19 | Classification of Coverings

19.1 Definition. Let $p_1\colon T_1 \to X$, $p_2\colon T_2 \to X$ be coverings over the same base space X. A *map of coverings* is a continuous function $f\colon T_1 \to T_2$ such that the following diagram commutes:

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

For a given space X by $\mathbf{Cov}(X)$ we will denote the category whose objects are all coverings over X and whose morphisms are maps of coverings.

In $\mathbf{Cov}(X)$, as in any category, we have the notion of an isomorphism of coverings. It is easy to check that the following holds:

19.2 Proposition. *Let $p_1\colon T_1 \to X$ and $p_2\colon T_2 \to X$ be coverings of X. A map of coverings $f\colon T_1 \to T_2$ is an isomorphism in $\mathbf{Cov}(X)$ if and only if f is a homeomorphism.*

Proof. Exercise. □

19.3 Note. If $p_1\colon T_1 \to X$ and $p_2\colon T_2 \to X$ are coverings and the spaces T_1 and T_2 are homeomorphic then this does not imply that p_1 and p_2 are isomorphic coverings since it may happen than no homeomorphism between T_1 and T_2 is a map of coverings. For example, recall the for $n = 1, 2, \ldots$ we have an n-fold covering $p_n\colon S^1 \to S^1$ given by $p_n(z) = z^n$ (where we consider S^1 as the set of unit complex numbers: $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$). The total space of each of these coverings is S^1, but these coverings are non-isomorphic to one another since an n-fold covering cannot be isomorphic to an m-fold covering for $n \neq m$ (exercise).

Our goal in this chapter is to show that under some mild conditions the problem of determining if two coverings of X are isomorphic or not can be reduced to a purely algebraic problem. Recall that a space X is *locally path connected* if for any point $x \in X$ and any open neighborhood $U \subseteq X$ of x there exists an open neighborhood V of x such that $V \subseteq U$ and V is path connected.

19.4 Theorem. *Let X be a locally path connected space, and for $i = 1, 2$ let $p_i\colon T_i \to X$ be a covering such that T_i is a path connected space. Let $x_0 \in X$ and let $\tilde{x}_i \in p_i^{-1}(x_0)$. The coverings p_1 and p_2 are isomorphic if and only if the subgroup $p_{1*}(\pi_1(T_1, \tilde{x}_1)) \subseteq \pi_1(X, x_0)$ is conjugate to the subgroup $p_{2*}(\pi_1(T_2, \tilde{x}_2))$.*

One implication of Theorem 19.4 is straightforward: if the coverings p_1 and p_2 are isomorphic then we have a homeomorphism $f\colon T_1 \to T_2$ that we can use to relate the subgroups coming from these coverings. The other implication is more challenging, since it says that based on some information about fundamental groups we can produce a map $T_1 \to T_2$. Existence of such map will follow from the next theorem which is very useful in many applications:

19.5 Theorem (Lifting Criterion). *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Assume that Y is a connected and locally path connected space and let $y_0 \in Y$. A map $f\colon (Y, y_0) \to (X, x_0)$ has a lift $\tilde{f}\colon (Y, y_0) \to (T, \tilde{x}_0)$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$.*

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd} \tag{$*$}$$

19.6 Note. By Lemma 17.11 if a lift $\tilde{f}$ exists then it is unique.

Proof of Theorem 19.5. ($\Rightarrow$) If the lift $\tilde{f}$ exists then the diagram ($*$) induces a commutative diagram of fundamental groups:

$$\begin{tikzcd} & \pi_1(T, \tilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(X, x_0) \end{tikzcd}$$

Therefore we obtain

$$f_*(\pi_1(Y, y_0)) = p_*\tilde{f}_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$$

($\Leftarrow$) We will construct the function $\tilde{f}\colon Y \to X$ as follows. Take $y \in Y$. Since the space Y is connected and locally path connected thus it is path connected, and so there exists a path $\omega\colon [0, 1] \to Y$ such that $\omega(0) = y_0$ and $\omega(1) = y$. By Corollary 17.10 the path $f\omega\colon [0, 1] \to X$ admits a unique lift $\widetilde{f\omega}\colon [0, 1] \to T$ such that $\widetilde{f\omega}(0) = \tilde{x}_0$. We set $\tilde{f}(y) := \widetilde{f\omega}(1)$.

In order to see that $\tilde{f}$ is well defined take ω' to be another path in Y joining y_0 with y, and let $\widetilde{f\omega}'$ be the lift of $f\omega'$ satisfying $\widetilde{f\omega}'(0) = \tilde{x}_0$. We need to show that $\widetilde{f\omega}(1) = \widetilde{f\omega}'(1)$. By Proposition 18.3 this is equivalent to showing that $[f\omega * \overline{f\omega'}] \in \pi_1(X, x_0)$ is an element of $p_*(\pi_1(T, \tilde{x}_0))$. Notice that we have

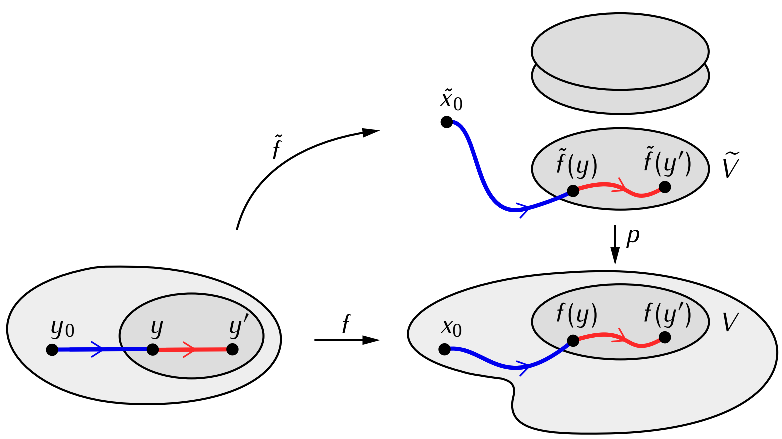
$$[f\omega * \overline{f\omega'}] = [f(\omega * \overline{\omega'})] = f_*([\omega * \overline{\omega'}])$$

where $[\omega * \overline{\omega'}] \in \pi_1(Y, y_0)$. By assumption $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$, so in particular $f_*([\omega * \overline{\omega'}]) \in p_*(\pi_1(T, \tilde{x}_0))$.

Directly from the construction of $\tilde{f}$ we get that $p\tilde{f} = f$ and that $\tilde{f}(y_0) = \tilde{x}_0$. We still need to check though that $\tilde{f}$ is a continuous function. It will suffice to show that for any $y \in Y$ there is an open neighborhood U of y such that $\tilde{f}|_U \colon U \to T$ is continuous. Let $V \subseteq X$ be an open neighborhood of $f(y)$ which is evenly covered, and take U to be an open neighborhood of y such that $U \subseteq f^{-1}(V)$ and that U is path connected. Such U exists by the assumption that Y is locally path connected. For any $y' \in U$ let $\tau_{y'}$ be a path in U joining y with y'. Also, let ω be a path in Y that joins y_0 with y. Notice that $\omega * \tau_{y'}$ joins y_0 and y', so $\tilde{f}(y') = \widetilde{\omega * \tau_{y'}}(1)$ where $\widetilde{\omega * \tau_{y'}}$ is the lift of $\omega * \tau_{y'}$ that starts at $\tilde{x}_0$. On the other hand $\widetilde{\omega * \tau_{y'}} = \widetilde{\omega} * \widetilde{\tau}_{y'}$ where $\widetilde{\omega}$ is the lift of ω that starts at $\tilde{x}_0$, and $\widetilde{\tau}_y$ is the lift of τ_y that starts at $\widetilde{\omega}(1) = \tilde{f}(y)$. In effect we obtain that $\tilde{f}(y') = \widetilde{\tau}_{y'}(1)$ for all $y' \in U$. Let $\widetilde{V}$ be the slice over V such that $\tilde{y} \in V$. The map $p|_{\widetilde{V}} \colon \widetilde{V} \to V$ is a homeomorphism. Notice that for any $y' \in U$ we have $\widetilde{\tau}_{y'} = (p|_{\widetilde{V}})^{-1} f\tau_{y'}$. This gives:

$$\tilde{f}(y') = \widetilde{\tau}_{y'}(1) = (p|_{\widetilde{V}})^{-1} f\tau_{y'}(1) = (p|_{\widetilde{V}})^{-1} f(y')$$

In other words we obtain $\tilde{f}|_U = (p|_{\widetilde{V}})^{-1} f$, which means that $\tilde{f}|_U$ is a continuous function.



$\square$

Proof of Theorem 19.4. ($\Rightarrow$) Assume that we have an isomorphism of coverings:

$$\begin{tikzcd} T_1 \arrow[rr, "f", "\cong"'] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

This gives a commutative diagram of fundamental groups:

$$\begin{tikzcd} \pi_1(T_1, \tilde{x}_1) \arrow[rr, "f_*", "\cong"'] \arrow[dr, "p_{1*}"'] & & \pi_1(T_2, f(\tilde{x}_1)) \arrow[dl, "p_{2*}"] \\ & \pi_1(X, x_0) & \end{tikzcd}$$

Since f_* is an isomorphism we obtain that $p_{1*}(\pi_1(T_1, \tilde{x}_1)) = p_{2*}(\pi_1(T_2, f(\tilde{x}_1)))$. It remains to notice that since $f(\tilde{x}_1), \tilde{x}_2 \in p_2^{-1}(x_0)$ thus by Proposition 18.5 the subgroups $p_{2*}(\pi_1(T_2, f(\tilde{x}_1)))$ and $p_{2*}(\pi_1(T_2, \tilde{x}_2))$ are conjugate in $\pi_1(X, x_0)$.

($\Leftarrow$) Assume that $p_{1*}(\pi_1(T_1, \tilde{x}_1))$ and $p_{2*}(\pi_1(T_2, \tilde{x}_2))$ are conjugate subgroups of $\pi_1(X, x_0)$. By Proposition 18.5 we can find $\tilde{x}_2' \in p_2^{-1}(x_0)$ such that $p_{1*}(\pi_1(T_1, \tilde{x}_1)) = p_{2*}(\pi_1(T_2, \tilde{x}_2'))$. Since the space X is locally path connected, thus so are T_1 and T_2. By the Lifting Criterion 19.5 there exists maps $\widetilde{p}_1\colon (T_1, \tilde{x}_1) \to (T_2, \tilde{x}_2')$ and $\widetilde{p}_2\colon (T_2, \tilde{x}_2') \to (T_1, \tilde{x}_1)$ such that the following diagrams commute:

$$\begin{tikzcd} & T_2 \arrow[d, "p_2"] \\ T_1 \arrow[ur, "\widetilde{p}_1"] \arrow[r, "p_1"'] & X \end{tikzcd} \qquad \begin{tikzcd} & T_1 \arrow[d, "p_1"] \\ T_2 \arrow[ur, "\widetilde{p}_2"] \arrow[r, "p_2"'] & X \end{tikzcd}$$

We will show that $\widetilde{p}_1$ and $\widetilde{p}_2$ are inverse isomorphisms of coverings. Notice that both $\widetilde{p}_2\widetilde{p}_1$ the identity map id_{T_1} fit into the commutative diagram

$$\begin{tikzcd} & T_1 \arrow[d, "p_1"] \\ T_1 \arrow[ur, shift left, "\widetilde{p}_2\widetilde{p}_1"] \arrow[ur, shift right, "\mathrm{id}_{T_1}"'] \arrow[r, "p_1"'] & X \end{tikzcd}$$

Moreover $\widetilde{p}_2\widetilde{p}_1(\tilde{x}_1) = \tilde{x}_1 = \mathrm{id}_{T_1}(\tilde{x}_1)$. By Lemma 17.11 this implies that $\widetilde{p}_2\widetilde{p}_1 = \mathrm{id}_{T_1}$. An analogous argument shows that $\widetilde{p}_1\widetilde{p}_2 = \mathrm{id}_{T_2}$. □

19.7 Note. Theorem 19.4 gives a correspondence between isomorphism classes of path connected coverings over X and conjugacy classes of subgroups of $\pi_1(X, x_0)$. There is also a variant of of this correspondence that takes values in the set of subgroups of $\pi_1(X, x_0)$ (rather that conjugacy classes of subgroups). Namely, let (X, x_0) be a pointed space. A pointed covering is a basepoint preserving map $p\colon (T, \tilde{x}_0) \to (X, x_0)$ where p is a covering. If $p_1\colon (T_1, \tilde{x}_1) \to (X, x_0)$ and $p_2\colon (T_1, \tilde{x}_2) \to (X, x_0)$ are pointed coverings then a map of pointed covering is a function $f\colon (T_1, \tilde{x}_1) \to (T_2, \tilde{x}_2)$ such that $p_2 f = p_1$. Pointed coverings of (X, x_0) and their maps form a category $\mathbf{Cov}_*(X, x_0)$. Modifying the proof of Theorem 19.4 we obtain:

19.8 Theorem. *Let (X, x_0) be a locally path connected space, and for $i = 1, 2$ let $p_i\colon (T_i, \tilde{x}_i) \to (X, x_0)$ be a pointed covering such that T_i is a path connected space. The coverings p_1 and p_2 are isomorphic in the category $\mathbf{Cov}_*(X, x_0)$ if and only if $p_{1*}(\pi_1(T_1, \tilde{x}_1)) = p_{2*}(\pi_1(T_2, \tilde{x}_2))$.*

Proof. Exercise. ☐

Exercises to Chapter 19

E19.1 Exercise. Prove Proposition 19.2.

E19.2 Exercise. Consider a commutative diagram

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

where p_1, p_2 are coverings and T_1, T_2 are path connected spaces. Show that f is onto.

E19.3 Exercise. Assume that we have a map of coverings

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

where T_1, T_2 are path connected spaces. Assume also that for some $x_0 \in X$ the map

$$f|_{p^{-1}(x_0)}\colon p_1^{-1}(x_0) \to p_2^{-1}(x_0)$$

is 1-1. Show that $f\colon T_1 \to T_2$ is 1-1.

E19.4 Exercise. Let X be a locally path connected space, and let $p\colon T \to X$ be a covering such that T is a path connected space and $\pi_1(T)$ is a finite group. Show that any map of coverings

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "p"'] & & T \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

is an isomorphism.

E19.5 Exercise. Recall that the n-th homotopy group of a pointed space (X, x_0) is a group whose elements are homotopy classes of basepoint preserving maps $(S^n, s_0) \to (X, x_0)$. Let $x_0 \in S^1$. Show that $\pi_n(S^1, x_0)$ is a trival group for $n > 1$.

E19.6 Exercise. Let X be a locally path connected space, $x_0 \in X$, and let $p\colon T \to X$ be a covering with a path connected space T. Show that the following conditions are equivalent:

(i) The covering $p\colon T \to X$ is regular

(ii) For any points $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ there exists a map of coverings

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "p"'] & & T \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

such that $f(\tilde{x}_1) = \tilde{x}_2$.

E19.7 Exercise. Let $p\colon T \to X$ be a covering such that T, X are connected and locally path connected spaces. Assume that there exists a function $s\colon X \to T$ such that $ps = \mathrm{id}_X$. Show that T is homeomorphic to X.

E19.8 Exercise. Let $p\colon T \to X$ be a covering such that X is locally path connected, and T is path connected. Given maps $f, g\colon Y \to X$ let $p'\colon f^*T \to Y$ and $p''\colon g^*T \to Y$ be the coverings of Y obtained from p using f and g, respectively, as in Exercise 18.3. Show that if $f \simeq g$ then p' and p'' are isomorphic coverings.