21 | Equivalences of Categories

Results of Chapters 19 and 20 can be summarized as follows:

21.1 Theorem. *Let X be a connected, locally path connected, and semi-locally simply connected space, and let $x_0 \in X$. The map*

$$\Omega\colon \left(\begin{array}{c}\text{isomorphism classes}\\ \text{of path connected}\\ \text{coverings of } X\end{array}\right) \longrightarrow \left(\begin{array}{c}\text{conjugacy classes}\\ \text{of subgroups}\\ \text{of } \pi_1(X, x_0)\end{array}\right)$$

given by $\Omega(p\colon T \to X) = p_*(\pi_1(T, \tilde{x}))$ for some $\tilde{x} \in p^{-1}(x_0)$ is a bijection.

Proof. The map Ω is 1-1 by Theorem 19.4, and it is onto by Theorems 20.3 and 20.8. □

Theorem 21.1 translates the topological problem of classifying coverings into an algebraic one, of identifying conjugacy classes of subgroups of a group. However, since coverings over X form a category **Cov**(X), with morphisms given by maps of coverings, a more complete correspondence between topology and algebra would be obtained if we could find some algebraic category **D** and a functor

$$F\colon \mathbf{Cov}(X) \to \mathbf{D}$$

that would let us restate problems about coverings and maps of coverings as problems about objects and morphism of the category **D**. In Chapter 22 we will show that such category **D** and a functor F exist. Before we get to this though, we need to consider what properties the functor F should have so that it would allow us to go back and forth between categories **Cov**(X) and **D** without losing any essential information. The most obvious requirement is that F should be an isomorphism of categories, i.e. that there should exist a functor $G\colon \mathbf{D} \to \mathbf{Cov}(X)$ such the compositions $GF\colon \mathbf{Cov}(X) \to \mathbf{Cov}(X)$

and $FG\colon \mathbf{D} \to \mathbf{D}$ are identies on all objects and morphisms. It turns out however, that isomorphisms of categories appear very rarely in practical applications. A somewhat weaker but much more useful notion is an equivalence of categories:

21.2 Definition. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an *equivalence of categories* if there exists a functor $G\colon \mathbf{D} \to \mathbf{C}$ for which the following conditions hold:

1) For each object $c \in \mathbf{C}$ there exists an isomorphism $\eta_c\colon c \to GF(c)$ such that for any morphism $f\colon c \to c'$ the following diagram commutes:

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\eta_c"', "\cong"] & c' \arrow[d, "\cong"', "\eta_{c'}"] \\ GF(c) \arrow[r, "GF(f)"'] & GF(c') \end{tikzcd}$$

2) For each object $d \in \mathbf{D}$ there exists an isomorphism $\tau_d\colon d \to FG(d)$ such that for any morphism $g\colon d \to d'$ the following diagram commutes:

$$\begin{tikzcd} d \arrow[r, "g"] \arrow[d, "\tau_d"', "\cong"] & d' \arrow[d, "\cong"', "\tau_{d'}"] \\ FG(d) \arrow[r, "FG(g)"'] & FG(d') \end{tikzcd}$$

We will say that $\mathbf{C}$ and $\mathbf{D}$ are *equivalent categories* if there exists an equivalence $\mathbf{C} \to \mathbf{D}$.

The following fact is often useful, since it allows us to check if a functor is an equivalence of categories without constructing the inverse functor G.

21.3 Proposition. *A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an equivalence of categories if and only if the following conditions hold.*

(i) For each object $d \in \mathbf{D}$ there exists an object $c \in \mathbf{C}$ such that $d \cong F(c)$.

(ii) For any objects $c, c' \in \mathbf{C}$ the map $\mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$ given by $f \mapsto F(f)$ is a bijection.

Proof. Exercise. □

21.4 Example. Let $\mathbf{FVect}(\mathbb{R})$ denote the category of finitely dimensional real vector spaces with linear transformations as morphisms. Also, let $\mathbf{M}(\mathbb{R})$ denote the category whose objects are natural numbers

$0, 1, 2, \ldots$. The set of morphisms $\mathrm{Mor}_{\mathbf{M}(\mathbb{R})}(n, m)$ consists of all $n \times m$ matrices with real coefficients. Composition of morphisms is given by matrix multiplication. We have a functor

$$F \colon \mathbf{M}(\mathbb{R}) \to \mathbf{FVect}(\mathbb{R})$$

defined as follows. On objects $F(n) = \mathbb{R}^n$. If A is an $n \times m$ matrix (i.e. a morphism $n \to m$ in $\mathbf{M}(\mathbb{R})$) then $F(A) \colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by $F(A)(v) = Av$ for $v \in \mathbb{R}^n$. One can show that F is an equivalence of categories (exercise).

21.5 Example. Recall (4.8) that the fundamental groupoid of a space X is a category $\Pi_1(X)$ whose objects are points of X. For $x, x' \in X$ morphisms $x \to x'$ are homotopy classes of paths that begin at x and end at x'. Composition of morphisms is given by concatenation of paths. A map of spaces $f \colon X \to X'$ induces a functor of fundamental groupoids $f_* \colon \Pi_1(X) \to \Pi_1(X')$. One can show that if f is a homotopy equivalence of spaces then the functor f_* is an equivalence of categories (exercise).

Exercises to Chapter 21

E21.1 Exercise. Prove Proposition 21.3.