16 | Compact Metric Spaces

We have seen previously that many questions related to metric spaces (e.g. whether a subset of a metric space is closed or whether a function between metric spaces is continuous) can be resolved by looking at convergence of sequences. Our main goal in this chapter the proof of Theorem 16.2 which says that also compactness of metric spaces can be characterized in terms convergence of sequences.

16.1 Definition. A topological space X is *sequentially compact* if every sequence $\{x_n\} \subseteq X$ contains a convergent subsequence.

16.2 Theorem. *A metric space (X, ϱ) is compact if and only if it is sequentially compact.*

16.3 Note. The statement of Theorem 16.2 is not true for general topological spaces: there exist spaces that are compact but not sequentially compact, and there exist spaces that are sequentially compact but not compact.

16.4 Lemma. *Let (X, ϱ) be a metric space. If a sequence $\{x_n\} \subseteq X$ does not contain any convergent subsequence then $\{x_n\}$ is a closed set in X.*

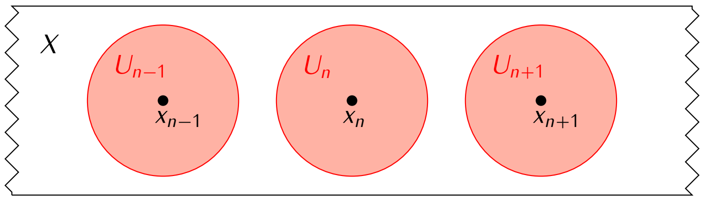
Proof. Exercise. $\square$

16.5 Lemma. *Let (X, ϱ) be a metric space. If a sequence $\{x_n\} \subseteq X$ does not contain any convergent subsequence then for each $k = 1, 2, \ldots$ there exists $\varepsilon_k > 0$ such that $B(x_k, \varepsilon_k) \cap \{x_n\} = x_k$.*

Proof. Exercise. $\square$

Proof of Theorem 16.2 (⇒). Assume that (X, ϱ) is a metric space and that $\{x_n\} \subseteq X$ is a sequence without a convergent subsequence. By Lemma 16.4 the set $U_0 = X \smallsetminus \{x_n\}$ is open. For $k = 1, 2, \ldots$

denote $U_k := B(x_k, \varepsilon_k)$ where $B(x_k, \varepsilon_k)$ is the open ball given by Lemma 16.5. The the family of sets $\{U_0, U_1, U_2, \dots\}$ is an open cover of X that has no finite subcover. Therefore X is not compact.



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16.6 Definition. Let (X, ϱ) be a metric space, and let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of X. A *Lebesgue number* for $\mathfrak{U}$ is a number $\lambda_{\mathfrak{U}} > 0$ such that for every $x \in X$ we have $B(x, \lambda_{\mathfrak{U}}) \subseteq U_i$ for some $U_i \in \mathfrak{U}$.

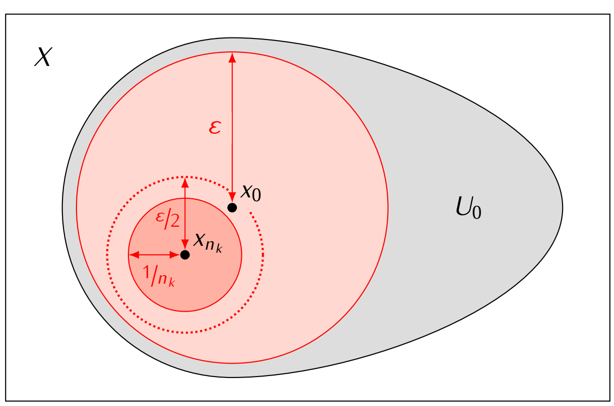
16.7 Note. For a general metric space (X, ϱ) and an open cover $\mathfrak{U}$ of X a Lebesgue number for $\mathfrak{U}$ may not exist (exercise).

16.8 Lemma. *If (X, ϱ) is a sequentially compact metric space then for any open cover $\mathfrak{U}$ of X there exists a Lebesgue number for $\mathfrak{U}$.*

Proof. We argue by contradiction. Assume that $\mathfrak{U}$ is an open cover of X without a Lebesgue number. This implies that for any $n \geq 1$ there is $x_n \in X$ such that $B(x_n, \frac{1}{n})$ is not contained in any element of $\mathfrak{U}$. Since X is sequentially compact the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$. Let $x_{n_k} \to x_0$ and let $U_0 \in \mathfrak{U}$ be a set such that $x_0 \in U_0$. We can find $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq U_0$ and $k > 0$ such that $\frac{1}{n_k} < \frac{\varepsilon}{2}$ and $\varrho(x_0, x_{n_k}) < \frac{\varepsilon}{2}$. This gives:

$$B(x_{n_k}, \tfrac{1}{n_k}) \subseteq B(x_{n_k}, \tfrac{\varepsilon}{2}) \subseteq B(x_0, \varepsilon) \subseteq U_0$$

which is impossible by the choice of x_{n_k}.



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16.9 Definition. Let (X, ϱ) be a metric space. For $\varepsilon > 0$ an *ε-net* in X is a set of points $\{x_i\}_{i \in I} \subseteq X$ such that $X = \bigcup_{i \in I} B(x_i, \varepsilon)$.

16.10 Note. A set $\{x_i\}_{i \in I}$ is an ε-net in X if and only if for every $x \in X$ there is $i \in I$ such that $\varrho(x, x_i) < \varepsilon$.

16.11 Lemma. *Let (X, ϱ) be a sequentially compact metric space. For every $\varepsilon > 0$ there exists a finite ε-net in X.*

Proof. Assume that for some $\varepsilon > 0$ the space X does not have a finite ε-net. Choose any point $x_1 \in X$. We have $B(x_1, \varepsilon) \neq X$ (since otherwise the set $\{x_1\}$ would be an ε-net in X), so we can find $x_2 \in X$ such that $x_2 \notin B(x_1, \varepsilon)$. Next, since $\{x_1, x_2\}$ is not an ε-net there exists $x_3 \in X$ such that $x_3 \notin \bigcup_{i=1}^{2} B(x_i, \varepsilon)$. Arguing by induction we get an infinite sequence $\{x_n\} \subseteq X$ such that

$$x_n \notin \bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$$

for $n = 1, 2, \ldots$ This means that for any $n \neq m$ we have $\varrho(x_n, x_m) > \varepsilon$. As a consequence $\{x_n\}$ does not contain any convergent subsequence (exercise), and so the space X is not sequentially compact. $\square$

Proof of Theorem 16.2 ($\Leftarrow$) . Assume that the space (X, ϱ) is sequentially compact and let $\mathcal{U}$ be an open cover of X. We need to show that $\mathcal{U}$ contains a finite subcover. By Lemma 16.8 there exists a Lebesgue number $\lambda_{\mathcal{U}}$ for $\mathcal{U}$. Also, by Lemma 16.11, we can find in X a finite $\lambda_{\mathcal{U}}$-net $\{x_1, \ldots, x_n\}$. For $i = 1, \ldots, n$ let $U_i \in \mathcal{U}$ be a set such that $B(x_i, \lambda_{\mathcal{U}}) \subseteq U_i$. We have:

$$X = \bigcup_{i=1}^{n} B(x_i, \lambda_{\mathcal{U}}) \subseteq \bigcup_{i=1}^{n} U_i$$

Therefore $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$. $\square$

16.12 Corollary. *If (X, ϱ) is a compact metric space then for any open cover $\mathcal{U}$ of X there exists a Lebesgue number for $\mathcal{U}$.*

Proof. Follows from Theorem 16.2 and Lemma 16.8. $\square$

Exercises to Chapter 16

E16.1 Exercise. Prove Lemma 16.4.

E16.2 Exercise. Prove Lemma 16.5.

E16.3 Exercise. Give an example of an open covering $\mathcal{U}$ of the open interval $(0, 1)$ (with the usual metric) such that there does not exist a Lebesgue number for $\mathcal{U}$.

E16.4 Exercise. The goal of this exercise is to fill one missing detail in the proof of Lemma 16.11. Let (X, ϱ) be a metric space and let $\{x_n\}$ be a sequence in X. Assume that for some $\varepsilon > 0$ we have $\varrho(x_n, x_m) > \varepsilon$ for all $m \neq n$. Show that $\{x_n\}$ does not contain any convergent subsequence.

E16.5 Exercise. Let (X, ϱ) be a metric space and let $A \subseteq X$ be a set such that $A \cap K$ is compact for every compact set $K \subseteq X$. Show that A is closed in X.

E16.6 Exercise. Let (X, ϱ) be a metric space and $A_1 \subseteq A_2 \subseteq \dots$ be subsets of X. Assume for each $n = 1, 2, \dots$ the set A_n is compact and connected. Show that $A = \bigcup_{n=1}^{\infty} A_n$ is compact and connected.

E16.7 Exercise. Recall that if (X, ϱ) is a metric space then a sequence $\{x_n\}$ in X is a Cauchy sequence if for each $\varepsilon > 0$ there exists $N > 0$ such that $\varrho(x_n, x_m) < \varepsilon$ for all $n, m > N$. The space (X, ϱ) is a *complete metric space* if each Cauchy sequence in X is convergent.

Let (X, ϱ) be a metric space. Show that the following conditions are equivalent.

(i) X is compact

(ii) The space X is a complete metric space and for any $\varepsilon > 0$ there exists a finite ε-net in X.

E16.8 Exercise. Let (X, ϱ) be a metric space. We will say that a function $f\colon X \to \mathbb{R}$ is *bounded* if there is $K > 0$ such that $|f(x)| < K$ for all $x \in X$. Show that the following conditions are equivalent:

(i) X is compact

(ii) every continuous function $f\colon X \to \mathbb{R}$ is bounded.

(Hint: Show that if X is non-compact then it contains a sequence $\{x_n\}$ with no convergent subsequence and such that $x_n \neq x_m$ for all $n \neq m$. Let A be the subspace of X consisting of all points of this sequence. Show the function $f\colon A \to \mathbb{R}$ given by $f(x_n) = n$ is continuous).

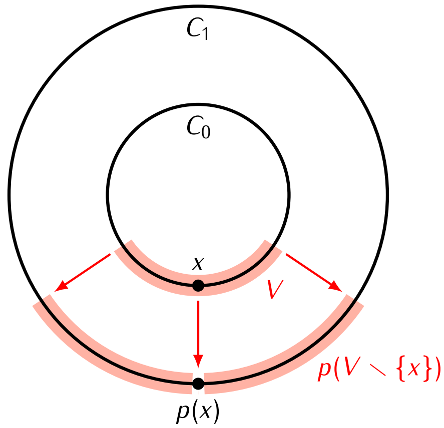
E16.9 Exercise. Theorem 16.2 characterizes compactness in metric spaces. One can ask if every compact Hausdorff space is metrizable. The goal of this exercise is to show that this is not true in general.

a) Recall that a space X is separable if it contains a countable dense subset. Show that any compact metric space is separable.

b) The *Alexandroff double circle* is a topological space X defined as follows. The points of X are the points of two concentric circles: C_0 (the inner circle) and C_1 (the outer circle). Let $p\colon C_0 \to C_1$ denote the radial projection map. A basis $\mathcal{B}$ of the topology on X consists of two types of sets:

(i) If $y \in C_1$ then $\{y\} \in \mathcal{B}$.

(ii) If $V \subseteq C_0$ is an open arch with center at the point x then then $V \cup p(V \smallsetminus \{x\}) \in \mathcal{B}$.



Show that X is a compact Hausdorff space, but that it is not separable. By part a) this will imply that X is not metrizable.

E16.10 Exercise. Let X be the Alexandroff double circle defined in Exercise 16.9. Is X sequentially compact? Justify your answer.

E16.11 Exercise. Let (X, ϱ) be a compact metric space, Y be a topological space, and let $f\colon X \to Y$ be a continuous function. Show that for family $\{A_1, A_2, \dots\}$ of closed sets in X such that $A_1 \subseteq A_2 \subseteq \dots$ we have

$$f\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} f(A_n)$$

E16.12 Exercise. Let (X, ϱ) be a compact metric space and let $f\colon X \to X$ be a continuous function such that $\varrho(f(x), f(y)) \geq \varrho(x, y)$ for all $x, y \in X$. Show that f is a homeomorphism.

E16.13 Exercise. Let (X, ϱ) be a compact metric space, and let $f\colon X \to X$ be a function such that $\varrho(f(x), f(y)) < \varrho(x, y)$ for all $x, y \in X$, $x \neq y$. By Exercise 14.7 there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$. Let x be an arbitrary point in X and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and $x_n = f(x_{n-1})$ for $n > 1$. Show that the sequence $\{x_n\}$ converges to the point x_0.

E16.14 Exercise. Let (X, ϱ), (Y, μ) be metric spaces. We say that a function $f\colon X \to Y$ is *uniformly continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $\varrho(x_1, x_2) < \delta$ then $\mu(f(x_1), f(x_2)) < \varepsilon$.

a) Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ which is not uniformly continuous. Justify your answer.

b) Show that if $f\colon X \to Y$ is continuous function and X is a compact space then f is uniformly continuous.

E16.15 Exercise. Let $U \subseteq \mathbb{R}^n$ be an open set and let $D \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be the set consisting of all pairs

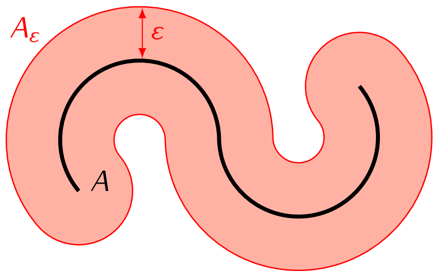
$(x, y) \in U \times U$ for which the whole line segment joining x and y is contained in U:

$$D = \{(x, y) \in U \times U \mid tx + (1 - t)y \in U \text{ for all } t \in [0, 1]\}$$

Show that D is open in $\mathbb{R}^n \times \mathbb{R}^n$.

E16.16 Exercise. For $A \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ define

$$A_\varepsilon := \{x \in \mathbb{R}^n \mid d(x, y) < \varepsilon \text{ for some } y \in A\}$$



Let $A \subseteq U \subseteq \mathbb{R}^n$ where A is compact and U is open in $\mathbb{R}^n$. Show that there exists $\varepsilon > 0$ such that $A_\varepsilon \subseteq U$.

E16.17 Exercise. Let (X, ϱ) be a metric space, and let $a, b \in X$. For $\varepsilon > 0$ we will say that a sequence of points $(x_1, \dots, x_n)$ is an *ε-chain connecting a and b* if $x_1 = a$, $x_n = b$, and $\varrho(x_i, x_{i+1}) < \varepsilon$ for $i = 1, \dots, n - 1$.

Let (X, ϱ) be a compact metric space. Show that the following conditions are equivalent:

1) the space X is connected;
2) for any points $a, b \in X$ and any $\varepsilon > 0$ there exists *ε-chain* connecting a and b.