 - 4| + |3 - 1| = 5$$



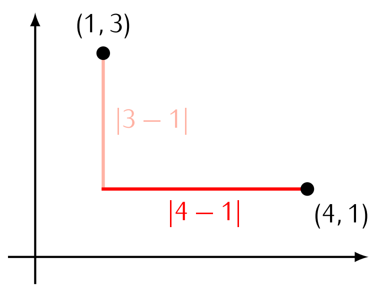
2.8 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define:

$$\varrho_{max}(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

The metric ϱ_{max} is called the *maximum metric* on $\mathbb{R}^n$.

For example, if $x = (1, 3)$ and $y = (4, 1)$ are points in $\mathbb{R}^2$ then

$$\varrho_{max}(x, y) = \max\{|1 - 4|, |3 - 1|\} = 3$$



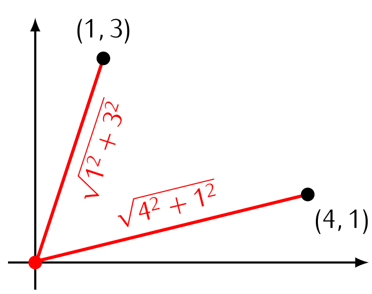
2.9 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define $\varrho_h(x, y)$ as follows. If $x = y$ then $\varrho_h(x, y) = 0$. If $x \neq y$ then

$$\varrho_h(x, y) = \sqrt{x_1^2 + \cdots + x_n^2} + \sqrt{y_1^2 + \cdots + y_n^2}$$

The metric ϱ_h is called the *hub metric* on $\mathbb{R}^n$.

For example, if $x = (1, 3)$ and $y = (4, 1)$ are points in $\mathbb{R}^2$ then

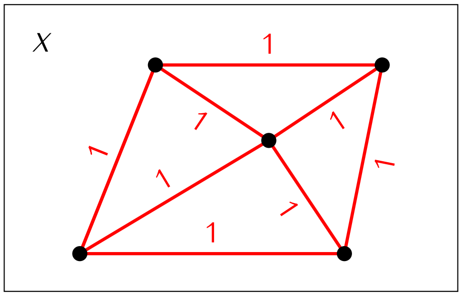
$$\varrho_h(x, y) = \sqrt{1^2 + 3^2} + \sqrt{4^2 + 1^2} = \sqrt{10} + \sqrt{17}$$



2.10 Example. Let X be any set. Define a metric ϱ_{disc} on X by

$$\varrho_{disc}(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The metric ϱ_{disc} is called the *discrete metric* on X.



2.11 Example. If (X, ϱ) is a metric space and $A \subseteq X$ then A is a metric space with the metric induced from X.

Exercises to Chapter 2

E2.1 Exercise. Verify the ϱ_{max} is a metric on $\mathbb{R}^n$.

E2.2 Exercise. For points $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ define

$$\varrho_{min}(x, y) = \min\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

Does this define a metric on $\mathbb{R}^n$? Justify your answer.

E2.3 Exercise. Let $\mathbb{Z}$ be a set of all integers, and let p be some fixed prime number. For $m, n \in \mathbb{Z}$ define

$$\varrho_p(m, n) := \begin{cases} 0 & \text{if } m = n \\ p^{-k} & \text{if } m - n = p^k r \text{ where } r \in \mathbb{Z},\ p \nmid r \end{cases}$$

Verify that ϱ_p is a metric on $\mathbb{Z}$. It is called the *p-adic metric*.

E2.4 Exercise. Let S be a set and let $\mathcal{F}(S)$ denote the set of all non-empty finite subsets of S. For $A, B \in \mathcal{F}(S)$ define

$$\varrho(A, B) = 1 - \frac{|A \cap B|}{|A \cup B|}$$

where $|A|$ denotes the number of elements of the set A. Show that ϱ is a metric on $\mathcal{F}(S)$.

E2.5 Exercise. Draw the following open balls in $\mathbb{R}^2$ defined by the specified metrics:

a) $B(x_0, 1)$ for $x_0 = (0, 0)$ and the orthogonal metric ϱ_{ort}.
b) $B(x_0, 1)$ for $x_0 = (0, 0)$ and the maximum metric ϱ_{max}.
c) $B(x_0, 1)$ for $x_0 = (0, 0)$ and the hub metric ϱ_h.
d) $B(x_0, 6)$ for $x_0 = (3, 4)$ and the hub metric ϱ_h.
e) $B(x_0, 1)$ for $x_0 = (3, 4)$ and the hub metric ϱ_h.

E2.6 Exercise. Let (X, ϱ) be a metric space, and let $x_0 \in X$, Show that if $x \in B(x_0, r)$ then exists $s > 0$ such that $B(x, s) \subseteq B(x_0, r)$.

E2.7 Exercise. a) Let (X, ϱ) be a metric space and let $B(x, r)$, $B(y, s)$ be open balls in X such that $B(y, s) \subseteq B(x, r)$ but $B(y, s) \neq B(x, r)$. Show that $s < 2r$.

b) Give an example of a metric space (X, ϱ) and open balls $B(x, r)$, $B(y, s)$ in X that satisfy the assumptions of part a) and such that $s > r$.

E2.8 Exercise. Let (X, ϱ_{disc}) be a discrete metric space and let (Y, μ) be some metric space. Show that every function $f \colon X \to Y$ is continuous.

E2.9 Exercise. Consider $\mathbb{R}^2$ as a metric space with the hub metric ϱ_h and $\mathbb{R}^1$ as a metric space with the Euclidean metric d.

a) Show that the function $f \colon \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) = (0, 0) \\ 1 & \text{otherwise} \end{cases}$$

is not continuous.

b) Show that the function $g \colon \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$g(x_1, x_2) = \begin{cases} 0 & \text{if } x_1^2 + x_2^2 < 1 \\ 1 & \text{otherwise} \end{cases}$$

is continuous.