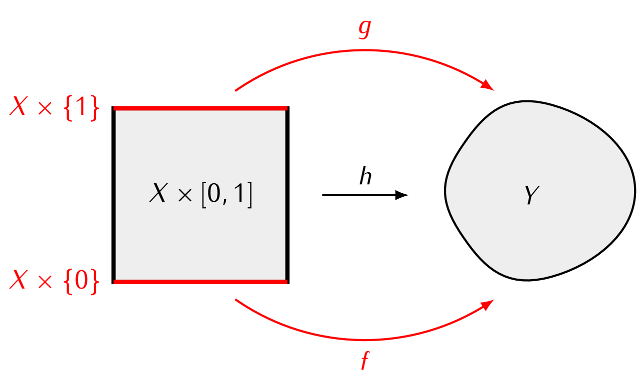
8 | Homotopy Invariance

So far we computed the fundamental group for very few spaces. In order to extend these computations to other spaces we will use three basic tools: homotopy invariance of π_1, the product formula for π_1, and the van Kampen theorem. In this chapter we discuss the first of these topics and in the subsequent ones we deal with the other two.

8.1 Definition. Let $f, g\colon X \to Y$ be continuous functions. A *homotopy* between f and g is a continuous function $h\colon X \times [0,1] \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$:



If such homotopy exists then we say that the functions f and g are *homotopic* and we write $f \simeq g$. We will also write $h\colon f \simeq g$ to indicate that h is a homotopy between f and g.

8.2 Note. Given a homotopy $h\colon X \times [0,1] \to Y$ it will be convenient denote by $h_t\colon X \to Y$ the function defined by $h_t(x) = h(x, t)$. If $h\colon f \simeq g$ then $h_0 = f$ and $h_1 = g$.

8.3 Example. Any two functions $f, g\colon X \to \mathbb{R}^n$ are homotopic. Indeed, define $h\colon X \times [0,1] \to \mathbb{R}$ by $h(x,t) = (1-t)f(x) + tg(x)$. Then $h_0 = f$ and $h_1 = g$.

A useful generalization of Definition 8.1 is the notion of a relative homotopy:

8.4 Definition. Let X be a space and let $A \subseteq X$. If $f, g\colon X \to Y$ are functions such that $f|_A = g|_A$ then we say that f and g are *homotopic relative to* A if there exists a homotopy $h\colon X \times [0,1] \to Y$ such that $h_0 = f$, $h_1 = g$ and $h_t|_A = f|_A = g|_A$ for all $t \in [0,1]$. In such case we write $f \simeq g$ (rel A).

8.5 Example. Let $\omega, \tau\colon [0,1] \to X$ be paths in X. Recall that path homotopy is defined only if $\omega|_{\{0,1\}} = \tau|_{\{0,1\}}$ and it is given by a map $h\colon [0,1] \times [0,1] \to X$ such that $h_0 = \omega$, $h_1 = \tau$ and $h_t|_{\{0,1\}} = \omega|_{\{0,1\}} = \tau|_{\{0,1\}}$ for each $t \in [0,1]$. Thus, in the paths ω and τ are path homotopic if and only if $\omega \simeq \tau$ (rel $\{0,1\}$).

8.6 Definition. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. If such maps exist we say that the spaces X and Y are *homotopy equivalent* and we write $X \simeq Y$.

8.7 Note. If f and g are maps as in Definition 8.6 then we say that g is a *homotopy inverse* of f.

8.8 Example. We will show $\mathbb{R}^n$ is homotopy equivalent to the space $\{*\}$ consisting of a single point. Let $f\colon \mathbb{R}^n \to \{*\}$ be the constant function and let $g\colon \{*\} \to \mathbb{R}^n$ be given by $f(*) = x_0$ for some $x_0 \in \mathbb{R}^n$. We have $fg = \mathrm{id}_{\{*\}}$ so $fg \simeq \mathrm{id}_{\{*\}}$. On the other hand by Example 8.3 any two functions into $\mathbb{R}^n$ are homotopic, so in particular $gf \simeq \mathrm{id}_{\mathbb{R}^n}$

8.9 Note. Example 8.8 shows that a homotopy inverse of a homotopy equivalence $f\colon X \to Y$ in general is not unique: any function $g\colon \{*\} \to \mathbb{R}^n$ is a homotopy inverse of the constant function $f\colon \mathbb{R}^n \to \{*\}$.

8.10 Definition. If X is a space such that $X \simeq \{*\}$ then we say that X is a *contractible space*.

8.11 Proposition. *Let X be a topological space. The following conditions are equivalent:*

1) *X is contractible;*
2) *the identify map id_X is homotopic to a constant map;*
3) *for each space Y and any maps $f, g\colon Y \to X$ we have $f \simeq g$.*

Proof. Exercise. □

Many examples of homotopy equivalences can be obtained using deformation retractions:

8.12 Definition. A subspace $A \subseteq X$ is a *deformation retract* of a space X if there exists a homotopy $h\colon X \times [0,1] \to X$ such that

1) $h_0 = \mathrm{id}_X$
2) $h_t|_A = \mathrm{id}_A$ for all $t \in [0,1]$

3) $h_1(x) \in A$ for all $x \in X$

In such case we say that h is a *deformation retraction* of X onto A.

8.13 Proposition. *If $A \subseteq X$ is a deformation retract of X then $A \simeq X$.*

Proof. Let $h\colon X \times [0,1] \to X$ be a deformation retraction, let $r\colon X \to A$ be given by $r(x) = h_1(x)$ and let $j\colon A \to X$ be the inclusion map. We have $rj = \mathrm{id}_A$. Also, h is a homotopy between id_X and jr. $\square$

8.14 Example. For any $n > 0$ the sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \smallsetminus \{0\}$. Indeed, a deformation retraction $h\colon \mathbb{R}^n \smallsetminus \{0\} \to S^{n-1}$ is given by

$$h(x,t) = \frac{x}{(1-t) + t\|x\|}$$

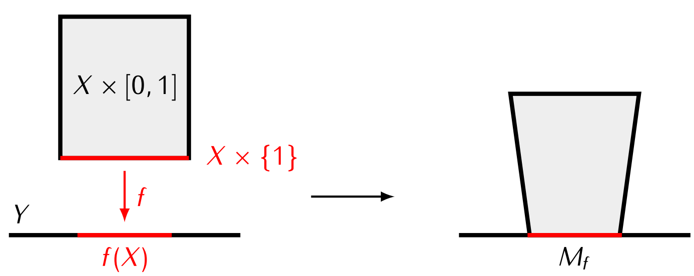
As a consequence $\mathbb{R}^n \smallsetminus \{0\} \simeq S^{n-1}$.

Interesting examples of homotopy equivalences can be also obtained using the constructions of a mapping cylinder and a mapping cone:

8.15 Definition. Let $f\colon X \to Y$ be a continuous function. The *mapping cylinder* of f is the space

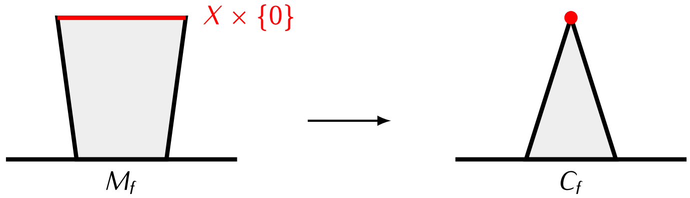
$$M_f = (X \times [0,1] \sqcup Y)/{\sim}$$

where $\sim$ is the equivalence relation given by $(x,1) \sim f(x)$ for all $x \in X$.



The *mapping cone* of f is the space obtained from M_f by collapsing the subspace $X \times \{0\} \subseteq M_f$ to a point:

$$C_f = M_f/(X \times \{0\})$$



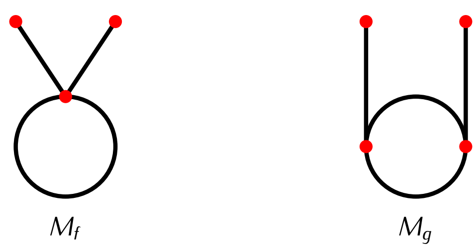
8.16 Proposition. *For any map $f: X \to Y$ we have $M_f \simeq Y$.*

Proof. Exercise. ☐

8.17 Proposition. *Let $f, g: X \to Y$ be continuous functions. If $f \simeq g$ then $C_f \simeq C_g$.*

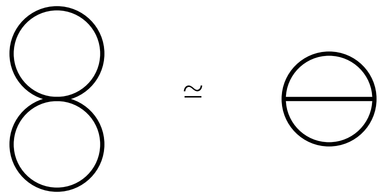
Proof. Exercise. ☐

8.18 Example. Consider maps $f, g: \{-1, 1\} \to S^1$ where f is a constant map and g is non-constant (e.g. g maps 1 and -1 to antipodal points of S^1). Mapping cylinders of the these functions can be depicted as follows:



The mapping cones, in turn, look as follows:

Notice that $f \simeq g$, and so $C_f \simeq C_g$. Notice also that the space C_f is homeomorphic to $S^1 \vee S^1$ while C_g is homeomorphic to the space obtained as a union of S^1 and one of its diagonals. In effect we obtain a homotopy equivalence:



Our next goal is to examine how the fundamental group behaves with respect to homotopic maps and homotopy equivalent spaces. First, recall that a map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ induces a homomorphism of fundamental groups $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ which is given by $f_*([\omega]) = [f \circ \omega]$. We have:

8.19 Proposition. *If $f, g\colon (X, x_0) \to (Y, y_0)$ are maps of pointed spaces such that $f \simeq g$ $(\mathrm{rel}\ \{x_0\})$ then $f_* = g_*$.*

Proof. For $[\omega] \in \pi_1(X, x_0)$ we want to show that $f_*([\omega]) = g_*([\omega])$, or equivalently that

$$f \circ \omega \simeq g \circ \omega \quad (\mathrm{rel}\ \{0, 1\})$$

Let $h\colon X \times [0, 1] \to Y$ be a homotopy between f and g $(\mathrm{rel}\ \{x_0\})$. Then the map

$$h \circ (\omega \times \mathrm{id}_{[0,1]})\colon [0, 1] \times [0, 1] \to Y$$

gives a path homotopy between $f \circ \omega$ and $g \circ \omega$. □

Proposition 8.19 can be generalized to the setting where we do not assume that homotopy preserves basepoints. Recall (4.2) that if Y is a space, $y_0, y_1 \in Y$ then a path τ in Y with $\tau(0) = y_0$ and $\tau(1) = y_1$ induces an isomorphism $s_\tau\colon \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ given by $s_\tau([\omega]) = [\bar{\tau} * \omega * \tau]$.

8.20 Proposition. *Let $f, g\colon X \to Y$ be homotopic maps and let $h\colon f \simeq g$. For $x_0 \in X$ let τ be the path in Y given by $\tau(t) = h(x_0, t)$. The following diagram commutes:*

$$\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[dd, "s_\tau", "\cong"'] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(Y, g(x_0))
\end{tikzcd}$$

Proof. Exercise. □

8.21 Corollary. *If $f, g\colon X \to Y$ are maps such that $f \simeq g$ then the homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism (or is trivial or is 1-1 or onto) if and only if the homomorphism $g_*\colon \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$ has the same property.*

8.22 Proposition. *If $f\colon X \to Y$ is a homotopy equivalence then for any $x_0 \in X$ the homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. Consider the sequence of homomorphisms

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, gf(x_0)) \xrightarrow{f_*} \pi_1(Y, fgf(x_0))$$

Composition of the first two homomorphisms satisfies $g_*f_* = (gf)_*$. Since $gf \simeq \mathrm{id}_X$ and id_{X*} is an isomorphism by Proposition 8.21 we obtain that g_*f_* is an isomorphism. This implies in particular that g_* is onto. Similarly, composing the last two homomorphisms we obtain $f_*g_* = (fg)_*$ and since $fg \simeq \mathrm{id}_Y$ we get that f_*g_* is an isomorphism. This means that g_* is 1-1. As a consequence g_* is an isomorphism. It follows that the first homomorphism f_* is a composition of two isomorphisms: $f_* = g_*^{-1}(g_*f_*)$, and so f_* is an isomorphism.

□

8.23 Corollary. *If X, Y are path connected spaces and $X \simeq Y$ then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ for any $x_0 \in X$, $y_0 \in Y$.*

Proof. Let $f\colon X \to Y$ be a homotopy equivalence. By Proposition 8.22 we get an isomorphism $f_*\colon \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, f(x_0))$. Since Y is path connected by Corrolary 4.3 we also have $\pi_1(Y, f(x_0)) \cong \pi_1(Y, y_0)$. □

8.24 Note. In the proof above we used only that Y is path connected, so the assumption in Corollary 8.23 that both X and Y are path connected may seem too strong. However, if Y is path connected and $X \simeq Y$ then X must be path connected as well (exercise).

8.25 Example. As we have seen before (8.14) the space $\mathbb{R}^n \smallsetminus \{0\}$ is homotopy equivalent to the sphere S^{n-1}. This gives $\pi_1(\mathbb{R}^n \smallsetminus \{0\}) \cong \pi_1(S^{n-1})$. In particular $\pi_1(\mathbb{R}^2 \smallsetminus \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$.

8.26 Example. Let Θ be the space obtained as a union of S^1 and one of its diagonals. By Example 8.18 this space is homotopy equivalent to $S^1 \vee S^1$, so $\pi_1(\Theta) \cong \pi_1(S^1 \vee S^1)$.

Exercises to Chapter 8

E8.1 Exercise. Recall that if X is a space then $\pi_0(X)$ denotes the set of path connected components of X. If $x \in X$ then by $[x] \in \pi_0(X)$ we will denote the path connected component of the point x. Recall that a continuous function $f\colon X \to Y$ induces a map of sets $f_*\colon \pi_0(X) \to \pi_0(Y)$ given by $\pi_0([x]) = [f(x)]$. Show that if f is a homotopy equivalence then f_* is a bijection.

E8.2 Exercise. Prove Proposion 8.11.

E8.3 Exercise. Let $f, g\colon X \to Y$ be two homeomorphisms and let $f^{-1}, g^{-1}\colon Y \to X$ be their respective inverses. Show that if $f \simeq g$ then $f^{-1} \simeq g^{-1}$.

E8.4 Exercise. a) For $i = 1, 2$ let X_i be a topological space and let $Y_i \subseteq X_i$. Assume that we have a commutative diagram:

$$\begin{tikzcd} Y_1 \arrow[r, "j_1"] \arrow[d, "f'"'] & X_1 \arrow[r, "r_1"] \arrow[d, "f"] & Y_1 \arrow[d, "f'"] \\ Y_2 \arrow[r, "j_2"'] & X_2 \arrow[r, "r_2"'] & Y_2 \end{tikzcd}$$

where $j_i\colon Y_i \to X_i$ is the inclusion map, and $r_i\colon X_i \to Y_i$ is a retraction. Show that if f is a homotopy equivalence then f' is a homotopy equivalence as well.

b) Let X be a contractible space and let $A \subseteq X$ be a retract of X. Show that A is contractible.

E8.5 Exercise. Let X, Y be topological spaces. Show that any map $f\colon X \times \mathbb{R}^k \to Y \times \mathbb{R}^k$ is homotopic to a map $g \times \mathrm{id}_{\mathbb{R}^k}\colon X \times \mathbb{R}^k \to Y \times \mathbb{R}^k$ for some $g\colon X \to Y$.

E8.6 Exercise. For spaces X and Y let $[X, Y]$ denote the set of homotopy classes of maps $X \to Y$. That is, each map $f\colon X \to Y$ defines an element $[f] \in [X, Y]$ and $[f] = [f']$ if $f \simeq f'$. Notice that any map $g\colon X \to X'$ defines a function $g^*\colon [X', Y] \to [X, Y]$ given by $g^*([f]) = [fg]$.

Given a map $g\colon X \to X'$ show that the following conditions are equivalent:

1) The map g is a homotopy equivalence.
2) For each space Z the function $g^*\colon [X', Z] \to [X, Z]$ is a bijection.

E8.7 Exercise. The antipodal map $f\colon S^n \to S^n$ is the map given by $f(x) = -x$. Show that if $g\colon S^n \to S^n$ is any map such that $g(x) \neq x$ for all $x \in S^n$ then $g \simeq f$.

E8.8 Exercise. Let X be a topological space. Assume that $f, g\colon X \to S^n$ are maps such that for some non-empty open set $U \subseteq S^n$ we have $f^{-1}(U) = g^{-1}(U) = V \subseteq X$ and $f|_V = g|_V$. Show that $f \simeq g$.

E8.9 Exercise. Prove Proposition 8.17.

E8.10 Exercise. Prove Proposition 8.20.

E8.11 Exercise. Let M be the Möbius band and let ∂M denote the boundary of M. Show that ∂M is not a retract of M.

E8.12 Exercise. Recall (8.15) that the cone of a map $f\colon X \to Y$ is the space

$$C_f = (X \times [0, 1] \sqcup Y)/\!\sim$$

where $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. We can consider Y as a subspace of C_f. Show that Y is contractible if any only if for every map $f\colon X \to Y$ the space Y is a retract of C_f.

E8.13 Exercise. a) Let $f\colon S^1 \to X$ be a continuous function. Show that f is homotopic to a constant map if and only if there exists $\bar{f}\colon D^2 \to X$ such that $\bar{f}|_{S^1} = f$.

b) Show that if $f\colon S^1 \to S^1$ is homotopic to a constant map then there exists $x_0 \in S^1$ such that $f(x_0) = x_0$.

E8.14 Exercise. Let $F\colon D^2 \to D^2$ be a function such that $F(S^1) \subseteq S^1$, and let $f\colon S^1 \to S^1$ be given by $f(x) = F(x)$ for all $x \in S^1$. Show that if f is not homotopic to a constant map, then for each function $G\colon D^2 \to D^2$ there is a point $x_0 \in D^2$ such that $F(x_0) = G(x_0)$.

E8.15 Exercise. Recall that for $n \geq 1$ multiplication in the group $\pi_n(X, x_0)$ can be defined using the pinch map $p\colon S^n \to S^n \vee S^n$: if $[\omega], [\tau] \in \pi_n(X, x_0)$ then $[\omega]\cdot[\sigma] = [(\omega \vee \sigma)\circ p]$. The goal of this exercise is to generalize this observation.

For pointed spaces (X, x_0) and (Y, y_0) let $[X, Y]_*$ denote the set of pointed homotopy classes of maps $X \to Y$. That is, each pointed map $f\colon (X, x_0) \to (Y, y_0)$ defines an element $[f] \in [X, Y]_*$ and $[f] = [g]$ if $f \simeq g$ relative the basepoint. Let (X, x_0) be a space such that

(i) for each space (Y, y_0) the set $[X, Y]_*$ has the structure of a group;
(ii) for each pointed map $f\colon (Y, y_0) \to (Y', y_0')$ the induced function $f_*\colon [X, Y]_* \to [X, Y']_*$ is a group homomorphism.

a) Show that for any space space (Y, y_0) there exists a bijection of sets $\varphi_Y\colon [X\vee X, Y]_* \to [X, Y]_*\times[X, Y]_*$ such that for any pointed map $f\colon (Y, y_0) \to (Y', y_0')$ the following diagram commutes:

$$\begin{tikzcd}
{[X \vee X, Y]_*} \arrow[r, "\varphi_Y", "\cong"'] \arrow[d, "f_*"'] & {[X, Y]_* \times [X, Y]_*} \arrow[d, "f_* \times f_*"] \\
{[X \vee X, Y']_*} \arrow[r, "\cong", "\varphi_{Y'}"'] & {[X, Y']_* \times [X, Y']_*}
\end{tikzcd}$$

b) Show that there exists a map $p\colon X \to X \vee X$ such that for each space (Y, y_0) the multiplication in the group $[X, Y]_*$ is given by $[f]\cdot[g] = [(f \vee g)\circ p]$.

Hint: For a space (Y, y_0) let μ_Y denote the multiplication in the group $[X, Y]_*$:

$$\mu_Y\colon [X, Y]_* \times [X, Y]_* \to [X, Y]_*$$

Notice that the condition (ii) above is equivalent to saying that for any map $f\colon (Y, y_0) \to (Y', y_0')$ the following diagram commutes:

$$\begin{tikzcd}
{[X, Y]_* \times [X, Y]_*} \arrow[r, "\mu_Y"] \arrow[d, "f_* \times f_*"'] & {[X, Y]_*} \arrow[d, "f_*"] \\
{[X, Y']_* \times [X, Y']_*} \arrow[r, "\mu_{Y'}"'] & {[X, Y']_*}
\end{tikzcd}$$