

5 | First Computations

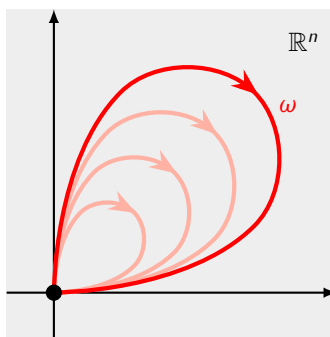
In this chapter we describe some basic examples of computations of the fundamental group. Later on we will see that these examples and a few additional tools let us calculate fundamental groups of many spaces.

5.1 Proposition. *If $X = \{*\}$ is a space consisting of only one point then $\pi_1(X)$ is the trivial group.*

Proof. It is enough to notice that the only loop in X is the constant loop. □

5.2 Proposition. *For any $n \geq 1$ the group $\pi_1(\mathbb{R}^n)$ is trivial.*

Proof. Choose $0 \in \mathbb{R}^n$ as the basepoint. Let $\omega: [0, 1] \rightarrow \mathbb{R}^n$ be a loop based at 0. We need to show that ω is homotopic to the constant loop c_0 . Such homotopy $h: [0, 1] \times [0, 1] \rightarrow \mathbb{R}^n$ is given by $h(s, t) = t \cdot \omega(s)$.



□

Let $D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$ be the n -dimensional closed unit disc. Using the the same argument as above we obtain:

5.3 Proposition. For any $n \geq 1$ the group $\pi_1(D^n)$ is trivial.

5.4 Note. As we have seen before homeomorphic spaces have isomorphic fundamental groups (3.17). The above calculations show that the converse is not true, e.g. $\mathbb{R}^n \not\cong D^n$ for $n \geq 1$ but $\pi_1(\mathbb{R}^n) \cong \pi_1(D^n)$.

5.5 Definition. A space X is *simply connected* if it is path connected and $\pi_1(X)$ is trivial.

For example $\{*\}$, \mathbb{R}^n , and D^n are simply connected spaces.

5.6 Proposition. A space X is simply connected if and only if X is path connected and for any two paths $\omega, \tau: [0, 1] \rightarrow X$ satisfying $\omega(0) = \tau(0)$ and $\omega(1) = \tau(1)$ we have $\omega \simeq \tau$.

Proof. Exercise. □

Our next goal will be to show that the fundamental group is not always trivial:

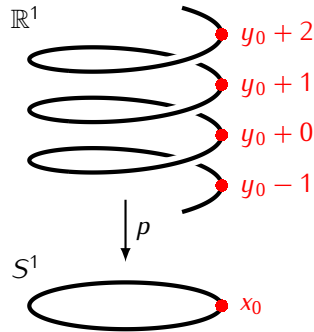
5.7 Theorem. $\pi_1(S^1) \cong \mathbb{Z}$.

The proof of Theorem 5.7 will require some technical preparation.

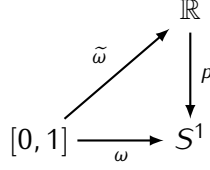
5.8 Definition. The *universal covering* of S^1 is the map $p: \mathbb{R}^1 \rightarrow S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$.

Geometrically p is the map that wraps \mathbb{R}^1 infinitely many times around the circle.

5.9 Note. For $y, y' \in \mathbb{R}$ we have $p(y) = p(y')$ if and only if $y' = y + n$ for some $n \in \mathbb{Z}$. As a consequence if $x_0 \in S^1$ and if $y_0 \in \mathbb{R}$ is a point such that $p(y_0) = x_0$ then $p^{-1}(x_0) = \{y_0 + n \mid n \in \mathbb{Z}\}$.

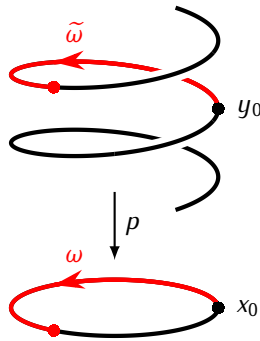


5.10 Definition. Let ω be a path in S^1 . We say that a path $\tilde{\omega}$ in \mathbb{R} is a *lift* of ω if $p \circ \tilde{\omega} = \omega$.



5.11 Proposition. Let $p: \mathbb{R}^1 \rightarrow S^1$ be the universal covering of S^1 , let $x_0 \in S^1$, and let $y_0 \in \mathbb{R}^1$ be a point such that $p(y_0) = x_0$.

1) For any path $\omega: [0, 1] \rightarrow S^1$ such that $\omega(0) = x_0$ there exists a lift $\tilde{\omega}: [0, 1] \rightarrow \mathbb{R}^1$ satisfying $\tilde{\omega}(0) = y_0$. Moreover, such lift is unique.



2) Let $\omega, \tau: [0, 1] \rightarrow S^1$ be paths such that $\omega(0) = \tau(0) = x_0$, $\omega(1) = \tau(1)$ and $\omega \simeq \tau$. If $\tilde{\omega}, \tilde{\tau}$ are lifts of ω, τ , respectively, such that $\tilde{\omega}(0) = \tilde{\tau}(0) = y_0$ then $\tilde{\omega}(1) = \tilde{\tau}(1)$ and $\tilde{\omega} \simeq \tilde{\tau}$.

We postpone the proof of Proposition 5.11 for now. We will get back to it in Chapter 17 where we will show that it is a special case of a more general statement. Meanwhile we will show how it can be used to obtain Theorem 5.7.

5.12 Definition. Let $x_0 \in S^1$ and $y_0 \in \mathbb{R}$ be points such that $p(y_0) = x_0$. Let ω be a loop in S^1 based at x_0 and let $\tilde{\omega}$ be the unique lift of ω such that $\tilde{\omega}(0) = y_0$. The *degree* of ω is the integer $\deg(\omega)$ such that $\tilde{\omega}(1) = y_0 + \deg(\omega)$.

In other words $\deg(\omega) = \tilde{\omega}(1) - \tilde{\omega}(0)$.

5.13 Note. Notice that $\deg(\omega)$ does not depend on the choice of the point y_0 , i.e. it does not depend on the choice of the lift of ω . Indeed, if $y'_0 \in \mathbb{R}$ is another point satisfying $p(y'_0) = x_0$ then $y'_0 = y_0 + n$

for some $n \in \mathbb{Z}$. Also, if $\tilde{\omega}$ is the lift of a loop ω with $\tilde{\omega}(0) = y_0$ then the lift $\tilde{\omega}'$ of ω with $\tilde{\omega}'(0) = y'_0$ is given by $\tilde{\omega}'(s) = \tilde{\omega}(s) + n$. This gives

$$\tilde{\omega}'(1) - \tilde{\omega}'(0) = (\tilde{\omega}(1) + n) - (\tilde{\omega}(0) + n) = \tilde{\omega}(1) - \tilde{\omega}(0)$$

In addition, by part 2) of Proposition 5.11 $\deg(\omega)$ depends only on the homotopy class of ω , and thus we obtain a well-defined function

$$\deg: \pi_1(S^1, x_0) \rightarrow \mathbb{Z}$$

Proof of Theorem 5.7. Let $x_0 \in S^1$. We will show that the function $\deg: \pi_1(S^1, x_0) \rightarrow \mathbb{Z}$ is an isomorphism of groups.

First, we will show that \deg is onto. Let $y_0 \in p^{-1}(x_0)$. Given $n \in \mathbb{Z}$ consider the path $\tilde{\omega}_n: [0, 1] \rightarrow \mathbb{R}$ given by $\tilde{\omega}_n(s) = y_0 + ns$ and let $\omega_n = p \circ \tilde{\omega}_n$. Since $\tilde{\omega}_n$ is the lift of ω_n such that $\tilde{\omega}_n(0) = y_0$ and since $\tilde{\omega}_n(1) = y_0 + n$ we obtain $\deg[\omega_n] = n$.

Next, we will check that \deg is a 1-1 function. Let $[\omega], [\tau] \in \pi_1(S^1, x_0)$ be elements such that $\deg[\omega] = \deg[\tau]$. We need to show that $[\omega] = [\tau]$. Let $\tilde{\omega}, \tilde{\tau}$ be the lifts of ω, τ , respectively, such that $\tilde{\omega}(0) = \tilde{\tau}(0) = y_0$. By assumption we get

$$\tilde{\omega}(1) = y_0 + \deg[\omega] = y_0 + \deg[\tau] = \tilde{\tau}(1)$$

Since \mathbb{R} is a simply connected space using Proposition 5.6 we obtain that $\tilde{\omega} \simeq \tilde{\tau}$. Therefore

$$\omega = p \circ \tilde{\omega} \simeq p \circ \tilde{\tau} = \tau$$

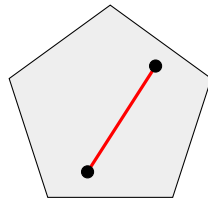
which gives $[\omega] = [\tau]$.

It remains to show that \deg is a homomorphism of groups (exercise). □

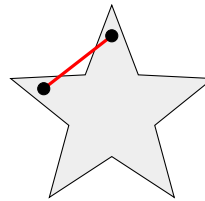
Exercises to Chapter 5

E5.1 Exercise. Prove Proposition 5.6.

E5.2 Exercise. a) A subspace $X \subseteq \mathbb{R}^n$ is *convex* if for any points $x_1, x_2 \in X$ the straight line segment joining these points is contained in X :



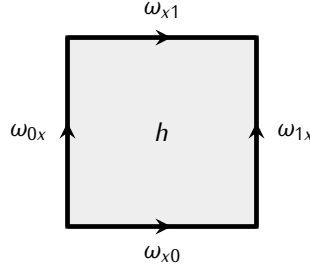
convex



not convex

Show that if X is convex then it is simply connected.

b) Let Y be a topological space and let $h: [0, 1] \times [0, 1] \rightarrow Y$ be a continuous function. Consider paths $\omega_{x0}, \omega_{x1}, \omega_{0x}, \omega_{1x}$ in Y which are defined by restricting h to the four edges of the square $[0, 1] \times [0, 1]$: $\omega_{x0}(s) = h(s, 0)$, $\omega_{x1}(s) = h(s, 1)$, $\omega_{0x}(s) = h(0, s)$ and $\omega_{1x}(s) = h(1, s)$.



Show that the path $\omega_{x0} * \omega_{1x}$ is path homotopic to $\omega_{0x} * \omega_{x1}$.

E5.3 Exercise. Let S^n be the n -dimensional sphere:

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

Show that S^n is simply connected. (Hint: show that any path $\omega: [0, 1] \rightarrow S^n$ is path homotopic to a path which is not onto.)

E5.4 Exercise. Recall that the Peano curve is a continuous function $\tau: [0, 1] \rightarrow [0, 1] \times [0, 1]$ which is onto, and such that $\tau(0) = (0, 0)$ and $\tau(1) = (1, 1)$. Let $\text{pr}_1: [0, 1] \times [0, 1] \rightarrow [0, 1]$ be the projection onto the first factor, $\text{pr}_1(s, t) = s$ and let $\omega: [0, 1] \rightarrow S^1$ be standard the degree 1 loop, $\omega(s) = (\cos(2\pi s), \sin(2\pi s))$. The composition $\omega \circ \text{pr}_1 \circ \tau: [0, 1] \rightarrow S^1$ is a loop with the basepoint at $x_0 = (1, 0) \in S^1$. Compute the degree of $\omega \circ \text{pr}_1 \circ \tau$.

E5.5 Exercise. Show that the degree function $\deg: \pi_1(S^1, x_0) \rightarrow \mathbb{Z}$ defined in Note 5.13 is a group homomorphism.

E5.6 Exercise. Let $\omega: [0, 1] \rightarrow S^1$ be a loop based at $x_0 \in S^1$. Assume that there exists a point $x \in S^1$ such that the set $\omega^{-1}(x)$ consists of n points. Show that $-n \leq \deg(\omega) \leq n$.