22 | Mapping Spaces

22.1 Definition. Let X, Y be topological spaces. By Map(X, Y) we will denote the set of all continuous functions $f: X \to Y$.

- **22.2 Definition.** Let X, Y be a topological spaces, and let \mathcal{T} be a topology on Map(X, Y).
 - 1) We will say that the topology $\mathfrak T$ is *lower admissible* if for any continuous function $F: Z \times X \to Y$ the function $F_*: Z \to \operatorname{Map}(X,Y)$ is continuous.
 - 2) We will say that the topology $\mathfrak T$ is *upper admissible* if for any function $F: Z \times X \to Y$ if the function $F_*: Z \to \mathsf{Map}(X,Y)$ is continuous then F is continuous.
 - 3) We will say that the topology T is *admissible* if it is both lower and upper admissible.

22.3 Definition. Let X, Y be topological spaces. The *evaluation map* is the function

ev:
$$Map(X, Y) \times X \rightarrow Y$$

given by ev((f, x)) = f(x).

- **22.4 Lemma.** Let X, Y be topological spaces, and let $\mathfrak T$ be a topology on $\mathsf{Map}(X, Y)$. The following conditions are equivalent:
 - 1) The topology ${\mathfrak T}$ is upper admissible.
 - 2) The evaluation map ev: $Map(X, Y) \times X \rightarrow Y$ is continuous.

22.6 Proposition. Let X, Y be topological spaces. 1) If \(\frac{1}{2} \) are two topologics on \(\text{Man}(X, Y) \) such that \(\frac{1}{2} \) and \(\frac{1}{2} \) is upper admissible, then \(\frac{1}{2} \).
1) If \mathcal{U} , \mathcal{U}' are two topologies on Map(X , Y) such that $\mathcal{U} \subseteq \mathcal{U}'$ and \mathcal{U} is upper admissible, then \mathcal{U}' also is upper admissible.
2) If \mathcal{L} , \mathcal{L}' are two topologies on Map(X , Y) such that $\mathcal{L}' \subseteq \mathcal{L}$ and \mathcal{L} is lower admissible, then \mathcal{L}' also is lower admissible.
3) If \mathcal{U} , \mathcal{L} are two topologies on $Map(X,Y)$ such that \mathcal{U} is upper admissible and \mathcal{L} is lower admissible then $\mathcal{L} \subseteq \mathcal{U}$.
22.7 Corollary. Given spaces X and Y , if there exists an admissible topology on $Map(X, Y)$ then such topology is unique.
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22.8 Proposition.	Let X	be	completely	regular	space.	lf	there	exist	an	admissible	topology	on
$Map(X, \mathbb{R})$ then X	is local	lly d	compact.									

22.10 Definition. Let X, Y be topological spaces. For sets $A \subseteq X$ and $B \subseteq Y$ denote

$$P(A, B) = \{ f \in \mathsf{Map}(X, Y) \mid f(A) \subseteq B \}$$

22.11 Lemma. Let X, Y topological spaces, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Let \mathcal{T} be a topology on $\mathsf{Map}(X,Y)$ with subbasis given by all sets P(A,V) where $A \subseteq X$ is a closed set such that $A \subseteq U_i$ for some $i \in I$, and $V \subseteq Y$ is an open set. If X is a regular space then \mathcal{T} upper admissible.

Proof. Exercise.

Proof of Proposition 22.8.

22.12 Definition. Let X, Y be topological spaces. The *compact-open* topology on Map(X, Y) is the topology defined by the subbasis consisting of all sets P(A, U) where $A \subseteq X$ is compact and $U \subseteq Y$ is an open set.

22.13 Theorem. For any spaces X, Y the compact-open topology on Map(X, Y) is lower admissible.

2.14 Theorem. Let X, Y be topological spaces. If X is locally compact Hausdorff space ompact-open topology on $Map(X, Y)$ is upper admisible.	then the
22.15 Corollary. If X is a locally compact Hausdorff space and Y is any space then the compopology on Map(X,Y) is admissible.	act-open

22.17 Proposition. Let X be a top space. There exists a homeomorph	ological space, and let S be a set considered as a discrete to hism	pological
	$Map(S,X) \cong \prod_{s \in S} X$	
where $Map(S, X)$ is taken with th	e compact-open topology, and $\prod_{s\in S}X$ with the product to	ppology.
Proof. Exercise.		
22.19 Proposition. Let X be a $G, g \in Map(X, Y)$ define	compact Hausdorff space, and let (Υ, ϱ) be a metric sp	ace. For
	$I(f,g) = \max\{\varrho(f(x),g(x)) \mid x \in X\}$	
Then d is a metric on $Map(X, Y)$. sopology.	Moreover, in the topology induced by this metric is the comp	oact-open
Proof. Exercise.		

22.20 Theorem.	Let X, Y, Z	be topological	spaces.	Let
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$$\Phi \colon \mathsf{Map}(X,Y) \times \mathsf{Map}(Y,Z) \to \mathsf{Map}(X,Z)$$

be a function given by $\Phi(f,g) = g \circ f$. If Y is a locally compact Hausdorff space, and all mapping spaces are equipped with the compact-open topology then Φ is continuous.

22.21 Lemma. Let X be a locally compact Hausdorff space, and let $A, W \subseteq X$ be sets such that A is compact, W is open, and $A \subseteq W$. Then there exists an open set $V \subseteq X$ such that $A \subseteq V$, $\overline{V} \subseteq W$, and \overline{V} is compact.

Proof.	Exercise.	

Proof of Theorem 22.20.