9 | The Product Formula

Recall that if G, H are groups then their direct product is the group $G \times H$ whose elements are pairs (g, h) where $g \in G, h \in H$ and multiplication is given by $(g, h) \cdot (g', h') = (gg', hh')$. We have:

9.1 Theorem. *If* (X_1, x_1)*,* (X_2, x_2) *are pointed spaces then*

$$\pi_1(X_1 \times X_2, (x_1, x_2)) \cong \pi_1(X_1, x_1) \times \pi_1(X_2, x_2)$$

Proof. For $i = 1, 2$ let $p_i \colon X_1 \times X_2 \to X_i$ be the projection map $p_i(x_1, x_2) = x_i$. These maps induce homomorphisms $p_{i*} \colon \pi_1(X_1 \times X_2, (x_1, x_2)) \to \pi_1(X_i, x_i)$. This defines a homomorphism

$$p_{1*} \times p_{2*} \colon \pi_1(X_1 \times X_2, (x_1, x_2)) \to \pi_1(X_1, x_1) \times \pi_1(X_2, x_2)$$

where $p_{1*} \times p_{2*}([\omega]) = (p_{1*}([\omega]), p_{2*}([\omega]))$. Next, for $i = 1, 2$ let ω_i be a loop in (X_i, x_i), and let $\omega_1 \times \omega_2$ be the loop in $X_1 \times X_2$ given by $\omega_1 \times \omega_2(s) = (\omega_1(s), \omega_2(s))$. One can check that the map

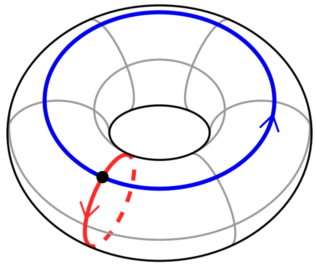
$$g \colon \pi_1(X_1, x_1) \times \pi_1(X_2, x_2) \to \pi_1(X_1 \times X_2, (x_1, x_2))$$

defined by $g([\omega_1], [\omega_2]) = [\omega_1 \times \omega_2]$ is a homomorphism of groups, and that $g = (p_{1*} \times p_{2*})^{-1}$. $\square$

9.2 Example. Let $T^2 = S^1 \times S^1$ be the torus. Then

$$\pi_1(T^2) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$$

Notice that under this isomorphism the elements $(1, 0)$ and $(0, 1) \in \mathbb{Z} \times \mathbb{Z}$ correspond to loops in T^2 that traverse the longitudinal and the meridional circles of T^2:



Theorem 9.1 can be generalized to products of arbitrary (finite or infinite) families of spaces. Recall that if $\{G_i\}_{i\in I}$ is a family of groups then the direct product $\prod_{i\in I} G_i$ is the group whose set of elements is the Cartesian product of G_i's with multiplication defined componentwise: $(g_i)_{i\in I}\cdot(h_i)_{i\in I} = (g_ih_i)_{i\in I}$

9.3 Theorem. *If $(X_i, x_i)_{i\in I}$ is a family of pointed spaces then*

$$\pi_1\left(\prod_{i\in I} X_i,\ (x_i)_{i\in I}\right) \cong \prod_{i\in I}\pi_1(X_i, x_i)$$

The proof is the same as in the case of two spaces.

Intuitively, one could rephrase Theorem 9.3 by saying that the fundamental group takes products (of topological spaces) to products (of groups). It is possible to make this statement more precise by using the language of categories and functors. This starts with the following definition:

9.4 Definition. Let $\mathbf{C}$ be a category and let $\{c_i\}_{i\in I}$ be a family of objects in $\mathbf{C}$. The *categorical product* of this family is an object $d \in \mathbf{C}$ equipped with a morphism $p_i\colon d \to c_i$ for each $i \in I$ and such that for any object a and morphisms $f_i\colon a \to c_i$ there exists a unique morphism $g\colon a \to d$ satisfying $p_ig = f_i$ for all $i \in I$:

$$\begin{tikzcd}
a \arrow[dr, dashed, "g"] \arrow[drr, bend left, "f_i"] \arrow[ddr, bend right, "f_j"'] & & \\
& d \arrow[r, "p_i"] \arrow[d, "p_j"'] & c_i \\
& c_j &
\end{tikzcd}$$

9.5 Example. If $\{X_i\}_{i\in I}$ is a family of topological spaces then the product space $\prod_{i\in I} X_i$ together with the projection maps $p_j\colon \prod_{i\in I} X_i \to X_j$ given by $p_j((x_i)_{i\in I}) = x_j$ is the categorical product of the family $\{X_i\}_{i\in I}$ in the category **Top** of topological spaces. Indeed, for any maps $f_i\colon Z \to X_i$ we have a map $g\colon Z \to \prod_{i\in I} X_i$ defined by $g(z) = (f_i(x))_{i\in I}$. Moreover g is the unique map that satisfies $p_ig = f_i$ for all $i \in I$.

9.6 Example. Similarly as above one can check that If $\{G_i\}_{i\in I}$ is a family of groups then the direct product $\prod_{i\in I} G_i$ taken together with the homomorphisms $p_j\colon \prod_{i\in I} G_i \to G_j$ given by $p_j((g_i)_{i\in I}) = g_j$ is the categorical product of the family $\{G_i\}_{i\in I}$ in the category **Gr** of groups.

9.7 Note. In general, in an arbitrary category **C** categorical products may not exists. However, if a product of a family $\{c_i\}_{i\in I}$ exists then it is unique up to an isomorphism (exercise).

9.8 Definition. Let $F\colon \mathbf{C} \to \mathbf{C}'$ be a functor. Assume that F has the property that if an object d with morphisms $p_i\colon d \to c_i$ is the categorical product of a family $\{c_i\}_{i\in I}$ in **C** then the object $F(d)$ with morphisms $F(p_i)\colon F(d) \to F(c_i)$ is the categorical product of the family $\{F(c_i)\}_{i\in I}$ in $\mathbf{C}'$. In such situation we say the the functor F *preserves products*.

9.9 Example. Let $F\colon \mathbf{Top} \to \mathbf{Gr}$ be the functor such that $F(X) = \mathbb{Z}$ for each space X and that for every map $f\colon X \to Y$ the homomorphism $F(f)\colon F(X) \to F(Y)$ is the identity function $\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$. The functor F does not preserve products since for spaces $X, Y \in \mathbf{Top}$ we have

$$F(X \times Y) = \mathbb{Z} \ncong \mathbb{Z} \times \mathbb{Z} = F(X) \times F(Y)$$

In the category $\mathbf{Top}_*$ of pointed topological spaces the categorical products of a family $\{(X_i, x_i)\}_{i\in I}$ is given by the pointed space $(\prod_{i\in I} X_i, (x_i)_{\in I})$. In view of this Theorem 9.3 can be restated as follows:

9.10 Theorem. *The fundamental group functor* $\pi_1\colon \mathbf{Top}_* \to \mathbf{Gr}$ *preserves products.*

Exercises to Chapter 9

E9.1 Exercise. The notion of a categorical coproduct is dual to the that of a product. Let **C** be a category and $\{c_i\}_{i\in I}$ be a family of objects in **C**. A *coproduct* of this family in the category **C** is an object $d \in \mathbf{C}$ equipped with a morphism $h_i\colon c_i \to d$ for each $i \in I$ such that for any object a and morphisms $f_i\colon c_i \to a$ there exists a unique morphism $g\colon d \to a$ satisfying $gh_i = f_i$ for all $i \in I$:

$$\begin{tikzcd}
 & c_i \arrow[d, "h_i"] \arrow[ddr, bend left, "f_i"] & \\
c_j \arrow[r, "h_j"'] \arrow[drr, bend right, "f_j"'] & d \arrow[dr, dashed, "g"] & \\
 & & a
\end{tikzcd}$$

a) Let **Ab** denote the category of abelian groups (with abelian groups as objects and homomorphisms of abelian groups as morphisms). Show that if $\{G_i\}_{i \in I}$ is a family of abelian groups, then the direct sum $\bigoplus_{i \in I} G_i$ is the coproduct of this family in the category **Ab**.

b) Show that in the category **Gr** of all groups the direct sum $\mathbb{Z} \oplus \mathbb{Z}$ is not the coproduct of the family $\{\mathbb{Z}, \mathbb{Z}\}$.