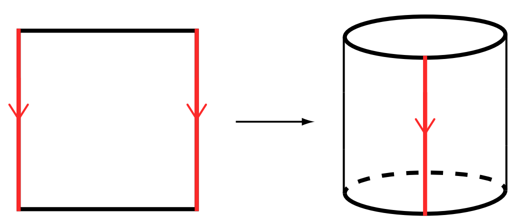
19 | Quotient Spaces

So far we have encountered two methods of constructing new topological spaces from old ones:

- given a space X we can obtain new spaces by taking subspaces of X;
- given two (or more) spaces X_1, X_2 we can obtain a new space by taking their product $X_1 \times X_2$.

Here we will consider another, very useful construction of a *quotient space* of a given topological space. This construction will let us produce, in particular, interesting examples of manifolds. Intuitively, a quotient space of a space X is a space Y which is obtained by identifying some points of X. For example, if we take the square $X = [0, 1] \times [0, 1]$ and identify each point $(0, t)$ with the point $(1, t)$ for $t \in [0, 1]$ we obtain a space Y that looks like a cylinder:



In order to make this precise we need to specify the following:

1) what are the points of Y;
2) what is the topology on Y.

The first part is done by considering Y as the set of *equivalence classes* of some *equivalence relation* on X. The second part is done by defining the *quotient topology*. We explain these notions below.

19.1 Definition. Let X be a set. An *equivalence relation on* X is a binary relation $\sim$ satisfying three properties:

1) $x \sim x$ for all $x \in X$ (reflexivity)
2) if $x \sim y$ then $y \sim x$ (symmetry)
3) if $x \sim y$ and $y \sim z$ then $x \sim z$ (transitivity)

19.2 Example. Let $X = [0,1] \times [0,1]$. Define a relation on X as follows. For any $(s,t) \in X$ we set $(s,t) \sim (s,t)$. Also, for any $t \in [0,1]$ we set $(0,t) \sim (1,t)$ and $(1,t) \sim (0,t)$. This relation is an equivalence relation that identifies corresponding points of the vertical edges of the square $[0,1]\times[0,1]$.

19.3 Example. Define a relation $\sim$ on the set of real numbers $\mathbb{R}$ as follows: $r \sim s$ if $s = r + n$ for some $n \in \mathbb{Z}$. One can check that this is an equivalence relation (exercise).

19.4 Definition. Let X we a set with an equivalence relation $\sim$ and let $x \in X$. The *equivalence class* of x is the subset $[x] \subseteq X$ consisting of all elements that are in the relation with x:

$$[x] = \{y \in X \mid x \sim y\}$$

19.5 Example. Take $X = [0,1] \times [0,1]$ with the equivalence relation defined as in Example 19.2. If $(s,t) \in X$ and $s \neq 0,1$ then $[(s,t)]$ consists of a single point: $[(s,t)] = \{(s,t)\}$. If $s = 0,1$ then $[(s,0)]$ consists of two points: $[(0,t)] = [(1,t)] = \{(0,t),(1,t)\}$.

19.6 Example. Take $\mathbb{R}$ with the equivalence relation defined as in Example 19.3. For $r \in \mathbb{R}$ we have:

$$[r] = \{r + n \mid n \in \mathbb{Z}\}$$

For example: $[1] = \{1 + n \mid n \in \mathbb{Z}\} = \mathbb{Z}$. Notice that $[1] = [2]$ and $[\sqrt{2}] = [\sqrt{2} + 1]$.

19.7 Proposition. *Let X be a set with an equivalence relation $\sim$, and let $x, y \in X$.*

1) *If $x \sim y$ then $[x] = [y]$.*
2) *If $x \not\sim y$ then $[x] \cap [y] = \varnothing$.*

Proof. 1) Assume that $x \sim y$ and that $z \in [x]$. This gives $z \sim x$ and by transitivity $z \sim y$. Therefore $z \in [y]$. This shows that $[x] \subseteq [y]$. In the same way we can show that $[y] \subseteq [x]$. Therefore we get $[x] = [y]$.

2) Assume that $[x] \cap [y] \neq \varnothing$, and let $z \in [x] \cap [y]$. Then $x \sim z$ and $y \sim z$, so by transitivity $x \sim y$ which contradicts our assumption. ☐

19.8 Note. Proposition 19.7 shows that an equivalence relation $\sim$ on a set X splits X into a disjoint union of distinct equivalence classes of $\sim$. The opposite is also true. Namely, assume that we have a family $\{A_i\}_{i \in I}$ of subsets of X such that $A_i \cap A_j = \varnothing$ for $i \neq j$ and $\bigcup_{i\in I} A_i = X$. We can define a relation $\sim$ on X such that $x \sim y$ if and only if both x and y are elements of the same subset A_i. This relation is an equivalence relation and its equivalence classes are the sets A_i.

19.9 Definition. Let X be a set with an equivalence relation $\sim$. The *quotient set* of X is the set $X/{\sim}$ whose elements are all distinct equivalence classes of $\sim$. The function

$$\pi \colon X \to X/{\sim}$$

given by $\pi(x) = [x]$ is called the *quotient map*.

19.10 Note. Let X be a set with an equivalence relation $\sim$, and let $f\colon X \to Y$ be a function. Assume that for each $x, x' \in X$ such that $x \sim x'$ we have $f(x) = f(x')$. Then we can define a function $\overline{f}\colon X/{\sim} \to Y$ by $\overline{f}([x]) = f(x)$. We have $f = \overline{f}\pi$, i.e. the following diagram commutes:

$$\begin{tikzcd}
 & X/{\sim} \arrow[d, "\overline{f}"] \\
X \arrow[ur, "\pi"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

19.11 Definition. Let X be a topological space and let $\sim$ be an equivalence relation on X. The *quotient topology* on the set $X/{\sim}$ is the topology where a set $U \subseteq X/{\sim}$ is open if the set $\pi^{-1}(U)$ is open in X. The set $X/{\sim}$ with this topology is called the *quotient space* of X taken with respect to the relation $\sim$.

19.12 Proposition. *Let X be a topological space and let $\sim$ be an equivalence relation on X. A set $A \subseteq X/{\sim}$ is closed if and only the set $\pi^{-1}(A)$ is closed in X.*

Proof. Exercise. $\square$

19.13 Proposition. *Let X, Y be a topological spaces and let $\sim$ be an equivalence relation on X. A function $f\colon X/{\sim} \to Y$ is continuous if and only if the function $f\pi\colon X \to Y$ is continuous.*

Proof. Exercise. $\square$

19.14 Note. Let X be a space with an equivalence relation $\sim$ and let $f\colon X \to Y$ be a continuous function. If for each $x, x' \in X$ such that $x \sim x'$ we have $f(x) = f(x')$ then as in (19.10) we obtain a function $\overline{f}\colon X/{\sim} \to Y$, $\overline{f}([x]) = f(x)$. Since the function $\overline{f}\pi = f$ is continuous thus by Proposition 19.13 $\overline{f}$ is a continuous function.

19.15 Example. Take the closed interval $[-1, 1]$ with the equivalence relation $\sim$ such that $(-1) \sim 1$ (and $t \sim t$ for all $t \in [-1, 1]$). We will show that the quotient space $[-1.1]/{\sim}$ is homeomorphic to the circle S^1. Consider the function $f\colon [-1, 1] \to S^1$ given by $f(x) = (\sin \pi x, -\cos \pi x)$:

Since $f(1) = f(-1)$ by (19.14) we get the induced continuous function $\overline{f}\colon [-1, 1]/{\sim} \to S^1$. We will prove that $\overline{f}$ is a homeomorphism. First, notice that $\overline{f}$ is a bijection. Next, since $[-1, 1]$ is a compact space and the quotient map $\pi\colon [-1, 1] \to [-1, 1]/{\sim}$ is onto by Proposition 14.9 we obtain that the space $[-1, 1]/{\sim}$ is compact. Therefore we can use Proposition 14.18 which says that any continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

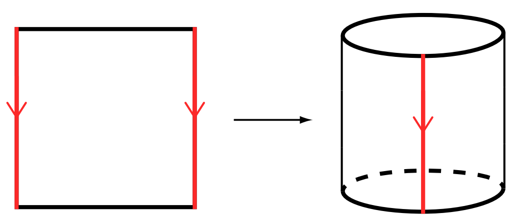
This example can be generalized as follows. Take the closed unit ball

$$\overline{B}^n = \{x \in \mathbb{R}^n \mid d(0, x) \leq 1\}$$

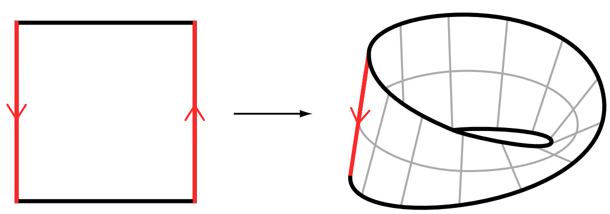
The unit sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid d(0,x) = 1\}$ is a subspace of $\bar{B}^n$. Consider the equivalence relation $\sim$ on $\bar{B}^n$ that identifies all points of S^{n-1}: $x \sim x'$ for all $x, x' \in S^{n-1}$. Using similar arguments as above one can show that $\bar{B}^n/{\sim}$ is homeomorphic to the sphere S^n (exercise). Notice that for $n = 1$ we have $\bar{B}^1 = [-1,1]$ and $S^0 = \{-1,1\}$ so in this case we recover the homeomorphism $[-1,1]/{\sim} \cong S^1$.

19.16 Note. Let X be a space and let $A \subseteq X$. Consider the equivalence relation on X that identifies all points of A: $x \sim x'$ for all $x, x' \in A$. The quotient space $X/{\sim}$ is usually denoted by X/A. Using this notation the homeomorphism given in Example 19.15 can be written as $\bar{B}^n/S^{n-1} \cong S^n$.

19.17 Example. Take the square $[0,1] \times [0,1]$ with the equivalence relation defined as in Example 19.2: $(0,t) \sim (1,t)$ for all $t \in [0,1]$. Using arguments similar as in Example 19.15 we can show that the quotient space is homeomorphic to the cylinder $S^1 \times [0,1]$:

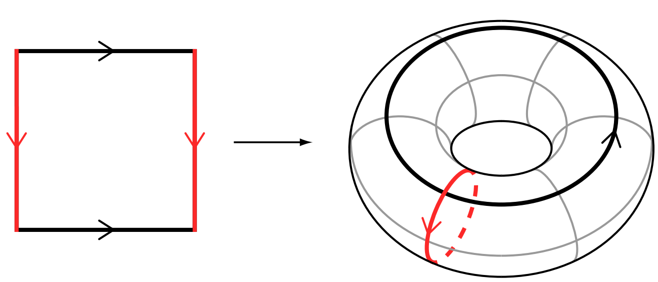


19.18 Example. Take the square $[0,1] \times [0,1]$ with the equivalence relation given by $(0,t) \sim (1, 1-t)$ for all $t \in [0,1]$. The space obtained as a quotient space is called the *Möbius band*:

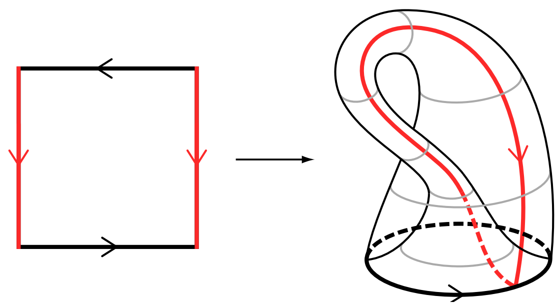


The Möbius band is a 2-dimensional manifold with boundary, and its boundary is homeomorphic to S^1.

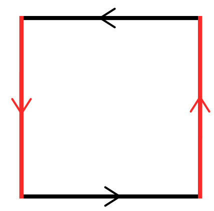
19.19 Example. Take the square $[0,1] \times [0,1]$ with the equivalence relation given by $(0,t) \sim (1,t)$ for all $t \in [0,1]$ and $(s,0) \sim (s,1)$ for all $s \in [0,1]$. Using arguments similar to these given in Example 19.15 one can show that the quotient space in this case is homeomorphic to the torus:



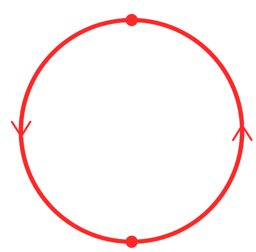
19.20 Example. Take the square $[0,1]\times[0,1]$ with the equivalence relation given by $(0,t)\sim(1,t)$ for all $t\in[0,1]$ and $(s,0)\sim(1-s,1)$ for all $s\in[0,1]$. The resulting quotient space is called the *Klein bottle*. One can show that the Klein bottle is a two dimensional manifold.



19.21 Example. Following the scheme of the last two examples we can consider the square $[0,1]\times[0,1]$ with the equivalence relation given by $(0,t)\sim(1,1-t)$ and $(s,0)\sim(1-s,1)$ for all $s,t\in[0,1]$:



The resulting quotient space is homeomorphic to the space $\mathbb{RP}^2$ which is defined as follows. Take the the 2-dimensional closed unit ball $\overline{B}^2$. The boundary of $\overline{B}^2$ is the circle S^1. Consider the equivalence relation $\sim$ on $\overline{B}^2$ that identifies each point $(x_1,x_2)\in S^1$ with its antipodal point $(-x_1,-x_2)$:



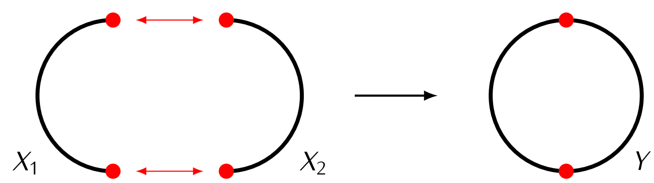
We define $\mathbb{RP}^2=\overline{B}^2/{\sim}$. This space is called the *2-dimensional real projective space* and it is a 2-dimensional manifold. One can show that $\mathbb{RP}^2$ (and also the Klein bottle) cannot be embedded into $\mathbb{R}^3$. For this reason it is harder to visualize it.

19.22 Example. The construction of $\mathbb{RP}^2$ given in Example 19.21 can be generalized to higher dimensions. Consider the n-dimensional closed unit ball $\overline{B}^n$. The boundary $\overline{B}^n$ is the sphere S^{n-1}. Similarly as before we can consider the equivalence relation $\sim$ on $\overline{B}^n$ that identifies antipodal points of S^{n-1}:

$$(x_1, \dots, x_n) \sim (-x_1, \dots, -x_n)$$

for all $(x_1, \dots, x_n) \in S^{n-1}$. The quotient space $\overline{B}^n/{\sim}$ is denoted by $\mathbb{RP}^n$ and is called the *n-dimensional real projective space*. The space $\mathbb{RP}^n$ is an n-dimensional manifold. For another perspective on projective spaces see Exercise 19.8.

Many constructions in topology involve the following setup. We start with two topological spaces X_1, X_2, and we build a new space Y by identifying certain points of X_1 with certain points of X_2:

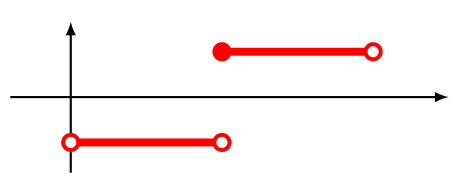


An example of a setting that uses such assembly process is described in Chapter 20.

The first step in constructions of this kind it to create a new space $X_1 \sqcup X_2$ which contains X_1 and X_2 as its subspaces. The space Y can be then described as a quotient space of $X_1 \sqcup X_2$. The space $X_1 \sqcup X_2$ is defined as follows. If $X_1 \cap X_2 = \varnothing$ then $X_1 \sqcup X_2 = X_1 \cup X_2$ as a set. A set $U \subseteq X_1 \sqcup X_2$ is open if and only if $U \cap X_i$ is open in X_i for $i = 1, 2$. If $X_1 \cap X_2 \neq \varnothing$ then we first replace X_i with a homeomorphic space X_i' such that $X_1' \cap X_2' = \varnothing$ (e.g. we can take $X_i' = \{i\} \times X_i$) and then we set $X_1 \sqcup X_2$ to be equal to $X_i' \sqcup X_2'$.

19.23 Definition. The space $X_1 \sqcup X_2$ is called the *disjoint union* (or the *coproduct*) of spaces X_1 and X_2.

19.24 Example. Take $X_1 = (0, 1)$ and $X_1 = [1, 2)$. Since $X_1 \cap X_2 = \varnothing$ we can construct the space $(0, 1) \sqcup [1, 2)$ so that it consists of the points of the interval $(0, 2)$. However, the disjoint union $(0, 1) \sqcup [1, 2)$ is not homeomorphic to the interval $(0, 2)$ taken with the usual topology. For example, the set $U = [1, \frac{1}{2})$ is not open in the interval $(0, 2)$, but it is open in $(0, 1) \sqcup [1, 2)$ since $U \cap (0, 1) = \varnothing$ is open in $(0, 1)$ and $U \cap [1, 2) = [1, \frac{1}{2})$ is open in $[1, 2)$. In general, in the disjoint union $X_1 \sqcup X_2$ the spaces X_1 and X_2 can be imagined as being far apart from each other so that an arbitrary combination of an open set in X_1 and and open set in X_1 gives an open set in $X_1 \sqcup X_2$. For example, the space $(0, 1) \sqcup [1, 2)$ is homeomorphic to the subspace of $\mathbb{R}^2$ given by $(0, 1) \times \{-a\} \cup [1, 2) \times \{a\}$ for some $a > 0$.



The construction of a disjoint union can be extended to arbitrary families of topological spaces. Given a family $\{X_i\}_{i\in I}$ such that $X_i \cap X_j = \varnothing$ for all $i \neq j$, we define $\bigsqcup_{i\in I} X_i = \bigcup_{i\in I} X_i$ as a set. A set $U \subseteq \bigsqcup_{i\in I} X_i$ is open if and only if the set $U \cap X_i$ is open in X_i for each $i \in I$. If the family $\{X\}_{i\in I}$ does not consist of disjoint spaces, then we first replace it with a family $\{X_i'\}_{i\in I}$ such that $X_i' \cong X_i$ for each $i \in I$, and $X_i' \cap X_j' = \varnothing$ for all $i \neq j$.

If $\bigsqcup_{i\in I} X_i$ is the disjoint union of a family $\{X_i\}_{i\in I}$, then for each $j \in I$ we have an embedding $k_j \colon X_j \to \bigsqcup_{i\in I} X_i$. The following fact is an essential property of the space $\bigsqcup_{i\in I} X_i$:

19.25 Proposition. *For any family of continuous functions $\{f_i \colon X_i \to Y\}_{i\in I}$, there exists a unique continuous function $f \colon \bigsqcup_{i\in I} X_i \to Y$ such that $k_j f = f_j$ for each $j \in I$.*

Proof. Exercise. □

19.26 Note. The function $f \colon \bigsqcup_{i\in I} X_i \to Y$ in Proposition 19.25 is usually denoted by $\bigsqcup_{i\in I} f_i$.

Exercises to Chapter 19

E19.1 Exercise. Prove Proposition 19.12.

E19.2 Exercise. Prove Proposition 19.13.

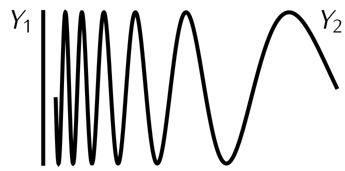
E19.3 Exercise. Consider the real line $\mathbb{R}$ with the equivalence relation defined as in Example 19.3. Show that the quotient space $\mathbb{R}/{\sim}$ is homeomorphic with S^1.

E19.4 Exercise. Take the closed interval $[0, 1]$ with the equivalence relation $\sim$ defined as in Example 19.15. Let $\pi \colon [0, 1] \to [0, 1]/{\sim}$ be the quotient map. The set $U = [0, \frac{1}{2})$ which is open subset of $[0, 1]$. Show that $\pi(U)$ is not open in $[0, 1]/{\sim}$.

E19.5 Exercise. Let $\overline{B}^n \subseteq \mathbb{R}^n$ be the closed unit ball (see Example 19.15). Show that $\overline{B}^n/S^{n-1}$ is homeomorphic to S^n.

E19.6 Exercise. Let X be a compact Hausdorff space, and let $U \subseteq X$ be an open set. Show that the one-point compactification U^+ of U (18.14) is homeomorphic to the quotient space $X/(X \smallsetminus U)$.

E19.7 Exercise. Recall that the topologists sine curve Y is the subspace of $\mathbb{R}^2$ consisting of the vertical line segment $Y_1 = \{(0, y) \mid -1 \leq y \leq 1\}$ and the curve $Y_2 = \{(x, \sin(\frac{1}{x})) \mid x > 0\}$:



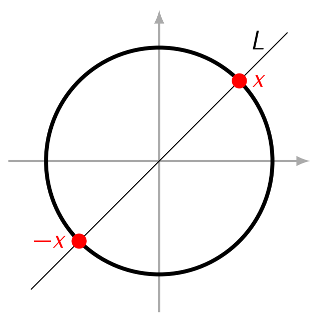
Show that the space Y/Y_1 is homeomorphic to the half line $[0, +\infty)$.

E19.8 Exercise. Consider the unit sphere S^n with the equivalence relation that identifies antipodal points of S^n:

$$(x_1, \dots, x_{n+1}) \sim (-x_1, \dots, -x_{n+1})$$

for all $(x_1, \dots, x_{n+1})$. Show that the quotient space $S^n/\sim$ is homeomorphic to the projective space $\mathbb{RP}^n$ (19.22).

Note: This construction lets us interpret $\mathbb{RP}^n$ as the space of straight lines in $\mathbb{R}^{n+1}$ that pass through the origin. Indeed, any such line L intersects the sphere S^n at two points: some point x and its antipodal point $-x$:



Since $\mathbb{RP}^n$ is obtained by identifying antipodal points we get a bijective correspondence between elements of $\mathbb{RP}^n$ and lines in $\mathbb{R}^{n+1}$ passing through the origin.

E19.9 Exercise. A *pointed topological space* is a pair (X, x_0) where X is a topological space and $x_0 \in X$. The *smash product* of pointed spaces (X, x_0) and (Y, y_0) is the quotient space

$$X \wedge Y = (X \times Y)/A$$

where $A = (X \times \{y_0\}) \cup (\{x_0\} \times Y)$

a) Let X, Y be a locally compact spaces (18.17). Show that the space $X \times Y$ is locally compact.

b) By part a) and Corrollary 17.17 if X, Y are locally compact Hausdorff spaces then the space $X \times Y$ is also locally compact and Hausdorff. By Theorem 18.19 we have in such case one-point compactifications X^+, Y^+, and $(X \times Y)^+$ of the spaces X, Y, and $X \times Y$ respectively. Recall that

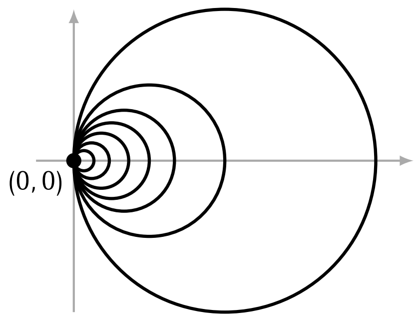
$X^+ = X \cup \{\infty\}$ and $Y^+ = Y \cup \{\infty\}$. Consider (X^+, ∞) and (Y^+, ∞) as pointed spaces. Show that there is a homeomorphism:

$$X^+ \wedge Y^+ \cong (X \times Y)^+$$

E19.10 Exercise. Prove Proposition 19.25.

E19.11 Exercise. Let $\{X_i\}_{i\in I}$ be a family of topological spaces, let Z be a topological space and for each $i \in I$ let $g_i\colon X_i \to Z$ let be a continuous function. Assume that for each family of continuous function functions $\{f_i\colon X_i \to Z\}_{i\in I}$ there exists a unique function $f\colon Z \to Y$ such that $g_i f = f_i$ for each $i \in I$. Show that the space Z is homeomorphic to $\coprod_{i\in I} X_i$.

E19.12 Exercise. The *Hawaiian earring* space is a subspace $X \subseteq \mathbb{R}^2$ given by $X = \bigcup_{n=1}^{\infty} C_n$ where C_n is the circle with radius $\frac{1}{n}$ and center at the point $(0, \frac{1}{n})$:



Notice that the point $(0,0)$ is the intersection of all circles C_n.

For $n = 1, 2, \ldots$ let C_n be the circle defined as above, and let Y be the quotient space of the disjoint union $\coprod_{i=1}^{\infty} C_n$ obtained by identifying points $(0,0) \in C_n$ for all n. Show that Y is not homeomorphic to X.

E19.13 Exercise. Let $\mathbb{R}^n_+$, $\mathbb{R}^n_-$, $\mathbb{R}^n_0$ be subspaces of $\mathbb{R}^n$ given by

$$\begin{aligned}
\mathbb{R}^n_+ &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\} \\
\mathbb{R}^n_- &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \leq 0\} \\
\mathbb{R}^n_0 &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}
\end{aligned}$$

Notice that $\mathbb{R}^n_0$ is contained in both $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$. Given a homeomorphism $h\colon \mathbb{R}^n_0 \to \mathbb{R}^n_0$ let $\mathbb{R}^n_+ \cup_h \mathbb{R}^n_-$ denote the quotient space $(\mathbb{R}^n_+ \sqcup \mathbb{R}^n_-)/\sim$ where $\sim$ is the equivalence relation which identifies each point $(x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^n_+$ with $h(x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^n_-$. Show that $\mathbb{R}^n_+ \cup_h \mathbb{R}^n_-$ is homeomorphic to $\mathbb{R}^n$.