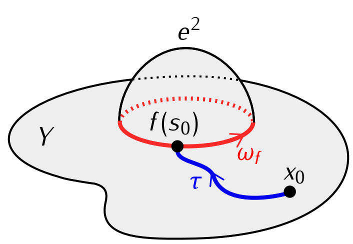
15 | Fundamental Group and 2-Cells

We return to the problem of computing fundamental groups of CW complexes. Building on results of Chapter 13 we will consider here CW complexes of dimension 2. Any such complex X is obtained by glueing some 2-cells to its 1-skeleton $X^{(1)}$. Since $X^{(1)}$ is a CW complex of dimension 1 its fundamental group is described by Theorems 13.11 and 13.13. In effect, we only need to determine how the fundamental group of a space changes if we attach to it 2-cells. We will consider first the case when only one 2-cell is attached. We will use the following setup. Let Y be a space and let $X = Y \cup_f e^2$ where $f\colon S^1 \to Y$ is the attaching map. Let $\omega\colon [0,1] \to S^1$ be the map given by $\omega(s) = (\cos(2\pi s), \sin(2\pi s))$. Since ω is a loop in S^1 based at the point $s_0 = (1,0)$ thus $f\omega$ is a loop in Y based at the point $f(s_0)$. We will denote the loop $f\omega$ by ω_f.

15.1 Theorem. *Let Y be a path connected space and let $X = Y \cup_f e^2$. Let $x_0 \in Y$ and let τ be a path in Y such that $\tau(0) = x_0$ and $\tau(1) = f(s_0)$. The homomorphism*

$$j_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$$

induced by the inclusion map $j\colon Y \to X$ is onto, and so $\pi_1(X, x_0) \cong \pi_1(Y, y_0)/\operatorname{Ker}(j_*)$. Moreover, $\operatorname{Ker}(j_*)$ is the normal subgroup of $\pi_1(Y, x_0)$ generated by the element $[\tau * \omega_f * \overline{\tau}]$.



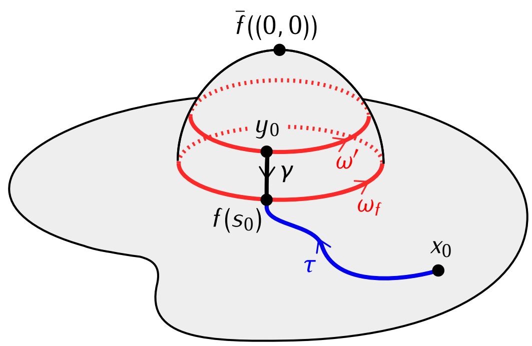
15.2 Note. In general the element $[\tau * \omega_f * \overline{\tau}] \in \pi_1(X, x_0)$ depends on the choice of the path τ. However, as the statement of Theorem 15.1 implies the normal subgroup of $\pi_1(Y, x_0)$ generated by such element does not depend on the choice of τ.

Proof of Theorem 15.1. Let $\bar{f}\colon D^2 \to X$ be the characteristic map of the cell e^2 and let $y_0 = \bar{f}((\frac{1}{2}, 0))$. Let $U_1 := X \smallsetminus \bar{f}((0,0))$ and $U_2 := X \smallsetminus Y$. The sets U_1, U_2, and $U_1 \cap U_2$ are path connected and open in X. Moreover, $U_1 \cup U_2 = X$ and $y_0 \in U_1 \cap U_2$. By van Kampen's Theorem we obtain

$$\pi_1(X, y_0) = \operatorname{colim}(\pi_1(U_1, y_0) \xleftarrow{i_{1*}} \pi_1(U_1 \cap U_2, y_0) \xrightarrow{i_{2*}} \pi_1(U_2, y_0))$$

where for $k = 1, 2$ the map $i_k\colon U_1 \cap U_2 \to U_k$ is the inclusion. The space U_2 is contractible, so $\pi_1(U_2, y_0) \cong \{1\}$. Also, $U_1 \cap U_2 \simeq S^1$, so $\pi_1(U_1 \cap U_2, y_0) \cong \mathbb{Z}$. The generator of $\pi_1(U_1 \cap U_2, y_0)$ is represented by the loop $\omega'\colon [0,1] \to U_1 \cap U_2$ where $\omega'(s) = \bar{f}((\frac{1}{2}\cos(2\pi s), \frac{1}{2}\sin(2\pi s)))$. These observations show that the homomorphism $j_*\colon \pi_1(U_1, y_0) \to \pi_1(X, y_0)$ induced by the inclusion $j\colon U_1 \to X$ is onto, and $\ker j_*$ is the normal subgroup of $\pi_1(U_1, y_0)$ generated by $[\omega']$.

Let $\gamma\colon [0,1] \to X$ be the path defined by $\gamma(s) = \bar{f}(\frac{1}{2}(1+s), 0)$. We have $\gamma(0) = y_0$ and $\gamma(1) = f(s_0)$. This path defines the change-of–basepoint isomorphisms $s_\gamma\colon \pi_1(X, y_0) \to \pi_1(X, f(s_0))$ and $s_\gamma\colon \pi_1(U_1, y_0) \to \pi_1(U_1, f(s_0))$. Notice that in $\pi_1(U_1, f(s_0))$ we have $s_\gamma([\omega']) = [\bar{\gamma} * \omega * \gamma] = [\omega_f]$:



Consider the following diagram:

$$\begin{tikzcd}
\pi_1(U_1, y_0) \arrow[r, "s_\gamma", "\cong"'] \arrow[d, "j_*"'] & \pi_1(U_1, f(s_0)) \arrow[d, "j_*"'] & \pi_1(Y, f(s_0)) \arrow[l, "i_*"', "\cong"] \arrow[r, "s_{\bar{\tau}}", "\cong"'] \arrow[d, "j_*"] & \pi_1(Y, x_0) \arrow[d, "j_*"] \\
\pi_1(X, y_0) \arrow[r, "\cong", "s_\gamma"'] & \pi_1(X, f(s_0)) & \pi_1(X, f(s_0)) \arrow[l, "\cong"', "\mathrm{id}"] \arrow[r, "\cong", "s_{\bar{\tau}}"'] & \pi_1(X, x_0)
\end{tikzcd}$$

Each square in this diagram commutes. The vertical homomorphisms are induced by inclusion maps. The homomorphisms s_γ and $s_{\bar{\tau}}$ are the change-of-basepoint isomorphisms defined by the paths γ and $\bar{\tau}$. The homomorphism $i_*\colon \pi_1(Y, f(s_0)) \to \pi_1(U_1, f(s_0))$ is induced by the inclusion map $Y \to U_1$, and it is an isomorphism since Y is a deformation retract of U_1. Since all horizontal homomorphisms are isomorphisms and $j_*\colon \pi_1(U_1, d_0) \to \pi_1(X, d_0)$ is onto we obtain that $j_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$ is also onto. Moreover, since the kernel of the first of these two homomorphisms is generated by the element $[\omega'] \in \pi_1(U_1, d_0)$ the kernel of the second is generated by the element $s_{\bar{\tau}}(i_*^{-1}(s_\gamma([\omega']))) = s_{\bar{\tau}}(i_*^{-1}([\omega_f])) = s_{\bar{\tau}}([\omega_f]) = [\tau * \omega_f * \bar{\tau}]$.

$\square$

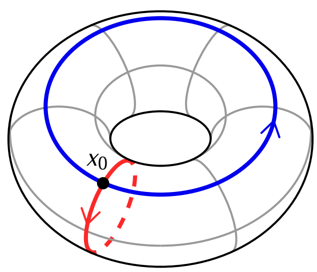
Theorem 15.1 can be generalized to the case when we are attaching an arbitrary (finite or infinite) number of 2-cells. In the statement below we use the same notation as in Theorem 15.1: if f is an attaching map of a 2-cell then by ω_f we denote the loop defined by f and by $f(s_0)$ the starting/ending point of this loop.

15.3 Theorem. *Let Y be a path connected space with basepoint $x_0 \in Y$ and let X be a space obtained by attaching to Y a collection of 2-cells: $X = Y \cup \{e^2_i\}_{i\in I}$. Let $f_i\colon S^1 \to X$ be the attaching map of the cell e^2_i and let $\tau_i\colon [0,1] \to Y$ be a path such that $\tau_i(0) = x_0$ and $\tau_i(1) = f_i(s_0)$. The homomorphism*

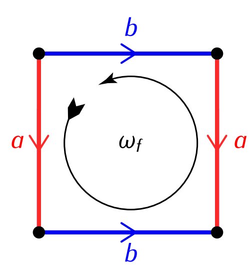
$$j_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$$

induced by the inclusion map $j\colon Y \to X$ is onto, and so $\pi_1(X, x_0) \cong \pi_1(Y, y_0)/\operatorname{Ker}(j_*)$. Moreover, $\operatorname{Ker}(j_*)$ is the normal subgroup of $\pi_1(Y, x_0)$ generated by set $\{[\tau_i * \omega_{f_i} * \overline{\tau}_i]\}_{i\in I}$.

15.4 Example. Recall that the torus T has a CW complex structure with one 0-cell, two 1-cells and one 2-cell:



We will take the 0-cell x_0 as the basepoint of T. The 1-skeleton $T^{(1)}$ of the torus is homeomorphic to $S^1 \vee S^1$, so $\pi_1(T^{(1)}, x_0) \cong \mathbb{Z} * \mathbb{Z}$. Denote by a the element of $\pi_1(T^{(1)}, x_0)$ represented by the loop that goes once around one 1-cell and by b the element represented by the loop that goes once around the second 1-cell. The elements a, b freely generate the group $\pi_1(T^{(1)}, x_0)$, i.e. we have an isomorphism $\pi_1(T^{(1)}, x_0) \cong \langle a, b\rangle$. Let $f\colon S^1 \to T^{(1)}$ be the attaching map of the 2-cell. The loop ω_f represents the element $aba^{-1}b^{-1} \in \pi_1(T^{(1)}, x_0)$:

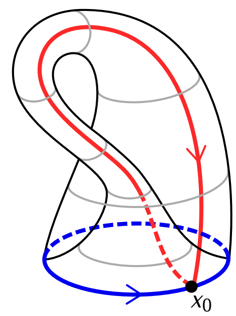


By Theorem 15.1 we get $\pi_1(T, x_0) \cong \langle a, b\rangle/N$ where N is the normal subgroup of $\langle a, b\rangle$ generated by the element $aba^{-1}b^{-1}$. In other words we obtain a presentation of $\pi_1(T, x_0)$:

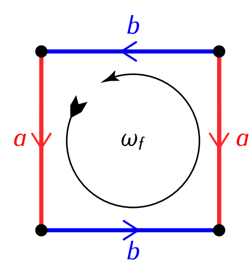
$$\pi_1(T, x_0) \cong \langle a, b \mid aba^{-1}b^{-1}\rangle$$

Since $\langle a, b \mid aba^{-1}b^{-1}\rangle \cong \mathbb{Z} \times \mathbb{Z}$ we recover the result we obtained before (9.2) using the product formula for the fundamental group.

15.5 Example. Recall the CW complex structure on the Klein bottle K:

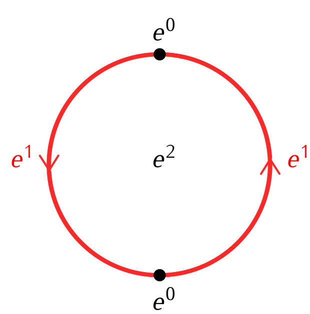


Again, we will take the 0-cell in this complex as the basepoint. The 1-skeleton $K^{(1)}$ is homeomorphic to $S^1 \vee S^1$, so $\pi_1(K^{(1)}, x_0) \cong \langle a, b\rangle$ where a, b are represented by the loops traversing each of the 1-cells. The attaching map of the 2-cell represents the element $aba^{-1}b \in \langle a, b\rangle$:



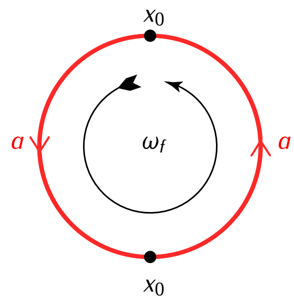
It follows that $\pi_1(K, x_0) \cong \langle a, b \mid aba^{-1}b\rangle$.

15.6 Example. Recall that 2-dimensional projective space $\mathbb{RP}^2$ has a cell structure with one 0-cell, one 1-cell and one 2-cell:



We will take the 0-cell as the basepoint $x_0 \in \mathbb{RP}^2$. The one skeleton $(\mathbb{RP}^2)^{(1)}$ is homeomorphic to S^1, so $\pi_1((\mathbb{RP}^2)^{(1)}, x_0) \cong \langle a\rangle$ where a denotes the generator represented by the loop that traverses the

1-cell. The attaching map $f\colon S^1 \to (\mathbb{RP}^2)^{(1)}$ for the 2-cell corresponds to the element a^2:



As a consequence $\pi_1(\mathbb{RP}^2, x_0) \cong \langle a \mid a^2 \rangle \cong \mathbb{Z}/2$.

15.7 Note. The computations above show right away that the torus T is not homotopy equivalent to $\mathbb{RP}^2$ since $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{RP}^2)$. It may be less clear though whether the fundamental group of the Klein bottle is isomorphic or not to one of these groups. We can resolve this problems as follows. If G is a group and $g, h \in G$ then the *commutator* of g and h is the element $[g, h] = ghg^{-1}h^{-1}$. Notice that $[g, h]$ is the trivial element if and only if $gh = hg$. The *commutator subgroup* of G is the subgroup $[G, G]$ generated by the set $\{[g, h] \mid g, h \in G\}$. Since $[G, G]$ is a normal subgroup of G (exercise) we can consider the quotient group $G_{ab} = G/[G, G]$ which is called the *abelianization* of G. The construction of G_{ab} has the following properties (exercise):

- G_{ab} is an abelian group.
- If G is an abelian then $G_{ab} \cong G$.
- If $f\colon G \to H$ is a group homomorphism then $f([G, G]) \subseteq [H, H]$, and so f induces a homomorphism $f_{ab}\colon G_{ab} \to H_{ab}$.
- Recall that **Gr** denotes the category of all groups. Denote by **Ab** the category of abelian groups whose elements are all abelian groups and morphisms are homomorphisms of such groups. The assignments $F(G) = G_{ab}$ and $F(f) = f_{ab}$ define a functor

$$F\colon \mathbf{Gr} \to \mathbf{Ab}$$

  This implies in particular that if $G \cong G'$ then $G_{ab} \cong G'_{ab}$.

Since the groups $\pi_1(T)$ and $\pi_1(\mathbb{RP}^2)$ are abelian we have $\pi_1(T)_{ab} \cong \pi_1(T)$ and $\pi_1(\mathbb{RP}^2)_{ab} \cong \pi_1(\mathbb{RP}^2)$. On the other hand abelianization of the fundamental group of the Klein bottle $\pi_1(K) \cong \langle a, b \mid aba^{-1}b \rangle$ is obtained by imposing the condition that $ab = ba$, or equivalently adding the relation $aba^{-1}b^{-1}$ to the presentation of the group:

$$\pi_1(K)_{ab} \cong \langle a, b \mid aba^{-1}b,\ aba^{-1}b^{-1} \rangle$$

Notice that if $ab = ba$ then $aba^{-1}b = b^2$. Therefore we obtain:

$$\pi_1(K)_{ab} \cong \langle a, b \mid b^2,\ aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

Since $\pi_1(K)_{ab}$ is isomorphic to neither $\pi_1(T)_{ab}$ nor $\pi_1(\mathbb{RP}^2)_{ab}$ we get that the $K \not\simeq T$ and $K \not\simeq \mathbb{RP}^2$.

As the final result of this chapter we will show that any group can be realized as the fundamental group of some CW complex:

15.8 Theorem. *For any group G there exists a CW complex X such that $\pi_1(X) \cong G$.*

Proof. We will construct the complex X as follows. The 0-skeleton of X consists of a single 0-cell that we will take as the basepoint x_0. The 1-skeleton of X is the wedge of circles, one copy of S^1 for each element of G:

$$X^{(1)} = \bigvee_{g \in G} S^1$$

Let ω_g denote the loop in $X^{(1)}$ that traverses the copy of S^1 corresponding the element $g \in G$. The group $\pi_1(X^{(1)}, x_0)$ is the free group generated by the set $T = \{[\omega_g] \in \pi_1(X^{(1)}, x_0 \mid g \in G\}$. Consider the function of sets $f\colon T \to G$ given by $f([\omega_g]) = g$. By the universal property of free groups (14.4) this function defines a homomorphims of groups $\bar{f}\colon \pi_1(X^{(1)}, x_0) \to G$. Moreover, since f is onto thus so is $\bar{f}$. As a consequence we obtain that $G \cong \pi_1(X^{(1)}, x_0)/\operatorname{Ker}(\bar{f})$. For each element $r \in \operatorname{Ker}(\bar{f})$ let $\sigma_r\colon [0,1] \to X^{(1)}$ be a loop representing r. Recall (3.18) that such loop can be identified with a map $S^1 \to X^{(1)}$. By abuse of notation we will denote this map also by σ_r. Let X be the CW complex obtained from $X^{(1)}$ by attaching one 2-cell for each element $r \in \operatorname{Ker}(\bar{f})$ with the attaching map given by σ_r. By Theorem 15.3 we obtain

$$\pi_1(X, x_0) \cong \pi_1(X^{(1)}, x_0)/\operatorname{Ker}(\bar{f}) \cong G$$

$\square$

15.9 Note. By modifying the proof Theorem 15.8 we can show that if a group G has a presentation $G \cong \langle S \mid R \rangle$ then G is isomorphic to the fundamental group of a CW complex X which has one 1-cell for each element of S and one 2-cell for each element of R. In particular G is finitely generated if and only if $G \cong \pi_1(X)$ for some CW complex X such that $X^{(1)}$ is finite, and and G is finitely presented if and only if $G \cong \pi_1(X)$ for some finite CW complex X.

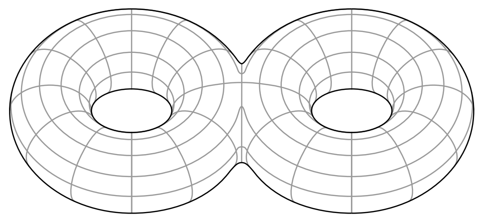
Exercises to Chapter 15

E15.1 Exercise. Let $x_1, x_2, x_3 \in S^2$ and let X be the quotient space obtained by identifying these three points. Put a CW complex structure on X and use it to compute $\pi_1(X)$.

E15.2 Exercise. Let T be the torus. The *connected sum* $T\#T$ is the space obtained as follows. Take $D \subseteq T$ to be a subspace homeomorphic to the closed disk D^2 and let $\operatorname{Int}(D)$, $\operatorname{Bd}(D)$ denote, respectively, the interior and the boundary of D. The space $\operatorname{Int}(D)$ is homeomorphic to the open disc and $\operatorname{Bd}(D)$ to the circle S^1. Take the space $\{0,1\}$ with the discrete topology. We set:

$$T\#T = (T \smallsetminus D) \times \{0,1\}/{\sim}$$

where $(x, 0) \sim (x, 1)$ for each $x \in \mathrm{Bd}(D)$. In other words $T\#T$ is obtained by taking two copies of the torus, cutting out a hole in each copy, and glueing the boundaries of the holes together:



a) Put a CW complex structure on $T\#T$ and use it to find a presentation of the fundamental group of $T\#T$.

b) Show that $T\#T$ is not homotopy equivalent to T.

Note: the homeomorphism type of $T\#T$ does not depend on which embedding of the disk into T one chooses, so you can work with whichever embedding is most convenient.

E15.3 Exercise. Consider S^1 as the unit circle in the complex plane

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

For $n = 1, 2, \ldots$ let T_n be a space given by

$$T_n = S^1 \times [0, 1]/\sim$$

where $(z, 0) \sim (z^n, 1)$ (in particular T_1 is just a torus). Find the fundamental group of T_n, and show that if $n \neq m$ then T_n is not homotopy equivalent to T_m.

E15.4 Exercise. Let X be a 2-dimensional CW complex. Assume that X has only one 0-cell $x_0 \in X$. Let (Y, y_0) be a pointed space. Show that for any group homomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ there exists a map $f\colon (X, x_0) \to (Y, y_0)$ such that $f_* = \varphi$.