## 14 Presentations of Groups

In this chapter we make here a brief algebraic interlude from the task of computing fundamental groups in order to discuss how groups can be described by means their *presentations*. This concept will be used in the next chapter where we will consider fundamental groups of 2-dimensional CW complexes.

**14.1 Definition.** Let S be a set. A *word* in S is a finite sequence of the form  $a_1^{k_1}a_2^{k_2}...a_n^{k_n}$  where  $n \ge 0$ ,  $a_i \in S$  and  $k_i \in \mathbb{Z}$ . The *free group generated by* S is the group F(S) whose elements are words in S with the following identifications:

• if 
$$a_i = a_{i+1}$$
 then

$$a_1^{k_1} \dots a_i^{k_i} a_{i+1}^{k_{i+1}} \dots a_n^{k_n} = a_1^{k_1} \dots a_i^{(k_i + k_{i+1})} \dots a_n^{k_n}$$

• if  $k_i = 0$  then

$$a_1^{k_1} \dots a_{i-1}^{k_{i-1}} a_i^{k_i} a_{i+1}^{k_{i+1}} \dots a_n^{k_n} = a_1^{k_1} \dots a_{i-1}^{k_{i-1}} a_{i+1}^{k_{i+1}} \dots a_n^{k_n}$$

Multiplication in F(S) is given by concatenation of words:

$$(a_1^{k_1} \dots a_n^{k_n}) \cdot (b_1^{l_1} \dots b_m^{l_m}) = a_1^{k_1} \dots a_n^{k_n} b_1^{l_1} \dots b_m^{l_m}$$

The identity element in F(S) is given by the empty word (i.e. the word of length 0).

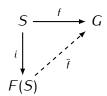
**14.2 Note.** If  $S = \emptyset$  then F(S) is the trivial group. If  $S = \{a\}$  is a set consisting of one element then  $F(S) \cong \mathbb{Z}$ . In general, the group F(S) isomorphic to the free product of free groups generated by the elements of S:

$$F(S) \cong *_{a \in S} F(\{a\}) \cong *_{a \in S} \mathbb{Z}$$

**14.3 Note.** We will say that a group G is free if G is isomorphic to the group F(S) for some set S. Notice that by Theorem 13.13 the fundamental group of any 1-dimensional CW complex is free.

For any set S we have a map of sets:  $i: S \to F(S)$  given by f(a) = a (where we consider  $a \in F(S)$  as a word of length 1). The statement of the following fact is called the *universal property of free groups*:

**14.4 Theorem.** Let S be a set and G be a group. For any map of sets  $f: S \to G$  there exists a unique homomorphism of groups  $\bar{f}: F(S) \to G$  such that the following diagram commutes:



*Proof.* The homomorphism  $\bar{f}$  is given by  $\bar{f}(a_1^{k_1}a_2^{k_2}\dots a_n^{k_n}):=f(a_1)^{k_1}\cdot f(a_2)^{k_2}\cdot \dots \cdot f(a_n)^{k_n}$ .

**14.5 Definition.** Let S be a set, and let R be a subset of elements of the free group F(S). By  $\langle S \mid R \rangle$  we denote the group given by

$$\langle S \mid R \rangle = F(S)/N$$

where N is the smallest normal subgroup of F(S) such that  $R \subseteq N$ . We say that elements of S are generators of  $S \mid R$  and elements of S are relations in  $S \mid R$ .

**14.6 Example.** For any set we have S is a set  $F(S) \cong \langle S \mid \varnothing \rangle$ .

**14.7** Example.  $\langle a \mid a^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$ .

14.8 Example.  $\langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z}$ .

**14.9 Definition.** If G is a group and  $G \cong \langle S \mid R \rangle$  for some set S and some  $R \subseteq F(S)$  then we say that  $\langle S \mid R \rangle$  is a *presentation* of G.

**14.10 Definition.** If a group G has a presentation  $\langle S \mid R \rangle$  such that S is a finite set then we say that G is *finitely generated* and if it has a presentations such that both S and R are finite sets then we say that G is *finitely presented*.

14.11 Proposition. Every group has a presentation.

*Proof.* Let G be a group and let  $f: S \to G$  be a map of sets which is onto. By Theorem 14.4 the function f defines a homomorphism  $\bar{f}: F(S) \to G$ . Since f is onto thus so is  $\bar{f}$ . This gives an isomorphism  $G \cong F(S)/\ker(\bar{f})$ . If follows that  $G \cong \langle S \mid R \rangle$  where R is the set of elements of  $\ker(\bar{f})$ .  $\square$ 

14.12 Note. 1) Every group has inifinitely many different presentations. For example

$$\mathbb{Z} \cong \langle a \rangle \cong \langle a, b \mid b \rangle \cong \langle a, b \mid ab^{-1} \rangle \cong \langle a, b \mid b^2, b^3 \rangle$$

2) In general if we know a presentation of a group it may be very difficult to say anything about the properties of the group (even if the group is trivial or not).

## **Exercises to Chapter 14**

**E14.1 Exercise.** Below are three groups described by their presentations. For each group decide if it is abelian and if it is finite. Justify your answers.

a) 
$$G_1 = \langle a, b | a^3, b^3, aba^2b^2 \rangle$$

b) 
$$G_2 = \langle a, b \mid a^2, aba \rangle$$

c) 
$$G_3 = \langle a, b | a^4, b^4, a^2b^2 \rangle$$