14 | Presentations of Groups

In this chapter we make here a brief algebraic interlude from the task of computing fundamental groups in order to discuss how groups can be described by means their *presentations*. This concept will be used in the next chapter where we will consider fundamental groups of 2-dimensional CW complexes.

14.1 Definition. Let S be a set. A *word* in S is a finite sequence of the form $a_1^{k_1}a_2^{k_2}\dots a_n^{k_n}$ where $n \geq 0$, $a_i \in S$ and $k_i \in \mathbb{Z}$. The *free group generated by* S is the group $F(S)$ whose elements are words in S with the the following identifications:

- if $a_i = a_{i+1}$ then
$$a_1^{k_1}\dots a_i^{k_i}a_{i+1}^{k_{i+1}}\dots a_n^{k_n} = a_1^{k_1}\dots a_i^{(k_i+k_{i+1})}\dots a_n^{k_n}$$
- if $k_i = 0$ then
$$a_1^{k_1}\dots a_{i-1}^{k_{i-1}}a_i^{k_i}a_{i+1}^{k_{i+1}}\dots a_n^{k_n} = a_1^{k_1}\dots a_{i-1}^{k_{i-1}}a_{i+1}^{k_{i+1}}\dots a_n^{k_n}$$

Multiplication in $F(S)$ is given by concatenation of words:

$$(a_1^{k_1}\dots a_n^{k_n})\cdot(b_1^{l_1}\dots b_m^{l_m}) = a_1^{k_1}\dots a_n^{k_n}b_1^{l_1}\dots b_m^{l_m}$$

The identity element in $F(S)$ is given by the empty word (i.e. the word of length 0).

14.2 Note. If $S = \varnothing$ then $F(S)$ is the trivial group. If $S = \{a\}$ is a set consisting of one element then $F(S) \cong \mathbb{Z}$. In general, the group $F(S)$ isomorphic to the free product of free groups generated by the elements of S:

$$F(S) \cong *_{a\in S}F(\{a\}) \cong *_{a\in S}\mathbb{Z}$$

14.3 Note. We will say that a group G is free if G is isomorphic to the group $F(S)$ for some set S. Notice that by Theorem 13.13 the fundamental group of any 1-dimensional CW complex is free.

For any set S we have a map of sets: $i\colon S \to F(S)$ given by $f(a) = a$ (where we consider $a \in F(S)$ as a word of length 1). The statement of the following fact is called the *universal property of free groups*:

14.4 Theorem. *Let S be a set and G be a group. For any map of sets $f\colon S \to G$ there exists a unique homomorphism of groups $\bar{f}\colon F(S) \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i"'] & G \\ F(S) \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

Proof. The homomorphism $\bar{f}$ is given by $\bar{f}(a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n}) := f(a_1)^{k_1} \cdot f(a_2)^{k_2} \cdot \ldots \cdot f(a_n)^{k_n}$. $\square$

14.5 Definition. Let S be a set, and let R be a subset of elements of the free group $F(S)$. By $\langle S \mid R \rangle$ we denote the group given by

$$\langle S \mid R \rangle = F(S)/N$$

where N is the smallest normal subgroup of $F(S)$ such that $R \subseteq N$. We say that elements of S are *generators* of $\langle S \mid R \rangle$ and elements of R are *relations* in $\langle S \mid R \rangle$.

14.6 Example. For any set we have S is a set $F(S) \cong \langle S \mid \varnothing \rangle$.

14.7 Example. $\langle a \mid a^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$.

14.8 Example. $\langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z}$.

14.9 Definition. If G is a group and $G \cong \langle S \mid R \rangle$ for some set S and some $R \subseteq F(S)$ then we say that $\langle S \mid R \rangle$ is a *presentation* of G.

14.10 Definition. If a group G has a presentation $\langle S \mid R \rangle$ such that S is a finite set then we say that G is *finitely generated* and if it has a presentations such that both S and R are finite sets then we say that G is *finitely presented*.

14.11 Proposition. *Every group has a presentation.*

Proof. Let G be a group and let $f\colon S \to G$ be a map of sets which is onto. By Theorem 14.4 the function f defines a homomorphism $\bar{f}\colon F(S) \to G$. Since f is onto thus so is $\bar{f}$. This gives an isomorphism $G \cong F(S)/\operatorname{Ker}(\bar{f})$. If follows that $G \cong \langle S \mid R \rangle$ where R is the set of elements of $\operatorname{Ker}(\bar{f})$. $\square$

14.12 Note. 1) Every group has inifinitely many different presentations. For example

$$\mathbb{Z} \cong \langle a \rangle \cong \langle a, b \mid b \rangle \cong \langle a, b \mid ab^{-1} \rangle \cong \langle a, b \mid b^2, b^3 \rangle$$

2) In general if we know a presentation of a group it may be very difficult to say anything about the properties of the group (even if the group is trivial or not).

Exercises to Chapter 14

E14.1 Exercise. Below are three groups described by their presentations. For each group decide if it is abelian and if it is finite. Justify your answers.

a) $G_1 = \langle a, b \mid a^3, b^3, aba^2b^2 \rangle$

b) $G_2 = \langle a, b \mid a^2, aba \rangle$

c) $G_3 = \langle a, b \mid a^4, b^4, a^2b^2 \rangle$