11 | Proof of van Kampen's Theorem

In the last chapter have seen the statement of van Kampen's theorem and some of examples of its applications. Our main goal in this chapter is to prove this result. For reference we state it here again:

10.17 **van Kampen Theorem.** *Let (X, x_0) be a pointed topological space and let $U_1, U_2 \subseteq X$ be open sets such that $X = U_1 \cup U_2$. If the sets U_1, U_2, and $U_1 \cap U_2$ are path connected and $x_0 \in U_1 \cap U_2$ then*

$$\pi_1(X, x_0) \cong \operatorname{colim}(\pi_1(U_1, x_0) \xleftarrow{i_{1*}} \pi_1(U_1 \cap U_2, x_0) \xrightarrow{i_{2*}} \pi_1(U_2, x_0))$$

where for $k = 1, 2$ the homomorphism i_{k*} is induced by the inclusion map $i_k \colon U_1 \cap U_2 \to U_k$.

Proof. Here is some notation that will be useful.

- For simplicity we will denote $U_1 \cap U_2$ by U_0.
- For $k = 1, 2$ by $i_k \colon U_0 \to U_k$ and $j_k \colon U_k \to X$ we will denote the inclusion maps.
- If ω is a loop in U_1 then it represents an element of $\pi_1(U_1, x_0)$ and an element of $\pi_1(X, x_0)$. In order to avoid such ambiguities we will write $[\omega]_k$ to indicate an element of $\pi_1(U_k, x_0)$ and $[\omega]$ to indicate an element of $\pi_1(X, x_0)$.

The strategy of the proof will be as follows. Let $P = \operatorname{colim}(\pi_1(U_1, x_0) \xleftarrow{i_{1*}} \pi_1(U_0, x_0) \xrightarrow{i_{2*}} \pi_1(U_2, x_0))$. Recall that by Proposition 10.3 P is the unique (up to an isomorphism) group that satisfies the following conditions:

1) for $k = 1, 2$ there exists exists a homomorphism $g_k \colon \pi_1(U_k, x_0) \to P$ such that $g_1 i_{1*} = g_2 i_*$;
2) for any group G and any homomorphisms $h_k \colon \pi_1(U_k, x_0) \to P$ satisfying $h_1 i_{1*} = h_2 i_*$ there exists a unique homomorphism $h \colon P \to G$ such that $h g_k = h_k$ for $k = 1, 2$.

If follows that in order to prove van Kampen's theorem it will suffice to show that the group $\pi_1(X, x_0)$ satisfies conditions 1) and 2). The first condition is satisfied by taking $g_k = j_{k*}$ for $k = 1, 2$. In order to verify the second condition let $h_1 \colon \pi_1(U_1, x_0) \to G$ and $h_2 \colon \pi_1(U_2, x_0) \to G$ be homomorphisms satisfying $h_1 i_{1*} = h_2 i_{2*}$. We need to construct a homomorphism $h \colon \pi_1(X, x_0) \to G$ such that the following diagram commutes:

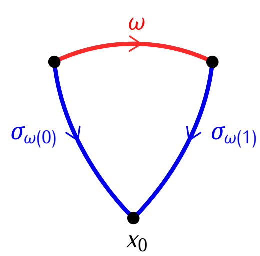
$$\begin{tikzcd}
\pi_1(U_0, x_0) \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"'] & \pi_1(U_1, x_0) \arrow[d, "j_{1*}"] \arrow[ddr, bend left, "h_1"] & \\
\pi_1(U_2, x_0) \arrow[r, "j_{2*}"'] \arrow[drr, bend right, "h_2"'] & \pi_1(X, x_0) \arrow[dr, dashed, "h"] & \\
 & & G
\end{tikzcd} \tag{$*$}$$

Moreover, we need to show that there is only one such homomorphism h.

The construction of h will use the following setup. For each point $x \in X$ choose a path σ_x such that

- $\sigma_x(0) = x$ and $\sigma_x(1) = x_0$;
- if $x \in U_k$ for $k \in \{0, 1, 2\}$ then σ_x is contained in U_k;
- σ_{x_0} is the constant path.

Such paths exist since by assumption U_0, U_1, U_2 are path connected sets. If ω is a path in X then the concatenation $\bar{\sigma}_{\omega(0)} * \omega * \sigma_{\omega(1)}$ is a loop based at x_0:



We will denote this loop by ω° and call it the *loop completion* of ω. Loop completion has the following properties:

(i) If ω is a path in U_k for $k \in \{0, 1, 2\}$ then ω° is a loop in U_k. This holds since in such case $\sigma_{\omega(0)}$ and $\sigma_{\omega(1)}$ are paths in U_k.

(ii) For any path ω we have $(\bar{\omega})^\circ \simeq \overline{\omega^\circ}$.

(iii) If ω, τ have the same endpoints and $\omega \simeq \tau$ then $\omega^\circ \simeq \tau^\circ$, and so in $\pi_1(X, x_0)$ we have $[\omega^\circ] = [\tau^\circ]$. Moreover, if ω, τ are paths in U_k and the homotopy between them is contained in U_k then in $\pi_1(U_k, x_0)$ we have $[\omega^\circ]_k = [\tau^\circ]_k$.

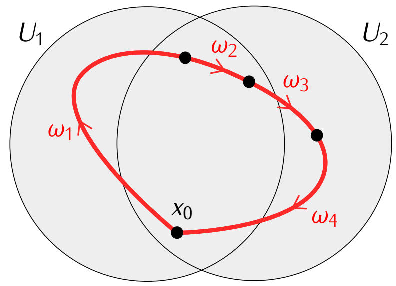
(iv) If ω, τ are paths such that $\omega(1) = \tau(0)$ then $(\omega * \tau)^\circ \simeq \omega^\circ * \tau^\circ$. Indeed, we have

$$\omega^\circ * \tau^\circ = (\bar{\sigma}_{\omega(0)} * \omega * \sigma_{\omega(1)}) * (\bar{\sigma}_{\omega(1)} * \tau * \sigma_{\tau(1)}) \simeq \bar{\sigma}_{\omega(0)} * \omega * \tau * \sigma_{\tau(1)} \simeq (\omega * \tau)^\circ$$

Notice also that if both ω and τ are paths in U_k then the above homotopy of loops is contained in U_k. This means that $[\omega^\circ]_k \cdot [\tau^\circ]_k = [(\omega * \tau)^\circ]_k$ in $\pi_1(U_k, x_0)$.

(v) If ω is a loop based at x_0 then $\omega \simeq \omega^\circ$. To see this notice that in this case $\sigma_{\omega(0)} = \sigma_{\omega(1)} = \sigma_{x_0}$, and by assumption σ_{x_0} is the constant path. Therefore we have $\omega^\circ = \bar{\sigma}_{x_0} * \omega * \sigma_{x_0} \simeq \omega$.

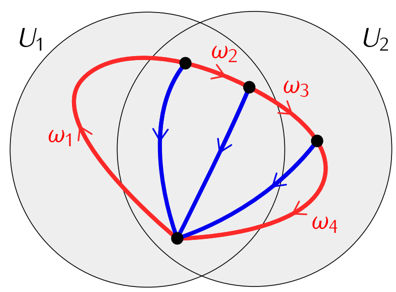
We are now ready to describe the construction of the homomorphism h in the diagram $(*)$. Let $\omega\colon [0,1] \to X$ be a loop. The sets $\omega^{-1}(U_1)$ and $\omega^{-1}(U_2)$ form an open cover of the interval $[0,1]$. Using the Lebesgue number of this cover we obtain that there exists an n-tuple of numbers $(s_0, \dots, s_n)$ such that $0 = s_0 < s_1 < \ldots < s_n = 1$ and $\omega([s_{i-1}, s_i])$ is contained in either U_1 or U_2 for each $i = 1, \dots, n$. Let $\omega_i\colon [0,1] \to X$ be the path given by $\omega_i(s) = \omega(s_{i-1}s + s_i(1-s))$. This path coincides with the restriction of ω to the subinterval $[s_{i-1} : s_i]$:



We have $\omega \simeq \omega_1 * \omega_2 * \cdots * \omega_n$. By the properties of loop completion we get:

$$\omega \simeq \omega^\circ \simeq \omega_1^\circ * \omega_2^\circ * \ldots * \omega_n^\circ$$

Moreover, if ω_i is contained in U_1 then ω_i° is a loop in U_1, and likewise for U_2.



Let $k(i)$ denote either 1 or 2 depending if ω_i° is a loop in U_1 or U_2. Define an element $\tilde{h}(\omega) \in G$ by

$$\tilde{h}(\omega) = h_{k(1)}([\omega_1^\circ]_{k(1)}) \cdot h_{k(2)}([\omega_2^\circ]_{k(2)}) \cdot \ldots \cdot h_{k(n)}([\omega_n^\circ]_{k(n)}) \qquad (**)$$

There are two ambiguities in this formula. First, if ω_i is a path entirely contained in U_0 then ω_i° is a loop in U_0 and in such case we can take $k(i)$ to be either 1 or 2. This however does not matter since in such case we have $[\omega_i]_1 = i_{1*}([\omega_i]_0)$ and $[\omega_i]_2 = i_{2*}([\omega_i]_0)$, and since $h_1 i_{1*} = h_2 i_{2*}$ we get

$$h_1([\omega_i^\circ]_1) = h_1 i_{1*}([\omega_i^\circ]_0) = h_2 i_{2*}([\omega_i^\circ]_0) = h_2([\omega_i^\circ]_1)$$

The second ambiguity comes from the fact that the formula $(**)$ uses subdivision of ω into paths ω_i, and the value of $\tilde{h}(\omega)$ could change if we change the subdivision. To see that this is not the case consider the subdivision $\omega \simeq \omega_1 * \ldots * \omega_n$ that comes from an n-tuple of numbers $\underline{s} = (s_0, \ldots, s_n)$, and let s_+ be a number such that $s_{i-1} < s_+ < s_i$ for some $1 \leq i \leq n$. The $(n+1)$-tuple $\underline{s}' = (s_1, \ldots, s_{i-1}, s_+, s_{i+1}, \ldots, s_n)$ produces the subdivision

$$\omega \simeq \omega_1 * \ldots * \omega_{i-1} * \tau_1 * \tau_2 * \omega_{i+1} * \ldots * \omega_n$$

where $\tau_1 * \tau_2 \simeq \omega_i$. By the properties of loop completion $[\omega^\circ]_{k(i)} = [\tau_1^\circ]_{k(i)} \cdot [\tau_2^\circ]_{k(i)}$, and thus we get

$$h_{k(i)}([\omega_i^\circ]_{k(i)}) = h_{k(i)}([\tau_1^\circ]_{k(i)} \cdot [\tau_2^\circ]_{k(i)}) = h_{k(i)}([\tau_1^\circ]_{k(i)}) \cdot h_{k(i)}([\tau_2^\circ]_{k(i)})$$

This shows the the value of $\tilde{h}(\omega)$ does not depend on whether we compute it using $\underline{s}$ or $\underline{s}'$. Arguing inductively we obtain that if an m-tuple $\underline{s}'$ is obtained by adding some number of elements to an n-tuple $\underline{s}$ then the value of $\tilde{h}(\omega)$ computed using $\underline{s}'$ is the same as the value computed using $\underline{s}$. Finally, given an arbitrary n-tuple $\underline{s}$ and an m-tuple $\underline{s}'$ that produce two subdivisions of ω we can find an r-tuple $\underline{s}''$ that can be obtained from each of $\underline{s}$ and $\underline{s}'$ by inserting some additional elements. The argument above shows then that the values of $\tilde{h}(\omega)$ computed using $\underline{s}$ and $\underline{s}'$ must be equal since they are both equal to the value computed using $\underline{s}''$.

Our goal will be to show that $\tilde{h}(\omega)$ depends only on the homotopy class of ω:

Claim. If ω, τ are loops in X and $\omega \simeq \tau$ then $\tilde{h}(\omega) = \tilde{h}(\tau)$.

Assuming for a moment this claim holds notice that it allows us to define a function $h\colon \pi_1(X, x_0) \to G$ by $h([\omega]) = \tilde{h}(\omega)$. Notice also that this function has the following properties:

1) h is a homomorphism;
2) h makes the diagram $(*)$ into a commutative diagram;
3) h is the only homomorphism that makes the diagram $(*)$ commute.

Indeed, to see 1) observe that if ω and ω' are two loops in X with subdivisions $\omega \simeq \omega_1 * \ldots * \omega_n$ and $\omega' \simeq \omega_1' * \ldots * \omega_m'$ then $\omega * \omega'$ has a subdivision

$$\omega * \omega' \simeq \omega_1 * \ldots * \omega_n * \omega_1' * \ldots * \omega_m'$$

Using this observation and the definition of h we get that $h([\omega] \cdot [\omega']) = h([\omega]) \cdot h([\omega'])$.

To verify 2) notice that if ω is a loop in U_k for either $k = 1$ or $k = 2$ then we don't need to subdivide it, i.e. we can take $\omega = \omega_1$. Since $\omega \simeq \omega^\circ$ the formula $(**)$ gives

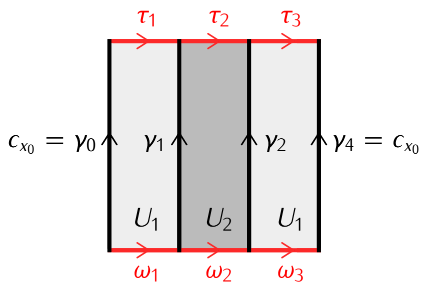
$$h([\omega]) = \tilde{h}(\omega) = h_k([\omega]_k)$$

and since $[\omega] = j_{k*}([\omega]_k)$ we obtain $hj_{k*} = h_k$.

Finally, to see 3) notice that we have shown that any element $[\omega] \in \pi_1(X, x_0)$ can be written as a product $[\omega] = [\omega_1^\circ] \cdot \ldots \cdot [\omega_n^\circ]$ where ω_i is a loop in $U_{k(i)}$ for either $k(i) = 1$ or $k(i) = 2$. For each

$i = 1, \ldots, n$ we have $[\omega_i^\circ] = j_{k(i)*}([\omega_i^\circ]_{k(i)})$. Thus in order to get the identity $hj_{k(i)*} = h_{k(i)}$ we must set $h([\omega_i^\circ]) = h_{k(i)}([\omega_i^\circ]_{k(i)})$. Since homomorphisms preserve products we are forced to define $h([\omega])$ by the formula $(**)$.

The above observations show that once we verify that $\tilde{h}$ is a homotopy invariant function the proof of van Kampen's Theorem will be complete. Assume then that ω, τ are loops in X and that there exists a homotopy $h\colon [0,1] \times [0,1] \to X$ between them: $h_0 = \omega$, $h_1 = \tau$. We will consider first a special case, and assume in addition that there exists numbers $0 = s_0 < s_1 < \ldots < s_n = 1$ such that the homotopy h maps each rectangle $[s_{i-1}, s_i] \times [0,1]$ either into U_1 or into U_2. In particular this gives a subdivisions $\omega \simeq \omega_1 * \ldots * \omega_n$ and $\tau \simeq \tau_1 * \ldots * \tau_n$ where ω_i and τ_i are restrictions of ω and τ to the subinterval $[s_{i-1}, s_i]$. For $i = 0, \ldots, n$ denote by γ_i the path given by $\gamma(t) = h(s_i, t)$. Notice that γ_0 and γ_n are constant paths at x_0.



Let $k(i)$ denote either 1 or 2 depending if $h([s_{i-1}, s_i] \times [0,1])$ is contained in U_1 or U_2. Notice that for each i the paths ω_i, τ_i, γ_{i-1}, and γ_i are paths in $U_{k(i)}$. Moreover, using the restriction of h to $[s_{i-1}, s_i] \times [0,1]$ we obtain that the path ω_i is homotopic to $\gamma_{i-1} * \tau_i * \bar{\gamma}_i$ via a homotopy contained in $U_{k(i)}$. Using the properties of loop completion we obtain

$$[\omega_i^\circ]_{k(i)} = [(\gamma_{i-1} * \tau_i * \bar{\gamma}_i)^\circ]_{k(i)} = [\gamma_{i-1}^\circ]_{k(i)} \cdot [\tau_i^\circ]_{k(i)} \cdot [\gamma_i^\circ]_{k(i)}^{-1}$$

This identity and the formula $(**)$ gives:

$$\begin{aligned} \tilde{h}(\omega) &= h_{k(1)}([\omega_1^\circ]_{k(1)}) \cdot h_{k(2)}([\omega_2^\circ]_{k(2)}) \cdot \ldots \cdot h_{k(n)}([\omega_n^\circ]_{k(n)}) \qquad (***) \\ &= h_{k(1)}[(\gamma_0^\circ]_{k(1)}) \cdot h_{k(1)}([\tau_1^\circ]_{k(1)}) \cdot h_{k(1)}([\gamma_1^\circ]_{k(1)})^{-1} \\ &\quad \cdot h_{k(2)}[(\gamma_1^\circ]_{k(2)}) \cdot h_{k(2)}([\tau_2^\circ]_{k(2)}) \cdot h_{k(2)}([\gamma_1^\circ]_{k(2)})^{-1} \cdot \ldots \\ &\quad \cdot h_{k(n)}[(\gamma_{n-1}^\circ]_{k(n)}) \cdot h_{k(n)}([\tau_n^\circ]_{k(n)}) \cdot h_{k(n)}([\gamma_n^\circ]_{k(n)})^{-1} \end{aligned}$$

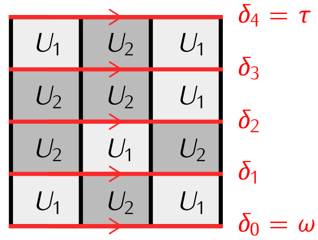
Notice that $h_{k(1)}([\gamma_1^\circ]_{k(1)})$ and $h_{k(n)}([\gamma_n^\circ]_{k(n)})$ are trivial elements of G since γ_1° and γ_n° are constant loops. Also, for $i = 1, \ldots n-1$ we have $h_{k(i)}([\gamma_i^\circ]_{k(i)}) = h_{k(i+1)}([\gamma_i^\circ]_{k(i+1)})$. This is obvious if $k(i) = k(i+1)$. If $k(i) \neq k(i+1)$ then this identity still holds since in such case γ_i (and thus also γ_i°) is contained in U_0 and then

$$h_1([\gamma_i^\circ]_1) = h_1 i_{1*}([\gamma_i^\circ]_0) = h_2 i_{2*}([\gamma_i^\circ]_0) = h_2([\gamma_i^\circ]_2)$$

Using these observations we can simplify the equation $(***)$ to

$$\tilde{h}(\omega) = h_{k(1)}([\tau_1^\circ]_{k(1)}) \cdot h_{k(2)}([\tau_2^\circ]_{k(2)}) \cdot \ldots \cdot h_{k(n)}([\tau_n^\circ]_{k(n)}) = \tilde{h}(\tau)$$

It remains to consider the general case when we are given two loops ω and τ in X such that $\omega \simeq \tau$. Let $h\colon [0,1]\times[0,1] \to X$ be a homotopy with $h_0 = \omega$ and $h_1 = \tau$. The sets $h^{-1}(U_1)$ and $h^{-1}(U_2)$ form an open cover of X. Using the Lebesgue number of this cover we can find numbers $0 = s_0 < s_1 < \ldots < s_n = 1$ and $0 = t_0 < t_1 < \ldots < t_m = 1$ such that h maps each rectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ either into U_1 or into U_2. Let δ_i be the path given by $\delta_i(s) = h(s, t_i)$:



Notice that the restriction of h to the rectangle $[0,1]\times[t_{i-1}, t_i]$ gives a homotopy between δ_{i-1} and δ_i, and moreover this homotopy is of the form considered in the special case above. Therefore for each $i = 1, \ldots, m$ we have $\tilde{h}(\delta_{i-1}) = \tilde{h}(\delta_i)$. As a consequence we obtain

$$\tilde{h}(\omega) = \tilde{h}(\delta_0) = \tilde{h}(\delta_1) = \ldots = \tilde{h}(\delta_m) = \tilde{h}(\tau)$$

$\square$

Theorem 10.17 can be generalized to the case where the space X is covered by more that two (and possibly inifinitely many) open sets:

11.1 Theorem. *Let (X, x_0) be a pointed topological space and let $\{U_i\}_{i\in I}$ be an open cover of X such that $x_0 \in U_i$ for all $i \in I$. For $i, j \in I$ let $f_{ij}\colon U_i \cap U_j \to U_i$ denote the inclusion map. If the set $U_i \cap U_j \cap U_k$ is path connected for all $i, j, k \in I$ then*

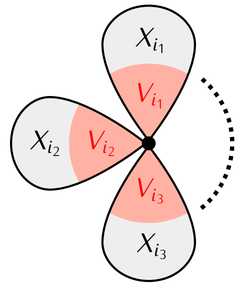
$$\pi_1(X, x_0) \cong *_{i\in I}\pi_1(U_i, x_0)/N$$

where N is the normal subgroup of $*_{i\in I}\pi_1(U_i, x_0)$ generated by all elements of the form $f_{ij}([\omega])\cdot f_{ji}([\omega])^{-1}$ for $i, j \in I$ and $[\omega] \in \pi_1(U_i \cap U_j, x_0)$.

Here is one application of the generalized van Kampen Theorem:

11.2 Proposition. *Let $\{(X_i, x_i)\}_{i\in I}$ be a family of path connected pointed spaces, and let $\bigvee_{i\in I} X_i$ be the space obtained by identifying the basepoints $x_i \sim x_j$ for all $i, j \in I$. Assume that for each $i \in I$ there exists a set $V_i \subseteq X_i$ such that V_i is open in X_i, $x_i \in V_i$, and the one-point space $\{x_i\}$ is a deformation retract of V_i. Then*

$$\pi_1(\textstyle\bigvee_i X_i) \cong *_{i\in I}\pi_1(X_i, x_i)$$



Proof. We will denote by $\bar{x}$ the point of $\bigvee_{i\in I} X_i$ obtained by identifying the points $x_i \in X_i$. For $i \in I$ let $U_i = X_i \vee \bigvee_{i\in} V_i$. The sets U_i are path connected and the form an open cover $\bigvee_i X_i$. Moreover for $j, k.l \in I$ we have

$$U_j \cap U_k \cap U_l = \begin{cases} U_j & \text{if } j = k = l \\ \bigvee_{i\in I} V_i & \text{otherwise} \end{cases}$$

so $U_j \cap U_k \cap U_l$ is path connected for all j, k, l. Using Theorem 11.1 we obtain

$$\pi_1(\bigvee_{i\in I} X_i, \bar{x}) \cong *_{i\in I}\pi_1(U_i, x_i)/N$$

where N is the normal group generated by elements $f_{ij*}([\omega]) \cdot f_{ji*}([\omega])^{-1}$ for $[\omega] \in \pi_1(U_i \cap U_j, \bar{x})$. As in Theorem 11.1 by f_{ij} we denote here the inclusion map $f_{ij}\colon U_i \cap U_j \to U_{ij}$. For $i \in I$ let $h_i\colon V_i \times [0, 1] \to V_i$ the deformation retraction of V_i onto $\{x_i\}$. These maps define a deformation retraction of the space $\bigvee_{i\in I} V_i$ onto $\{\bar{x}\}$. This shows that the space $\bigvee_{i\in I} V_i$ is contractible and so $\pi_1(\bigvee_{i\in I} V_i, \bar{x}) = \{1\}$. Since for all $i, j \in I$ such that $i \neq j$ we have $U_i \cap U_j = \bigvee_{i\in I} V_i$ it follows that the group N is trivial and so $\pi_1(\bigvee_{i\in I} X_i, \bar{x}) \cong *_{i\in I}\pi_1(U_i, x_i)$. Finally, using again the deformation retractions h_i we can construct for each $j \in I$ a deformations retraction of U_j onto X_j. This shows that $\pi_1(U_j, \bar{x}) \cong \pi_1(X_j, \bar{x})$, and so we get

$$\pi_1(\bigvee_{i\in I} X_i, \bar{x}) \cong *_{i\in I}\pi_1(U_i, \bar{x}) \cong *_{i\in I}\pi_1(X_i, \bar{x})$$

□

11.3 Example. Let s_0 be a basepoint of S^1. Taking $V = S^1 \smallsetminus \{s_1\}$ were $s_1 \in S^1$ is a point different from s_0 we obtain an open neighborhood of s_0 which deformation retracts onto $\{x_0\}$. Thus using Proposition 11.2 we get

$$\pi_1(\bigvee_{i\in I} S^1) \cong *_{i\in I}\pi_1(S^1) \cong *_{i\in I}\mathbb{Z}$$

Exercises to Chapter 11