17 | Covering Spaces

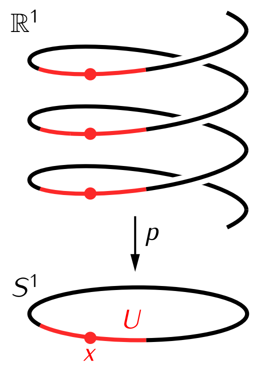
17.1 Definition. A map $p\colon T \to X$ is a *covering* of X if for every point $x \in X$ there exists an open neighborhood $U_x \subseteq X$ and a homeomorphism $h_{U_x}\colon p^{-1}(U_x) \to U_x \times D_x$ where D_x is some discrete space, such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U_x) \arrow[rr, "h_{U_x}"] \arrow[dr, "p"'] & & U_x \times D_x \arrow[dl, "\mathrm{pr}_1"] \\ & U_x & \end{tikzcd}$$

Here $\mathrm{pr}_1\colon U_x \times D_x \to U_x$ is the projection map $\mathrm{pr}_1(y, d) = y$.

17.3 Example. Let D be a discrete space. The projection map $\mathrm{pr}_1\colon X \times D \to X$ is a covering of X. In this case the whole space X is evenly covered. We say that $\mathrm{pr}_1\colon X \times D \to X$ is a *trivial covering* of X.

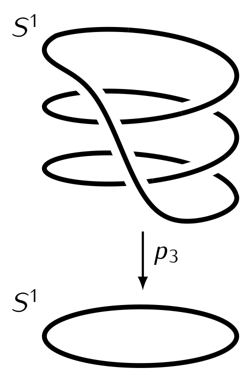
17.4 Example. Recall that the universal covering of S^1 is the map $p\colon \mathbb{R}^1 \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$. If $U \subseteq S^1$ is any open set such that $U \neq S^1$, then $p^{-1}(U)$ is evenly covered and $p^{-1}(U) \cong U \times \mathbb{Z}$ (exercise).



17.5 Example. Consider S^1 as a subset of the complex plane:

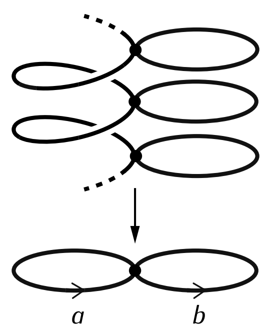
$$S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$$

For $n = 1, 2, \dots$ the map $p_n\colon S^1 \to S^1$ given by $p_n(z) = z^n$ is an n-fold covering of S^1. Similarly as in the case of the universal covering of S^1 any open set $U \subseteq S^1$ such that $U \neq S^1$ is evenly covered by p_n (exercise).

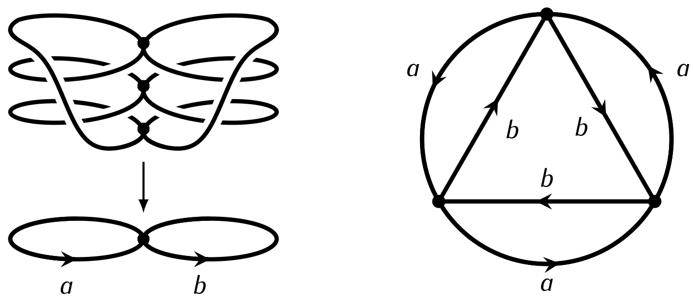


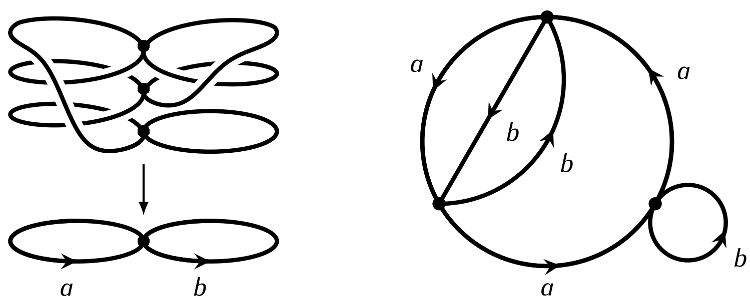
17.6 Example. If $p_1\colon T_1 \to X_1$ and $p_2\colon T_2 \to X_2$ are coverings then the map $p_1 \times p_2\colon T_1 \times T_2 \to X_1 \times X_2$ is also a covering (exercise). For example, starting with the universal covering $p\colon \mathbb{R}^1 \to S^1$ of the circle we obtain a covering $p \times p\colon \mathbb{R}^1 \times \mathbb{R}^1 \to S^1 \times S^1$ of the torus.

17.7 Example. Using the coverings of S^1 described above we can construct many coverings of $S^1 \vee S^1$. For example, here is a covering obtained by combining the universal covering over one copy of S^1 and a trivial covering over the second copy:



Here are two different 3-fold coverings of $S^1 \vee S^1$:





17.8 Definition. If $p\colon T \to X$ is a covering and $f\colon Y \to X$ is a map then a *lift* of f is a map $\tilde{f}\colon Y \to T$ such that the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

17.9 Theorem (Homotopy Lifting Property). *Let $p\colon T \to X$ be a covering. Let $F\colon Y \times [0,1] \to X$ and $\tilde{f}\colon Y \times \{0\} \to T$ be functions satisfying $p\tilde{f} = F|_{Y\times\{0\}}$. There exists a function $\widetilde{F}\colon Y \times [0,1] \to T$ such that $p\widetilde{F} = F$ and $\widetilde{F}|_{Y\times\{0\}} = \tilde{f}$:*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & T \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X \end{tikzcd} \tag{$*$}$$

Moreover, such function $\widetilde{F}$ is unique.

17.10 Corollary. *Let* $p\colon T \to X$ *be a covering. Let* $x_0 \in X$*, and let* $\tilde{x}_0 \in T$ *be a point such that* $p(\tilde{x}_0) = x_0$*.*

1) For any path $\omega\colon [0,1] \to X$ *such that* $\omega(0) = x_0$ *there exists a lift* $\widetilde{\omega}\colon [0,1] \to T$ *satisfying* $\widetilde{\omega}(0) = \tilde{x}_0$*. Moreover, such lift is unique.*

2) Let $\omega, \tau\colon [0,1] \to X$ *be paths such that* $\omega(0) = \tau(0) = x_0$*,* $\omega(1) = \tau(1)$ *and* $\omega \simeq \tau$*. If* $\widetilde{\omega}$*,* $\widetilde{\tau}$ *are lifts of* ω*,* τ*, respectively, such that* $\widetilde{\omega}(0) = \widetilde{\tau}(0) = \tilde{x}_0$ *then* $\widetilde{\omega}(1) = \widetilde{\tau}(1)$ *and* $\widetilde{\omega} \simeq \widetilde{\tau}$*.*

17.11 Lemma. *Let $p\colon T \to X$ be a covering, and let $\tilde{f}_1, \tilde{f}_2\colon Y \to T$ be two lifts of a map $f\colon Y \to X$. If Y is a connected space and there exists $y_0 \in Y$ such that $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$ then $\tilde{f}_1(y) = \tilde{f}_2(y)$ for all $y \in Y$.*

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, shift left, "\tilde{f}_1"] \arrow[ur, shift right, "\tilde{f}_2"'] \arrow[r, "f"'] & X \end{tikzcd}$$

17.12 Lemma. *Let $p\colon T \to X$ be a covering. Let $F\colon Y \times [a,b] \to X$ and $\tilde{f}\colon Y \times \{a\} \to T$ be functions satisfying $p\tilde{f} = F|_{Y\times\{a\}}$. Assume also that $F(Y \times [a,b]) \subseteq U$ where $U \subseteq X$ is an evenly covered open set. There exists a function $\widetilde{F}\colon Y \times [a,b] \to T$ such that $p\widetilde{F} = F$ and $\widetilde{F}|_{Y\times\{a\}} = \tilde{f}$:*

$$\begin{tikzcd}
Y \times \{a\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & T \arrow[d, "p"] \\
Y \times [a,b] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X
\end{tikzcd}$$

Proof of Theorem 17.9. □