# 23 | Deck Transformations

**23.1 Definition.** Let $p\colon T \to X$ be a covering. A *deck transformation* of $p$ is an isomorphism of coverings

$$\begin{tikzcd} T \arrow[rr, "f", "\cong"'] \arrow[dr, "p"'] & & T \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Deck transformations form a group under composition of isomorphisms. We will denote this group by $D(p)$. In this chapter we will compute the group $D(p)$ for a path connected covering $p$ in terms of fundamental groups of $X$ and $T$. Recall that in Chapter 22 we constructed a functor

$$\Lambda\colon \mathbf{PCov}(X) \to \mathbf{TSet}_{\pi_1(X,x_0)}$$

from the category of path connected coverings of a space $X$ to the category of transitive $\pi_1(X, x_0)$-sets. We also showed (22.15) that if $X$ is a connected and locally path connected space then this functor is a bijection of sets of morphisms. Since any functor preserves isomorphism, if $f : T_1 \to T_2$ is an isomorphism of coverings of $X$, then $\Lambda(f)$ is an isomorphism of $\pi_1(X, x_0)$-sets. The following fact implies that the converse is also true:

**23.2 Lemma.** *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor such that for any $c, c' \in \mathbf{C}$ the map the map* $\mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$ *given by $f \mapsto F(f)$ is a bijection. A morphism $f\colon c \to c'$ i in $\mathbf{C}$ is an isomorphism if and only if $F(f)\colon F(c) \to F(c')$ is an isomorphism.*

*Proof.* Exercise. □

As a consequence we obtain:

**23.3 Corollary.** *Let $X$ be a connected and locally path connected space, $x_0 \in X$, and let $p\colon T \to X$ be a path connected covering. The group of deck transformations $D(p)$ is isomorphic to the group of $\pi_1(X, x_0)$-equivariant isomorphisms $p^{-1}(x_0) \to p^{-1}(x_0)$.*

*Proof.* Exercise. □

In view of Corollary 23.3 the problem of computing the group of deck transformations reduces to the problem of computing the group of $G$-equivariant isomorphisms of a $G$-set $S$. Denote this group by $\mathrm{Iso}_G(S)$.

**23.4 Definition.** Let $G$ be a group, and let $H \subseteq G$ be a subgroup. The *normalizer* of $H$ in $G$ is the subgroup $N_G(H) \subseteq G$ defined by

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

**23.5 Note.** $N_G(H)$ is the largest subgroup of $G$ that contains $H$ as its normal subgroup. In particular $H$ is a normal subgroup of $G$ if and only if $N_G(H) = G$.

Recall that if $S$ is a $G$-set and $s \in S$ then by $G_s$ we denote the stabilizer of $s$.

**23.6 Proposition.** *Let $G$ be a group, and let $S$ is a transitive $G$-set. For any $s \in S$ there exists an isomorphism of groups*

$$\mathrm{Iso}_G(S) \cong N_G(G_s)/G_s$$

*Proof.* Let $f\colon S \to S$ be a $G$-equivariant isomorphism. Since the action of $G$ on $S$ is transitive we we have $f(s) = sg_f$ for some $g_f \in G$ (depending on $s$). We claim that $g_f \in N_G(G_s)$. Indeed, for any $h \in G_s$ we have

$$s(g_f h g_f^{-1}) = f(s)(hg_f^{-1}) = f(sh)g_f^{-1} = f(s)g_f^{-1} = s(g_f g_f^{-1}) = s$$

which shows that $g_f h g_f^{-1} \in G_s$.

Define a map

$$\varphi\colon \mathrm{Iso}_G(S) \to N_G(G_s)/G_s$$

by $\varphi(f) := g_f G_s$. To verify that $\varphi$ is well defined we need to check that if $\bar{g}_f \in G$ is another element such that $f(s) = s\bar{g}_f$ then $g_f G_s = \bar{g}_f G_s$. Since $sg_f = f(s) = s\bar{g}_f$ we get $s = s\bar{g}_f g_f^{-1}$ which gives $\bar{g}_f g_f^{-1} \in G_s$. By the observation above $g_f \in N_G(G_s)$, so $(\bar{g}_f g_f^{-1})g_f = g_f h$ for some $h \in G_s$. This gives:

$$\bar{g}_f G_s = \bar{g}_f g_f^{-1} g_f G_s = g_f h G_s = g_f G_s$$

Next, we claim that $\varphi$ is a group homomorphism. Indeed, if $f, f' \in \mathrm{Iso}_G(S)$, $f(s) = sg_f$, $f'(s) = sg_{f'}$ then

$$f' \circ f(s) = f'(sg_f) = f'(s)g_f = sg_{f'}g_f$$

and so $\varphi(f' \circ f) = (g_{f'}g_f)G_s = \varphi(f') \cdot \varphi(f)$. It remains to show that $\varphi$ is an isomorphism (exercise). □

**23.7 Proposition.** *Let $X$ be a connected and locally path connected space, and let $x_0 \in X$. For a path connected covering $p\colon T \to X$ and $\tilde{x} \in p^{-1}(x_0)$ there exists an isomorphism of groups:*

$$D(p) \cong N_{\pi_1(X,x_0)}(p_*(\pi_1(T,\tilde{x})))/p_*(\pi_1(T,\tilde{x}))$$

**23.8 Note.** Recall that a covering $p\colon T \to X$ is regular if $p_*(\pi_1(T,\tilde{x}))$ is a normal subgroup of $\pi_1(X,x_0)$. In such case the isomorphism in Proposition 23.7 gives

$$D(p) \cong \pi_1(X,x_0)/p_*(\pi_1(T,\tilde{x}))$$

In particular, for the universal covering $\tilde{p}\colon \widetilde{X} \to X$ we obtain $D(\tilde{p}) \cong \pi_1(X,x_0)$.

## Exercises to Chapter 23

**E23.1 Exercise.** For a function $f\colon X \to X$ by $\mathrm{Fix}(f)$ we will denote the set of fixed points of $f$:

$$\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$$

Let $X$ be a connected and locally path connected space, let $\widetilde{p}\colon \widetilde{X} \to X$ be the universal covering of $X$, and let $f\colon X \to X$ be a map. We will say that a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ is a lift of $f$ if the following diagram commutes:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{p}"'] & \widetilde{X} \arrow[d, "\widetilde{p}"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

Let $S$ denote the set of all lifts of $f$.

a) Show that $\mathrm{Fix}(f) = \bigcup_{\widetilde{f}\in S} \widetilde{p}(\mathrm{Fix}(\widetilde{f}))$.

b) Let $\widetilde{f}_1, \widetilde{f}_2 \in S$. Show that the following conditions are equivalent:

(i) $\widetilde{p}(\mathrm{Fix}(\widetilde{f}_1)) \cap \widetilde{p}(\mathrm{Fix}(\widetilde{f}_2)) \neq \varnothing$

(ii) There exists a deck transformation $g\colon \widetilde{X} \to \widetilde{X}$ such that $\widetilde{f}_2 = g\widetilde{f}_1 g^{-1}$

(iii) $\widetilde{p}(\mathrm{Fix}(\widetilde{f}_1)) = \widetilde{p}(\mathrm{Fix}(\widetilde{f}_2))$

c) Let $f\colon (S^1, x_0) \to (S^1, x_0)$ be a map such that the homomorphism $f_*\colon \pi_1(S^1,x_0) \to \pi_1(S^1,x_0)$ is given by $f_*([\omega]) = n\cdot[\omega]$ for some $n \in \mathbb{Z}$. Show that $\mathrm{Fix}(f)$ consists of at least $|n-1|$ points.