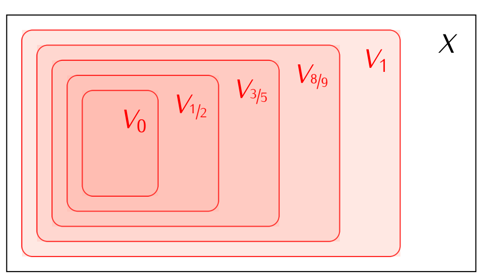
10 | Urysohn Lemma

The separation axioms introduced in the last chapter can be seen as a tool to constructing closer and closer approximations of the class of metrizable spaces. However, even normal spaces, i.e. spaces that satisfy the of the strongest of these axioms need not be metrizable. For example, take the real line $\mathbb{R}$ with the arrow topology (4.8). One can show that it is a normal space (exercise), but by Exercise 5.15 this space is not metrizable. The Urysohn Lemma, which is the main result of this chapter, shows however that normal spaces retain some useful properties of metrizable spaces. Recall that in the last chapter we have seen that for any metric space X, and any pair of disjoint closed sets in X we can find is a continuous function $f\colon X \to [0,1]$ which maps one set to 0 and the other set to 1. The Urysohn lemma says that the same property holds for any normal space:

10.1 Urysohn Lemma. *Let X be a normal space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$. There exists a continuous function $f\colon X \to [0,1]$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$.*

The proof of this fact will use a couple of lemmas:

10.2 Lemma. *Let X be a topological space. Assume that for each $r \in [0,1] \cap \mathbb{Q}$ we are given an open set $V_r \subseteq X$ such that $\overline{V}_r \subseteq V_{r'}$ if $r < r'$. There exists a continuous function $f\colon X \to [0,1]$ such that if $x \in V_0$ then $f(x) = 0$ and if $x \notin V_1$ then $f(x) = 1$.*



Proof. Define the function $f\colon X \to [0,1]$ by:

$$f(x) = \begin{cases} 1 & \text{if } x \notin V_1 \\ \inf\{r \mid x \in V_r\} & \text{if } x \in V_1 \end{cases}$$

We need to show that f is continuous. Notice that the set

$$\mathcal{S} = \{U \subseteq [0,1] \mid U = [0,a) \text{ or } U = (a,1] \text{ for some } a \in [0,1]\}$$

is a subbasis of the topology on $[0,1]$, so it will suffice to show that for any $a \in [0,1]$ the sets $f^{-1}([0,a))$ and $f^{-1}((a,1])$ are open in X.

We have:

$$f^{-1}([0,a)) = \bigcup_{r<a} V_r$$

so $f^{-1}([0,a))$ is an open set.

Next, we claim that

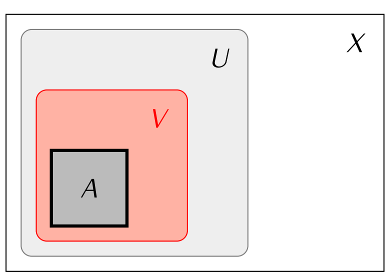
$$f^{-1}((a,1]) = \bigcup_{r>a} (X \smallsetminus \overline{V}_r)$$

Indeed, if $x \in X \smallsetminus \overline{V}_r$ for some $r > a$ then $x \notin V_r$. This gives $f(x) \geq r > a$, and so $x \in f^{-1}((a,1])$. Conversely, assume that $x \in f^{-1}((a,1])$. Then $f(x) > a$ so there exist $r > a$ such that $x \notin V_r$ Take $r' \in [0,1] \cap \mathbb{Q}$ such that $a < r' < r$. Since $\overline{V}_{r'} \subseteq V_r$ we get that $x \notin \overline{V}_{r'}$, or equivalently $x \in X \smallsetminus \overline{V}_{r'}$. Therefore $x \in \bigcup_{r>a} X \smallsetminus \overline{V}_r$.

Since the sets $X \smallsetminus \overline{V}_r$ are open it follows that $f^{-1}((a,1])$ is an open set.

□

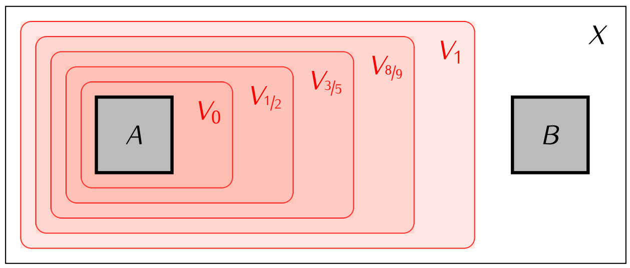
10.3 Lemma. *Let X be a normal space, let $A \subseteq X$ be a closed set and let $U \subseteq X$ be an open set such that $A \subseteq U$. There exists an open set V such that $A \subseteq V$ and $\overline{V} \subseteq U$.*



Proof. Exercise. □

Proof of Urysohn Lemma 10.1. We will show that for each $r \in [0,1] \cap \mathbb{Q}$ there exists an open set $V_r \subseteq X$ such that

1) $A \subseteq V_0$
2) $B \subseteq X \smallsetminus V_1$
3) if $r < r'$ then $\overline{V}_r \subseteq V_{r'}$.



By Lemma 10.2 this will give a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ for all $x \in V_0$ and $f(x) = 1$ for all $x \notin V_1$. By 1) we will get then that $f(x) = 0$ for all $x \in A$ and by 2) that $f(x) = 1$ for all $x \in B$.

Construction of sets V_r proceeds as follows. Since the set $[0,1] \cap \mathbb{Q}$ is countable we can arrange its elements into a sequence:

$$[0,1] \cap \mathbb{Q} = \{r_0, r_1, r_2, \dots\}$$

We can assume that $r_0 = 0$ and $r_1 = 1$. We will construct the sets V_{r_k} by induction with respect to k.

Take $V_{r_1} = X \smallsetminus B$. Since V_{r_1} is open and $A \subseteq V_{r_1}$ by Lemma 10.3 there exists an open set V such that $A \subseteq V$ and $\overline{V} \subseteq V_{r_1}$. Define $V_{r_0} = V$.

Next, assume that we have already constructed sets $V_{r_0}, \dots, V_{r_n}$. We obtain the set $V_{r_{n+1}}$ as follows. Let r_p be the biggest number in the set $\{r_0, \dots, r_n\}$ satisfying $r_p < r_{n+1}$, and let r_q be the smallest number in $\{r_0, \dots, r_n\}$ satisfying $r_{n+1} < r_q$. Since $r_p < r_q$ we have $\overline{V}_{r_p} \subseteq V_{r_q}$. By Lemma 10.3 there exists an open set V such that $\overline{V}_{r_p} \subseteq V$ and $\overline{V} \subseteq V_{r_q}$. We set $V_{r_{n+1}} := V$.

□

One can ask whether an analog of Urysohn Lemma holds for regular spaces: given a regular space X, a point $x \in X$, and a closed set $A \subseteq X$ such that $x \notin A$ is there a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(A) \subseteq \{1\}$? It turns out that this is not true, but it provides motivation for one more separation axiom:

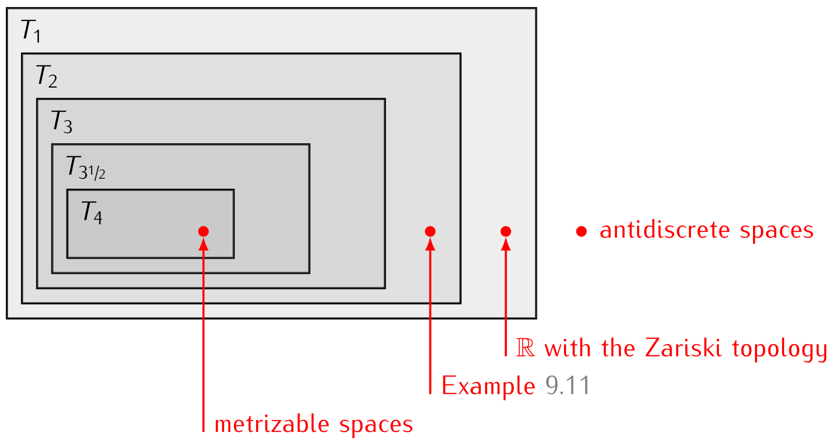
10.4 Definition. A topological space X *satisfies the axiom* $T_{3\frac{1}{2}}$ if X satisfies T_1 and if for each point $x \in X$ and each closed set $A \subseteq X$ such that $x \notin A$ there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f|_A = 0$.

A space that satisfies the axiom $T_{3\frac{1}{2}}$ is called a *completely regular space* or a *Tychonoff space*.

While Definition 10.4 may seem a bit artificial at the moment, there is a different context which makes the class of completely regular spaces interesting. We will get back to this in Chapter 18.

10.5 Note. By Urysohn Lemma every normal space is completely regular. Also, if X is a completely regular space then X is regular. Indeed, for a point $x \in X$ and a closed set $A \subseteq X$ such that $x \notin A$ let $f\colon X \to [0,1]$ be a function as in Definition 10.4. Let $U = f^{-1}([0, \frac{1}{2}))$ and let $V = f^{-1}((\frac{1}{2}, 1])$. Then the sets U, V are open in X, $A \subseteq U$, $x \in V$, and $U \cap V = \varnothing$.

The diagram in Note 9.20 can be now extended as follows:



No area of this diagram is empty: there exist regular spaces that are not completely regular and there exist completely regular spaces that are not normal.

Exercises to Chapter 10

E10.1 Exercise. Let $\mathbb{R}_{Ar}$ denote the set of real numbers with the arrow topology (4.8). Show that this space is normal.

E10.2 Exercise. Prove Lemma 10.3.

E10.3 Exercise. By Corollary 9.19 metric spaces satisfy a stronger version of the Urysohn Lemma 10.2: for any pair of disjoint, closed subsets A, B in a metric space X there exists a continuous function $f\colon X \to [0,1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$. One can ask if the same is true for all normal spaces. The goal of this exercise is to resolve this question.

a) A set $A \subseteq X$ is called a G_δ*-set* if there exists a countable family of open sets $U_1, U_2, \dots$ such that $A = \bigcap_{n=1}^{\infty} U_n$. Let X be a topological space and let $A, B \subseteq X$ be disjoint, closed subsets such that there exists a continuous function $f\colon X \to [0,1]$ with $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$. Show that both A and B are G_δ-sets.

Note: One can also show that the converse holds: if X is a normal space and A, B are closed, disjoint G_δ-sets in X then such function f exists (see Exercise 11.4).

b) Let X be be a topological space defined as follows. As a set $X = \mathbb{R} \cup \{\infty\}$ where ∞ is an extra point. Any set $U \subseteq X$ such that $\infty \notin U$ is open in X. If $\infty \in U$ then U is open if $X \smallsetminus U$ is a finite set. Show that X is a normal space, but that not every closed set in X is a G_δ-set. Thus the stronger version of Urysohn Lemma does not hold in X.

Notice that as a consequence the space X described in part b) gives another example of a space which is normal but not metrizable.