2 | Categories and Functors

Before we get into the details of algebraic invariants of topological spaces, it will be worth a while to have a look at the general framework used to construct such invariants. In this chapter we introduce the notions of a *category* and a *functor*, which underlie such constructions.

2.1 Definition. A *category* $\mathbf{C}$ consists of the following ingredients:

1) a class of *objects* $\mathrm{Ob}(\mathbf{C})$
2) for each pair of objects $c, c' \in \mathrm{Ob}(\mathbf{C})$ a set of *morphisms* $\mathrm{Mor}_{\mathbf{C}}(c, c')$
3) for each object $c \in \mathrm{Ob}(\mathbf{C})$ a distinguished *identity morphism* $\mathrm{id}_c \in \mathrm{Mor}_{\mathbf{C}}(c, c)$
4) for each triple of objects $c, c', c'' \in \mathrm{Ob}(\mathbf{C})$ a *composition of morphisms* function

$$\circ\colon \mathrm{Mor}_{\mathbf{C}}(c, c') \times \mathrm{Mor}_{\mathbf{C}}(c', c'') \to \mathrm{Mor}_{\mathbf{C}}(c, c'')$$

Moreover, the composition of morphisms satisfies the following conditions:

(i) $f \circ (g \circ h) = (f \circ g) \circ h$, whenever morphisms f, g, h are composable
(ii) if $f \in \mathrm{Mor}_{\mathbf{C}}(c, c')$ then $f \circ \mathrm{id}_c = f = \mathrm{id}_{c'} \circ f$.

2.2 Example. By **Set** we will denote the category of sets. Its objects are sets, and for any sets A, B the set of morphisms $\mathrm{Mor}_{\mathbf{Set}}(A, B)$ consists of all functions $f\colon A \to B$. Composition of morphism is the usual composition of functions, and for a set A the identity morphism $\mathrm{id}_A\colon A \to A$ is given by the identity function: $\mathrm{id}_A(x) = x$ for all $x \in A$.

2.3 Example. Let **Gr** denote the category of groups. The objects of **Gr** are groups. Given two groups G, H the set $\mathrm{Mor}_{\mathbf{Gr}}(G, H)$ consists of all group homomorphisms $f\colon G \to H$.

2.4 Example. By **Top** we will denote the category of topological spaces. Its objects are topological spaces. For $X, Y \in \mathrm{Ob}(\mathbf{Top})$ the set $\mathrm{Mor}_{\mathbf{Top}}(X, Y)$ consists of all continuous functions $f\colon X \to Y$.

2.5 Example. The previous examples may suggest that categories are very large structures, and that each category corresponds to a whole area of mathematics (set theory, group theory, topology etc.) However, categories can be also small. For example, given any group G we can construct a category $\mathbf{C}_G$ as follows. The only object of $\mathbf{C}_G$ will be denoted by $*$. For every element $g \in G$ there is a morphism $f_g \colon * \to *$. Composition of morphisms is defined by multiplication in G: $f_g \circ f_h = f_{gh}$. The identity morphisms on $*$ corresponds to the identity element of G.

2.6 Note. To simplify notation we will frequently write $c \in \mathbf{C}$ instead of $c \in \mathrm{Ob}(\mathbf{C})$ to indicate that c is an object of a category $\mathbf{C}$.

2.7 Note. The definition of a category (2.1) deliberately says that objects of a category form a *class*. The notion of a class is more general that that of a set: every set is a class, but not every class is a set. This distinction lets us avoid some logical problems. For example, while defining the category **Set** (2.2) we cannot say that its objects form "the set of all sets" since this would lead to contradictions (e.g. Russell's paradox). On the other hand, we can talk about the class of all sets. As this suggests a class can be though of intuitively as something that can be bigger than any set.

Since every set is a class some categories have a set of objects. Such categories are called *small categories*. For example, the category $\mathbf{C}_G$ defined in Example 2.5 is small since its objects form a set with one element: $\mathrm{Ob}(\mathbf{C}_G) = \{*\}$.

2.8 Definition. Let $\mathbf{C}$, $\mathbf{D}$ be categories. A *(covariant) functor* $F \colon \mathbf{C} \to \mathbf{D}$ consists of

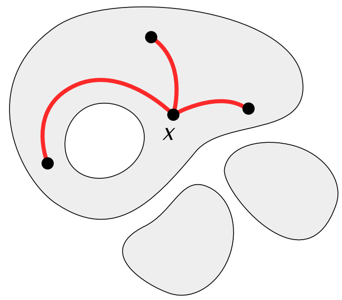
1) an assignment $F \colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$
2) for each $c, c' \in \mathrm{Ob}(\mathbf{C})$ a function

$$F \colon \mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$$

such that $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for all $c \in \mathrm{Ob}(\mathbf{C})$ and $F(f \circ g) = F(f) \circ F(g)$ for each pair of composable morphisms f, g in $\mathbf{C}$.

2.9 Example. For a topological space X let $U(X)$ denote the sets of points of X. Also, given a continuous map of topological spaces $f \colon X \to Y$ denote by $U(f) \colon U(X) \to U(Y)$ the underlying map of sets. These assignments define a functor $U \colon \mathbf{Top} \to \mathbf{Set}$ which is called the *forgetful functor*.

2.10 Example. A more interesting example of a functor $\mathbf{Top} \to \mathbf{Set}$ can be obtained as follows. Let X be a topological space. Recall that a *path* in X is a continuous function $\omega \colon [0, 1] \to X$. Recall also that a *path connected component* of a point $x \in X$ is the subspace of X consisting of all points that can be connected to x by a path:



Denote this subspace by $[x]$. Notice that for $x, x' \in X$ we have $[x] = [x']$ if and only if there is a path joining x and x'. Let $\pi_0(X)$ denote the set whose elements are path connected components of the space X. Given a continuous function of topological spaces $f\colon X \to Y$ consider the function of sets

$$f_*\colon \pi_0(X) \to \pi_0(Y)$$

given by $f_*([x]) = [f(x)]$. The function f_* is well defined. Indeed, if $x, x' \in X$ are points such that $[x] = [x']$ then there exists a path $\omega\colon [0,1] \to X$ such that $\omega(0) = x$ and $\omega(1) = x'$. Then $f\omega\colon [0,1] \to Y$ is a path joining $f(x)$ with $f(x')$ which shows that $[f(x)] = [f(x')]$. The assignments $X \mapsto \pi_0(X)$ and $f \mapsto f_*$ define a functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$.

2.11 Definition. Let $\mathbf{C}$ be a category. A morphism $f\colon c \to c'$ in $\mathbf{C}$ is an *isomorphism* if there exists a morphism $g\colon c' \to c$ such that $gf = \mathrm{id}_c$ and $fg = \mathrm{id}_{c'}$. In such case we say that g is the *inverse* of f and we write $g = f^{-1}$.

If there exists an isomorphism between $c, c' \in \mathbf{C}$ then we say that these objects are *isomorphic* and we write $c \cong c'$.

2.12 Example. A morphism $f\colon X \to Y$ in $\mathbf{Top}$ is an isomorphism if and only if f is a homeomorphism.

2.13 Example. A morphism $f\colon A \to B$ in $\mathbf{Set}$ is an isomorphism if and only if f is a bijection of sets.

2.14 Example. A morphism $f\colon G \to H$ in $\mathbf{Gr}$ is an isomorphism if and only if f is a group isomorphism.

2.15 Proposition. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. If $f\colon c \to c'$ is an isomorphism in $\mathbf{C}$ then $F(f)\colon F(c) \to F(c')$ is an isomorphism in $\mathbf{D}$ and $F(f)^{-1} = F(f^{-1})$.*

Proof. Let $f^{-1}\colon c' \to c$ be the inverse of f. We have

$$F(f^{-1})F(f) = F(f^{-1}f) = F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$$

Similarly, using that $ff^{-1} = \mathrm{id}_{c'}$ we obtain $F(f)F(f^{-1}) = \mathrm{id}_{F(c')}$. Thus $F(f^{-1})$ is the inverse of $F(f)$. $\square$

2.16 Corollary. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor and $c, c' \in \mathbf{C}$. If $F(c) \not\cong F(c')$ then $c \not\cong c'$.*

2.17 Example. Consider the functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ (2.10). In **Top** take the spaces $\mathbb{R}$ and $\mathbb{R} \smallsetminus \{0\}$. The space $\mathbb{R}$ has only one path connected component while $\mathbb{R} \smallsetminus \{0\}$ has two path connected components: $(-\infty, 0)$ and $(0, +\infty)$. It follows that $\pi_0(\mathbb{R})$ consists of one element while $\pi_0(\mathbb{R} \smallsetminus \{0\})$ is a set with two elements, so $\pi_0(\mathbb{R}) \ncong \pi_0(\mathbb{R} \smallsetminus \{0\})$ in **Set**. This shows that $\mathbb{R} \ncong \mathbb{R} \smallsetminus \{0\}$ in **Top**, i.e. that these two spaces are not homeomorphic.

2.18 Note. For a general functor $F\colon \mathbf{C} \to \mathbf{D}$ and $c, c' \in \mathbf{C}$ it may happen that $F(c) \cong F(c')$ even though $c \ncong c'$. Take for example the functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ and let $X = \{*\}$ be a space consisting of a single point. We have $\pi_0(X) \cong \pi_0(\mathbb{R})$, since both $\pi_0(X)$ and $\pi_0(\mathbb{R})$ are sets with only one element, but $X \ncong \mathbb{R}$.

Exercises to Chapter 2

E2.1 Exercise. Let **C** be a category. An object $c \in \mathbf{C}$ is initial in **C** if for each object $d \in \mathbf{C}$ there is exactly one morphism $c \to d$.

a) Show that if c is an initial object in **C** and $c' \in \mathbf{C}$ is an object isomorphic to c then c' is also an initial object.

b) Show that if c and c' are objects of **C** such that each of them is initial then $c \cong c'$.

E2.2 Exercise. Given a morphism $f\colon c \to c'$ and an object d in a category **C**, consider the functions

$$f_*\colon \mathrm{Mor}_{\mathbf{C}}(d, c) \to \mathrm{Mor}_{\mathbf{C}}(d, c') \quad \text{ and } \quad f^*\colon \mathrm{Mor}_{\mathbf{C}}(c', d) \to \mathrm{Mor}_{\mathbf{C}}(c, d)$$

given by $f_*(g) = f \circ g$ for $g \in \mathrm{Mor}_{\mathbf{C}}(d, c)$, and $f^*(h) = h \circ f$ for $h \in \mathrm{Mor}_{\mathbf{C}}(c', d)$.

Show that for a morphism $f\colon c \to c'$ the following conditions are equivalent:

1) The morphism f is is an isomorphism.
2) The function f_* is a bijection for every $d \in \mathbf{C}$.
3) The function f^* is a bijection for every $d \in \mathbf{C}$.

E2.3 Exercise. Cosider a sequence of five morphisms in a category **C**:

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3 \xrightarrow{f_4} c_4 \xrightarrow{f_5} c_5$$

Assume that the composition of any three morphisms in this sequence $f_3 \circ f_2 \circ f_1$, $f_4 \circ f_3 \circ f_2$, and $f_5 \circ f_4 \circ f_3$ is an isomorphism. Show that f_i is an isomorphism for $i = 1, \dots, 5$.

E2.4 Exercise. Find two different functors $F, F'\colon \mathbf{Gr} \to \mathbf{Gr}$ such that $F(G) = F'(G) = G$ for each group $G \in \mathbf{Gr}$.

E2.5 Exercise. Let $\mathbf{C}$, $\mathbf{D}$ be categories. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is called *full* if for each $c, c' \in \mathbf{C}$ the function

$$F\colon \mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$$

is onto, and if is *faithful* if for each $c, c' \in \mathbf{C}$ this function is 1-1.

a) Give an example of a functor $F\colon \mathbf{Gr} \to \mathbf{Gr}$ which is full but not faithful.

b) Give an example of a functor $F'\colon \mathbf{Gr} \to \mathbf{Gr}$ which is faithful but not full.

E2.6 Exercise. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor which is both full and faithful (Exercise 2.5), and let $c, c' \in \mathbf{C}$. Show that $c \cong c'$ if and only if $F(c) \cong F(c')$.