22 | Coverings and Group Actions

22.1 Definition. Let G be a group and S be a set. We say that G *acts on* X on the right if there exists a function

$$\mu\colon S \times G \to S$$

such that

(i) $\mu(s, e) = s$ for any $s \in S$, where $e \in G$ is the trivial element;
(ii) $\mu(\mu(s, g), h) = \mu(s, gh)$ for all $s \in S$, $h, g \in G$.

22.4 Definition. We say that a group G acts on set S *transitively* if for any $s, s' \in S$ there exists $g \in G$ such that $sg = s'$.

22.5 Proposition. *Let $p\colon T \to X$ be a covering, and let $x_0 \in X$. If T is path connected then the monodromy action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is transitive.*

Proof. Exercise. $\square$

22.6 Definition. Let G be a group and let S, S' be G-sets. A function $f\colon S \to S'$ is *G-equivariant* if $f(sg) = f(s)g$ for all $s \in S$ and $g \in G$.

22.7 Note. G-sets and G-equivariant functions form a category which we will denote by $\mathbf{Set}_G$.

22.8 Proposition. *Let X be a space, and let*

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

be a map of coverings. For any $x_0 \in X$ the induced map of fibers $f\colon p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ is $\pi_1(X, x_0)$-equivariant.

Proof. Exercise. $\square$

22.9 Corollary. *Let X be a space and let $x_0 \in X$. The assignment which associates to each path connected covering $p\colon T \to X$ the $\pi_1(X, x_0)$-set $p^{-1}(x_0)$ and to each map of coverings the map of fibers defines a functor*

$$\Lambda\colon \mathbf{Cov}(X) \to \mathbf{Set}_{\pi_1(X,x_0)}$$

Proof. Exercise. $\square$

22.10 Theorem. *Let X be a connected, locally path connected, and semi-locally simply connected space, and let $x_0 \in X$. The functor*

$$\Lambda\colon \mathbf{PCov}(X) \to \mathbf{TSet}_{\pi_1(X,x_0)}$$

is an equivalence of categories.

Outline of proof of Theorem 22.10:

1) We need to show that any set with a transitive action of the group $\pi_1(X, x_0)$ is isomorphic to a $\pi_1(X, x_0)$-set $\Lambda(p\colon T \to X) = p^{-1}(x_0)$ for some path connected covering p.

2) We also need to show that maps of path connected coverings of X are in a bijective correspondence with $\pi_1(X, x_0)$-equivariant maps of their fibers.

22.11 Proposition. *Let X be a connected, locally path connected, and semi-locally simply connected space and let $x_0 \in X$. The map*

$$\Lambda\colon \left(\begin{array}{c}\textit{isomorphism classes}\\ \textit{of path connected}\\ \textit{coverings of } X\end{array}\right) \longrightarrow \left(\begin{array}{c}\textit{isomorphism classes}\\ \textit{of sets with transitive}\\ \textit{action of } \pi_1(X, x_0)\end{array}\right)$$

given by $\Lambda(p\colon T \to X) = p^{-1}(x_0)$ is a bijection.

22.12 Definition. Let G be a group, and S be a G-set. The *stabilizer* of en element $s \in S$ is the subgroup $G_s \subseteq G$ given by:

$$G_s = \{g \in G \mid sg = s\}$$

22.13 Proposition. *Let $p\colon T \to X$ be a covering, and let $x_0 \in X$. The stabilizer of an element $\tilde{x} \in p^{-1}(x_0)$ under the monodromy action is the subgroup $p_*(\pi_1(T, \tilde{x})) \subseteq \pi_1(X, x_0)$.*

Proof. Exercise. $\square$

22.14 Lemma. *Let G be a group.*

1) *If G acts transitively on a set S and $s, s' \in S$ then the stabilizers G_s and $G_{s'}$ are conjugate subgroups of the group G.*
2) *Let S be a set with an action of G and let $s \in S$. The assignment $S \mapsto G_s$ defines a bijective correspondence:*

$$\Phi\colon \begin{pmatrix} \textit{isomorphism classes} \\ \textit{of sets with a transitive} \\ \textit{action of } G \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{conjugacy classes} \\ \textit{of subgroups} \\ \textit{of } G \end{pmatrix}$$

22.15 Proposition. *Let X be a connected and locally path connected space, and let $x_0 \in X$. For any path connected coverings $p_i \colon T_i \to X$, $i = 1, 2$ the assignment*

$$\Lambda\colon \begin{pmatrix} \textit{maps of coverings} \\ T_1 \to T_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \pi_1(X, x_0)\textit{-equivariant maps} \\ p_1^{-1}(x_0) \to p_2^{-1}(x_0) \end{pmatrix}$$

is a bijection.

22.16 Lemma. *Let S, T be sets with a transitive action of a group G, and let $s_0 \in S$, $t_0 \in T$. A G-equivariant map $f \colon S \to T$ such that $f(s_0) = t_0$ exists if and only if $G_{s_0} \subseteq G_{t_0}$. Moreover, if such map exists then it is unique.*