18 | Coverings and the Fundamental Group

The first example of a covering we have encountered, the universal covering of S^1, was introduced as a tool for computing the fundamental group of S^1. It turns out that coverings in general are very closely related to the fundamental group. In this chapter we explore some results illustrating this. Ultimately these results will let us give an algebraic description of all possible coverings of a given space X.

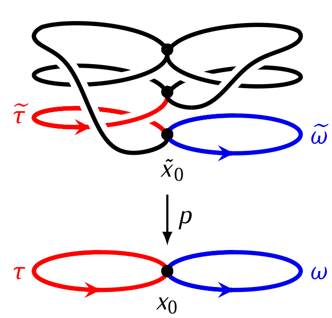
Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Recall that by Corollary 17.10 a path $\omega\colon [0,1] \to X$ such that $\omega(0) = x_0$ admits a unique lift $\widetilde{\omega}\colon [0,1] \to T$ that satisfies $\widetilde{\omega}(0) = \tilde{x}_0$. We have:

18.1 Theorem. *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$.*

1) The homomorphism $p_*\colon \pi_1(T, \tilde{x}_0) \to \pi_1(X, x_0)$ is 1-1.

2) An element $[\omega] \in \pi_1(X, x_0)$ is in the subgroup $p_*(\pi_1(T, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$ if and only if the lift $\widetilde{\omega}$ such that $\widetilde{\omega}(0) = \tilde{x}_0$ is a loop in T.

18.2 Example. Consider the following 3-fold covering $p\colon T \to S^1 \vee S^1$:



The lift $\widetilde{\omega}$ of the loop ω that satisfies $\widetilde{\omega}(0) = \tilde{x}_0$ is a loop in T, so $[\omega] \in p_*(\pi_1(T, \tilde{x}_0))$. On the other hand the lift $\widetilde{\tau}$ of the loop τ that satisfies $\widetilde{\tau}(0) = \tilde{x}_0$ is an open path in T, which means that $[\tau] \notin p_*(\pi_1(T, \tilde{x}_0))$.

Proof of Theorem 18.1. 1) Assume that $[\tau], [\tau'] \in \pi_1(T, \tilde{x}_0)$ are elements such that $p_*([\tau]) = p_*([\tau'])$. This means that $p \circ \tau \simeq p \circ \tau'$. Since τ and τ' are the unique lifts of $p \circ \tau$ and $p \circ \tau'$ that begin at $\tilde{x}_0$ by part 2) of Corollary 17.10 we obtain that $\tau \simeq \tau'$, and so $[\tau] = [\tau']$.

2) Let $[\omega] \in \pi_1(X, x_0)$. If the lift $\widetilde{\omega}$ of ω is a loop then it represents an element $[\widetilde{\omega}] \in \pi_1(T, \tilde{x}_0)$. We have $p_*([\widetilde{\omega}]) = [p \circ \widetilde{\omega}] = [\omega]$, so $[\omega] \in p_*(\pi_1(T, \tilde{x}_0))$. Conversely, if $[\omega] \in p_*(\pi_1(T, \tilde{x}_0))$ then $\omega \simeq p \circ \tau$ where τ is a loop in T based at $\tilde{x}_0$. Since τ is the lift of $p \circ \tau$ that starts at $\tilde{x}_0$, thus by part 2) of Corollary 17.10 we obtain that $\widetilde{\omega}(1) = \tau(1) = \tilde{x}_0$ which shows that $\widetilde{\omega}$ is a loop in T. ☐

The second part of Theorem 18.1 can be generalized as follows:

18.3 Proposition. *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Assume that ω_1 and ω_2 are paths in X such that $\omega_1(0) = \omega_2(0) = x_0$ and $\omega_1(1) = \omega_2(1)$. For $i = 1, 2$ let $\widetilde{\omega}_i\colon [0, 1] \to T$ be the lift of ω_i such that $\widetilde{\omega}_i(0) = \tilde{x}_0$. Then $\widetilde{\omega}_1(1) = \widetilde{\omega}_2(1)$ if and only if $[\omega_1 * \overline{\omega}_2] \in p_*(\pi_1(T, \tilde{x}_0))$.*

Proof. Exercise. ☐

Recall that if G is a group and $H \subseteq G$ is a subgroup then the *left coset* of an element $g \in G$ is the set $gH = \{gh \in G \mid h \in H\}$ and that the *index* of H in G is the number $[G : H]$ of distinct left cosets.

18.4 Corollary. *Let $p\colon T \to X$ be a covering such that T is a path connected space, let $x_0 \in X$, and let $\tilde{x}_0 \in p^{-1}(x_0)$. Denote $H = p_*(\pi_1(T, \tilde{x}_0))$.*

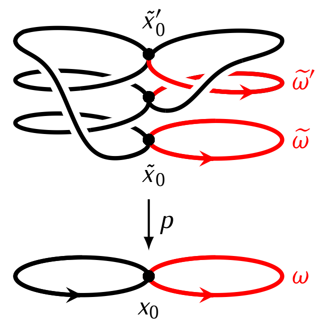
1) For $i = 1, 2$ let ω_i be a loop in X based at x_0 and let $\widetilde{\omega}_i$ be the lift of ω such that $\widetilde{\omega}_i(0) = \tilde{x}_0$. We have $\widetilde{\omega}_1(1) = \widetilde{\omega}_2(1)$ if and only if $[\omega_1]H = [\omega_2]H$.

2) The index $[\pi_1(X, x_0) : H]$ is equal to the number of elements of the fiber $p^{-1}(x_0)$.

Proof. 1) Notice that $[\omega_1]H = [\omega_2]H$ if and only if $[\omega_1] \cdot [\omega_2]^{-1} = [\omega_1 * \overline{\omega}_2] \in H$. Therefore it suffices to apply Proposition 18.3.

2) Let $\pi_1(X, x_0)/H$ denote the set of left cosets of H. Consider the function $\varphi\colon \pi_1(X, x_0)/H \to p^{-1}(x_0)$ given by $\varphi([\omega]H) = \widetilde{\omega}(1)$ where $\widetilde{\omega}$ is the lift of ω such that $\widetilde{\omega}(0) = \tilde{x}_0$. By part 1) the function φ is well defined and it is 1-1. To see that φ it is also onto take $\tilde{x}'_0 \in \pi^{-1}(x_0)$. By assumption T is path connected, so there exists a path $\tau\colon [0, 1] \to T$ such that $\tau(0) = \tilde{x}_0$ and $\tau(1) = \tilde{x}'_0$. Notice that $p\tau$ is a loop in X and that $\varphi([p\tau]H) = \tilde{x}'_0$. ☐

Theorem 18.1 lets us associate to each covering p of a space X the subgroup $p_*(\pi_1(T, \tilde{x}_0))$ of the fundamental group $\pi_1(X, x_0)$. However, this subgroup depends not only on the covering p but also on the choice of the basepoint $\tilde{x}_0 \in p^{-1}(x_0)$. Indeed, consider the same covering $p\colon T \to S^1 \vee S^1$ as before:



The lift of the loop ω that starts at the point $\tilde{x}_0$ is the loop $\widetilde{\omega}$. This means that $[\omega] \in p_*(\pi_1(T, \tilde{x}_0))$. On the other hand the lift of ω that starts at $\tilde{x}'_0$ is the open path $\widetilde{\omega}'$, which shows that $[\omega] \notin p_*(\pi_1(T, \tilde{x}'_0))$. As a consequence $p_*(\pi_1(T, \tilde{x}_0)) \neq p_*(\pi_1(T, \tilde{x}'_0))$.

Our next goal will be to describe the relationship between the subgroups $p_*(\pi_1(T, \tilde{x}_0))$ of $\pi_1(X, x_0)$ that come from different choices of points $\tilde{x}_0 \in p^{-1}(x_0)$. Recall that we say that subgroups H, H' of a group G are *conjugate* if $H' = gHg^{-1}$ for some $g \in G$.

18.5 Proposition. *Let $p\colon T \to X$ be a covering such that T is a path connected space and let $x_0 \in X$.*

1) For any $\tilde{x}_0, \tilde{x}'_0 \in p^{-1}(x_0)$ the subgroups $p_*(\pi_1(T, \tilde{x}_0))$ and $p_*(\pi_1(T, \tilde{x}'_0))$ of $\pi_1(X, x_0)$ are conjugate.

2) If $\tilde{x}_0 \in p^{-1}(x_0)$ and $H \subseteq \pi_1(X, x_0)$ is a subgroup conjugate to $p_*(\pi_1(T, \tilde{x}_0))$ then $H = p_*(\pi_1(T, \tilde{x}'_0))$ for some $\tilde{x}'_0 \in p^{-1}(x_0)$.

Proof. 1) Since T is path connected there exists a path τ in T such that $\tau(0) = \tilde{x}_0$ and $\tau(1) = \tilde{x}'_0$. Recall that such path defines an isomorphism $s_\tau\colon \pi_1(T, \tilde{x}_0) \to \pi_1(T, \tilde{x}'_0)$ given by $s_\tau([\omega]) = [\overline{\tau} * \omega * \tau]$. This gives:

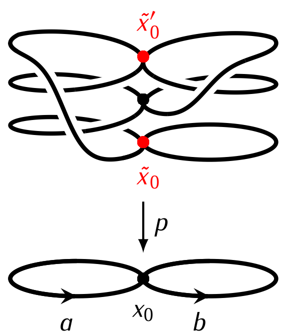
$$\begin{aligned} p_*(\pi_1(T, \tilde{x}'_0)) &= p_*(s_\tau(\pi_1(T, \tilde{x}_0))) \\ &= p_*(\{[\overline{\tau} * \omega * \tau] \mid [\omega] \in \pi_1(T, \tilde{x}_0)\}) \\ &= \{[(p \circ \overline{\tau}) * (p \circ \omega) * (p \circ \tau)] \mid [\omega] \in \pi_1(T, \tilde{x}_0)\} \end{aligned}$$

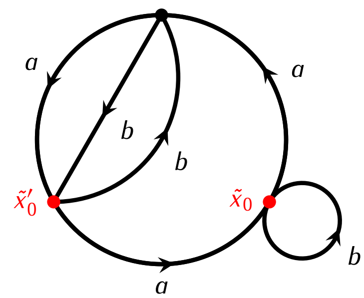
Since $p \circ \tau$ is a loop in X based at x_0 it represents an element $[p \circ \tau] \in \pi_1(X, x_0)$ and we have $[(p \circ \overline{\tau}) * (p \circ \omega) * (p \circ \tau)] = [p \circ \tau]^{-1} \cdot p_*[\omega] \cdot [p \circ \tau]$. Therefore we obtain:

$$p_*(\pi_1(T, \tilde{x}'_0)) = [p \circ \tau]^{-1} \cdot p_*(\pi_1(T, \tilde{x}'_0)) \cdot [p \circ \tau]$$

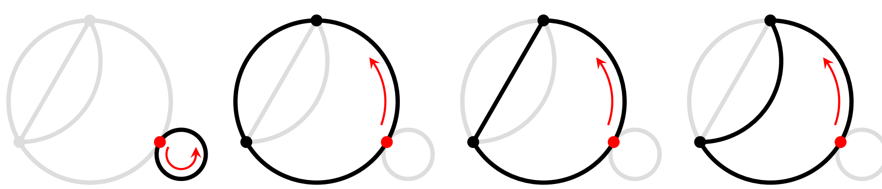
2) Exercise. $\square$

18.6 Example. Lets have a look again at the covering $p\colon T \to S^1 \vee S^1$:

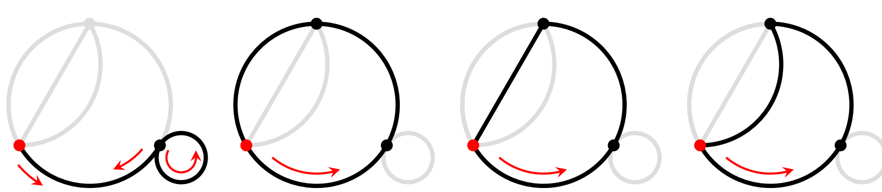




The fundamental group of the base space is a free group on two generators: $\pi_1(S^1 \vee S^1, x_0) \cong \langle a, b\rangle$ where a is represented by the loop that traverses one copy of S^1, and b by the loop that traverses the second copy of S^1. The total space T of this covering has the structure of a 1-dimensional path connected CW complex with three 0-cells and six 1-cells, so by Theorem 13.11 the group $\pi_1(T)$ is a free group on 4 generators. Free generators of the group $\pi_1(T, \tilde{x}_0)$ can be selected so that they are represented by the following loops based at $\tilde{x}_0$:



Arrows indicate orientations of these loops. The images of these loops in $S^1 \vee S^1$ represent the following elements of the group $\langle a, b\rangle$: b, a^3, aba, $ab^{-1}a$. It follows that the subgroup $p_*(\pi_1(T, \tilde{x}_0))$ of $\pi_1(S^1 \vee S^1, x_0)$ corresponds to the subgroup $H \subseteq \langle a, b\rangle$ generated by these four elements. On the other hand free generators of the group $\pi_1(T, \tilde{x}_0')$ can be selected so that they are represented by the following loops based at $\tilde{x}_0'$:



In effect, the subgroup $p_*(\pi_1(T, \tilde{x}_0')) \subseteq \pi_1(S^1 \vee S^1, x_0)$ can be identified with the subgroup of $H' \subseteq \langle a, b\rangle$ which is generated by the elements aba^{-1}, a^3, a^2b, and a^2b^{-1}. Since each of these elements is obtained by conjugating the corresponding generator of the subgroup H by a we get: $H' = aHa^{-1}$. Notice that the conjugation by a comes from the fact that there is a path τ in T joining $\tilde{x}_0$ with $\tilde{x}_0'$ such that the loop $p \circ \tau$ represents the element $a^{-1} \in \langle a, b \rangle$

It is interesting to consider the case when the subgroup $p_*(\pi_1(T, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$ does not depend on the choice of the point $\tilde{x}_0 \in p^{-1}(x_0)$. This motivates the following definition:

18.7 Definition. Let $p\colon T \to X$ be a covering with a path connected total space T and let $x_0 \in X$. The covering p is *regular* if $p_*(\pi_1(T, \tilde{x}_0)) = p_*(\pi_1(T, \tilde{x}_0'))$ for any $\tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0)$.

18.8 Proposition. *Let $p\colon T \to X$ be a covering such that T is a path connected space and let $x_0 \in X$. The following conditions are equivalent:*

1) The covering p is regular.

2) For any $\tilde{x}_0 \in p^{-1}(x_0)$ the group $p_*(\pi_1(T, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$.

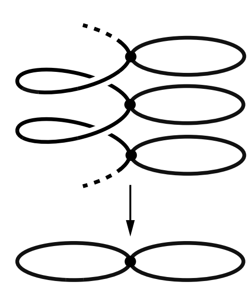
3) Let ω be a loop in X based at x_0. If ω has a lift which is a loop then every lift of ω is a loop, and if ω has a lift which is an open path then every lift of ω is an open path.

Proof. Exercise. □

So far the objective of our study of the fundamental group functor was to use properties of groups to get information about topological spaces. The relationship between fundamental groups and covering spaces can be used to show that we can also work in the opposite direction: sometimes we can use properties of topological spaces to derive facts about groups. The next result illustrates this approach.

18.9 Proposition. *Every free group on two or more generators contains a free subgroup on an infinite number of generators.*

Proof. It suffices to consider the case of a free group on two generators. Such group is isomorphic to $\pi_1(S^1 \vee S^1)$. Consider the covering $p\colon T \to S^1 \vee S^1$ obtained by gluing the universal covering over one copy of S^1 to the trivial covering over the second copy:



The total space T of this covering is homotopy equivalent to the space $\bigvee_{n\in\mathbb{Z}} S^1$, and so $\pi_1(T)$ is a free group on an infinite number of generators. By Theorem 18.1 $\pi_1(T)$ can be identified with a subgroup of $\pi_1(S^1 \vee S^1)$. □

Exercises to Chapter 18

E18.1 Exercise. Prove Proposition 18.3.

E18.2 Exercise. Prove Proposition 18.8.

E18.3 Exercise. Let $p\colon T \to X$ be a covering and let $f\colon Y \to X$ be an arbitrary continuous function. Let f^*T be a subspace of $Y \times T$ defined by

$$f^*T = \{(y, \tilde{x}) \in Y \times T \mid f(y) = p(\tilde{x})\}$$

Let $p'\colon f^*T \to Y$ be the map given by $p'(y, \tilde{x}) = y$.

a) Show that $p'\colon f^*T \to Y$ is a covering.

b) Assume that $(y_0, \tilde{x}_0) \in f^*T$. Show that $p'_*(\pi_1(f^*T, (y_0, \tilde{x}_0))) = f_*^{-1}(p_*(\pi_1(T, \tilde{x}_0)))$ where $f_*\colon \pi_1(Y, y_0) \to \pi_1(X, f(x_0))$ is the homomorphism induced by f.

E18.4 Exercise. A map $p\colon E \to B$ is a fibration if it has the homotopy lifting property, i.e. if for any maps $F\colon X \times [0, 1] \to B$ and $\widetilde{f}\colon X \times \{0\} \to E$ satisfying $F|_{X\times\{0\}} = p\widetilde{f}$ there exists $\widetilde{F}\colon X \times [0, 1] \to E$ such that $\widetilde{F}|_{X\times\{0\}} = \widetilde{f}$ and $p\widetilde{F} = F$:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widetilde{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B
\end{tikzcd}$$

For example, any covering is a fibration.

Let $p\colon E \to B$ be a fibration, let $b_0 \in B$, $F = p^{-1}(b_0)$, and $e_0 \in F$. Consider the homomorphisms

$$\pi_1(F, e_0) \xrightarrow{i_*} \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0)$$

where $i\colon F \to E$ is the inclusion map. Show that $\mathrm{Im}(i_*) = \mathrm{Ker}(p_*)$.

Note: if p is a covering then $\pi_1(F, e_0)$ is the trivial group (since F is a discrete space), so the above formula implies that $\mathrm{Ker}(p_*)$ is trivial. This gives another proof that for a covering p the induced homomorphism p_* is 1-1.

E18.5 Exercise. Let $p\colon E \to B$ be a fibration, let $b_0 \in B$, and $F = p^{-1}(b_0)$. Show that if B is a contractible space then the inclusion map $i\colon F \to E$ is a homotopy equivalence.