20 | From Subgroups to Coverings

In the last chapter we have seen that if X is a locally path connected space and $x_0 \in X$ then there are 1-1 functions:

$$\begin{pmatrix}\text{isomorphism classes}\\ \text{of path connected}\\ \text{coverings of } X\end{pmatrix} \xrightarrow{\Omega} \begin{pmatrix}\text{conjugacy classes}\\ \text{of subgroups}\\ \text{of } \pi_1(X, x_0)\end{pmatrix}$$

$$\begin{pmatrix}\text{isomorphism classes of}\\ \text{pointed path connected}\\ \text{coverings of } (X, x_0)\end{pmatrix} \xrightarrow{\Omega} \begin{pmatrix}\text{subgroups}\\ \text{of}\\ \pi_1(X, x_0)\end{pmatrix}$$

In both cases the function Ω associates to a covering $p\colon T \to X$ with $\tilde{x}_0 \in p^{-1}(x_0)$ the (conjugacy class of) subgroup $p_*(\pi_1(T, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$. The natural question is for which subgroups $H \subseteq \pi_1(X, x_0)$ there exists a covering $p\colon T \to X$ such that $\Omega(p) = H$. Our goal here will be to prove that under some assumptions on X such covering p exists for any subgroup H, and so the maps Ω given above are bijections. As the first step we will show that Ω is a bijection provided that there exists a covering of X corresponding to the trivial subgroup of $\pi_1(X, x_0)$.

20.1 Definition. Let X be a locally path connected space. A *universal covering* of X is a covering $\tilde{p}\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is a simply connected space.

Directly from the Lifting Criterion 19.5 we obtain:

20.2 Proposition. *Let X be a locally path connected space and $\tilde{p}\colon \widetilde{X} \to X$ be a universal covering of X. For any covering $q\colon T \to X$ there exists a map of coverings:*

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "\tilde{p}"'] & & T \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

Notice that by Exercise 19.2 if T is path connected then the map f in Proposition 20.2 is onto. This suggests that if X has a universal covering then any path connected covering of X may be obtained as a quotient space of the universal covering space $\widetilde{X}$. This is the main idea in the proof of the following fact:

20.3 Theorem. *Let X be a locally path connected space and let $x_0 \in X$. If there exists a universal covering $\tilde{p}\colon \widetilde{X} \to X$ then for each subgroup $H \subseteq \pi_1(X, x_0)$ there exists a covering $p_H\colon T_H \to X$ and $\tilde{x}_H \in p_H^{-1}(x_0)$ such that $p_{H*}(\pi_1(T_H, \tilde{x}_H)) = H$.*

Proof. Let $H \subseteq \pi_1(X, x_0)$ be a subgroup. Let $\tilde{p}\colon \widetilde{X} \to X$ be a universal covering of X and let $y_0 \in \tilde{p}^{-1}(x_0)$. For each point $y \in \widetilde{X}$ let τ_y be a path in $\widetilde{X}$ joining y_0 with y. Notice that if $\tilde{p}(y) = \tilde{p}(y')$ then the path $\tilde{p}\tau_y * \tilde{p}\overline{\tau_{y'}}$ is loop in X based at x_0. Notice also that the homotopy class of this loop does not depend on the choice of paths τ_y and $\tau_{y'}$. Indeed, if σ_y and $\sigma_{y'}$ are some other paths in $\widetilde{X}$ joining y_0 with, respectively, y and y' then, since $\widetilde{X}$ is simply connected, by Proposition 5.6 we obtain $\tau_y \simeq \sigma_y$ and $\tau_{y'} \simeq \sigma_{y'}$ which gives a homotopy $\tilde{p}\tau_y * \tilde{p}\overline{\tau_{y'}} \simeq \tilde{p}\sigma_y * \tilde{p}\overline{\sigma_{y'}}$.

Define a relation $\sim$ on $\widetilde{X}$ such that $y \sim y'$ if

(i) $\tilde{p}(y) = \tilde{p}(y')$
(ii) $[\tilde{p}\tau_y * \tilde{p}\overline{\tau_{y'}}] \in H$

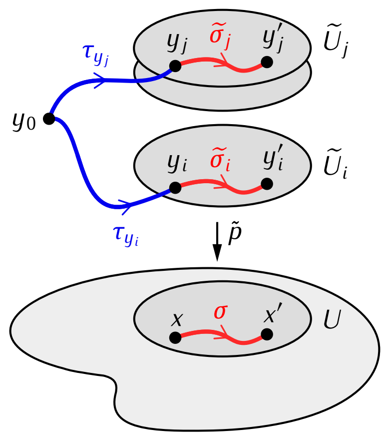
One can check that $\sim$ is an equivalence relation on $\widetilde{X}$ (exercise). Denote the quotient space by X_H and let $q\colon \widetilde{X} \to X_H$ be the quotient map. We get a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "q"] \arrow[dr, "\tilde{p}"'] & & X_H \arrow[dl, "p_H"] \\ & X & \end{tikzcd}$$

where p_H is given by $p_H([y]) = \tilde{p}(y)$. We will prove that $p_H\colon X_H \to X$ is a covering. Let $x \in X$ and let $U \subseteq X$ be an open neighborhood of x which is U is path connected and evenly covered by $\tilde{p}$. Such U exists by the assumption that X is locally path connected. We will show that U is evenly covered by p_H. We have $\tilde{p}^{-1}(U) = \bigcup_{i\in I} \widetilde{U}_i$ where $\{\widetilde{U}_i\}_{i\in I}$ is the set of all distinct slices of $\tilde{p}$ over U. Notice that if y, y' are points in the same slice $\widetilde{U}_i$ and $y \neq y'$ then $y \not\sim y'$ since $\tilde{p}(y) \neq \tilde{p}(y')$. On the other hand we claim that the following holds:

Claim. If $\widetilde{U}_i$, $\widetilde{U}_j$ are two slices, and there exist points $y_i \in \widetilde{U}_i$, $y_j \in \widetilde{U}_j$ such that $y_i \sim y_j$ then for every $y'_i \in \widetilde{U}_i$, $y'_j \in \widetilde{U}_j$ such that $\tilde{p}(y'_i) = \tilde{p}(y'_j)$ we have $y'_i \sim y'_j$.

To see this denote $x = \tilde{p}(y_i) = \tilde{p}(y_j)$ and $x' = \tilde{p}(y'_i) = \tilde{p}(y'_j)$. Since $x, x' \in U$ and U is path connected we can find a path σ in U such that $\sigma(0) = x$ and $\sigma(1) = x'$. Denote by $\widetilde{\sigma}_i$ and $\widetilde{\sigma}_j$ the lifts of σ to, respectively $\widetilde{U}_i$ and $\widetilde{U}_j$. Notice that $\widetilde{\sigma}_i(0) = y_i$, $\widetilde{\sigma}_i(1) = y'_i$, and likewise $\widetilde{\sigma}_j(0) = y_j$, $\widetilde{\sigma}_j(1) = y'_j$. Denote also by τ_{y_i}, τ_{y_j} paths in $\widetilde{X}$ that connect the point y_0 to, respectively y_i and y_j:



By the definition of the relation $\sim$ in order to show that $y'_i \sim y'_j$ we only need to verify that $[\tilde{p}(\tau_{y_i} * \widetilde{\sigma}_i) * \tilde{p}\overline{(\tau_{y_j} * \widetilde{\sigma}_j)}] \in H$. This holds since

$$[\tilde{p}(\tau_{y_i} * \widetilde{\sigma}_i) * \tilde{p}\overline{(\tau_{y_j} * \widetilde{\sigma}_j)}] = [\tilde{p}\tau_{y_i} * \sigma * \overline{\sigma} * \tilde{p}\overline{\tau_{y_j}}] = [\tilde{p}\tau_{y_i} * \tilde{p}\overline{\tau_{y_j}}]$$

and $[\tilde{p}\tau_{y_i} * \tilde{p}\overline{\tau_{y_j}}] \in H$, since by assumption $y_i \sim y_j$.

The statement of the claim implies that for any slice $\widetilde{U}_i$ the set $q^{-1}(q(\widetilde{U}_i))$ is a union of some number of slices of $\tilde{p}$ over U, and so it is an open set in $\widetilde{X}$. This shows that the set $q(\widetilde{U}_i)$ is open in X_H. It also shows that if $V \subseteq \widetilde{U}_i$ is an open set then $q(V)$ is open in X_H. Indeed, it is enough to check that $q^{-1}(q(V))$ is open in $\widetilde{X}$, but this holds since $q^{-1}(q(V)) = \tilde{p}^{-1}(\tilde{p}(V)) \cap q^{-1}(q(\widetilde{U}_i))$.

The claim also implies that we can select a subset $\{\widetilde{U}_{i_k}\}_{k\in K}$ of the set of slices of $\tilde{p}$ over U such that the map $q' \colon \bigcup_{k\in K} U_{i_k} \to p_H^{-1}(U)$ obtained as a restriction of q is a continuous bijection. Since by the observation above q' maps open sets to open sets, the inverse function q'^{-1} is also continuous, and so q' is a homeomorphism. Finally, since $\bigcup_{k\in K} U_{i_k} \cong U \times K$ (where the set K is taken with the discrete topology) we obtain a homeomorphism $U \times K \cong p_H^{-1}(U)$.

Let $\tilde{x}_H = q(y_0)$. It remains to prove that $p_{H*}(\pi_1(T_H, \tilde{x}_H)) = H$. Let ω be a loop in X based at x_0, and let $\widetilde{\omega} \colon [0,1] \to X_H$ be the lift of ω satisfying $\widetilde{\omega}(0) = \tilde{x}_H$ Recall that by Theorem 18.1 $[\omega]$ is an element of $p_{H*}(\pi_1(T_H, \tilde{x}_H))$ if and only if $\widetilde{\omega}$ is a loop in X_H. Therefore it will suffice to show that $\widetilde{\omega}$ is a loop if and only if $[\omega] \in H$. Notice that $\widetilde{\omega} = q\widetilde{\omega}'$ where $\widetilde{\omega}' \colon [0,1] \to \widetilde{X}$ is the lift of ω to $\widetilde{X}$ satisfying $\widetilde{\omega}'(0) = y_0$. From the construction of X_H it follows that $\widetilde{\omega}$ is a loop if and only if $\widetilde{\omega}'(1) \sim \widetilde{\omega}'(0) = y_0$

where $\sim$ is the equivalence relation on $\widetilde{X}$ defined before. Take $\widetilde{\omega}'$ to be a path joining y_0 with $\widetilde{\omega}'(1)$ and take the constant path c_{y_0} as a path joining y_0 with itself. Using the definition of $\sim$ we obtain that $\widetilde{\omega}'(1) \sim \widetilde{\omega}'(0)$ if and only if $[\tilde{p}\widetilde{\omega}' * \tilde{p}\overline{c_{y_0}}] \in H$. Since $[\tilde{p}\widetilde{\omega}' * \tilde{p}\overline{c_{y_0}}] = [\omega]$ we obtain that $\widetilde{\omega}'(1) \sim \widetilde{\omega}'(0)$ if and only if $[\omega] \in H$

$\square$

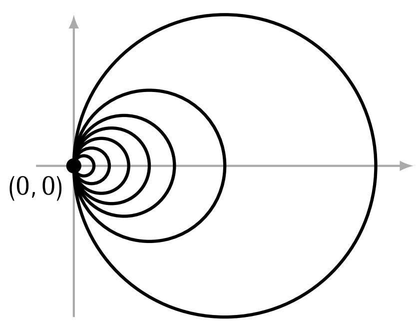
The remaining task is to determine for which spaces a universal covering exists. We will need the following definition:

20.4 Definition. A space X is *semi-locally simply connected* if every point $x \in X$ has an open neighborhood $U \subseteq X$ such that the homomorphism $i_*\colon \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion map $i\colon U \to X$ is the trivial homomorphism.

Equivalently, X is semi-locally simply connected if each point in X has an open neighborhood U such that any loop based at x and contained in U is homotopic to the constant loop, but though a homotopy contained in X (and not necessarily a homotopy contained in U).

20.5 Example. If X is a space such that each point $x \in X$ has an open neighborhood U where $\pi_1(U, x)$ is the trivial group, then X is semi-locally simply connected. One can use this to show, for example, that every topological manifold is semi-locally simply connected. On the other hand, it is possible to find a semi-locally simply connected space X, such that for some point $x \in X$ every open neighborhood of x has a non-trivial fundamental group.

20.6 Example. The *Hawaiian earring* space is a subspace $X \subseteq \mathbb{R}^2$ given by $X = \bigcup_{n=1}^{\infty} C_n$ where C_n is the circle with radius $\frac{1}{n}$ and center at the point $(0, \frac{1}{n})$:



This space is not semi-locally simply connected since for each open neighborhood U of the point $x_0 = (0, 0)$ the homomorphism $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is non-trivial.

Semi-local simple connectedness is a necessary condition for existence of a universal covering:

20.7 Proposition. *If X is space such that there exists a universal covering $p\colon \widetilde{X} \to X$ then X is semi-locally simply connected.*

Proof. Exercise. □

Conversely, we will show that the following holds:

20.8 Theorem. *If X is a space which is connected, locally path connected, and semi-locally simply connected then there exists a universal covering $p\colon \widetilde{X} \to X$.*

Proof. Let X be a space satisfying assumptions of the theorem. We will say that an open set $U \subseteq X$ is *trivial* if U is path connected and for any $x \in U$ the homomorphism $i_*\colon \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion map $i\colon U \to X$ is trivial. Since X is locally path connected and semi-locally simply connected trivial sets form a basis of the topology on X, that is any open set in X is a union of trivial sets.

The first step in the construction of a universal covering $p\colon \widetilde{X} \to X$ is to describe the set of points of the space $\widetilde{X}$. This description will be based on the following reasoning. Assume that we already have a universal covering $p\colon \widetilde{X} \to X$, let $x_0 \in X$ and let $\widetilde{x}_0 \in p^{-1}(x_0)$. Since the space $\widetilde{X}$ is path connected, for any point $\tilde{x} \in \widetilde{X}$ there exists a path $\widetilde{\omega}$ such that $\omega(0) = \widetilde{x}_0$ and $\widetilde{\omega}(1) = \tilde{x}$. Moreover, since $\widetilde{X}$ is simply connected any two such path in $\widetilde{X}$ are homotopic. In effect the assignment $[\widetilde{\omega}] \mapsto \widetilde{\omega}(1)$ gives a bijection:

$$\begin{pmatrix}\text{homotopy classes of paths}\\ \widetilde{\omega}\colon [0,1] \to \widetilde{X}\\ \text{with } \widetilde{\omega}(0) = \widetilde{x}_0\end{pmatrix} \cong \begin{pmatrix}\text{points of } \widetilde{X}\end{pmatrix}$$

Notice that we also have a bijection:

$$\begin{pmatrix}\text{homotopy classes of paths}\\ \widetilde{\omega}\colon [0,1] \to \widetilde{X}\\ \text{with } \widetilde{\omega}(0) = \widetilde{x}_0\end{pmatrix} \cong \begin{pmatrix}\text{homotopy classes of paths}\\ \omega\colon [0,1] \to X\\ \text{with } \omega(0) = x_0\end{pmatrix}$$

which assigns to the homotopy class of a path $\widetilde{\omega}$ in $\widetilde{X}$ the homotopy class of $p\widetilde{\omega}$. The inverse function sends the homotopy class of a path ω in X to the homotopy class of $\widetilde{\omega}$, where $\widetilde{\omega}$ is the unique lift of ω satisfying $\widetilde{\omega}(0) = \widetilde{x}_0$.

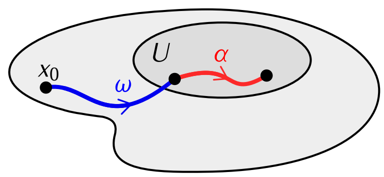
In effect we get a bijective correspondence:

$$\begin{pmatrix}\text{points of } \widetilde{X}\end{pmatrix} \cong \begin{pmatrix}\text{homotopy classes of paths}\\ \omega\colon [0,1] \to X\\ \text{with } \omega(0) = x_0\end{pmatrix}$$

The upshot of this argument is that if we are given a space X then we can define $\widetilde{X}$ to be the set on the right hand side of the above bijection.

Next, we need to define a topology on the set $\widetilde{X}$. Let $[\omega] \in \widetilde{X}$, and let U be a trivial set such that $\omega(1) \in U$. Define:

$$U[\omega] = \{[\omega * \alpha] \mid \alpha\colon [0,1] \to U,\ \alpha(0) = \omega(1)\}$$



One can check that the collection of all sets $U[\omega]$ defined in this way is a basis of a topology on $\widetilde{X}$ (exercise). We will consider $\widetilde{X}$ as a topological space with topology defined by this basis.

Consider the function $p\colon \widetilde{X} \to X$ given by $p([\omega]) = \omega(1)$. We will show that this is a universal covering of X. We will use the following observations, proofs of which are left as an exercise:

(i) For any trivial set $U \subseteq X$ and any path $[\omega] \in \widetilde{X}$ such that $\omega(1) \in U$ the map

$$p|_{U[\omega]}\colon U[\omega] \to U$$

is a homeomorphism.

(ii) Let $U \subseteq X$ be a trivial set, let $x \in U$ and let $H(x_0, x) = \{[\omega] \in \widetilde{X} \mid \omega(1) = x\}$. Then

$$p^{-1}(U) = \bigcup_{[\omega]\in H(x_0,x)} U[\omega]$$

Moreover $U[\omega] \cap U[\omega'] = \varnothing$ for all $[\omega], [\omega'] \in H(x_0, x)$, $[\omega] \neq [\omega']$.

(iii) For a path $\omega\colon [0,1] \to X$ such that $\omega(0) = x_0$ and for $s \in [0,1]$ let ω_s be the path in X defined by $\omega_s(t) = \omega(st)$. The function $h_\omega\colon [0,1] \to \widetilde{X}$ given by $h_\omega(s) = [\omega_s]$ is continuous.

Directly from (ii) is follows that the function p is continuous. Furthermore, combining (ii) and (i) we obtain that p is covering and that each trivial set in X is evenly covered by p.

Next, by (iii) the space $\widetilde{X}$ is path connected. Indeed, for any $[\omega] \in \widetilde{X}$ the function h_ω is a path in $\widetilde{X}$ joining $[\omega]$ with $[c_{x_0}]$, the homotopy class of the constant path at x_0. It remains then to show that the fundamental group $\pi_1(\widetilde{X}, [c_{x_0}])$ is trivial, or equivalently that $p_*(\pi_1(\widetilde{X}, [c_{x_0}]))$ is the trivial subgroup of $\pi_1(X, x_0)$. Assume then that ω is a loop in X such that $[\omega] \in p_*(\pi_1(\widetilde{X}, [c_{x_0}]))$. By Theorem 18.1 this means that the lift of ω to $\widetilde{X}$ that starts at $[c_{x_0}]$ is a loop in $\widetilde{X}$. Notice, however, that this lift is given the path h_s defined in (iii). This path is a loop only when $[c_{x_0}] = h_\omega(0) = h_\omega(1) = [\omega]$ i.e. only when $[\omega]$ is the trivial element of $\pi_1(X, x_0)$.

$\square$

Exercises to Chapter 20

E20.1 Exercise. Prove Proposition 20.7.

E20.2 Exercise. Let X, Y be connected and locally path connected spaces, and let $\tilde{p}_X\colon \widetilde{X} \to X$, and $\tilde{p}_Y\colon \widetilde{Y} \to Y$ be their universal coverings. Show that if $X \simeq Y$ then $\widetilde{X} \simeq \widetilde{Y}$.

E20.3 Exercise. Describe explicitly all non-isomorphic coverings of the space $\mathbb{R}P^2 \times \mathbb{R}P^2$

E20.4 Exercise. Let X be a space, and let $A \subseteq X$. Assume that both X and A are connected and locally path connected, and that the inclusion map $i\colon A \to X$ induces an isomorphism of the fundamental groups

$$i_*\colon \pi_1(A, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

for $x_0 \in A$. Show that if $\tilde{p}\colon \widetilde{X} \to X$ is a universal covering of X then the map $\tilde{p}|_{\tilde{p}^{-1}(A)}\colon \tilde{p}^{-1}(A) \to A$ is a universal covering of A.

E20.5 Exercise. a) Let X be a finite, path connected, 1-dimensional CW complex. Show that if $\tilde{p}\colon \widetilde{X} \to X$ is the universal covering of X then the space $\widetilde{X}$ has the structure of a 1-dimensional CW complex such that $\tilde{p}$ is a cellular map.

b) Use part a) to show that if F is a finitely generated free group then every subgroup of F is free.

c) Recall that $[G : H]$ denotes the index of a subgroup H in a group G. Let F be free group on n generators, and let H be a subgroup of F. Show that if $[F : H] = k$ then H is a free group on $(n-1)k + 1$ generators.