18 | Compactification

We have seen that compact Hausdorff spaces have several interesting properties that make this class of spaces especially important in topology. If we are working with a space X which is not compact we can ask if X can be embedded into some compact Hausdorff space Y. If such embedding exists we can identify X with a subspace of Y, and some arguments that work for compact Hausdorff spaces will still apply to X. This approach leads to the notion of a *compactification* of a space. Our goal in this chapter is to determine which spaces have compactifications. We will also show that compactifications of a given space X can be ordered, and we will look for the largest and smallest compactifications of X.

18.1 Proposition. *Let X be a topological space. If there exists an embedding $j\colon X \to Y$ such that Y is a compact Hausdorff space then there exists an embedding $j_1\colon X \to Z$ such that Z is compact Hausdorff and $\overline{j_1(X)} = Z$.*

Proof. Assume that we have an embedding $j\colon X \to Y$ where Y is a compact Hausdorff space. Let $\overline{j(X)}$ be the closure of $j(X)$ in Y. The space $\overline{j(X)}$ is compact (by Proposition 14.13) and Hausdorff, so we can take $Z = \overline{j(X)}$ and define $j_1\colon X \to Z$ by $j_1(x) = j(x)$ for all $x \in X$. □

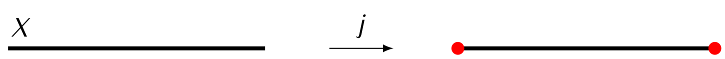
18.2 Definition. A space Z is a *compactification* of X if Z is compact Hausdorff and there exists an embedding $j\colon X \to Z$ such that $\overline{j(X)} = Z$.

18.3 Corollary. *Let X be a topological space. The following conditions are equivalent:*

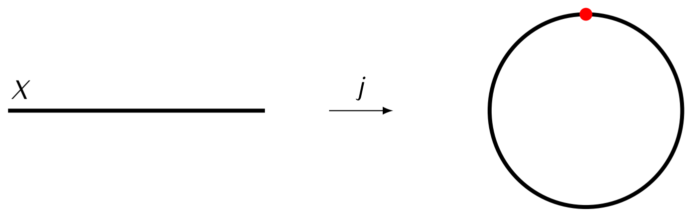
1) *There exists a compactification of X.*
2) *There exists an embedding $j\colon X \to Y$ where Y is a compact Hausdorff space.*

Proof. Follows from Proposition 18.1. □

18.4 Example. The closed interval $[-1, 1]$ is a compactification of the open interval $(-1, 1)$. with the embedding $j\colon (-1, 1) \to [-1, 1]$ is given by $j(t) = t$ for $t \in (-1, 1)$.



18.5 Example. The unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ is another compactification of the interval $(-1, 1)$. The embedding $j\colon (-1, 1) \to S^1$ is given by $j(t) = (\sin \pi t, -\cos \pi t)$.

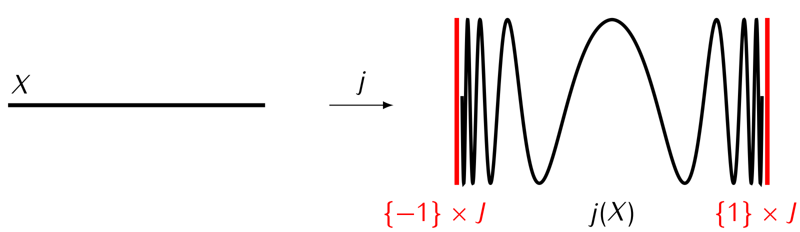


18.6 Example. A more complex compactification of the space $X = (-1, 1)$ can be obtained as follows. Let $J = [-1, 1]$. Consider the function $j\colon X \to J \times J$ given by

$$j(t) = \left(t,\, \cos\left(\tfrac{|t|}{1-|t|}\right)\right)$$

The map j is an embedding, and so $\overline{j(X)} \subseteq J \times J$ is a compactification of X. We have:

$$\overline{j(X)} = \{-1\} \times J \cup j(X) \cup \{1\} \times J$$



18.7 Theorem. *A space X has a compactification if and only if X is completely regular (i.e. it is a $T_{3\frac{1}{2}}$-space).*

Proof. ($\Rightarrow$) Assume that X has a compactification. Let $j\colon X \to Y$ be an embedding where Y is a compact Hausdorff space. By Theorem 14.19 the space Y is normal, so it is completely regular. Since subspaces of completely regular spaces are completely regular (exercise) we obtain that $j(X) \subseteq Y$ is completely regular. Finally, since $j(X) \cong X$ we get that X is completely regular.

($\Leftarrow$) Assume that X is completely regular. We need to show that there exists an embedding $j\colon X \to Y$ where Y is a compact Hausdorff space. Let $C(X)$ denote the set of all continuous functions $f\colon X \to [0, 1]$.

Complete regularity of X implies that $C(X)$ is a family of functions that separates points from closed sets in X (12.13). Consider the map

$$j_X \colon : X \to \prod_{f \in C(X)} [0,1]$$

given by $j_X(x) = (f(x))_{f\in C(X)}$. By the Embedding Lemma 12.14 we obtain that this map is an embedding. It remains to notice that by Corollary 17.17 the space $\prod_{f\in C(X)}[0,1]$ is compact Hausdorff. □

18.8 Note. In the part ($\Rightarrow$) of the proof of Theorem 18.7 we used the fact that subspaces of completely regular spaces are completely regular. An analogous property does not hold for normal spaces: a subspace of a normal space need not be normal. For this reason it is not true that a space that has a compactification must be a normal space.

18.9 Definition. Let X be a completely regular space and let $j_X \colon X \to \prod_{f\in C(X)}[0,1]$ be the embedding defined in the proof of Theorem 18.7 and let $\beta(X)$ be the closure of $j_X(X)$ in $\prod_{f\in C(X)}[0,1]$. The compactification $j_X \colon X \to \beta(X)$ is called the *Čech-Stone compactification* of X.

The Čech-Stone compactification is the largest compactification of a space X in the following sense:

18.10 Definition. Let X be a space and let $i_1 \colon X \to Y_1$, $i_2 \colon X \to Y_2$ be compactifications of X. We will write $Y_1 \geq Y_2$ if there exists a continuous function $g \colon Y_1 \to Y_2$ such that $i_2 = gi_1$:

$$\begin{tikzcd} & X \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ Y_1 \arrow[rr, "g"'] & & Y_2 \end{tikzcd}$$

18.11 Proposition. *Let $i_1 \colon X \to Y_1$, $i_2 \colon X \to Y_2$ be compactifications of a space X.*

1) If $Y_1 \geq Y_2$ then there exists only one map $g \colon Y_1 \to Y_2$ satisfying $i_2 = gi_1$. Moreover g is onto.

2) $Y_1 \geq Y_2$ and $Y_2 \geq Y_1$ if and only if the map $g \colon Y_1 \to Y_2$ is a homeomorphism.

Proof. Exercise. □

18.12 Theorem. *Let X be a completely regular space and let $j_X \colon X \to \beta(X)$ be the Čech-Stone compactification of X. For any compactification $i \colon X \to Y$ of X we have $\beta(X) \geq Y$.*

The proof Theorem 18.12 will use the following fact:

18.13 Lemma. *If $f\colon X_1 \to X_2$ is a continuous map of compact Hausdorff spaces then $f(\overline{A}) = \overline{f(A)}$ for any $A \subseteq X_1$.*

Proof. Exercise. □

Proof of Theorem 18.12. Let $i\colon X \to Y$ be a compactification of X. We need to show that there exists a map $g\colon \beta(X) \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "j_X"'] \arrow[dr, "i"] & \\ \beta(X) \arrow[rr, "g"'] & & Y \end{tikzcd}$$

Let $C(X)$, $C(Y)$ denote the sets of all continuous functions $X \to [0,1]$ and $Y \to [0,1]$ respectively. Consider the continuous functions $j_X\colon X \to \prod_{f\in C(X)}[0,1]$ and $j_Y\colon Y \to \prod_{f'\in C(Y)}[0,1]$ defined as in the proof of Theorem 18.7. Notice that we have a continuous function

$$i_*\colon \prod_{f\in C(X)} [0,1] \to \prod_{f'\in C(Y)} [0,1]$$

given by $i_*((t_f)_{f\in C(X)}) = (s_{f'})_{f'\in C(Y)}$ where $s_{f'} = t_{if'}$. Moreover, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j_X"'] & Y \arrow[d, "j_Y"] \\ \prod_{f\in C(X)}[0,1] \arrow[r, "i_*"'] & \prod_{f'\in C(Y)}[0,1] \end{tikzcd}$$

We have:

$$i_*(\beta(X)) = i_*(\overline{j_X(X)}) = \overline{i_* j_X(X)} = \overline{j_Y i(X)} = j_Y(\overline{i(X)}) = j_Y(Y)$$

Here the first equality comes from the definition of $\beta(X)$, the second from Lemma 18.13, the third from commutativity of the diagram above, the fourth again from Lemma 18.13, and the last from the assumption that $i\colon X \to Y$ is a compactification. Since the map $j_Y\colon Y \to \prod_{f'\in C(Y)}[0,1]$ is embedding the map $j_Y\colon Y \to j_Y(Y)$ is a homeomorphism. We can take $g = j_Y^{-1} i_*\colon \beta(X) \to Y$.

□

Motivated by the fact that Čech-Stone compactification is the largest compactification of a space X one can ask if the smallest compactification of X also exists. If X is a non-compact space then we need to add at least one point to X to compactify it. If adding only one point suffices then it gives an obvious candidate for the smallest compactification:

18.14 Definition. A space Z is a *one-point compactification* of a space X if Z is a compactification of X with embedding $j\colon X \to Z$ such that the set $Z \smallsetminus j(X)$ consists of only one point.

18.15 Example. The unit circle S^1 is a one-point compactification of the open interval $(0, 1)$.

18.16 Proposition. *If a space X has a one-point compactification $j\colon X \to Z$ then this compactification is unique up to homeomorphism. That is, if $j'\colon X \to Z'$ is another one-point compactification of X then there exists a homeomorphism $h\colon Z \to Z'$ such that $j' = hj$.*

Proof. Exercise. □

Our next goal is to determine which spaces admit a one-point compactification.

18.17 Definition. A topological space X is *locally compact* if every point $x \in X$ has an open neighborhood $U_x \subseteq X$ such that the the closure $\overline{U_x}$ is compact.

18.18 Note. 1) If X is a compact space then X is locally compact since for any $x \in X$ we can take $U_x = X$.

2) The real line $\mathbb{R}$ is not compact but it is locally compact. For $x \in \mathbb{R}$ we can take $U_x = (x - 1, x + 1)$, and then $\overline{U_x} = [x - 1, x + 1]$ is compact. Similarly, for each $n \geq 0$ the space $\mathbb{R}^n$ is a non-compact but locally compact.

3) The set $\mathbb{Q}$ of rational numbers, considered as a subspace of the real line, is not locally compact (exercise).

18.19 Theorem. *Let X be a non-compact topological space. The following conditions are equivalent:*

1) *The space X is locally compact and Hausdorff.*
2) *There exists a one-point compactification of X.*

Proof. 1) ⇒ 2) Assume that X locally compact and Hausdorff. We define a space X^+ as follows. Points of X^+ are points of X and one extra point that we will denote by ∞:

$$X^+ := X \cup \{\infty\}$$

A set $U \subseteq X^+$ is open if either of the following conditions holds:

(i) $U \subseteq X$ and U is open in X

(ii) $U = \{\infty\} \cup (X \smallsetminus K)$ where $K \subseteq X$ is a compact set.

The collection of subsets of X^+ defined in this way is a topology on X^+ (exercise). One can check that the function $j\colon X \to X^+$ given by $j(x) = x$ is continuous and that it is an embedding (exercise). Moreover, since X is not compact for every open neighborhood U of ∞ we have $U \cap X \neq \varnothing$, so $\overline{j(X)} = X^+$.

To see that X^+ is a compact space assume that $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of X^+. Let $U_{i_0} \in \mathcal{U}$ be a set such that $\infty \in U_{i_0}$. By the definition of the topology on X^+ we have $X^+ \smallsetminus U_{i_0} = K$ where $K \subseteq X$ is a compact set. Compactness of K gives that

$$K \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

for some $U_1, \dots, U_{i_n} \in \mathcal{U}$. It follows that $\{U_{i_0}, U_{i_1}, \dots, U_{i_n}\}$ is a finite cover of X^+.

It remains to check that X^+ is a Hausdorff space (exercise).

2) $\Rightarrow$ 1) Let $j\colon X \to Z$ be a one-point compactification of X. Since $X \cong j(X)$ it suffices to show that the space $j(X)$ is locally compact and Hausdorff. We will denote by ∞ the unique point in $Z \smallsetminus j(X)$.

Since Z is a Hausdorff space and subspaces of a Hausdorff space are Hausdorff we get that $j(X)$ is a Hausdorff space.

Next, we will show that $j(X)$ is locally compact. Let $x \in j(X)$. Since Z is Hausdorff there are sets $U, V \subseteq Z$ open in Z such that $x \in U$, $\infty \in V$, and $U \cap V = \varnothing$. Since $\infty \notin U$ the set U is open in X. Let $\overline{U}$ denote the closure of U in X. We will show that $\overline{U}$ is a compact set. Notice that we have

$$\overline{U} \subseteq Z \smallsetminus V \subseteq Z$$

Since $Z \smallsetminus V$ is closed in the compact space Z thus it is compact by Proposition 14.13. Also, since $\overline{U}$ is a closed subset of $Z \smallsetminus V$, thus $\overline{U}$ is compact by the same result. $\square$

18.20 Corollary. *If X is a locally compact Hausdorff space then X is completely regular.*

Proof. Follows from Theorem 18.7 and Theorem 18.19. $\square$

18.21 Corollary. *Let X be a topological space. The following conditions are equivalent:*

1) *The space X is locally compact and Hausdorff .*
2) *There exists an embedding $i\colon X \to Y$ where Y is compact Hausdorff space and $i(X)$ is an open set in Y.*

Proof. 1) $\Rightarrow$ 2) If X is compact then we can take i to be the identity map $\mathrm{id}_X\colon X \to X$. If X is not compact take the one-point compactification $j\colon X \to X^+$. By the definition of topology on X^+ the set $j(X)$ is open in X^+.

2) $\Rightarrow$ 1) exercise. $\square$

The next proposition says that one-point compactification, when it exists, is the smallest compactification of a space in the sense of Definition 18.10:

18.22 Proposition. *Let X be a non-compact, locally compact space and let $j\colon X \to X^+$ be the one-point compactification of X. For every compactification $i\colon X \to Y$ of X we have $Y \geq X^+$.*

Proof. Exercise. □

One can also show that if a space X is not locally compact (and so it does not have a one-point compactification) then no compactification of X has the property of being the smallest (see Exercise 18.15).

Exercises to Chapter 18

E18.1 Exercise. Show that a subspace of a completely regular space is completely regular (this will complete the proof of Theorem 18.7).

E18.2 Exercise. Prove Proposition 18.11.

E18.3 Exercise. Prove Lemma 18.13.

E18.4 Exercise. Consider the set $\mathbb{Q}$ of rational numbers with the subspace topology of the real line. Show that $\mathbb{Q}$ is not locally compact.

E18.5 Exercise. Let X be a locally compact Hausdorff space, let $x_0 \in X$, and let $U \subseteq X$ be an open neighborhood of x_0. Show that there exists an open neighborhood W of x_0 such that $\overline{W} \subseteq U$ and $\overline{W}$ is compact.

E18.6 Exercise. Prove Proposition 18.16.

E18.7 Exercise. The goal of this exercise is to fill one of the gaps left in the proof of Theorem 18.19. Let X be a locally compact Hausdorff space and let $X^+ = X \cup \{\infty\}$ be the space defined in part 1) $\Rightarrow$ 2) of the proof of (18.19). Show that X^+ is a Hausdorff space.

E18.8 Exercise. Prove the implication 2) $\Rightarrow$ 1) of Corollary 18.21.

E18.9 Exercise. A continuous function $f\colon X \to Y$ is *proper* if for every compact set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is compact. Let X, Y be locally compact, Hausdorff spaces and let X^+, Y^+ be their one-point compactifications. Let $f\colon X \to Y$ be a continuous function. Show that the following conditions are equivalent:

1) The function f is proper.
2) The function $f^+\colon X^+ \to Y^+$ given by $f^+(x) = f(x)$ for $x \in X$ and $f^+(\infty) = \infty$ is continuous.

E18.10 Exercise. Let (X, ϱ), (Y, μ) be metric spaces and let $f\colon X \to Y$ be a continuous function. Show that the following conditions are equivalent:

1) f is proper (Exercise 18.9)
2) If $\{x_n\} \subseteq X$ is a sequence such that $\{f(x_n)\} \subseteq Y$ converges then $\{x_n\} \subseteq X$ has a convergent subsequence.

E18.11 Exercise. Let X, Y be locally compact Hausdorff spaces, and let $j\colon X \to Y$ be an embedding such that $j(X)$ is an open in Y. Define $j^\sharp : Y^+ \to X^+$ as follows:

$$j^\sharp(y) = \begin{cases} j^{-1}(y) & \text{if } y \in j(X) \\ \infty & \text{otherwise} \end{cases}$$

Show that $j^\sharp$ is a continuous function.

E18.12 Exercise. Let X, Y be locally compact, Hausdorff spaces and let X^+, Y^+ be their one-point compactifications. Let $f\colon X^+ \to Y^+$ be a continuous function such $f(\infty) = \infty$. Show that there exists an open set $U \subseteq X$ such $f = g^+ j^\sharp$ where $j\colon U \to X$ is the inclusion map, $g = f|_U\colon U \to Y$ is a proper map, $j^\sharp\colon X^+ \to U^+$ is obtained form j as in Exercise 18.11, and $g^+\colon U^+ \to Y^+$ obtained from g as in Exercise 18.9.

E18.13 Exercise. Let X be topological space and let $j\colon X \to Y$ be a compactification of X. Show that if X is locally compact the set $j(X)$ is open in Y.

E18.14 Exercise. Prove Proposition 18.22.

E18.15 Exercise. The goal of this exercise is to show that the smallest compactification of a non-compact space X exists only if X has a one-point compactification (i.e. if X is a locally compact space).

Let X be a completely regular non-compact space. Assume that there exists a compactification $j\colon X \to Y$ of X such that for any other compactification $i\colon X \to Z$ we have $Z \geq Y$. Show that Y is a one-point compactification of X. As a consequence X must be locally compact. (Hint: Assume that Y is not a one-point compactification of X and let $y_1, y_2 \in Y \smallsetminus j(X)$. Show that the space $W = Y \smallsetminus \{y_1, y_2\}$ has a one-point compactification $k\colon W \to W^+$ and that $kj\colon : X \to W^+$ is a compactification of X. Show that it is not true that $W^+ \geq Y$).