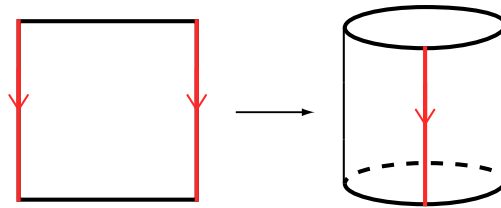


19 | Quotient Spaces

So far we have encountered two methods of constructing new topological spaces from old ones:

- given a space X we can obtain new spaces by taking subspaces of X ;
- given two (or more) spaces X_1, X_2 we can obtain a new space by taking their product $X_1 \times X_2$.

Here we will consider another, very useful construction of a *quotient space* of a given topological space. This construction will let us produce, in particular, interesting examples of manifolds. Intuitively, a quotient space of a space X is a space Y which is obtained by identifying some points of X . For example, if we take the square $X = [0, 1] \times [0, 1]$ and identify each point $(0, t)$ with the point $(1, t)$ for $t \in [0, 1]$ we obtain a space Y that looks like a cylinder:



In order to make this precise we need to specify the following:

- 1) what are the points of Y ;
- 2) what is the topology on Y .

The first part is done by considering Y as the set of *equivalence classes* of some *equivalence relation* on X . The second part is done by defining the *quotient topology*. We explain these notions below.

19.1 Definition. Let X be a set. An *equivalence relation* on X is a binary relation \sim satisfying three properties:

- 1) $x \sim x$ for all $x \in X$ (reflexivity)
- 2) if $x \sim y$ then $y \sim x$ (symmetry)
- 3) if $x \sim y$ and $y \sim z$ then $x \sim z$ (transitivity)

19.2 Example. Let $X = [0, 1] \times [0, 1]$. Define a relation on X as follows. For any $(s, t) \in X$ we set $(s, t) \sim (s, t)$. Also, for any $t \in [0, 1]$ we set $(0, t) \sim (1, t)$ and $(1, t) \sim (0, t)$. This relation is an equivalence relation that identifies corresponding points of the vertical edges of the square $[0, 1] \times [0, 1]$.

19.3 Example. Define a relation \sim on the set of real numbers \mathbb{R} as follows: $r \sim s$ if $s = r + n$ for some $n \in \mathbb{Z}$. One can check that this is an equivalence relation (exercise).

19.4 Definition. Let X be a set with an equivalence relation \sim and let $x \in X$. The *equivalence class* of x is the subset $[x] \subseteq X$ consisting of all elements that are in the relation with x :

$$[x] = \{y \in X \mid x \sim y\}$$

19.5 Example. Take $X = [0, 1] \times [0, 1]$ with the equivalence relation defined as in Example 19.2. If $(s, t) \in X$ and $s \neq 0, 1$ then $[(s, t)]$ consists of a single point: $[(s, t)] = \{(s, t)\}$. If $s = 0, 1$ then $[(s, 0)]$ consists of two points: $[(0, t)] = [(1, t)] = \{(0, t), (1, t)\}$.

19.6 Example. Take \mathbb{R} with the equivalence relation defined as in Example 19.3. For $r \in \mathbb{R}$ we have:

$$[r] = \{r + n \mid n \in \mathbb{Z}\}$$

For example: $[1] = \{1 + n \mid n \in \mathbb{Z}\} = \mathbb{Z}$. Notice that $[1] = [2]$ and $[\sqrt{2}] = [\sqrt{2} + 1]$.

19.7 Proposition. Let X be a set with an equivalence relation \sim , and let $x, y \in X$.

- 1) If $x \sim y$ then $[x] = [y]$.
- 2) If $x \not\sim y$ then $[x] \cap [y] = \emptyset$.

Proof. 1) Assume that $x \sim y$ and that $z \in [x]$. This gives $z \sim x$ and by transitivity $z \sim y$. Therefore $z \in [y]$. This shows that $[x] \subseteq [y]$. In the same way we can show that $[y] \subseteq [x]$. Therefore we get $[x] = [y]$.

2) Assume that $[x] \cap [y] \neq \emptyset$, and let $z \in [x] \cap [y]$. Then $x \sim z$ and $y \sim z$, so by transitivity $x \sim y$ which contradicts our assumption. \square

19.8 Note. Proposition 19.7 shows that an equivalence relation \sim on a set X splits X into a disjoint union of distinct equivalence classes of \sim . The opposite is also true. Namely, assume that we have a family $\{A_i\}_{i \in I}$ of subsets of X such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i \in I} A_i = X$. We can define a relation \sim on X such that $x \sim y$ if and only if both x and y are elements of the same subset A_i . This relation is an equivalence relation and its equivalence classes are the sets A_i .

19.9 Definition. Let X be a set with an equivalence relation \sim . The *quotient set* of X is the set X/\sim whose elements are all distinct equivalence classes of \sim . The function

$$\pi: X \rightarrow X/\sim$$

given by $\pi(x) = [x]$ is called the *quotient map*.

19.10 Note. Let X be a set with an equivalence relation \sim , and let $f: X \rightarrow Y$ be a function. Assume that for each $x, x' \in X$ such that $x \sim x'$ we have $f(x) = f(x')$. Then we can define a function $\bar{f}: X/\sim \rightarrow Y$ by $\bar{f}([x]) = f(x)$. We have $f = \bar{f}\pi$, i.e. the following diagram commutes:

$$\begin{array}{ccc} & X/\sim & \\ \pi \nearrow & \downarrow \bar{f} & \\ X & \xrightarrow{f} & Y \end{array}$$

19.11 Definition. Let X be a topological space and let \sim be an equivalence relation on X . The *quotient topology* on the set X/\sim is the topology where a set $U \subseteq X/\sim$ is open if the set $\pi^{-1}(U)$ is open in X . The set X/\sim with this topology is called the *quotient space* of X taken with respect to the relation \sim .

19.12 Proposition. Let X be a topological space and let \sim be an equivalence relation on X . A set $A \subseteq X/\sim$ is closed if and only if the set $\pi^{-1}(A)$ is closed in X .

Proof. Exercise. □

19.13 Proposition. Let X, Y be topological spaces and let \sim be an equivalence relation on X . A function $f: X/\sim \rightarrow Y$ is continuous if and only if the function $f\pi: X \rightarrow Y$ is continuous.

Proof. Exercise. □

19.14 Note. Let X be a space with an equivalence relation \sim and let $f: X \rightarrow Y$ be a continuous function. If for each $x, x' \in X$ such that $x \sim x'$ we have $f(x) = f(x')$ then as in (19.10) we obtain a function $\bar{f}: X/\sim \rightarrow Y$, $\bar{f}([x]) = f(x)$. Since the function $\bar{f}\pi = f$ is continuous thus by Proposition 19.13 \bar{f} is a continuous function.

19.15 Example. Take the closed interval $[-1, 1]$ with the equivalence relation \sim such that $(-1) \sim 1$ (and $t \sim t$ for all $t \in [-1, 1]$). We will show that the quotient space $[-1, 1]/\sim$ is homeomorphic to the circle S^1 . Consider the function $f: [-1, 1] \rightarrow S^1$ given by $f(x) = (\sin \pi x, -\cos \pi x)$:

Since $f(1) = f(-1)$ by (19.14) we get the induced continuous function $\bar{f}: [-1, 1]/\sim \rightarrow S^1$. We will prove that \bar{f} is a homeomorphism. First, notice that \bar{f} is a bijection. Next, since $[-1, 1]$ is a compact space and the quotient map $\pi: [-1, 1] \rightarrow [-1, 1]/\sim$ is onto by Proposition 14.8 we obtain that the space $[-1, 1]/\sim$ is compact. Therefore we can use Proposition 14.18 which says that any continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

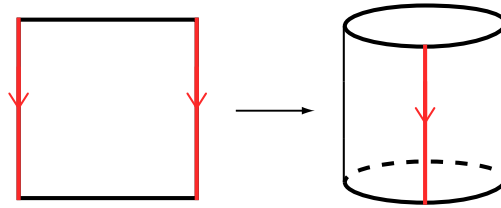
This example can be generalized as follows. Take the closed unit ball

$$\bar{B}^n = \{x \in \mathbb{R}^n \mid d(0, x) \leq 1\}$$

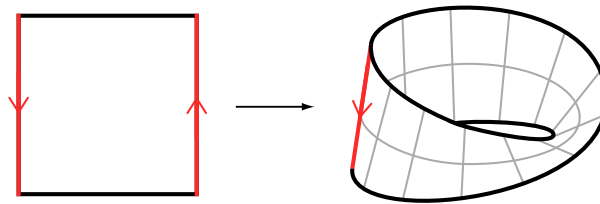
The unit sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid d(0, x) = 1\}$ is a subspace of \bar{B}^n . Consider the equivalence relation \sim on \bar{B}^n that identifies all points of S^{n-1} : $x \sim x'$ for all $x, x' \in S^{n-1}$. Using similar arguments as above one can show that \bar{B}^n/\sim is homeomorphic to the sphere S^n (exercise). Notice that for $n = 1$ we have $\bar{B}^1 = [-1, 1]$ and $S^0 = \{-1, 1\}$ so in this case we recover the homeomorphism $[-1, 1]/\sim \cong S^1$.

19.16 Note. Let X be a space and let $A \subseteq X$. Consider the equivalence relation on X that identifies all points of A : $x \sim x'$ for all $x, x' \in A$. The quotient space X/\sim is usually denoted by X/A . Using this notation the homeomorphism given in Example 19.15 can be written as $\bar{B}^n/S^{n-1} \cong S^n$.

19.17 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation defined as in Example 19.2: $(0, t) \sim (1, t)$ for all $t \in [0, 1]$. Using arguments similar as in Example 19.15 we can show that the quotient space is homeomorphic to the cylinder $S^1 \times [0, 1]$:

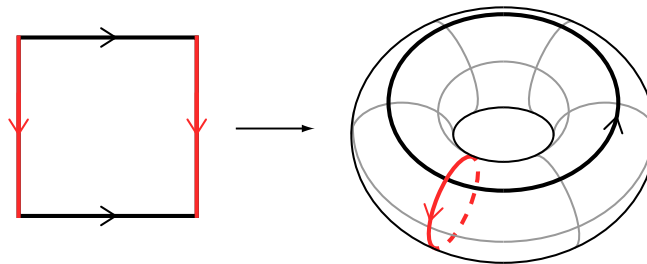


19.18 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, 1 - t)$ for all $t \in [0, 1]$. The space obtained as a quotient space is called the *Möbius band*:

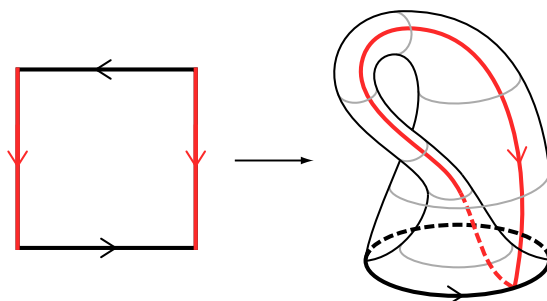


The Möbius band is a 2-dimensional manifold with boundary, and its boundary is homeomorphic to S^1 .

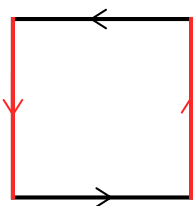
19.19 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, t)$ for all $t \in [0, 1]$ and $(s, 0) \sim (s, 1)$ for all $s \in [0, 1]$. Using arguments similar to these given in Example 19.15 one can show that the quotient space in this case is homeomorphic to the torus:



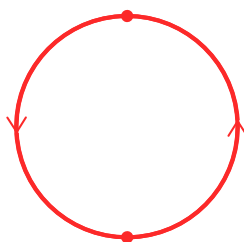
19.20 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, t)$ for all $t \in [0, 1]$ and $(s, 0) \sim (1 - s, 1)$ for all $s \in [0, 1]$. The resulting quotient space is called the *Klein bottle*. One can show that the Klein bottle is a two dimensional manifold.



19.21 Example. Following the scheme of the last two examples we can consider the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, 1 - t)$ and $(s, 0) \sim (1 - s, 1)$ for all $s, t \in [0, 1]$:



The resulting quotient space is homeomorphic to the space \mathbb{RP}^2 which is defined as follows. Take the the 2-dimensional closed unit ball \bar{B}^2 . The boundary of \bar{B}^2 is the circle S^1 . Consider the equivalence relation \sim on \bar{B}^2 that identifies each point $(x_1, x_2) \in S^1$ with its antipodal point $(-x_1, -x_2)$:



We define $\mathbb{RP}^2 = \bar{B}^2 / \sim$. This space is called the *2-dimensional real projective space* and it is a 2-dimensional manifold. One can show that \mathbb{RP}^2 (and also the Klein bottle) cannot be embedded into \mathbb{R}^3 . For this reason it is harder to visualize it.

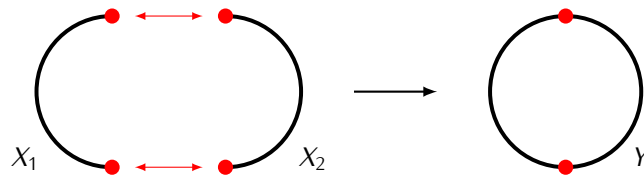
19.22 Example. The construction of \mathbb{RP}^2 given in Example 19.21 can be generalized to higher dimensions. Consider the n -dimensional closed unit ball \bar{B}^n . The boundary \bar{B}^n is the sphere S^{n-1} . Similarly as before we can consider the equivalence relation \sim on \bar{B}^n that identifies antipodal points

of S^{n-1} :

$$(x_1, \dots, x_n) \sim (-x_1, \dots, -x_n)$$

for all $(x_1, \dots, x_n) \in S^{n-1}$. The quotient space \bar{B}^n / \sim is denoted by \mathbb{RP}^n and is called the *n-dimensional real projective space*. The space \mathbb{RP}^n is an *n*-dimensional manifold. For another perspective on projective spaces see Exercise 19.8.

Many constructions in topology involve the following setup. We start with two topological spaces X_1 , X_2 , and we build a new space Y by identifying certain points of X_1 with certain points of X_2 :

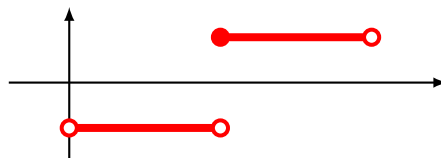


An example of a setting that uses such assembly process is described in Chapter 20.

The first step in constructions of this kind is to create a new space $X_1 \sqcup X_2$ which contains X_1 and X_2 as its subspaces. The space Y can be then described as a quotient space of $X_1 \sqcup X_2$. The space $X_1 \sqcup X_2$ is defined as follows. If $X_1 \cap X_2 = \emptyset$ then $X_1 \sqcup X_2 = X_1 \cup X_2$ as a set. A set $U \subseteq X_1 \sqcup X_2$ is open if and only if $U \cap X_i$ is open in X_i for $i = 1, 2$. If $X_1 \cap X_2 \neq \emptyset$ then we first replace X_i with a homeomorphic space X'_i such that $X'_1 \cap X'_2 = \emptyset$ (e.g. we can take $X'_i = \{i\} \times X_i$) and then we set $X_1 \sqcup X_2$ to be equal to $X'_1 \sqcup X'_2$.

19.23 Definition. The space $X_1 \sqcup X_2$ is called the *disjoint union* (or the *coproduct*) of spaces X_1 and X_2 .

19.24 Example. Take $X_1 = (0, 1)$ and $X_2 = [1, 2)$. Since $X_1 \cap X_2 = \emptyset$ we can construct the space $(0, 1) \sqcup [1, 2)$ so that it consists of the points of the interval $(0, 2)$. However, the disjoint union $(0, 1) \sqcup [1, 2)$ is not homeomorphic to the interval $(0, 2)$ taken with the usual topology. For example, the set $U = [1, \frac{1}{2})$ is not open in the interval $(0, 2)$, but it is open in $(0, 1) \sqcup [1, 2)$ since $U \cap (0, 1) = \emptyset$ is open in $(0, 1)$ and $U \cap [1, 2) = [1, \frac{1}{2})$ is open in $[1, 2)$. In general, in the disjoint union $X_1 \sqcup X_2$ the spaces X_1 and X_2 can be imagined as being far apart from each other so that an arbitrary combination of an open set in X_1 and an open set in X_2 gives an open set in $X_1 \sqcup X_2$. For example, the space $(0, 1) \sqcup [1, 2)$ is homeomorphic to the subspace of \mathbb{R}^2 given by $(0, 1) \times \{-a\} \cup [1, 2) \times \{a\}$ for some $a > 0$.



The construction of a disjoint union can be extended to arbitrary families of topological spaces. Given a family $\{X_i\}_{i \in I}$ such that $X_i \cap X_j = \emptyset$ for all $i \neq j$, we define $\bigsqcup_{i \in I} X_i = \bigcup_{i \in I} X_i$ as a set. A set $U \subseteq \bigsqcup_{i \in I} X_i$ is open if and only if the set $U \cap X_i$ is open in X_i for each $i \in I$. If the family $\{X_i\}_{i \in I}$ does not consist of disjoint spaces, then we first replace it with a family $\{X'_i\}_{i \in I}$ such that $X'_i \cong X_i$ for each $i \in I$, and $X'_i \cap X'_j = \emptyset$ for all $i \neq j$.

If $\bigsqcup_{i \in I} X_i$ is the disjoint union of a family $\{X_i\}_{i \in I}$, then for each $j \in I$ we have an embedding $k_j: X_j \rightarrow \bigsqcup_{i \in I} X_i$. The following fact is an essential property of the space $\bigsqcup_{i \in I} X_i$:

19.25 Proposition. *For any family of continuous functions $\{f_i: X_i \rightarrow Y\}_{i \in I}$, there exists a unique continuous function $f: \bigsqcup_{i \in I} X_i \rightarrow Y$ such that $k_j f = f_j$ for each $j \in I$.*

Proof. Exercise. □

19.26 Note. The function $f: \bigsqcup_{i \in I} X_i \rightarrow Y$ in Proposition 19.25 is usually denoted by $\bigsqcup_{i \in I} f_i$.

Exercises to Chapter 19

E19.1 Exercise. Prove Proposition 19.12.

E19.2 Exercise. Prove Proposition 19.13.

E19.3 Exercise. Consider the real line \mathbb{R} with the equivalence relation defined as in Example 19.3. Show that the quotient space \mathbb{R}/\sim is homeomorphic with S^1 .

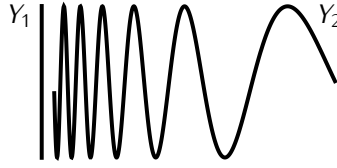
E19.4 Exercise. Take the closed interval $[0, 1]$ with the equivalence relation \sim defined as in Example 19.15. Let $\pi: [0, 1] \rightarrow [0, 1]/\sim$ be the quotient map. The set $U = [0, \frac{1}{2})$ which is open subset of $[0, 1]$. Show that $\pi(U)$ is not open in $[0, 1]/\sim$.

E19.5 Exercise. Let $\bar{B}^n \subseteq \mathbb{R}^n$ be the closed unit ball (see Example 19.15). Show that \bar{B}^n/S^{n-1} is homeomorphic to S^n .

E19.6 Exercise. Let X be a compact Hausdorff space, and let $U \subseteq X$ be an open set. Show that the one-point compactification U^+ of U (18.14) is homeomorphic to the quotient space $X/(X \setminus U)$.

E19.7 Exercise. Recall that the topologists sine curve Y is the subspace of \mathbb{R}^2 consisting of the

vertical line segment $Y_1 = \{(0, y) \mid -1 \leq y \leq 1\}$ and the curve $Y_2 = \{(x, \sin(\frac{1}{x})) \mid x > 0\}$:



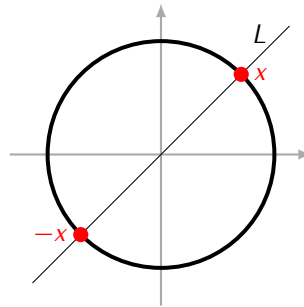
Show that the space Y/Y_1 is homeomorphic to the half line $[0, +\infty)$.

E19.8 Exercise. Consider the unit sphere S^n with the equivalence relation that identifies antipodal points of S^n :

$$(x_1, \dots, x_{n+1}) \sim (-x_1, \dots, -x_{n+1})$$

for all (x_1, \dots, x_{n+1}) . Show that the quotient space S^n/\sim is homeomorphic to the projective space \mathbb{RP}^n (19.22).

Note: This construction lets us interpret \mathbb{RP}^n as the space of straight lines in \mathbb{R}^{n+1} that pass through the origin. Indeed, any such line L intersects the sphere S^n at two points: some point x and its antipodal point $-x$:



Since \mathbb{RP}^n is obtained by identifying antipodal points we get a bijective correspondence between elements of \mathbb{RP}^n and lines in \mathbb{R}^{n+1} passing through the origin.

E19.9 Exercise. A *pointed topological space* is a pair (X, x_0) where X is a topological space and $x_0 \in X$. The *smash product* of pointed spaces (X, x_0) and (Y, y_0) is the quotient space

$$X \wedge Y = (X \times Y)/A$$

where $A = (X \times \{y_0\}) \cup (\{x_0\} \times Y)$

a) Let X, Y be a locally compact spaces (18.17). Show that the space $X \times Y$ is locally compact.

b) By part a) and Corollary 17.17 if X, Y are locally compact Hausdorff spaces then the space $X \times Y$ is also locally compact and Hausdorff. By Theorem 18.19 we have in such case one-point compactifications X^+, Y^+ , and $(X \times Y)^+$ of the spaces X, Y , and $X \times Y$ respectively. Recall that

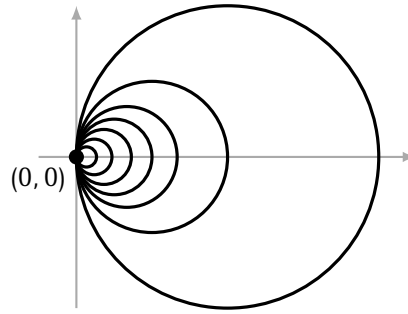
$X^+ = X \cup \{\infty\}$ and $Y^+ = Y \cup \{\infty\}$. Consider (X^+, ∞) and (Y^+, ∞) as pointed spaces. Show that there is a homeomorphism:

$$X^+ \wedge Y^+ \cong (X \times Y)^+$$

E19.10 Exercise. Prove Proposition 19.25.

E19.11 Exercise. Let $\{X_i\}_{i \in I}$ be a family of topological spaces, let Z be a topological space and for each $i \in I$ let $g_i: X_i \rightarrow Z$ let be a continuous function. Assume that for each family of continuous function functions $\{f_i: X_i \rightarrow Z\}_{i \in I}$ there exists a unique function $f: Z \rightarrow Y$ such that $g_i f = f_i$ for each $i \in I$. Show that the space Z is homeomorphic to $\bigsqcup_{i \in I} X_i$.

E19.12 Exercise. The *Hawaiian earring* space is a subspace $X \subseteq \mathbb{R}^2$ given by $X = \bigcup_{n=1}^{\infty} C_n$ where C_n is the circle with radius $\frac{1}{n}$ and center at the point $(0, \frac{1}{n})$:



Notice that the point $(0, 0)$ is the intersection of all circles C_n .

For $n = 1, 2, \dots$ let C_n be the circle defined as above, and let Y be the quotient space of the disjoint union $\bigsqcup_{i=1}^{\infty} C_n$ obtained by identifying points $(0, 0) \in C_n$ for all n . Show that Y is not homeomorphic to X .

E19.13 Exercise. Let $\mathbb{R}_+^n, \mathbb{R}_-^n, \mathbb{R}_0^n$ be subspaces of \mathbb{R}^n given by

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

$$\mathbb{R}_-^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \leq 0\}$$

$$\mathbb{R}_0^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$$

Notice that \mathbb{R}_0^n is contained in both \mathbb{R}_+^n and \mathbb{R}_-^n . Given a homeomorphism $h: \mathbb{R}_0^n \rightarrow \mathbb{R}_0^n$ let $\mathbb{R}_+^n \cup_h \mathbb{R}_-^n$ denote the quotient space $(\mathbb{R}_+^n \sqcup \mathbb{R}_-^n) / \sim$ where \sim is the equivalence relation which identifies each point $(x_1, \dots, x_{n-1}, 0) \in \mathbb{R}_+^n$ with $h(x_1, \dots, x_{n-1}, 0) \in \mathbb{R}_-^n$. Show that $\mathbb{R}_+^n \cup_h \mathbb{R}_-^n$ is homeomorphic to \mathbb{R}^n .

