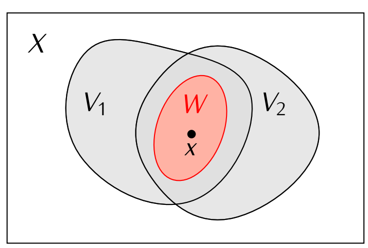
4 | Basis, Subbasis, Subspace

Our main goal in this chapter is to develop some tools that make it easier to construct examples of topological spaces. By Definition 3.12 in order to define a topology on a set X we need to specify which subsets of X are open sets. It can difficult to describe all open sets explicitly, so topological spaces are often defined by giving either a *basis* or a *subbasis* of a topology. Interesting topological spaces can be also obtained by considering *subspaces* of topological spaces. We explain these notions below.

4.1 Definition. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. The collection $\mathcal{B}$ is a *basis* on X if it satisfies the following conditions:

1) $X = \bigcup_{V\in\mathcal{B}} V$;
2) for any $V_1, V_2 \in \mathcal{B}$ and $x \in V_1 \cap V_2$ there exists $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq V_1 \cap V_2$.



4.2 Example. If (X, ϱ) is a metric space then the set $\mathcal{B} = \{B(x, r) \mid x \in X,\ r > 0\}$ consisting of all open balls in X is a basis on X (exercise).

4.3 Proposition. *Let X be a set, and let $\mathcal{B}$ be a basis on X. Let $\mathcal{T}$ denote the collection of all subsets $U \subseteq X$ that can be obtained as the union of some elements of $\mathcal{B}$: $U = \bigcup_{V\in\mathcal{B}_1} V$ for some $\mathcal{B}_1 \subseteq \mathcal{B}$. Then $\mathcal{T}$ is a topology on X.*

Proof. Exercise. □

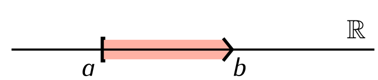
4.4 Definition. Let $\mathcal{B}$ be a basis on a set X and let $\mathcal{T}$ be the topology defined as in Proposition 4.3. In such case we will say that $\mathcal{B}$ is a *basis of the topology* $\mathcal{T}$ and that $\mathcal{T}$ is the *topology defined by the basis* $\mathcal{B}$.

4.5 Example. Let (X, ϱ) be a metric space, let $\mathcal{T}$ be the topology on X induced by ϱ, and let $\mathcal{B}$ be the collection of all open balls in X. Directly from the definition of the topology $\mathcal{T}$ (3.14) it follows that $\mathcal{B}$ is a basis of $\mathcal{T}$.

4.6 Example. Consider $\mathbb{R}^n$ with the Euclidean metric d. Let $\mathcal{B}$ be the collection of all open balls $B(x, r) \subseteq \mathbb{R}^n$ such that $r \in \mathbb{Q}$ and $x = (x_1, x_2, \ldots, x_n)$ where $x_1, \ldots, x_n \in \mathbb{Q}$. Then $\mathcal{B}$ is a basis of the Euclidean topology on $\mathbb{R}^n$ (exercise).

4.7 Note. If a topological space X has a basis consisting of countably many sets then we say that X satisfies the *2*[nd] *countability axiom* or that X is *second countable*. Since the set of rational numbers is countable it follows that the basis of the Euclidean topology given in Example 4.6 is countable. Thus, $\mathbb{R}^n$ with the Euclidean topology is a second countable space. Second countable spaces have some interesting properties, some of which we will encounter later on.

4.8 Example. The set $\mathcal{B} = \{[a, b) \mid a, b \in \mathbb{R}\}$ is a basis of a certain topology on $\mathbb{R}$. We will call it the *arrow topology*.



4.9 Example. Let $\mathcal{B} = \{[a, b] \mid a, b \in \mathbb{R}\}$. The set $\mathcal{B}$ is a basis of the discrete topology on $\mathbb{R}$ (exercise).

4.10 Example. Let $X = \{a, b, c, d\}$ and let $\mathcal{B} = \{\{a, b, c\}, \{b, c, d\}\}$. The set $\mathcal{B}$ is not a basis of any topology on X since $b \in \{a, b, c\} \cap \{b, c, d\}$, and $\mathcal{B}$ does not contain any subset W such that $b \in W$ and $W \subseteq \{a, b, c\} \cap \{b, c, d\}$.

4.11 Proposition. *Let X be a set and let $\mathcal{S}$ be any collection of subsets of X such that $X = \bigcup_{V \in \mathcal{S}} V$. Let $\mathcal{T}$ denote the collection of all subsets of X that can be obtained using two operations:*

1) *taking finite intersections of sets in $\mathcal{S}$;*
2) *taking arbitrary unions of sets obtained in 1).*

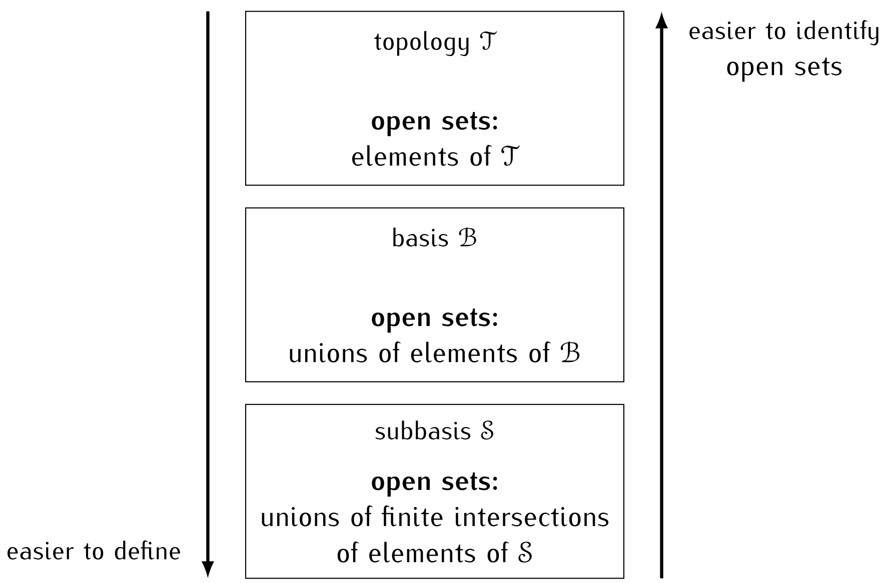
Then $\mathcal{T}$ is a topology on X.

Proof. Exercise. □

4.12 Definition. Let X be a set and let $\mathcal{S}$ be any collection of subsets of X such that $X = \bigcup_{V \in \mathcal{S}} V$. The topology $\mathcal{T}$ defined by Proposition 4.11 is called the *topology generated by* $\mathcal{S}$, and the collection $\mathcal{S}$ is called a *subbasis* of $\mathcal{T}$.

4.13 Example. If $X = \{a, b, c, d\}$ and $\mathcal{S} = \{\{a, b, c\}, \{b, c, d\}\}$ then the topology generated by $\mathcal{S}$ is $\mathcal{T} = \{\{a, b, c\}, \{b, c, d\}, \{b.c\}, \{a, b, c, d\}, \varnothing\}$.

The notions of a basis and a subbasis provide shortcuts for defining topologies: it is easier to specify a basis of a topology than to define explicitly the whole topology (i.e. to describe all open sets). Specifying a subbasis is even easier. The price we pay for this convenience is that it is more difficult to identify which sets are open if we know only a basis or a subbasis of a topology:



The next proposition often simplifies checking if a function between topological spaces is continuous:

4.14 Proposition. *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}$ be a basis (or a subbasis) of $\mathcal{T}_Y$. A function $f\colon X \to Y$ is continuous if and only if $f^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{B}$.*

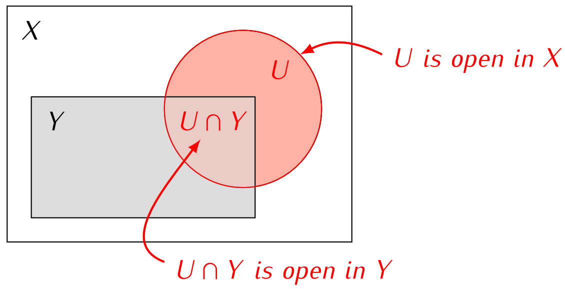
Proof. Exercise. □

A useful way of obtaining new examples of topological spaces is by considering subspaces of existing spaces:

4.15 Definition. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. The collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

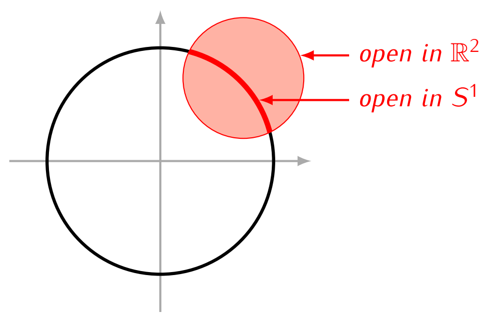
is a topology on Y called the *subspace topology*. We say that $(Y, \mathcal{T}_Y)$ is a *subspace* of the topological space $(X, \mathcal{T})$.



4.16 Example. The unit circle S^1 is defined by

$$S^1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

The circle S^1 is a topological space considered as a subspace of $\mathbb{R}^2$.



In general the n-dimensional sphere S^n is defined by

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

It is a topological space considered as a subspace of $\mathbb{R}^{n+1}$.

4.17 Example. Consider $\mathbb{Z}$ as a subspace of $\mathbb{R}$. The subspace topology on $\mathbb{Z}$ is the same as the discrete topology (exercise).

4.18 Proposition. *Let X be a topological space and let Y be its subspace.*

1) The inclusion map $j: Y \to X$ is a continuous function.
2) If Z is a topological space then a function $f: Z \to Y$ is continuous if and only if the composition $jf: Z \to X$ is continuous.

Proof. Exercise. □

4.19 Proposition. *Let X be a topological space and let Y be its subspace. If $\mathcal{B}$ is a basis (or a subbasis) of X then the set $\mathcal{B}_Y = \{U \cap Y \mid U \in \mathcal{B}\}$ is a basis (resp. a subbasis) of Y.*

Proof. Exercise. □

Exercises to Chapter 4

E4.1 Exercise. Prove Proposition 4.3

E4.2 Exercise. Verify the statement of Example 4.6.

E4.3 Exercise. Verify the statement of Example 4.9.

E4.4 Exercise. Prove Proposition 4.11.

E4.5 Exercise. Prove Proposition 4.14.

E4.6 Exercise. Consider the interval $[0, 1]$ as a subspace of $\mathbb{R}$. Determine which of the following sets are open in $[0, 1]$. Justify your answers.

a) $\left(\frac{1}{2}, 1\right)$ b) $\left(\frac{1}{2}, 1\right]$ c) $\left(\frac{1}{3}, \frac{2}{3}\right)$ d) $\left(\frac{1}{3}, \frac{2}{3}\right]$

E4.7 Exercise. Verify the statement of Example 4.17.

E4.8 Exercise. Prove Proposition 4.18.

E4.9 Exercise. The goal of this exercise if to show that the subspace topology is uniquely determined by the properties listed in Proposition 4.18. Let X be a topological space, let $Y \subseteq X$ and let $j\colon Y \to X$ be the inclusion map. Let $\mathcal{T}$ be a topology on Y, and let $Y_\mathcal{T}$ denote Y considered as a topological space with respect to the topology $\mathcal{T}$. Assume that $Y_\mathcal{T}$ satisfies the following conditions:

1) The map $j\colon Y_\mathcal{T} \to X$ is a continuous function.
2) If Z is a topological space then a function $f\colon Z \to Y_\mathcal{T}$ is continuous if and only if the composition $jf\colon Z \to X$ is continuous.

Show that $\mathcal{T}$ is the subspace topology on Y. That is, show that $U \in \mathcal{T}$ if and only if $U = Y \cap U'$ where U' is some open set in X.

E4.10 Exercise. Recall that a topological space X is second countable if the topology on X has a countable basis. Show that the discrete topology on a set X is second countable if and only if X is a countable set.

E4.11 Exercise. Show that $\mathbb{R}$ with the arrow topology is not second countable. (Hint: Assume by contradiction that $\mathcal{B} = \{V_1, V_2, \dots\}$ is a countable basis of the arrow topology. Let $\alpha_i = \inf V_i$. Take $\alpha_0 \in \mathbb{R} \smallsetminus \{\alpha_1, \alpha_2, \dots\}$. Show that the set $[\alpha_0, \alpha_0 + 1)$ is not a union of sets from $\mathcal{B}$).

E4.12 Exercise. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same set X. We say that the topology $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ (e.i. if every open set in $\mathcal{T}_1$ is also open in $\mathcal{T}_2$). Let $\mathcal{T}_{Ar}$ be the arrow topology on $\mathbb{R}$ and let $\mathcal{T}_{Eu}$ be the Euclidean topology on $\mathbb{R}$. Show that $\mathcal{T}_{Ar}$ is finer than $\mathcal{T}_{Eu}$.