3 | The Fundamental Group

3.1 Definition. A *pointed topological space* is a pair (X, x_0), where X is a topological space and $x_0 \in X$. We say that x_0 is the *basepoint* of X. Given two pointed spaces (X, x_0) and (Y, y_0) a *basepoint preserving map* $f\colon (X, x_0) \to (Y, y_0)$ is a continuous function $f\colon X \to Y$ such that $f(x_0) = y_0$.

Let $\mathbf{Top}_*$ denote the category the objects of which are pointed spaces and morphisms are basepoint preserving maps. Our goal in this chapter will be to construct the *fundamental group functor*

$$\pi_1\colon \mathbf{Top}_* \to \mathbf{Gr}$$

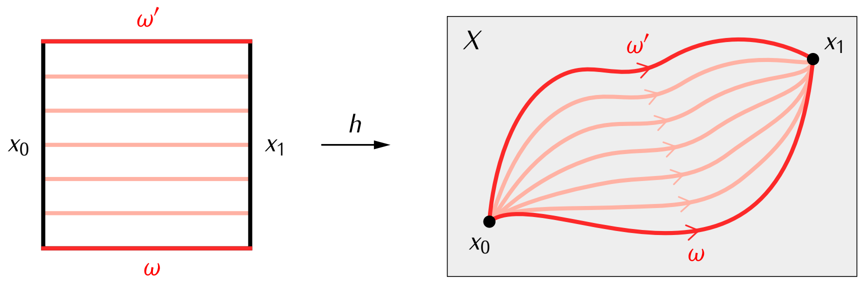
That is, we will construct an assignment that associates to every pointed space (X, x_0) a group $\pi_1(X, x_0)$ and to every basepoint preserving map $f\colon (X, x_0) \to (Y, y_0)$ a group homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ in a way that preserves identity functions and compositions of functions.

Recall that a path in a space X is a continuous function $\omega\colon [0, 1] \to X$.

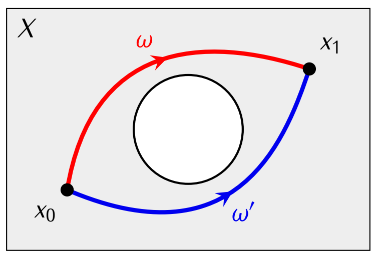
3.2 Definition. Let $\omega, \omega'\colon [0, 1] \to X$ be paths such that $\omega(0) = \omega'(0) = x_0$ and $\omega(1) = \omega'(1) = x_1$ for some $x_0, x_1 \in X$. We say that the paths ω and ω' are *path homotopic* if for every $t \in [0, 1]$ there exists a path $h_t\colon [0, 1] \to X$ such that:

1) $h_t(0) = x_0$, and $h_t(1) = x_1$ for all $t \in [0, 1]$
2) $h_0 = \omega$, and $h_1 = \omega'$
3) the function $h\colon [0, 1] \times [0, 1] \to X$ given by $h(s, t) = h_t(s)$ is continuous.

In this case we write $\omega \simeq \omega'$ and we say that h is a *path homotopy* between ω and ω'.



Intuitively, path homotopy is a device for detecting holes in topological spaces. If ω and ω' are paths in X with the same endpoints but such that $\omega \not\simeq \omega'$ then it indicates that there is a hole in X that prevents us from deforming ω to ω':



3.3 Lemma. *Let X be a space and let $x_0, x_1 \in X$. Path homotopy defines an equivalence relation on the set of paths in X that start at x_0 and terminate at x_1.*

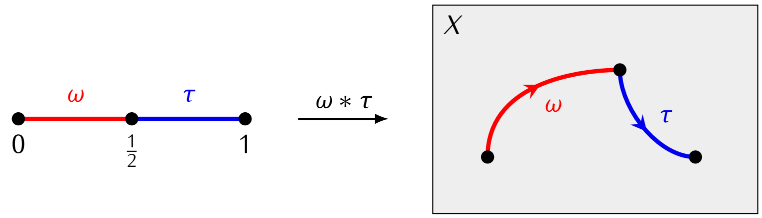
Proof. Exercise. □

3.4 Definition. For a path ω we will denote by $[\omega]$ the equivalence class of ω taken with respect to the equivalence relation given by path homotopy. We will say that $[\omega]$ is the *homotopy class* of ω.

3.5 Notation. Let X be a space and let $x_0, x_1 \in X$. By $\pi_1(X, x_0, x_1)$ we will denote the set of homotopy classes of paths that begin at x_0 and terminate at x_1. If $x_0 = x_1$ then we will write $\pi_1(X, x_0)$ instead of $\pi_1(X, x_0, x_0)$. Notice that elements of $\pi_1(X, x_0)$ are homotopy classes $[\omega]$ where ω is a path such that $\omega(0) = \omega(1) = x_0$. We call such ω a *loop* based at x_0.

3.6 Definition. Let $\omega, \tau\colon [0,1] \to X$ be paths such that $\omega(1) = \tau(0)$. The *concatenation* of ω and τ is the path $\omega * \tau\colon [0,1] \to X$ given by

$$(\omega * \tau)(s) = \begin{cases} \omega(2s) & \text{for } s \in [0, \frac{1}{2}] \\ \tau(2s-1) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

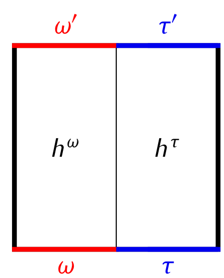


3.7 Proposition. *Let ω ,τ be paths in X such that $\omega(1) = \tau(0)$. If ω', τ' are paths such that $\omega \simeq \omega'$ and $\tau \simeq \tau'$ then $\omega * \tau \simeq \omega' * \tau'$.*

Proof. Let $h^\omega\colon [0,1]\times[0,1] \to X$ be a path homotopy between ω and ω' and $h^\tau\colon [0,1]\times[0,1]\to X$ be a path homotopy between τ and τ'. Define $h\colon [0,1]\times[0,1]\to X$ by

$$h(s,t) = \begin{cases} h^\omega(2s,t) & \text{for } s\in[0,\frac{1}{2}] \\ h^\tau(2s-1,t) & \text{for } s\in[\frac{1}{2},1]\end{cases}$$

The map h is a path homotopy between $\omega * \tau$ and $\omega' * \tau'$.



□

Notice that by Proposition 3.7 the homotopy class of $\omega * \tau$ depends only on the homotopy classes of ω and τ. Therefore for any $x_0, x_1, x_2 \in X$ we obtain a well defined function

$$\mu\colon \pi_1(X, x_0, x_1) \times \pi_1(X, x_1, x_2) \longrightarrow \pi_1(X, x_0, x_2)$$

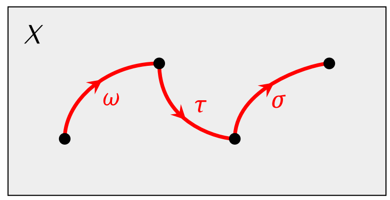
where $\mu([\omega],[\tau]) = [\omega * \tau]$. To simplify notation we will write $[\omega]\cdot[\tau]$ instead of $\mu([\omega],[\tau])$. In the case when $x_0 = x_1 = x_2$ this gives a multiplication on the set $\pi_1(X, x_0)$:

$$\pi_1(X,x_0)\times\pi_1(X,x_0)\longrightarrow\pi_1(X,x_0), \quad [\omega]\cdot[\tau] = [\omega * \tau]$$

Our next goal will be to show that the set $\pi_1(X, x_0)$ taken with this multiplication is a group.

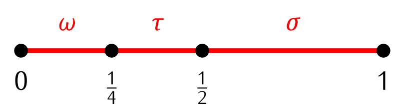
3.8 Lemma. *If ω, τ, σ are paths in a space X such that $\omega(1) = \tau(0)$ and $\tau(1) = \sigma(0)$ then*

$$([\omega] \cdot [\tau]) \cdot [\sigma] = [\omega] \cdot ([\tau] \cdot [\sigma])$$



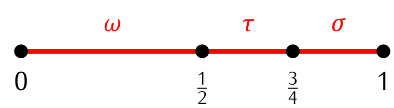
Proof. We have:

$$([\omega] \cdot [\tau]) \cdot [\sigma] = [\omega * \tau] \cdot [\sigma] = [(\omega * \tau) * \sigma]$$

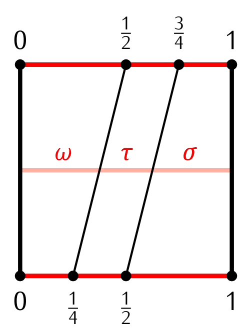


Similarly:

$$[\omega] \cdot ([\tau] \cdot [\sigma]) = [\omega] \cdot [\tau * \sigma] = [\omega * (\tau * \sigma)]$$



We need to show that $(\omega * \tau) * \sigma \simeq \omega * (\tau * \sigma)$. Graphically a homotopy $h\colon [0,1] \times [0,1] \to X$ between these paths can be represented as follows:



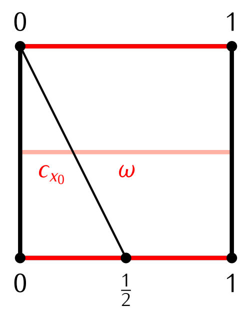
More precisely, h is given by the following formula:

$$h(s,t) = \begin{cases} \omega\left(\frac{4s}{t+1}\right) & \text{for } s \in [0, \frac{t+1}{4}] \\ \tau(4s-t-1) & \text{for } s \in [\frac{t+1}{4}, \frac{t+2}{4}] \\ \sigma\left(\frac{4s-t-2}{2-t}\right) & \text{for } s \in [\frac{t+2}{4}, 1] \end{cases}$$

□

3.9 Lemma. *Let X be a space, and let $x_0 \in X$. Let $c_{x_0}\colon [0,1] \to X$ denote the constant path at the point x_0: $c_{x_0}(s) = x_0$ for all $t \in [0,1]$. If ω is a path in X such that $\omega(0) = x_0$ then $[c_{x_0}] \cdot [\omega] = [\omega]$. Also, if τ is a path such that $\tau(1) = x_0$ then $[\tau] \cdot [c_{x_0}] = [\tau]$.*

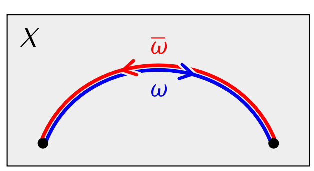
Proof. To obtain the first equality we need to check that $c_{x_0} * \omega \simeq \omega$. A homotopy between these paths can be represented as follows:



The second equality comes from a homotopy $\tau * c_{x_0} \simeq \tau$ that can be obtained in a similar way.

□

Let $\omega\colon [0,1] \to X$ be a path. By $\overline{\omega}$ we will denote the path given by $\overline{\omega}(s) = \omega(1-s)$ for $s \in [0,1]$. In other words $\overline{\omega}$ is obtained by reversing the orientation of ω::



We will say that $\overline{\omega}$ is the *inverse* of ω. This name is justified by the following fact:

3.10 Lemma. *Let ω be a path in a space X such that $\omega(0) = x_0$ and $\omega(1) = x_1$. We have:*

$$[\omega] \cdot [\overline{\omega}] = [c_{x_0}], \qquad [\overline{\omega}] \cdot [\omega] = [c_{x_1}]$$

Proof. Intuitively, a homotopy h between c_{x_0} and $\omega * \overline{\omega}$ can be obtained by taking h_t to be the path that goes from x_0 to the point $\omega(t) = \overline{\omega}(1-t)$ along ω, and the follows $\overline{\omega}$ back to x_0. Formally, we can define h as follows:

$$h(s,t) = \begin{cases} \omega(2st) & \text{for } s \in [0, \frac{1}{2}] \\ \omega((2-2s)t) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

□

3.11 Proposition. *Let X be a topological space and let $x_0 \in X$. The set $\pi_1(X, x_0)$ taken with the multiplication given by*

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

for $[\omega], [\tau] \in \pi_1(X, x_0)$ is a group. The trivial element in this group is the homotopy class of the constant path $[c_{x_0}]$, and for $[\omega] \in \pi_1(X, x_0)$ we have $[\omega]^{-1} = [\overline{\omega}]$.

Proof. The multiplication is associative by Lemma 3.8. The element $[c_{x_0}]$ is trivial with respect to this multiplication by Lemma 3.9, and $[\overline{\omega}]$ is the multiplicative inverse of $[\omega]$ by Lemma 3.10. □

3.12 Definition. Let (X, x_0) be a pointed space. The group $\pi_1(X, x_0)$ is called the *fundamental group* of (X, x_0).

3.13 Lemma. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. If $\omega\colon [0,1] \to X$ is a loop in X based at x_0 then $f \circ \omega\colon [0,1] \to Y$ is a loop in Y based at y_0. Moreover, if ω' is another loop in X based at x_0 such that $\omega \simeq \omega'$ then $f \circ \omega \simeq f \circ \omega'$.*

Proof. If $h\colon [0,1] \times [0,1] \to X$ is a homotopy between ω and ω' then $f \circ h\colon [0,1] \times [0,1] \to Y$ gives a homotopy between $f\omega$ and $f\omega'$. □

By Lemma 3.13 each map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ defines a function

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

given by $f_*([\omega]) = [f \circ \omega]$. In addition we have:

3.14 Proposition. *If $f\colon (X, x_0) \to (Y, y_0)$ is a map of pointed spaces then the function $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group homomorphism.*

Proof. First, notice that $f \circ c_{x_0} = c_{y_0}$, so $f_*([c_{x_0}]) = [c_{y_0}]$. Also, if ω, τ are loops in X then $f \circ (\omega * \tau) = (f \circ \omega) * (f \circ \tau)$. This gives:

$$f_*([\omega] \cdot [\tau]) = [f \circ (\omega * \tau)] = [(f \circ \omega) * (f \circ \tau)] = [f \circ \omega] \cdot [f \circ \tau] = f_*([\omega]) \cdot f_*([\tau])$$

□

3.15 Corollary. *The assignments $(X, x_0) \mapsto \pi_1(X, x_0)$ and $f \mapsto f_*$ define a functor*

$$\pi_1 \colon \mathbf{Top}_* \to \mathbf{Gr}$$

Proof. We need to check that

1) if $\mathrm{id}\colon (X, x_0) \to (X, x_0)$ is the identity map, then $\mathrm{id}_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identify homomorphism;
2) if $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ are maps of pointed spaces then $(g \circ f)_* = g_* \circ f_*$.

Property 1) holds since $\mathrm{id}_*([\omega]) = [\mathrm{id} \circ \omega] = [\omega]$. Similarly, property 2) holds since

$$(g \circ f)_*[\omega] = [g \circ f \circ \omega] = g_*([f \circ \omega]) = g_*(f_*([\omega])) = g_* \circ f_*([\omega])$$

□

Notice that an isomorphism in $\mathbf{Top}_*$ is a homeomorphism that preserves basepoints. As a consequence of Proposition 2.15 we obtain:

3.16 Corollary. *If $(X, x_0), (Y, y_0)$ are pointed spaces and $f\colon X \to Y$ is a homeomorphism such that $f(x_0) = y_0$, then $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

3.17 Note. If $f\colon X \to Y$ is any homeomorphism of topological spaces and $x_0 \in X$ then we get a homeomorphism of pointed spaces $f\colon (X, x_0) \to (Y, f(x_0))$, which gives an isomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

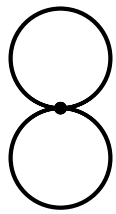
3.18 Note. In some settings it is convenient to use a somewhat different construction of the fundamental group than the one described above. Recall the every element of $\pi_1(X, x_0)$ can be represented by a function $\omega\colon [0, 1] \to X$ that satisfies $\omega(0) = \omega(1) = x_0$. Such function uniquely determines a map $[0, 1]/\!\sim\, \to X$ from the quotient space $[0, 1]/\!\sim$ where $\sim$ is the equivalence relation identifying the endpoints of the interval: $0 \sim 1$. The space $[0, 1]/\!\sim$ is homeomorphic to the circle S^1. Under such homeomorphism the point $[0] \in [0, 1]/\!\sim$ is mapped to some point $s_0 \in S^1$ that we can consider as a basepoint of S^1. As a consequence we obtain a bijection between two sets of maps:

$$\begin{pmatrix} \text{maps } \omega\colon [0,1] \to X \\ \text{with } \omega(0) = \omega(1) = x_0 \end{pmatrix} \cong \begin{pmatrix} \text{basepoint preserving maps} \\ \omega\colon (S^1, s_0) \to (X, x_0) \end{pmatrix}$$

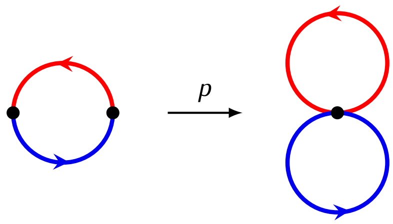
Next, given two maps $\omega, \tau\colon (S^1, s_0) \to (X, x_0)$ we will say that ω and τ are homotopic if there is a continuous function $h\colon S^1 \times [0, 1] \to X$ such that $h(s, 0) = \omega(s)$, $h(s, 1) = \tau(s)$ for all $s \in S^1$ and $h(s_0, t) = x_0$ for all $t \in [0, 1]$. The above bijection maps homotopic functions on one side to homotopic functions on the other side, so we obtain a bijection of sets:

$$\begin{pmatrix} \text{elements} \\ \text{of the group} \\ \pi_1(X, x_0) \end{pmatrix} = \begin{pmatrix} \text{homotopy classes} \\ \text{of maps } \omega\colon [0,1] \to X \\ \text{with } \omega(0) = \omega(1) = x_0 \end{pmatrix} \cong \begin{pmatrix} \text{homotopy classes} \\ \text{of basepoint preserving maps} \\ \omega\colon (S^1, s_0) \to (X, x_0) \end{pmatrix}$$

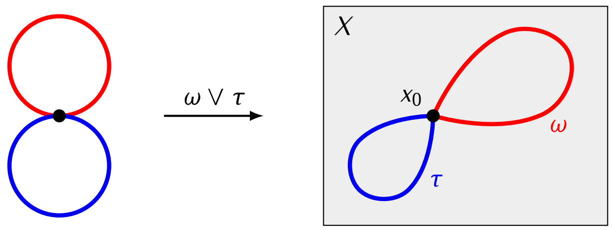
In effect we can think of elements $\pi_1(X, x_0)$ as homotopy classes of basepoint preserving maps $(S^1, s_0) \to (X, x_0)$. Using this interpretation the trivial element in $\pi_1(X, x_0)$ is given by the homotopy class of the constant map $S^1 \to X$. Multiplication in $\pi_1(X, x_0)$ can be described as follows. Let $S^1 \vee S^1$ denote the space obtained by taking two copies of S^1 and identifying a basepoint of one copy with the basepoint of the other copy:



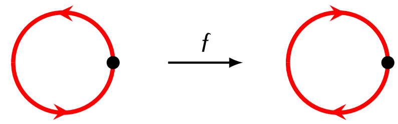
The *pinch map* is a function $p\colon S^1 \to S^1 \vee S^1$ that wraps half of the circle around one copy of S^1 and the other half around the other copy:



Given two functions $\omega, \tau\colon (S^1, s_0) \to (X, x_0)$ define $\omega \vee \tau\colon S^1 \vee S^1 \to X$ to be the function that maps one copy of S^1 by ω and the other by τ:



We have: $[\omega] \cdot [\tau] = [(\omega \vee \tau) \circ p]$. Finally, in order to describe multiplicative inverses in $\pi_1(X, x_0)$ consider the *flip map* $f\colon S^1 \to S^1$ that reflects the circle about its diagonal that passes through the basepoint:



For $\omega\colon (S^1, s_0) \to (X, x_0)$ we have $[\omega]^{-1} = [\omega \circ f]$.

Exercises to Chapter 3

E3.1 Exercise. Prove Lemma 3.3.