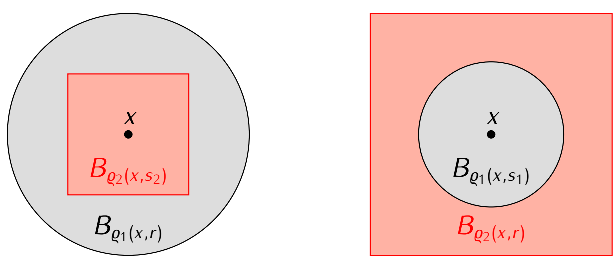
3 | Open Sets

We have seen that by equipping sets X, Y with metrics we can specify what it means that a function $f\colon X \to Y$ is continuous. In general continuity of functions depends on the choice of metrics: if we have two different metrics on X (or on Y) then a function $f\colon X \to Y$ that is continuous with respect to one of these metrics may be not continuous with respect to the other. This is however not always the case. Our first goal in this chapter will be to show that if two metrics on X or Y are *equivalent* then functions continuous with respect to one of them are continuous with respect to the other and vice versa.

3.1 Definition. Let ϱ_1 and ϱ_2 be two metrics on the same set X. We say that the metrics ϱ_1 and ϱ_2 are *equivalent* if for every $x \in X$ and for every $r > 0$ there exist $s_1, s_2 > 0$ such that $B_{\varrho_1}(x, s_1) \subseteq B_{\varrho_2}(x, r)$ and $B_{\varrho_2}(x, s_2) \subseteq B_{\varrho_1}(x, r)$.



3.2 Proposition. *Let ϱ_1, ϱ_2 be equivalent metrics on a set X, and let μ_1, μ_2 be equivalent metrics on a set Y. A function $f\colon X \to Y$ is continuous with respect to the metrics ϱ_1 and μ_1 if and only if it is continuous with respect to the metrics ϱ_2 and μ_2.*

Proof. Assume that f is continuous with respect to ϱ_1 and μ_1. We will show that it is also continuous with respect to ϱ_2 and μ_2 (the argument in the other direction is the same). Let $x \in X$ and let $\varepsilon > 0$. We need to show that there is $\delta > 0$ such that $B_{\varrho_2}(x, \delta) \subseteq f^{-1}(B_{\mu_2}(f(x), \varepsilon)))$. Since the metrics μ_1 and μ_2 are equivalent there exists $\varepsilon_1 > 0$ such that $B_{\mu_1}(f(x), \varepsilon_1)) \subseteq B_{\mu_2}(f(x), \varepsilon))$, and so $f^{-1}(B_{\mu_1}(f(x), \varepsilon_1))) \subseteq f^{-1}(B_{\mu_2}(f(x), \varepsilon)))$. Also, since by assumption f is continuous with respect to ϱ_1

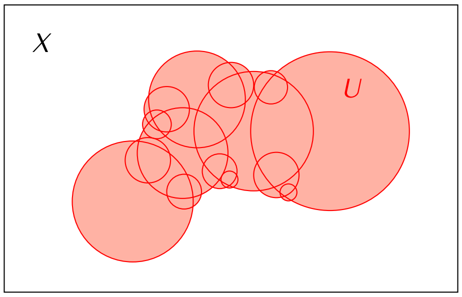
and μ_1, there is δ_1 such that $B_{\varrho_1}(x, \delta_1) \subseteq f^{-1}(B_{\mu_1}(f(x), \varepsilon_1)))$. Finally, using equivalence of metrics ϱ_1 and ϱ_2 we obtain that there exists $\delta > 0$ such that $B_{\varrho_2}(x, \delta) \subseteq B_{\varrho_1}(x, \delta_1)$. Combining these inclusions we get $B_{\varrho_2}(x, \delta) \subseteq f^{-1}(B_{\mu_2}(f(x), \varepsilon)))$. $\square$

3.3 Example. The Euclidean metric d, the orthogonal metric ϱ_{ort} and the maximum metric ϱ_{max} are equivalent metrics on $\mathbb{R}^n$ (exercise).

3.4 Example. The following metrics on $\mathbb{R}^2$ are not equivalent to one another: the Euclidean metric d, the hub metric ϱ_h, and the discrete metric ϱ_{disc} (exercise).

Every metric defines open balls, but even if metrics are equivalent their open balls may look very differently (compare e.g. open balls in $\mathbb{R}^2$ taken with respect to d and ϱ_{ort}). It turns out, however, that each metric defines also a collection of so-called *open sets*, and that open sets defined by two metrics are the same precisely when these metrics are equivalent.

3.5 Definition. Let (X, ϱ) be a metric space. A subset $U \subseteq X$ is an *open set* if U is a union of (perhaps infinitely many) open balls in X: $U = \bigcup_{i \in I} B(x_i, r_i)$.



3.6 Proposition. *Let (X, ϱ) be a metric space and let $U \subseteq X$. The following conditions are equivalent:*

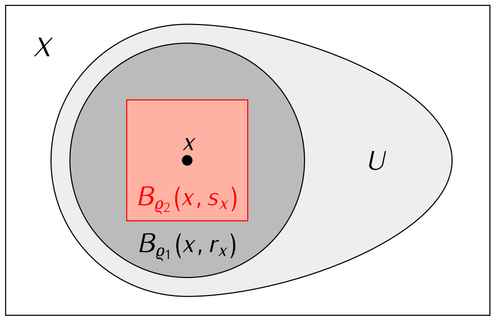
1) *The set U is open.*
2) *For every $x \in U$ there exists $r_x > 0$ such that $B(x, r_x) \subseteq U$.*

Proof. Exercise. $\square$

3.7 Proposition. *Let X be a set and let ϱ_1, ϱ_2 be two metrics on X. The following conditions are equivalent:*

1) *The metrics ϱ_1 and ϱ_2 are equivalent.*
2) *A set $U \subseteq X$ is open with respect to the metric ϱ_1 if and only if it is open with respect to the metric ϱ_2.*

Proof. 1) $\Rightarrow$ 2) Assume that ϱ_1 and ϱ_2 are equivalent and that the set U is open with respect to ϱ_1. By Proposition 3.6 this means that for every $x \in U$ there exists $r_x > 0$ such that $B_{\varrho_1}(x, r_x) \subseteq U$. Since the metric ϱ_1 is equivalent to ϱ_2 we can find $s_x > 0$ such that $B_{\varrho_2}(x, s_x) \subseteq B_{\varrho_1}(x, r_x)$. As a consequence for every $x \in U$ we have $B(x, s_x) \subseteq U$.



Using Proposition 3.6 again we get that the set U is open with respect to ϱ_2. By the same argument we obtain that if U is open with respect to ϱ_2 then it is open with respect to ϱ_1.

2) $\Rightarrow$ 1) Exercise.

□

Here are some basic properties of open sets in metric spaces:

3.8 Proposition. *Let (X, ϱ) be a metric space.*

1) *The sets X and $\varnothing$ are open sets.*
2) *If U_i is an open set for $i \in I$ then the set $\bigcup_{i \in I} U_i$ is open.*
3) *If U_1, U_2 are open sets then the set $U_1 \cap U_2$ is open.*

Proof. Exercise. □

3.9 Note. From part 3) of Proposition 3.8 is follows that if $\{U_1, \dots, U_n\}$ is a finite family of open sets then $U_1 \cap \dots \cap U_n$ is open. However, if $\{U_i\}_{i \in I}$ is an infinite family of open sets then in general the set $\bigcap_{i \in I} U_i$ need not be open (exercise).

Our original definition of a continuous function between metric spaces stated that continuous functions behave well with respect to open balls. The next proposition says that in order to check if a function is continuous it is enough to know how it behaves with respect to open sets:

3.10 Proposition. *Let (X, ϱ), (Y, μ) be metric spaces and let $f \colon X \to Y$ be a function. The following conditions are equivalent:*

1) *The function f is continuous.*
2) *For every open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X.*

The proof of Proposition 3.10 will use the following fact:

3.11 Lemma. *Let (X, ϱ), (Y, μ) be metric spaces and let $f\colon X \to Y$ be a continuous function. If $B := B(y_0, r)$ is an open ball in Y then the set $f^{-1}(B)$ is open in X.*

Proof. Exercise. □

Proof of Proposition 3.10. 1) ⇒ 2) Assume that $f\colon X \to Y$ is a continuous function and that $U \subseteq Y$ is an open set. By definition this means that U is a union of some collection of open balls in Y:

$$U = \bigcup_{i \in I} B_\mu(y_i, r_i)$$

This gives:

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i \in I} B_\mu(y_i, r_i)\right) = \bigcup_{i \in I} f^{-1}(B_\mu(y_i, r_i))$$

Since by Lemma 3.11 each of the sets $f^{-1}(B_\mu(y_i, r_i)$ is open in X and by Proposition 3.8 a union of open sets is open we obtain that the set $f^{-1}(U)$ is open in X.

2) ⇒ 1) Assume that $f^{-1}(U)$ is open in X for every open set $U \subseteq Y$. Given $x \in X$ and $\varepsilon > 0$ take $U = B_\mu(f(x), \varepsilon)$. By assumption the set $f^{-1}(B_\mu(f(x), \varepsilon)) \subseteq X$ is open. Since $x \in f^{-1}(B_\mu(f(x), \varepsilon))$ this implies that there exists $\delta > 0$ such that $B_\varrho(x, \delta) \subseteq f^{-1}(B_\mu(f(x), \varepsilon))$. This shows that f is a continuous function. □

Recall that we introduced metric spaces in order to be able to define continuity of functions. Proposition 3.10 says however that to define continuity we don't really need to use metrics, it is enough to know which sets are open. This observation leads to the following generalization of the notion of a metric space:

3.12 Definition. Let X be a set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X satisfying the following conditions:

1) $X, \varnothing \in \mathcal{T}$;
2) If $U_i \in \mathcal{T}$ for $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$;
3) If $U_1, U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called *open sets*.

A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

In the setting of topological spaces we can define continuous functions as follows:

3.13 Definition. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f\colon X \to Y$ is *continuous* if for every $U \in \mathcal{T}_Y$ we have $f^{-1}(U) \in \mathcal{T}_X$.

3.14 Example. If (X, ϱ) is a metric space then X is a topological space with the topology

$$\mathcal{T} = \{U \subseteq X \mid U \text{ is a union of open balls}\}$$

We say that the topology $\mathcal{T}$ is *induced by the metric* ϱ.

3.15 Note. From now on, unless indicated otherwise, we will consider $\mathbb{R}^n$ as a topological space with the topology induced by the Euclidean metric.

3.16 Example. Let X be an arbitrary set and let

$$\mathcal{T} = \{\text{all subsets of } X\}$$

The topology $\mathcal{T}$ is called the *discrete topology* on X. If X is equipped with this topology then we say that it is a *discrete topological space*.

Note that the discrete topology is induced by the discrete metric ϱ_{disc} on X. Indeed, for $x \in X$ we have

$$B_{\varrho_{disc}}\left(x, \tfrac{1}{2}\right) = \{x\}$$

so for any subset $U \subseteq X$ we get

$$U = \bigcup_{x \in U} B_{\varrho_{disc}}\left(x, \tfrac{1}{2}\right)$$

3.17 Example. Let X be an arbitrary set and let

$$\mathcal{T} = \{X, \varnothing\}$$

The topology $\mathcal{T}$ is called the *antidiscrete topology* on X.

3.18 Example. Let $X = \mathbb{R}$ and let

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \smallsetminus S) \text{ for some finite set } S \subseteq \mathbb{R}\}$$

The topology $\mathcal{T}$ is called the *Zariski topology* on $\mathbb{R}$.

One can ask whether for every topological space $(X, \mathcal{T})$ we can find a metric ϱ on X such that the topology $\mathcal{T}$ is induced by ϱ. Our next goal is to show that this is not the case: some topologies do not come from any metric. Thus, the notion of a topological space is more general than that of a metric space.

3.19 Definition. A topological space $(X, \mathcal{T})$ is *metrizable* if there exists a metric ϱ on X such that $\mathcal{T}$ is the topology induced by ϱ.

3.20 Lemma. *If $(X, \mathcal{T})$ is a metrizable topological space and $x, y \in X$ are points such that $x \neq y$ then there exists an open set $U \subseteq X$ such that $x \in U$ and $y \notin U$.*

Proof. Exercise. ☐

3.21 Proposition. *If X is a set containing more than one point then the antidiscrete topology on X is not metrizable.*

Proof. This follows directly from Lemma 3.20. ☐

Exercises to Chapter 3

E3.1 Exercise. Verify the statement of Example 3.3.

E3.2 Exercise. Verify the statement of Example 3.4.

E3.3 Exercise. The goal of this exercise is to show that the converse of Proposition 3.2 is also true. Let X be a set and let ϱ_1, ϱ_2 be two metrics on X.

a) Assume that for each metric space (Y, μ) and for each function $f\colon X \to Y$ the function f is continuous with respect to ϱ_1 and μ if and only if it is continuous respect to ϱ_2 and μ. Show that ϱ_1 and ϱ_2 must be equivalent metrics.

b) Assume that for each metric space (Y, μ) and for each function $g\colon Y \to X$ the function g is continuous with respect to μ and ϱ_1 if and only if it is continuous respect to μ and ϱ_2. Show that ϱ_1 and ϱ_2 must be equivalent metrics.

E3.4 Exercise. Prove Proposition 3.6.

E3.5 Exercise. Consider the set $\mathbb{R}^2$ with the Euclidean metric.

a) Show that the open half plane $H = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ is an open set in $\mathbb{R}^2$

b) Show that the closed half plane $\overline{H} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$ is not an open set in $\mathbb{R}^2$

E3.6 Exercise. Consider the set $\mathbb{R}^2$ with the hub metric ϱ_h. Show that the set

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq -1\}$$

is an open set in $\mathbb{R}^2$

E3.7 Exercise. Consider the set $\mathbb{R}$ with the Euclidean metric. Show that every open set in $\mathbb{R}$ is a *disjoint* union of open intervals (a, b) (where possibly $a = -\infty$ or $b = +\infty$).

E3.8 Exercise. Prove the implication 2) $\Rightarrow$ 1) of Proposition 3.7.

E3.9 Exercise. Prove Proposition 3.8.

E3.10 Exercise. Consider the set $\mathbb{R}^2$ with the Euclidean metric. Give an example of open sets $U_1, U_2, \dots,$ in $\mathbb{R}^2$ such that the set $\bigcap_{n=1}^{\infty} U_i$ is not open.

E3.11 Exercise. Prove Lemma 3.11.

E3.12 Exercise. Prove Lemma 3.20.

E3.13 Exercise. Show that the set $\mathbb{R}$ with the Zariski topology is not metrizable.

E3.14 Exercise. Let X be a topological space consisting of a finite number of points. Show that if X is metrizable then it is a discrete space.

E3.15 Exercise. Let $\mathbb{R}_{Eu}$ denote the set $\mathbb{R}$ with the Euclidean topology and $\mathbb{R}_{Za}$ the set $\mathbb{R}$ with the Zariski topology. Show that every continuous function $f\colon \mathbb{R}_{Za} \to \mathbb{R}_{Eu}$ must be constant.