22 | Coverings and Group Actions

Let X be a topological space, and let $x_0 \in X$. Our goal in this chapter is to show that under some assumptions on X the category of path connected coverings of X is equivalent to the category of sets equipped with a transitive action of the group $\pi_1(X, x_0)$.

22.1 Definition. Let G be a group and S be a set. We say that G *acts on* X on the right if there exists a function

$$\mu\colon S \times G \to S$$

such that

(i) $\mu(s, e) = s$ for any $s \in S$, where $e \in G$ is the trivial element;
(ii) $\mu(\mu(s, g), h) = \mu(s, gh)$ for all $s \in S$, $h, g \in G$.

22.2 Note. From now on we will write sg instead of $\mu(s, g)$ in order to describe the action of g on s. We will also refer to sets with an action of a group G as G-sets.

22.3 Example. Let $p\colon T \to X$ be a covering and let $x_0 \in X$. We can define a right action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ as follows. For $[\omega] \in \pi_1(X, x_0)$ and $y \in p^{-1}(x_0)$ let $\widetilde{\omega}\colon [0, 1] \to T$ be the lift of ω such that $\widetilde{\omega}(0) = y$. Define:

$$y[\omega] := \widetilde{\omega}(1)$$

One can check that this satisfies the conditions of Definition 22.1 (exercise). The action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ defined in this way is called the *monodromy action* associated to the covering p.

22.4 Definition. We say that a group G acts on set S *transitively* if for any $s, s' \in S$ there exists $g \in G$ such that $sg = s'$.

22.5 Proposition. *Let $p\colon T \to X$ be a covering, and let $x_0 \in X$. If T is path connected then the monodromy action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is transitive.*

Proof. Exercise. □

22.6 Definition. Let G be a group and let S, S' be G-sets. A function $f\colon S \to S'$ is *G-equivariant* if $f(sg) = f(s)g$ for all $s \in S$ and $g \in G$.

22.7 Note. G-sets and G-equivariant functions form a category which we will denote by $\mathbf{Set}_G$.

22.8 Proposition. *Let X be a space, and let*

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

be a map of coverings. For any $x_0 \in X$ the induced map of fibers $f\colon p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ is $\pi_1(X, x_0)$-equivariant.

Proof. Exercise. □

22.9 Corollary. *Let X be a space and let $x_0 \in X$. The assignment which associates to each path connected covering $p\colon T \to X$ the $\pi_1(X, x_0)$-set $p^{-1}(x_0)$ and to each map of coverings the map of fibers defines a functor*

$$\Lambda\colon \mathbf{Cov}(X) \to \mathbf{Set}_{\pi_1(X,x_0)}$$

Proof. Exercise. □

For the reminder of this chapter we will be restrict attention to coverings $T \to X$ where T is a path connected space. Let $\mathbf{PCov}(X)$ denote the category of all such covering of X. Also, for a group G let $\mathbf{TSet}_G$ denote the category of all G-sets with a transitive action of G. By Proposition 22.5 the functor Λ restricts to a functor

$$\Lambda\colon \mathbf{PCov}(X) \to \mathbf{TSet}_{\pi_1(X,x_0)}$$

Our next goal is to show that the following holds:

22.10 Theorem. *Let X be a connected, locally path connected, and semi-locally simply connected space, and let $x_0 \in X$. The functor*

$$\Lambda\colon \mathbf{PCov}(X) \to \mathbf{TSet}_{\pi_1(X,x_0)}$$

is an equivalence of categories.

By Proposition 21.3 the proof of Theorem 22.10 can be split into two parts:

1) We need to show that any set with a transitive action of the group $\pi_1(X, x_0)$ is isomorphic to a $\pi_1(X, x_0)$-set $\Lambda(p\colon T \to X) = p^{-1}(x_0)$ for some path connected covering p.

2) We also need to show that maps of path connected coverings of X are in a bijective correspondence with $\pi_1(X, x_0)$-equivariant maps of their fibers.

Part 1) will follow immediately from the following result:

22.11 Proposition. *Let X be a connected, locally path connected, and semi-locally simply connected space and let $x_0 \in X$. The map*

$$\Lambda\colon \begin{pmatrix} \textit{isomorphism classes} \\ \textit{of path connected} \\ \textit{coverings of } X \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{isomorphism classes} \\ \textit{of sets with transitive} \\ \textit{action of } \pi_1(X, x_0) \end{pmatrix}$$

given by $\Lambda(p\colon T \to X) = p^{-1}(x_0)$ is a bijection.

The proof of Proposition 22.11 will use some properties of transitive G-sets that we develop below.

22.12 Definition. Let G be a group, and S be a G-set. The *stabilizer* of en element $s \in S$ is the subgroup $G_s \subseteq G$ given by:

$$G_s = \{g \in G \mid sg = s\}$$

22.13 Proposition. *Let $p\colon T \to X$ be a covering, and let $x_0 \in X$. The stabilizer of an element $\tilde{x} \in p^{-1}(x_0)$ under the monodromy action is the subgroup $p_*(\pi_1(T, \tilde{x})) \subseteq \pi_1(X, x_0)$.*

Proof. Exercise. $\square$

22.14 Lemma. *Let G be a group.*

1) *If G acts transitively on a set S and $s, s' \in S$ then the stabilizers G_s and $G_{s'}$ are conjugate subgroups of the group G.*
2) *Let S be a set with an action of G and let $s \in S$. The assignment $S \mapsto G_s$ defines a bijective correspondence:*

$$\Phi\colon \begin{pmatrix} \textit{isomorphism classes} \\ \textit{of sets with a transitive} \\ \textit{action of } G \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{conjugacy classes} \\ \textit{of subgroups} \\ \textit{of } G \end{pmatrix}$$

Proof. 1) Since G acts transitively we have $s' = sh$ for some $h \in G$. We will show that $G_{s'} = h^{-1}G_s h$.

For $g \in G_s$ we have:

$$s'(h^{-1}gh) = sgh = sh = s'$$

Therefore $h^{-1}G_sh \subseteq G_{s'}$. Conversely, if $g \in G_{s'}$ then

$$sh = s' = s'g = shg$$

This implies that $s = s(hgh)^{-1}$, so $hgh^{-1} \in G_s$, or equivalently $g \in h^{-1}G_sh$. Thus $G_{s'} \subseteq h^{-1}G_sh$.

2) We will construct a function

$$\Psi\colon \begin{pmatrix} \text{conjugacy classes} \\ \text{of subgroups} \\ \text{of } G \end{pmatrix} \longrightarrow \begin{pmatrix} \text{isomorphism classes} \\ \text{of sets with transitive} \\ \text{action of } G \end{pmatrix}$$

which is the inverse of Φ. For a subgroup $H \subseteq G$ let $H\backslash G$ denote the set of right cosets of H in G. Define an action of G on $H\backslash G$ by $(Hg)g' = H(gg')$. This action is transitive since for any $Hg, Hg' \in H\backslash G$ we have $Hg' = (Hg)(g^{-1}g')$. Let $\Psi(H) = H\backslash G$. In order to show that Ψ is well defined on conjugacy classes we need to check that if $H' \subseteq G$ is a subgroup conjugate to H then the G-sets $H\backslash G$ and $H'\backslash G$ are isomorphic. Assume then that $H' = kHk^{-1}$ for some $k \in G$. Define $f\colon H\backslash G \to H'\backslash G$ by $f(Hg) = H'kg$. One can check that this is a well defined isomorphism of G-sets (exercise). Let $e \in G$ be the trivial element. Since the stabilizer of $He \in H\backslash G$ is the subgroup H, we obtain that $\Phi\Psi([H]) = [H]$ where $[H]$ denotes the conjugacy class of H, and so $\Phi\Psi$ is an identity function. One can check that the composition $\Psi\Phi$ also is an identity (exercise).

□

Proof of Proposition 22.11. Consider the diagram:

$$\begin{tikzcd}
\begin{pmatrix} \text{isomorphism classes} \\ \text{of path connected} \\ \text{coverings of } X \end{pmatrix} \arrow[rr, "\Lambda"] \arrow[dr, "\Omega"'] & & \begin{pmatrix} \text{isomorphism classes} \\ \text{of sets with transitive} \\ \text{action of } \pi_1(X, x_0) \end{pmatrix} \arrow[dl, "\Phi"] \\
& \begin{pmatrix} \text{conjugacy classes} \\ \text{of subgroups} \\ \text{of } \pi_1(X, x_0) \end{pmatrix} &
\end{tikzcd}$$

The map Φ is defined as in proposition 22.14, and and Ω is defined as in Theorem 21.1. By Proposition 22.13 this diagram commutes. Since Φ is a bijection by Proposition 22.14, and Ω is a bijection by Theorem 21.1 we obtain that Λ is a bijection. □

Next, we turn to properties of the functor Λ related to maps of coverings. We will show that the following holds.

22.15 Proposition. *Let X be a connected and locally path connected space, and let $x_0 \in X$. For any path connected coverings $p_i\colon T_i \to X$, $i = 1, 2$ the assignment*

$$\Lambda\colon \begin{pmatrix} \textit{maps of coverings} \\ T_1 \to T_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \pi_1(X, x_0)\textit{-equivariant maps} \\ p_1^{-1}(x_0) \to p_2^{-1}(x_0) \end{pmatrix}$$

is a bijection.

The proof of Proposition 22.15 will use the following fact. Recall that for a G-set S by G_s we denote the stabilizer of an element $s \in S$

22.16 Lemma. *Let S, T be sets with a transitive action of a group G, and let $s_0 \in S$, $t_0 \in T$. A G-equivariant map $f\colon S \to T$ such that $f(s_0) = t_0$ exists if and only if $G_{s_0} \subseteq G_{t_0}$. Moreover, if such map exists then it is unique.*

Proof. Exercise. □

Proof of Proposition 22.15. We will prove first that Λ is onto. Let $f\colon p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ be a $\pi_1(X, x_0)$-equivariant map. We need to show that there exists a map of coverings $\bar{f}\colon T_1 \to T_2$ such that $\Lambda(\bar{f}) = f$. Let $\tilde{x}_1 \in p_1^{-1}(x_0)$, and let $\tilde{x}_2 = f(\tilde{x}_1)$. Combining Proposition 22.13 and Lemma 22.16 we obtrain

$$p_{1*}(\pi_1(T_1, \tilde{x}_1)) \subseteq p_{2*}(\pi_1(T_2, \tilde{x}_2))$$

Therefore, by the lifting criterion (19.5) there exists a map of coverings $\bar{f}\colon T_1 \to T_2$ such that $f(\tilde{x}_1) = \tilde{x_2}$. Since the map $\Lambda(\bar{f})\colon p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ satisfies $\Lambda(\bar{f})(\tilde{x}_1) = \tilde{x}_2$ the uniqueness part of Lemma 22.16 gives $\Lambda(\bar{f}) = f$.

Next, assume that $f, f'\colon T_1 \to T_2$ are maps of coverings such that $\Lambda(f) = \Lambda(f')$. This implies that for $\tilde{x} \in p^{-1}(x_0)$ we have

$$f(\tilde{x}) = \Lambda(f)(\tilde{x}) = \Lambda(f')(\tilde{x}) = f'(\tilde{x})$$

By Lemma 17.11 this gives $f = f'$.

□

Proof of Theorem 22.10. Follows directly from Propositions 21.3, 22.11 and 22.15. □

Exercises to Chapter 22

E22.1 Exercise. Prove Proposition 22.5.

E22.2 Exercise. Prove Proposition 22.8.

E22.3 Exercise. Prove Proposition 22.13.

E22.4 Exercise. Prove Lemma 22.16.