4 | Dependence on the Basepoint

By construction, the fundamental group of a space depends not only on the space itself, but also on the choice of a basepoint. In some applications a space may come equipped with a preferred basepoint, but in other situations we may need to choose a basepoint arbitrarily to compute the fundamental group. In this chapter we examine how the choice of a basepoint impacts the fundamental group of a space. We will also see how the construction of the fundamental group can be modified so that it does not involve a choice of a basepoint.

We start with the observation that the fundamental group of a pointed space depends only on the path connected component of the basepoint:

4.1 Proposition. *Let X be a space, let $x_0 \in X$, and let $Y \subseteq X$ be the path connected component of x_0. If $i\colon Y \to X$ is the inclusion map then the induced homomorphism*

$$i_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$$

is an isomorphism of groups.

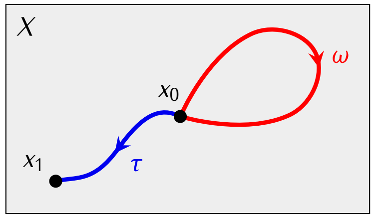
Proof. Exercise. $\square$

Proposition 4.1 implies that if we change the basepoint from one path connected component of X to another we can get entirely different fundamental groups, since in general path connected components need not be related in any way. It remains then to consider the situation when we are given a space X with two different basepoints x_0 and x_1, that belong to the same path connected component of X. In this case there exists a path in X joining these points. We have:

4.2 Proposition. *Let X be a space and let $x_0, x_1 \in X$. For any path $\tau\colon [0, 1] \to X$ such that $\tau(0) = x_0$ and $\tau(1) = x_1$ the function*

$$s_\tau\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

given by $s_\tau([\omega]) = [\bar{\tau} * \omega * \tau]$ *is an isomorphism of groups.*



Proof. Exercise. □

4.3 Corollary. *If* X *is a space and* $x_0, x_1 \in X$ *are points than belong to the same path connected component of* X *then* $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

Proof. Follows from Proposition 4.2. □

4.4 Note. In general the isomorphism s_τ given in Proposition 4.2 depends on the choice of the path τ. However, if $\pi_1(X, x_0)$ is an abelian group then for any paths τ, τ' joining x_0 and x_1 we have $s_\tau = s_{\tau'}$ (exercise). Thus in such case we obtain a canonical isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

4.5 Note. Given a path connected space X we will sometimes write $\pi_1(X)$ to denote the fundamental group of X taken with respect to some unspecified basepoint of X. By Corollary 4.3 this will not create problems as long as we are interested in the isomorphism type of the fundamental group only.

Recall that any continuous function $f\colon X \to Y$ defines a homomorphism of fundamental groups $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$. The next proposition describes how this homomorphism changes with a change of the basepoint:

4.6 Proposition. *Let* $x_0, x_1 \in X$ *and let* $f\colon X \to Y$ *be a continuous function. Given a path* τ *in* X *such that* $\tau(0) = x_0$ *and* $\tau(1) = x_1$ *consider the isomorphisms* $s_\tau\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ *and* $s_{f\tau}\colon \pi_1(Y, f(x_0)) \to \pi_1(Y, f(x_1))$ *defined as in Proposition 4.2. Then following diagram commutes:*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "s_\tau"', "\cong"] & \pi_1(Y, f(x_0)) \arrow[d, "\cong"', "s_{f\tau}"] \\
\pi_1(X, x_1) \arrow[r, "f_*"'] & \pi_1(Y, f(x_1))
\end{tikzcd}$$

Proof. Exercise. □

4.7 Corollary. *Let X be a path connected space, $x_0, x_1 \in X$, and let $f\colon X \to Y$ be a continuous function. The homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism (or is the trivial homomorphism or is 1-1 or onto) if and only if the homomorphism $f_*\colon \pi_1(X, x_1) \to \pi_1(Y, f(x_1))$ has the same property.*

Proof. Follows from Proposition 4.6. □

In most applications it is sufficient to work with the fundamental group associated to some choice of a basepoint, using Proposition 4.2 whenever we need to change the basepoint. However, it is also possible to modify the construction of the fundamental group in a way that does not involve any choice of a basepoint. This is done as follows. Given a space X in place of the group $\pi_1(X, x_0)$ we take the category $\Pi_1(X)$ whose objects are points of X. The set of morphisms between points $x_0, x_1 \in X$ is the set of homotopy classes of paths joining these points:

$$\mathrm{Mor}_{\Pi_1(X)}(x_0, x_1) = \pi_1(X, x_1, x_0)$$

Composition of morphisms is given by concatenation of paths: for $[\omega] \in \mathrm{Mor}_{\Pi_1(X)}(x_0, x_1)$ and $[\tau] \in \mathrm{Mor}_{\Pi_1(X)}(x_1, x_2)$ we set $[\tau] \circ [\omega] = [\omega * \tau]$. By Lemma 3.8 this composition is associative, and by Lemma 3.9 the homotopy class $[c_{x_0}]$ of the constant path at x_0 plays the role of the identity morphism in $\mathrm{Mor}_{\Pi_1(X)}(x_0, x_0)$.

4.8 Definition. Let X be a topological space. The category $\Pi_1(X)$ is called the fundamental groupoid of X.

4.9 Note. In general, a *groupoid* is a category where every morphism is an isomorphism. The category $\Pi_1(X)$ is a groupoid since by Lemma 3.10 any $[\omega] \in \mathrm{Mor}_{\Pi_1(X)}(x_0, x_1)$ is an isomorphism with the inverse given by $[\overline{\omega}] \in \mathrm{Mor}_{\Pi_1(X)}(x_1, x_0)$.

Notice that the category $\Pi_1(X)$ contains information about fundamental groups of X taken with respect to all possible basepoints, since for any $x_0 \in X$ we have $\mathrm{Mor}_{\Pi_1(X)}(x_0, x_0) \cong \pi_1(X, x_0)$.

As we have seen any pointed map $f\colon (X, x_0) \to (Y, y_0)$ defines a homomorphism of fundamental groups $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Similarly, any map of spaces $f\colon X \to Y$ defines a functor of fundamental groupoids

$$f_*\colon \Pi_1(X) \to \Pi_1(Y)$$

defined as follows. For $x \in X$ we set $f_*(x) = f(x)$ (where we consider x as and object of $\Pi_1(X)$ and $f(x)$ as an object of $\Pi_1(Y)$). For $[\omega] \in \mathrm{Mor}_{\Pi_1(X)}(x_0, x_1)$ we define $f_*([\omega]) = [f \circ \omega]$.

Recall that by a small category is a category whose objects form a set. Let **Cat** denote a category whose objects are small categories and morphisms are functors. We have:

4.10 Corollary. *The assignments $X \mapsto \Pi_1(X)$ and $f \mapsto f_*$ define a functor*

$$\Pi_1\colon \mathbf{Top} \to \mathbf{Cat}$$

from the category of unpointed topological spaces to the category of small categories

Proof. Exercise. □

Exercises to Chapter 4

E4.1 Exercise. Prove Proposition 4.2.

E4.2 Exercise. Recall that if X is a topological space, $x_0, x_1 \in X$, and $\tau\colon [0,1] \to X$ is a path such that $\tau(0) = x_0$, $\tau(1) = x_1$ then τ defines an isomorphism

$$s_\tau\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

Show that if $\pi_1(X, x_0)$ is an abelian group then this isomorphism does not depend on the choice of the path τ. That is, if τ' is another path in X such that $\tau'(0) = x_0$ and $\tau'(1) = x_1$ then $s_\tau = s_{\tau'}$.

E4.3 Exercise. Recall that $S^1 \vee S^1$ is the space consisting of two circles joined at one point x_0 (the eight-figure space). Assume that there exists a space (Y, y_0) such that the group $\pi_1(Y, y_0)$ is non-abelian. Show that this implies that $\pi_1(S^1 \vee S^1, x_0)$ must be a non-abelian group.

E4.4 Exercise. A *topological group* G is a group that is also a topological space, and such that the maps $\mu\colon G \times G \to G$, $\mu(g,h) = gh$, and $\eta\colon G \to G$, $\eta(g) = g^{-1}$ are continuous. Let e denote the identity element in G.

a) Show that for any $g_0 \in G$ we have $\pi_1(G, g_0) \cong \pi_1(G, e)$ even if G is not path connected.

b) Let ω, τ be loops in G based at $e \in G$. Since $\omega(s)$ and $\tau(s)$ are elements of the group G for $s \in [0,1]$, we can use group multiplication to obtain an element $\omega(s) \cdot \tau(s) \in G$. Let

$$\omega \odot \tau\colon [0,1] \to G$$

denote the loop defined by $\omega \odot \tau(s) = \omega(s) \cdot \tau(s)$. Show that $[\omega \odot \tau] = [\omega * \tau]$ (where $*$ denotes the concatenation of loops). It follows that for a topological group G we can describe multiplication in $\pi_1(G, e)$ in two different but equivalent ways: as a loop concatenation and as a pointwise multiplication of loops.

c) Show that the fundamental group $\pi_1(G, e)$ is abelian.