16 | Cellular Approximation Theorem

Results of the last few chapters tell us how to compute the fundamental group of a CW complex of dimension 2 or lower. In this chapter we show that this actually suffices to compute the fundamental group of any CW complex X, since the fundamental group of X is always isomorphic to the fundamental group of its 2-skeleton $X^{(2)}$. This fact is a consequence of the Cellular Approximation Theorem which, in general, is one of the main tools used when working with CW complexes.

16.1 Definition. Let X, Y be CW complexes. A map $f\colon X \to Y$ is *cellular* if $f(X^{(n)}) \subseteq Y^{(n)}$ for all $n \geq 0$.

16.2 Cellular Approximation Theorem. *Let X, Y be CW complexes. For any map $f\colon X \to Y$ there exists a cellular map $g\colon X \to Y$ such that $f \simeq g$. Moreover, if $A \subseteq X$ is a subcomplex and $f|_A\colon A \to Y$ is a cellular map then g can be selected so that $f|_A = g|_A$ and $f \simeq g$ (rel A).*

Before proving this result we will show how it lets us identify the fundamental group of any CW complex with the fundamental group of its 2-skeleton.

16.3 Theorem. *Let X be a CW complex and let $x_0 \in X^{(2)}$. The inclusion map $i\colon X^{(2)} \to X$ induces an isomorphism $i_*\colon \pi_1(X^{(2)}, x_0) \to \pi_1(X, x_0)$.*

Proof. We can assume that x_0 is a 0-cell in X. We will prove first that i_* is onto. Let $\omega\colon [0,1] \to X$ be a loop based at x_0. We need to show that there exists a loop $\omega'\colon [0,1] \to X$ such that $\omega'([0,1]) \subseteq X^{(2)}$ and that $\omega \simeq \omega'$ (rel $\{0,1\}$). Consider the interval $[0,1]$ as a CW complex with two 0-cells joined by one 1-cell. The 0-skeleton of $[0,1]$ is the subspace $\{0,1\} \subseteq [0,1]$. Since $\omega(0) = \omega(1) = x_0 \in X^{(0)}$ the map $\omega|_{\{0,1\}}$ is cellular. By Theorem 16.2 there exists a cellular map $\omega'\colon [0,1] \to X$ such that $\omega' \simeq \omega$ (rel $\{0,1\}$). This means that $[\omega] = [\omega']$ in $\pi_1(X, x_0)$. Moreover, since $[0,1]$ is a CW complex of dimension 1 thus ω' is a loop in $X^{(1)} \subseteq X^{(2)}$.

Next, we will show that i_* is 1-1. Let $[\omega], [\tau] \in \pi_1(X^{(2)}, x_0)$. Using the same argument as above we can assume that $\omega, \tau\colon [0,1] \to X^{(2)}$ are cellular maps. Assume that $i_*([\omega]) = i_*([\tau])$. This means that there exists a path homotopy $h\colon [0,1] \times [0,1] \to X$ with $h_0 = \omega$ and $h_1 = \tau$. The square $I^2 = [0,1] \times [0,1]$ can be considered as a CW complex whose 0-cells are vertices of the square and whose 1-cells are the edges. The 1-skeleton of I^2 is the boundary ∂I^2. Notice that $h|_{\partial I^2}$ is a cellular map. Using Theorem 16.2 we obtain that there exists a cellular map $h'\colon [0,1] \times [0,1] \to X$ such that $h'|_{\partial I^2} = h|_{\partial I^2}$. The map h' gives another path homotopy between ω and τ. Moreover, since $\dim I^2 = 2$ thus h' is a homotopy contained in $X^{(2)}$. This shows that $[\omega] = [\tau]$ in $\pi_1(X^{(2)}, x_0)$

□

The rest of this chapter will be devoted to a proof of Theorem 16.2. The proof will be split into several lemmas.

16.4 Lemma. *Let Y be a space, and let Y' be obtained from Y by attaching a single n-cell:*

$$Y' = Y \cup e^n$$

Let $f\colon D^m \to Y'$ be a map such that $f(S^{m-1}) \subseteq Y$. If $m < n$ then there exists a map $g\colon D^m \to Y'$ such that $f|_{S^{m-1}} = g|_{S^{m-1}}$, $f \simeq g$ (rel S^{m-1}) and that for some point $y_0 \in e^n$ we have $y_0 \notin g(D^n)$.

Idea of the proof. This is the most technical step in the proof of Theorem 16.2. Let $\varphi\colon D^n \to Y'$ be the characteristic map of the cell e^n. Let $B_{1/2} \subseteq D^n$ be the open ball with the center at the origin and radius 1/2, and let $U = \varphi(B_{1/2}) \subseteq Y'$. The map φ restricts to a homeomorphism $U \cong B_{1/2}$, so we can identify U with an open set in $\mathbb{R}^n$.

Since the disc D^m is homeomorphic to the cube $K = [0,1]^m$, we can consider f as a function $f\colon K \to Y'$. One can show that the cube K can be subdivided into a finite number m-dimensional polyhedra $K_1, \dots, K_N$ in such way that there exists a function $g\colon K \to Y'$ satisfying the following conditions:

(i) $g \simeq f$ (rel ∂K) (where ∂K is the boundary of the cube K)

(ii) For each polyhedron $K_i \subseteq K$ such that $g(K_i) \cap U \neq \varnothing$, the restriction $g|_{K_i}\colon K_i \to U$ is a linear function. We use here the identification of U with an open set in $\mathbb{R}^n$ to make sense of linearity of these maps.

Property (ii) implies that the set $g(K) \cap U$ is contained in the set $\bigcup_{K_{i_j}} g(K_{i_j})$ where the union is taken over all polyhedra K_{i_j} on which g is linear. Since the union of images of finitely many linear (or, more precisely, affine) functions $\mathbb{R}^m \to \mathbb{R}^n$ with $m < n$ does not contain any open set in $\mathbb{R}^n$, we obtain that $g(K)$ does not contain the whole set U, and so it does not contain the whole cell e^n.

□

16.5 Lemma. *Let Y be a space, and let Y' be obtained from Y by attaching a single n-cell:*

$$Y' = Y \cup e^n$$

Let $f\colon D^m \to Y'$ be a map such that $f(S^{m-1}) \subseteq Y$. If $m < n$ then there exists a map $g\colon D^m \to Y'$ such that $g(D^m) \subseteq Y$, $f|_{S^{m-1}} = g|_{S^{m-1}}$ and $f \simeq g$ (rel S^{m-1}).

Proof. By Lemma 16.4 there exists a function $g'\colon D^m \to Y'$ such that $f \simeq g'$ (rel S^{m-1}) and such that $y_0 \notin g'(D^m)$ for some $y_0 \in e^n$. We can consider g' as a map $g'\colon D^m \to Y' \smallsetminus \{y_0\}$. One can show (exercise) that there exists a map $h\colon (Y' \smallsetminus \{y_0\}) \times [0, 1] \to Y' \smallsetminus \{y_0\}$ which is a deformation retraction of $Y' \smallsetminus \{y_0\}$ onto Y. The function $h_1 g$ is homotopic to g' (rel S^{m-1}) and $h_1 g'(D^m) \subseteq Y$. Thus we can take $g = h_1 g'$. □

16.6 Lemma. *Let Y be a space, and let Y' be obtained from Y by attaching n-cells:*

$$Y' = Y \cup \bigcup_{i \in I} e^n_i$$

Let $f\colon D^m \to Y'$ be a map such that $f(S^{m-1}) \subseteq Y$. If $m < n$ then there exists a map $g\colon D^m \to Y'$ such that $g(D^m) \subseteq Y$, $f|_{S^{m-1}} = g|_{S^{m-1}}$ and $f \simeq g$ (rel S^{m-1}).

Proof. Since D^m is a compact space, by Proposition 12.18 the set $f(D^m)$ has a non-empty intersection with only finitely many n-cells $e^n_{i_1}, \dots, e^n_{i_k}$. We will prove the lemma by induction with respect to the number k of these cells.

If $k = 0$ then $f(D^m) \subseteq Y$ and we can take $g = f$. Next, assume that the lemma is true for some $k \geq 0$, and let $f\colon D^m \to Y'$ be a functiom such that $f(D^m)$ has non-empty intersections with $k + 1$ cells $e^n_{i_1}, \dots, e^n_{i_{k+1}}$. Let Z be the subcomplex of Y' consisting of Y and these cells. We can consider f as a function $f\colon D^m \to Z$. Notice that Z can be viewed as space obtained by attaching a single cell e^n_{k+1} to $Z' = Y \cup \bigcup_{i=1}^{k} e^n_i$. Therefore, by Lemma 16.5 the function f is homotopic relative S^{m-1} to a function $f'\colon D^m \to Z$ such that $f'(D^m) \subseteq Z'$. Since $f'(D^m)$ intersects non-trivially with at most k cells of dimension n, by the inductive assumption it is homotopic (rel S^{m-1}) to a function g such that $g(D^m) \subseteq Y$. Therefore we obtain $f \simeq f' \simeq g$ (rel S^{m-1}). □

16.7 Lemma. *Let Y be a CW complex, and $f\colon D^m \to Y$ be a map such that $f(S^{m-1}) \subseteq Y^{(m-1)}$. Then there exists a map $g\colon D^m \to Y$ such $g(D^m) \subseteq Y^{(m)}$, $f|_{S^{m-1}} = g|_{S^{m-1}}$ and $f \simeq g$* (rel S^{m-1}).

Proof. Since D^m is a compact space, by Proposition 12.18 the set $f(D^m)$ has a non-empty intersection with finitely many cells of Y only. In particular, $f(D^m)$ is contained in an n-skeleton $Y^{(n)}$ of Y for some $n \geq 0$. If $n > m$, then using Lemma 16.6 we get that f is homotopic (rel S^{m-1}) to a function f' such that $f'(D^m) \subseteq Y^{(n-1)}$. Arguing inductively, we obtain the statement of the lemma. □

16.8 Lemma. *Let X, Y be CW complexes and $A \subseteq X$ be a subcomplex. Also, let $f\colon X \to Y$ be a map which is cellular on $A \cup X^{(m)}$ for some $m \geq -1$. Then there exists a map $g\colon X \to Y$ such that g is cellular on $A \cup X^{(m+1)}$, $f|_{A \cup X^{(m)}} = g|_{A \cup X^{(m)}}$ and $f \simeq g$* (rel $A \cup X^{(m)}$).

Proof. Assume first that $m = -1$. Since $X^{(-1)} = \varnothing$, thus f is a map cellular on A. We want to show that there exists a function $g\colon X \to Y$ such that $f \simeq g$ (rel A) and that g is cellular on $A \cup X^0$. The complex $A \cup X^0$ is a disjoint union

$$A \cup X^0 = A \sqcup \{e_i^0\}_{\in I}$$

where e_i^0 are 0-cells of X not contained in A. Since every path connected component of Y contains some 0-cell, for each $i \in I$ we can find a path $\omega_i\colon [0,1] \to Y$ such that $\omega_i(0) = f(e_i^0)$ and $\omega_i(1) \in Y^{(0)}$. Define a homotopy $h\colon (A \cup X^0) \times [0,1] \to Y$ by $h(x,t) = f(x)$ for $x \in A$ and $h(e_i^0, t) = \omega_i(t)$. By Theorem 13.7 this homotopy can be extended to a homotopy $\bar{h}\colon X \times [0,1] \to Y$ between f and a certain function $g\colon X \to Y$. Directly from this construction it follows that $f \simeq g$ (rel A) and that g is cellular on $A \cup X^{(0)}$.

Next, assume that $m \geq 0$. Then f is a function cellular on $A \cup X^{(m)}$, and we want to obtain a function g cellular on $A \cup X^{(m+1)}$. We have

$$A \cup X^{(m+1)} = (A \cup X^{(m)}) \cup \bigcup_{i \in I} e_i^{m+1}$$

where e_i^{m+1} are $(m+1)$-cells of X not contained in A. Let $\varphi_i\colon D^{m+1} \to X$ be the characteristic map of the cell e_i^{m+1} (12.2). Since $\varphi_i(S^m) \subseteq X^{(m)}$ and f is cellular on $X^{(m)}$ we obtain that $f\varphi_i(S^m) \subseteq Y^{(m)}$. Therefore, by Lemma 16.7 there exists a homotopy $h_i\colon D^{m+1} \times [0,1] \to Y$ (rel S^m) between $f\varphi_i$ and some map $\psi_i\colon D^{m+1} \to Y$ such that $\psi_i(D^{m+1}) \subseteq Y^{(m+1)}$. Define a homotopy $h\colon (A \cup X^{(m+1)}) \times [0,1] \to Y$ by $h(x,t) = f(x)$ for $x \in A \cup X^{(m)}$ and $h(x,t) = h_i(y,t)$ for $x = \varphi_i(y) \in e_i^{m+1}$. Using Theorem 13.7 again, we can extend this homotopy to a homotopy $\bar{h}\colon X \times [0,1] \to Y$ between f and some function $g\colon X \to Y$. The construction of the homotopy $\bar{h}$ implies that g is cellular on $A \cup X^{(m+1)}$ and that $f \simeq g$ (rel $A \cup X^{(m)}$). □

Proof of Theorem 16.2. Let X, Y be CW complexes, let $A \subseteq X$ be a subcomplex, and let $f\colon X \to Y$ be a map which is cellular on A. Using Lemma 16.8 inductively we can construct functions $f_i\colon X \to Y$ and homotopies $h_i\colon X \times [0,1] \to Y$ for $m = 0, 1, 2, \dots$ such that:

- the function f_m is cellular on $A \cup X^{(m)}$
- h_0 is a homotopy (rel A) between f and f_0
- h_m is a homotopy (rel $A \cup X^{(m-1)}$) between f_{m-1} and f_m for $m = 1, 2, \dots$

Notice that if $\dim X = n < \infty$ then $X = X^{(n)}$, and so f_n is a cellular map such that $f \simeq f_n$ (rel A). Thus we can take $g = f_n$.

If $\dim X = \infty$ define $g\colon X \to Y$ by $g(x) = f_m(x)$ if $x \in X^{(m)}$. Notice that since $f_n|_{X^{(m)}} = f_m|_{X^{(m)}}$ for all $n > m$ this function is well defined, and it is continuous by (12.8) since for each m the function $g|_{X^{(m)}} = f_m|_{X^{(m)}}$ is continuous. In addition, g is a cellular function since for each m we have $g(X^{(m)}) = f_m(X^{(m)}) \subseteq Y^{(m)}$, and it satisfies $f|_A = g|_A$ since $f|_A = f_m|_A$ for all m.

To obtain a homotopy $h\colon X \times [0,1] \to Y$ between f and g, choose numbers $t_m \in [0,1]$ for $m = 0, 1, \dots$ such that $t_0 = 0$, $t_m < t_{m+1}$ for all m, and that the sequence t_m converges to 1. On the subinterval

$[t_m, t_{m+1}]$ define h by reparametrizing the homotopy h_m:

$$h(x, t) = h_m(x, (t - t_m)/(t_{m+1} - t_m))$$

for $t \in [t_m, t_{m+1}]$. Also, set $h(x, 1) = g(x)$ for $x \in X$. To verify that h is continuous, it suffices to show that it is continuous on $X^{(m)} \times [0, 1]$ for each m. This holds since $h(x, t) = f_m(x)$ for $(x, t) \in X^{(m)} \times [t_{m+1}, 1]$, and $h|_{X \times [0, t_{m+1}]}$ is continuous as a concatenation of a finite number of homotopies $h_0, \dots, h_m$.

□

Exercises to Chapter 16

E16.1 Exercise. Recall that the n-th homotopy group of a pointed space (X, x_0) is a group whose elements are homotopy classes of basepoint preserving maps $(S^n, s_0) \to (X, x_0)$. Let S^m be an m-dimensional sphere with a basepoint $s_0' \in S^m$. Show that if $n < m$ then the group $\pi_n(S^m, s_0')$ is trivial.

E16.2 Exercise. Recall that the n-dimensional sphere is given by

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

For $0 \leq m < n$ consider the embedding $i \colon S^m \to S^n$ given by

$$i((x_1, \dots, x_{m+1})) = (x_1, \dots, x_{m+1}, 0, \dots 0)$$

Using this embedding we can consider S^m as a subspace of S^n. Show that the quotient space S^n/S^m is homotopy equivalent to $S^n \vee S^{m+1}$. (Hint: Proposition 13.4 may be useful.)

E16.3 Exercise. Let (Y, y_0) be a pointed space. Show that there exists a pointed 2-dimensional CW complex (X, x_0) and a function $f \colon (X, x_0) \to (Y, y_0)$ such that the induced homomorphism of fundamental groups $f_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

E16.4 Exercise. The goal of this exercise is to complete a missing step in the proof of Lemma 16.5. Let $Y' = Y \cup e^n$ be a space obtained by attaching a single n-dimensional cell to a space Y. Show that for any point $y_0 \in e^n$ the space Y is a deformation retract of $Y' \smallsetminus \{y_0\}$