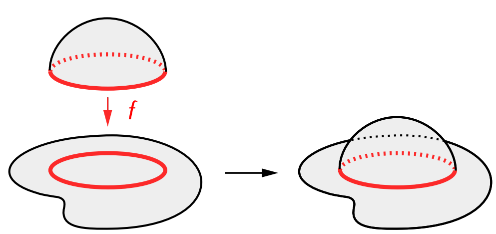
12 | CW Complexes

Van Kampen's theorem tells us that in order to compute the fundamental group of a complicated space we may try to decompose the space into simpler pieces. Information about these pieces and how they fit together suffices to determine the fundamental group of the whole space. A similar approach works in many settings, and for this reason it is useful if we can represent a given space as a collection of some simple building blocks that are assembled together. In this chapter we discuss the notion of a CW complex that provides a scheme for constructing such representations of spaces. In this scheme as the building blocks one uses closed discs of various dimensions. We will see that many interesting spaces can be given the structure of a CW complex. In the next chapters we will see how to compute the fundamental group of an arbitrary CW complex. Building on this result we will also show that any group can be realized as the fundamental group of some space: given a group G we will construct a CW complex X such that $\pi_1(X) \cong G$.

12.1 Definition. Let X be a space and let $f\colon S^{n-1} \to X$ be a continuous function. We say that a space Y is obtained by *attaching an n-cell* to X if $Y = X \sqcup D^n/{\sim}$ where $\sim$ is the equivalence relation given by $x \sim f(x)$ for all $x \in S^{n-1} \subseteq D^n$. We write $Y = X \cup_f e^n$.



Notice that $X \cup_f e^n = \operatorname{colim}(D^n \xleftarrow{j} S^{n-1} \xrightarrow{f} X)$ where $j\colon S^{n-1} \to D^n$ is the inclusion map.

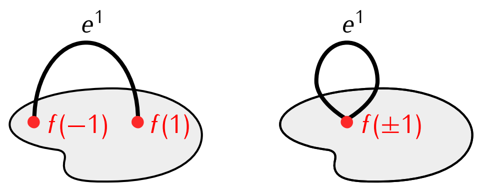
12.2 Here is some terminology associated to the operation of cell attachment:

- The map $f\colon S^{n-1} \to X$ is called the *attaching map* of the cell e^n.
- The map $\bar{f}\colon D^n \to X \sqcup D^n \to X \cup_f e^n$ is called the *characteristic map* of the cell e^n.

- The subspace $e^n = \bar{f}(D^n \smallsetminus S^{n-1}) \subseteq X \cup_f e^n$ is called the *open cell*.
- The subspace $\bar{e}^n = \bar{f}(D^n) \subseteq X \cup_f e^n$ is called the *closed cell*.

12.3 Example. For $n = 0$ we have $D^0 = \{*\}$ and $S^{-1} = \varnothing$. Therefore $X \cup e^0$ is a disjoint union of X and a point.

12.4 Example. For $n = 1$ we have $D^1 = [-1, 1]$ and $S^0 = \{-1, 1\}$. The space $X \cup_f e^1$ is obtained by attaching to X an arch or a loop, depending if $f: S^0 \to X$ is a 1-1 function or not:



In general, the operation of cell attachment can be viewed as a special case of the construction of a mapping cone:

12.5 Lemma. *For any map $f: S^{n-1} \to X$ the space $X \cup_f e^n$ is homeomorphic to the mapping cone C_f.*

Proof. Exercise. $\square$

This immediately gives the following fact:

12.6 Proposition. *If $f, g: S^{n-1} \to X$ are maps such that $f \simeq g$ then $X \cup_f e^n \simeq X \cup_g e^n$.*

Proof of Proposition 12.6. Follows from Lemma 12.5 and Proposition 8.17. $\square$

12.7 Definition. Let X be topological space and let $Y \subseteq X$. The pair (X, Y) is a *relative CW complex* if $X = \bigcup_{n=-1}^{\infty} X^{(n)}$ where

1) $X^{(-1)} = Y$;
2) for $n \geq 0$ the space $X^{(n)}$ is obtained by attaching n-cells to $X^{(n-1)}$;
3) the topology on X is defined so that a set $U \subseteq X$ is open if and only if $U \cap X^{(n)}$ is open in $X^{(n)}$ for all n.

12.8 Note. By part 3) of Definition 12.7 if (X, Y) is a relative CW complex then a function $f: X \to Z$ is continuous if and only if $f|_{X^{(n)}}: X^{(n)} \to Z$ is continuous for all $n \geq -1$.

12.9 Note. If (X, Y) is a relative CW complex then the space $X^{(n)}$ is called *the n-skeleton* of X.

12.10 Definition. A *CW complex* is a space X such that $(X, \varnothing)$ is a relative CW complex.

12.11 Definition. 1) A CW complex X is *finite* if it consists of finitely many cells.

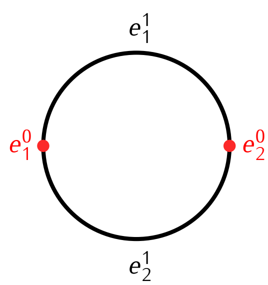
2) A CW complex X is *finite dimensional* if $X = X^{(n)}$ for some n.

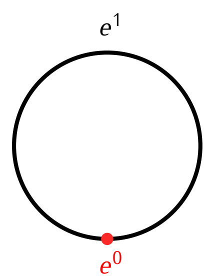
3) The *dimension* of a CW complex X is defined by

$$\dim X = \begin{cases} \min\{n \mid X = X^{(n)}\} & \text{if } X \text{ is finite dimensional} \\ \infty & \text{otherwise} \end{cases}$$

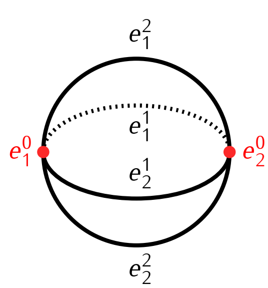
12.12 Example. The only CW complex of dimension -1 is the empty space. A CW complex of dimension 0 is a discrete topological space (with each point defining a 0-cell).

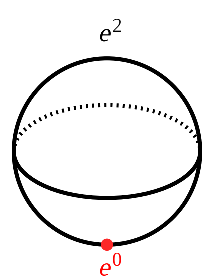
12.13 Example. If a space can be equipped with a structure of a CW complex of dimension greater than 0 then such structure is not unique. Here are two different CW complex structures on S^1, one with two 0-cells and two 1-cells, and the other with one 0-cell and one 1-cell:



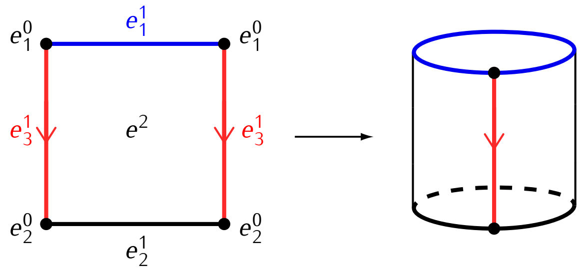


12.14 Example. Here are two examples of CW complex structures on S^2. The first has two cells in in each of the dimensions 0, 1, and 2. The second has one 0-cell e^0 and one 2-cell which is attached using the constant attaching map $S^1 \to e^0$:

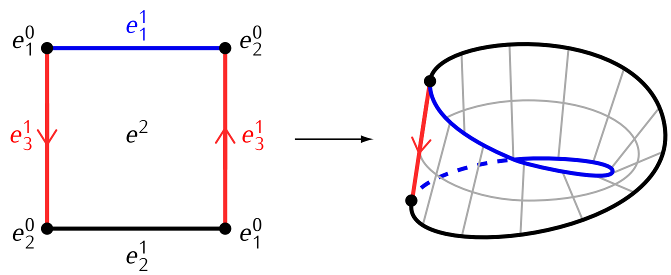


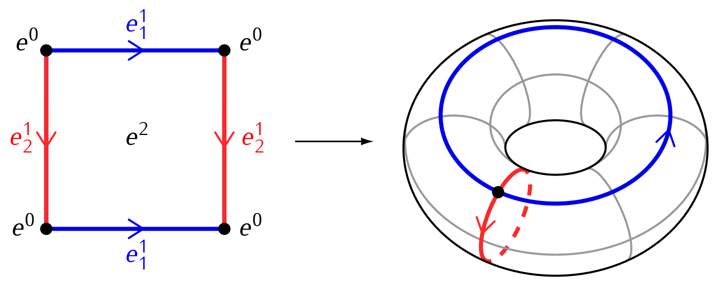


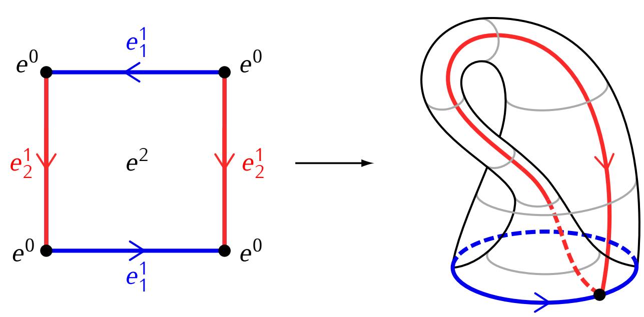
12.15 Example. The cylinder $S^1 \times [0, 1]$ can be given a CW complex structure with two 0-cells, three 1-cells, and one 2-cell. It is easier to visualize this structure if we consider the cylinder as a quotient space obtained by gluing together two vertical edges of a square. The pair of the upper vertices of the square represents one 0-cell of the cylinder, and the pair of lower vertices the second 0-cell. The three 1-cells come from each of the horizontal edges and the pair of vertical edges. The interior of the square corresponds to the interior of the 2-cell of the cylinder:



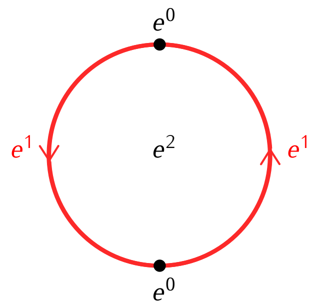
Since the Möbius band, the torus, and the Klein bottle also can be constructed by gluing together some edges of the square we can describe CW complex structures on these spaces in a similar way:







12.16 Example. Recall that the 2-dimensional real projective space $\mathbb{RP}^2$ can be constructed as a quotient space of the disc D^2 obtained by identifying antipodal points on the boundary of D^2: $x \sim (-x)$ for $x \in S^1$. This space can be given a CW complex structure with one 0-cell, one 1-cell, and one 2-cell:



12.17 Note. It is not true that every space can be given a structure of a CW complex. For example, consider the following subspace of the real line:

$$X = \{\tfrac{1}{n} \in \mathbb{R} \mid n = 1, 2, \dots\} \cup \{0\}$$

This space is not homeomorphic or even homotopy equivalent to a CW complex. To show this we will use the fact that if Y is a CW complex then each path connected component of Y is open in Y (exercise). Assume that there exists a homotopy equivalence $f\colon X \to Y$ where Y is a CW complex. This implies that the induced map $f_*\colon \pi_0(X) \to \pi_0(Y)$ is a bijection. On the other hand let $\{Y_i\}_{i\in I}$ be the family of all path connected components of Y. The sets Y_i are open in Y, so the sets $f^{-1}(Y_i)$ are open in X and they define an open cover of X. The space X is compact, so this cover has a finite subcover $\{f^{-1}(Y_{i_1}), \dots, f^{-1}(Y_{i_k})\}$. Since X is an infinite space this implies that that there exist distinct points $x_1, x_2 \in X$ such that $f(x_1)$ and $f(x_2)$ belong to the same path connected component of Y. On the other hand x_1 and x_2 belong to different path connected components in X since every path connected component of X consists of a single point. This shows that $f_*\colon \pi_0(X) \to \pi_0(Y)$ is not a bijection, and so we obtain a contradiction.

The following fact if often useful:

12.18 Proposition. *1) Let (X, Y) be a relative CW complex. If $A \subseteq X$ is a compact set then A is closed in X and it has a non-empty intersection with finitely many open cells of X only.*

2) If X is a CW complex and $A \subseteq X$ is a closed set which has a non-empty intersection with only finitely many open cells of X then A is compact.

Proof. Exercise. □

12.19 Corollary. *A CW complex is compact if and only if it is a finite.*

Proof. Follows from Proposition 12.18 □

Exercises to Chapter 12

E12.1 Exercise. Show that a CW complex X is path connected if and only if its 1-skeleton $X^{(1)}$ is path connected.

E12.2 Exercise. Prove Proposition 12.18.

E12.3 Exercise. Let X be a space, and let $Y = X \cup_f e^n$ be obtained by attaching one n-dimensional cell to X. Show that if X is a retract of Y then $Y \simeq X \vee S^n$.