7 | Higher Homotopy Groups

Let $n \geq 1$. Recall that $D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$ is the n-dimensional closed disc and $S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$ is the $(n-1)$-dimensional sphere. While most of the results proved in Chapter 6 are stated in terms of spheres and discs of dimension 1 or 2 it is easy to formulate their possible generalizations to higher dimensions:

1) For any $n \geq 1$ the sphere S^n is not a retract of the disc D^{n+1} (cf. 6.1).

2) Let $n \geq 1$. For each map $f\colon D^n \to D^n$ there exists a point $x_0 \in D^n$ such that $f(x_0) = x_0$ (cf. 6.2).

3) Let $n \geq 1$. For each map $f\colon S^n \to \mathbb{R}^n$ there exists $x \in S^n$ such that $f(x) = f(-x)$ (cf. 6.3).

4) If $n \geq 1$ and $A_1, \ldots, A_{n+1} \subseteq S^n$ are closed sets such that $A_1 \cup \cdots \cup A_{n+1} = S^n$ then one of these sets contains a pair of antipodal points $\{x, -x\}$ (cf. 6.6).

All these generalizations are in fact true. However, if one tries to prove them mimicking the proofs used in the low dimensional cases it turns out that some additional machinery is needed. For example, the main ingredient of the proof of the fact that S^1 is not a retract of D^2 was the observation that the fundamental group is a functor $\pi_1\colon \mathbf{Top}_* \to \mathbf{Gr}$ such that $\pi_1(D^2) \cong \{1\}$ and $\pi_1(S^1) \ncong \{1\}$. Analogously, in order to prove that S^n is not a retract of D^{n+1} it would be useful to have a functor $F\colon \mathbf{Top}_* \to \mathbf{Gr}$ satisfying $F(D^{n+1}) \cong \{1\}$ and $F(S^n) \ncong \{1\}$. Assuming that there exists a retraction $r\colon D^{n+1} \to S^n$ we would get a commutative diagram of pointed spaces:

$$\begin{tikzcd} S^n \arrow[rr, "\mathrm{id}_{S^n}"] \arrow[dr, "i"'] & & S^n \\ & D^{n+1} \arrow[ur, "r"'] & \end{tikzcd}$$

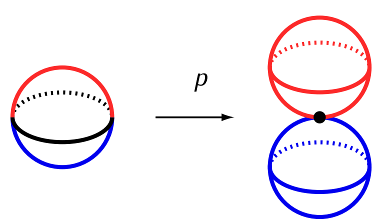
where $i\colon S^n \to D^{n+1}$ is the inclusion map. Applying the functor F we would obtain a commutative diagram of groups:

$$\begin{tikzcd} F(S^n) \arrow[rr, "\mathrm{id}_{F(S^n)}"] \arrow[dr, "F(i)"'] & & F(S^n) \\ & F(D^{n+1}) \arrow[ur, "F(r)"'] & \end{tikzcd}$$

This would imply that $F(r)$ is onto which is impossible since $F(D^{n+1})$ a trivial group and $F(S^n)$ is non-trivial.

In the above argument we cannot take F to be the fundamental group functor since the group $\pi_1(S^n)$ is trivial for $n > 1$ (exercise). A functor which is useful in this context is the n-th homotopy group functor π_n. Its construction can be described similarly to the construction of π_1 given in (3.18). Take $s_0 = (1, 0, \dots, 0)$ to be the basepoint of S^n. Given a pointed space (X, x_0) we will say that basepoint preserving maps $\omega, \tau \colon (S^n, s_0) \to (X, x_0)$ are homotopic if there exists a continuous function $h \colon S^n \times [0, 1] \to X$ such that $h(s, 0) = \omega(s)$, $h(s, 1) = \tau(s)$ for all $s \in S^n$ and $h(s_0, t) = x_0$ for all $t \in [0, 1]$. Let $\pi_n(X, x_0)$ be the set of homotopy classes $[\omega]$ of basepoint preserving maps $\omega \colon (S^n, s_0) \to (X, x_0)$.

In order to describe multiplication in $\pi_n(X, x_0)$ denote by $S^n \vee S^n$ the space obtained by taking two copies of S^n and identifying their basepoints. There is a *pinch map* $p \colon S^n \to S^n \vee S^n$ that maps the upper hemisphere of S^n onto one copy of $S^n \subseteq S^n \vee S^n$, the lower hemisphere onto the second copy, and the equator of S^n to the basepoint of $S^n \vee S^n$:



Given two basepoint preserving maps $\omega, \tau \colon (S^n, s_0) \to (X, x_0)$ we can define a map $\omega \vee \tau \colon S^n \vee S^n \to X$ that maps the first copy of S^n using ω and the second copy using τ. We set: $[\omega] \cdot [\tau] = [(\omega \vee \tau) \circ p]$.

One can check that $\pi_n(X, x_0)$ taken with this multiplication is a group. The trivial element in this group is given by the homotopy class of the constant map $(S^n, s_0) \to (X, x_0)$. For $[\omega] \in \pi_n(X, x_0)$ we have $[\omega]^{-1} = [\omega \circ f]$ where $f \colon S^n \to S^n$ is the map given by $f(x_1, x_2, \dots, x_n, x_{n+1}) = (x_1, x_2, \dots, x_n, -x_{n+1})$.

The assignments $(X, x_0) \mapsto \pi_n(X, x_0)$ define a functor $\pi_n \colon \mathbf{Top}_* \to \mathbf{Gr}$.

7.1 Theorem.

1) *For any $m, n \geq 1$ and $x_0 \in D^m$ the group $\pi_n(D^m, x_0)$ is trivial.*
2) *For any $n \geq 1$ and $x_0 \in S^n$ there is an isomorphism $\pi_n(S^n) \cong \mathbb{Z}$.*

The proof of part 1) is easy and similar to the proof that $\pi_1(D^m)$ is trivial for all $m \geq 1$ (see Proposition 5.3). The second part is harder and requires more work that the proof that $\pi_1(S^1) \cong \mathbb{Z}$. Since our main focus in these notes is the fundamental group we will skip this proof.

Theorem 7.1 combined with the argument outlined above implies that statement 1) on page 32 holds. Using this, by the same argument as in the proof of Theorem 6.2 we obtain statement 2). Higher homotopy groups can be also used to prove statement 3) which in turn implies statement 4).

7.2 Note. Recall that in Example 2.10 we defined a functor π_0 that assigns to a space X the set $\pi_0(X)$ of path connected components of X. This functor is related to the functors π_n constructed above as follows. Recall that the 0-dimesional sphere is a discrete space consisting of two points $S^0 = \{-1, 1\}$. Choose $1 \in S^0$ as the basepoint. If (X, x_0) is a pointed space then any basepoint preserving map $f\colon (S^0, 1) \to (X, x_0)$ is determined by the value of $f(-1)$, and this value can be an arbitrary point of X. This gives a bijection:

$$\begin{array}{ccc} \begin{pmatrix} \text{basepoint preserving maps} \\ f\colon (S^0,1) \to (X, x_0) \end{pmatrix} & \cong & \begin{pmatrix} \text{points} \\ \text{of } X \end{pmatrix} \\ f & \mapsto & f(-1) \end{array}$$

It is also easy to see that giving a homotopy between maps $f, g\colon (S^0, 1) \to (X, x_0)$ is the same as giving a path between the points $f(-1)$ and $g(-1)$. This means that maps f, g are homotopic if and only if the points $f(-1)$ and $g(-1)$ are in the same path connected component of X. As a consequence we obtain a bijection:

$$\begin{pmatrix} \text{homotopy classes of maps} \\ f\colon (S^0,1) \to (X, x_0) \end{pmatrix} \cong \begin{pmatrix} \text{path connected} \\ \text{components of } X \end{pmatrix} = \pi_0(X)$$

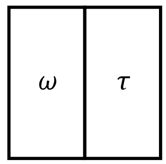
The difference between the functors π_0 and π_n for $n > 0$ is that $\pi_0(X)$ is in general just a set, not a group. However, for any pointed space (X, x_0) the set $\pi_0(X)$ has a natural choice of a basepoint given by the path connected component of x_0 (or equivalently, by the homotopy class of the constant map $(S^0, 1) \to (X, x_0)$). This means that we can consider π_0 as a functor

$$\pi_0\colon \mathbf{Top}_* \to \mathbf{Set}_*$$

where $\mathbf{Set}_*$ denotes the category of pointed sets i.e. sets equipped with a basepoint.

The above description of higher homotopy groups generalizes the construction of the fundamental group given in (3.18), using maps $S^1 \to X$. We can also describe groups $\pi_n(X, x_0)$ in a way paralleling the construction of $\pi_1(X, x_0)$ that used loops $[0, 1] \to X$. Namely, let $I^n = \prod_{i=1}^n [0, 1]$ be the n-dimensional cube, and let $\partial I^n = I^n \smallsetminus \prod_{i=1}^n (0, 1)$. Given a pointed space (X, x_0) by $\omega\colon (I^n, \partial I^n) \to (X, x_0)$ we will denote a map $\omega\colon I^n \to X$ such that $\omega(\partial I^n) = x_0$. We will say that to maps $\omega, \tau\colon (I^n, \partial I^n) \to (X, x_0)$ are homotopic if there exists a map $h\colon I^n \times [0, 1] \to X$ such that $h(s, 0) = \omega(s)$, $h(s, 1) = \tau(s)$ for any $s \in I^n$, and $h(\partial I^n \times [0, 1]) = x_0$. We define $\pi_n(X, x_0)$ as the set of homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. For $\omega, \tau\colon (I^n, \partial I^n) \to (X, x_0)$ define $\omega * \tau\colon (I^n, \partial I^n) \to (X, x_0)$ by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$



One can check that this induces a well-defined associative multiplication

$$\pi_n(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)$$

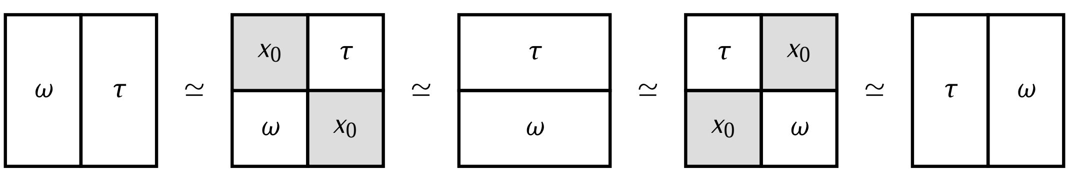
given by $[\omega] \cdot [\tau] = [\omega * \tau]$ which makes $\pi_n(X, x_0)$ into a group. The trivial element of $\pi_n(X, x_0)$ is the homotopy class of the constant map $c_{x_0} \colon I^n \to X$. Also, for $[\omega] \in \pi_1(X, x_0)$ we have $[\omega]^{-1} = [\overline{\omega}]$ where $\overline{\omega} \colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$\overline{\omega}(s_1, s_2, \dots, s_n) = (1 - s_1, s_2, \dots, s_n)$$

We will see later that the fundamental group of a space need not be commutative. By contrast we have:

7.3 Theorem. *For $n \geq 2$ then the group $\pi_n(X, x_0)$ is abelian for any pointed space (X, x_0).*

Proof. A homotopy $\omega * \tau \simeq \tau * \omega$ can be depicted as follows:



The shaded squares in the pictures are mapped to the basepoint $x_0 \in X$.

□

Exercises to Chapter 7

E7.1 Exercise. Show that the fundamental group of the sphere S^n is trivial for $n > 1$. (Hint: show first that any loop $\omega \colon [0, 1] \to S^n$ is homotopic to a loop $\omega' \colon [0, 1] \to S^n$ which is not onto.)