11 | Proof of van Kampen's Theorem

10.17 van Kampen Theorem. *Let (X, x_0) be a pointed topological space and let $U_1, U_2 \subseteq X$ be open sets such that $X = U_1 \cup U_2$. If the sets U_1, U_2, and $U_1 \cap U_2$ are path connected and $x_0 \in U_1 \cap U_2$ then*

$$\pi_1(X, x_0) \cong \operatorname{colim}(\pi_1(U_1, x_0) \xleftarrow{i_{1*}} \pi_1(U_1 \cap U_2, x_0) \xrightarrow{i_{2*}} \pi_1(U_2, x_0))$$

where for $k = 1, 2$ the homomorphism i_{k*} is induced by the inclusion map $i_k\colon U_1 \cap U_2 \to U_k$.

11.1 Theorem. *Let (X, x_0) be a pointed topological space and let $\{U_i\}_{i\in I}$ be an open cover of X such that $x_0 \in U_i$ for all $i \in I$. For $i, j \in I$ let $f_{ij}\colon U_i \cap U_j \to U_i$ denote the inclusion map. If the set $U_i \cap U_j \cap U_k$ is path connected for all $i, j, k \in I$ then*

$$\pi_1(X, x_0) \cong *_{i\in I}\pi_1(U_i, x_0)/N$$

where N is the normal subgroup of $*_{i\in I}\pi_1(U_i, x_0)$ generated by all elements of the form $f_{ij}([\omega]) \cdot f_{ji}([\omega])^{-1}$ for $i, j \in I$ and $[\omega] \in \pi_1(U_i \cap U_j, x_0)$.

11.2 Proposition. *Let $\{(X_i, x_i)\}_{i\in I}$ be a family of path connected pointed spaces, and let $\bigvee_{i\in I} X_i$ be the space obtained by identifying the basepoints $x_i \sim x_j$ for all $i, j \in I$. Assume that for each $i \in I$ there exists a set $V_i \subseteq X_i$ such that V_i is open in X_i, $x_i \in V_i$, and the one-point space $\{x_i\}$ is a deformation retract of V_i. Then*

$$\pi_1(\bigvee_i X_i) \cong *_{i\in I}\pi_1(X_i, x_i)$$