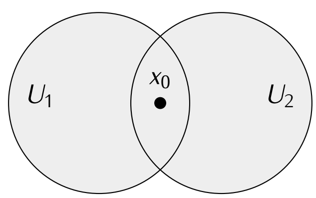
10 | Pushouts and van Kampen's Theorem

Our main goal in this and the next chapter is to explain and prove van Kampen's theorem, which is one of the most useful tools for computing fundamental groups of spaces. This theorem says that if a space (X, x_0) can be decomposed into a union of two subspaces $X = U_1 \cup U_2$ in a way that satisfies a few conditions then it is possible to describe $\pi_1(X, x_0)$ in terms of the groups $\pi_1(U_1, x_0)$, $\pi_1(U_2, x_0)$, and $\pi_1(U_1 \cap U_2, x_0)$.



In Chapter 9 we have seen that the product formula for the fundamental group can be interpreted as a statement saying that the functor $\pi_1 \colon \mathbf{Top}_* \to \mathbf{Gr}$ preserves categorical products. Similarly, one way to state van Kampen's theorem is to say that under certain assumptions the functor π_1 preserves categorical structures called *pushouts*. In this chapter we will discuss the notion of a pushout in a general category. We will then describe how pushouts look like in the category of topological spaces and in the category of groups. This will allow us to state van Kampen's theorem precisely and show a few of its applications. The proof of this theorem will be given in the next chapter.

10.1 Definition. Let $\mathbf{C}$ be a category. Assume that we are given a diagram of objects and morphisms in $\mathbf{C}$ of the following form:

$$c_1 \xleftarrow{f_1} c_0 \xrightarrow{f_2} c_2$$

A *pushout* of this diagram is an object $p \in \mathsf{C}$ equipped with morphisms $g_i \colon c_i \to p$ for $i = 1, 2$ that satisfy the following conditions:

1) $g_1 f_1 = g_2 f_2$
2) for any morphisms $h_i \colon c_i \to a$ ($i = 1, 2$) that satisfy $h_1 f_1 = h_2 f_2$ there exists a unique morphism $h \colon p \to a$ such that $hg_i = h_i$ for $i = 1, 2$.

$$
\begin{tikzcd}
c_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & c_1 \arrow[d, "g_1"] \arrow[ddr, bend left, "h_1"] & \\
c_2 \arrow[r, "g_2"'] \arrow[drr, bend right, "h_2"'] & p \arrow[dr, dashed, "h"] & \\
& & a
\end{tikzcd}
$$

If such p exists then we write

$$p = \operatorname{colim}(c_1 \xleftarrow{f_1} c_0 \xrightarrow{f_2} c_2)$$

10.2 Note. Pushout is a special instance of a more general notion of a *colimit* (or a *direct limit*). This is where the notation colim comes from.

In a general category pushouts may not exist. However, if a pushout of a diagram $c_1 \leftarrow c_0 \rightarrow c_2$ exists then it is defined uniquely up to an isomorphism:

10.3 Proposition. *Let $c_1 \overset{f_1}{\leftarrow} c_0 \overset{f_2}{\rightarrow} c_2$ be a diagram in a category C and let $p, p' \in \mathsf{C}$.*

1) If p is a pushout of this diagram and $p' \cong p$ then p' is also a pushout.

2) Conversely, both p and p' are pushouts of the above diagram then $p \cong p'$.

Proof. Exercise. □

10.4 Note. Proposition 10.3 shows that the notation $p = \operatorname{colim}(c_1 \leftarrow c_0 \rightarrow c_2)$ is not entirely precise. It would be more accurate to write $p \cong \operatorname{colim}(c_1 \leftarrow c_0 \rightarrow c_2)$.

We will now look at constructions of pushouts in two categories that are of interests for us: the category of topological spaces and the category of groups.

Pushouts of topological spaces. Pushout is in the category **Top** can be described as follows:

10.5 Proposition. *For any diagram of topological spaces* $X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2$ *the pushout exists and it is given by*

$$\operatorname{colim}(X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2) = (X_1 \sqcup X_2)/\sim$$

where $\sim$ *is the equivalence relation defined by* $f_1(x) \sim f_2(x)$ *for all* $x \in X_0$.

Proof. Exercise □

10.6 Example. Assume that $X_0 = \varnothing$. We have $\operatorname{colim}(X_1 \leftarrow \varnothing \rightarrow X_2) = X_1 \sqcup X_2$.

10.7 Example. If X is a space, $A \subseteq X$ is a subspace and $*$ is a space consisting of one point then $\operatorname{colim}(* \leftarrow A \rightarrow X) = X/A$.

10.8 Example. Recall that given a map $f\colon X \to Y$ by M_f we denote the mapping cylinder of f (8.15). We can view M_f as a pushout as follows:

$$M_f = \operatorname{colim}(Y \xleftarrow{f} X \xrightarrow{j} X \times [0,1])$$

where j is given by $j(x) = (x, 1)$.

10.9 Example. The following fact will be used later on. If X is a topological space and $U, V \subseteq X$ are open sets such that $X = U \cup V$ then we have a homeomorphism

$$X \cong \operatorname{colim}(U \longleftarrow U \cap V \longrightarrow V)$$

(exercise). Note that this is not true in general, if U, V are not open in X.

Pushouts of groups. In the case of a diagram of topological spaces $X_1 \leftarrow X_0 \rightarrow X_2$ the pushout was constructed starting with the disjoint union $X_1 \sqcup X_2$ and then performing certain identifications in this space. Pushouts in the category of groups are built in a similar way, but the disjoint union of spaces is replaced by the free product of groups. We will describe free products first and then proceed to the construction of pushouts.

10.10 Definition. The *free product* of groups G and H is a group $G{*}H$ described as follows.

- Elements of $G{*}H$ are sequences $(g_1, g_2, \ldots, g_n)$ where $n \geq 0$ and $g_i \in G$ or $g_i \in H$ for each $i = 1, \ldots, n$.
- If for some i the elements g_i, g_{i+1} are either both in G or both in H (so that we can take their product in one of these groups) then

$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n)$$

Also, if g_i is the identity element in either G or H then

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_n) = (g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$$

- Multiplication in $G{*}H$ is defined by concatenation of sequences:

$$(g_1, \dots, g_n) \cdot (g'_1, \dots, g'_m) = (g_1, \dots, g_n, g'_1, \dots, g'_m)$$

10.11 Note. Definition 10.10 can be extended to a free product of an arbitrary family $\{G_j\}_{j\in J}$ of groups. In this case the free product is the group $*_{j\in J} G_j$ whose elements are sequences $(g_1, g_2, \dots, g_n)$ where for each $i = 1, \dots, n$ we have $g_i \in G_j$ for some $j \in J$. Identities between such sequences and multiplication is defined analogously as in the case of two groups.

10.12 Note. From now on we will write elements of $G{*}H$ as $g_1 g_2 \dots g_n$ instead of $(g_1, g_2, \dots, g_n)$.

10.13 Proposition. *For any diagram of groups $G_1 \stackrel{f_1}{\leftarrow} G_0 \stackrel{f_2}{\rightarrow} G_2$ the pushout exists and it is given by*

$$\operatorname{colim}(G_1 \xleftarrow{f_1} G_0 \xrightarrow{f_2} G_2) = (G_1{*}G_2)/N$$

where N is the normal subgroup of $G_1{*}G_2$ generated by all elements of the form $f_1(g)f_2(g)^{-1}$ for $g \in G_0$.

Proof. Exercise. $\square$

10.14 Note. The group $(G_1{*}G_2)/N$ described above is called the *free amalgamated product* of G_1 and G_2 and it is denoted by $G_1{*}_{G_0}G_2$

10.15 Example. Let $\{1\}$ denote the trivial group. We have $\operatorname{colim}(G_1 \leftarrow \{1\} \rightarrow G_2) = G_1{*}G_2$.

10.16 Example. If H is a subgroup of G then $\operatorname{colim}(\{1\} \leftarrow H \rightarrow G) = G/N$ where N is the smallest normal subgroup of G generated by H. In particular is H is a normal subgroup then $N = H$.

We are now ready to state the main result of this chapter:

10.17 van Kampen Theorem. *Let (X, x_0) be a pointed topological space and let $U_1, U_2 \subseteq X$ be open sets such that $X = U_1 \cup U_2$. If the sets U_1, U_2, and $U_1 \cap U_2$ are path connected and $x_0 \in U_1 \cap U_2$ then*

$$\pi_1(X, x_0) \cong \operatorname{colim}(\pi_1(U_1, x_0) \xleftarrow{i_{1*}} \pi_1(U_1 \cap U_2, x_0) \xrightarrow{i_{2*}} \pi_1(U_2, x_0))$$

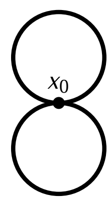
where for $k = 1, 2$ the homomorphism i_{k*} is induced by the inclusion map $i_k\colon U_1 \cap U_2 \rightarrow U_k$.

10.18 Note. By Example 10.9 under the assumptions of van Kampen's theorem we have a homeomorphism $X \cong \mathrm{colim}(U_1 \leftarrow U_1 \cap U_2 \rightarrow U_2)$. Thus van Kampen's theorem says that under some conditions the functor π_1 preserves pushouts:

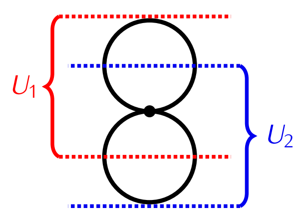
$$\pi_1(\mathrm{colim}(U_1 \longleftarrow U_1 \cap U_2 \longrightarrow U_2)) \cong \mathrm{colim}(\pi_1(U_1) \longleftarrow \pi_1(U_1 \cap U_2) \longrightarrow \pi_1(U_2))$$

We will now have a look at a few examples showing how van Kampen's Theorem can be used in computations of the fundamental group.

10.19 Example. We will compute the fundamental group of the space $S^1 \vee S^1$. It will be convenient to choose as the basepoint of $S^1 \vee S^1$ the point x_0 where the two copies of S^1 are glued together:



While $S^1 \vee S^1$ is a union of two circles we cannot use these circles as sets U_1 and U_2 in van Kampen's Theorem since they are not open in $S^1 \vee S^1$. Instead we will decompose $S^1 \vee S^1$ as follows:



Applying van Kampen's Theorem we obtain

$$\pi_1(S^1 \vee S^1, x_0) \;\cong\; \mathrm{colim}(\pi_1(U_1, x_0) \longleftarrow \pi_1(U_1 \cap U_2, x_0) \longrightarrow \pi_1(U_2, x_0))$$

Notice that $U_1 \simeq S^1 \simeq U_2$ and and $U_1 \cap U_2 \simeq *$. It follows that $\pi(U_1, x_0) \cong \pi_1(U_2, x_0) \cong \mathbb{Z}$ and $\pi_1(U_1 \cap U_2, x_0) \cong \{1\}$. This gives:

$$\pi_1(S^1 \vee S^1, x_0) \cong \mathrm{colim}(\mathbb{Z} \longleftarrow \{1\} \longrightarrow \mathbb{Z}) \cong \mathbb{Z} * \mathbb{Z}$$

Arguing by induction with respect to n we obtain that $\pi_1(\bigvee_{i=1}^n S^1) \cong *_{i=1}^n \mathbb{Z}$.

The following fact will be useful in the next example:

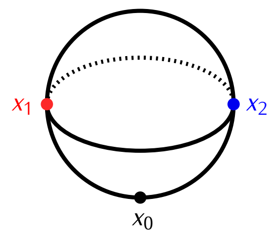
10.20 Lemma. *Let X be a space and let $U_1, U_2 \subseteq X$ be open sets such that $X = U_1 \cup U_2$ and $U_1, U_2, U_1 \cap U_2$ are path connected. If $\pi_1(U_1) \cong \{1\}$ and $\pi_1(U_2) \cong \{1\}$ then $\pi_1(X) \cong \{1\}$.*

Proof. Choose $x_0 \in U_1 \cap U_2$. By van Kampen's Theorem we get

$$\pi_1(X, x_0) \cong \mathrm{colim}(\{1\} \longleftarrow \pi_1(U_1 \cap U_2, x_0) \longrightarrow \{1\}) \cong \{1\}$$

□

10.21 Example. Take a sphere S^n, $n > 1$. Let $x_0 \in S^n$ be a basepoint, and let x_1, x_2 be two points distinct from x_0:



Take $U_1 = S^n \smallsetminus \{x_1\}$, $U_2 = S^n \smallsetminus \{x_2\}$. We have $U_1 \cong \mathbb{R}^n \cong U_2$, and so $\pi_1(U_1, x_0) \cong \pi_1(U_2, x_0) \cong \{1\}$. By Proposition 10.20 we obtain that $\pi_1(S^n, x_0) \cong \{1\}$ for $n > 1$.

Exercises to Chapter 10

E10.1 Exercise. Prove Proposition 10.3

E10.2 Exercise. Assume that we have the following commutative diagram in a category $\mathbf{C}$:

$$\begin{tikzcd}
c_1 \arrow[d, "g_1"'] & c_0 \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "g_0"'] & c_2 \arrow[d, "g_2"] \\
c_1' & c_0' \arrow[l, "f_1'"] \arrow[r, "f_2'"'] & c_2'
\end{tikzcd}$$

Show that if the morphisms g_0, g_1, g_2 are isomorphisms, then there exists an isomorphism

$$\mathrm{colim}(c_1 \xleftarrow{f_1} c_0 \xrightarrow{f_2} c_2) \xrightarrow{\cong} \mathrm{colim}(c_1' \xleftarrow{f_1'} c_0' \xrightarrow{f_2'} c_2')$$

E10.3 Exercise. Prove Proposition 10.5.

E10.4 Exercise. Let X be a topological space, and let $U, V \subseteq X$ be sets such that $U \cup V = X$.

a) Show that if U and V are open sets then there exists a homeomorphism

$$X \cong \operatorname{colim}(U \longleftarrow U \cap V \longrightarrow V)$$

b) Give an example of a space X and (non-open) sets $U, V \subseteq X$ such that the above homeomorphism does not hold. Justify your answer.

E10.5 Exercise. Let U, V be open sets in $\mathbb{R}^n$ such that $U, V, U \cap V$ are path connected, and $U \cup V = \mathbb{R}^n$. Let $x_0 \in U \cap V$. Show that if $\pi_1(U, x_0) \ncong \{1\}$ then $\pi_1(U \cap V, x_0) \ncong \{1\}$.

E10.6 Exercise. Compute the fundamental group of each of the following spaces.

a) The space X_1 obtained from $\mathbb{R}^3$ by removing n straight lines intersecting at the origin.

b) The space X_2 obtained from $\mathbb{R}^3$ by removing the circle

$$S^1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ z = 0\}$$

and the z-axis.

c) The space X_3 obtained from two copies of the torus $S^1 \times S^1$ by identifying the circle $S^1 \times \{x_0\}$ in one torus with the circle $\{x_0\} \times S^1$ in the other torus.

d) The space X_4 obtained by removing a straight line from $\mathbb{R}^4$.

e) The space X_5 obtained by removing a 2-dimensional vector subspace from $\mathbb{R}^4$.

E10.7 Exercise. Let S^3 be the 3-dimensional sphere, let $A \subset S^3$ be a closed subspace of S^3, and let $x_0 \in S^3$. Assuming that the space $S^3 - (A \cup \{x_0\})$ is path connected show that the inclusion map

$$j \colon S^3 - (A \cup \{x_0\}) \hookrightarrow S^3 - A$$

induces an isomorphism of fundamental groups.

E10.8 Exercise. Let $f \colon X \to Y_1$ and $g \colon X \to Y_2$ be maps of topological spaces. The *double mapping cylinder* of f and g is the space

$$M(f, g) = (Y_1 \sqcup X \times [0, 1] \sqcup Y_2)/\sim$$

where $\sim$ is the equivalence relation given by $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that if X, Y_1, Y_2 are path connected spaces and $x_0 \in X$ then there exists an isomorphism

$$\pi_1(M(f, g)) \cong \operatorname{colim}(\pi_1(Y_1, f(x_0)) \xleftarrow{f_*} \pi_1(X, x_0) \xrightarrow{g_*} \pi_1(Y_2, g(x_0)))$$

E10.9 Exercise. The *join* of spaces X and Y is the space $X * Y$ given by

$$X * Y = X \times Y \times [0, 1]/\sim$$

where $(x, y, 0) \sim (x, y', 0)$ for all $x \in X$, $y, y' \in Y$, and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$, $y \in Y$. Show that if X and Y are non-empty path connected spaces then $\pi_1(X * Y) \cong \{1\}$.

E10.10 Exercise. Recall that a topological group is a topological space X equipped with a group structure such that the maps $\mu\colon X \times X \to X$, $\mu(g, h) = gh$ and $\eta\colon X \to X$, $\eta(g) = g^{-1}$ are continuous. For example, the space $\mathbb{R}^2$ is a topological group with the addition

$$(x, y) + (x', y') = (x + x', y + y')$$

defining the group structure.

a) Show that for any $x_1 \in \mathbb{R}^2$ the space $X = \mathbb{R}^2 \smallsetminus \{x_1\}$ can be given the structure of a topological group.

b) Show that for $n > 1$ the space $X = \mathbb{R}^2 \smallsetminus \{x_1, \dots, x_n\}$ does not have the structure of a topological group.