# 13 | Homotopy Extension Property

In this chapter we begin work toward computing fundamental groups of CW complexes. Since a 0-dimensional CW complex is a discrete space, the fundamental group of any such complex is trivial. The first non-trivial case we will develop a formula for the fundamental group of a CW complex of dimension 1. Our main tool will be the homotopy extension property, which is one of the most important notions of algebraic topology.

**13.1 Definition.** Let $X$ be a topological space, and let $A \subseteq X$. The pair $(X, A)$ has the *homotopy extension property* if any map

$$h\colon X \times \{0\} \cup A \times [0,1] \to Y$$

can be extended to a map $\bar{h}\colon X \times [0,1] \to Y$.

The following proposition is often useful when we want to verify that the homotopy extension property holds for a given pair of $(X, A)$:

**13.2 Proposition.** *A pair $(X, A)$ has the homotopy extension property if and only if $X \times \{0\} \cup A \times [0,1]$ is a retract of $X \times [0,1]$.*

*Proof.* Exercise. □

The next fact implies that the homotopy extension property does not hold for arbitrary pairs of spaces:

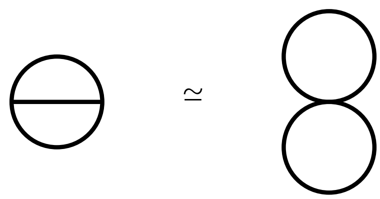
**13.3 Proposition.** *If a pair $(X, A)$ has the homotopy extension property and $X$ is a Hausdorff space then $A$ is closed in $X$.*

*Proof.* Exercise. □

**13.4 Proposition.** *If a pair $(X, A)$ has the homotopy extension property and the space $A$ is contractible then the quotient map $q\colon X \to X/A$ is a homotopy equivalence.*

*Proof.* Exercise ☐

**13.5 Example.** In Example 8.18 we have shown that the space $X$ consisting of a circle and its diagonal is homotopy equivalent to a wedge of two circles:



We can obtain the same result as follows. Let $A \subseteq X$ be the diagonal of the circle (together with its enpoints). It will follow from Theorem 13.7 that the pair $(X, A)$ has the homotopy extension property. Since the space $A$ is contractible, using Proposition 13.4 we get a homotopy equivalence $X \simeq X/A$. It remains to notice that $X/A$ is homeomorphic to $S^1 \vee S^1$.

**13.6 Example.** Here is an example which shows that Proposition 13.4 is not true in general, if $(X, A)$ does not have the homotopy extension property. The *Warsaw circle* is a subspace $W$ of $\mathbb{R}^2$ consisting of three subsets:

$$W = A \cup B \cup C$$

where:

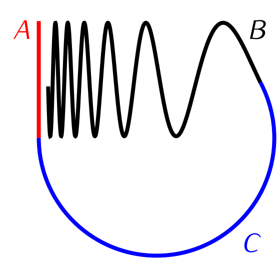
- $A$ is a segment of the $y$-axis:

$$A = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$$

- $B$ is a part of the graph of the function $f(x) = \sin\left(\frac{1}{x}\right)$:

$$B = \{(x, \sin\left(\tfrac{1}{x}\right)) \in \mathbb{R}^2 \mid 0 < x \leq \tfrac{1}{2\pi}\}$$

- $C$ is an arc joining points $(0, -1) \in A$ and $(\frac{1}{2\pi}, 0) \in B$, and disjoint from $A \cup B$ at all other points.



Consider the pair $(W, A)$. One can show that the quotient space $W/A$ is homeomorphic to the circle $S^1$ (exercise), so in particular $\pi_1(W/A) \cong \mathbb{Z}$. On the other hand, $\pi_1(W) \cong \{1\}$ (exercise). Therefore $W/A$ is not homotopy equivalent to $W$.

**13.7 Theorem.** *Any relative CW complex $(X, Y)$ has the homotopy extension property.*

**13.8 Lemma.** *For any $n > 0$ the pair $(D^n, S^{n-1})$ has the homotopy extension property.*

While it is not difficult to prove Lemma 13.8 directly, we will show that it follows from a more general fact. Recall (8.15) that for a map $f\colon X \to Y$ the mapping cylinder of $f$ is the space $M_f = (X \times [0, 1] \sqcup Y)/\sim$ where $(x, 1) \sim f(x)$ for all $x \in X$. Notice that the space $X$ is homeomorphic with the subspace $X \times \{0\} \subseteq M_f$.

**13.9 Proposition.** *For any continuous function $f\colon X \to Y$ the pair $(M_f, X \times \{0\})$ has the homotopy extension property.*

*Proof.* Exercise. □

*Proof of Lemma 13.8.* Let $c\colon S^{n-1} \to \{*\}$ be the constant function. We have a homeomorphism $f\colon M_c \to D^n$ given by $f(x, t) = (1 - t)x$. Moreover, $f(S^{n-1} \times \{0\}) = S^{n-1} \subseteq D^n$. Since by Proposition 13.9 the pair $(M_c, S^{n-1} \times \{0\})$ has the homotopy extension property it follows that $(D^n, S^{n-1})$ also has this property. □

**13.10 Lemma.** *Let $Y$ be any space an let $X = Y \cup \{e^n_\alpha\}_{\alpha \in I}$ be a space obtained from by attaching some number of $n$-cells to $X$. Then the pair $(X, Y)$ has the homotopy extension property.*

*Proof.* To simplify notation we will assume that $X$ is obtained from $Y$ by attaching a single $n$-cell: $X = Y \cup e^n$. The proof in the general case is essentially the same. By Proposition 13.2 it will suffice to show that $X \times \{0\} \cup Y \times [0, 1]$ is a retract of $X \times [0, 1]$. Let $f\colon S^{n-1} \to Y$ be the attaching map of the cell $e^n$. We have a homeomorphisms

$$X \times [0, 1] \simeq (D^n \times [0, 1] \sqcup Y \times [0, 1])/\sim$$

and

$$X \times \{0\} \cup Y \times [0, 1] \simeq ((D^n \times \{0\} \cup S^{n-1} \times [0, 1]) \sqcup Y \times [0, 1])/\sim$$

where $(x, t) \sim (f(x), t)$ for $x \in S^{n-1}$. By Lemma 13.8 there is a retraction

$$r\colon D^n \times [0, 1] \to D^n \times \{0\} \cup S^{n-1} \times [0, 1]$$

The map

$$r \sqcup \mathrm{id}_{Y \times [0,1]}\colon ((D^n \times \{0\} \cup S^{n-1} \times [0, 1]) \sqcup Y \times [0, 1])/\sim\ \to\ (D^n \times [0, 1] \sqcup Y \times [0, 1])/\sim$$

gives the desired retraction $X \times [0, 1] \to X \times \{0\} \cup Y \times [0, 1]$. □

*Proof of Theorem 13.7.* Recall (12.7) that if $(X, Y)$ is a relative CW complex then $X = \bigcup_{n=-1}^{\infty} X^{(n)}$ where $X^{(-1)} = Y$ and for $n \geq 0$ the subspace of $X^{(n)} \subseteq X$ obtained by attaching $n$-cells to $X^{(n-1)}$. By Lemma 13.10 for each $n \geq 0$ there exists a retraction

$$r_n \colon X^{(n)} \times [0,1] \to X^{(n)} \times \{0\} \cup X^{(n-1)} \times [0,1]$$

We can extend $r_n$ to a map

$$\bar{r}_n \colon X \times \{0\} \cup X^{(n)} \times [0,1] \to X \times \{0\} \cup X^{(n-1)} \times [0,1]$$

by setting $\bar{r}_n(x,0) = (x,0)$ for $x \in X$. Define:

$$r \colon X \times [0,1] \to X \times \{0\} \cup Y \times [0,1]$$

by $r(x,t) = \bar{r}_0 \circ \bar{r}_1 \circ \ldots \circ \bar{r}_n(x,t)$ if $x \in X^{(n)}$, $n \geq 0$, and $r(x,t) = (x,t)$ if $x \in X^{(-1)} = Y$. One can check that $r$ is a well defined, continuous retraction (exercise).

□

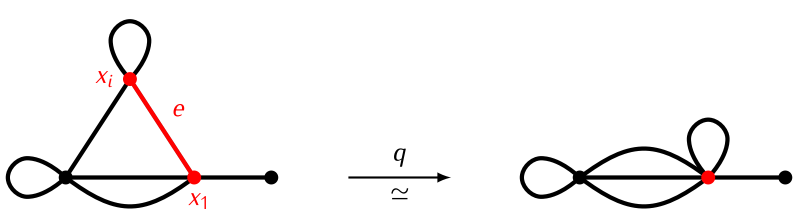
**13.11 Theorem.** *If $X$ is a path connected finite CW complex of dimension 1 then $X \simeq \bigvee_{i=1}^{n} S^1$ where*

$$n = \begin{pmatrix} \textit{number of} \\ \textit{1-cells of } X \end{pmatrix} - \begin{pmatrix} \textit{number of} \\ \textit{0-cells of } X \end{pmatrix} + 1$$

**13.12 Corollary.** *If $X$ is a path connected finite CW complex of dimension 1 then $\pi_1(X) \cong *_{i=1}^{n}\mathbb{Z}$ where $n$ is defined as in Theorem 13.11.*

*Proof.* This follows from Theorem 13.11 and Example 10.19. □

*Proof of Theorem 13.11.* We will argue by induction with respect to the number $k$ of 0-cells in $X$. If $k = 1$ then the statement is obvious. Assume then that the statement of theorem is true for all complexes whose number of 0-cells is $k$, and let $X$ be a path connected finite 1-dimensional CW complex whose set of 0-cells is $\{x_1, x_2, \ldots, x_{k+1}\}$ for some $k \geq 1$. Since $X$ is path connected there exists a 1-cell $e$ in $X$ that joins $x_1$ with some other 0-cell $x_i$. Let $A$ denote the subcomplex of $X$ consisting of the cells $x_1$, $x_i$ and $e$. Notice that $A$ is homeomorphic to the closed interval $[0,1]$. The pair $(X, A)$ is a relative CW complex, so by Theorem 13.7 it satisfies the homotopy extension property. Since $A$ is contractible, by Proposition 13.4 the quotient map $q \colon X \to X/A$ is a homotopy equivalence.



The space $X/A$ has the structure of a 1-dimensional CW complex with one 0-cell and one 1-cell less than $X$. Therefore, by the inductive assumption the statement of the theorem holds for $X/A$, and so it also holds for $X$.

□

Theorem 13.11 can be generalized to infinite 1-dimensional complexes:

**13.13 Theorem.** *If $X$ is a path connected 1-dimensional CW complex then $X \simeq \bigvee_{I \in I} S^1$ for some set $I$. As a consequence $\pi_1(X) \cong *_{i \in I}\mathbb{Z}$.*

**13.14 Note.** For a finite CW complex $X$, let $c_n(X)$ denote the number of $n$-cells of $X$. Theorem 13.11 implies that if $X$ is a path connected CW complex of dimension 1, then the number $c_1(X) - c_0(X)$ depends only on the homotopy type of $X$: if $Y$ is another such CW complex and $X \simeq Y$ then $c_1(X) - c_0(X) = c_1(Y) - c_0(Y)$. This observation can be generalized as follows. The *Euler characteristic* of a finite CW complex $X$ is the integer $\chi(X) = \sum_n (-1)^n c_n(X)$. One can show that if $X$ and $Y$ are finite CW complexes and $X \simeq Y$ then $\chi(X) = \chi(Y)$.

## Exercises to Chapter 13

**E13.1 Exercise.** Prove Proposition 13.2.

**E13.2 Exercise.** Prove Proposition 13.4.

**E13.3 Exercise.** Show that if a pair $(X, A)$ has the homotopy extension property then for any space $Y$ the pair $(X \times Y, A \times Y)$ also has the homotopy extension property.

**E13.4 Exercise.** Prove Proposition 13.9.

**E13.5 Exercise.** Given spaces $X, Y$ let $[X, Y]$ denote the set of homotopy classes of maps $f\colon X \to Y$. A map of spaces $g\colon X \to X'$ induces a map of sets $g^*\colon [X', Y] \to [X, Y]$ given by $g^*([f]) = [fg]$. Let $A \subseteq X$, let $j\colon A \to X$ be the inclusion and $q\colon X \to X/A$ be the quotient map. For any $Y$ this induces maps of sets

$$[X/A, Y] \xrightarrow{q^*} [X, Y] \xrightarrow{j^*} [A, Y]$$

Show that if the pair $(X, A)$ has the homotopy extension property then $j^*[f]$ is the homotopy class of a constant map $A \to Y$ if and only if $[f] = q^*[f']$ for some $f'\colon X/A \to Y$.

**E13.6 Exercise.** Let $(X, x_0)$, $(Y, y_0)$ be pointed spaces. Denote by $[X, Y]_*$ the set of pointed homotopy classes of basepoint preserving maps $X \to Y$. That is, any map $f\colon (X, x_0) \to (Y, y_0)$ defines an element $[f]_* \in [X, Y]_*$, and $[f]_* = [g]_*$ if $f \simeq g$ (rel $\{x_0\}$). Also, let $[X, Y]$ be the set of homotopy classes of all functions $X \to Y$. Thus, any map $f\colon X \to Y$ defines an element $[f] \in [X, Y]$, and $[f] = [g]$ if $f \simeq g$ (there is no assumption that maps or homotopies preserve the basepoints). Let

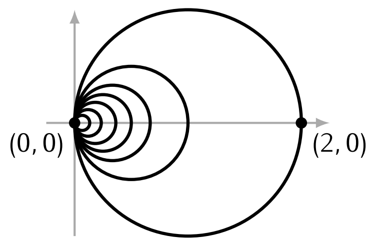
$$\Phi\colon [X, Y]_* \to [X, Y]$$

be a function given by $\Phi([f]_*) = [f]$.

a) Assume that the pair $(X, x_0)$ has the homotopy extension property, and $Y$ is a path connected space. Show that $\Phi$ is onto.

b) Assume that in addition the group $\pi_1(X, x_0)$ is trivial. Show that $\Phi$ is a bijection.

**E13.7 Exercise.** The *Hawaiian earring* space is a subspace $X \subseteq \mathbb{R}^2$ given by $X = \bigcup_{n=1}^{\infty} C_n$ where $C_n$ is the circle with radius $\frac{1}{n}$ and center at the point $(0, \frac{1}{n})$:




Denote $x_0 = (0, 0)$ and $y_0 = (2, 0)$. Let $\mathrm{id}_X\colon X \to X$ be the identity map.

a) Show that there does not exist a map $g\colon X \to X$ such that $\mathrm{id}_X \simeq g$ and $g(x_0) = y_0$.

b) Show that the pair $(X, x_0)$ does not have the homotopy extension property. (Hint: use Exercise 13.6).

**E13.8 Exercise.** Let $(X, Y)$ be a relative CW complex, let $j\colon Y \to X$ be the inclusion map, and let $C_j$ be the mapping cone of $j$. Show that $C_j$ is homotopy equivalent to the space $X/Y$.

**E13.9 Exercise.** Assume that $(X, A)$ is a pair with the homotopy extension property such that the inclusion map $i\colon A \hookrightarrow X$ is a homotopy equivalence.

a) Show that $A$ is a retract of $X$.

b) Show that $A$ is a strong deformation retract of $X$.

**E13.10 Exercise.** Let $f\colon (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. Show that if $X$ is a path connected 1-dimensional CW complex and $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is the trivial homomorphism then $f$ is homotopic to a constant map.

**E13.11 Exercise.** Let $X$ be a finite, path connected CW complex.

a) Show that $X$ is homotopy equivalent to a CW complex $X'$ which has only one 0-cell.

b) Show that if $\pi_1(X) = \{1\}$ then $X$ is homotopy equivalent to a CW complex $X''$ which has only one 0-cell and no 1-cells.