MTH 428/528

Introduction to Topology II

Elements of Algebraic Topology

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Contents

1 | Some Motivation

The main idea behind algebraic topology is that, in order to solve problems involving topological spaces one can try to translate them into problems about algebraic objects (groups, vectors spaces, rings, modules etc.) and then solve the resulting algebraic problems. The translation between topology and algebra is achieved by constructing assignments of the form:

$$\begin{array}{lcl} \text{topological spaces} & \longmapsto & \text{groups (rings, modules, \dots)} \\ \text{continuous functions} & \longmapsto & \text{homomorphisms of groups (or rings, modules, \dots)} \end{array}$$

For example, one of the main objectives of these notes is to study the assignment that associates to each space X a group $\pi_1(X)$ which is called the *fundamental group* of X [1]. Let S^1 denote the unit circle and D^2 the closed unit disc:

$$\begin{aligned} S^1 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} \\ D^2 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} \end{aligned}$$

We will see that $\pi_1(S^1) \cong \mathbb{Z}$ and that $\pi_1(D^2)$ is the trivial group. Since homeomorphic spaces have isomorphic fundamental groups an immediate consequence is that $S^1 \ncong D^2$. This is one typical application of algebraic invariants appearing in algebraic topology: they provide a tool for detecting if topological spaces are homeomorphic or not. However, these invariants can be also used to study more subtle relationships between spaces. Recall, for example, that if X is a topological space then we say that a subspace $A \subseteq X$ is a retract of X if there exists a continuous function $r\colon X \to A$ such that $r(x) = x$ for all $x \in A$.

1.1 Example. Let $\mathbf{0} = (0, 0)$ be the center of the disc D^2. Define $r\colon D^2 \smallsetminus \{\mathbf{0}\} \to S^1$ by

$$r(x) = \frac{x}{||x||}$$

[1] Technically $\pi_1(X)$ depends not only on the space X but also on the choice of a basepoint $x_0 \in X$, but we will disregard this for a moment.

where, for $x = (x_1, x_2)$ we set $||x|| = \sqrt{x_1^2 + x_2^2}$. Since $r(x) = x$ for all $x \in S^1$ this shows that S^1 is a retract of $D^2 \smallsetminus \{\mathbf{0}\}$.

On the other hand we have:

1.2 Proposition. *The circle S^1 is not a retract of D^2.*

Idea of a proof. We argue by contradiction. Let $i\colon S^1 \to D^2$ be the inclusion map. If S^1 is a retract of D^2, then there exists a map $r\colon D^2 \to S^1$ such that $ri = \mathrm{id}_{S^1}$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} S^1 \arrow[rr, "\mathrm{id}_{S^1}"] \arrow[dr, "i"'] & & S^1 \\ & D^2 \arrow[ur, "r"'] & \end{tikzcd}$$

The construction of the fundamental group associates to any continuous function of topological spaces $f\colon X \to Y$, a homomorphism of groups $f_*\colon \pi_1(X) \to \pi_1(Y)$ in a way that preserves compositions (i.e. $(fg)_* = f_*g_*$) and maps identity functions to identity group homomorphisms: $\mathrm{id}_{X*} = \mathrm{id}_{\pi_1(X)}$. As a result, the above commutative diagram of topological spaces gives a commutative diagram of groups:

$$\begin{tikzcd} \pi_1(S^1) \arrow[rr, "\mathrm{id}_{\pi_1(S^1)}"] \arrow[dr, "i_*"'] & & \pi_1(S^1) \\ & \pi_1(D^2) \arrow[ur, "r_*"'] & \end{tikzcd}$$

This implies, in particular, that r_* is onto, which is impossible since $\pi_1(D^2)$ is the trivial group and $\pi_1(S^1)$ is non-trivial. $\square$

2 | Categories and Functors

Before we get into the details of algebraic invariants of topological spaces, it will be worth a while to have a look at the general framework used to construct such invariants. In this chapter we introduce the notions of a *category* and a *functor*, which underlie such constructions.

2.1 Definition. A *category* $\mathbf{C}$ consists of the following ingredients:

1) a class of *objects* $\mathrm{Ob}(\mathbf{C})$
2) for each pair of objects $c, c' \in \mathrm{Ob}(\mathbf{C})$ a set of *morphisms* $\mathrm{Mor}_{\mathbf{C}}(c, c')$
3) for each object $c \in \mathrm{Ob}(\mathbf{C})$ a distinguished *identity morphism* $\mathrm{id}_c \in \mathrm{Mor}_{\mathbf{C}}(c, c)$
4) for each triple of objects $c, c', c'' \in \mathrm{Ob}(\mathbf{C})$ a *composition of morphisms* function

$$\circ\colon \mathrm{Mor}_{\mathbf{C}}(c, c') \times \mathrm{Mor}_{\mathbf{C}}(c', c'') \to \mathrm{Mor}_{\mathbf{C}}(c, c'')$$

Moreover, the composition of morphisms satisfies the following conditions:

(i) $f \circ (g \circ h) = (f \circ g) \circ h$, whenever morphisms f, g, h are composable
(ii) if $f \in \mathrm{Mor}_{\mathbf{C}}(c, c')$ then $f \circ \mathrm{id}_c = f = \mathrm{id}_{c'} \circ f$.

2.2 Example. By **Set** we will denote the category of sets. Its objects are sets, and for any sets A, B the set of morphisms $\mathrm{Mor}_{\mathbf{Set}}(A, B)$ consists of all functions $f\colon A \to B$. Composition of morphism is the usual composition of functions, and for a set A the identity morphism $\mathrm{id}_A\colon A \to A$ is given by the identity function: $\mathrm{id}_A(x) = x$ for all $x \in A$.

2.3 Example. Let **Gr** denote the category of groups. The objects of **Gr** are groups. Given two groups G, H the set $\mathrm{Mor}_{\mathbf{Gr}}(G, H)$ consists of all group homomorphisms $f\colon G \to H$.

2.4 Example. By **Top** we will denote the category of topological spaces. Its objects are topological spaces. For $X, Y \in \mathrm{Ob}(\mathbf{Top})$ the set $\mathrm{Mor}_{\mathbf{Top}}(X, Y)$ consists of all continuous functions $f\colon X \to Y$.

2.5 Example. The previous examples may suggest that categories are very large structures, and that each category corresponds to a whole area of mathematics (set theory, group theory, topology etc.) However, categories can be also small. For example, given any group G we can construct a category $\mathbf{C}_G$ as follows. The only object of $\mathbf{C}_G$ will be denoted by $*$. For every element $g \in G$ there is a morphism $f_g \colon * \to *$. Composition of morphisms is defined by multiplication in G: $f_g \circ f_h = f_{gh}$. The identity morphisms on $*$ corresponds to the identity element of G.

2.6 Note. To simplify notation we will frequently write $c \in \mathbf{C}$ instead of $c \in \mathrm{Ob}(\mathbf{C})$ to indicate that c is an object of a category $\mathbf{C}$.

2.7 Note. The definition of a category (2.1) deliberately says that objects of a category form a *class*. The notion of a class is more general that that of a set: every set is a class, but not every class is a set. This distinction lets us avoid some logical problems. For example, while defining the category **Set** (2.2) we cannot say that its objects form "the set of all sets" since this would lead to contradictions (e.g. Russell's paradox). On the other hand, we can talk about the class of all sets. As this suggests a class can be though of intuitively as something that can be bigger than any set.

Since every set is a class some categories have a set of objects. Such categories are called *small categories*. For example, the category $\mathbf{C}_G$ defined in Example 2.5 is small since its objects form a set with one element: $\mathrm{Ob}(\mathbf{C}_G) = \{*\}$.

2.8 Definition. Let $\mathbf{C}$, $\mathbf{D}$ be categories. A *(covariant) functor* $F \colon \mathbf{C} \to \mathbf{D}$ consists of

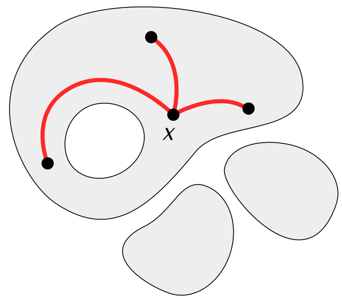
1) an assignment $F \colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$
2) for each $c, c' \in \mathrm{Ob}(\mathbf{C})$ a function

$$F \colon \mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$$

   such that $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for all $c \in \mathrm{Ob}(\mathbf{C})$ and $F(f \circ g) = F(f) \circ F(g)$ for each pair of composable morphisms f, g in $\mathbf{C}$.

2.9 Example. For a topological space X let $U(X)$ denote the sets of points of X. Also, given a continuous map of topological spaces $f \colon X \to Y$ denote by $U(f) \colon U(X) \to U(Y)$ the underlying map of sets. These assignments define a functor $U \colon \mathbf{Top} \to \mathbf{Set}$ which is called the *forgetful functor*.

2.10 Example. A more interesting example of a functor $\mathbf{Top} \to \mathbf{Set}$ can be obtained as follows. Let X be a topological space. Recall that a *path* in X is a continuous function $\omega \colon [0, 1] \to X$. Recall also that a *path connected component* of a point $x \in X$ is the subspace of X consisting of all points that can be connected to x by a path:



Denote this subspace by $[x]$. Notice that for $x, x' \in X$ we have $[x] = [x']$ if and only if there is a path joining x and x'. Let $\pi_0(X)$ denote the set whose elements are path connected components of the space X. Given a continuous function of topological spaces $f\colon X \to Y$ consider the function of sets

$$f_*\colon \pi_0(X) \to \pi_0(Y)$$

given by $f_*([x]) = [f(x)]$. The function f_* is well defined. Indeed, if $x, x' \in X$ are points such that $[x] = [x']$ then there exists a path $\omega\colon [0,1] \to X$ such that $\omega(0) = x$ and $\omega(1) = x'$. Then $f\omega\colon [0,1] \to Y$ is a path joining $f(x)$ with $f(x')$ which shows that $[f(x)] = [f(x')]$. The assignments $X \mapsto \pi_0(X)$ and $f \mapsto f_*$ define a functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$.

2.11 Definition. Let $\mathbf{C}$ be a category. A morphism $f\colon c \to c'$ in $\mathbf{C}$ is an *isomorphism* if there exists a morphism $g\colon c' \to c$ such that $gf = \mathrm{id}_c$ and $fg = \mathrm{id}_{c'}$. In such case we say that g is the *inverse* of f and we write $g = f^{-1}$.

If there exists an isomorphism between $c, c' \in \mathbf{C}$ then we say that these objects are *isomorphic* and we write $c \cong c'$.

2.12 Example. A morphism $f\colon X \to Y$ in $\mathbf{Top}$ is an isomorphism if and only if f is a homeomorphism.

2.13 Example. A morphism $f\colon A \to B$ in $\mathbf{Set}$ is an isomorphism if and only if f is a bijection of sets.

2.14 Example. A morphism $f\colon G \to H$ in $\mathbf{Gr}$ is an isomorphism if and only if f is a group isomorphism.

2.15 Proposition. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. If $f\colon c \to c'$ is an isomorphism in $\mathbf{C}$ then $F(f)\colon F(c) \to F(c')$ is an isomorphism in $\mathbf{D}$ and $F(f)^{-1} = F(f^{-1})$.*

Proof. Let $f^{-1}\colon c' \to c$ be the inverse of f. We have

$$F(f^{-1})F(f) = F(f^{-1}f) = F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$$

Similarly, using that $ff^{-1} = \mathrm{id}_{c'}$ we obtain $F(f)F(f^{-1}) = \mathrm{id}_{F(c')}$. Thus $F(f^{-1})$ is the inverse of $F(f)$. $\square$

2.16 Corollary. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor and $c, c' \in \mathbf{C}$. If $F(c) \not\cong F(c')$ then $c \not\cong c'$.*

2.17 Example. Consider the functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ (2.10). In **Top** take the spaces $\mathbb{R}$ and $\mathbb{R} \smallsetminus \{0\}$. The space $\mathbb{R}$ has only one path connected component while $\mathbb{R} \smallsetminus \{0\}$ has two path connected components: $(-\infty, 0)$ and $(0, +\infty)$. It follows that $\pi_0(\mathbb{R})$ consists of one element while $\pi_0(\mathbb{R} \smallsetminus \{0\})$ is a set with two elements, so $\pi_0(\mathbb{R}) \not\cong \pi_0(\mathbb{R} \smallsetminus \{0\})$ in **Set**. This shows that $\mathbb{R} \not\cong \mathbb{R} \smallsetminus \{0\}$ in **Top**, i.e. that these two spaces are not homeomorphic.

2.18 Note. For a general functor $F\colon \mathbf{C} \to \mathbf{D}$ and $c, c' \in \mathbf{C}$ it may happen that $F(c) \cong F(c')$ even though $c \not\cong c'$. Take for example the functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ and let $X = \{*\}$ be a space consisting of a single point. We have $\pi_0(X) \cong \pi_0(\mathbb{R})$, since both $\pi_0(X)$ and $\pi_0(\mathbb{R})$ are sets with only one element, but $X \not\cong \mathbb{R}$.

Exercises to Chapter 2

E2.1 Exercise. Let $\mathbf{C}$ be a category. An object $c \in \mathbf{C}$ is initial in $\mathbf{C}$ if for each object $d \in \mathbf{C}$ there is exactly one morphism $c \to d$.

a) Show that if c is an initial object in $\mathbf{C}$ and $c' \in \mathbf{C}$ is an object isomorphic to c then c' is also an initial object.

b) Show that if c and c' are objects of $\mathbf{C}$ such that each of them is initial then $c \cong c'$.

E2.2 Exercise. Given a morphism $f\colon c \to c'$ and an object d in a category $\mathbf{C}$, consider the functions

$$f_*\colon \mathrm{Mor}_{\mathbf{C}}(d, c) \to \mathrm{Mor}_{\mathbf{C}}(d, c') \quad \text{ and } \quad f^*\colon \mathrm{Mor}_{\mathbf{C}}(c', d) \to \mathrm{Mor}_{\mathbf{C}}(c, d)$$

given by $f_*(g) = f \circ g$ for $g \in \mathrm{Mor}_{\mathbf{C}}(d, c)$, and $f^*(h) = h \circ f$ for $h \in \mathrm{Mor}_{\mathbf{C}}(c', d)$.

Show that for a morphism $f\colon c \to c'$ the following conditions are equivalent:

1) The morphism f is is an isomorphism.
2) The function f_* is a bijection for every $d \in \mathbf{C}$.
3) The function f^* is a bijection for every $d \in \mathbf{C}$.

E2.3 Exercise. Cosider a sequence of five morphisms in a category $\mathbf{C}$:

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3 \xrightarrow{f_4} c_4 \xrightarrow{f_5} c_5$$

Assume that the composition of any three morphisms in this sequence $f_3 \circ f_2 \circ f_1$, $f_4 \circ f_3 \circ f_2$, and $f_5 \circ f_4 \circ f_3$ is an isomorphism. Show that f_i is an isomorphism for $i = 1, \dots, 5$.

E2.4 Exercise. Find two different functors $F, F'\colon \mathbf{Gr} \to \mathbf{Gr}$ such that $F(G) = F'(G) = G$ for each group $G \in \mathbf{Gr}$.

E2.5 Exercise. Let $\mathbf{C}$, $\mathbf{D}$ be categories. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is called *full* if for each $c, c' \in \mathbf{C}$ the function

$$F\colon \mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$$

is onto, and if is *faithful* if for each $c, c' \in \mathbf{C}$ this function is 1-1.

a) Give an example of a functor $F\colon \mathbf{Gr} \to \mathbf{Gr}$ which is full but not faithful.

b) Give an example of a functor $F'\colon \mathbf{Gr} \to \mathbf{Gr}$ which is faithful but not full.

E2.6 Exercise. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor which is both full and faithful (Exercise 2.5), and let $c, c' \in \mathbf{C}$. Show that $c \cong c'$ if and only if $F(c) \cong F(c')$.

3 | The Fundamental Group

3.1 Definition. A *pointed topological space* is a pair (X, x_0), where X is a topological space and $x_0 \in X$. We say that x_0 is the *basepoint* of X. Given two pointed spaces (X, x_0) and (Y, y_0) a *basepoint preserving map* $f\colon (X, x_0) \to (Y, y_0)$ is a continuous function $f\colon X \to Y$ such that $f(x_0) = y_0$.

Let $\mathbf{Top}_*$ denote the category the objects of which are pointed spaces and morphisms are basepoint preserving maps. Our goal in this chapter will be to construct the *fundamental group functor*

$$\pi_1\colon \mathbf{Top}_* \to \mathbf{Gr}$$

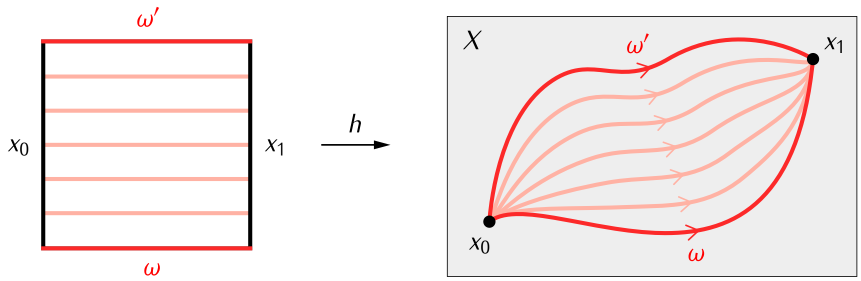
That is, we will construct an assignment that associates to every pointed space (X, x_0) a group $\pi_1(X, x_0)$ and to every basepoint preserving map $f\colon (X, x_0) \to (Y, y_0)$ a group homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ in a way that preserves identity functions and compositions of functions.

Recall that a path in a space X is a continuous function $\omega\colon [0, 1] \to X$.

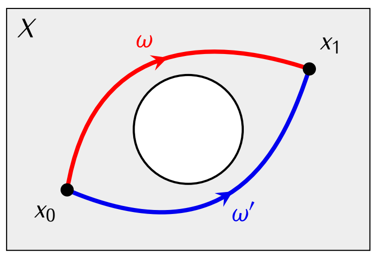
3.2 Definition. Let $\omega, \omega'\colon [0, 1] \to X$ be paths such that $\omega(0) = \omega'(0) = x_0$ and $\omega(1) = \omega'(1) = x_1$ for some $x_0, x_1 \in X$. We say that the paths ω and ω' are *path homotopic* if for every $t \in [0, 1]$ there exists a path $h_t\colon [0, 1] \to X$ such that:

1) $h_t(0) = x_0$, and $h_t(1) = x_1$ for all $t \in [0, 1]$
2) $h_0 = \omega$, and $h_1 = \omega'$
3) the function $h\colon [0, 1] \times [0, 1] \to X$ given by $h(s, t) = h_t(s)$ is continuous.

In this case we write $\omega \simeq \omega'$ and we say that h is a *path homotopy* between ω and ω'.



Intuitively, path homotopy is a device for detecting holes in topological spaces. If ω and ω' are paths in X with the same endpoints but such that $\omega \not\simeq \omega'$ then it indicates that there is a hole in X that prevents us from deforming ω to ω':



3.3 Lemma. *Let X be a space and let $x_0, x_1 \in X$. Path homotopy defines an equivalence relation on the set of paths in X that start at x_0 and terminate at x_1.*

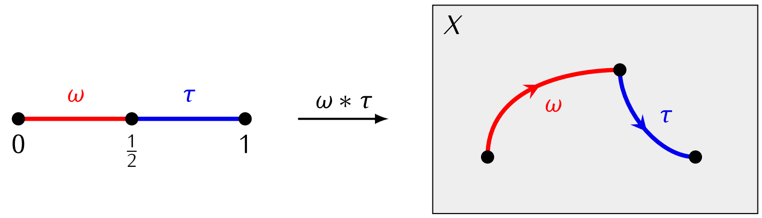
Proof. Exercise. □

3.4 Definition. For a path ω we will denote by $[\omega]$ the equivalence class of ω taken with respect to the equivalence relation given by path homotopy. We will say that $[\omega]$ is the *homotopy class* of ω.

3.5 Notation. Let X be a space and let $x_0, x_1 \in X$. By $\pi_1(X, x_0, x_1)$ we will denote the set of homotopy classes of paths that begin at x_0 and terminate at x_1. If $x_0 = x_1$ then we will write $\pi_1(X, x_0)$ instead of $\pi_1(X, x_0, x_0)$. Notice that elements of $\pi_1(X, x_0)$ are homotopy classes $[\omega]$ where ω is a path such that $\omega(0) = \omega(1) = x_0$. We call such ω a *loop* based at x_0.

3.6 Definition. Let $\omega, \tau\colon [0,1] \to X$ be paths such that $\omega(1) = \tau(0)$. The *concatenation* of ω and τ is the path $\omega * \tau\colon [0,1] \to X$ given by

$$(\omega * \tau)(s) = \begin{cases} \omega(2s) & \text{for } s \in [0, \frac{1}{2}] \\ \tau(2s-1) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

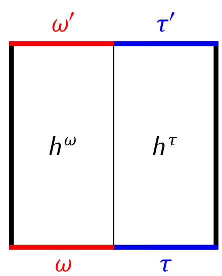


3.7 Proposition. *Let ω ,τ be paths in X such that $\omega(1) = \tau(0)$. If ω', τ' are paths such that $\omega \simeq \omega'$ and $\tau \simeq \tau'$ then $\omega * \tau \simeq \omega' * \tau'$.*

Proof. Let $h^\omega\colon [0,1]\times[0,1] \to X$ be a path homotopy between ω and ω' and $h^\tau\colon [0,1]\times[0,1] \to X$ be a path homotopy between τ and τ'. Define $h\colon [0,1]\times[0,1] \to X$ by

$$h(s,t) = \begin{cases} h^\omega(2s,t) & \text{for } s \in [0, \frac{1}{2}] \\ h^\tau(2s-1,t) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

The map h is a path homotopy between $\omega * \tau$ and $\omega' * \tau'$.



□

Notice that by Proposition 3.7 the homotopy class of $\omega * \tau$ depends only on the homotopy classes of ω and τ. Therefore for any $x_0, x_1, x_2 \in X$ we obtain a well defined function

$$\mu\colon \pi_1(X, x_0, x_1) \times \pi_1(X, x_1, x_2) \longrightarrow \pi_1(X, x_0, x_2)$$

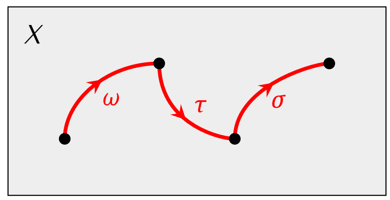
where $\mu([\omega], [\tau]) = [\omega * \tau]$. To simplify notation we will write $[\omega]\cdot[\tau]$ instead of $\mu([\omega], [\tau])$. In the case when $x_0 = x_1 = x_2$ this gives a multiplication on the set $\pi_1(X, x_0)$:

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0), \quad [\omega]\cdot[\tau] = [\omega * \tau]$$

Our next goal will be to show that the set $\pi_1(X, x_0)$ taken with this multiplication is a group.

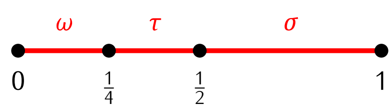
3.8 Lemma. *If ω, τ, σ are paths in a space X such that $\omega(1) = \tau(0)$ and $\tau(1) = \sigma(0)$ then*

$$([\omega]\cdot[\tau])\cdot[\sigma] = [\omega]\cdot([\tau]\cdot[\sigma])$$



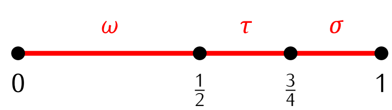
Proof. We have:

$$([\omega]\cdot[\tau])\cdot[\sigma] = [\omega * \tau]\cdot[\sigma] = [(\omega * \tau) * \sigma]$$

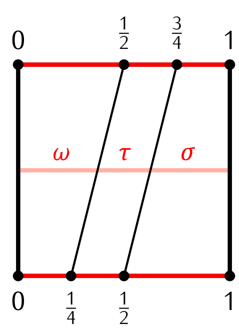


Similarly:

$$[\omega]\cdot([\tau]\cdot[\sigma]) = [\omega]\cdot[\tau * \sigma] = [\omega * (\tau * \sigma)]$$



We need to show that $(\omega * \tau) * \sigma \simeq \omega * (\tau * \sigma)$. Graphically a homotopy $h\colon [0,1]\times[0,1] \to X$ between these paths can be represented as follows:



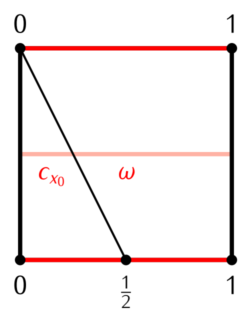
More precisely, h is given by the following formula:

$$h(s,t) = \begin{cases} \omega\left(\frac{4s}{t+1}\right) & \text{for } s \in [0, \frac{t+1}{4}] \\ \tau(4s - t - 1) & \text{for } s \in [\frac{t+1}{4}, \frac{t+2}{4}] \\ \sigma\left(\frac{4s-t-2}{2-t}\right) & \text{for } s \in [\frac{t+2}{4}, 1] \end{cases}$$

□

3.9 Lemma. *Let X be a space, and let $x_0 \in X$. Let $c_{x_0}\colon [0,1] \to X$ denote the constant path at the point x_0: $c_{x_0}(s) = x_0$ for all $t \in [0,1]$. If ω is a path in X such that $\omega(0) = x_0$ then $[c_{x_0}] \cdot [\omega] = [\omega]$. Also, if τ is a path such that $\tau(1) = x_0$ then $[\tau] \cdot [c_{x_0}] = [\tau]$.*

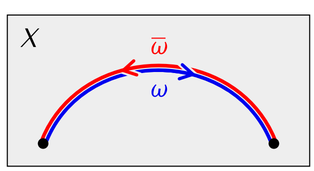
Proof. To obtain the first equality we need to check that $c_{x_0} * \omega \simeq \omega$. A homotopy between these paths can be represented as follows:



The second equality comes from a homotopy $\tau * c_{x_0} \simeq \tau$ that can be obtained in a similar way.

□

Let $\omega\colon [0,1] \to X$ be a path. By $\overline{\omega}$ we will denote the path given by $\overline{\omega}(s) = \omega(1-s)$ for $s \in [0,1]$. In other words $\overline{\omega}$ is obtained by reversing the orientation of ω::



We will say that $\overline{\omega}$ is the *inverse* of ω. This name is justified by the following fact:

3.10 Lemma. *Let ω be a path in a space X such that $\omega(0) = x_0$ and $\omega(1) = x_1$. We have:*

$$[\omega] \cdot [\overline{\omega}] = [c_{x_0}], \qquad [\overline{\omega}] \cdot [\omega] = [c_{x_1}]$$

Proof. Intuitively, a homotopy h between c_{x_0} and $\omega * \overline{\omega}$ can be obtained by taking h_t to be the path that goes from x_0 to the point $\omega(t) = \overline{\omega}(1-t)$ along ω, and the follows $\overline{\omega}$ back to x_0. Formally, we can define h as follows:

$$h(s,t) = \begin{cases} \omega(2st) & \text{for } s \in [0, \frac{1}{2}] \\ \omega((2-2s)t) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

$\square$

3.11 Proposition. *Let X be a topological space and let $x_0 \in X$. The set $\pi_1(X, x_0)$ taken with the multiplication given by*

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

for $[\omega], [\tau] \in \pi_1(X, x_0)$ is a group. The trivial element in this group is the homotopy class of the constant path $[c_{x_0}]$, and for $[\omega] \in \pi_1(X, x_0)$ we have $[\omega]^{-1} = [\overline{\omega}]$.

Proof. The multiplication is associative by Lemma 3.8. The element $[c_{x_0}]$ is trivial with respect to this multiplication by Lemma 3.9, and $[\overline{\omega}]$ is the multiplicative inverse of $[\omega]$ by Lemma 3.10. $\square$

3.12 Definition. Let (X, x_0) be a pointed space. The group $\pi_1(X, x_0)$ is called the *fundamental group* of (X, x_0).

3.13 Lemma. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. If $\omega\colon [0,1] \to X$ is a loop in X based at x_0 then $f \circ \omega\colon [0,1] \to Y$ is a loop in Y based at y_0. Moreover, if ω' is another loop in X based at x_0 such that $\omega \simeq \omega'$ then $f \circ \omega \simeq f \circ \omega'$.*

Proof. If $h\colon [0,1] \times [0,1] \to X$ is a homotopy between ω and ω' then $f \circ h\colon [0,1] \times [0,1] \to Y$ gives a homotopy between $f\omega$ and $f\omega'$. $\square$

By Lemma 3.13 each map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ defines a function

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

given by $f_*([\omega]) = [f \circ \omega]$. In addition we have:

3.14 Proposition. *If $f\colon (X, x_0) \to (Y, y_0)$ is a map of pointed spaces then the function $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group homomorphism.*

Proof. First, notice that $f \circ c_{x_0} = c_{y_0}$, so $f_*([c_{x_0}]) = [c_{y_0}]$. Also, if ω, τ are loops in X then $f \circ (\omega * \tau) = (f \circ \omega) * (f \circ \tau)$. This gives:

$$f_*([\omega] \cdot [\tau]) = [f \circ (\omega * \tau)] = [(f \circ \omega) * (f \circ \tau)] = [f \circ \omega] \cdot [f \circ \tau] = f_*([\omega]) \cdot f_*([\tau])$$

$\square$

3.15 Corollary. *The assignments $(X, x_0) \mapsto \pi_1(X, x_0)$ and $f \mapsto f_*$ define a functor*

$$\pi_1 \colon \mathbf{Top}_* \to \mathbf{Gr}$$

Proof. We need to check that

1) if $\mathrm{id}\colon (X, x_0) \to (X, x_0)$ is the identity map, then $\mathrm{id}_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identify homomorphism;
2) if $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ are maps of pointed spaces then $(g \circ f)_* = g_* \circ f_*$.

Property 1) holds since $\mathrm{id}_*([\omega]) = [\mathrm{id} \circ \omega] = [\omega]$. Similarly, property 2) holds since

$$(g \circ f)_*[\omega] = [g \circ f \circ \omega] = g_*([f \circ \omega]) = g_*(f_*([\omega])) = g_* \circ f_*([\omega])$$

□

Notice that an isomorphism in $\mathbf{Top}_*$ is a homeomorphism that preserves basepoints. As a consequence of Proposition 2.15 we obtain:

3.16 Corollary. *If $(X, x_0), (Y, y_0)$ are pointed spaces and $f\colon X \to Y$ is a homeomorphism such that $f(x_0) = y_0$, then $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

3.17 Note. If $f\colon X \to Y$ is any homeomorphism of topological spaces and $x_0 \in X$ then we get a homeomorphism of pointed spaces $f\colon (X, x_0) \to (Y, f(x_0))$, which gives an isomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

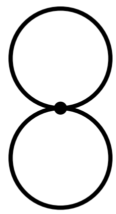
3.18 Note. In some settings it is convenient to use a somewhat different construction of the fundamental group than the one described above. Recall the every element of $\pi_1(X, x_0)$ can be represented by a function $\omega\colon [0,1] \to X$ that satisfies $\omega(0) = \omega(1) = x_0$. Such function uniquely determines a map $[0,1]/{\sim} \to X$ from the quotient space $[0,1]/{\sim}$ where $\sim$ is the equivalence relation identifying the endpoints of the interval: $0 \sim 1$. The space $[0,1]/{\sim}$ is homeomorphic to the circle S^1. Under such homeomorphism the point $[0] \in [0,1]/{\sim}$ is mapped to some point $s_0 \in S^1$ that we can consider as a basepoint of S^1. As a consequence we obtain a bijection between two sets of maps:

$$\begin{pmatrix} \text{maps } \omega\colon [0,1] \to X \\ \text{with } \omega(0) = \omega(1) = x_0 \end{pmatrix} \cong \begin{pmatrix} \text{basepoint preserving maps} \\ \omega\colon (S^1, s_0) \to (X, x_0) \end{pmatrix}$$

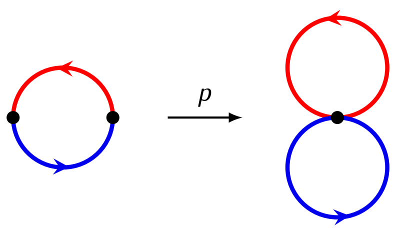
Next, given two maps $\omega, \tau\colon (S^1, s_0) \to (X, x_0)$ we will say that ω and τ are homotopic if there is a continuous function $h\colon S^1 \times [0,1] \to X$ such that $h(s, 0) = \omega(s)$, $h(s, 1) = \tau(s)$ for all $s \in S^1$ and $h(s_0, t) = x_0$ for all $t \in [0,1]$. The above bijection maps homotopic functions on one side to homotopic functions on the other side, so we obtain a bijection of sets:

$$\begin{pmatrix} \text{elements} \\ \text{of the group} \\ \pi_1(X, x_0) \end{pmatrix} = \begin{pmatrix} \text{homotopy classes} \\ \text{of maps } \omega\colon [0,1] \to X \\ \text{with } \omega(0) = \omega(1) = x_0 \end{pmatrix} \cong \begin{pmatrix} \text{homotopy classes} \\ \text{of basepoint preserving maps} \\ \omega\colon (S^1, s_0) \to (X, x_0) \end{pmatrix}$$

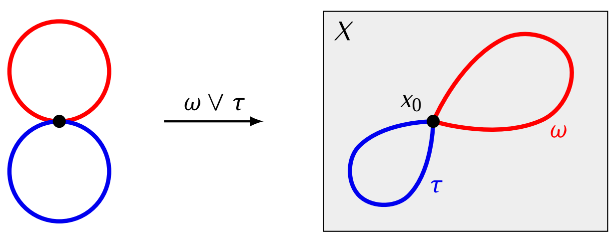
In effect we can think of elements $\pi_1(X, x_0)$ as homotopy classes of basepoint preserving maps $(S^1, s_0) \to (X, x_0)$. Using this interpretation the trivial element in $\pi_1(X, x_0)$ is given by the homotopy class of the constant map $S^1 \to X$. Multiplication in $\pi_1(X, x_0)$ can be described as follows. Let $S^1 \vee S^1$ denote the space obtained by taking two copies of S^1 and identifying a basepoint of one copy with the basepoint of the other copy:



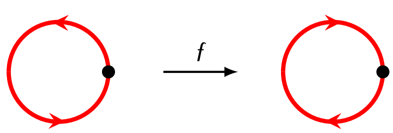
The *pinch map* is a function $p\colon S^1 \to S^1 \vee S^1$ that wraps half of the circle around one copy of S^1 and the other half around the other copy:



Given two functions $\omega, \tau\colon (S^1, s_0) \to (X, x_0)$ define $\omega \vee \tau\colon S^1 \vee S^1 \to X$ to be the function that maps one copy of S^1 by ω and the other by τ:



We have: $[\omega] \cdot [\tau] = [(\omega \vee \tau) \circ p]$. Finally, in order to describe multiplicative inverses in $\pi_1(X, x_0)$ consider the *flip map* $f\colon S^1 \to S^1$ that reflects the circle about its diagonal that passes through the basepoint:



For $\omega\colon (S^1, s_0) \to (X, x_0)$ we have $[\omega]^{-1} = [\omega \circ f]$.

Exercises to Chapter 3

E3.1 Exercise. Prove Lemma 3.3.

4 | Dependence on the Basepoint

By construction, the fundamental group of a space depends not only on the space itself, but also on the choice of a basepoint. In some applications a space may come equipped with a preferred basepoint, but in other situations we may need to choose a basepoint arbitrarily to compute the fundamental group. In this chapter we examine how the choice of a basepoint impacts the fundamental group of a space. We will also see how the construction of the fundamental group can be modified so that it does not involve a choice of a basepoint.

We start with the observation that the fundamental group of a pointed space depends only on the path connected component of the basepoint:

4.1 Proposition. *Let X be a space, let $x_0 \in X$, and let $Y \subseteq X$ be the path connected component of x_0. If $i\colon Y \to X$ is the inclusion map then the induced homomorphism*

$$i_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$$

is an isomorphism of groups.

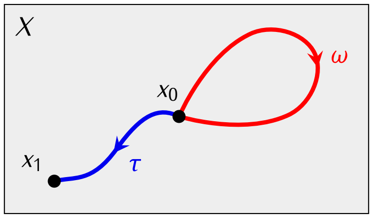
Proof. Exercise. □

Proposition 4.1 implies that if we change the basepoint from one path connected component of X to another we can get entirely different fundamental groups, since in general path connected components need not be related in any way. It remains then to consider the situation when we are given a space X with two different basepoints x_0 and x_1, that belong to the same path connected component of X. In this case there exists a path in X joining these points. We have:

4.2 Proposition. *Let X be a space and let $x_0, x_1 \in X$. For any path $\tau\colon [0,1] \to X$ such that $\tau(0) = x_0$ and $\tau(1) = x_1$ the function*

$$s_\tau\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

given by $s_\tau([\omega]) = [\bar{\tau} * \omega * \tau]$ is an isomorphism of groups.



Proof. Exercise. □

4.3 Corollary. *If X is a space and $x_0, x_1 \in X$ are points than belong to the same path connected component of X then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. Follows from Proposition 4.2. □

4.4 Note. In general the isomorphism s_τ given in Proposition 4.2 depends on the choice of the path τ. However, if $\pi_1(X, x_0)$ is an abelian group then for any paths τ, τ' joining x_0 and x_1 we have $s_\tau = s_{\tau'}$ (exercise). Thus in such case we obtain a canonical isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

4.5 Note. Given a path connected space X we will sometimes write $\pi_1(X)$ to denote the fundamental group of X taken with respect to some unspecified basepoint of X. By Corollary 4.3 this will not create problems as long as we are interested in the isomorphism type of the fundamental group only.

Recall that any continuous function $f\colon X \to Y$ defines a homomorphism of fundamental groups $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$. The next proposition describes how this homomorphism changes with a change of the basepoint:

4.6 Proposition. *Let $x_0, x_1 \in X$ and let $f\colon X \to Y$ be a continuous function. Given a path τ in X such that $\tau(0) = x_0$ and $\tau(1) = x_1$ consider the isomorphisms $s_\tau\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $s_{f\tau}\colon \pi_1(Y, f(x_0)) \to \pi_1(Y, f(x_1))$ defined as in Proposition 4.2. Then following diagram commutes:*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "s_\tau"', "\cong"] & \pi_1(Y, f(x_0)) \arrow[d, "\cong"', "s_{f\tau}"] \\
\pi_1(X, x_1) \arrow[r, "f_*"'] & \pi_1(Y, f(x_1))
\end{tikzcd}$$

Proof. Exercise. □

4.7 Corollary. *Let X be a path connected space, $x_0, x_1 \in X$, and let $f\colon X \to Y$ be a continuous function. The homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism (or is the trivial homomorphism or is 1-1 or onto) if and only if the homomorphism $f_*\colon \pi_1(X, x_1) \to \pi_1(Y, f(x_1))$ has the same property.*

Proof. Follows from Proposition 4.6. □

In most applications it is sufficient to work with the fundamental group associated to some choice of a basepoint, using Proposition 4.2 whenever we need to change the basepoint. However, it is also possible to modify the construction of the fundamental group in a way that does not involve any choice of a basepoint. This is done as follows. Given a space X in place of the group $\pi_1(X, x_0)$ we take the category $\Pi_1(X)$ whose objects are points of X. The set of morphisms between points $x_0, x_1 \in X$ is the set of homotopy classes of paths joining these points:

$$\mathrm{Mor}_{\Pi_1(X)}(x_0, x_1) = \pi_1(X, x_1, x_0)$$

Composition of morphisms is given by concatenation of paths: for $[\omega] \in \mathrm{Mor}_{\Pi_1(X)}(x_0, x_1)$ and $[\tau] \in \mathrm{Mor}_{\Pi_1(X)}(x_1, x_2)$ we set $[\tau]\circ[\omega] = [\omega * \tau]$. By Lemma 3.8 this composition is associative, and by Lemma 3.9 the homotopy class $[c_{x_0}]$ of the constant path at x_0 plays the role of the identity morphism in $\mathrm{Mor}_{\Pi_1(X)}(x_0, x_0)$.

4.8 Definition. Let X be a topological space. The category $\Pi_1(X)$ is called the fundamental groupoid of X.

4.9 Note. In general, a *groupoid* is a category where every morphism is an isomorphism. The category $\Pi_1(X)$ is a groupoid since by Lemma 3.10 any $[\omega] \in \mathrm{Mor}_{\Pi_1(X)}(x_0, x_1)$ is an isomorphism with the inverse given by $[\overline{\omega}] \in \mathrm{Mor}_{\Pi_1(X)}(x_1, x_0)$.

Notice that the category $\Pi_1(X)$ contains information about fundamental groups of X taken with respect to all possible basepoints, since for any $x_0 \in X$ we have $\mathrm{Mor}_{\Pi_1(X)}(x_0, x_0) \cong \pi_1(X, x_0)$.

As we have seen any pointed map $f\colon (X, x_0) \to (Y, y_0)$ defines a homomorphism of fundamental groups $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Similarly, any map of spaces $f\colon X \to Y$ defines a functor of fundamental groupoids

$$f_*\colon \Pi_1(X) \to \Pi_1(Y)$$

defined as follows. For $x \in X$ we set $f_*(x) = f(x)$ (where we consider x as and object of $\Pi_1(X)$ and $f(x)$ as an object of $\Pi_1(Y)$). For $[\omega] \in \mathrm{Mor}_{\Pi_1(X)}(x_0, x_1)$ we define $f_*([\omega]) = [f \circ \omega]$.

Recall that by a small category is a category whose objects form a set. Let **Cat** denote a category whose objects are small categories and morphisms are functors. We have:

4.10 Corollary. *The assignments $X \mapsto \Pi_1(X)$ and $f \mapsto f_*$ define a functor*

$$\Pi_1\colon \mathbf{Top} \to \mathbf{Cat}$$

from the category of unpointed topological spaces to the category of small categories

Proof. Exercise. □

Exercises to Chapter 4

E4.1 Exercise. Prove Proposition 4.2.

E4.2 Exercise. Recall that if X is a topological space, $x_0, x_1 \in X$, and $\tau\colon [0,1] \to X$ is a path such that $\tau(0) = x_0$, $\tau(1) = x_1$ then τ defines an isomorphism

$$s_\tau\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

Show that if $\pi_1(X, x_0)$ is an abelian group then this isomorphism does not depend on the choice of the path τ. That is, if τ' is another path in X such that $\tau'(0) = x_0$ and $\tau'(1) = x_1$ then $s_\tau = s_{\tau'}$.

E4.3 Exercise. Recall that $S^1 \vee S^1$ is the space consisting of two circles joined at one point x_0 (the eight-figure space). Assume that there exists a space (Y, y_0) such that the group $\pi_1(Y, y_0)$ is non-abelian. Show that this implies that $\pi_1(S^1 \vee S^1, x_0)$ must be a non-abelian group.

E4.4 Exercise. A *topological group* G is a group that is also a topological space, and such that the maps $\mu\colon G \times G \to G$, $\mu(g,h) = gh$, and $\eta\colon G \to G$, $\eta(g) = g^{-1}$ are continuous. Let e denote the identity element in G.

a) Show that for any $g_0 \in G$ we have $\pi_1(G, g_0) \cong \pi_1(G, e)$ even if G is not path connected.

b) Let ω, τ be loops in G based at $e \in G$. Since $\omega(s)$ and $\tau(s)$ are elements of the group G for $s \in [0,1]$, we can use group multiplication to obtain an element $\omega(s) \cdot \tau(s) \in G$. Let

$$\omega \odot \tau\colon [0,1] \to G$$

denote the loop defined by $\omega \odot \tau(s) = \omega(s) \cdot \tau(s)$. Show that $[\omega \odot \tau] = [\omega * \tau]$ (where $*$ denotes the concatenation of loops). It follows that for a topological group G we can describe multiplication in $\pi_1(G, e)$ in two different but equivalent ways: as a loop concatenation and as a pointwise multiplication of loops.

c) Show that the fundamental group $\pi_1(G, e)$ is abelian.

5 | First Computations

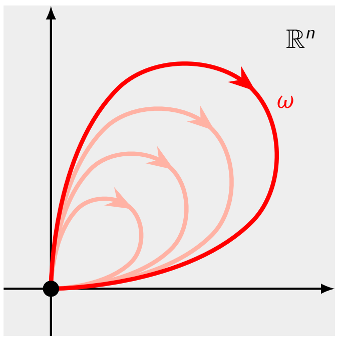
In this chapter we describe some basic examples of computations of the fundamental group. Later on we will see that these examples and a few additional tools let us calculate fundamental groups of many spaces.

5.1 Proposition. *If $X = \{*\}$ is a space consisting of only one point then $\pi_1(X)$ is the trivial group.*

Proof. It is enough to notice that the only loop in X is the constant loop. $\square$

5.2 Proposition. *For any $n \geq 1$ the group $\pi_1(\mathbb{R}^n)$ is trivial.*

Proof. Choose $0 \in \mathbb{R}^n$ as the basepoint. Let $\omega\colon [0,1] \to \mathbb{R}^n$ be a loop based at 0. We need to show that ω is homotopic to the constant loop c_0. Such homotopy $h\colon [0,1] \times [0,1] \to \mathbb{R}^n$ is given by $h(s,t) = t \cdot \omega(s)$.



$\square$

Let $D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$ be the n-dimensional closed unit disc. Using the the same argument as above we obtain:

5.3 Proposition. *For any $n \geq 1$ the group $\pi_1(D^n)$ is trivial.*

5.4 Note. As we have seen before homeomorphic spaces have isomorphic fundamental groups (3.17). The above calculations show that the converse is not true, e.g. $\mathbb{R}^n \not\cong D^n$ for $n \geq 1$ but $\pi_1(\mathbb{R}^n) \cong \pi_1(D^n)$.

5.5 Definition. A space X is *simply connected* if it is path connected and $\pi_1(X)$ is trivial.

For example $\{*\}$, $\mathbb{R}^n$, and D^n are simply connected spaces.

5.6 Proposition. *A space X is simply connected if and only if X is path connected and for any two paths $\omega, \tau\colon [0,1] \to X$ satisfying $\omega(0) = \tau(0)$ and $\omega(1) = \tau(1)$ we have $\omega \simeq \tau$.*

Proof. Exercise. □

Our next goal will be to show that the fundamental group is not always trivial:

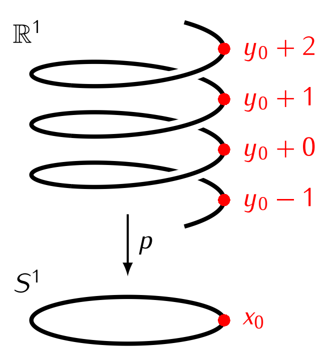
5.7 Theorem. $\pi_1(S^1) \cong \mathbb{Z}$.

The proof of Theorem 5.7 will require some technical preparation.

5.8 Definition. The *universal covering* of S^1 is the map $p\colon \mathbb{R}^1 \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$.

Geometrically p is the map that wraps $\mathbb{R}^1$ infinitely many times around the circle.

5.9 Note. For $y, y' \in \mathbb{R}$ we have $p(y) = p(y')$ if and only if $y' = y + n$ for some $n \in \mathbb{Z}$. As a consequence if $x_0 \in S^1$ and if $y_0 \in \mathbb{R}$ is a point such that $p(y_0) = x_0$ then $p^{-1}(x_0) = \{y_0 + n \mid n \in \mathbb{Z}\}$.

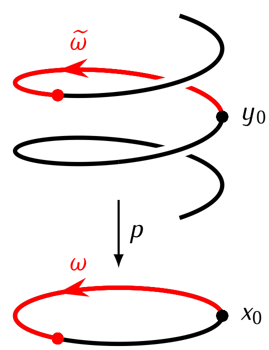


5.10 Definition. Let ω be a path in S^1. We say that a path $\widetilde{\omega}$ in $\mathbb{R}$ is a *lift* of ω if $p \circ \widetilde{\omega} = \omega$.

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, "p"] \\
{[0,1]} \arrow[ur, "\widetilde{\omega}"] \arrow[r, "\omega"'] & S^1
\end{tikzcd}$$

5.11 Proposition. *Let $p\colon \mathbb{R}^1 \to S^1$ be the universal covering of S^1, let $x_0 \in S^1$, and let $y_0 \in \mathbb{R}^1$ be a point such that $p(y_0) = x_0$.*

1) For any path $\omega\colon [0,1] \to S^1$ such that $\omega(0) = x_0$ there exists a lift $\widetilde{\omega}\colon [0,1] \to \mathbb{R}^1$ satisfying $\widetilde{\omega}(0) = y_0$. Moreover, such lift is unique.



2) Let $\omega, \tau\colon [0,1] \to S^1$ be paths such that $\omega(0) = \tau(0) = x_0$, $\omega(1) = \tau(1)$ and $\omega \simeq \tau$. If $\widetilde{\omega}$, $\widetilde{\tau}$ are lifts of ω, τ, respectively, such that $\widetilde{\omega}(0) = \widetilde{\tau}(0) = y_0$ then $\widetilde{\omega}(1) = \widetilde{\tau}(1)$ and $\widetilde{\omega} \simeq \widetilde{\tau}$.

We postpone the proof of Proposition 5.11 for now. We will get back to it in Chapter 17 where we will show that it is a special case of a more general statement. Meanwhile we will show how it can be used to obtain Theorem 5.7.

5.12 Definition. Let $x_0 \in S^1$ and $y_0 \in \mathbb{R}$ be points such that $p(y_0) = x_0$. Let ω be a loop in S^1 based at x_0 and let $\widetilde{\omega}$ be the unique lift of ω such that $\widetilde{\omega}(0) = y_0$. The *degree* of ω is the integer $\deg(\omega)$ such that $\widetilde{\omega}(1) = y_0 + \deg(\omega)$.

In other words $\deg(\omega) = \widetilde{\omega}(1) - \widetilde{\omega}(0)$.

5.13 Note. Notice that $\deg(\omega)$ does not depend on the choice of the point y_0, i.e. it does not depend on the choice of the lift of ω. Indeed, if $y_0' \in \mathbb{R}$ is another point satisfying $p(y_0') = x_0$ then $y_0' = y_0 + n$

for some $n \in \mathbb{Z}$. Also, if $\widetilde{\omega}$ is the lift of a loop ω with $\widetilde{\omega}(0) = y_0$ then the lift $\widetilde{\omega}'$ of ω with $\widetilde{\omega}'(0) = y_0'$ is given by $\widetilde{\omega}'(s) = \widetilde{\omega}(s) + n$. This gives

$$\widetilde{\omega}'(1) - \widetilde{\omega}'(0) = (\widetilde{\omega}(1) + n) - (\widetilde{\omega}(0) + n) = \widetilde{\omega}(1) - \widetilde{\omega}(0)$$

In addition, by part 2) of Proposition 5.11 $\deg(\omega)$ depends only on the homotopy class of ω, and thus we obtain a well-defined function

$$\deg\colon \pi_1(S^1, x_0) \to \mathbb{Z}$$

Proof of Theorem 5.7. Let $x_0 \in S^1$. We will show that the function $\deg\colon \pi_1(S^1, x_0) \to \mathbb{Z}$ is an isomorphism of groups.

First, we will show that deg is onto. Let $y_0 \in p^{-1}(x_0)$. Given $n \in \mathbb{Z}$ consider the path $\widetilde{\omega}_n\colon [0,1] \to \mathbb{R}$ given by $\widetilde{\omega}_n(s) = y_0 + ns$ and let $\omega_n = p \circ \widetilde{\omega}_n$. Since $\widetilde{\omega}_n$ is the lift of ω_n such that $\widetilde{\omega}_n(0) = y_0$ and since $\widetilde{\omega}_n(1) = y_0 + n$ we obtain $\deg[\omega_n] = n$.

Next, we will check that deg is a 1-1 function. Let $[\omega], [\tau] \in \pi_1(S^1, x_0)$ be elements such that $\deg[\omega] = \deg[\tau]$. We need to show that $[\omega] = [\tau]$. Let $\widetilde{\omega}$, $\widetilde{\tau}$ be the lifts of ω, τ, respectively, such that $\widetilde{\omega}(0) = \widetilde{\tau}(0) = y_0$. By assumption we get

$$\widetilde{\omega}(1) = y_0 + \deg[\omega] = y_0 + \deg[\tau] = \widetilde{\tau}(1)$$

Since $\mathbb{R}$ is a simply connected space using Proposition 5.6 we obtain that $\widetilde{\omega} \simeq \widetilde{\tau}$. Therefore

$$\omega = p \circ \widetilde{\omega} \simeq p \circ \widetilde{\tau} = \tau$$

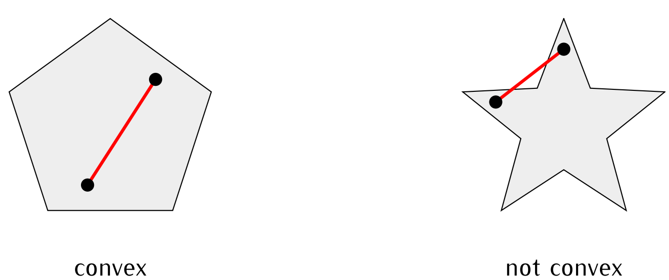
which gives $[\omega] = [\tau]$.

It remains to show that deg is a homomorphism of groups (exercise). $\square$

Exercises to Chapter 5

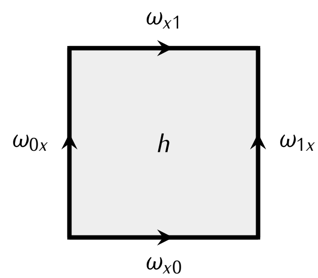
E5.1 Exercise. Prove Proposition 5.6.

E5.2 Exercise. a) A subspace $X \subseteq \mathbb{R}^n$ is *convex* is for any points $x_1, x_2 \in X$ the straight line segment joining these points is contained in X:



Show that if X is convex then it is simply connected.

b) Let Y be a topological space and let $h\colon [0,1]\times[0,1]\to Y$ be a continuous function. Consider paths ω_{x0}, ω_{x1}, ω_{0x}, ω_{1x} in Y which are defined by restricting h to the four edges of the square $[0,1]\times[0,1]$: $\omega_{x0}(s)=h(s,0)$, $\omega_{x1}(s)=h(s,1)$, $\omega_{0x}(s)=h(0,s)$ and $\omega_{1x}=h(1,s)$.



Show that the path $\omega_{x0}*\omega_{1x}$ is path homotopic to $\omega_{0x}*\omega_{x1}$.

E5.3 Exercise. Let S^n be the n-dimensional sphere:

$$S^n=\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1}\mid x_1^2+\ldots+x_{n+1}^2=1\}$$

Show that S^n is simply connected. (Hint: show that any path $\omega\colon[0,1]\to S^n$ is path homotopic to a path which is not onto.)

E5.4 Exercise. Recall that the Peano curve is a continuous function $\tau\colon[0,1]\to[0,1]\times[0.1]$ which is onto, and such that $\tau(0)=(0,0)$ and $\tau(1)=(1,1)$. Let $\mathrm{pr}_1\colon[0,1]\times[0,1]$ be the projection onto the first factor, $\mathrm{pr}_1(s,t)=s$ and let $\omega\colon[0,1]\to S^1$ be standard the degree 1 loop, $\omega(s)=(\cos(2\pi s),\sin(2\pi s))$. The composition $\omega\circ\mathrm{pr}_1\circ\tau\colon[0,1]\to S^1$ is a loop with the basepoint at $x_0=(1,0)\in S^1$. Compute the degree of $\omega\circ\mathrm{pr}_1\circ\tau$.

E5.5 Exercise. Show that the degree function $\deg\colon\pi_1(S^1,x_0)\to\mathbb{Z}$ defined in Note 5.13 is a group homomorphism.

E5.6 Exercise. Let $\omega\colon[0,1]\to S^1$ be a loop based at $x_0\in S^1$. Assume that there exists a point $x\in S^1$ such that the set $\omega^{-1}(x)$ consists of n points. Show that $-n\le\deg(\omega)\le n$.

6 | Some Applications

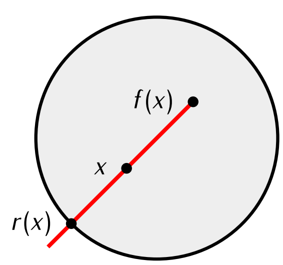
In this chapter we will use the computations of the fundamental group we completed so far to obtain a few interesting results. We start with the fact that was already mentioned in Chapter 1.

6.1 Proposition. *The circle S^1 is not a retract of the disc D^2.*

Proof. See the proof of Proposition 1.2. $\square$

6.2 Brouwer Fixed Point Theorem. *For each map $f\colon D^2 \to D^2$ there exists a point $x_0 \in D^2$ such that $f(x_0) = x_0$.*

Proof. We argue by contradiction. Assume that $f\colon D^2 \to D^2$ is a continuous function such that $f(x) \neq x$ for all $x \in D^2$. Define a function $r\colon D^2 \to S^1$ as follows. For a point $x \in D^2$ let $L_x \subseteq \mathbb{R}^2$ be the half-line that begins at $f(x)$ and that passes through the point x. This half-line intersects with S^1 at exactly one point. We set $r(x)$ to be the point of intersection:

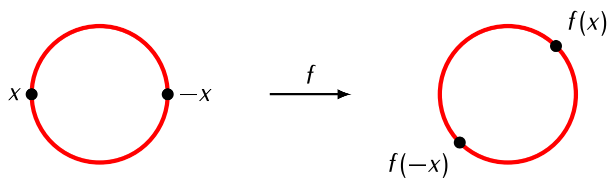


One can check that r is a continuous function (exercise). Since for $x \in S^1$ we have $r(x) = x$ the function r is a retraction of D^2 onto S^1. This contradicts Proposition 6.1.

$\square$

6.3 Borsuk-Ulam Theorem. *For each map $f\colon S^2 \to \mathbb{R}^2$ there exists $x \in S^2$ such that $f(x) = f(-x)$.*

6.4 Lemma. *Let $f\colon S^1 \to S^1$ be a function such that $f(-x) = -f(x)$ for all $x \in S^1$:*



For any $x_0 \in X$ the homomorphism $f_*\colon \pi_1(S^1, x_0) \to \pi_1(S^1, f(x_0))$ is non-trivial.

Proof. Exercise. ☐

Proof of Theorem 6.3. We argue by contradiction. Assume that $f\colon S^2 \to \mathbb{R}^2$ is a function such that $f(x) \neq f(-x)$ for all $x \in S^2$ and let $g\colon S^2 \to S^1$ be the function given by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

Notice that by assumption $f(x) - f(-x) \neq 0$ for all $x \in S^2$, so g is well defined. Notice also that $g(-x) = -g(x)$ for all $x \in S^2$. Let $j\colon S^1 \to S^2$ be the inclusion of S^1 onto the equator of S^2: $j(x, y) = (x, y, 0)$. The composition $gj\colon S^1 \to S^1$ satisfies the assumption of Lemma 6.4, so for any $x_0 \in S^1$ the homomorphism

$$(gj)_*\colon \pi_1(S^1, x_0) \to \pi_1(S^1, gj(x_0))$$

is non-trivial. On the other hand we have $gj = g|_{S^2_+} j$ where $g|_{S^2_+}$ is the restriction of g to the upper hemisphere $S^2_+ \subseteq S^2$. This gives a commutative diagram:

$$\begin{tikzcd}
\pi_1(S^1, x_0) \arrow[rr, "(gj)_*"] \arrow[dr, "j_*"'] & & \pi_1(S^1, gj(x_0)) \\
& \pi_1(S^2_+, j(x_0)) \arrow[ur, "(g|_{S^2_+})_*"'] &
\end{tikzcd}$$

Since $S^2_+ \cong D^2$ by Proposition 5.3 we get that the group $\pi_1(S^2_+)$ is trivial, and so $(gj)_*$ is the trivial homomorphism. Thus we obtain a contradiction. ☐

6.5 Corollary. *There does not exist an embedding of S^2 into $\mathbb{R}^2$.*

Proof. An embedding $S^2 \to \mathbb{R}^2$ would be a 1-1 map which by Theorem 6.3 does not exist. ☐

6.6 Corollary. *If $A_1, A_2, A_3 \subseteq S^2$ are closed sets such that $A_1 \cup A_2 \cup A_3 = S^2$ then one of these sets contains a pair of antipodal points $\{x, -x\}$.*

Proof. For $x \in S^2$ let $d_i(x)$ denote the distance from x to the set A_i:

$$d_i(x) = \inf\{\|x - y\| \mid y \in A_i\}$$

The function $d_i\colon S^2 \to \mathbb{R}$ is continuous. Also, since A_i is closed we have $d_i(x) = 0$ if and only if $x \in A_i$. Consider the function $d_{12}\colon S^2 \to \mathbb{R}^2$ given by $d_{12}(x) = (d_1(x), d_2(x))$. By Theorem 6.3 there exists a point $x_0 \in S^2$ such that $d_{12}(x_0) = d_{12}(-x_0)$, i.e. $d_1(x_0) = d_1(-x_0)$ and $d_2(x_0) = d_2(-x_0)$. It follows that if $d_1(x_0) = 0$ then also $d_1(-x_0) = 0$, and so $\{x_0, -x_0\} \subseteq A_1$. Likewise, if $d_2(x_0) = 0$ then $\{x_0, -x_0\} \subseteq A_2$. If $d_1(x_0) > 0$ and $d_2(x_0) > 0$ then $\{x_0, -x_0\} \subseteq S^2 \smallsetminus (A_1 \cup A_2) \subseteq A_3$. □

6.7 The Fundamental Theorem of Algebra. *If $P(x)$ is a polynomial with coefficients in $\mathbb{C}$ and $\deg P(x) > 0$ then $P(z_0) = 0$ for some $z_0 \in \mathbb{C}$.*

Proof. We start with a few preliminary observations. We will consider S^1 as the subspace of the complex plane:

$$S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$$

and we will take $1 \in \mathbb{C}$ as the basepoint of S^1. Consider the degree isomorphism $\deg\colon \pi_1(S^1, 1) \to \mathbb{Z}$. Notice that if $\omega_n\colon [0,1] \to S^1$ is the loop given by $\omega_n(s) = e^{2\pi i n s}$ then $\deg([\omega_n]) = n$.

We will prove Theorem 6.7 by contradiction. Assume that $P(x) = x^n + a_{n-1}x^n + \cdots + a_0$ is a polynomial with complex coefficients such that $n > 0$ and that $P(z) \neq 0$ for all $z \in \mathbb{C}$. For $r \geq 0$ let $\sigma_r\colon [0,1] \to S^1$ be a loop based at $1 \in S^1$ given by

$$\sigma_r(s) = \frac{P(re^{2\pi i s})/P(r)}{\left\|P(re^{2\pi i s})/P(r)\right\|}$$

We have $[\sigma_r] \in \pi_1(S^1, 1) \cong \mathbb{Z}$.

Claim 1. For each $r \geq 0$ we have $[\sigma_r] = 0$.

For $r = 0$ this is true since σ_0 is the constant loop. For $r > 0$ the map $h\colon [0,1] \times [0,1] \to S^1$ defined by $h(s,t) = \sigma_{tr}(s)$ gives a homotopy between σ_0 and σ_r so $[\sigma_r] = [\sigma_0] = 0$.

Claim 2. If $r > \max\left\{1, \left\|a_{n-1}\right\| + \cdots + \left\|a_0\right\|\right\}$ then $[\sigma_r] \neq 0$.

Indeed, assume that r satisfies the assumption of Claim 2. If $\|z\| = r$ then

$$\begin{aligned}
\|z^n\| &= r \cdot \|z^{n-1}\| \\
&\geq (\|a_{n-1}\| + \cdots + \|a_0\|) \cdot \|z^{n-1}\| \\
&\geq \|a_{n-1}\| \cdot \|z^{n-1}\| + \cdots + \|a_1\| \cdot \|z\| + \|a_0\| \\
&\geq \|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\|
\end{aligned}$$

For $t \in [0,1]$ take the polynomial $P_t(x) = x^n + t(a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$. The inequality above shows that $P_t(z) \neq 0$ for all $z \in \mathbb{C}$ such that $\|z\| = r$. Define $h\colon [0,1] \times [0,1] \to S^1$ by

$$h(s,t) = \frac{P_t(re^{2\pi i s})/P_t(r)}{\left\|P_t(re^{2\pi i s})/P_t(r)\right\|}$$

The map h gives a path homotopy between σ_r and the loop ω_n defined above. Therefore $\deg([\sigma_r]) = \deg([\omega_n]) = n \neq 0$.

Since Claim 1 and Claim 2 contradict each other we are done. $\square$

Exercises to Chapter 6

E6.1 Exercise. Prove Lemma 6.4

E6.2 Exercise. Let $f\colon S^2 \to \mathbb{R}^2$ be a function such that $f(-x) = -f(x)$ for all $x \in S^2$. Show that there exists a point $x_0 \in S^2$ such that $f(x_0) = 0$

E6.3 Exercise. Let $\omega, \tau\colon [0,1] \to [0,1] \times [0,1]$ be paths in the square such that $\omega(0) = (0,0)$, $\omega(1) = (1,1)$, $\tau(0) = (1,0)$, and $\tau(1) = (0,1)$. Show that $\omega(s) = \tau(t)$ for some $s, t \in [0,1]$. (Hint: use Brouwer fixed point theorem.)

7 | Higher Homotopy Groups

Let $n \geq 1$. Recall that $D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$ is the n-dimensional closed disc and $S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1\}$ is the $(n-1)$-dimensional sphere. While most of the results proved in Chapter 6 are stated in terms of spheres and discs of dimension 1 or 2 it is easy to formulate their possible generalizations to higher dimensions:

1) For any $n \geq 1$ the sphere S^n is not a retract of the disc D^{n+1} (cf. 6.1).

2) Let $n \geq 1$. For each map $f\colon D^n \to D^n$ there exists a point $x_0 \in D^n$ such that $f(x_0) = x_0$ (cf. 6.2).

3) Let $n \geq 1$. For each map $f\colon S^n \to \mathbb{R}^n$ there exists $x \in S^n$ such that $f(x) = f(-x)$ (cf. 6.3).

4) If $n \geq 1$ and $A_1, \dots, A_{n+1} \subseteq S^n$ are closed sets such that $A_1 \cup \dots \cup A_{n+1} = S^n$ then one of these sets contains a pair of antipodal points $\{x, -x\}$ (cf. 6.6).

All these generalizations are in fact true. However, if one tries to prove them mimicking the proofs used in the low dimensional cases it turns out that some additional machinery is needed. For example, the main ingredient of the proof of the fact that S^1 is not a retract of D^2 was the observation that the fundamental group is a functor $\pi_1\colon \mathbf{Top}_* \to \mathbf{Gr}$ such that $\pi_1(D^2) \cong \{1\}$ and $\pi_1(S^1) \ncong \{1\}$. Analogously, in order to prove that S^n is not a retract of D^{n+1} it would be useful to have a functor $F\colon \mathbf{Top}_* \to \mathbf{Gr}$ satisfying $F(D^{n+1}) \cong \{1\}$ and $F(S^n) \ncong \{1\}$. Assuming that there exists a retraction $r\colon D^{n+1} \to S^n$ we would get a commutative diagram of pointed spaces:

$$\begin{tikzcd} S^n \arrow[rr, "\mathrm{id}_{S^n}"] \arrow[dr, "i"'] & & S^n \\ & D^{n+1} \arrow[ur, "r"'] & \end{tikzcd}$$

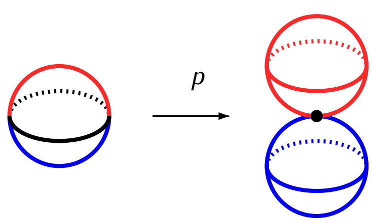
where $i\colon S^n \to D^{n+1}$ is the inclusion map. Applying the functor F we would obtain a commutative diagram of groups:

$$\begin{tikzcd} F(S^n) \arrow[rr, "\mathrm{id}_{F(S^n)}"] \arrow[dr, "F(i)"'] & & F(S^n) \\ & F(D^{n+1}) \arrow[ur, "F(r)"'] & \end{tikzcd}$$

This would imply that $F(r)$ is onto which is impossible since $F(D^{n+1})$ a trivial group and $F(S^n)$ is non-trivial.

In the above argument we cannot take F to be the fundamental group functor since the group $\pi_1(S^n)$ is trivial for $n > 1$ (exercise). A functor which is useful in this context is the n-th homotopy group functor π_n. Its construction can be described similarly to the construction of π_1 given in (3.18). Take $s_0 = (1, 0, \dots, 0)$ to be the basepoint of S^n. Given a pointed space (X, x_0) we will say that basepoint preserving maps $\omega, \tau\colon (S^n, s_0) \to (X, x_0)$ are homotopic if there exists a continuous function $h\colon S^n \times [0,1] \to X$ such that $h(s, 0) = \omega(s)$, $h(s, 1) = \tau(s)$ for all $s \in S^n$ and $h(s_0, t) = x_0$ for all $t \in [0, 1]$. Let $\pi_n(X, x_0)$ be the set of homotopy classes $[\omega]$ of basepoint preserving maps $\omega\colon (S^n, s_0) \to (X, x_0)$.

In order to describe multiplication in $\pi_n(X, x_0)$ denote by $S^n \vee S^n$ the space obtained by taking two copies of S^n and identifying their basepoints. There is a *pinch map* $p\colon S^n \to S^n \vee S^n$ that maps the upper hemisphere of S^n onto one copy of $S^n \subseteq S^n \vee S^n$, the lower hemisphere onto the second copy, and the equator of S^n to the basepoint of $S^n \vee S^n$:



Given two basepoint preserving maps $\omega, \tau\colon (S^n, s_0) \to (X, x_0)$ we can define a map $\omega \vee \tau\colon S^n \vee S^n \to X$ that maps the first copy of S^n using ω and the second copy using τ. We set: $[\omega] \cdot [\tau] = [(\omega \vee \tau) \circ p]$.

One can check that $\pi_n(X, x_0)$ taken with this multiplication is a group. The trivial element in this group is given by the homotopy class of the constant map $(S^n, s_0) \to (X, x_0)$. For $[\omega] \in \pi_n(X, x_0)$ we have $[\omega]^{-1} = [\omega \circ f]$ where $f\colon S^n \to S^n$ is the map given by $f(x_1, x_2, \dots, x_n, x_{n+1}) = (x_1, x_2, \dots, x_n, -x_{n+1})$.

The assignments $(X, x_0) \mapsto \pi_n(X, x_0)$ define a functor $\pi_n\colon \mathbf{Top}_* \to \mathbf{Gr}$.

7.1 Theorem.

1) *For any $m, n \geq 1$ and $x_0 \in D^m$ the group $\pi_n(D^m, x_0)$ is trivial.*
2) *For any $n \geq 1$ and $x_0 \in S^n$ there is an isomorphism $\pi_n(S^n) \cong \mathbb{Z}$.*

The proof of part 1) is easy and similar to the proof that $\pi_1(D^m)$ is trivial for all $m \geq 1$ (see Proposition 5.3). The second part is harder and requires more work that the proof that $\pi_1(S^1) \cong \mathbb{Z}$. Since our main focus in these notes is the fundamental group we will skip this proof.

Theorem 7.1 combined with the argument outlined above implies that statement 1) on page 32 holds. Using this, by the same argument as in the proof of Theorem 6.2 we obtain statement 2). Higher homotopy groups can be also used to prove statement 3) which in turn implies statement 4).

7.2 Note. Recall that in Example 2.10 we defined a functor π_0 that assigns to a space X the set $\pi_0(X)$ of path connected components of X. This functor is related to the functors π_n constructed above as follows. Recall that the 0-dimesional sphere is a discrete space consisting of two points $S^0 = \{-1, 1\}$. Choose $1 \in S^0$ as the basepoint. If (X, x_0) is a pointed space then any basepoint preserving map $f\colon (S^0, 1) \to (X, x_0)$ is determined by the value of $f(-1)$, and this value can be an arbitrary point of X. This gives a bijection:

$$\begin{array}{ccc} \begin{pmatrix} \text{basepoint preserving maps} \\ f\colon (S^0,1) \to (X, x_0) \end{pmatrix} & \cong & \begin{pmatrix} \text{points} \\ \text{of } X \end{pmatrix} \\ f & \mapsto & f(-1) \end{array}$$

It is also easy to see that giving a homotopy between maps $f, g\colon (S^0, 1) \to (X, x_0)$ is the same as giving a path between the points $f(-1)$ and $g(-1)$. This means that maps f, g are homotopic if and only if the points $f(-1)$ and $g(-1)$ are in the same path connected component of X. As a consequence we obtain a bijection:

$$\begin{pmatrix} \text{homotopy classes of maps} \\ f\colon (S^0,1) \to (X, x_0) \end{pmatrix} \cong \begin{pmatrix} \text{path connected} \\ \text{components of } X \end{pmatrix} = \pi_0(X)$$

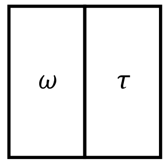
The difference between the functors π_0 and π_n for $n > 0$ is that $\pi_0(X)$ is in general just a set, not a group. However, for any pointed space (X, x_0) the set $\pi_0(X)$ has a natural choice of a basepoint given by the path connected component of x_0 (or equivalently, by the homotopy class of the constant map $(S^0, 1) \to (X, x_0)$). This means that we can consider π_0 as a functor

$$\pi_0\colon \mathbf{Top}_* \to \mathbf{Set}_*$$

where $\mathbf{Set}_*$ denotes the category of pointed sets i.e. sets equipped with a basepoint.

The above description of higher homotopy groups generalizes the construction of the fundamental group given in (3.18), using maps $S^1 \to X$. We can also describe groups $\pi_n(X, x_0)$ in a way paralleling the construction of $\pi_1(X, x_0)$ that used loops $[0, 1] \to X$. Namely, let $I^n = \prod_{i=1}^n [0, 1]$ be the n-dimensional cube, and let $\partial I^n = I^n \smallsetminus \prod_{i=1}^n (0, 1)$. Given a pointed space (X, x_0) by $\omega\colon (I^n, \partial I^n) \to (X, x_0)$ we will denote a map $\omega\colon I^n \to X$ such that $\omega(\partial I^n) = x_0$. We will say that to maps $\omega, \tau\colon (I^n, \partial I^n) \to (X, x_0)$ are homotopic if there exists a map $h\colon I^n \times [0, 1] \to X$ such that $h(s, 0) = \omega(s)$, $h(s, 1) = \tau(s)$ for any $s \in I^n$, and $h(\partial I^n \times [0, 1]) = x_0$. We define $\pi_n(X, x_0)$ as the set of homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. For $\omega, \tau\colon (I^n, \partial I^n) \to (X, x_0)$ define $\omega * \tau\colon (I^n, \partial I^n) \to (X, x_0)$ by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$



One can check that this induces a well-defined associative multiplication

$$\pi_n(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)$$

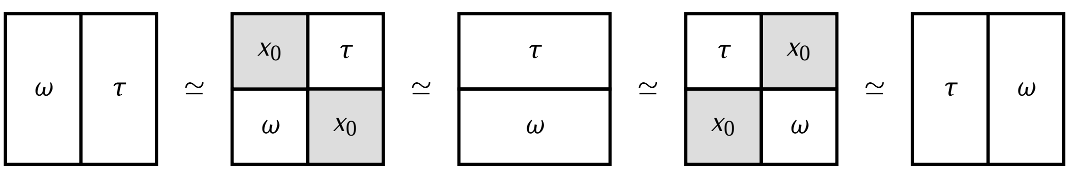
given by $[\omega] \cdot [\tau] = [\omega * \tau]$ which makes $\pi_n(X, x_0)$ into a group. The trivial element of $\pi_n(X, x_0)$ is the homotopy class of the constant map $c_{x_0} \colon I^n \to X$. Also, for $[\omega] \in \pi_1(X, x_0)$ we have $[\omega]^{-1} = [\overline{\omega}]$ where $\overline{\omega} \colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$\overline{\omega}(s_1, s_2, \dots, s_n) = (1 - s_1, s_2, \dots, s_n)$$

We will see later that the fundamental group of a space need not be commutative. By contrast we have:

7.3 Theorem. *For $n \geq 2$ then the group $\pi_n(X, x_0)$ is abelian for any pointed space (X, x_0).*

Proof. A homotopy $\omega * \tau \simeq \tau * \omega$ can be depicted as follows:



The shaded squares in the pictures are mapped to the basepoint $x_0 \in X$.

□

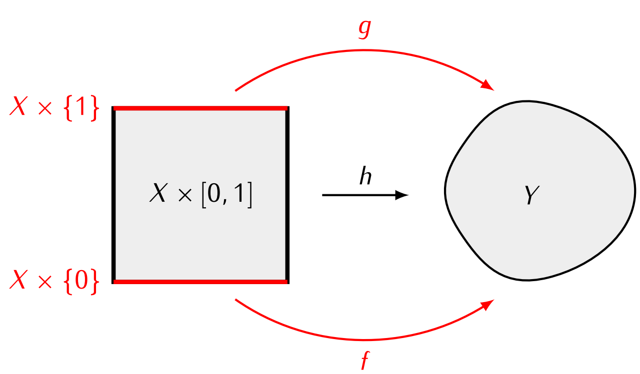
Exercises to Chapter 7

E7.1 Exercise. Show that the fundamental group of the sphere S^n is trivial for $n > 1$. (Hint: show first that any loop $\omega \colon [0, 1] \to S^n$ is homotopic to a loop $\omega' \colon [0, 1] \to S^n$ which is not onto.)

8 | Homotopy Invariance

So far we computed the fundamental group for very few spaces. In order to extend these computations to other spaces we will use three basic tools: homotopy invariance of π_1, the product formula for π_1, and the van Kampen theorem. In this chapter we discuss the first of these topics and in the subsequent ones we deal with the other two.

8.1 Definition. Let $f, g\colon X \to Y$ be continuous functions. A *homotopy* between f and g is a continuous function $h\colon X \times [0,1] \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$:



If such homotopy exists then we say that the functions f and g are *homotopic* and we write $f \simeq g$. We will also write $h\colon f \simeq g$ to indicate that h is a homotopy between f and g.

8.2 Note. Given a homotopy $h\colon X \times [0,1] \to Y$ it will be convenient denote by $h_t\colon X \to Y$ the function defined by $h_t(x) = h(x, t)$. If $h\colon f \simeq g$ then $h_0 = f$ and $h_1 = g$.

8.3 Example. Any two functions $f, g\colon X \to \mathbb{R}^n$ are homotopic. Indeed, define $h\colon X \times [0,1] \to \mathbb{R}$ by $h(x, t) = (1 - t)f(x) + tg(x)$. Then $h_0 = f$ and $h_1 = g$.

A useful generalization of Definition 8.1 is the notion of a relative homotopy:

8.4 Definition. Let X be a space and let $A \subseteq X$. If $f, g\colon X \to Y$ are functions such that $f|_A = g|_A$ then we say that f and g are *homotopic relative to* A if there exists a homotopy $h\colon X \times [0,1] \to Y$ such that $h_0 = f$, $h_1 = g$ and $h_t|_A = f|_A = g|_A$ for all $t \in [0,1]$. In such case we write $f \simeq g$ (rel A).

8.5 Example. Let $\omega, \tau\colon [0,1] \to X$ be paths in X. Recall that path homotopy is defined only if $\omega|_{\{0,1\}} = \tau|_{\{0,1\}}$ and it is given by a map $h\colon [0,1] \times [0,1] \to X$ such that $h_0 = \omega$, $h_1 = \tau$ and $h_t|_{\{0,1\}} = \omega|_{\{0,1\}} = \tau|_{\{0,1\}}$ for each $t \in [0,1]$. Thus, in the paths ω and τ are path homotopic if and only if $\omega \simeq \tau$ (rel $\{0,1\}$).

8.6 Definition. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. If such maps exist we say that the spaces X and Y are *homotopy equivalent* and we write $X \simeq Y$.

8.7 Note. If f and g are maps as in Definition 8.6 then we say that g is a *homotopy inverse* of f.

8.8 Example. We will show $\mathbb{R}^n$ is homotopy equivalent to the space $\{*\}$ consisting of a single point. Let $f\colon \mathbb{R}^n \to \{*\}$ be the constant function and let $g\colon \{*\} \to \mathbb{R}^n$ be given by $f(*) = x_0$ for some $x_0 \in \mathbb{R}^n$. We have $fg = \mathrm{id}_{\{*\}}$ so $fg \simeq \mathrm{id}_{\{*\}}$. On the other hand by Example 8.3 any two functions into $\mathbb{R}^n$ are homotopic, so in particular $gf \simeq \mathrm{id}_{\mathbb{R}^n}$

8.9 Note. Example 8.8 shows that a homotopy inverse of a homotopy equivalence $f\colon X \to Y$ in general is not unique: any function $g\colon \{*\} \to \mathbb{R}^n$ is a homotopy inverse of the constant function $f\colon \mathbb{R}^n \to \{*\}$.

8.10 Definition. If X is a space such that $X \simeq \{*\}$ then we say that X is a *contractible space*.

8.11 Proposition. *Let X be a topological space. The following conditions are equivalent:*

1) *X is contractible;*
2) *the identify map id_X is homotopic to a constant map;*
3) *for each space Y and any maps $f, g\colon Y \to X$ we have $f \simeq g$.*

Proof. Exercise. □

Many examples of homotopy equivalences can be obtained using deformation retractions:

8.12 Definition. A subspace $A \subseteq X$ is a *deformation retract* of a space X if there exists a homotopy $h\colon X \times [0,1] \to X$ such that

1) $h_0 = \mathrm{id}_X$
2) $h_t|_A = \mathrm{id}_A$ for all $t \in [0,1]$

3) $h_1(x) \in A$ for all $x \in X$

In such case we say that h is a *deformation retraction* of X onto A.

8.13 Proposition. *If $A \subseteq X$ is a deformation retract of X then $A \simeq X$.*

Proof. Let $h\colon X \times [0,1] \to X$ be a deformation retraction, let $r\colon X \to A$ be given by $r(x) = h_1(x)$ and let $j\colon A \to X$ be the inclusion map. We have $rj = \mathrm{id}_A$. Also, h is a homotopy between id_X and jr. $\square$

8.14 Example. For any $n > 0$ the sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \smallsetminus \{0\}$. Indeed, a deformation retraction $h\colon \mathbb{R}^n \smallsetminus \{0\} \to S^{n-1}$ is given by

$$h(x,t) = \frac{x}{(1-t) + t\|x\|}$$

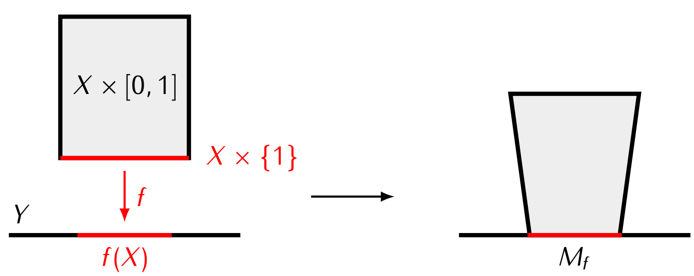
As a consequence $\mathbb{R}^n \smallsetminus \{0\} \simeq S^{n-1}$.

Interesting examples of homotopy equivalences can be also obtained using the constructions of a mapping cylinder and a mapping cone:

8.15 Definition. Let $f\colon X \to Y$ be a continuous function. The *mapping cylinder* of f is the space

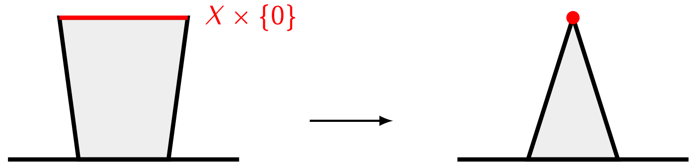
$$M_f = (X \times [0,1] \sqcup Y)/\sim$$

where $\sim$ is the equivalence relation given by $(x,1) \sim f(x)$ for all $x \in X$.



The *mapping cone* of f is the space obtained from M_f by collapsing the subspace $X \times \{0\} \subseteq M_f$ to a point:

$$C_f = M_f/(X \times \{0\})$$



M_f C_f

8.16 Proposition. *For any map $f\colon X \to Y$ we have $M_f \simeq Y$.*

Proof. Exercise. □

8.17 Proposition. *Let $f, g\colon X \to Y$ be continuous functions. If $f \simeq g$ then $C_f \simeq C_g$.*

Proof. Exercise. □

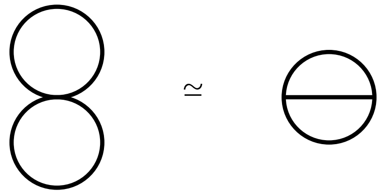
8.18 Example. Consider maps $f, g\colon \{-1, 1\} \to S^1$ where f is a constant map and g is non-constant (e.g. g maps 1 and -1 to antipodal points of S^1). Mapping cylinders of the these functions can be depicted as follows:

M_f M_g

The mapping cones, in turn, look as follows:

C_f C_g

Notice that $f \simeq g$, and so $C_f \simeq C_g$. Notice also that the space C_f is homeomorphic to $S^1 \vee S^1$ while C_g is homeomorphic to the space obtained as a union of S^1 and one of its diagonals. In effect we obtain a homotopy equivalence:



Our next goal is to examine how the fundamental group behaves with respect to homotopic maps and homotopy equivalent spaces. First, recall that a map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ induces a homomorphism of fundamental groups $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ which is given by $f_*([\omega]) = [f \circ \omega]$. We have:

8.19 Proposition. *If $f, g\colon (X, x_0) \to (Y, y_0)$ are maps of pointed spaces such that $f \simeq g$ (rel $\{x_0\}$) then $f_* = g_*$.*

Proof. For $[\omega] \in \pi_1(X, x_0)$ we want to show that $f_*([\omega]) = g_*([\omega])$, or equivalently that

$$f \circ \omega \simeq g \circ \omega \quad (\text{rel } \{0, 1\})$$

Let $h\colon X \times [0, 1] \to Y$ be a homotopy between f and g (rel $\{x_0\}$). Then the map

$$h \circ (\omega \times \mathrm{id}_{[0,1]})\colon [0, 1] \times [0, 1] \to Y$$

gives a path homotopy between $f \circ \omega$ and $g \circ \omega$. ☐

Proposition 8.19 can be generalized to the setting where we do not assume that homotopy preserves basepoints. Recall (4.2) that if Y is a space, $y_0, y_1 \in Y$ then a path τ in Y with $\tau(0) = y_0$ and $\tau(1) = y_1$ induces an isomorphism $s_\tau\colon \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ given by $s_\tau([\omega]) = [\overline{\tau} * \omega * \tau]$.

8.20 Proposition. *Let $f, g\colon X \to Y$ be homotopic maps and let $h\colon f \simeq g$. For $x_0 \in X$ let τ be the path in Y given by $\tau(t) = h(x_0, t)$. The following diagram commutes:*

$$\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[dd, "s_\tau", "\cong"'] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(Y, g(x_0))
\end{tikzcd}$$

Proof. Exercise. ☐

8.21 Corollary. *If $f, g\colon X \to Y$ are maps such that $f \simeq g$ then the homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism (or is trivial or is 1-1 or onto) if and only if the homomorphism $g_*\colon \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$ has the same property.*

8.22 Proposition. *If $f\colon X \to Y$ is a homotopy equivalence then for any $x_0 \in X$ the homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. Consider the sequence of homomorphisms

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, gf(x_0)) \xrightarrow{f_*} \pi_1(Y, fgf(x_0))$$

Composition of the first two homomorphisms satisfies $g_*f_* = (gf)_*$. Since $gf \simeq \mathrm{id}_X$ and id_{X*} is an isomorphism by Proposition 8.21 we obtain that g_*f_* is an isomorphism. This implies in particular that g_* is onto. Similarly, composing the last two homomorphisms we obtain $f_*g_* = (fg)_*$ and since $fg \simeq \mathrm{id}_Y$ we get that f_*g_* is an isomorphism. This means that g_* is 1-1. As a consequence g_* is an isomorphism. It follows that the first homomorphism f_* is a composition of two isomorphisms: $f_* = g_*^{-1}(g_*f_*)$, and so f_* is an isomorphism.

□

8.23 Corollary. *If X, Y are path connected spaces and $X \simeq Y$ then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ for any $x_0 \in X$, $y_0 \in Y$.*

Proof. Let $f\colon X \to Y$ be a homotopy equivalence. By Proposition 8.22 we get an isomorphism $f_*\colon \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, f(x_0))$. Since Y is path connected by Corrolary 4.3 we also have $\pi_1(Y, f(x_0)) \cong \pi_1(Y, y_0)$. □

8.24 Note. In the proof above we used only that Y is path connected, so the assumption in Corollary 8.23 that both X and Y are path connected may seem too strong. However, if Y is path connected and $X \simeq Y$ then X must be path connected as well (exercise).

8.25 Example. As we have seen before (8.14) the space $\mathbb{R}^n \smallsetminus \{0\}$ is homotopy equivalent to the sphere S^{n-1}. This gives $\pi_1(\mathbb{R}^n \smallsetminus \{0\}) \cong \pi_1(S^{n-1})$. In particular $\pi_1(\mathbb{R}^2 \smallsetminus \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$.

8.26 Example. Let Θ be the space obtained as a union of S^1 and one of its diagonals. By Example 8.18 this space is homotopy equivalent to $S^1 \vee S^1$, so $\pi_1(\Theta) \cong \pi_1(S^1 \vee S^1)$.

Exercises to Chapter 8

E8.1 Exercise. Recall that if X is a space then $\pi_0(X)$ denotes the set of path connected components of X. If $x \in X$ then by $[x] \in \pi_0(X)$ we will denote the path connected component of the point x. Recall that a continuous function $f\colon X \to Y$ induces a map of sets $f_*\colon \pi_0(X) \to \pi_0(Y)$ given by $\pi_0([x]) = [f(x)]$. Show that if f is a homotopy equivalence then f_* is a bijection.

E8.2 Exercise. Prove Proposion 8.11.

E8.3 Exercise. Let $f, g\colon X \to Y$ be two homeomorphisms and let $f^{-1}, g^{-1}\colon Y \to X$ be their respective inverses. Show that if $f \simeq g$ then $f^{-1} \simeq g^{-1}$.

E8.4 Exercise. a) For $i = 1, 2$ let X_i be a topological space and let $Y_i \subseteq X_i$. Assume that we have a commutative diagram:

$$\begin{tikzcd} Y_1 \arrow[r, "j_1"] \arrow[d, "f'"'] & X_1 \arrow[r, "r_1"] \arrow[d, "f"] & Y_1 \arrow[d, "f'"] \\ Y_2 \arrow[r, "j_2"'] & X_2 \arrow[r, "r_2"'] & Y_2 \end{tikzcd}$$

where $j_i\colon Y_i \to X_i$ is the inclusion map, and $r_i\colon X_i \to Y_i$ is a retraction. Show that if f is a homotopy equivalence then f' is a homotopy equivalence as well.

b) Let X be a contractible space and let $A \subseteq X$ be a retract of X. Show that A is contractible.

E8.5 Exercise. Let X, Y be topological spaces. Show that any map $f\colon X \times \mathbb{R}^k \to Y \times \mathbb{R}^k$ is homotopic to a map $g \times \mathrm{id}_{\mathbb{R}^k}\colon X \times \mathbb{R}^k \to Y \times \mathbb{R}^k$ for some $g\colon X \to Y$.

E8.6 Exercise. For spaces X and Y let $[X, Y]$ denote the set of homotopy classes of maps $X \to Y$. That is, each map $f\colon X \to Y$ defines an element $[f] \in [X, Y]$ and $[f] = [f']$ if $f \simeq f'$. Notice that any map $g\colon X \to X'$ defines a function $g^*\colon [X', Y] \to [X, Y]$ given by $g^*([f]) = [fg]$.

Given a map $g\colon X \to X'$ show that the following conditions are equivalent:

1) The map g is a homotopy equivalence.
2) For each space Z the function $g^*\colon [X', Z] \to [X, Z]$ is a bijection.

E8.7 Exercise. The antipodal map $f\colon S^n \to S^n$ is the map given by $f(x) = -x$. Show that if $g\colon S^n \to S^n$ is any map such that $g(x) \neq x$ for all $x \in S^n$ then $g \simeq f$.

E8.8 Exercise. Let X be a topological space. Assume that $f, g\colon X \to S^n$ are maps such that for some non-empty open set $U \subseteq S^n$ we have $f^{-1}(U) = g^{-1}(U) = V \subseteq X$ and $f|_V = g|_V$. Show that $f \simeq g$.

E8.9 Exercise. Prove Proposition 8.17.

E8.10 Exercise. Prove Proposition 8.20.

E8.11 Exercise. Let M be the Möbius band and let ∂M denote the boundary of M. Show that ∂M is not a retract of M.

E8.12 Exercise. Recall (8.15) that the cone of a map $f\colon X \to Y$ is the space

$$C_f = (X \times [0, 1] \sqcup Y)/\sim$$

where $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. We can consider Y as a subspace of C_f. Show that Y is contractible if any only if for every map $f\colon X \to Y$ the space Y is a retract of C_f.

E8.13 Exercise. a) Let $f\colon S^1 \to X$ be a continuous function. Show that f is homotopic to a constant map if and only if there exists $\bar{f}\colon D^2 \to X$ such that $\bar{f}|_{S^1} = f$.

b) Show that if $f\colon S^1 \to S^1$ is homotopic to a constant map then there exists $x_0 \in S^1$ such that $f(x_0) = x_0$.

E8.14 Exercise. Let $F\colon D^2 \to D^2$ be a function such that $F(S^1) \subseteq S^1$, and let $f\colon S^1 \to S^1$ be given by $f(x) = F(x)$ for all $x \in S^1$. Show that if f is not homotopic to a constant map, then for each function $G\colon D^2 \to D^2$ there is a point $x_0 \in D^2$ such that $F(x_0) = G(x_0)$.

E8.15 Exercise. Recall that for $n \geq 1$ multiplication in the group $\pi_n(X, x_0)$ can be defined using the pinch map $p\colon S^n \to S^n \vee S^n$: if $[\omega], [\tau] \in \pi_n(X, x_0)$ then $[\omega]\cdot[\sigma] = [(\omega \vee \sigma)\circ p]$. The goal of this exercise is to generalize this observation.

For pointed spaces (X, x_0) and (Y, y_0) let $[X, Y]_*$ denote the set of pointed homotopy classes of maps $X \to Y$. That is, each pointed map $f\colon (X, x_0) \to (Y, y_0)$ defines an element $[f] \in [X, Y]_*$ and $[f] = [g]$ if $f \simeq g$ relative the basepoint. Let (X, x_0) be a space such that

(i) for each space (Y, y_0) the set $[X, Y]_*$ has the structure of a group;
(ii) for each pointed map $f\colon (Y, y_0) \to (Y', y_0')$ the induced function $f_*\colon [X, Y]_* \to [X, Y']_*$ is a group homomorphism.

a) Show that for any space space (Y, y_0) there exists a bijection of sets $\varphi_Y\colon [X\vee X, Y]_* \to [X, Y]_*\times[X, Y]_*$ such that for any pointed map $f\colon (Y, y_0) \to (Y', y_0')$ the following diagram commutes:

$$\begin{tikzcd}
{[X \vee X, Y]_*} \arrow[r, "\varphi_Y", "\cong"'] \arrow[d, "f_*"'] & {[X, Y]_* \times [X, Y]_*} \arrow[d, "f_* \times f_*"] \\
{[X \vee X, Y']_*} \arrow[r, "\cong", "\varphi_{Y'}"'] & {[X, Y']_* \times [X, Y']_*}
\end{tikzcd}$$

b) Show that there exists a map $p\colon X \to X \vee X$ such that for each space (Y, y_0) the multiplication in the group $[X, Y]_*$ is given by $[f]\cdot[g] = [(f \vee g)\circ p]$.

Hint: For a space (Y, y_0) let μ_Y denote the multiplication in the group $[X, Y]_*$:

$$\mu_Y\colon [X, Y]_* \times [X, Y]_* \to [X, Y]_*$$

Notice that the condition (ii) above is equivalent to saying that for any map $f\colon (Y, y_0) \to (Y', y_0')$ the following diagram commutes:

$$\begin{tikzcd}
{[X, Y]_* \times [X, Y]_*} \arrow[r, "\mu_Y"] \arrow[d, "f_* \times f_*"'] & {[X, Y]_*} \arrow[d, "f_*"] \\
{[X, Y']_* \times [X, Y']_*} \arrow[r, "\mu_{Y'}"'] & {[X, Y']_*}
\end{tikzcd}$$

9 | The Product Formula

Recall that if G, H are groups then their direct product is the group $G \times H$ whose elements are pairs (g, h) where $g \in G, h \in H$ and multiplication is given by $(g, h) \cdot (g', h') = (gg', hh')$. We have:

9.1 Theorem. *If* (X_1, x_1), (X_2, x_2) *are pointed spaces then*

$$\pi_1(X_1 \times X_2, (x_1, x_2)) \cong \pi_1(X_1, x_1) \times \pi_1(X_2, x_2)$$

Proof. For $i = 1, 2$ let $p_i \colon X_1 \times X_2 \to X_i$ be the projection map $p_i(x_1, x_2) = x_i$. These maps induce homomorphisms $p_{i*} \colon \pi_1(X_1 \times X_2, (x_1, x_2)) \to \pi_1(X_i, x_i)$. This defines a homomorphism

$$p_{1*} \times p_{2*} \colon \pi_1(X_1 \times X_2, (x_1, x_2)) \to \pi_1(X_1, x_1) \times \pi_1(X_2, x_2)$$

where $p_{1*} \times p_{2*}([\omega]) = (p_{1*}([\omega]), p_{2*}([\omega]))$. Next, for $i = 1, 2$ let ω_i be a loop in (X_i, x_i), and let $\omega_1 \times \omega_2$ be the loop in $X_1 \times X_2$ given by $\omega_1 \times \omega_2(s) = (\omega_1(s), \omega_2(s))$. One can check that the map

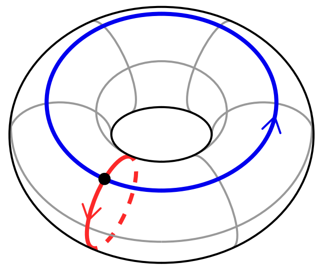
$$g \colon \pi_1(X_1, x_1) \times \pi_1(X_2, x_2) \to \pi_1(X_1 \times X_2, (x_1, x_2))$$

defined by $g([\omega_1], [\omega_2]) = [\omega_1 \times \omega_2]$ is a homomorphism of groups, and that $g = (p_{1*} \times p_{2*})^{-1}$. $\square$

9.2 Example. Let $T^2 = S^1 \times S^1$ be the torus. Then

$$\pi_1(T^2) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$$

Notice that under this isomorphism the elements $(1, 0)$ and $(0, 1) \in \mathbb{Z} \times \mathbb{Z}$ correspond to loops in T^2 that traverse the longitudinal and the meridional circles of T^2:



Theorem 9.1 can be generalized to products of arbitrary (finite or infinite) families of spaces. Recall that if $\{G_i\}_{i\in I}$ is a family of groups then the direct product $\prod_{i\in I} G_i$ is the group whose set of elements is the Cartesian product of G_i's with multiplication defined componentwise: $(g_i)_{i\in I}\cdot(h_i)_{i\in I}=(g_ih_i)_{i\in I}$

9.3 Theorem. *If $(X_i, x_i)_{i\in I}$ is a family of pointed spaces then*

$$\pi_1\left(\prod_{i\in I} X_i,\ (x_i)_{i\in I}\right)\cong\prod_{i\in I}\pi_1(X_i,x_i)$$

The proof is the same as in the case of two spaces.

Intuitively, one could rephrase Theorem 9.3 by saying that the fundamental group takes products (of topological spaces) to products (of groups). It is possible to make this statement more precise by using the language of categories and functors. This starts with the following definition:

9.4 Definition. Let $\mathbf{C}$ be a category and let $\{c_i\}_{i\in I}$ be a family of objects in $\mathbf{C}$. The *categorical product* of this family is an object $d\in\mathbf{C}$ equipped with a morphism $p_i\colon d\to c_i$ for each $i\in I$ and such that for any object a and morphisms $f_i\colon a\to c_i$ there exists a unique morphism $g\colon a\to d$ satisfying $p_ig=f_i$ for all $i\in I$:

$$\begin{tikzcd}
a \arrow[dr, dashed, "g"] \arrow[drr, bend left, "f_i"] \arrow[ddr, bend right, "f_j"'] & & \\
 & d \arrow[r, "p_i"] \arrow[d, "p_j"] & c_i \\
 & c_j &
\end{tikzcd}$$

9.5 Example. If $\{X_i\}_{i\in I}$ is a family of topological spaces then the product space $\prod_{i\in I} X_i$ together with the projection maps $p_j\colon \prod_{i\in I} X_i\to X_j$ given by $p_j((x_i)_{i\in I})=x_j$ is the categorical product of the family $\{X_i\}_{i\in I}$ in the category **Top** of topological spaces. Indeed, for any maps $f_i\colon Z\to X_i$ we have a map $g\colon Z\to\prod_{i\in I} X_i$ defined by $g(z)=(f_i(x))_{i\in I}$. Moreover g is the unique map that satisfies $p_ig=f_i$ for all $i\in I$.

9.6 Example. Similarly as above one can check that If $\{G_i\}_{i\in I}$ is a family of groups then the direct product $\prod_{i\in I} G_i$ taken together with the homomorphisms $p_j\colon \prod_{i\in I} G_i \to G_j$ given by $p_j((g_i)_{i\in I}) = g_j$ is the categorical product of the family $\{G_i\}_{i\in I}$ in the category **Gr** of groups.

9.7 Note. In general, in an arbitrary category **C** categorical products may not exists. However, if a product of a family $\{c_i\}_{i\in I}$ exists then it is unique up to an isomorphism (exercise).

9.8 Definition. Let $F\colon \mathbf{C} \to \mathbf{C}'$ be a functor. Assume that F has the property that if an object d with morphisms $p_i\colon d \to c_i$ is the categorical product of a family $\{c_i\}_{i\in I}$ in **C** then the object $F(d)$ with morphisms $F(p_i)\colon F(d) \to F(c_i)$ is the categorical product of the family $\{F(c_i)\}_{i\in I}$ in $\mathbf{C}'$. In such situation we say the the functor F *preserves products*.

9.9 Example. Let $F\colon \mathbf{Top} \to \mathbf{Gr}$ be the functor such that $F(X) = \mathbb{Z}$ for each space X and that for every map $f\colon X \to Y$ the homomorphism $F(f)\colon F(X) \to F(Y)$ is the identity function $\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$. The functor F does not preserve products since for spaces $X, Y \in \mathbf{Top}$ we have

$$F(X \times Y) = \mathbb{Z} \ncong \mathbb{Z} \times \mathbb{Z} = F(X) \times F(Y)$$

In the category $\mathbf{Top}_*$ of pointed topological spaces the categorical products of a family $\{(X_i, x_i)\}_{i\in I}$ is given by the pointed space $(\prod_{i\in I} X_i, (x_i)_{\in I})$. In view of this Theorem 9.3 can be restated as follows:

9.10 Theorem. *The fundamental group functor* $\pi_1\colon \mathbf{Top}_* \to \mathbf{Gr}$ *preserves products.*

Exercises to Chapter 9

E9.1 Exercise. The notion of a categorical coproduct is dual to the that of a product. Let **C** be a category and $\{c_i\}_{i\in I}$ be a family of objects in **C**. A *coproduct* of this family in the category **C** is an object $d \in \mathbf{C}$ equipped with a morphism $h_i\colon c_i \to d$ for each $i \in I$ such that for any object a and morphisms $f_i\colon c_i \to a$ there exists a unique morphism $g\colon d \to a$ satisfying $gh_i = f_i$ for all $i \in I$:

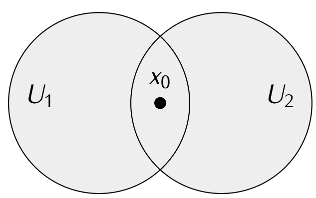
$$\begin{tikzcd}
& c_i \arrow[d, "h_i"] \arrow[ddr, bend left, "f_i"] & \\
c_j \arrow[r, "h_j"'] \arrow[drr, bend right, "f_j"'] & d \arrow[dr, dashed, "g"] & \\
& & a
\end{tikzcd}$$

a) Let **Ab** denote the category of abelian groups (with abelian groups as objects and homomorphisms of abelian groups as morphisms). Show that if $\{G_i\}_{i\in I}$ is a family of abelian groups, then the direct sum $\bigoplus_{i\in I} G_i$ is the coproduct of this family in the category **Ab**.

b) Show that in the category **Gr** of all groups the direct sum $\mathbb{Z}\oplus\mathbb{Z}$ is not the coproduct of the family $\{\mathbb{Z},\mathbb{Z}\}$.

10 | Pushouts and van Kampen's Theorem

Our main goal in this and the next chapter is to explain and prove van Kampen's theorem, which is one of the most useful tools for computing fundamental groups of spaces. This theorem says that if a space (X, x_0) can be decomposed into a union of two subspaces $X = U_1 \cup U_2$ in a way that satisfies a few conditions then it is possible to describe $\pi_1(X, x_0)$ in terms of the groups $\pi_1(U_1, x_0)$, $\pi_1(U_2, x_0)$, and $\pi_1(U_1 \cap U_2, x_0)$.



In Chapter 9 we have seen that the product formula for the fundamental group can be interpreted as a statement saying that the functor $\pi_1\colon \mathbf{Top}_* \to \mathbf{Gr}$ preserves categorical products. Similarly, one way to state van Kampen's theorem is to say that under certain assumptions the functor π_1 preserves categorical structures called *pushouts*. In this chapter we will discuss the notion of a pushout in a general category. We will then describe how pushouts look like in the category of topological spaces and in the category of groups. This will allow us to state van Kampen's theorem precisely and show a few of its applications. The proof of this theorem will be given in the next chapter.

10.1 Definition. Let $\mathbf{C}$ be a category. Assume that we are given a diagram of objects and morphisms in $\mathbf{C}$ of the following form:

$$c_1 \xleftarrow{f_1} c_0 \xrightarrow{f_2} c_2$$

A *pushout* of this diagram is an object $p \in \mathbf{C}$ equipped with morphisms $g_i\colon c_i \to p$ for $i = 1, 2$ that satisfy the following conditions:

1) $g_1 f_1 = g_2 f_2$
2) for any morphisms $h_i\colon c_i \to a$ $(i = 1, 2)$ that satisfy $h_1 f_1 = h_2 f_2$ there exists a unique morphism $h\colon p \to a$ such that $hg_i = h_i$ for $i = 1, 2$.

$$
\begin{tikzcd}
c_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & c_1 \arrow[d, "g_1"] \arrow[ddr, bend left, "h_1"] & \\
c_2 \arrow[r, "g_2"'] \arrow[drr, bend right, "h_2"'] & p \arrow[dr, dashed, "h"] & \\
& & a
\end{tikzcd}
$$

If such p exists then we write

$$p = \operatorname{colim}(c_1 \xleftarrow{f_1} c_0 \xrightarrow{f_2} c_2)$$

10.2 Note. Pushout is a special instance of a more general notion of a *colimit* (or a *direct limit*). This is where the notation colim comes from.

In a general category pushouts may not exist. However, if a pushout of a diagram $c_1 \leftarrow c_0 \rightarrow c_2$ exists then it is defined uniquely up to an isomorphism:

10.3 Proposition. *Let $c_1 \xleftarrow{f_1} c_0 \xrightarrow{f_2} c_2$ be a diagram in a category $\mathbf{C}$ and let $p, p' \in \mathbf{C}$.*

1) If p is a pushout of this diagram and $p' \cong p$ then p' is also a pushout.

2) Conversely, both p and p' are pushouts of the above diagram then $p \cong p'$.

Proof. Exercise. □

10.4 Note. Proposition 10.3 shows that the notation $p = \operatorname{colim}(c_1 \leftarrow c_0 \rightarrow c_2)$ is not entirely precise. It would be more accurate to write $p \cong \operatorname{colim}(c_1 \leftarrow c_0 \rightarrow c_2)$.

We will now look at constructions of pushouts in two categories that are of interests for us: the category of topological spaces and the category of groups.

Pushouts of topological spaces. Pushout is in the category **Top** can be described as follows:

10.5 Proposition. *For any diagram of topological spaces $X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2$ the pushout exists and it is given by*

$$\operatorname{colim}(X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2) = (X_1 \sqcup X_2)/\!\sim$$

where $\sim$ is the equivalence relation defined by $f_1(x) \sim f_2(x)$ for all $x \in X_0$.

Proof. Exercise □

10.6 Example. Assume that $X_0 = \varnothing$. We have $\operatorname{colim}(X_1 \leftarrow \varnothing \rightarrow X_2) = X_1 \sqcup X_2$.

10.7 Example. If X is a space, $A \subseteq X$ is a subspace and $*$ is a space consisting of one point then $\operatorname{colim}(* \leftarrow A \rightarrow X) = X/A$.

10.8 Example. Recall that given a map $f\colon X \to Y$ by M_f we denote the mapping cylinder of f (8.15). We can view M_f as a pushout as follows:

$$M_f = \operatorname{colim}(Y \xleftarrow{f} X \xrightarrow{j} X \times [0,1])$$

where j is given by $j(x) = (x, 1)$.

10.9 Example. The following fact will be used later on. If X is a topological space and $U, V \subseteq X$ are open sets such that $X = U \cup V$ then we have a homeomorphism

$$X \cong \operatorname{colim}(U \longleftarrow U \cap V \longrightarrow V)$$

(exercise). Note that this is not true in general, if U, V are not open in X.

Pushouts of groups. In the case of a diagram of topological spaces $X_1 \leftarrow X_0 \rightarrow X_2$ the pushout was constructed starting with the disjoint union $X_1 \sqcup X_2$ and then performing certain identifications in this space. Pushouts in the category of groups are built in a similar way, but the disjoint union of spaces is replaced by the free product of groups. We will describe free products first and then proceed to the construction of pushouts.

10.10 Definition. The *free product* of groups G and H is a group $G*H$ described as follows.

- Elements of $G*H$ are sequences $(g_1, g_2, \dots, g_n)$ where $n \geq 0$ and $g_i \in G$ or $g_i \in H$ for each $i = 1, \dots, n$.
- If for some i the elements g_i, g_{i+1} are either both in G or both in H (so that we can take their product in one of these groups) then

$$(g_1, \dots, g_i, g_{i+1}, \dots, g_n) = (g_1, \dots, g_i g_{i+1}, \dots, g_n)$$

Also, if g_i is the identity element in either G or H then

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_n) = (g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$$

- Multiplication in $G*H$ is defined by concatenation of sequences:

$$(g_1, \dots, g_n) \cdot (g'_1, \dots, g'_m) = (g_1, \dots, g_n, g'_1, \dots, g'_m)$$

10.11 Note. Definition 10.10 can be extended to a free product of an arbitrary family $\{G_j\}_{j\in J}$ of groups. In this case the free product is the group $*_{j\in J}G_j$ whose elements are sequences $(g_1, g_2, \dots, g_n)$ where for each $i = 1, \dots, n$ we have $g_i \in G_j$ for some $j \in J$. Identities between such sequences and multiplication is defined analogously as in the case of two groups.

10.12 Note. From now on we will write elements of $G*H$ as $g_1 g_2 \dots g_n$ instead of $(g_1, g_2, \dots, g_n)$.

10.13 Proposition. *For any diagram of groups $G_1 \stackrel{f_1}{\leftarrow} G_0 \stackrel{f_2}{\rightarrow} G_2$ the pushout exists and it is given by*

$$\mathrm{colim}(G_1 \xleftarrow{f_1} G_0 \xrightarrow{f_2} G_2) = (G_1 * G_2)/N$$

where N is the normal subgroup of $G_1 * G_2$ generated by all elements of the form $f_1(g)f_2(g)^{-1}$ for $g \in G_0$.

Proof. Exercise. □

10.14 Note. The group $(G_1*G_2)/N$ described above is called the *free amalgamated product* of G_1 and G_2 and it is denoted by $G_1 *_{G_0} G_2$

10.15 Example. Let $\{1\}$ denote the trivial group. We have $\mathrm{colim}(G_1 \leftarrow \{1\} \rightarrow G_2) = G_1 * G_2$.

10.16 Example. If H is a subgroup of G then $\mathrm{colim}(\{1\} \leftarrow H \rightarrow G) = G/N$ where N is the smallest normal subgroup of G generated by H. In particular is H is a normal subgroup then $N = H$.

We are now ready to state the main result of this chapter:

10.17 van Kampen Theorem. *Let (X, x_0) be a pointed topological space and let $U_1, U_2 \subseteq X$ be open sets such that $X = U_1 \cup U_2$. If the sets U_1, U_2, and $U_1 \cap U_2$ are path connected and $x_0 \in U_1 \cap U_2$ then*

$$\pi_1(X, x_0) \cong \mathrm{colim}(\pi_1(U_1, x_0) \xleftarrow{i_{1*}} \pi_1(U_1 \cap U_2, x_0) \xrightarrow{i_{2*}} \pi_1(U_2, x_0))$$

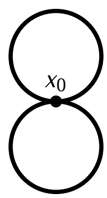
where for $k = 1, 2$ the homomorphism i_{k*} is induced by the inclusion map $i_k \colon U_1 \cap U_2 \to U_k$.

10.18 Note. By Example 10.9 under the assumptions of van Kampen's theorem we have a homeomorphism $X \cong \mathrm{colim}(U_1 \leftarrow U_1 \cap U_2 \rightarrow U_2)$. Thus van Kampen's theorem says that under some conditions the functor π_1 preserves pushouts:

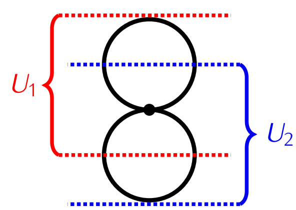
$$\pi_1(\mathrm{colim}(U_1 \longleftarrow U_1 \cap U_2 \longrightarrow U_2)) \cong \mathrm{colim}(\pi_1(U_1) \longleftarrow \pi_1(U_1 \cap U_2) \longrightarrow \pi_1(U_2))$$

We will now have a look at a few examples showing how van Kampen's Theorem can be used in computations of the fundamental group.

10.19 Example. We will compute the fundamental group of the space $S^1 \vee S^1$. It will be convenient to choose as the basepoint of $S^1 \vee S^1$ the point x_0 where the two copies of S^1 are glued together:



While $S^1 \vee S^1$ is a union of two circles we cannot use these circles as sets U_1 and U_2 in van Kampen's Theorem since they are not open in $S^1 \vee S^1$. Instead we will decompose $S^1 \vee S^1$ as follows:



Applying van Kampen's Theorem we obtain

$$\pi_1(S^1 \vee S^1, x_0) \ \cong \ \mathrm{colim}(\pi_1(U_1, x_0) \longleftarrow \pi_1(U_1 \cap U_2, x_0) \longrightarrow \pi_1(U_2, x_0))$$

Notice that $U_1 \simeq S^1 \simeq U_2$ and and $U_1 \cap U_2 \simeq *$. It follows that $\pi(U_1, x_0) \cong \pi_1(U_2, x_0) \cong \mathbb{Z}$ and $\pi_1(U_1 \cap U_2, x_0) \cong \{1\}$. This gives:

$$\pi_1(S^1 \vee S^1, x_0) \cong \mathrm{colim}(\mathbb{Z} \longleftarrow \{1\} \longrightarrow \mathbb{Z}) \cong \mathbb{Z} * \mathbb{Z}$$

Arguing by induction with respect to n we obtain that $\pi_1(\bigvee_{i=1}^n S^1) \cong *_{i=1}^n \mathbb{Z}$.

The following fact will be useful in the next example:

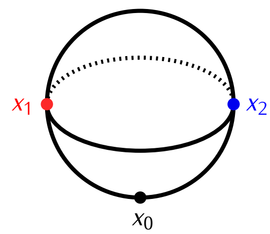
10.20 Lemma. *Let X be a space and let $U_1, U_2 \subseteq X$ be open sets such that $X = U_1 \cup U_2$ and $U_1, U_2, U_1 \cap U_2$ are path connected. If $\pi_1(U_1) \cong \{1\}$ and $\pi_1(U_2) \cong \{1\}$ then $\pi_1(X) \cong \{1\}$.*

Proof. Choose $x_0 \in U_1 \cap U_2$. By van Kampen's Theorem we get

$$\pi_1(X, x_0) \cong \operatorname{colim}(\{1\} \longleftarrow \pi_1(U_1 \cap U_2, x_0) \longrightarrow \{1\}) \cong \{1\}$$

$\square$

10.21 Example. Take a sphere S^n, $n > 1$. Let $x_0 \in S^n$ be a basepoint, and let x_1, x_2 be two points distinct from x_0:



Take $U_1 = S^n \smallsetminus \{x_1\}$, $U_2 = S^n \smallsetminus \{x_2\}$. We have $U_1 \cong \mathbb{R}^n \cong U_2$, and so $\pi_1(U_1, x_0) \cong \pi_1(U_2, x_0) \cong \{1\}$. By Proposition 10.20 we obtain that $\pi_1(S^n, x_0) \cong \{1\}$ for $n > 1$.

Exercises to Chapter 10

E10.1 Exercise. Prove Proposition 10.3

E10.2 Exercise. Assume that we have the following commutative diagram in a category $\mathbf{C}$:

$$\begin{tikzcd}
c_1 \arrow[d, "g_1"'] & c_0 \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "g_0"'] & c_2 \arrow[d, "g_2"] \\
c'_1 & c'_0 \arrow[l, "f'_1"] \arrow[r, "f'_2"'] & c'_2
\end{tikzcd}$$

Show that if the morphisms g_0, g_1, g_2 are isomorphisms, then there exists an isomorphism

$$\operatorname{colim}(c_1 \xleftarrow{f_1} c_0 \xrightarrow{f_2} c_2) \xrightarrow{\cong} \operatorname{colim}(c'_1 \xleftarrow{f'_1} c'_0 \xrightarrow{f'_2} c'_2)$$

E10.3 Exercise. Prove Proposition 10.5.

E10.4 Exercise. Let X be a topological space, and let $U, V \subseteq X$ be sets such that $U \cup V = X$.

a) Show that if U and V are open sets then there exists a homeomorphism

$$X \cong \operatorname{colim}(U \longleftarrow U \cap V \longrightarrow V)$$

b) Give an example of a space X and (non-open) sets $U, V \subseteq X$ such that the above homeomorphism does not hold. Justify your answer.

E10.5 Exercise. Let U, V be open sets in $\mathbb{R}^n$ such that $U, V, U \cap V$ are path connected, and $U \cup V = \mathbb{R}^n$. Let $x_0 \in U \cap V$. Show that if $\pi_1(U, x_0) \ncong \{1\}$ then $\pi_1(U \cap V, x_0) \ncong \{1\}$.

E10.6 Exercise. Compute the fundamental group of each of the following spaces.

a) The space X_1 obtained from $\mathbb{R}^3$ by removing n straight lines intersecting at the origin.

b) The space X_2 obtained from $\mathbb{R}^3$ by removing the circle

$$S^1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ z = 0\}$$

and the z-axis.

c) The space X_3 obtained from two copies of the torus $S^1 \times S^1$ by identifying the circle $S^1 \times \{x_0\}$ in one torus with the circle $\{x_0\} \times S^1$ in the other torus.

d) The space X_4 obtained by removing a straight line from $\mathbb{R}^4$.

e) The space X_5 obtained by removing a 2-dimensional vector subspace from $\mathbb{R}^4$.

E10.7 Exercise. Let S^3 be the 3-dimensional sphere, let $A \subset S^3$ be a closed subspace of S^3, and let $x_0 \in S^3$. Assuming that the space $S^3 - (A \cup \{x_0\})$ is path connected show that the inclusion map

$$j \colon S^3 - (A \cup \{x_0\}) \hookrightarrow S^3 - A$$

induces an isomorphism of fundamental groups.

E10.8 Exercise. Let $f \colon X \to Y_1$ and $g \colon X \to Y_2$ be maps of topological spaces. The *double mapping cylinder* of f and g is the space

$$M(f, g) = (Y_1 \sqcup X \times [0, 1] \sqcup Y_2)/\!\sim$$

where $\sim$ is the equivalence relation given by $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that if X, Y_1, Y_2 are path connected spaces and $x_0 \in X$ then there exists an isomorphism

$$\pi_1(M(f, g)) \cong \operatorname{colim}(\pi_1(Y_1, f(x_0)) \xleftarrow{f_*} \pi_1(X, x_0) \xrightarrow{g_*} \pi_1(Y_2, g(x_0)))$$

E10.9 Exercise. The *join* of spaces X and Y is the space $X * Y$ given by

$$X * Y = X \times Y \times [0, 1]/\!\sim$$

where $(x, y, 0) \sim (x, y', 0)$ for all $x \in X$, $y, y' \in Y$, and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$, $y \in Y$. Show that if X and Y are non-empty path connected spaces then $\pi_1(X * Y) \cong \{1\}$.

E10.10 Exercise. Recall that a topological group is a topological space X equipped with a group structure such that the maps $\mu\colon X \times X \to X$, $\mu(g, h) = gh$ and $\eta\colon X \to X$, $\eta(g) = g^{-1}$ are continuous. For example, the space $\mathbb{R}^2$ is a topological group with the addition

$$(x, y) + (x', y') = (x + x', y + y')$$

defining the group structure.

a) Show that for any $x_1 \in \mathbb{R}^2$ the space $X = \mathbb{R}^2 \smallsetminus \{x_1\}$ can be given the structure of a topological group.

b) Show that for $n > 1$ the space $X = \mathbb{R}^2 \smallsetminus \{x_1, \dots, x_n\}$ does not have the structure of a topological group.

11 | Proof of van Kampen's Theorem

In the last chapter have seen the statement of van Kampen's theorem and some of examples of its applications. Our main goal in this chapter is to prove this result. For reference we state it here again:

10.17 **van Kampen Theorem.** *Let (X, x_0) be a pointed topological space and let $U_1, U_2 \subseteq X$ be open sets such that $X = U_1 \cup U_2$. If the sets U_1, U_2, and $U_1 \cap U_2$ are path connected and $x_0 \in U_1 \cap U_2$ then*

$$\pi_1(X, x_0) \cong \operatorname{colim}(\pi_1(U_1, x_0) \xleftarrow{i_{1*}} \pi_1(U_1 \cap U_2, x_0) \xrightarrow{i_{2*}} \pi_1(U_2, x_0))$$

where for $k = 1, 2$ the homomorphism i_{k*} is induced by the inclusion map $i_k \colon U_1 \cap U_2 \to U_k$.

Proof. Here is some notation that will be useful.

- For simplicity we will denote $U_1 \cap U_2$ by U_0.
- For $k = 1, 2$ by $i_k \colon U_0 \to U_k$ and $j_k \colon U_k \to X$ we will denote the inclusion maps.
- If ω is a loop in U_1 then it represents an element of $\pi_1(U_1, x_0)$ and an element of $\pi_1(X, x_0)$. In order to avoid such ambiguities we will write $[\omega]_k$ to indicate an element of $\pi_1(U_k, x_0)$ and $[\omega]$ to indicate an element of $\pi_1(X, x_0)$.

The strategy of the proof will be as follows. Let $P = \operatorname{colim}(\pi_1(U_1, x_0) \overset{i_{1*}}{\leftarrow} \pi_1(U_0, x_0) \overset{i_{2*}}{\rightarrow} \pi_1(U_2, x_0))$. Recall that by Proposition 10.3 P is the unique (up to an isomorphism) group that satisfies the following conditions:

1) for $k = 1, 2$ there exists exists a homomorphism $g_k \colon \pi_1(U_k, x_0) \to P$ such that $g_1 i_{1*} = g_2 i_*$;
2) for any group G and any homomorphisms $h_k \colon \pi_1(U_k, x_0) \to P$ satisfying $h_1 i_{1*} = h_2 i_*$ there exists a unique homomorphism $h \colon P \to G$ such that $h g_k = h_k$ for $k = 1, 2$.

If follows that in order to prove van Kampen's theorem it will suffice to show that the group $\pi_1(X, x_0)$ satisfies conditions 1) and 2). The first condition is satisfied by taking $g_k = j_{k*}$ for $k = 1, 2$. In order to verify the second condition let $h_1 \colon \pi_1(U_1, x_0) \to G$ and $h_2 \colon \pi_1(U_2, x_0) \to G$ be homomorphisms satisfying $h_1 i_{1*} = h_2 i_{2*}$. We need to construct a homomorphism $h \colon \pi_1(X, x_0) \to G$ such that the following diagram commutes:

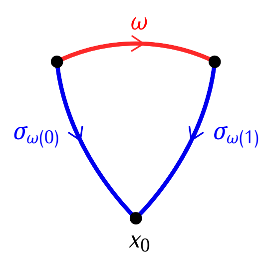
$$
\begin{tikzcd}
\pi_1(U_0,x_0) \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"'] & \pi_1(U_1,x_0) \arrow[d, "j_{1*}"] \arrow[ddr, bend left, "h_1"] & \\
\pi_1(U_2,x_0) \arrow[r, "j_{2*}"'] \arrow[drr, bend right, "h_2"'] & \pi_1(X,x_0) \arrow[dr, dashed, "h"] & \\
& & G
\end{tikzcd}
\tag{$*$}
$$

Moreover, we need to show that there is only one such homomorphism h.

The construction of h will use the following setup. For each point $x \in X$ choose a path σ_x such that

- $\sigma_x(0) = x$ and $\sigma_x(1) = x_0$;
- if $x \in U_k$ for $k \in \{0, 1, 2\}$ then σ_x is contained in U_k;
- σ_{x_0} is the constant path.

Such paths exist since by assumption U_0, U_1, U_2 are path connected sets. If ω is a path in X then the concatenation $\overline{\sigma}_{\omega(0)} * \omega * \sigma_{\omega(1)}$ is a loop based at x_0:



We will denote this loop by ω° and call it the *loop completion* of ω. Loop completion has the following properties:

(i) If ω is a path in U_k for $k \in \{0, 1, 2\}$ then ω° is a loop in U_k. This holds since in such case $\sigma_{\omega(0)}$ and $\sigma_{\omega(1)}$ are paths in U_k.

(ii) For any path ω we have $(\overline{\omega})^\circ \simeq \overline{\omega^\circ}$.

(iii) If ω, τ have the same endpoints and $\omega \simeq \tau$ then $\omega^\circ \simeq \tau^\circ$, and so in $\pi_1(X, x_0)$ we have $[\omega^\circ] = [\tau^\circ]$. Moreover, if ω, τ are paths in U_k and the homotopy between them is contained in U_k then in $\pi_1(U_k, x_0)$ we have $[\omega^\circ]_k = [\tau^\circ]_k$.

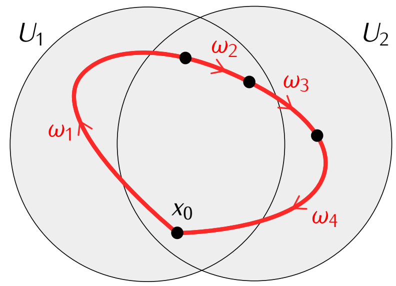
(iv) If ω, τ are paths such that $\omega(1) = \tau(0)$ then $(\omega * \tau)^\circ \simeq \omega^\circ * \tau^\circ$. Indeed, we have

$$\omega^\circ * \tau^\circ = (\overline{\sigma}_{\omega(0)} * \omega * \sigma_{\omega(1)}) * (\overline{\sigma}_{\omega(1)} * \tau * \sigma_{\tau(1)}) \simeq \overline{\sigma}_{\omega(0)} * \omega * \tau * \sigma_{\tau(1)} \simeq (\omega * \tau)^\circ$$

Notice also that if both ω and τ are paths in U_k then the above homotopy of loops is contained in U_k. This means that $[\omega^\circ]_k \cdot [\tau^\circ]_k = [(\omega * \tau)^\circ]_k$ in $\pi_1(U_k, x_0)$.

(v) If ω is a loop based at x_0 then $\omega \simeq \omega^\circ$. To see this notice that in this case $\sigma_{\omega(0)} = \sigma_{\omega(1)} = \sigma_{x_0}$, and by assumption σ_{x_0} is the constant path. Therefore we have $\omega^\circ = \bar{\sigma}_{x_0} * \omega * \sigma_{x_0} \simeq \omega$.

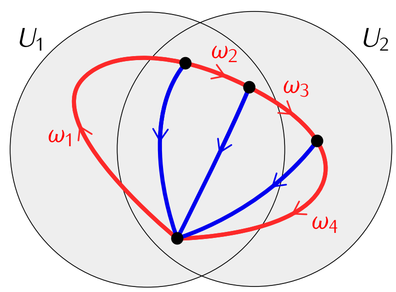
We are now ready to describe the construction of the homomorphism h in the diagram ($*$). Let $\omega\colon [0,1] \to X$ be a loop. The sets $\omega^{-1}(U_1)$ and $\omega^{-1}(U_2)$ form an open cover of the interval $[0,1]$. Using the Lebesgue number of this cover we obtain that there exists an n-tuple of numbers $(s_0, \ldots, s_n)$ such that $0 = s_0 < s_1 < \ldots < s_n = 1$ and $\omega([s_{i-1}, s_i])$ is contained in either U_1 or U_2 for each $i = 1, \ldots, n$. Let $\omega_i\colon [0,1] \to X$ be the path given by $\omega_i(s) = \omega(s_{i-1}s + s_i(1-s))$. This path coincides with the restriction of ω to the subinterval $[s_{i-1} : s_i]$:



We have $\omega \simeq \omega_1 * \omega_2 * \cdots * \omega_n$. By the properties of loop completion we get:

$$\omega \simeq \omega^\circ \simeq \omega_1^\circ * \omega_2^\circ * \ldots * \omega_n^\circ$$

Moreover, if ω_i is contained in U_1 then ω_i° is a loop in U_1, and likewise for U_2.



Let $k(i)$ denote either 1 or 2 depending if ω_i° is a loop in U_1 or U_2. Define an element $\tilde{h}(\omega) \in G$ by

$$\tilde{h}(\omega) = h_{k(1)}([\omega_1^\circ]_{k(1)}) \cdot h_{k(2)}([\omega_2^\circ]_{k(2)}) \cdot \ldots \cdot h_{k(n)}([\omega_n^\circ]_{k(n)}) \qquad (**)$$

There are two ambiguities in this formula. First, if ω_i is a path entirely contained in U_0 then ω_i° is a loop in U_0 and in such case we can take $k(i)$ to be either 1 or 2. This however does not matter since in such case we have $[\omega_i]_1 = i_{1*}([\omega_i]_0)$ and $[\omega_i]_2 = i_{2*}([\omega_i]_0)$, and since $h_1 i_{1*} = h_2 i_{2*}$ we get

$$h_1([\omega_i^\circ]_1) = h_1 i_{1*}([\omega_i^\circ]_0) = h_2 i_{2*}([\omega_i^\circ]_0) = h_2([\omega_i^\circ]_1)$$

The second ambiguity comes from the fact that the formula (∗∗) uses subdivision of ω into paths ω_i, and the value of $\tilde{h}(\omega)$ could change if we change the subdivision. To see that this is not the case consider the subdivision $\omega \simeq \omega_1 * \ldots * \omega_n$ that comes from an n-tuple of numbers $\underline{s} = (s_0, \ldots, s_n)$, and let s_+ be a number such that $s_{i-1} < s_+ < s_i$ for some $1 \leq i \leq n$. The $(n+1)$-tuple $\underline{s}' = (s_1, \ldots, s_{i-1}, s_+, s_{i+1}, \ldots, s_n)$ produces the subdivision

$$\omega \simeq \omega_1 * \ldots * \omega_{i-1} * \tau_1 * \tau_2 * \omega_{i+1} * \ldots * \omega_n$$

where $\tau_1 * \tau_2 \simeq \omega_i$. By the properties of loop completion $[\omega^\circ]_{k(i)} = [\tau_1^\circ]_{k(i)} \cdot [\tau_2^\circ]_{k(i)}$, and thus we get

$$h_{k(i)}([\omega_i^\circ]_{k(i)}) = h_{k(i)}([\tau_1^\circ]_{k(i)} \cdot [\tau_2^\circ]_{k(i)}) = h_{k(i)}([\tau_1^\circ]_{k(i)}) \cdot h_{k(i)}([\tau_2^\circ]_{k(i)})$$

This shows the the value of $\tilde{h}(\omega)$ does not depend on whether we compute it using $\underline{s}$ or $\underline{s}'$. Arguing inductively we obtain that if an m-tuple $\underline{s}'$ is obtained by adding some number of elements to an n-tuple $\underline{s}$ then the value of $\tilde{h}(\omega)$ computed using $\underline{s}'$ is the same as the value computed using $\underline{s}$. Finally, given an arbitrary n-tuple $\underline{s}$ and an m-tuple $\underline{s}'$ that produce two subdivisions of ω we can find an r-tuple $\underline{s}''$ that can be obtained from each of $\underline{s}$ and $\underline{s}'$ by inserting some additional elements. The argument above shows then that the values of $\tilde{h}(\omega)$ computed using $\underline{s}$ and $\underline{s}'$ must be equal since they are both equal to the value computed using $\underline{s}''$.

Our goal will be to show that $\tilde{h}(\omega)$ depends only on the homotopy class of ω:

Claim. If ω, τ are loops in X and $\omega \simeq \tau$ then $\tilde{h}(\omega) = \tilde{h}(\tau)$.

Assuming for a moment this claim holds notice that it allows us to define a function $h\colon \pi_1(X, x_0) \to G$ by $h([\omega]) = \tilde{h}(\omega)$. Notice also that this function has the following properties:

1) h is a homomorphism;
2) h makes the diagram (∗) into a commutative diagram;
3) h is the only homomorphism that makes the diagram (∗) commute.

Indeed, to see 1) observe that if ω and ω' are two loops in X with subdivisions $\omega \simeq \omega_1 * \ldots * \omega_n$ and $\omega' \simeq \omega'_1 * \ldots * \omega'_m$ then $\omega * \omega'$ has a subdivision

$$\omega * \omega' \simeq \omega_1 * \ldots * \omega_n * \omega'_1 * \ldots * \omega'_m$$

Using this observation and the definition of h we get that $h([\omega] \cdot [\omega']) = h([\omega]) \cdot h([\omega'])$.

To verify 2) notice that if ω is a loop in U_k for either $k = 1$ or $k = 2$ then we don't need to subdivide it, i.e. we can take $\omega = \omega_1$. Since $\omega \simeq \omega^\circ$ the formula (∗∗) gives

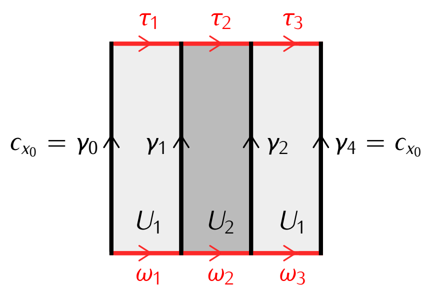
$$h([\omega]) = \tilde{h}(\omega) = h_k([\omega]_k)$$

and since $[\omega] = j_{k*}([\omega]_k)$ we obtain $hj_{k*} = h_k$.

Finally, to see 3) notice that we have shown that any element $[\omega] \in \pi_1(X, x_0)$ can be written as a product $[\omega] = [\omega_1^\circ] \cdot \ldots \cdot [\omega_n^\circ]$ where ω_i is a loop in $U_{k(i)}$ for either $k(i) = 1$ or $k(i) = 2$. For each

$i = 1, \ldots, n$ we have $[\omega_i^\circ] = j_{k(i)*}([\omega_i^\circ]_{k(i)})$. Thus in order to get the identity $h j_{k(i)*} = h_{k(i)}$ we must set $h([\omega_i^\circ]) = h_{k(i)}([\omega_i^\circ]_{k(i)})$. Since homomorphisms preserve products we are forced to define $h([\omega])$ by the formula ($**$).

The above observations show that once we verify that $\tilde{h}$ is a homotopy invariant function the proof of van Kampen's Theorem will be complete. Assume then that ω, τ are loops in X and that there exists a homotopy $h\colon [0,1] \times [0,1] \to X$ between them: $h_0 = \omega$, $h_1 = \tau$. We will consider first a special case, and assume in addition that there exists numbers $0 = s_0 < s_1 < \ldots < s_n = 1$ such that the homotopy h maps each rectangle $[s_{i-1}, s_i] \times [0,1]$ either into U_1 or into U_2. In particular this gives a subdivisions $\omega \simeq \omega_1 * \ldots * \omega_n$ and $\tau \simeq \tau_1 * \ldots * \tau_n$ where ω_i and τ_i are restrictions of ω and τ to the subinterval $[s_{i-1}, s_i]$. For $i = 0, \ldots, n$ denote by γ_i the path given by $\gamma(t) = h(s_i, t)$. Notice that γ_0 and γ_n are constant paths at x_0.



Let $k(i)$ denote either 1 or 2 depending if $h([s_{i-1}, s_i] \times [0,1])$ is contained in U_1 or U_2. Notice that for each i the paths ω_i, τ_i, γ_{i-1}, and γ_i are paths in $U_{k(i)}$. Moreover, using the restriction of h to $[s_{i-1}, s_i] \times [0,1]$ we obtain that the path ω_i is homotopic to $\gamma_{i-1} * \tau_i * \bar{\gamma}_i$ via a homotopy contained in $U_{k(i)}$. Using the properties of loop completion we obtain

$$[\omega_i^\circ]_{k(i)} = [(\gamma_{i-1} * \tau_i * \bar{\gamma}_i)^\circ]_{k(i)} = [\gamma_{i-1}^\circ]_{k(i)} \cdot [\tau_i^\circ]_{k(i)} \cdot [\gamma_i^\circ]_{k(i)}^{-1}$$

This identity and the formula ($**$) gives:

$$\begin{aligned} \tilde{h}(\omega) &= h_{k(1)}([\omega_1^\circ]_{k(1)}) \cdot h_{k(2)}([\omega_2^\circ]_{k(2)}) \cdot \ldots \cdot h_{k(n)}([\omega_n^\circ]_{k(n)}) \\ &= h_{k(1)}[(\gamma_0^\circ]_{k(1)}) \cdot h_{k(1)}([\tau_1^\circ]_{k(1)}) \cdot h_{k(1)}([\gamma_1^\circ]_{k(1)})^{-1} \\ &\quad \cdot h_{k(2)}[(\gamma_1^\circ]_{k(2)}) \cdot h_{k(2)}([\tau_2^\circ]_{k(2)}) \cdot h_{k(2)}([\gamma_1^\circ]_{k(2)})^{-1} \cdot \ldots \\ &\quad \cdot h_{k(n)}[(\gamma_{n-1}^\circ]_{k(n)}) \cdot h_{k(n)}([\tau_n^\circ]_{k(n)}) \cdot h_{k(n)}([\gamma_n^\circ]_{k(n)})^{-1} \end{aligned} \tag{$***$}$$

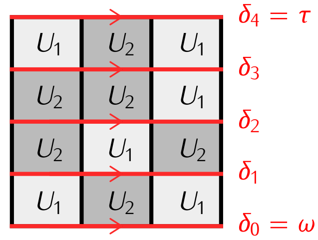
Notice that $h_{k(1)}([\gamma_1^\circ]_{k(1)})$ and $h_{k(n)}([\gamma_n^\circ]_{k(n)})$ are trivial elements of G since γ_1° and γ_n° are constant loops. Also, for $i = 1, \ldots n-1$ we have $h_{k(i)}([\gamma_i^\circ]_{k(i)}) = h_{k(i+1)}([\gamma_i^\circ]_{k(i+1)})$. This is obvious if $k(i) = k(i+1)$. If $k(i) \neq k(i+1)$ then this identity still holds since in such case γ_i (and thus also γ_i°) is contained in U_0 and then

$$h_1([\gamma_i^\circ]_1) = h_1 i_{1*}([\gamma_i^\circ]_0) = h_2 i_{2*}([\gamma_i^\circ]_0) = h_2([\gamma_i^\circ]_2)$$

Using these observations we can simplify the equation ($***$) to

$$\tilde{h}(\omega) = h_{k(1)}([\tau_1^\circ]_{k(1)}) \cdot h_{k(2)}([\tau_2^\circ]_{k(2)}) \cdot \ldots \cdot h_{k(n)}([\tau_n^\circ]_{k(n)}) = \tilde{h}(\tau)$$

It remains to consider the general case when we are given two loops ω and τ in X such that $\omega \simeq \tau$. Let $h\colon [0,1]\times[0,1] \to X$ be a homotopy with $h_0 = \omega$ and $h_1 = \tau$. The sets $h^{-1}(U_1)$ and $h^{-1}(U_2)$ form an open cover of X. Using the Lebesgue number of this cover we can find numbers $0 = s_0 < s_1 < \ldots < s_n = 1$ and $0 = t_0 < t_1 < \ldots < t_m = 1$ such that h maps each rectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ either into U_1 or into U_2. Let δ_i be the path given by $\delta_i(s) = h(s, t_i)$:



Notice that the restriction of h to the rectangle $[0,1]\times[t_{i-1}, t_i]$ gives a homotopy between δ_{i-1} and δ_i, and moreover this homotopy is of the form considered in the special case above. Therefore for each $i = 1, \ldots, m$ we have $\tilde{h}(\delta_{i-1}) = \tilde{h}(\delta_i)$. As a consequence we obtain

$$\tilde{h}(\omega) = \tilde{h}(\delta_0) = \tilde{h}(\delta_1) = \ldots = \tilde{h}(\delta_m) = \tilde{h}(\tau)$$

□

Theorem 10.17 can be generalized to the case where the space X is covered by more that two (and possibly inifinitely many) open sets:

11.1 Theorem. *Let (X, x_0) be a pointed topological space and let $\{U_i\}_{i\in I}$ be an open cover of X such that $x_0 \in U_i$ for all $i \in I$. For $i, j \in I$ let $f_{ij}\colon U_i \cap U_j \to U_i$ denote the inclusion map. If the set $U_i \cap U_j \cap U_k$ is path connected for all $i, j, k \in I$ then*

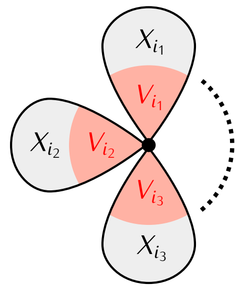
$$\pi_1(X, x_0) \cong *_{i\in I}\pi_1(U_i, x_0)/N$$

where N is the normal subgroup of $*_{i\in I}\pi_1(U_i, x_0)$ generated by all elements of the form $f_{ij}([\omega])\cdot f_{ji}([\omega])^{-1}$ for $i, j \in I$ and $[\omega] \in \pi_1(U_i \cap U_j, x_0)$.

Here is one application of the generalized van Kampen Theorem:

11.2 Proposition. *Let $\{(X_i, x_i)\}_{i\in I}$ be a family of path connected pointed spaces, and let $\bigvee_{i\in I} X_i$ be the space obtained by identifying the basepoints $x_i \sim x_j$ for all $i, j \in I$. Assume that for each $i \in I$ there exists a set $V_i \subseteq X_i$ such that V_i is open in X_i, $x_i \in V_i$, and the one-point space $\{x_i\}$ is a deformation retract of V_i. Then*

$$\pi_1(\textstyle\bigvee_i X_i) \cong *_{i\in I}\pi_1(X_i, x_i)$$



Proof. We will denote by $\bar{x}$ the point of $\bigvee_{i\in I} X_i$ obtained by identifying the points $x_i \in X_i$. For $i \in I$ let $U_i = X_i \vee \bigvee_{i\in} V_i$. The sets U_i are path connected and the form an open cover $\bigvee_i X_i$. Moreover for $j, k.l \in I$ we have

$$U_j \cap U_k \cap U_l = \begin{cases} U_j & \text{if } j = k = l \\ \bigvee_{i\in I} V_i & \text{otherwise} \end{cases}$$

so $U_j \cap U_k \cap U_l$ is path connected for all j, k, l. Using Theorem 11.1 we obtain

$$\pi_1(\bigvee_{i\in I} X_i, \bar{x}) \cong *_{i\in I}\pi_1(U_i, x_i)/N$$

where N is the normal group generated by elements $f_{ij*}([\omega]) \cdot f_{ji*}([\omega])^{-1}$ for $[\omega] \in \pi_1(U_i \cap U_j, \bar{x})$. As in Theorem 11.1 by f_{ij} we denote here the inclusion map $f_{ij}\colon U_i \cap U_j \to U_{ij}$. For $i \in I$ let $h_i\colon V_i \times [0,1] \to V_i$ the deformation retraction of V_i onto $\{x_i\}$. These maps define a deformation retraction of the space $\bigvee_{i\in I} V_i$ onto $\{\bar{x}\}$. This shows that the space $\bigvee_{i\in I} V_i$ is contractible and so $\pi_1(\bigvee_{i\in I} V_i, \bar{x}) = \{1\}$. Since for all $i, j \in I$ such that $i \neq j$ we have $U_i \cap U_j = \bigvee_{i\in I} V_i$ it follows that the group N is trivial and so $\pi_1(\bigvee_{i\in I} X_i, \bar{x}) \cong *_{i\in I}\pi_1(U_i, x_i)$. Finally, using again the deformation retractions h_i we can construct for each $j \in I$ a deformations retraction of U_j onto X_j. This shows that $\pi_1(U_j, \bar{x}) \cong \pi_1(X_j, \bar{x})$, and so we get

$$\pi_1(\bigvee_{i\in I} X_i, \bar{x}) \cong *_{i\in I}\pi_1(U_i, \bar{x}) \cong *_{i\in I}\pi_1(X_i, \bar{x})$$

□

11.3 Example. Let s_0 be a basepoint of S^1. Taking $V = S^1 \smallsetminus \{s_1\}$ were $s_1 \in S^1$ is a point different from s_0 we obtain an open neighborhood of s_0 which deformation retracts onto $\{x_0\}$. Thus using Proposition 11.2 we get

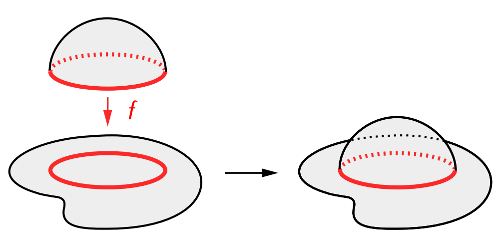
$$\pi_1(\bigvee_{i\in I} S^1) \cong *_{i\in I}\pi_1(S^1) \cong *_{i\in I}\mathbb{Z}$$

Exercises to Chapter 11

12 | CW Complexes

Van Kampen's theorem tells us that in order to compute the fundamental group of a complicated space we may try to decompose the space into simpler pieces. Information about these pieces and how they fit together suffices to determine the fundamental group of the whole space. A similar approach works in many settings, and for this reason it is useful if we can represent a given space as a collection of some simple building blocks that are assembled together. In this chapter we discuss the notion of a CW complex that provides a scheme for constructing such representations of spaces. In this scheme as the building blocks one uses closed discs of various dimensions. We will see that many interesting spaces can be given the structure of a CW complex. In the next chapters we will see how to compute the fundamental group of an arbitrary CW complex. Building on this result we will also show that any group can be realized as the fundamental group of some space: given a group G we will construct a CW complex X such that $\pi_1(X) \cong G$.

12.1 Definition. Let X be a space and let $f\colon S^{n-1} \to X$ be a continuous function. We say that a space Y is obtained by *attaching an n-cell* to X if $Y = X \sqcup D^n/{\sim}$ where $\sim$ is the equivalence relation given by $x \sim f(x)$ for all $x \in S^{n-1} \subseteq D^n$. We write $Y = X \cup_f e^n$.



Notice that $X \cup_f e^n = \operatorname{colim}(D^n \xleftarrow{j} S^{n-1} \xrightarrow{f} X)$ where $j\colon S^{n-1} \to D^n$ is the inclusion map.

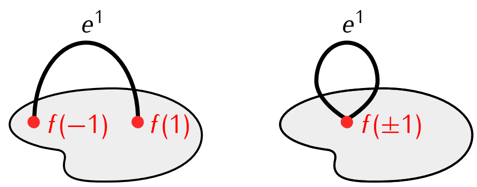
12.2 Here is some terminology associated to the operation of cell attachment:

- The map $f\colon S^{n-1} \to X$ is called the *attaching map* of the cell e^n.
- The map $\bar{f}\colon D^n \to X \sqcup D^n \to X \cup_f e^n$ is called the *characteristic map* of the cell e^n.

- The subspace $e^n = \bar{f}(D^n \setminus S^{n-1}) \subseteq X \cup_f e^n$ is called the *open cell*.
- The subspace $\bar{e}^n = \bar{f}(D^n) \subseteq X \cup_f e^n$ is called the *closed cell*.

12.3 Example. For $n = 0$ we have $D^0 = \{*\}$ and $S^{-1} = \varnothing$. Therefore $X \cup e^0$ is a disjoint union of X and a point.

12.4 Example. For $n = 1$ we have $D^1 = [-1, 1]$ and $S^0 = \{-1, 1\}$. The space $X \cup_f e^1$ is obtained by attaching to X an arch or a loop, depending if $f\colon S^0 \to X$ is a 1-1 function or not:



In general, the operation of cell attachment can be viewed as a special case of the construction of a mapping cone:

12.5 Lemma. *For any map $f\colon S^{n-1} \to X$ the space $X \cup_f e^n$ is homeomorphic to the mapping cone C_f.*

Proof. Exercise. $\square$

This immediately gives the following fact:

12.6 Proposition. *If $f, g\colon S^{n-1} \to X$ are maps such that $f \simeq g$ then $X \cup_f e^n \simeq X \cup_g e^n$.*

Proof of Proposition 12.6. Follows from Lemma 12.5 and Proposition 8.17. $\square$

12.7 Definition. Let X be topological space and let $Y \subseteq X$. The pair (X, Y) is a *relative CW complex* if $X = \bigcup_{n=-1}^{\infty} X^{(n)}$ where

1) $X^{(-1)} = Y$;
2) for $n \geq 0$ the space $X^{(n)}$ is obtained by attaching n-cells to $X^{(n-1)}$;
3) the topology on X is defined so that a set $U \subseteq X$ is open if and only if $U \cap X^{(n)}$ is open in $X^{(n)}$ for all n.

12.8 Note. By part 3) of Definition 12.7 if (X, Y) is a relative CW complex then a function $f\colon X \to Z$ is continuous if and only if $f|_{X^{(n)}}\colon X^{(n)} \to Z$ is continuous for all $n \geq -1$.

12.9 Note. If (X, Y) is a relative CW complex then the space $X^{(n)}$ is called *the n-skeleton* of X.

12.10 Definition. A *CW complex* is a space X such that $(X, \varnothing)$ is a relative CW complex.

12.11 Definition. 1) A CW complex X is *finite* if it consists of finitely many cells.

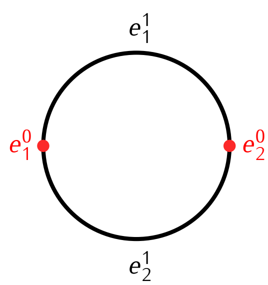
2) A CW complex X is *finite dimensional* if $X = X^{(n)}$ for some n.

3) The *dimension* of a CW complex X is defined by

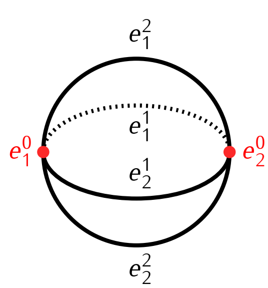
$$\dim X = \begin{cases} \min\{n \mid X = X^{(n)}\} & \text{if } X \text{ is finite dimensional} \\ \infty & \text{otherwise} \end{cases}$$

12.12 Example. The only CW complex of dimension -1 is the empty space. A CW complex of dimension 0 is a discrete topological space (with each point defining a 0-cell).

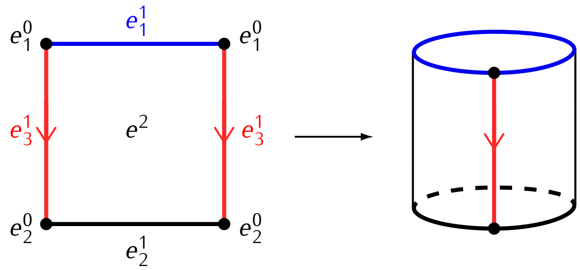
12.13 Example. If a space can be equipped with a structure of a CW complex of dimension greater than 0 then such structure is not unique. Here are two different CW complex structures on S^1, one with two 0-cells and two 1-cells, and the other with one 0-cell and one 1-cell:



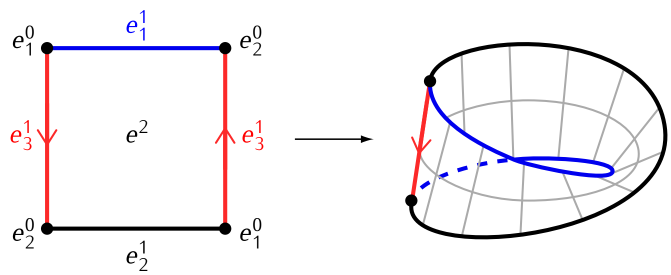
12.14 Example. Here are two examples of CW complex structures on S^2. The first has two cells in in each of the dimensions 0, 1, and 2. The second has one 0-cell e^0 and one 2-cell which is attached using the constant attaching map $S^1 \to e^0$:

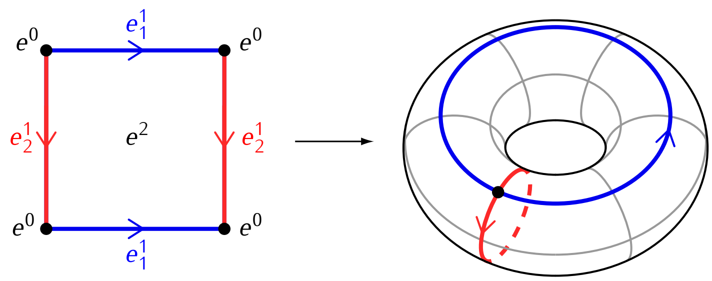


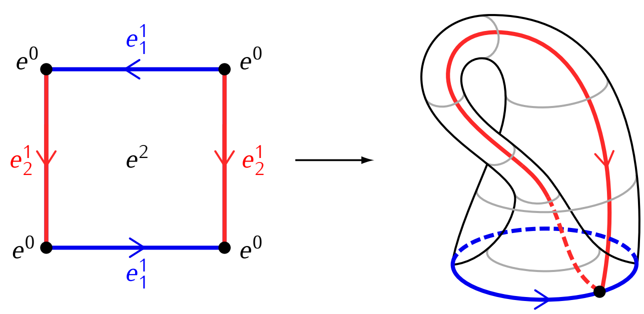
12.15 Example. The cylinder $S^1 \times [0, 1]$ can be given a CW complex structure with two 0-cells, three 1-cells, and one 2-cell. It is easier to visualize this structure if we consider the cylinder as a quotient space obtained by gluing together two vertical edges of a square. The pair of the upper vertices of the square represents one 0-cell of the cylinder, and the pair of lower vertices the second 0-cell. The three 1-cells come from each of the horizontal edges and the pair of vertical edges. The interior of the square corresponds to the interior of the 2-cell of the cylinder:



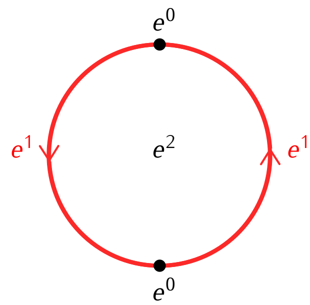
Since the Möbius band, the torus, and the Klein bottle also can be constructed by gluing together some edges of the square we can describe CW complex structures on these spaces in a similar way:







12.16 Example. Recall that the 2-dimensional real projective space $\mathbb{RP}^2$ can be constructed as a quotient space of the disc D^2 obtained by identifying antipodal points on the boundary of D^2: $x \sim (-x)$ for $x \in S^1$. This space can be given a CW complex structure with one 0-cell, one 1-cell, and one 2-cell:



12.17 Note. It is not true that every space can be given a structure of a CW complex. For example, consider the following subspace of the real line:

$$X = \{\tfrac{1}{n} \in \mathbb{R} \mid n = 1, 2, \dots\} \cup \{0\}$$

This space is not homeomorphic or even homotopy equivalent to a CW complex. To show this we will use the fact that if Y is a CW complex then each path connected component of Y is open in Y (exercise). Assume that there exists a homotopy equivalence $f\colon X \to Y$ where Y is a CW complex. This implies that the induced map $f_*\colon \pi_0(X) \to \pi_0(Y)$ is a bijection. On the other hand let $\{Y_i\}_{i\in I}$ be the family of all path connected components of Y. The sets Y_i are open in Y, so the sets $f^{-1}(Y_i)$ are open in X and they define an open cover of X. The space X is compact, so this cover has a finite subcover $\{f^{-1}(Y_{i_1}), \dots, f^{-1}(Y_{i_k})\}$. Since X is an infinite space this implies that that there exist distinct points $x_1, x_2 \in X$ such that $f(x_1)$ and $f(x_2)$ belong to the same path connected component of Y. On the other hand x_1 and x_2 belong to different path connected components in X since every path connected component of X consists of a single point. This shows that $f_*\colon \pi_0(X) \to \pi_0(Y)$ is not a bijection, and so we obtain a contradiction.

The following fact if often useful:

12.18 Proposition. *1) Let (X, Y) be a relative CW complex. If $A \subseteq X$ is a compact set then A is closed in X and it has a non-empty intersection with finitely many open cells of X only.*

2) If X is a CW complex and $A \subseteq X$ is a closed set which has a non-empty intersection with only finitely many open cells of X then A is compact.

Proof. Exercise. □

12.19 Corollary. *A CW complex is compact if and only if it is a finite.*

Proof. Follows from Proposition 12.18 □

Exercises to Chapter 12

E12.1 Exercise. Show that a CW complex X is path connected if and only if its 1-skeleton $X^{(1)}$ is path connected.

E12.2 Exercise. Prove Proposition 12.18.

E12.3 Exercise. Let X be a space, and let $Y = X \cup_f e^n$ be obtained by attaching one n-dimensional cell to X. Show that if X is a retract of Y then $Y \simeq X \vee S^n$.

13 | Homotopy Extension Property

In this chapter we begin work toward computing fundamental groups of CW complexes. Since a 0-dimensional CW complex is a discrete space, the fundamental group of any such complex is trivial. The first non-trivial case we will develop a formula for the fundamental group of a CW complex of dimension 1. Our main tool will be the homotopy extension property, which is one of the most important notions of algebraic topology.

13.1 Definition. Let X be a topological space, and let $A \subseteq X$. The pair (X, A) has the *homotopy extension property* if any map

$$h\colon X \times \{0\} \cup A \times [0,1] \to Y$$

can be extended to a map $\bar{h}\colon X \times [0,1] \to Y$.

The following proposition is often useful when we want to verify that the homotopy extension property holds for a given pair of (X, A):

13.2 Proposition. *A pair (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times [0,1]$ is a retract of $X \times [0,1]$.*

Proof. Exercise. □

The next fact implies that the homotopy extension property does not hold for arbitrary pairs of spaces:

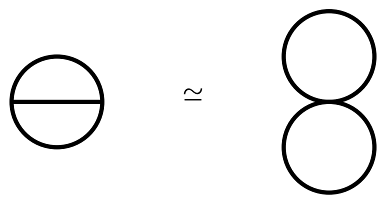
13.3 Proposition. *If a pair (X, A) has the homotopy extension property and X is a Hausdorff space then A is closed in X.*

Proof. Exercise. □

13.4 Proposition. *If a pair (X, A) has the homotopy extension property and the space A is contractible then the quotient map $q\colon X \to X/A$ is a homotopy equivalence.*

Proof. Exercise $\square$

13.5 Example. In Example 8.18 we have shown that the space X consisting of a circle and its diagonal is homotopy equivalent to a wedge of two circles:



We can obtain the same result as follows. Let $A \subseteq X$ be the diagonal of the circle (together with its enpoints). It will follow from Theorem 13.7 that the pair (X, A) has the homotopy extension property. Since the space A is contractible, using Proposition 13.4 we get a homotopy equivalence $X \simeq X/A$. It remains to notice that X/A is homeomorphic to $S^1 \vee S^1$.

13.6 Example. Here is an example which shows that Proposition 13.4 is not true in general, if (X, A) does not have the homotopy extension property. The *Warsaw circle* is a subspace W of $\mathbb{R}^2$ consisting of three subsets:

$$W = A \cup B \cup C$$

where:

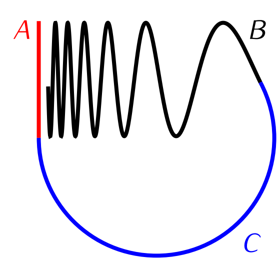
- A is a segment of the y-axis:

$$A = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$$

- B is a part of the graph of the function $f(x) = \sin\left(\frac{1}{x}\right)$:

$$B = \{(x, \sin\left(\tfrac{1}{x}\right)) \in \mathbb{R}^2 \mid 0 < x \leq \tfrac{1}{2\pi}\}$$

- C is an arc joining points $(0, -1) \in A$ and $(\frac{1}{2\pi}, 0) \in B$, and disjoint from $A \cup B$ at all other points.



Consider the pair (W, A). One can show that the quotient space W/A is homeomorphic to the circle S^1 (exercise), so in particular $\pi_1(W/A) \cong \mathbb{Z}$. On the other hand, $\pi_1(W) \cong \{1\}$ (exercise). Therefore W/A is not homotopy equivalent to W.

13.7 Theorem. *Any relative CW complex (X, Y) has the homotopy extension property.*

13.8 Lemma. *For any $n > 0$ the pair (D^n, S^{n-1}) has the homotopy extension property.*

While it is not difficult to prove Lemma 13.8 directly, we will show that it follows from a more general fact. Recall (8.15) that for a map $f: X \to Y$ the mapping cylinder of f is the space $M_f = (X \times [0,1] \sqcup Y)/\sim$ where $(x, 1) \sim f(x)$ for all $x \in X$. Notice that the space X is homeomorphic with the subspace $X \times \{0\} \subseteq M_f$.

13.9 Proposition. *For any continuous function $f: X \to Y$ the pair $(M_f, X \times \{0\})$ has the homotopy extension property.*

Proof. Exercise. □

Proof of Lemma 13.8. Let $c: S^{n-1} \to \{*\}$ be the constant function. We have a homeomorphism $f: M_c \to D^n$ given by $f(x, t) = (1 - t)x$. Moreover, $f(S^{n-1} \times \{0\}) = S^{n-1} \subseteq D^n$. Since by Proposition 13.9 the pair $(M_c, S^{n-1} \times \{0\})$ has the homotopy extension property it follows that (D^n, S^{n-1}) also has this property. □

13.10 Lemma. *Let Y be any space an let $X = Y \cup \{e^n_\alpha\}_{\alpha \in I}$ be a space obtained from by attaching some number of n-cells to X. Then the pair (X, Y) has the homotopy extension property.*

Proof. To simplify notation we will assume that X is obtained from Y by attaching a single n-cell: $X = Y \cup e^n$. The proof in the general case is essentially the same. By Proposition 13.2 it will suffice to show that $X \times \{0\} \cup Y \times [0,1]$ is a retract of $X \times [0,1]$. Let $f: S^{n-1} \to Y$ be the attaching map of the cell e^n. We have a homeomorphisms

$$X \times [0,1] \simeq (D^n \times [0,1] \sqcup Y \times [0,1])/\sim$$

and

$$X \times \{0\} \cup Y \times [0,1] \simeq ((D^n \times \{0\} \cup S^{n-1} \times [0,1]) \sqcup Y \times [0,1])/\sim$$

where $(x, t) \sim (f(x), t)$ for $x \in S^{n-1}$. By Lemma 13.8 there is a retraction

$$r: D^n \times [0,1] \to D^n \times \{0\} \cup S^{n-1} \times [0,1]$$

The map

$$r \sqcup \mathrm{id}_{Y \times [0,1]}: ((D^n \times \{0\} \cup S^{n-1} \times [0,1]) \sqcup Y \times [0,1])/\sim \;\to\; (D^n \times [0,1] \sqcup Y \times [0,1])/\sim$$

gives the desired retraction $X \times [0,1] \to X \times \{0\} \cup Y \times [0,1]$. □

Proof of Theorem 13.7. Recall (12.7) that if (X, Y) is a relative CW complex then $X = \bigcup_{n=-1}^{\infty} X^{(n)}$ where $X^{(-1)} = Y$ and for $n \geq 0$ the subspace of $X^{(n)} \subseteq X$ obtained by attaching n-cells to $X^{(n-1)}$. By Lemma 13.10 for each $n \geq 0$ there exists a retraction

$$r_n\colon X^{(n)} \times [0,1] \to X^{(n)} \times \{0\} \cup X^{(n-1)} \times [0,1]$$

We can extend r_n to a map

$$\bar{r}_n\colon X \times \{0\} \cup X^{(n)} \times [0,1] \to X \times \{0\} \cup X^{(n-1)} \times [0,1]$$

by setting $\bar{r}_n(x, 0) = (x, 0)$ for $x \in X$. Define:

$$r\colon X \times [0,1] \to X \times \{0\} \cup Y \times [0,1]$$

by $r(x, t) = \bar{r}_0 \circ \bar{r}_1 \circ \ldots \circ \bar{r}_n(x, t)$ if $x \in X^{(n)}$, $n \geq 0$, and $r(x, t) = (x, t)$ if $x \in X^{(-1)} = Y$. One can check that r is a well defined, continuous retraction (exercise).

□

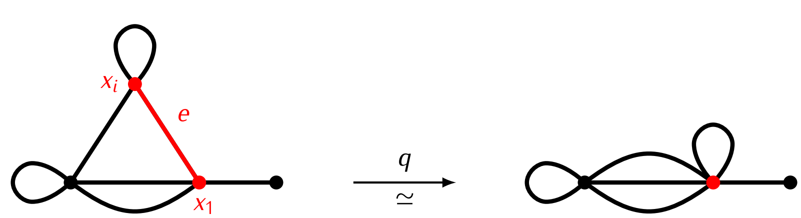
13.11 Theorem. *If X is a path connected finite CW complex of dimension 1 then $X \simeq \bigvee_{i=1}^{n} S^1$ where*

$$n = \begin{pmatrix} \textit{number of} \\ \textit{1-cells of } X \end{pmatrix} - \begin{pmatrix} \textit{number of} \\ \textit{0-cells of } X \end{pmatrix} + 1$$

13.12 Corollary. *If X is a path connected finite CW complex of dimension 1 then $\pi_1(X) \cong *_{i=1}^{n}\mathbb{Z}$ where n is defined as in Theorem 13.11.*

Proof. This follows from Theorem 13.11 and Example 10.19. □

Proof of Theorem 13.11. We will argue by induction with respect to the number k of 0-cells in X. If $k = 1$ then the statement is obvious. Assume then that the statement of theorem is true for all complexes whose number of 0-cells is k, and let X be a path connected finite 1-dimensional CW complex whose set of 0-cells is $\{x_1, x_2, \ldots, x_{k+1}\}$ for some $k \geq 1$. Since X is path connected there exists a 1-cell e in X that joins x_1 with some other 0-cell x_i. Let A denote the subcomplex of X consisting of the cells x_1, x_i and e. Notice that A is homeomorphic to the closed interval $[0, 1]$. The pair (X, A) is a relative CW complex, so by Theorem 13.7 it satisfies the homotopy extension property. Since A is contractible, by Proposition 13.4 the quotient map $q\colon X \to X/A$ is a homotopy equivalence.



The space X/A has the structure of a 1-dimensional CW complex with one 0-cell and one 1-cell less than X. Therefore, by the inductive assumption the statement of the theorem holds for X/A, and so it also holds for X.

□

Theorem 13.11 can be generalized to infinite 1-dimensional complexes:

13.13 Theorem. *If X is a path connected 1-dimensional CW complex then $X \simeq \bigvee_{I \in I} S^1$ for some set I. As a consequence $\pi_1(X) \cong *_{i \in I}\mathbb{Z}$.*

13.14 Note. For a finite CW complex X, let $c_n(X)$ denote the number of n-cells of X. Theorem 13.11 implies that if X is a path connected CW complex of dimension 1, then the number $c_1(X) - c_0(X)$ depends only on the homotopy type of X: if Y is another such CW complex and $X \simeq Y$ then $c_1(X) - c_0(X) = c_1(Y) - c_0(Y)$. This observation can be generalized as follows. The *Euler characteristic* of a finite CW complex X is the integer $\chi(X) = \sum_n (-1)^n c_n(X)$. One can show that if X and Y are finite CW complexes and $X \simeq Y$ then $\chi(X) = \chi(Y)$.

Exercises to Chapter 13

E13.1 Exercise. Prove Proposition 13.2.

E13.2 Exercise. Prove Proposition 13.4.

E13.3 Exercise. Show that if a pair (X, A) has the homotopy extension property then for any space Y the pair $(X \times Y, A \times Y)$ also has the homotopy extension property.

E13.4 Exercise. Prove Proposition 13.9.

E13.5 Exercise. Given spaces X, Y let $[X, Y]$ denote the set of homotopy classes of maps $f\colon X \to Y$. A map of spaces $g\colon X \to X'$ induces a map of sets $g^*\colon [X', Y] \to [X, Y]$ given by $g^*([f]) = [fg]$. Let $A \subseteq X$, let $j\colon A \to X$ be the inclusion and $q\colon X \to X/A$ be the quotient map. For any Y this induces maps of sets

$$[X/A, Y] \xrightarrow{q^*} [X, Y] \xrightarrow{j^*} [A, Y]$$

Show that if the pair (X, A) has the homotopy extension property then $j^*[f]$ is the homotopy class of a constant map $A \to Y$ if and only if $[f] = q^*[f']$ for some $f'\colon X/A \to Y$.

E13.6 Exercise. Let (X, x_0), (Y, y_0) be pointed spaces. Denote by $[X, Y]_*$ the set of pointed homotopy classes of basepoint preserving maps $X \to Y$. That is, any map $f\colon (X, x_0) \to (Y, y_0)$ defines an element $[f]_* \in [X, Y]_*$, and $[f]_* = [g]_*$ if $f \simeq g$ (rel $\{x_0\}$). Also, let $[X, Y]$ be the set of homotopy classes of all functions $X \to Y$. Thus, any map $f: X \to Y$ defines an element $[f] \in [X, Y]$, and $[f] = [g]$ if $f \simeq g$ (there is no assumption that maps or homotopies preserve the basepoints). Let

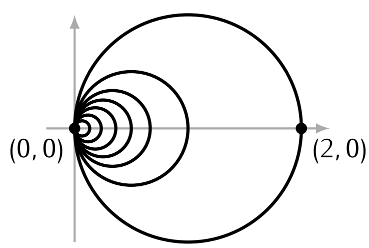
$$\Phi: [X, Y]_* \to [X, Y]$$

be a function given by $\Phi([f]_*) = [f]$.

a) Assume that the pair (X, x_0) has the homotopy extension property, and Y is a path connected space. Show that Φ is onto.

b) Assume that in addition the group $\pi_1(X, x_0)$ is trivial. Show that Φ is a bijection.

E13.7 Exercise. The *Hawaiian earring* space is a subspace $X \subseteq \mathbb{R}^2$ given by $X = \bigcup_{n=1}^{\infty} C_n$ where C_n is the circle with radius $\frac{1}{n}$ and center at the point $(0, \frac{1}{n})$:



Denote $x_0 = (0, 0)$ and $y_0 = (2, 0)$. Let $\mathrm{id}_X: X \to X$ be the identity map.

a) Show that there does not exist a map $g: X \to X$ such that $\mathrm{id}_X \simeq g$ and $g(x_0) = y_0$.

b) Show that the pair (X, x_0) does not have the homotopy extension property. (Hint: use Exercise 13.6).

E13.8 Exercise. Let (X, Y) be a relative CW complex, let $j: Y \to X$ be the inclusion map, and let C_j be the mapping cone of j. Show that C_j is homotopy equivalent to the space X/Y.

E13.9 Exercise. Assume that (X, A) is a pair with the homotopy extension property such that the inclusion map $i: A \hookrightarrow X$ is a homotopy equivalence.

a) Show that A is a retract of X.

b) Show that A is a strong deformation retract of X.

E13.10 Exercise. Let $f: (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. Show that if X is a path connected 1-dimensional CW complex and $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is the trivial homomorphism then f is homotopic to a constant map.

E13.11 Exercise. Let X be a finite, path connected CW complex.

a) Show that X is homotopy equivalent to a CW complex X' which has only one 0-cell.

b) Show that if $\pi_1(X) = \{1\}$ then X is homotopy equivalent to a CW complex X'' which has only one 0-cell and no 1-cells.

14 | Presentations of Groups

In this chapter we make here a brief algebraic interlude from the task of computing fundamental groups in order to discuss how groups can be described by means their *presentations*. This concept will be used in the next chapter where we will consider fundamental groups of 2-dimensional CW complexes.

14.1 Definition. Let S be a set. A *word* in S is a finite sequence of the form $a_1^{k_1}a_2^{k_2}\dots a_n^{k_n}$ where $n \geq 0$, $a_i \in S$ and $k_i \in \mathbb{Z}$. The *free group generated by* S is the group $F(S)$ whose elements are words in S with the the following identifications:

- if $a_i = a_{i+1}$ then
$$a_1^{k_1}\dots a_i^{k_i}a_{i+1}^{k_{i+1}}\dots a_n^{k_n} = a_1^{k_1}\dots a_i^{(k_i+k_{i+1})}\dots a_n^{k_n}$$
- if $k_i = 0$ then
$$a_1^{k_1}\dots a_{i-1}^{k_{i-1}}a_i^{k_i}a_{i+1}^{k_{i+1}}\dots a_n^{k_n} = a_1^{k_1}\dots a_{i-1}^{k_{i-1}}a_{i+1}^{k_{i+1}}\dots a_n^{k_n}$$

Multiplication in $F(S)$ is given by concatenation of words:

$$(a_1^{k_1}\dots a_n^{k_n})\cdot(b_1^{l_1}\dots b_m^{l_m}) = a_1^{k_1}\dots a_n^{k_n}b_1^{l_1}\dots b_m^{l_m}$$

The identity element in $F(S)$ is given by the empty word (i.e. the word of length 0).

14.2 Note. If $S = \varnothing$ then $F(S)$ is the trivial group. If $S = \{a\}$ is a set consisting of one element then $F(S) \cong \mathbb{Z}$. In general, the group $F(S)$ isomorphic to the free product of free groups generated by the elements of S:

$$F(S) \cong *_{a\in S}F(\{a\}) \cong *_{a\in S}\mathbb{Z}$$

14.3 Note. We will say that a group G is free if G is isomorphic to the group $F(S)$ for some set S. Notice that by Theorem 13.13 the fundamental group of any 1-dimensional CW complex is free.

For any set S we have a map of sets: $i\colon S \to F(S)$ given by $f(a) = a$ (where we consider $a \in F(S)$ as a word of length 1). The statement of the following fact is called the *universal property of free groups*:

14.4 Theorem. *Let S be a set and G be a group. For any map of sets $f\colon S \to G$ there exists a unique homomorphism of groups $\bar{f}\colon F(S) \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i"'] & G \\ F(S) \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

Proof. The homomorphism $\bar{f}$ is given by $\bar{f}(a_1^{k_1}a_2^{k_2}\ldots a_n^{k_n}) := f(a_1)^{k_1} \cdot f(a_2)^{k_2} \cdot \ldots \cdot f(a_n)^{k_n}$. $\square$

14.5 Definition. Let S be a set, and let R be a subset of elements of the free group $F(S)$. By $\langle S \mid R\rangle$ we denote the group given by

$$\langle S \mid R\rangle = F(S)/N$$

where N is the smallest normal subgroup of $F(S)$ such that $R \subseteq N$. We say that elements of S are *generators* of $\langle S \mid R\rangle$ and elements of R are *relations* in $\langle S \mid R\rangle$.

14.6 Example. For any set we have S is a set $F(S) \cong \langle S \mid \varnothing\rangle$.

14.7 Example. $\langle a \mid a^n\rangle \cong \mathbb{Z}/n\mathbb{Z}$.

14.8 Example. $\langle a, b \mid aba^{-1}b^{-1}\rangle \cong \mathbb{Z} \times \mathbb{Z}$.

14.9 Definition. If G is a group and $G \cong \langle S \mid R\rangle$ for some set S and some $R \subseteq F(S)$ then we say that $\langle S \mid R\rangle$ is a *presentation* of G.

14.10 Definition. If a group G has a presentation $\langle S \mid R\rangle$ such that S is a finite set then we say that G is *finitely generated* and if it has a presentations such that both S and R are finite sets then we say that G is *finitely presented*.

14.11 Proposition. *Every group has a presentation.*

Proof. Let G be a group and let $f\colon S \to G$ be a map of sets which is onto. By Theorem 14.4 the function f defines a homomorphism $\bar{f}\colon F(S) \to G$. Since f is onto thus so is $\bar{f}$. This gives an isomorphism $G \cong F(S)/\operatorname{Ker}(\bar{f})$. If follows that $G \cong \langle S \mid R\rangle$ where R is the set of elements of $\operatorname{Ker}(\bar{f})$. $\square$

14.12 Note. 1) Every group has inifinitely many different presentations. For example

$$\mathbb{Z} \cong \langle a\rangle \cong \langle a, b \mid b\rangle \cong \langle a, b \mid ab^{-1}\rangle \cong \langle a, b \mid b^2, b^3\rangle$$

2) In general if we know a presentation of a group it may be very difficult to say anything about the properties of the group (even if the group is trivial or not).

Exercises to Chapter 14

E14.1 Exercise. Below are three groups described by their presentations. For each group decide if it is abelian and if it is finite. Justify your answers.

a) $G_1 = \langle a, b \mid a^3, b^3, aba^2b^2 \rangle$

b) $G_2 = \langle a, b \mid a^2, aba \rangle$

c) $G_3 = \langle a, b \mid a^4, b^4, a^2b^2 \rangle$

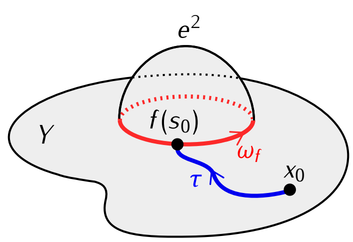
15 | Fundamental Group and 2-Cells

We return to the problem of computing fundamental groups of CW complexes. Building on results of Chapter 13 we will consider here CW complexes of dimension 2. Any such complex X is obtained by glueing some 2-cells to its 1-skeleton $X^{(1)}$. Since $X^{(1)}$ is a CW complex of dimension 1 its fundamental group is described by Theorems 13.11 and 13.13. In effect, we only need to determine how the fundamental group of a space changes if we attach to it 2-cells. We will consider first the case when only one 2-cell is attached. We will use the following setup. Let Y be a space and let $X = Y \cup_f e^2$ where $f\colon S^1 \to Y$ is the attaching map. Let $\omega\colon [0, 1] \to S^1$ be the map given by $\omega(s) = (\cos(2\pi s), \sin(2\pi s))$. Since ω is a loop in S^1 based at the point $s_0 = (1, 0)$ thus $f\omega$ is a loop in Y based at the point $f(s_0)$. We will denote the loop $f\omega$ by ω_f.

15.1 Theorem. *Let Y be a path connected space and let $X = Y \cup_f e^2$. Let $x_0 \in Y$ and let τ be a path in Y such that $\tau(0) = x_0$ and $\tau(1) = f(s_0)$. The homomorphism*

$$j_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$$

induced by the inclusion map $j\colon Y \to X$ is onto, and so $\pi_1(X, x_0) \cong \pi_1(Y, y_0)/\operatorname{Ker}(j_*)$. Moreover, $\operatorname{Ker}(j_*)$ *is the normal subgroup of $\pi_1(Y, x_0)$ generated by the element* $[\tau * \omega_f * \overline{\tau}]$.



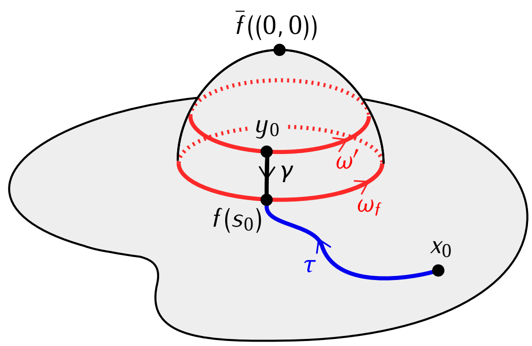
15.2 Note. In general the element $[\tau * \omega_f * \overline{\tau}] \in \pi_1(X, x_0)$ depends on the choice of the path τ. However, as the statement of Theorem 15.1 implies the normal subgroup of $\pi_1(Y, x_0)$ generated by such element does not depend on the choice of τ.

Proof of Theorem 15.1. Let $\bar{f}\colon D^2 \to X$ be the characteristic map of the cell e^2 and let $y_0 = \bar{f}((\frac{1}{2}, 0))$. Let $U_1 := X \smallsetminus \bar{f}((0,0))$ and $U_2 := X \smallsetminus Y$. The sets U_1, U_2, and $U_1 \cap U_2$ are path connected and open in X. Moreover, $U_1 \cup U_2 = X$ and $y_0 \in U_1 \cap U_2$. By van Kampen's Theorem we obtain

$$\pi_1(X, y_0) = \operatorname{colim}(\pi_1(U_1, y_0) \xleftarrow{i_{1*}} \pi_1(U_1 \cap U_2, y_0) \xrightarrow{i_{2*}} \pi_1(U_2, y_0))$$

where for $k = 1, 2$ the map $i_k\colon U_1 \cap U_2 \to U_k$ is the inclusion. The space U_2 is contractible, so $\pi_1(U_2, y_0) \cong \{1\}$. Also, $U_1 \cap U_2 \simeq S^1$, so $\pi_1(U_1 \cap U_2, y_0) \cong \mathbb{Z}$. The generator of $\pi_1(U_1 \cap U_2, y_0)$ is represented by the loop $\omega'\colon [0,1] \to U_1 \cap U_2$ where $\omega'(s) = \bar{f}((\frac{1}{2}\cos(2\pi s), \frac{1}{2}\sin(2\pi s)))$. These observations show that the homomorphism $j_*\colon \pi_1(U_1, y_0) \to \pi_1(X, y_0)$ induced by the inclusion $j\colon U_1 \to X$ is onto, and $\ker j_*$ is the normal subgroup of $\pi_1(U_1, y_0)$ generated by $[\omega']$.

Let $\gamma\colon [0,1] \to X$ be the path defined by $\gamma(s) = \bar{f}(\frac{1}{2}(1+s), 0)$. We have $\gamma(0) = y_0$ and $\gamma(1) = f(s_0)$. This path defines the change-of-basepoint isomorphisms $s_\gamma\colon \pi_1(X, y_0) \to \pi_1(X, f(s_0))$ and $s_\gamma\colon \pi_1(U_1, y_0) \to \pi_1(U_1, f(s_0))$. Notice that in $\pi_1(U_1, f(s_0))$ we have $s_\gamma([\omega']) = [\bar{\gamma} * \omega * \gamma] = [\omega_f]$:



Consider the following diagram:

$$\begin{tikzcd}
\pi_1(U_1, y_0) \arrow[r, "s_\gamma", "\cong"'] \arrow[d, "j_*"'] & \pi_1(U_1, f(s_0)) \arrow[d, "j_*"'] & \pi_1(Y, f(s_0)) \arrow[l, "i_*"', "\cong"] \arrow[r, "s_{\overline{\tau}}", "\cong"'] \arrow[d, "j_*"] & \pi_1(Y, x_0) \arrow[d, "j_*"] \\
\pi_1(X, y_0) \arrow[r, "\cong", "s_\gamma"'] & \pi_1(X, f(s_0)) & \pi_1(X, f(s_0)) \arrow[l, "\cong"', "\mathrm{id}"] \arrow[r, "\cong", "s_{\overline{\tau}}"'] & \pi_1(X, x_0)
\end{tikzcd}$$

Each square in this diagram commutes. The vertical homomorphisms are induced by inclusion maps. The homomorphisms s_γ and $s_{\overline{\tau}}$ are the change-of-basepoint isomorphisms defined by the paths γ and $\overline{\tau}$. The homomorphism $i_*\colon \pi_1(Y, f(s_0)) \to \pi_1(U_1, f(s_0))$ is induced by the inclusion map $Y \to U_1$, and it is an isomorphism since Y is a deformation retract of U_1. Since all horizontal homomorphisms are isomorphisms and $j_*\colon \pi_1(U_1, d_0) \to \pi_1(X, d_0)$ is onto we obtain that $j_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$ is also onto. Moreover, since the kernel of the first of these two homomorphisms is generated by the element $[\omega'] \in \pi_1(U_1, d_0)$ the kernel of the second is generated by the element $s_{\overline{\tau}}(i_*^{-1}(s_\gamma([\omega']))) = s_{\overline{\tau}}(i_*^{-1}([\omega_f])) = s_{\overline{\tau}}([\omega_f]) = [\tau * \omega_f * \overline{\tau}]$.

$\square$

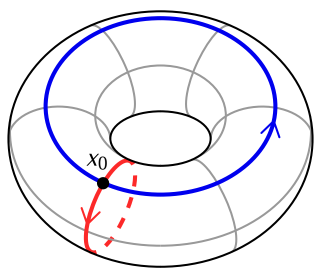
Theorem 15.1 can be generalized to the case when we are attaching an arbitrary (finite or infinite) number of 2-cells. In the statement below we use the same notation as in Theorem 15.1: if f is an attaching map of a 2-cell then by ω_f we denote the loop defined by f and by $f(s_0)$ the starting/ending point of this loop.

15.3 Theorem. *Let Y be a path connected space with basepoint $x_0 \in Y$ and let X be a space obtained by attaching to Y a collection of 2-cells: $X = Y \cup \{e^2_i\}_{i\in I}$. Let $f_i\colon S^1 \to X$ be the attaching map of the cell e^2_i and let $\tau_i\colon [0,1] \to Y$ be a path such that $\tau_i(0) = x_0$ and $\tau_i(1) = f_i(s_0)$. The homomorphism*

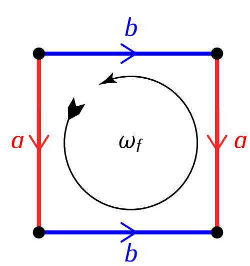
$$j_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$$

induced by the inclusion map $j\colon Y \to X$ is onto, and so $\pi_1(X, x_0) \cong \pi_1(Y, y_0)/\operatorname{Ker}(j_*)$. Moreover, $\operatorname{Ker}(j_*)$ is the normal subgroup of $\pi_1(Y, x_0)$ generated by set $\{[\tau_i * \omega_{f_i} * \overline{\tau}_i]\}_{i\in I}$.

15.4 Example. Recall that the torus T has a CW complex structure with one 0-cell, two 1-cells and one 2-cell:



We will take the 0-cell x_0 as the basepoint of T. The 1-skeleton $T^{(1)}$ of the torus is homeomorphic to $S^1 \vee S^1$, so $\pi_1(T^{(1)}, x_0) \cong \mathbb{Z} * \mathbb{Z}$. Denote by a the element of $\pi_1(T^{(1)}, x_0)$ represented by the loop that goes once around one 1-cell and by b the element represented by the loop that goes once around the second 1-cell. The elements a, b freely generate the group $\pi_1(T^{(1)}, x_0)$, i.e. we have an isomorphism $\pi_1(T^{(1)}, x_0) \cong \langle a, b\rangle$. Let $f\colon S^1 \to T^{(1)}$ be the attaching map of the 2-cell. The loop ω_f represents the element $aba^{-1}b^{-1} \in \pi_1(T^{(1)}, x_0)$:

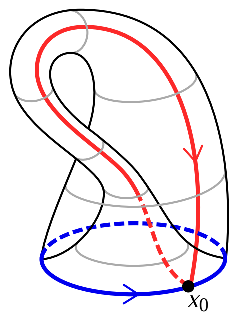


By Theorem 15.1 we get $\pi_1(T, x_0) \cong \langle a, b\rangle/N$ where N is the normal subgroup of $\langle a, b\rangle$ generated by the element $aba^{-1}b^{-1}$. In other words we obtain a presentation of $\pi_1(T, x_0)$:

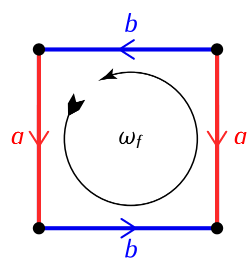
$$\pi_1(T, x_0) \cong \langle a, b \mid aba^{-1}b^{-1}\rangle$$

Since $\langle a, b \mid aba^{-1}b^{-1}\rangle \cong \mathbb{Z} \times \mathbb{Z}$ we recover the result we obtained before (9.2) using the product formula for the fundamental group.

15.5 Example. Recall the CW complex structure on the Klein bottle K:

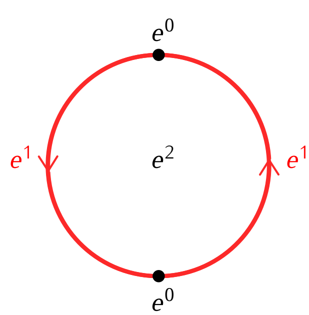


Again, we will take the 0-cell in this complex as the basepoint. The 1-skeleton $K^{(1)}$ is homeomorphic to $S^1 \vee S^1$, so $\pi_1(K^{(1)}, x_0) \cong \langle a, b\rangle$ where a, b are represented by the loops traversing each of the 1-cells. The attaching map of the 2-cell represents the element $aba^{-1}b \in \langle a, b\rangle$:



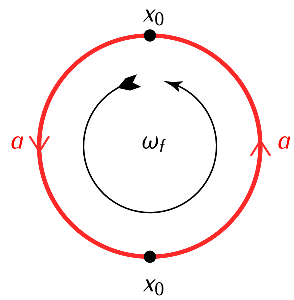
It follows that $\pi_1(K, x_0) \cong \langle a, b \mid aba^{-1}b\rangle$.

15.6 Example. Recall that 2-dimensional projective space $\mathbb{RP}^2$ has a cell structure with one 0-cell, one 1-cell and one 2-cell:



We will take the 0-cell as the basepoint $x_0 \in \mathbb{RP}^2$. The one skeleton $(\mathbb{RP}^2)^{(1)}$ is homeomorphic to S^1, so $\pi_1((\mathbb{RP}^2)^{(1)}, x_0) \cong \langle a\rangle$ where a denotes the generator represented by the loop that traverses the

1-cell. The attaching map $f\colon S^1 \to (\mathbb{RP}^2)^{(1)}$ for the 2-cell corresponds to the element a^2:



As a consequence $\pi_1(\mathbb{RP}^2, x_0) \cong \langle a \mid a^2\rangle \cong \mathbb{Z}/2$.

15.7 Note. The computations above show right away that the torus T is not homotopy equivalent to $\mathbb{RP}^2$ since $\pi_1(T) \cong \mathbb{Z}\oplus\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{RP}^2)$. It may be less clear though whether the fundamental group of the Klein bottle is isomorphic or not to one of these groups. We can resolve this problems as follows. If G is a group and $g, h \in G$ then the *commutator* of g and h is the element $[g, h] = ghg^{-1}h^{-1}$. Notice that $[g, h]$ is the trivial element if and only if $gh = hg$. The *commutator subgroup* of G is the subgroup $[G, G]$ generated by the set $\{[g, h] \mid g, h \in G\}$. Since $[G, G]$ is a normal subgroup of G (exercise) we can consider the quotient group $G_{ab} = G/[G, G]$ which is called the *abelianization* of G. The construction of G_{ab} has the following properties (exercise):

- G_{ab} is an abelian group.
- If G is an abelian then $G_{ab} \cong G$.
- If $f\colon G \to H$ is a group homomorphism then $f([G, G]) \subseteq [H, H]$, and so f induces a homomorphism $f_{ab}\colon G_{ab} \to H_{ab}$.
- Recall that **Gr** denotes the category of all groups. Denote by **Ab** the category of abelian groups whose elements are all abelian groups and morphisms are homomorphisms of such groups. The assignments $F(G) = G_{ab}$ and $F(f) = f_{ab}$ define a functor

$$F\colon \mathbf{Gr} \to \mathbf{Ab}$$

  This implies in particular that if $G \cong G'$ then $G_{ab} \cong G'_{ab}$.

Since the groups $\pi_1(T)$ and $\pi_1(\mathbb{RP}^2)$ are abelian we have $\pi_1(T)_{ab} \cong \pi_1(T)$ and $\pi_1(\mathbb{RP}^2)_{ab} \cong \pi_1(\mathbb{RP}^2)$. On the other hand abelianization of the fundamental group of the Klein bottle $\pi_1(K) \cong \langle a, b \mid aba^{-1}b\rangle$ is obtained by imposing the condition that $ab = ba$, or equivalently adding the relation $aba^{-1}b^{-1}$ to the presentation of the group:

$$\pi_1(K)_{ab} \cong \langle a, b \mid aba^{-1}b,\ aba^{-1}b^{-1}\rangle$$

Notice that if $ab = ba$ then $aba^{-1}b = b^2$. Therefore we obtain:

$$\pi_1(K)_{ab} \cong \langle a, b \mid b^2,\ aba^{-1}b^{-1}\rangle \cong \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$$

Since $\pi_1(K)_{ab}$ is isomorphic to neither $\pi_1(T)_{ab}$ nor $\pi_1(\mathbb{RP}^2)_{ab}$ we get that the $K \not\simeq T$ and $K \not\simeq \mathbb{RP}^2$.

As the final result of this chapter we will show that any group can be realized as the fundamental group of some CW complex:

15.8 Theorem. *For any group G there exists a CW complex X such that $\pi_1(X) \cong G$.*

Proof. We will construct the complex X as follows. The 0-skeleton of X consists of a single 0-cell that we will take as the basepoint x_0. The 1-skeleton of X is the wedge of circles, one copy of S^1 for each element of G:

$$X^{(1)} = \bigvee_{g \in G} S^1$$

Let ω_g denote the loop in $X^{(1)}$ that traverses the copy of S^1 corresponding the element $g \in G$. The group $\pi_1(X^{(1)}, x_0)$ is the free group generated by the set $T = \{[\omega_g] \in \pi_1(X^{(1)}, x_0 \mid g \in G\}$. Consider the function of sets $f\colon T \to G$ given by $f([\omega_g]) = g$. By the universal property of free groups (14.4) this function defines a homomorphims of groups $\bar{f}\colon \pi_1(X^{(1)}, x_0) \to G$. Moreover, since f is onto thus so is $\bar{f}$. As a consequence we obtain that $G \cong \pi_1(X^{(1)}, x_0)/\operatorname{Ker}(\bar{f})$. For each element $r \in \operatorname{Ker}(\bar{f})$ let $\sigma_r\colon [0,1] \to X^{(1)}$ be a loop representing r. Recall (3.18) that such loop can be identified with a map $S^1 \to X^{(1)}$. By abuse of notation we will denote this map also by σ_r. Let X be the CW complex obtained from $X^{(1)}$ by attaching one 2-cell for each element $r \in \operatorname{Ker}(\bar{f})$ with the attaching map given by σ_r. By Theorem 15.3 we obtain

$$\pi_1(X, x_0) \cong \pi_1(X^{(1)}, x_0)/\operatorname{Ker}(\bar{f}) \cong G$$

$\square$

15.9 Note. By modifying the proof Theorem 15.8 we can show that if a group G has a presentation $G \cong \langle S \mid R \rangle$ then G is isomorphic to the fundamental group of a CW complex X which has one 1-cell for each element of S and one 2-cell for each element of R. In particular G is finitely generated if and only if $G \cong \pi_1(X)$ for some CW complex X such that $X^{(1)}$ is finite, and and G is finitely presented if and only if $G \cong \pi_1(X)$ for some finite CW complex X.

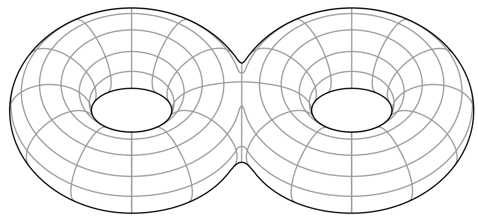
Exercises to Chapter 15

E15.1 Exercise. Let $x_1, x_2, x_3 \in S^2$ and let X be the quotient space obtained by identifying these three points. Put a CW complex structure on X and use it to compute $\pi_1(X)$.

E15.2 Exercise. Let T be the torus. The *connected sum* $T\#T$ is the space obtained as follows. Take $D \subseteq T$ to be a subspace homeomorphic to the closed disk D^2 and let $\operatorname{Int}(D)$, $\operatorname{Bd}(D)$ denote, respectively, the interior and the boundary of D. The space $\operatorname{Int}(D)$ is homeomorphic to the open disc and $\operatorname{Bd}(D)$ to the circle S^1. Take the space $\{0,1\}$ with the discrete topology. We set:

$$T\#T = (T \smallsetminus D) \times \{0,1\}/{\sim}$$

where $(x, 0) \sim (x, 1)$ for each $x \in \mathrm{Bd}(D)$. In other words $T\#T$ is obtained by taking two copies of the torus, cutting out a hole in each copy, and glueing the boundaries of the holes together:



a) Put a CW complex structure on $T\#T$ and use it to find a presentation of the fundamental group of $T\#T$.

b) Show that $T\#T$ is not homotopy equivalent to T.

Note: the homeomorphism type of $T\#T$ does not depend on which embedding of the disk into T one chooses, so you can work with whichever embedding is most convenient.

E15.3 Exercise. Consider S^1 as the unit circle in the complex plane

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

For $n = 1, 2, \ldots$ let T_n be a space given by

$$T_n = S^1 \times [0, 1]/\sim$$

where $(z, 0) \sim (z^n, 1)$ (in particular T_1 is just a torus). Find the fundamental group of T_n, and show that if $n \neq m$ then T_n is not homotopy equivalent to T_m.

E15.4 Exercise. Let X be a 2-dimensional CW complex. Assume that X has only one 0-cell $x_0 \in X$. Let (Y, y_0) be a pointed space. Show that for any group homomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ there exists a map $f\colon (X, x_0) \to (Y, y_0)$ such that $f_* = \varphi$.

16 | Cellular Approximation Theorem

Results of the last few chapters tell us how to compute the fundamental group of a CW complex of dimension 2 or lower. In this chapter we show that this actually suffices to compute the fundamental group of any CW complex X, since the fundamental group of X is always isomorphic to the fundamental group of its 2-skeleton $X^{(2)}$. This fact is a consequence of the Cellular Approximation Theorem which, in general, is one of the main tools used when working with CW complexes.

16.1 Definition. Let X, Y be CW complexes. A map $f\colon X \to Y$ is *cellular* if $f(X^{(n)}) \subseteq Y^{(n)}$ for all $n \geq 0$.

16.2 Cellular Approximation Theorem. *Let X, Y be CW complexes. For any map $f\colon X \to Y$ there exists a cellular map $g\colon X \to Y$ such that $f \simeq g$. Moreover, if $A \subseteq X$ is a subcomplex and $f|_A\colon A \to Y$ is a cellular map then g can be selected so that $f|_A = g|_A$ and $f \simeq g$ (rel A).*

Before proving this result we will show how it lets us identify the fundamental group of any CW complex with the fundamental group of its 2-skeleton.

16.3 Theorem. *Let X be a CW complex and let $x_0 \in X^{(2)}$. The inclusion map $i\colon X^{(2)} \to X$ induces an isomorphism $i_*\colon \pi_1(X^{(2)}, x_0) \to \pi_1(X, x_0)$.*

Proof. We can assume that x_0 is a 0-cell in X. We will prove first that i_* is onto. Let $\omega\colon [0,1] \to X$ be a loop based at x_0. We need to show that there exists a loop $\omega'\colon [0,1] \to X$ such that $\omega'([0,1]) \subseteq X^{(2)}$ and that $\omega \simeq \omega'$ (rel $\{0,1\}$). Consider the interval $[0,1]$ as a CW complex with two 0-cells joined by one 1-cell. The 0-skeleton of $[0,1]$ is the subspace $\{0,1\} \subseteq [0,1]$. Since $\omega(0) = \omega(1) = x_0 \in X^{(0)}$ the map $\omega|_{\{0,1\}}$ is cellular. By Theorem 16.2 there exists a cellular map $\omega'\colon [0,1] \to X$ such that $\omega' \simeq \omega$ (rel $\{0,1\}$). This means that $[\omega] = [\omega']$ in $\pi_1(X, x_0)$. Moreover, since $[0,1]$ is a CW complex of dimension 1 thus ω' is a loop in $X^{(1)} \subseteq X^{(2)}$.

Next, we will show that i_* is 1-1. Let $[\omega], [\tau] \in \pi_1(X^{(2)}, x_0)$. Using the same argument as above we can assume that $\omega, \tau \colon [0,1] \to X^{(2)}$ are cellular maps. Assume that $i_*([\omega]) = i_*([\tau])$. This means that there exists a path homotopy $h \colon [0,1] \times [0,1] \to X$ with $h_0 = \omega$ and $h_1 = \tau$. The square $I^2 = [0,1] \times [0,1]$ can be considered as a CW complex whose 0-cells are vertices of the square and whose 1-cells are the edges. The 1-skeleton of I^2 is the boundary ∂I^2. Notice that $h|_{\partial I^2}$ is a cellular map. Using Theorem 16.2 we obtain that there exists a cellular map $h' \colon [0,1] \times [0,1] \to X$ such that $h'|_{\partial I^2} = h|_{\partial I^2}$. The map h' gives another path homotopy between ω and τ. Moreover, since $\dim I^2 = 2$ thus h' is a homotopy contained in $X^{(2)}$. This shows that $[\omega] = [\tau]$ in $\pi_1(X^{(2)}, x_0)$

□

The rest of this chapter will be devoted to a proof of Theorem 16.2. The proof will be split into several lemmas.

16.4 Lemma. *Let Y be a space, and let Y' be obtained from Y by attaching a single n-cell:*

$$Y' = Y \cup e^n$$

Let $f \colon D^m \to Y'$ be a map such that $f(S^{m-1}) \subseteq Y$. If $m < n$ then there exists a map $g \colon D^m \to Y'$ such that $f|_{S^{m-1}} = g|_{S^{m-1}}$, $f \simeq g$ (rel S^{m-1}) and that for some point $y_0 \in e^n$ we have $y_0 \notin g(D^n)$.

Idea of the proof. This is the most technical step in the proof of Theorem 16.2. Let $\varphi \colon D^n \to Y'$ be the characteristic map of the cell e^n. Let $B_{1/2} \subseteq D^n$ be the open ball with the center at the origin and radius 1/2, and let $U = \varphi(B_{1/2}) \subseteq Y'$. The map φ restricts to a homeomorphism $U \cong B_{1/2}$, so we can identify U with an open set in $\mathbb{R}^n$.

Since the disc D^m is homeomorphic to the cube $K = [0,1]^m$, we can consider f as a function $f \colon K \to Y'$. One can show that the cube K can be subdivided into a finite number m-dimensional polyhedra $K_1, \dots, K_N$ in such way that there exists a function $g \colon K \to Y'$ satisfying the following conditions:

(i) $g \simeq f$ (rel ∂K) (where ∂K is the boundary of the cube K)

(ii) For each polyhedron $K_i \subseteq K$ such that $g(K_i) \cap U \neq \varnothing$, the restriction $g|_{K_i} \colon K_i \to U$ is a linear function. We use here the identification of U with an open set in $\mathbb{R}^n$ to make sense of linearity of these maps.

Property (ii) implies that the set $g(K) \cap U$ is contained in the set $\bigcup_{K_{i_j}} g(K_{i_j})$ where the union is taken over all polyhedra K_{i_j} on which g is linear. Since the union of images of finitely many linear (or, more precisely, affine) functions $\mathbb{R}^m \to \mathbb{R}^n$ with $m < n$ does not contain any open set in $\mathbb{R}^n$, we obtain that $g(K)$ does not contain the whole set U, and so it does not contain the whole cell e^n.

□

16.5 Lemma. *Let Y be a space, and let Y' be obtained from Y by attaching a single n-cell:*

$$Y' = Y \cup e^n$$

Let $f\colon D^m \to Y'$ be a map such that $f(S^{m-1}) \subseteq Y$. If $m < n$ then there exists a map $g\colon D^m \to Y'$ such that $g(D^m) \subseteq Y$, $f|_{S^{m-1}} = g|_{S^{m-1}}$ and $f \simeq g$ (rel S^{m-1}).

Proof. By Lemma 16.4 there exists a function $g'\colon D^m \to Y'$ such that $f \simeq g'$ (rel S^{m-1}) and such that $y_0 \notin g'(D^m)$ for some $y_0 \in e^n$. We can consider g' as a map $g'\colon D^m \to Y' \smallsetminus \{y_0\}$. One can show (exercise) that there exists a map $h\colon (Y' \smallsetminus \{y_0\}) \times [0,1] \to Y' \smallsetminus \{y_0\}$ which is a deformation retraction of $Y' \smallsetminus \{y_0\}$ onto Y. The function $h_1 g$ is homotopic to g' (rel S^{m-1}) and $h_1 g'(D^m) \subseteq Y$. Thus we can take $g = h_1 g'$. □

16.6 Lemma. *Let Y be a space, and let Y' be obtained from Y by attaching n-cells:*

$$Y' = Y \cup \bigcup_{i \in I} e^n_i$$

Let $f\colon D^m \to Y'$ be a map such that $f(S^{m-1}) \subseteq Y$. If $m < n$ then there exists a map $g\colon D^m \to Y'$ such that $g(D^m) \subseteq Y$, $f|_{S^{m-1}} = g|_{S^{m-1}}$ and $f \simeq g$ (rel S^{m-1}).

Proof. Since D^m is a compact space, by Proposition 12.18 the set $f(D^m)$ has a non-empty intersection with only finitely many n-cells $e^n_{i_1}, \dots, e^n_{i_k}$. We will prove the lemma by induction with respect to the number k of these cells.

If $k = 0$ then $f(D^m) \subseteq Y$ and we can take $g = f$. Next, assume that the lemma is true for some $k \geq 0$, and let $f\colon D^m \to Y'$ be a functiom such that $f(D^m)$ has non-empty intersections with $k+1$ cells $e^n_{i_1}, \dots, e^n_{i_{k+1}}$. Let Z be the subcomplex of Y' consisting of Y and these cells. We can consider f as a function $f\colon D^m \to Z$. Notice that Z can be viewed as space obtained by attaching a single cell e^n_{k+1} to $Z' = Y \cup \bigcup_{i=1}^k e^n_i$. Therefore, by Lemma 16.5 the function f is homotopic relative S^{m-1} to a function $f'\colon D^m \to Z$ such that $f'(D^m) \subseteq Z'$. Since $f'(D^m)$ intersects non-trivially with at most k cells of dimension n, by the inductive assumption it is homotopic (rel S^{m-1}) to a function g such that $g(D^m) \subseteq Y$. Therefore we obtain $f \simeq f' \simeq g$ (rel S^{m-1}). □

16.7 Lemma. *Let Y be a CW complex, and $f\colon D^m \to Y$ be a map such that $f(S^{m-1}) \subseteq Y^{(m-1)}$. Then there exists a map $g\colon D^m \to Y$ such $g(D^m) \subseteq Y^{(m)}$, $f|_{S^{m-1}} = g|_{S^{m-1}}$ and $f \simeq g$* (rel S^{m-1}).

Proof. Since D^m is a compact space, by Proposition 12.18 the set $f(D^m)$ has a non-empty intersection with finitely many cells of Y only. In particular, $f(D^m)$ is contained in an n-skeleton $Y^{(n)}$ of Y for some $n \geq 0$. If $n > m$, then using Lemma 16.6 we get that f is homotopic (rel S^{m-1}) to a function f' such that $f'(D^m) \subseteq Y^{(n-1)}$. Arguing inductively, we obtain the statement of the lemma. □

16.8 Lemma. *Let X, Y be CW complexes and $A \subseteq X$ be a subcomplex. Also, let $f\colon X \to Y$ be a map which is cellular on $A \cup X^{(m)}$ for some $m \geq -1$. Then there exists a map $g\colon X \to Y$ such that g is cellular on $A \cup X^{(m+1)}$, $f|_{A \cup X^{(m)}} = g|_{A \cup X^{(m)}}$ and $f \simeq g$* (rel $A \cup X^{(m)}$).

Proof. Assume first that $m = -1$. Since $X^{(-1)} = \varnothing$, thus f is a map cellular on A. We want to show that there exists a function $g\colon X \to Y$ such that $f \simeq g$ (rel A) and that g is cellular on $A \cup X^0$. The complex $A \cup X^0$ is a disjoint union

$$A \cup X^0 = A \sqcup \{e_i^0\}_{\in I}$$

where e_i^0 are 0-cells of X not contained in A. Since every path connected component of Y contains some 0-cell, for each $i \in I$ we can find a path $\omega_i\colon [0,1] \to Y$ such that $\omega_i(0) = f(e_i^0)$ and $\omega_i(1) \in Y^{(0)}$. Define a homotopy $h\colon (A \cup X^0) \times [0,1] \to Y$ by $h(x,t) = f(x)$ for $x \in A$ and $h(e_i^0, t) = \omega_i(t)$. By Theorem 13.7 this homotopy can be extended to a homotopy $\bar{h}\colon X \times [0,1] \to Y$ between f and a certain function $g\colon X \to Y$. Directly from this construction it follows that $f \simeq g$ (rel A) and that g is cellular on $A \cup X^{(0)}$.

Next, assume that $m \geq 0$. Then f is a function cellular on $A \cup X^{(m)}$, and we want to obtain a function g cellular on $A \cup X^{(m+1)}$. We have

$$A \cup X^{(m+1)} = (A \cup X^{(m)}) \cup \bigcup_{i \in I} e_i^{m+1}$$

where e_i^{m+1} are $(m+1)$-cells of X not contained in A. Let $\varphi_i\colon D^{m+1} \to X$ be the characteristic map of the cell e_i^{m+1} (12.2). Since $\varphi_i(S^m) \subseteq X^{(m)}$ and f is cellular on $X^{(m)}$ we obtain that $f\varphi_i(S^m) \subseteq Y^{(m)}$. Therefore, by Lemma 16.7 there exists a homotopy $h_i\colon D^{m+1} \times [0,1] \to Y$ (rel S^m) between $f\varphi_i$ and some map $\psi_i\colon D^{m+1} \to Y$ such that $\psi_i(D^{m+1}) \subseteq Y^{(m+1)}$. Define a homotopy $h\colon (A \cup X^{(m+1)}) \times [0,1] \to Y$ by $h(x,t) = f(x)$ for $x \in A \cup X^{(m)}$ and $h(x,t) = h_i(y,t)$ for $x = \varphi_i(y) \in e_i^{m+1}$. Using Theorem 13.7 again, we can extend this homotopy to a homotopy $\bar{h}\colon X \times [0,1] \to Y$ between f and some function $g\colon X \to Y$. The construction of the homotopy $\bar{h}$ implies that g is cellular on $A \cup X^{(m+1)}$ and that $f \simeq g$ (rel $A \cup X^{(m)}$). $\square$

Proof of Theorem 16.2. Let X, Y be CW complexes, let $A \subseteq X$ be a subcomplex, and let $f\colon X \to Y$ be a map which is cellular on A. Using Lemma 16.8 inductively we can construct functions $f_i\colon X \to Y$ and homotopies $h_i\colon X \times [0,1] \to Y$ for $m = 0, 1, 2, \dots$ such that:

- the function f_m is cellular on $A \cup X^{(m)}$
- h_0 is a homotopy (rel A) between f and f_0
- h_m is a homotopy (rel $A \cup X^{(m-1)}$) between f_{m-1} and f_m for $m = 1, 2, \dots$

Notice that if $\dim X = n < \infty$ then $X = X^{(n)}$, and so f_n is a cellular map such that $f \simeq f_n$ (rel A). Thus we can take $g = f_n$.

If $\dim X = \infty$ define $g\colon X \to Y$ by $g(x) = f_m(x)$ if $x \in X^{(m)}$. Notice that since $f_n|_{X^{(m)}} = f_m|_{X^{(m)}}$ for all $n > m$ this function is well defined, and it is continuous by (12.8) since for each m the function $g|_{X^{(m)}} = f_m|_{X^{(m)}}$ is continuous. In addition, g is a cellular function since for each m we have $g(X^{(m)}) = f_m(X^{(m)}) \subseteq Y^{(m)}$, and it satisfies $f|_A = g|_A$ since $f|_A = f_m|_A$ for all m.

To obtain a homotopy $h\colon X \times [0,1] \to Y$ between f and g, choose numbers $t_m \in [0,1]$ for $m = 0, 1, \dots$ such that $t_0 = 0$, $t_m < t_{m+1}$ for all m, and that the sequence t_m converges to 1. On the subinterval

$[t_m, t_{m+1}]$ define h by reparametrizing the homotopy h_m:

$$h(x, t) = h_m(x, (t - t_m)/(t_{m+1} - t_m))$$

for $t \in [t_m, t_{m+1}]$. Also, set $h(x, 1) = g(x)$ for $x \in X$. To verify that h is continuous, it suffices to show that it is continuous on $X^{(m)} \times [0, 1]$ for each m. This holds since $h(x, t) = f_m(x)$ for $(x, t) \in X^{(m)} \times [t_{m+1}, 1]$, and $h|_{X\times[0,t_{m+1}]}$ is continuous as a concatenation of a finite number of homotopies $h_0, \dots, h_m$.

□

Exercises to Chapter 16

E16.1 Exercise. Recall that the n-th homotopy group of a pointed space (X, x_0) is a group whose elements are homotopy classes of basepoint preserving maps $(S^n, s_0) \to (X, x_0)$. Let S^m be an m-dimensional sphere with a basepoint $s'_0 \in S^m$. Show that if $n < m$ then the group $\pi_n(S^m, s'_0)$ is trivial.

E16.2 Exercise. Recall that the n-dimensional sphere is given by

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\}$$

For $0 \leq m < n$ consider the embedding $i\colon S^m \to S^n$ given by

$$i((x_1, \dots, x_{m+1})) = (x_1, \dots, x_{m+1}, 0, \dots 0)$$

Using this embedding we can consider S^m as a subspace of S^n. Show that the quotient space S^n/S^m is homotopy equivalent to $S^n \vee S^{m+1}$. (Hint: Proposition 13.4 may be useful.)

E16.3 Exercise. Let (Y, y_0) be a pointed space. Show that there exists a pointed 2-dimensional CW complex (X, x_0) and a function $f\colon (X, x_0) \to (Y, y_0)$ such that the induced homomorphism of fundamental groups $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

E16.4 Exercise. The goal of this exercise is to complete a missing step in the proof of Lemma 16.5. Let $Y' = Y \cup e^n$ be a space obtained by attaching a single n-dimensional cell to a space Y. Show that for any point $y_0 \in e^n$ the space Y is a deformation retract of $Y' \smallsetminus \{y_0\}$

17 | Covering Spaces

All computations of non-trivial fundamental groups we have seen use the fact that the group $\pi_1(S^1)$ is isomorphic to the group of integers. The proof of this fact, however, is still incomplete since it relies on the path lifting property of the universal covering of S^1 (Proposition 5.11) that we left without justification. Our next goal is to fill this gap. In this chapter we define the notion of a *covering* of a space and we show that the path lifting property holds for any covering. Since the universal covering of S^1 is an example of a covering this will give in particular a proof of Proposition 5.11.

17.1 Definition. A map $p\colon T \to X$ is a *covering* of X if for every point $x \in X$ there exists an open neighborhood $U_x \subseteq X$ and a homeomorphism $h_{U_x}\colon p^{-1}(U_x) \to U_x \times D_x$ where D_x is some discrete space, such that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(U_x) \arrow[rr, "h_{U_x}"] \arrow[dr, "p"'] & & U_x \times D_x \arrow[dl, "\mathrm{pr}_1"] \\
& U_x &
\end{tikzcd}$$

Here $\mathrm{pr}_1\colon U_x \times D_x \to U_x$ is the projection map $\mathrm{pr}_1(y, d) = y$.

17.2 Here is some terminology and a few comments related to the notion of a covering.

1) Given a covering $p\colon T \to X$ we will say that T is the *total space* of p and that X is the *base space*.

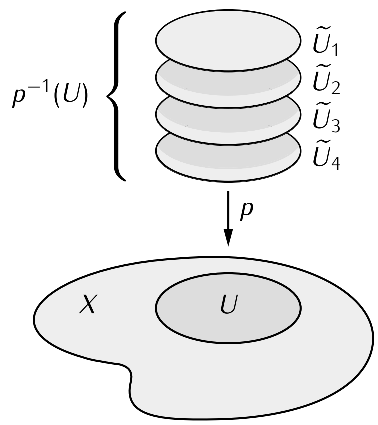
2) For $x \in X$ we have $p^{-1}(x) \cong \{x\} \times D_x \cong D_x$ which means that $p^{-1}(x)$ is a discrete space. We call $p^{-1}(x)$ the *fiber* of the covering p over the point x.

3) In general for $x, x' \in X$ the fibers $p^{-1}(x)$ and $p^{-1}(x')$ may have different numbers of points, and so they don't need to be homeomorphic. However, if the space X is connected then $p^{-1}(x) \cong p^{-1}(x')$ for all $x, x' \in X$ (exercise).

4) If $p^{-1}(x)$ consists of n points for all $x \in X$ then we say that p is an n-fold covering of X.

5) If $U \subseteq X$ is an open set such that for some discrete space D there exists a homeomorphism $h_U\colon p^{-1}(U) \to U \times D$ satisfying $\mathrm{pr}_1 h_U = p$ then we say that the set U is *evenly covered*. Definition of a covering can be rephrased by saying that every point $x \in X$ has an open neighborhood which is evenly covered.

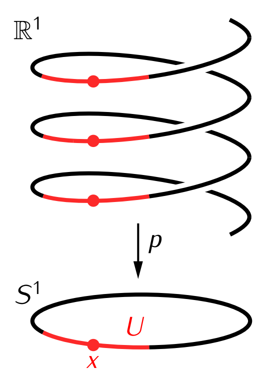
6) If $U \subseteq X$ is an evenly covered set and $h_U\colon p^{-1}(U) \to U \times D$ is a homeomorphism then for $d \in D$ we will say that the set $\widetilde{U}_d = h_U^{-1}(U \times \{d\}) \subseteq p^{-1}(U)$ is a *slice* over U. The set $p^{-1}(U)$ is then a disjoint union of slices:



Moreover, for each slice $\widetilde{U}_d$ the map $p|_{\widetilde{U}_d}\colon \widetilde{U}_d \to U$ is a homeomorphism.

17.3 Example. Let D be a discrete space. The projection map $\mathrm{pr}_1\colon X \times D \to X$ is a covering of X. In this case the whole space X is evenly covered. We say that $\mathrm{pr}_1\colon X \times D \to X$ is a *trivial covering* of X.

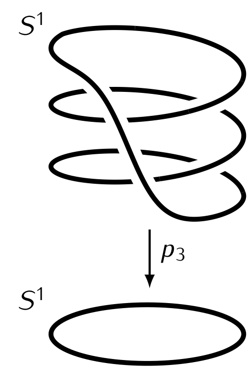
17.4 Example. Recall that the universal covering of S^1 is the map $p\colon \mathbb{R}^1 \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$. If $U \subseteq S^1$ is any open set such that $U \neq S^1$, then $p^{-1}(U)$ is evenly covered and $p^{-1}(U) \cong U \times \mathbb{Z}$ (exercise).



17.5 Example. Consider S^1 as a subset of the complex plane:

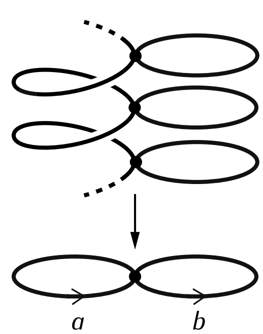
$$S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$$

For $n = 1, 2, \ldots$ the map $p_n\colon S^1 \to S^1$ given by $p_n(z) = z^n$ is an n-fold covering of S^1. Similarly as in the case of the universal covering of S^1 any open set $U \subseteq S^1$ such that $U \neq S^1$ is evenly covered by p_n (exercise).

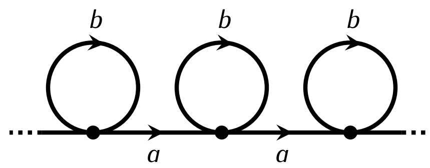


17.6 Example. If $p_1\colon T_1 \to X_1$ and $p_2\colon T_2 \to X_2$ are coverings then the map $p_1 \times p_2\colon T_1 \times T_2 \to X_1 \times X_2$ is also a covering (exercise). For example, starting with the universal covering $p\colon \mathbb{R}^1 \to S^1$ of the circle we obtain a covering $p \times p\colon \mathbb{R}^1 \times \mathbb{R}^1 \to S^1 \times S^1$ of the torus.

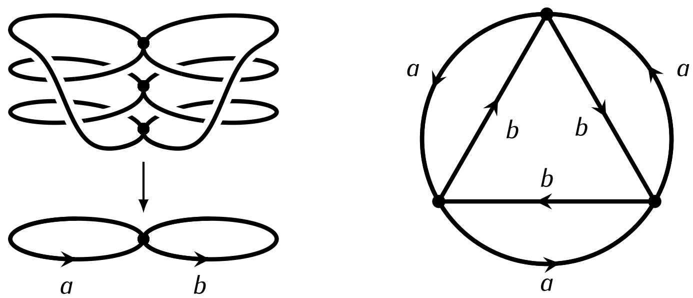
17.7 Example. Using the coverings of S^1 described above we can construct many coverings of $S^1 \vee S^1$. For example, here is a covering obtained by combining the universal covering over one copy of S^1 and a trivial covering over the second copy:

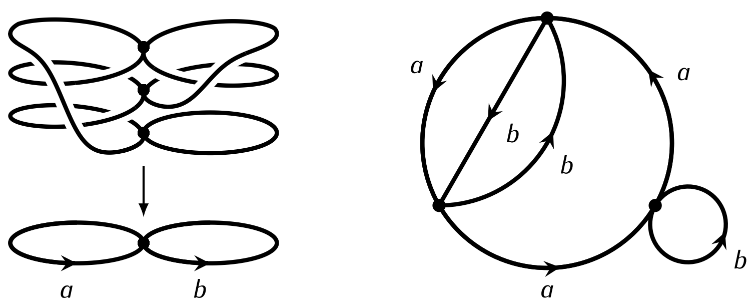


Coverings like this are easier to represent graphically if we untangle the total space and indicate which of its parts are being mapped to which copy of S^1. For the covering depicted above this gives:



Here are two different 3-fold coverings of $S^1 \vee S^1$:





17.8 Definition. If $p\colon T \to X$ is a covering and $f\colon Y \to X$ is a map then a *lift* of f is a map $\tilde{f}\colon Y \to T$ such that the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

The following fact describes one of the main properties of coverings:

17.9 Theorem (Homotopy Lifting Property). *Let $p\colon T \to X$ be a covering. Let $F\colon Y \times [0,1] \to X$ and $\tilde{f}\colon Y \times \{0\} \to T$ be functions satisfying $p\tilde{f} = F|_{Y\times\{0\}}$. There exists a function $\widetilde{F}\colon Y \times [0,1] \to T$ such that $p\widetilde{F} = F$ and $\widetilde{F}|_{Y\times\{0\}} = \tilde{f}$:*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & T \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X \end{tikzcd} \qquad (*)$$

Moreover, such function $\widetilde{F}$ is unique.

Before we get to the proof of this theorem we will show that it implies that any covering has path lifting properties analogous to the ones described in Proposition 5.11 for the universal covering of S^1:

17.10 Corollary. *Let $p\colon T \to X$ be a covering. Let $x_0 \in X$, and let $\tilde{x}_0 \in T$ be a point such that $p(\tilde{x}_0) = x_0$.*

1) For any path $\omega\colon [0,1] \to X$ such that $\omega(0) = x_0$ there exists a lift $\widetilde{\omega}\colon [0,1] \to T$ satisfying $\widetilde{\omega}(0) = \tilde{x}_0$. Moreover, such lift is unique.

2) Let $\omega, \tau\colon [0,1] \to X$ be paths such that $\omega(0) = \tau(0) = x_0$, $\omega(1) = \tau(1)$ and $\omega \simeq \tau$. If $\widetilde{\omega}$, $\widetilde{\tau}$ are lifts of ω, τ, respectively, such that $\widetilde{\omega}(0) = \widetilde{\tau}(0) = \tilde{x}_0$ then $\widetilde{\omega}(1) = \widetilde{\tau}(1)$ and $\widetilde{\omega} \simeq \widetilde{\tau}$.

Proof. For part 1) let $Y = \{*\}$ be the space consisting of one point. We can consider the path ω as a map $\omega\colon \{*\} \times [0,1] \to X$. Denote by $c_{\tilde{x}_0}\colon \{*\} \times \{0\} \to T$ the map given by $c_{\tilde{x}_0}(*, 0) = \tilde{x}_0$. We have a commutative diagram:

$$\begin{tikzcd} \{*\} \times \{0\} \arrow[r, "c_{\tilde{x}_0}"] \arrow[d, hook] & T \arrow[d, "p"] \\ \{*\} \times [0,1] \arrow[r, "\omega"'] & X \end{tikzcd}$$

By Theorem 17.9 there exists a unique map $\widetilde{\omega}\colon \{*\} \times [0,1] \to T$ which gives the desired lift of ω.

Part 2) is an exercise.

□

Proof of Theorem 17.9 will use a couple of lemmas. The first of them if of interest of its own right:

17.11 Lemma. *Let $p\colon T \to X$ be a covering, and let $\tilde{f}_1, \tilde{f}_2\colon Y \to T$ be two lifts of a map $f\colon Y \to X$. If Y is a connected space and there exists $y_0 \in Y$ such that $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$ then $\tilde{f}_1(y) = \tilde{f}_2(y)$ for all*

$y \in Y$.

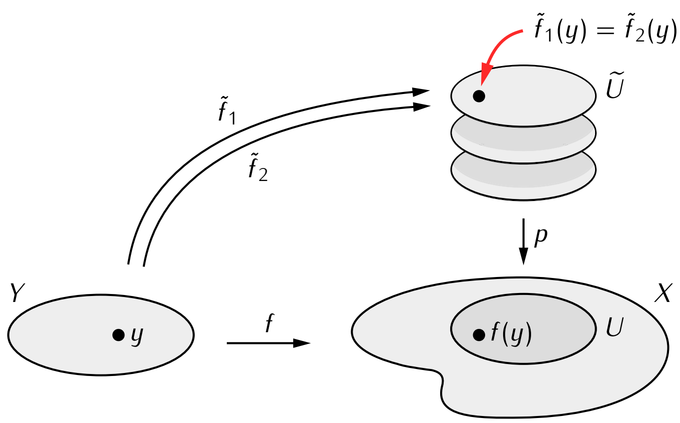
$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, shift left, "\tilde{f}_1"] \arrow[ur, shift right, "\tilde{f}_2"'] \arrow[r, "f"'] & X \end{tikzcd}$$

Proof. Let $Y_e, Y_n \subseteq X$ be sets defined by

$$Y_e = \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\} \quad \text{and} \quad Y_n = \{y \in Y \mid \tilde{f}_1(y) \neq \tilde{f}_2(y)\}$$

Notice that $Y_e \cup Y_n = Y$ and $Y_e \cap Y_n = \varnothing$. Notice also that $Y_e \neq \varnothing$ since $y_0 \in Y_e$. It will be enough to show that Y_e and Y_n are open in Y. By connectedness of Y this will imply that $Y_e = Y$.

To see that Y_e is open take $y \in Y_e$. It will suffice to show that there exists an open set $V \subseteq Y$ such that $y \in V$ and $V \subseteq Y_e$. Let $U \subseteq X$ be an evenly covered open neighborhood of $f(y)$ and let $\widetilde{U}$ be a slice over U such that $\tilde{f}_1(y) = \tilde{f}_2(y) \in \widetilde{U}$.



Take $V = \tilde{f}_1^{-1}(\widetilde{U}) \cap \tilde{f}_2^{-1}(\widetilde{U})$. The set V is an open neighborhood of y. Also, for $y' \in V$ we have

$$p|_{\widetilde{U}} \circ \tilde{f}_1(y') = f(y') = p|_{\widetilde{U}} \circ \tilde{f}_2(y')$$

Since $p|_{\widetilde{U}} \colon \widetilde{U} \to U$ is a homeomorphism this gives $\tilde{f}_1(y') = \tilde{f}_2(y')$, and so $y' \in Y_e$.

Openness of Y_n can be verified in a similar way (exercise). $\square$

The next lemma is a special case of Theorem 17.9:

17.12 Lemma. *Let $p \colon T \to X$ be a covering. Let $F \colon Y \times [a, b] \to X$ and $\tilde{f} \colon Y \times \{a\} \to T$ be functions satisfying $p\tilde{f} = F|_{Y \times \{a\}}$. Assume also that $F(Y \times [a, b]) \subseteq U$ where $U \subseteq X$ is an evenly covered open* *set. There exists a function* $\widetilde{F}\colon Y\times[a,b]\to T$ *such that* $p\widetilde{F}=F$ *and* $\widetilde{F}|_{Y\times\{a\}}=\tilde{f}$*:*

$$\begin{tikzcd} Y\times\{a\} \arrow[r, "\tilde f"] \arrow[d, hook] & T \arrow[d, "p"] \\ Y\times[a,b] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X \end{tikzcd}$$

Proof. Let $\{\widetilde{U}_i\}_{i\in I}$ be the set of slices of p over the set U, and let $V_i=\{y\in Y\mid \tilde{f}(y,a)\in\widetilde{U}_i\}$. The sets $V_i\times[a,b]$ form an open cover of $Y\times[a,b]$. Since for each $i\in I$ the map $p|_{\widetilde{U}_i}\colon\widetilde{U}_i\to U$ is a homeomorphism we can define

$$\widetilde{F}|_{V_i\times[a,b]}=(p|_{\widetilde{U}_i})^{-1}\circ F|_{V_i\times[a,b]}$$

Since for $i\neq j$ we have $V_i\cap V_j=\varnothing$ this gives a well defined continuous function $\widetilde{F}\colon Y\times[a,b]\to T$ □

Proof of Theorem 17.9. We will show first that if the map $\widetilde{F}$ exists then it must be unique. Assume that for $i=1,2$ we have a map $\widetilde{F}_i$ such that the following diagram commutes:

$$\begin{tikzcd} Y\times\{0\} \arrow[r, "\tilde f"] \arrow[d, hook] & T \arrow[d, "p"] \\ Y\times[0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}_i"] & X \end{tikzcd}$$

Take $y\in Y$. Notice that the set $\{y\}\times[0,1]\subseteq Y\times[0,1]$ is connected, for $i=1,2$ the map $\widetilde{F}_i|_{\{y\}\times[0,1]}$ is a lift of $F|_{\{y\}\times[0,1]}$, and $\widetilde{F}_1|_{\{y\}\times[0,1]}(y,0)=\tilde{f}(y,0)=\widetilde{F}_2|_{\{y\}\times[0,1]}(y,0)$. By Lemma 17.11 we obtain that $\widetilde{F}_1|_{\{y\}\times[0,1]}=\widetilde{F}_2|_{\{y\}\times[0,1]}$ for each $y\in Y$. Therefore $\widetilde{F}_1=\widetilde{F}_2$.

It remains to show that a map $\widetilde{F}$ in the diagram $(*)$ exists. Let $y\in Y$. We will construct first a map $\widetilde{F}_y\colon V_y\times[0,1]\to T$ where $V_y\subseteq Y$ is some open neighborhood of y, such that the following diagram commutes:

$$\begin{tikzcd} V_y\times\{0\} \arrow[r, "\tilde f|_{V_y\times\{0\}}"] \arrow[d, hook] & T \arrow[d, "p"] \\ V_y\times[0,1] \arrow[r, "F|_{V_y\times[0,1]}"'] \arrow[ur, dashed, "\widetilde{F}_y"] & X \end{tikzcd} \tag{$**$}$$

The construction of $\widetilde{F}_y$ proceeds as follows. Using compactness of the interval $[0,1]$ we can find an open neighborhood $V_y\subseteq Y$ of y and numbers $0=t_0<t_1<\ldots<t_n=1$ such that for each $i=1,\ldots,n$

we have $F(V_y \times [t_{i-1}, t_i]) \subseteq U_i$ where $U_i \subseteq X$ is an evenly covered open set. Using Lemma 17.12 we obtain a map $\widetilde{F}_{y,1} \colon V_y \times [t_0, t_1] \to T$ that gives a commutative diagram

$$\begin{tikzcd}
V_y \times \{0\} \arrow[r, "\tilde{f}|_{V_y \times \{0\}}"] \arrow[d, hook] & T \arrow[d, "p"] \\
V_y \times [t_0, t_1] \arrow[r, "F|_{V_y \times [t_0, t_1]}"'] \arrow[ur, dashed, "\widetilde{F}_{y,1}"] & X
\end{tikzcd}$$

Next, using Lemma 17.12 again we obtain a map $\widetilde{F}_{y,2} \colon V_y \times [t_1, t_2] \to T$ that fits into the commutative diagram

$$\begin{tikzcd}
V_y \times \{t_1\} \arrow[r, "\widetilde{F}_{y,1}|_{V_y \times \{t_1\}}"] \arrow[d, hook] & T \arrow[d, "p"] \\
V_y \times [t_1, t_2] \arrow[r, "F|_{V_y \times [t_0, t_1]}"'] \arrow[ur, dashed, "\widetilde{F}_{y,2}"] & X
\end{tikzcd}$$

Arguing inductively we obtain in this way for each $k = 1, \dots, n$ a map $\widetilde{F}_{y,k} \colon V_y \times [t_{k-1}, t_k] \to T$. By construction $\widetilde{F}_{y,k}|_{V_y \times \{t_k\}} = \widetilde{F}_{y,k+1}|_{V_y \times \{t_k\}}$ for all $k = 1, \dots, n-1$, so these maps taken together define a map $\widetilde{F}_y \colon V_y \times [0, 1] \to T$ that fits into the commutative diagram ($**$).

Next, we would like to take $\widetilde{F} \colon Y \times [0, 1] \to T$ to be the map such that $\widetilde{F}|_{V_y \times [0,1]} = \widetilde{F}_y$ for each $y \in Y$. In order to verify that such map is well defined we need to show that if $z \in V_y \cap V_{y'}$ for some $y, y' \in Y$ then $F_y|_{\{z\} \times [0,1]} = F_{y'}|_{\{z\} \times [0,1]}$. This however holds by Lemma 17.11 since both $F_y|_{\{z\} \times [0,1]}$ and $F_{y'}|_{\{z\} \times [0,1]}$ are lifts of the map $F|_{\{z\} \times [0,1]}$, the set $\{z\} \times [0, 1]$ is connected, and $F_y|_{\{z\} \times [0,1]}(z, 0) = \tilde{f}(z, 0) = F_{y'}|_{\{z\} \times [0,1]}(z, 0)$. $\square$

Exercises to Chapter 17

E17.1 Exercise. a) Let $p_i \colon T_i \to X_i$ be a covering for $i = 1, 2$. Show that the map $p_1 \times p_2 \colon T_1 \times T_2 \to X_1 \times X_2$ is a covering.

b) Give an example showing that if $p_i \colon T_i \to X_i$ is a covering for $i = 1, 2, \dots$ then the map

$$\prod_{i=1}^{\infty} p_i \colon \prod_{i=1}^{\infty} T_i \to \prod_{i=1}^{\infty} X_i$$

need not be a covering. Justify your answer.

E17.2 Exercise. Let $p\colon T \to X$ be a covering and let $U \subseteq T$ be an open set. Show that the set $p(U)$ is open in X.

E17.3 Exercise. Let $p\colon T \to X$ be a covering such that X is a connected space. Show that for any points $x, x' \in X$ there exists a bijection between $p^{-1}(x)$ and $p^{-1}(x')$.

E17.4 Exercise. Prove part 2) of Corollary 17.10.

18 | Coverings and the Fundamental Group

The first example of a covering we have encountered, the universal covering of S^1, was introduced as a tool for computing the fundamental group of S^1. It turns out that coverings in general are very closely related to the fundamental group. In this chapter we explore some results illustrating this. Ultimately these results will let us give an algebraic description of all possible coverings of a given space X.

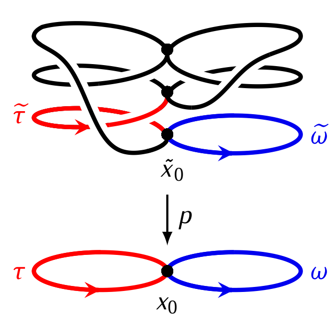
Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Recall that by Corollary 17.10 a path $\omega\colon [0,1] \to X$ such that $\omega(0) = x_0$ admits a unique lift $\widetilde{\omega}\colon [0,1] \to T$ that satisfies $\widetilde{\omega}(0) = \tilde{x}_0$. We have:

18.1 Theorem. *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$.*

1) The homomorphism $p_*\colon \pi_1(T, \tilde{x}_0) \to \pi_1(X, x_0)$ is 1-1.

2) An element $[\omega] \in \pi_1(X, x_0)$ is in the subgroup $p_*(\pi_1(T, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$ if and only if the lift $\widetilde{\omega}$ such that $\widetilde{\omega}(0) = \tilde{x}_0$ is a loop in T.

18.2 Example. Consider the following 3-fold covering $p\colon T \to S^1 \vee S^1$:



The lift $\widetilde{\omega}$ of the loop ω that satisfies $\widetilde{\omega}(0) = \tilde{x}_0$ is a loop in T, so $[\omega] \in p_*(\pi_1(T, \tilde{x}_0))$. On the other hand the lift $\widetilde{\tau}$ of the loop τ that satisfies $\widetilde{\tau}(0) = \tilde{x}_0$ is an open path in T, which means that $[\tau] \notin p_*(\pi_1(T, \tilde{x}_0))$.

Proof of Theorem 18.1. 1) Assume that $[\tau], [\tau'] \in \pi_1(T, \tilde{x}_0)$ are elements such that $p_*([\tau]) = p_*([\tau'])$. This means that $p \circ \tau \simeq p \circ \tau'$. Since τ and τ' are the unique lifts of $p \circ \tau$ and $p \circ \tau'$ that begin at $\tilde{x}_0$ by part 2) of Corollary 17.10 we obtain that $\tau \simeq \tau'$, and so $[\tau] = [\tau']$.

2) Let $[\omega] \in \pi_1(X, x_0)$. If the lift $\widetilde{\omega}$ of ω is a loop then it represents an element $[\widetilde{\omega}] \in \pi_1(T, \tilde{x}_0)$. We have $p_*([\widetilde{\omega}]) = [p \circ \widetilde{\omega}] = [\omega]$, so $[\omega] \in p_*(\pi_1(T, \tilde{x}_0))$. Conversely, if $[\omega] \in p_*(\pi_1(T, \tilde{x}_0))$ then $\omega \simeq p \circ \tau$ where τ is a loop in T based at $\tilde{x}_0$. Since τ is the lift of $p \circ \tau$ that starts at $\tilde{x}_0$, thus by part 2) of Corollary 17.10 we obtain that $\widetilde{\omega}(1) = \tau(1) = \tilde{x}_0$ which shows that $\widetilde{\omega}$ is a loop in T. □

The second part of Theorem 18.1 can be generalized as follows:

18.3 Proposition. *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Assume that ω_1 and ω_2 are paths in X such that $\omega_1(0) = \omega_2(0) = x_0$ and $\omega_1(1) = \omega_2(1)$. For $i = 1, 2$ let $\widetilde{\omega}_i\colon [0, 1] \to T$ be the lift of ω_i such that $\widetilde{\omega}_i(0) = \tilde{x}_0$. Then $\widetilde{\omega}_1(1) = \widetilde{\omega}_2(1)$ if and only if $[\omega_1 * \overline{\omega}_2] \in p_*(\pi_1(T, \tilde{x}_0))$.*

Proof. Exercise. □

Recall that if G is a group and $H \subseteq G$ is a subgroup then the *left coset* of an element $g \in G$ is the set $gH = \{gh \in G \mid h \in H\}$ and that the *index* of H in G is the number $[G : H]$ of distinct left cosets.

18.4 Corollary. *Let $p\colon T \to X$ be a covering such that T is a path connected space, let $x_0 \in X$, and let $\tilde{x}_0 \in p^{-1}(x_0)$. Denote $H = p_*(\pi_1(T, \tilde{x}_0))$.*

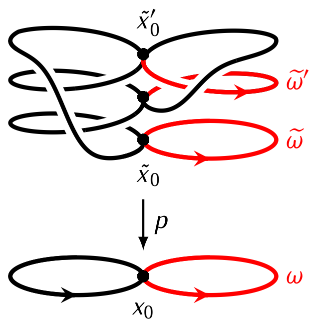
1) For $i = 1, 2$ let ω_i be a loop in X based at x_0 and let $\widetilde{\omega}_i$ be the lift of ω such that $\widetilde{\omega}_i(0) = \tilde{x}_0$. We have $\widetilde{\omega}_1(1) = \widetilde{\omega}_2(1)$ if and only if $[\omega_1]H = [\omega_2]H$.

2) The index $[\pi_1(X, x_0) : H]$ is equal to the number of elements of the fiber $p^{-1}(x_0)$.

Proof. 1) Notice that $[\omega_1]H = [\omega_2]H$ if and only if $[\omega_1] \cdot [\omega_2]^{-1} = [\omega_1 * \overline{\omega}_2] \in H$. Therefore it suffices to apply Proposition 18.3.

2) Let $\pi_1(X, x_0)/H$ denote the set of left cosets of H. Consider the function $\varphi\colon \pi_1(X, x_0)/H \to p^{-1}(x_0)$ given by $\varphi([\omega]H) = \widetilde{\omega}(1)$ where $\widetilde{\omega}$ is the lift of ω such that $\widetilde{\omega}(0) = \tilde{x}_0$. By part 1) the function φ is well defined and it is 1-1. To see that φ it is also onto take $\tilde{x}'_0 \in \pi^{-1}(x_0)$. By assumption T is path connected, so there exists a path $\tau\colon [0, 1] \to T$ such that $\tau(0) = \tilde{x}_0$ and $\tau(1) = \tilde{x}'_0$. Notice that $p\tau$ is a loop in X and that $\varphi([p\tau]H) = \tilde{x}'_0$. □

Theorem 18.1 lets us associate to each covering p of a space X the subgroup $p_*(\pi_1(T, \tilde{x}_0))$ of the fundamental group $\pi_1(X, x_0)$. However, this subgroup depends not only on the covering p but also on the choice of the basepoint $\tilde{x}_0 \in p^{-1}(x_0)$. Indeed, consider the same covering $p\colon T \to S^1 \vee S^1$ as before:



The lift of the loop ω that starts at the point $\tilde{x}_0$ is the loop $\tilde{\omega}$. This means that $[\omega] \in p_*(\pi_1(T, \tilde{x}_0))$. On the other hand the lift of ω that starts at $\tilde{x}'_0$ is the open path $\tilde{\omega}'$, which shows that $[\omega] \notin p_*(\pi_1(T, \tilde{x}'_0))$. As a consequence $p_*(\pi_1(T, \tilde{x}_0)) \neq p_*(\pi_1(T, \tilde{x}'_0))$.

Our next goal will be to describe the relationship between the subgroups $p_*(\pi_1(T, \tilde{x}_0))$ of $\pi_1(X, x_0)$ that come from different choices of points $\tilde{x}_0 \in p^{-1}(x_0)$. Recall that we say that subgroups H, H' of a group G are *conjugate* if $H' = gHg^{-1}$ for some $g \in G$.

18.5 Proposition. *Let $p\colon T \to X$ be a covering such that T is a path connected space and let $x_0 \in X$.*

1) For any $\tilde{x}_0, \tilde{x}'_0 \in p^{-1}(x_0)$ the subgroups $p_*(\pi_1(T, \tilde{x}_0))$ and $p_*(\pi_1(T, \tilde{x}'_0))$ of $\pi_1(X, x_0)$ are conjugate.

2) If $\tilde{x}_0 \in p^{-1}(x_0)$ and $H \subseteq \pi_1(X, x_0)$ is a subgroup conjugate to $p_*(\pi_1(T, \tilde{x}_0))$ then $H = p_*(\pi_1(T, \tilde{x}'_0))$ for some $\tilde{x}'_0 \in p^{-1}(x_0)$.

Proof. 1) Since T is path connected there exists a path τ in T such that $\tau(0) = \tilde{x}_0$ and $\tau(1) = \tilde{x}'_0$. Recall that such path defines an isomorphism $s_\tau\colon \pi_1(T, \tilde{x}_0) \to \pi_1(T, \tilde{x}'_0)$ given by $s_\tau([\omega]) = [\bar{\tau} * \omega * \tau]$. This gives:

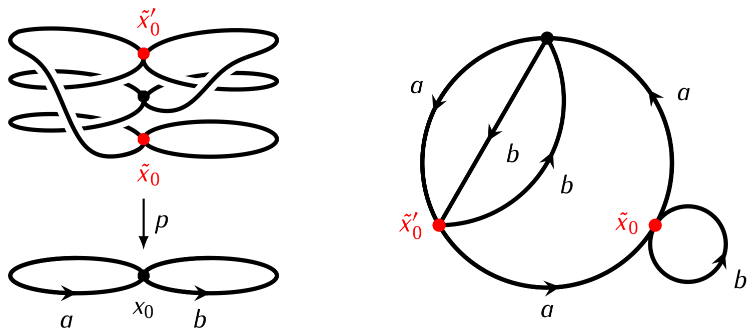
$$\begin{aligned} p_*(\pi_1(T, \tilde{x}'_0)) &= p_*(s_\tau(\pi_1(T, \tilde{x}_0))) \\ &= p_*(\{[\bar{\tau} * \omega * \tau] \mid [\omega] \in \pi_1(T, \tilde{x}_0)\}) \\ &= \{[(p \circ \bar{\tau}) * (p \circ \omega) * (p \circ \tau)] \mid [\omega] \in \pi_1(T, \tilde{x}_0)\} \end{aligned}$$

Since $p \circ \tau$ is a loop in X based at x_0 it represents an element $[p \circ \tau] \in \pi_1(X, x_0)$ and we have $[(p \circ \bar{\tau}) * (p \circ \omega) * (p \circ \tau)] = [p \circ \tau]^{-1} \cdot p_*[\omega] \cdot [p \circ \tau]$. Therefore we obtain:

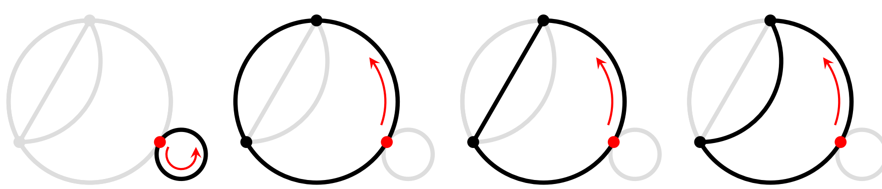
$$p_*(\pi_1(T, \tilde{x}'_0)) = [p \circ \tau]^{-1} \cdot p_*(\pi_1(T, \tilde{x}'_0)) \cdot [p \circ \tau]$$

2) Exercise. $\square$

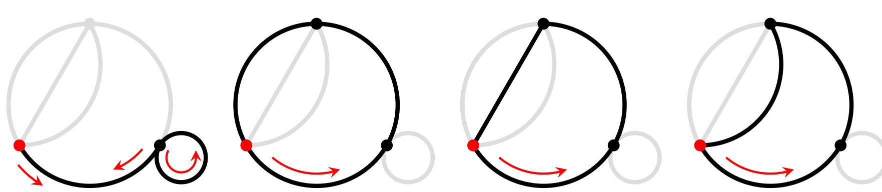
18.6 Example. Lets have a look again at the covering $p\colon T \to S^1 \vee S^1$:



The fundamental group of the base space is a free group on two generators: $\pi_1(S^1 \vee S^1, x_0) \cong \langle a, b\rangle$ where a is represented by the loop that traverses one copy of S^1, and b by the loop that traverses the second copy of S^1. The total space T of this covering has the structure of a 1-dimensional path connected CW complex with three 0-cells and six 1-cells, so by Theorem 13.11 the group $\pi_1(T)$ is a free group on 4 generators. Free generators of the group $\pi_1(T, \tilde{x}_0)$ can be selected so that they are represented by the following loops based at $\tilde{x}_0$:



Arrows indicate orientations of these loops. The images of these loops in $S^1 \vee S^1$ represent the following elements of the group $\langle a, b\rangle$: b, a^3, aba, $ab^{-1}a$. It follows that the subgroup $p_*(\pi_1(T, \tilde{x}_0))$ of $\pi_1(S^1 \vee S^1, x_0)$ corresponds to the subgroup $H \subseteq \langle a, b\rangle$ generated by these four elements. On the other hand free generators of the group $\pi_1(T, \tilde{x}_0')$ can be selected so that they are represented by the following loops based at $\tilde{x}_0'$:



In effect, the subgroup $p_*(\pi_1(T, \tilde{x}_0')) \subseteq \pi_1(S^1 \vee S^1, x_0)$ can be identified with the subgroup of $H' \subseteq \langle a, b\rangle$ which is generated by the elements aba^{-1}, a^3, a^2b, and a^2b^{-1}. Since each of these elements is obtained by conjugating the corresponding generator of the subgroup H by a we get: $H' = aHa^{-1}$. Notice that the conjugation by a comes from the fact that there is a path τ in T joining $\tilde{x}_0$ with $\tilde{x}_0'$ such that the loop $p \circ \tau$ represents the element $a^{-1} \in \langle a, b \rangle$

It is interesting to consider the case when the subgroup $p_*(\pi_1(T, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$ does not depend on the choice of the point $\tilde{x}_0 \in p^{-1}(x_0)$. This motivates the following definition:

18.7 Definition. Let $p\colon T \to X$ be a covering with a path connected total space T and let $x_0 \in X$. The covering p is *regular* if $p_*(\pi_1(T, \tilde{x}_0)) = p_*(\pi_1(T, \tilde{x}_0'))$ for any $\tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0)$.

18.8 Proposition. *Let $p\colon T \to X$ be a covering such that T is a path connected space and let $x_0 \in X$. The following conditions are equivalent:*

1) The covering p is regular.

2) For any $\tilde{x}_0 \in p^{-1}(x_0)$ the group $p_*(\pi_1(T, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$.

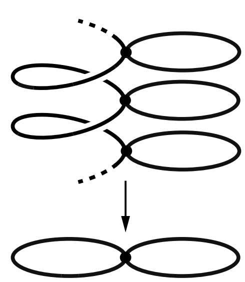
3) Let ω be a loop in X based at x_0. If ω has a lift which is a loop then every lift of ω is a loop, and if ω has a lift which is an open path then every lift of ω is an open path.

Proof. Exercise. □

So far the objective of our study of the fundamental group functor was to use properties of groups to get information about topological spaces. The relationship between fundamental groups and covering spaces can be used to show that we can also work in the opposite direction: sometimes we can use properties of topological spaces to derive facts about groups. The next result illustrates this approach.

18.9 Proposition. *Every free group on two or more generators contains a free subgroup on an infinite number of generators.*

Proof. It suffices to consider the case of a free group on two generators. Such group is isomorphic to $\pi_1(S^1 \vee S^1)$. Consider the covering $p\colon T \to S^1 \vee S^1$ obtained by gluing the universal covering over one copy of S^1 to the trivial covering over the second copy:



The total space T of this covering is homotopy equivalent to the space $\bigvee_{n\in\mathbb{Z}} S^1$, and so $\pi_1(T)$ is a free group on an infinite number of generators. By Theorem 18.1 $\pi_1(T)$ can be identified with a subgroup of $\pi_1(S^1 \vee S^1)$. □

Exercises to Chapter 18

E18.1 Exercise. Prove Proposition 18.3.

E18.2 Exercise. Prove Proposition 18.8.

E18.3 Exercise. Let $p\colon T \to X$ be a covering and let $f\colon Y \to X$ be an arbitrary continuous function. Let f^*T be a subspace of $Y \times T$ defined by

$$f^*T = \{(y, \tilde{x}) \in Y \times T \mid f(y) = p(\tilde{x})\}$$

Let $p'\colon f^*T \to Y$ be the map given by $p'(y, \tilde{x}) = y$.

a) Show that $p'\colon f^*T \to Y$ is a covering.

b) Assume that $(y_0, \tilde{x}_0) \in f^*T$. Show that $p'_*(\pi_1(f^*T, (y_0, \tilde{x}_0))) = f_*^{-1}(p_*(\pi_1(T, \tilde{x}_0)))$ where $f_*\colon \pi_1(Y, y_0) \to \pi_1(X, f(x_0))$ is the homomorphism induced by f.

E18.4 Exercise. A map $p\colon E \to B$ is a fibration if it has the homotopy lifting property, i.e. if for any maps $F\colon X \times [0,1] \to B$ and $\widetilde{f}\colon X \times \{0\} \to E$ satisfying $F|_{X\times\{0\}} = p\widetilde{f}$ there exists $\widetilde{F}\colon X \times [0,1] \to E$ such that $\widetilde{F}|_{X\times\{0\}} = \widetilde{f}$ and $p\widetilde{F} = F$:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widetilde{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B
\end{tikzcd}$$

For example, any covering is a fibration.

Let $p\colon E \to B$ be a fibration, let $b_0 \in B$, $F = p^{-1}(b_0)$, and $e_0 \in F$. Consider the homomorphisms

$$\pi_1(F, e_0) \xrightarrow{i_*} \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0)$$

where $i\colon F \to E$ is the inclusion map. Show that $\mathrm{Im}(i_*) = \mathrm{Ker}(p_*)$.

Note: if p is a covering then $\pi_1(F, e_0)$ is the trivial group (since F is a discrete space), so the above formula implies that $\mathrm{Ker}(p_*)$ is trivial. This gives another proof that for a covering p the induced homomorphism p_* is 1-1.

E18.5 Exercise. Let $p\colon E \to B$ be a fibration, let $b_0 \in B$, and $F = p^{-1}(b_0)$. Show that if B is a contractible space then the inclusion map $i\colon F \to E$ is a homotopy equivalence.

19 | Classification of Coverings

19.1 Definition. Let $p_1\colon T_1 \to X$, $p_2\colon T_2 \to X$ be coverings over the same base space X. A *map of coverings* is a continuous function $f\colon T_1 \to T_2$ such that the following diagram commutes:

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

For a given space X by $\mathbf{Cov}(X)$ we will denote the category whose objects are all coverings over X and whose morphisms are maps of coverings.

In $\mathbf{Cov}(X)$, as in any category, we have the notion of an isomorphism of coverings. It is easy to check that the following holds:

19.2 Proposition. *Let $p_1\colon T_1 \to X$ and $p_2\colon T_2 \to X$ be coverings of X. A map of coverings $f\colon T_1 \to T_2$ is an isomorphism in $\mathbf{Cov}(X)$ if and only if f is a homeomorphism.*

Proof. Exercise. □

19.3 Note. If $p_1\colon T_1 \to X$ and $p_2\colon T_2 \to X$ are coverings and the spaces T_1 and T_2 are homeomorphic then this does not imply that p_1 and p_2 are isomorphic coverings since it may happen than no homeomorphism between T_1 and T_2 is a map of coverings. For example, recall the for $n = 1, 2, \ldots$ we have an n-fold covering $p_n\colon S^1 \to S^1$ given by $p_n(z) = z^n$ (where we consider S^1 as the set of unit complex numbers: $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$). The total space of each of these coverings is S^1, but these coverings are non-isomorphic to one another since an n-fold covering cannot be isomorphic to an m-fold covering for $n \neq m$ (exercise).

Our goal in this chapter is to show that under some mild conditions the problem of determining if two coverings of X are isomorphic or not can be reduced to a purely algebraic problem. Recall that a space X is *locally path connected* if for any point $x \in X$ and any open neighborhood $U \subseteq X$ of x there exists an open neighborhood V of x such that $V \subseteq U$ and V is path connected.

19.4 Theorem. *Let X be a locally path connected space, and for $i = 1, 2$ let $p_i \colon T_i \to X$ be a covering such that T_i is a path connected space. Let $x_0 \in X$ and let $\tilde{x}_i \in p_i^{-1}(x_0)$. The coverings p_1 and p_2 are isomorphic if and only if the subgroup $p_{1*}(\pi_1(T_1, \tilde{x}_1)) \subseteq \pi_1(X, x_0)$ is conjugate to the subgroup $p_{2*}(\pi_1(T_2, \tilde{x}_2))$.*

One implication of Theorem 19.4 is straightforward: if the coverings p_1 and p_2 are isomorphic then we have a homeomorphism $f \colon T_1 \to T_2$ that we can use to relate the subgroups coming from these coverings. The other implication is more challenging, since it says that based on some information about fundamental groups we can produce a map $T_1 \to T_2$. Existence of such map will follow from the next theorem which is very useful in many applications:

19.5 Theorem (Lifting Criterion). *Let $p \colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Assume that Y is a connected and locally path connected space and let $y_0 \in Y$. A map $f \colon (Y, y_0) \to (X, x_0)$ has a lift $\tilde{f} \colon (Y, y_0) \to (T, \tilde{x}_0)$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$.*

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd} \qquad (*)$$

19.6 Note. By Lemma 17.11 if a lift $\tilde{f}$ exists then it is unique.

Proof of Theorem 19.5. ($\Rightarrow$) If the lift $\tilde{f}$ exists then the diagram ($*$) induces a commutative diagram of fundamental groups:

$$\begin{tikzcd} & \pi_1(T, \tilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(X, x_0) \end{tikzcd}$$

Therefore we obtain

$$f_*(\pi_1(Y, y_0)) = p_*\tilde{f}_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$$

($\Leftarrow$) We will construct the function $\tilde{f} \colon Y \to X$ as follows. Take $y \in Y$. Since the space Y is connected and locally path connected thus it is path connected, and so there exists a path $\omega \colon [0, 1] \to Y$ such that $\omega(0) = y_0$ and $\omega(1) = y$. By Corollary 17.10 the path $f\omega \colon [0, 1] \to X$ admits a unique lift $\widetilde{f\omega} \colon [0, 1] \to T$ such that $\widetilde{f\omega}(0) = \tilde{x}_0$. We set $\tilde{f}(y) := \widetilde{f\omega}(1)$.

In order to see that $\tilde{f}$ is well defined take ω' to be another path in Y joining y_0 with y, and let $\widetilde{f\omega'}$ be the lift of $f\omega'$ satisfying $\widetilde{f\omega'}(0) = \tilde{x}_0$. We need to show that $\widetilde{f\omega}(1) = \widetilde{f\omega'}(1)$. By Proposition 18.3 this is equivalent to showing that $[f\omega * \overline{f\omega'}] \in \pi_1(X, x_0)$ is an element of $p_*(\pi_1(T, \tilde{x}_0))$. Notice that we have

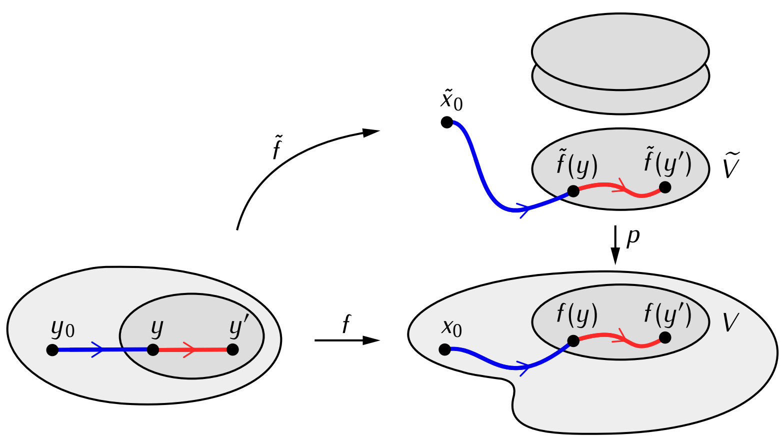
$$[f\omega * \overline{f\omega'}] = [f(\omega * \overline{\omega'})] = f_*([\omega * \overline{\omega'}])$$

where $[\omega * \overline{\omega'}] \in \pi_1(Y, y_0)$. By assumption $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$, so in particular $f_*([\omega * \overline{\omega'}]) \in p_*(\pi_1(T, \tilde{x}_0))$.

Directly from the construction of $\tilde{f}$ we get that $p\tilde{f} = f$ and that $\tilde{f}(y_0) = \tilde{x}_0$. We still need to check though that $\tilde{f}$ is a continuous function. It will suffice to show that for any $y \in Y$ there is an open neighborhood U of y such that $\tilde{f}|_U \colon U \to T$ is continuous. Let $V \subseteq X$ be an open neighborhood of $f(y)$ which is evenly covered, and take U to be an open neighborhood of y such that $U \subseteq f^{-1}(V)$ and that U is path connected. Such U exists by the assumption that Y is locally path connected. For any $y' \in U$ let $\tau_{y'}$ be a path in U joining y with y'. Also, let ω be a path in Y that joins y_0 with y. Notice that $\omega * \tau_{y'}$ joins y_0 and y', so $\tilde{f}(y') = \widetilde{\omega * \tau_{y'}}(1)$ where $\widetilde{\omega * \tau_{y'}}$ is the lift of $\omega * \tau_{y'}$ that starts at $\tilde{x}_0$. On the other hand $\widetilde{\omega * \tau_{y'}} = \widetilde{\omega} * \widetilde{\tau}_{y'}$ where $\widetilde{\omega}$ is the lift of ω that starts at $\tilde{x}_0$, and $\widetilde{\tau}_y$ is the lift of τ_y that starts at $\widetilde{\omega}(1) = \tilde{f}(y)$. In effect we obtain that $\tilde{f}(y') = \widetilde{\tau}_{y'}(1)$ for all $y' \in U$. Let $\widetilde{V}$ be the slice over V such that $\tilde{y} \in V$. The map $p|_{\widetilde{V}} \colon \widetilde{V} \to V$ is a homeomorphism. Notice that for any $y' \in U$ we have $\widetilde{\tau}_{y'} = (p|_{\widetilde{V}})^{-1} f\tau_{y'}$. This gives:

$$\tilde{f}(y') = \widetilde{\tau}_{y'}(1) = (p|_{\widetilde{V}})^{-1} f\tau_{y'}(1) = (p|_{\widetilde{V}})^{-1} f(y')$$

In other words we obtain $\tilde{f}|_U = (p|_{\widetilde{V}})^{-1} f$, which means that $\tilde{f}|_U$ is a continuous function.



□

Proof of Theorem 19.4. (⇒) Assume that we have an isomorphism of coverings:

$$\begin{tikzcd} T_1 \arrow[rr, "f", "\cong"'] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

This gives a commutative diagram of fundamental groups:

$$\begin{tikzcd} \pi_1(T_1, \tilde{x}_1) \arrow[rr, "f_*", "\cong"'] \arrow[dr, "p_{1*}"'] & & \pi_1(T_2, f(\tilde{x}_1)) \arrow[dl, "p_{2*}"] \\ & \pi_1(X, x_0) & \end{tikzcd}$$

Since f_* is an isomorphism we obtain that $p_{1*}(\pi_1(T_1, \tilde{x}_1)) = p_{2*}(\pi_1(T_2, f(\tilde{x}_1)))$. It remains to notice that since $f(\tilde{x}_1), \tilde{x}_2 \in p_2^{-1}(x_0)$ thus by Proposition 18.5 the subgroups $p_{2*}(\pi_1(T_2, f(\tilde{x}_1)))$ and $p_{2*}(\pi_1(T_2, \tilde{x}_2))$ are conjugate in $\pi_1(X, x_0)$.

($\Leftarrow$) Assume that $p_{1*}(\pi_1(T_1, \tilde{x}_1))$ and $p_{2*}(\pi_1(T_2, \tilde{x}_2))$ are conjugate subgroups of $\pi_1(X, x_0)$. By Proposition 18.5 we can find $\tilde{x}'_2 \in p_2^{-1}(x_0)$ such that $p_{1*}(\pi_1(T_1, \tilde{x}_1)) = p_{2*}(\pi_1(T_2, \tilde{x}'_2))$. Since the space X is locally path connected, thus so are T_1 and T_2. By the Lifting Criterion 19.5 there exists maps $\widetilde{p}_1 \colon (T_1, \tilde{x}_1) \to (T_2, \tilde{x}'_2)$ and $\widetilde{p}_2 \colon (T_2, \tilde{x}'_2) \to (T_1, \tilde{x}_1)$ such that the following diagrams commute:

$$\begin{tikzcd} & T_2 \arrow[d, "p_2"] \\ T_1 \arrow[ur, "\widetilde{p}_1"] \arrow[r, "p_1"'] & X \end{tikzcd} \qquad \begin{tikzcd} & T_1 \arrow[d, "p_1"] \\ T_2 \arrow[ur, "\widetilde{p}_2"] \arrow[r, "p_2"'] & X \end{tikzcd}$$

We will show that $\widetilde{p}_1$ and $\widetilde{p}_2$ are inverse isomorphisms of coverings. Notice that both $\widetilde{p}_2\widetilde{p}_1$ the identity map id_{T_1} fit into the commutative diagram

$$\begin{tikzcd} & T_1 \arrow[d, "p_1"] \\ T_1 \arrow[ur, shift left, "\widetilde{p}_2\widetilde{p}_1"] \arrow[ur, shift right, "\mathrm{id}_{T_1}"'] \arrow[r, "p_1"'] & X \end{tikzcd}$$

Moreover $\widetilde{p}_2\widetilde{p}_1(\tilde{x}_1) = \tilde{x}_1 = \mathrm{id}_{T_1}(\tilde{x}_1)$. By Lemma 17.11 this implies that $\widetilde{p}_2\widetilde{p}_1 = \mathrm{id}_{T_1}$. An analogous argument shows that $\widetilde{p}_1\widetilde{p}_2 = \mathrm{id}_{T_2}$. $\square$

19.7 Note. Theorem 19.4 gives a correspondence between isomorphism classes of path connected coverings over X and conjugacy classes of subgroups of $\pi_1(X, x_0)$. There is also a variant of of this correspondence that takes values in the set of subgroups of $\pi_1(X, x_0)$ (rather that conjugacy classes of subgroups). Namely, let (X, x_0) be a pointed space. A pointed covering is a basepoint preserving map $p\colon (T, \tilde{x}_0) \to (X, x_0)$ where p is a covering. If $p_1\colon (T_1, \tilde{x}_1) \to (X, x_0)$ and $p_2\colon (T_1, \tilde{x}_2) \to (X, x_0)$ are pointed coverings then a map of pointed covering is a function $f\colon (T_1, \tilde{x}_1) \to (T_2, \tilde{x}_2)$ such that $p_2 f = p_1$. Pointed coverings of (X, x_0) and their maps form a category $\mathbf{Cov}_*(X, x_0)$. Modifying the proof of Theorem 19.4 we obtain:

19.8 Theorem. *Let (X, x_0) be a locally path connected space, and for $i = 1, 2$ let $p_i\colon (T_i, \tilde{x}_i) \to (X, x_0)$ be a pointed covering such that T_i is a path connected space. The coverings p_1 and p_2 are isomorphic in the category $\mathbf{Cov}_*(X, x_0)$ if and only if $p_{1*}(\pi_1(T_1, \tilde{x}_1)) = p_{2*}(\pi_1(T_2, \tilde{x}_2))$.*

Proof. Exercise. □

Exercises to Chapter 19

E19.1 Exercise. Prove Proposition 19.2.

E19.2 Exercise. Consider a commutative diagram

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

where p_1, p_2 are coverings and T_1, T_2 are path connected spaces. Show that f is onto.

E19.3 Exercise. Assume that we have a map of coverings

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

where T_1, T_2 are path connected spaces. Assume also that for some $x_0 \in X$ the map

$$f|_{p^{-1}(x_0)}\colon p_1^{-1}(x_0) \to p_2^{-1}(x_0)$$

is 1-1. Show that $f\colon T_1 \to T_2$ is 1-1.

E19.4 Exercise. Let X be a locally path connected space, and let $p\colon T \to X$ be a covering such that T is a path connected space and $\pi_1(T)$ is a finite group. Show that any map of coverings

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "p"'] & & T \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

is an isomorphism.

E19.5 Exercise. Recall that the n-th homotopy group of a pointed space (X, x_0) is a group whose elements are homotopy classes of basepoint preserving maps $(S^n, s_0) \to (X, x_0)$. Let $x_0 \in S^1$. Show that $\pi_n(S^1, x_0)$ is a trival group for $n > 1$.

E19.6 Exercise. Let X be a locally path connected space, $x_0 \in X$, and let $p\colon T \to X$ be a covering with a path connected space T. Show that the following conditions are equivalent:

(i) The covering $p\colon T \to X$ is regular

(ii) For any points $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ there exists a map of coverings

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "p"'] & & T \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

such that $f(\tilde{x}_1) = \tilde{x}_2$.

E19.7 Exercise. Let $p\colon T \to X$ be a covering such that T, X are connected and locally path connected spaces. Assume that there exists a function $s\colon X \to T$ such that $ps = \mathrm{id}_X$. Show that T is homeomorphic to X.

E19.8 Exercise. Let $p\colon T \to X$ be a covering such that X is locally path connected, and T is path connected. Given maps $f, g\colon Y \to X$ let $p'\colon f^*T \to Y$ and $p''\colon g^*T \to Y$ be the coverings of Y obtained from p using f and g, respectively, as in Exercise 18.3. Show that if $f \simeq g$ then p' and p'' are isomorphic coverings.

20 | From Subgroups to Coverings

In the last chapter we have seen that if X is a locally path connected space and $x_0 \in X$ then there are 1-1 functions:

$$\begin{pmatrix} \text{isomorphism classes} \\ \text{of path connected} \\ \text{coverings of } X \end{pmatrix} \xrightarrow{\Omega} \begin{pmatrix} \text{conjugacy classes} \\ \text{of subgroups} \\ \text{of } \pi_1(X, x_0) \end{pmatrix}$$

$$\begin{pmatrix} \text{isomorphism classes of} \\ \text{pointed path connected} \\ \text{coverings of } (X, x_0) \end{pmatrix} \xrightarrow{\Omega} \begin{pmatrix} \text{subgroups} \\ \text{of} \\ \pi_1(X, x_0) \end{pmatrix}$$

In both cases the function Ω associates to a covering $p\colon T \to X$ with $\tilde{x}_0 \in p^{-1}(x_0)$ the (conjugacy class of) subgroup $p_*(\pi_1(T, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$. The natural question is for which subgroups $H \subseteq \pi_1(X, x_0)$ there exists a covering $p\colon T \to X$ such that $\Omega(p) = H$. Our goal here will be to prove that under some assumptions on X such covering p exists for any subgroup H, and so the maps Ω given above are bijections. As the first step we will show that Ω is a bijection provided that there exists a covering of X corresponding to the trivial subgroup of $\pi_1(X, x_0)$.

20.1 Definition. Let X be a locally path connected space. A *universal covering* of X is a covering $\tilde{p}\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is a simply connected space.

Directly from the Lifting Criterion 19.5 we obtain:

20.2 Proposition. *Let X be a locally path connected space and $\tilde{p}\colon \widetilde{X} \to X$ be a universal covering of X. For any covering $q\colon T \to X$ there exists a map of coverings:*

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "\tilde{p}"'] & & T \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

Notice that by Exercise 19.2 if T is path connected then the map f in Proposition 20.2 is onto. This suggests that if X has a universal covering then any path connected covering of X may be obtained as a quotient space of the universal covering space $\widetilde{X}$. This is the main idea in the proof of the following fact:

20.3 Theorem. *Let X be a locally path connected space and let $x_0 \in X$. If there exists a universal covering $\tilde{p}\colon \widetilde{X} \to X$ then for each subgroup $H \subseteq \pi_1(X, x_0)$ there exists a covering $p_H\colon T_H \to X$ and $\tilde{x}_H \in p_H^{-1}(x_0)$ such that $p_{H*}(\pi_1(T_H, \tilde{x}_H)) = H$.*

Proof. Let $H \subseteq \pi_1(X, x_0)$ be a subgroup. Let $\tilde{p}\colon \widetilde{X} \to X$ be a universal covering of X and let $y_0 \in \tilde{p}^{-1}(x_0)$. For each point $y \in \widetilde{X}$ let τ_y be a path in $\widetilde{X}$ joining y_0 with y. Notice that if $\tilde{p}(y) = \tilde{p}(y')$ then the path $\tilde{p}\tau_y * \tilde{p}\overline{\tau_{y'}}$ is loop in X based at x_0. Notice also that the homotopy class of this loop does not depend on the choice of paths τ_y and $\tau_{y'}$. Indeed, if σ_y and $\sigma_{y'}$ are some other paths in $\widetilde{X}$ joining y_0 with, respectively, y and y' then, since $\widetilde{X}$ is simply connected, by Proposition 5.6 we obtain $\tau_y \simeq \sigma_y$ and $\tau_{y'} \simeq \sigma_{y'}$ which gives a homotopy $\tilde{p}\tau_y * \tilde{p}\overline{\tau_{y'}} \simeq \tilde{p}\sigma_y * \tilde{p}\overline{\sigma_{y'}}$.

Define a relation $\sim$ on $\widetilde{X}$ such that $y \sim y'$ if

(i) $\tilde{p}(y) = \tilde{p}(y')$
(ii) $[\tilde{p}\tau_y * \tilde{p}\overline{\tau_{y'}}] \in H$

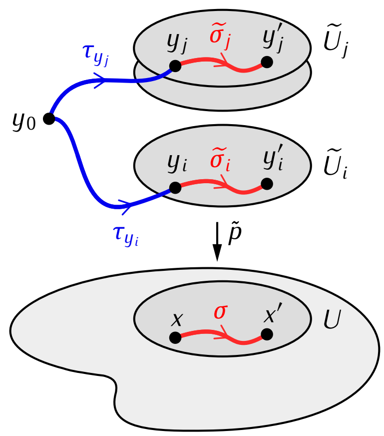
One can check that $\sim$ is an equivalence relation on $\widetilde{X}$ (exercise). Denote the quotient space by X_H and let $q\colon \widetilde{X} \to X_H$ be the quotient map. We get a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "q"] \arrow[dr, "\tilde{p}"'] & & X_H \arrow[dl, "p_H"] \\ & X & \end{tikzcd}$$

where p_H is given by $p_H([y]) = \tilde{p}(y)$. We will prove that $p_H\colon X_H \to X$ is a covering. Let $x \in X$ and let $U \subseteq X$ be an open neighborhood of x which is U is path connected and evenly covered by $\tilde{p}$. Such U exists by the assumption that X is locally path connected. We will show that U is evenly covered by p_H. We have $\tilde{p}^{-1}(U) = \bigcup_{i \in I} \widetilde{U}_i$ where $\{\widetilde{U}_i\}_{i\in I}$ is the set of all distinct slices of $\tilde{p}$ over U. Notice that if y, y' are points in the same slice $\widetilde{U}_i$ and $y \neq y'$ then $y \not\sim y'$ since $\tilde{p}(y) \neq \tilde{p}(y')$. On the other hand we claim that the following holds:

Claim. If $\widetilde{U}_i$, $\widetilde{U}_j$ are two slices, and there exist points $y_i \in \widetilde{U}_i$, $y_j \in \widetilde{U}_j$ such that $y_i \sim y_j$ then for every $y'_i \in \widetilde{U}_i$, $y'_j \in \widetilde{U}_j$ such that $\tilde{p}(y'_i) = \tilde{p}(y'_j)$ we have $y'_i \sim y'_j$.

To see this denote $x = \tilde{p}(y_i) = \tilde{p}(y_j)$ and $x' = \tilde{p}(y'_i) = \tilde{p}(y'_j)$. Since $x, x' \in U$ and U is path connected we can find a path σ in U such that $\sigma(0) = x$ and $\sigma(1) = x'$. Denote by $\widetilde{\sigma}_i$ and $\widetilde{\sigma}_j$ the lifts of σ to, respectively $\widetilde{U}_i$ and $\widetilde{U}_j$. Notice that $\widetilde{\sigma}_i(0) = y_i$, $\widetilde{\sigma}_i(1) = y'_i$, and likewise $\widetilde{\sigma}_j(0) = y_j$, $\widetilde{\sigma}_j(1) = y'_j$. Denote also by τ_{y_i}, τ_{y_j} paths in $\widetilde{X}$ that connect the point y_0 to, respectively y_i and y_j:



By the definition of the relation $\sim$ in order to show that $y'_i \sim y'_j$ we only need to verify that $[\tilde{p}(\tau_{y_i} * \widetilde{\sigma}_i) * \tilde{p}\overline{(\tau_{y_j} * \widetilde{\sigma}_j)}] \in H$. This holds since

$$[\tilde{p}(\tau_{y_i} * \widetilde{\sigma}_i) * \tilde{p}\overline{(\tau_{y_j} * \widetilde{\sigma}_j)}] = [\tilde{p}\tau_{y_i} * \sigma * \overline{\sigma} * \tilde{p}\overline{\tau_{y_j}}] = [\tilde{p}\tau_{y_i} * \tilde{p}\overline{\tau_{y_j}}]$$

and $[\tilde{p}\tau_{y_i} * \tilde{p}\overline{\tau_{y_j}}] \in H$, since by assumption $y_i \sim y_j$.

The statement of the claim implies that for any slice $\widetilde{U}_i$ the set $q^{-1}(q(\widetilde{U}_i))$ is a union of some number of slices of $\tilde{p}$ over U, and so it is an open set in $\widetilde{X}$. This shows that the set $q(\widetilde{U}_i)$ is open in X_H. It also shows that if $V \subseteq \widetilde{U}_i$ is an open set then $q(V)$ is open in X_H. Indeed, it is enough to check that $q^{-1}(q(V))$ is open in $\widetilde{X}$, but this holds since $q^{-1}(q(V)) = \tilde{p}^{-1}(\tilde{p}(V)) \cap q^{-1}(q(\widetilde{U}_i))$.

The claim also implies that we can select a subset $\{\widetilde{U}_{i_k}\}_{k\in K}$ of the set of slices of $\tilde{p}$ over U such that the map $q' \colon \bigcup_{k\in K} U_{i_k} \to p_H^{-1}(U)$ obtained as a restriction of q is a continuous bijection. Since by the observation above q' maps open sets to open sets, the inverse function q'^{-1} is also continuous, and so q' is a homeomorphism. Finally, since $\bigcup_{k\in K} U_{i_k} \cong U \times K$ (where the set K is taken with the discrete topology) we obtain a homeomorphism $U \times K \cong p_H^{-1}(U)$.

Let $\tilde{x}_H = q(y_0)$. It remains to prove that $p_{H*}(\pi_1(T_H, \tilde{x}_H)) = H$. Let ω be a loop in X based at x_0, and let $\widetilde{\omega} \colon [0,1] \to X_H$ be the lift of ω satisfying $\widetilde{\omega}(0) = \tilde{x}_H$ Recall that by Theorem 18.1 $[\omega]$ is an element of $p_{H*}(\pi_1(T_H, \tilde{x}_H))$ if and only if $\widetilde{\omega}$ is a loop in X_H. Therefore it will suffice to show that $\widetilde{\omega}$ is a loop if and only if $[\omega] \in H$. Notice that $\widetilde{\omega} = q\widetilde{\omega}'$ where $\widetilde{\omega}' \colon [0,1] \to \widetilde{X}$ is the lift of ω to $\widetilde{X}$ satisfying $\widetilde{\omega}'(0) = y_0$. From the construction of X_H it follows that $\widetilde{\omega}$ is a loop if and only if $\widetilde{\omega}'(1) \sim \widetilde{\omega}'(0) = y_0$

where $\sim$ is the equivalence relation on $\widetilde{X}$ defined before. Take $\widetilde{\omega}'$ to be a path joining y_0 with $\widetilde{\omega}'(1)$ and take the constant path c_{y_0} as a path joining y_0 with itself. Using the definition of $\sim$ we obtain that $\widetilde{\omega}'(1) \sim \widetilde{\omega}'(0)$ if and only if $[\tilde{p}\widetilde{\omega}' * \tilde{p}\overline{c_{y_0}}] \in H$. Since $[\tilde{p}\widetilde{\omega}' * \tilde{p}\overline{c_{y_0}}] = [\omega]$ we obtain that $\widetilde{\omega}'(1) \sim \widetilde{\omega}'(0)$ if and only if $[\omega] \in H$

□

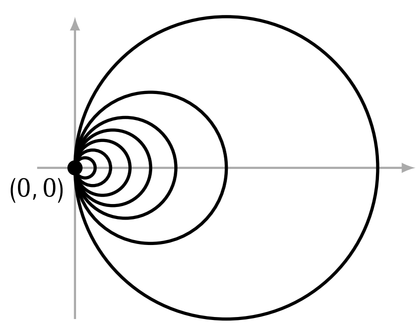
The remaining task is to determine for which spaces a universal covering exists. We will need the following definition:

20.4 Definition. A space X is *semi-locally simply connected* if every point $x \in X$ has an open neighborhood $U \subseteq X$ such that the homomorphism $i_*\colon \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion map $i\colon U \to X$ is the trivial homomorphism.

Equivalently, X is semi-locally simply connected if each point in X has an open neighborhood U such that any loop based at x and contained in U is homotopic to the constant loop, but though a homotopy contained in X (and not necessarily a homotopy contained in U).

20.5 Example. If X is a space such that each point $x \in X$ has an open neighborhood U where $\pi_1(U, x)$ is the trivial group, then X is semi-locally simply connected. One can use this to show, for example, that every topological manifold is semi-locally simply connected. On the other hand, it is possible to find a semi-locally simply connected space X, such that for some point $x \in X$ every open neighborhood of x has a non-trivial fundamental group.

20.6 Example. The *Hawaiian earring* space is a subspace $X \subseteq \mathbb{R}^2$ given by $X = \bigcup_{n=1}^{\infty} C_n$ where C_n is the circle with radius $\frac{1}{n}$ and center at the point $(0, \frac{1}{n})$:



This space is not semi-locally simply connected since for each open neighborhood U of the point $x_0 = (0, 0)$ the homomorphism $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is non-trivial.

Semi-local simple connectedness is a necessary condition for existence of a universal covering:

20.7 Proposition. *If X is space such that there exists a universal covering $p\colon \widetilde{X} \to X$ then X is semi-locally simply connected.*

Proof. Exercise. $\square$

Conversely, we will show that the following holds:

20.8 Theorem. *If X is a space which is connected, locally path connected, and semi-locally simply connected then there exists a universal covering $p\colon \widetilde{X} \to X$.*

Proof. Let X be a space satisfying assumptions of the theorem. We will say that an open set $U \subseteq X$ is *trivial* if U is path connected and for any $x \in U$ the homomorphism $i_*\colon \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion map $i\colon U \to X$ is trivial. Since X is locally path connected and semi-locally simply connected trivial sets form a basis of the topology on X, that is any open set in X is a union of trivial sets.

The first step in the construction of a universal covering $p\colon \widetilde{X} \to X$ is to describe the set of points of the space $\widetilde{X}$. This description will be based on the following reasoning. Assume that we already have a universal covering $p\colon \widetilde{X} \to X$, let $x_0 \in X$ and let $\widetilde{x}_0 \in p^{-1}(x_0)$. Since the space $\widetilde{X}$ is path connected, for any point $\tilde{x} \in \widetilde{X}$ there exists a path $\widetilde{\omega}$ such that $\omega(0) = \widetilde{x}_0$ and $\widetilde{\omega}(1) = \widetilde{x}$. Moreover, since $\widetilde{X}$ is simply connected any two such path in $\widetilde{X}$ are homotopic. In effect the assignment $[\widetilde{\omega}] \mapsto \widetilde{\omega}(1)$ gives a bijection:

$$\begin{pmatrix} \text{homotopy classes of paths} \\ \widetilde{\omega}\colon [0,1] \to \widetilde{X} \\ \text{with } \widetilde{\omega}(0) = \widetilde{x}_0 \end{pmatrix} \cong \begin{pmatrix} \text{points of } \widetilde{X} \end{pmatrix}$$

Notice that we also have a bijection:

$$\begin{pmatrix} \text{homotopy classes of paths} \\ \widetilde{\omega}\colon [0,1] \to \widetilde{X} \\ \text{with } \widetilde{\omega}(0) = \widetilde{x}_0 \end{pmatrix} \cong \begin{pmatrix} \text{homotopy classes of paths} \\ \omega\colon [0,1] \to X \\ \text{with } \omega(0) = x_0 \end{pmatrix}$$

which assigns to the homotopy class of a path $\widetilde{\omega}$ in $\widetilde{X}$ the homotopy class of $p\widetilde{\omega}$. The inverse function sends the homotopy class of a path ω in X to the homotopy class of $\widetilde{\omega}$, where $\widetilde{\omega}$ is the unique lift of ω satisfying $\widetilde{\omega}(0) = \widetilde{x}_0$.

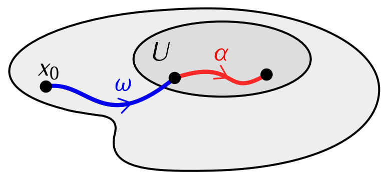
In effect we get a bijective correspondence:

$$\begin{pmatrix} \text{points of } \widetilde{X} \end{pmatrix} \cong \begin{pmatrix} \text{homotopy classes of paths} \\ \omega\colon [0,1] \to X \\ \text{with } \omega(0) = x_0 \end{pmatrix}$$

The upshot of this argument is that if we are given a space X then we can define $\widetilde{X}$ to be the set on the right hand side of the above bijection.

Next, we need to define a topology on the set $\widetilde{X}$. Let $[\omega] \in \widetilde{X}$, and let U be a trivial set such that $\omega(1) \in U$. Define:

$$U[\omega] = \{[\omega * \alpha] \mid \alpha\colon [0,1] \to U,\ \alpha(0) = \omega(1)\}$$



One can check that the collection of all sets $U[\omega]$ defined in this way is a basis of a topology on $\widetilde{X}$ (exercise). We will consider $\widetilde{X}$ as a topological space with topology defined by this basis.

Consider the function $p\colon \widetilde{X} \to X$ given by $p([\omega]) = \omega(1)$. We will show that this is a universal covering of X. We will use the following observations, proofs of which are left as an exercise:

(i) For any trivial set $U \subseteq X$ and any path $[\omega] \in \widetilde{X}$ such that $\omega(1) \in U$ the map

$$p|_{U[\omega]}\colon U[\omega] \to U$$

is a homeomorphism.

(ii) Let $U \subseteq X$ be a trivial set, let $x \in U$ and let $H(x_0, x) = \{[\omega] \in \widetilde{X} \mid \omega(1) = x\}$. Then

$$p^{-1}(U) = \bigcup_{[\omega] \in H(x_0,x)} U[\omega]$$

Moreover $U[\omega] \cap U[\omega'] = \varnothing$ for all $[\omega], [\omega'] \in H(x_0, x)$, $[\omega] \neq [\omega']$.

(iii) For a path $\omega\colon [0,1] \to X$ such that $\omega(0) = x_0$ and for $s \in [0,1]$ let ω_s be the path in X defined by $\omega_s(t) = \omega(st)$. The function $h_\omega\colon [0,1] \to \widetilde{X}$ given by $h_\omega(s) = [\omega_s]$ is continuous.

Directly from (ii) is follows that the function p is continuous. Furthermore, combining (ii) and (i) we obtain that p is covering and that each trivial set in X is evenly covered by p.

Next, by (iii) the space $\widetilde{X}$ is path connected. Indeed, for any $[\omega] \in \widetilde{X}$ the function h_ω is a path in $\widetilde{X}$ joining $[\omega]$ with $[c_{x_0}]$, the homotopy class of the constant path at x_0. It remains then to show that the fundamental group $\pi_1(\widetilde{X}, [c_{x_0}])$ is trivial, or equivalently that $p_*(\pi_1(\widetilde{X}, [c_{x_0}]))$ is the trivial subgroup of $\pi_1(X, x_0)$. Assume then that ω is a loop in X such that $[\omega] \in p_*(\pi_1(\widetilde{X}, [c_{x_0}]))$. By Theorem 18.1 this means that the lift of ω to $\widetilde{X}$ that starts at $[c_{x_0}]$ is a loop in $\widetilde{X}$. Notice, however, that this lift is given the path h_s defined in (iii). This path is a loop only when $[c_{x_0}] = h_\omega(0) = h_\omega(1) = [\omega]$ i.e. only when $[\omega]$ is the trivial element of $\pi_1(X, x_0)$.

$\square$

Exercises to Chapter 20

E20.1 Exercise. Prove Proposition 20.7.

E20.2 Exercise. Let X, Y be connected and locally path connected spaces, and let $\tilde{p}_X\colon \widetilde{X} \to X$, and $\tilde{p}_Y\colon \widetilde{Y} \to Y$ be their universal coverings. Show that if $X \simeq Y$ then $\widetilde{X} \simeq \widetilde{Y}$.

E20.3 Exercise. Describe explicitly all non-isomorphic coverings of the space $\mathbb{R}P^2 \times \mathbb{R}P^2$

E20.4 Exercise. Let X be a space, and let $A \subseteq X$. Assume that both X and A are connected and locally path connected, and that the inclusion map $i\colon A \to X$ induces an isomorphism of the fundamental groups

$$i_*\colon \pi_1(A, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

for $x_0 \in A$. Show that if $\tilde{p}\colon \widetilde{X} \to X$ is a universal covering of X then the map $\tilde{p}|_{\tilde{p}^{-1}(A)}\colon \tilde{p}^{-1}(A) \to A$ is a universal covering of A.

E20.5 Exercise. a) Let X be a finite, path connected, 1-dimensional CW complex. Show that if $\tilde{p}\colon \widetilde{X} \to X$ is the universal covering of X then the space $\widetilde{X}$ has the structure of a 1-dimensional CW complex such that $\tilde{p}$ is a cellular map.

b) Use part a) to show that if F is a finitely generated free group then every subgroup of F is free.

c) Recall that $[G : H]$ denotes the index of a subgroup H in a group G. Let F be free group on n generators, and let H be a subgroup of F. Show that if $[F : H] = k$ then H is a free group on $(n-1)k + 1$ generators.

21 | Equivalences of Categories

Results of Chapters 19 and 20 can be summarized as follows:

21.1 Theorem. *Let X be a connected, locally path connected, and semi-locally simply connected space, and let $x_0 \in X$. The map*

$$\Omega\colon \begin{pmatrix} \textit{isomorphism classes} \\ \textit{of path connected} \\ \textit{coverings of } X \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{conjugacy classes} \\ \textit{of subgroups} \\ \textit{of } \pi_1(X, x_0) \end{pmatrix}$$

given by $\Omega(p\colon T \to X) = p_*(\pi_1(T, \tilde{x}))$ for some $\tilde{x} \in p^{-1}(x_0)$ is a bijection.

Proof. The map Ω is 1-1 by Theorem 19.4, and it is onto by Theorems 20.3 and 20.8. □

Theorem 21.1 translates the topological problem of classifying coverings into an algebraic one, of identifying conjugacy classes of subgroups of a group. However, since coverings over X form a category $\mathbf{Cov}(X)$, with morphisms given by maps of coverings, a more complete correspondence between topology and algebra would be obtained if we could find some algebraic category $\mathbf{D}$ and a functor

$$F\colon \mathbf{Cov}(X) \to \mathbf{D}$$

that would let us restate problems about coverings and maps of coverings as problems about objects and morphism of the category $\mathbf{D}$. In Chapter 22 we will show that such category $\mathbf{D}$ and a functor F exist. Before we get to this though, we need to consider what properties the functor F should have so that it would allow us to go back and forth between categories $\mathbf{Cov}(X)$ and $\mathbf{D}$ without losing any essential information. The most obvious requirement is that F should be an isomorphism of categories, i.e. that there should exist a functor $G\colon \mathbf{D} \to \mathbf{Cov}(X)$ such the compositions $GF\colon \mathbf{Cov}(X) \to \mathbf{Cov}(X)$

and $FG\colon \mathbf{D} \to \mathbf{D}$ are identies on all objects and morphisms. It turns out however, that isomorphisms of categories appear very rarely in practical applications. A somewhat weaker but much more useful notion is an equivalence of categories:

21.2 Definition. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an *equivalence of categories* if there exists a functor $G\colon \mathbf{D} \to \mathbf{C}$ for which the following conditions hold:

1) For each object $c \in \mathbf{C}$ there exists an isomorphism $\eta_c\colon c \to GF(c)$ such that for any morphism $f\colon c \to c'$ the following diagram commutes:

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\eta_c"', "\cong"] & c' \arrow[d, "\cong"', "\eta_{c'}"] \\ GF(c) \arrow[r, "GF(f)"'] & GF(c') \end{tikzcd}$$

2) For each object $d \in \mathbf{D}$ there exists an isomorphism $\tau_d\colon d \to FG(d)$ such that for any morphism $g\colon d \to d'$ the following diagram commutes:

$$\begin{tikzcd} d \arrow[r, "g"] \arrow[d, "\tau_d"', "\cong"] & d' \arrow[d, "\cong"', "\tau_{d'}"] \\ FG(d) \arrow[r, "FG(g)"'] & FG(d') \end{tikzcd}$$

We will say that $\mathbf{C}$ and $\mathbf{D}$ are *equivalent categories* if there exists an equivalence $\mathbf{C} \to \mathbf{D}$.

The following fact is often useful, since it allows us to check if a functor is an equivalence of categories without constructing the inverse functor G.

21.3 Proposition. *A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an equivalence of categories if and only if the following conditions hold.*

(i) For each object $d \in \mathbf{D}$ there exists an object $c \in \mathbf{C}$ such that $d \cong F(c)$.

(ii) For any objects $c, c' \in \mathbf{C}$ the map $\mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$ given by $f \mapsto F(f)$ is a bijection.

Proof. Exercise. $\square$

21.4 Example. Let $\mathbf{FVect}(\mathbb{R})$ denote the category of finitely dimensional real vector spaces with linear transformations as morphisms. Also, let $\mathbf{M}(\mathbb{R})$ denote the category whose objects are natural numbers

$0, 1, 2, \ldots$. The set of morphisms $\mathrm{Mor}_{\mathbf{M}(\mathbb{R})}(n, m)$ consists of all $n \times m$ matrices with real coefficients. Composition of morphisms is given by matrix multiplication. We have a functor

$$F\colon \mathbf{M}(\mathbb{R}) \to \mathbf{FVect}(\mathbb{R})$$

defined as follows. On objects $F(n) = \mathbb{R}^n$. If A is an $n \times m$ matrix (i.e. a morphism $n \to m$ in $\mathbf{M}(\mathbb{R})$) then $F(A)\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by $F(A)(v) = Av$ for $v \in \mathbb{R}^n$. One can show that F is an equivalence of categories (exercise).

21.5 Example. Recall (4.8) that the fundamental groupoid of a space X is a category $\Pi_1(X)$ whose objects are points of X. For $x, x' \in X$ morphisms $x \to x'$ are homotopy classes of paths that begin at x and end at x'. Composition of morphisms is given by concatenation of paths. A map of spaces $f\colon X \to X'$ induces a functor of fundamental groupoids $f_*\colon \Pi_1(X) \to \Pi_1(X')$. One can show that if f is a homotopy equivalence of spaces then the functor f_* is an equivalence of categories (exercise).

Exercises to Chapter 21

E21.1 Exercise. Prove Proposition 21.3.

22 | Coverings and Group Actions

Let X be a topological space, and let $x_0 \in X$. Our goal in this chapter is to show that under some assumptions on X the category of path connected coverings of X is equivalent to the category of sets equipped with a transitive action of the group $\pi_1(X, x_0)$.

22.1 Definition. Let G be a group and S be a set. We say that G *acts on* X on the right if there exists a function

$$\mu\colon S \times G \to S$$

such that

(i) $\mu(s, e) = s$ for any $s \in S$, where $e \in G$ is the trivial element;

(ii) $\mu(\mu(s, g), h) = \mu(s, gh)$ for all $s \in S$, $h, g \in G$.

22.2 Note. From now on we will write sg instead of $\mu(s, g)$ in order to describe the action of g on s. We will also refer to sets with an action of a group G as G-sets.

22.3 Example. Let $p\colon T \to X$ be a covering and let $x_0 \in X$. We can define a right action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ as follows. For $[\omega] \in \pi_1(X, x_0)$ and $y \in p^{-1}(x_0)$ let $\widetilde{\omega}\colon [0, 1] \to T$ be the lift of ω such that $\widetilde{\omega}(0) = y$. Define:

$$y[\omega] := \widetilde{\omega}(1)$$

One can check that this satisfies the conditions of Definition 22.1 (exercise). The action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ defined in this way is called the *monodromy action* associated to the covering p.

22.4 Definition. We say that a group G acts on set S *transitively* if for any $s, s' \in S$ there exists $g \in G$ such that $sg = s'$.

22.5 Proposition. *Let $p\colon T \to X$ be a covering, and let $x_0 \in X$. If T is path connected then the monodromy action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is transitive.*

Proof. Exercise. □

22.6 Definition. Let G be a group and let S, S' be G-sets. A function $f\colon S \to S'$ is *G-equivariant* if $f(sg) = f(s)g$ for all $s \in S$ and $g \in G$.

22.7 Note. G-sets and G-equivariant functions form a category which we will denote by $\mathbf{Set}_G$.

22.8 Proposition. *Let X be a space, and let*

$$\begin{tikzcd} T_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

be a map of coverings. For any $x_0 \in X$ the induced map of fibers $f\colon p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ is $\pi_1(X, x_0)$-equivariant.

Proof. Exercise. □

22.9 Corollary. *Let X be a space and let $x_0 \in X$. The assignment which associates to each path connected covering $p\colon T \to X$ the $\pi_1(X, x_0)$-set $p^{-1}(x_0)$ and to each map of coverings the map of fibers defines a functor*

$$\Lambda\colon \mathbf{Cov}(X) \to \mathbf{Set}_{\pi_1(X,x_0)}$$

Proof. Exercise. □

For the reminder of this chapter we will be restrict attention to coverings $T \to X$ where T is a path connected space. Let $\mathbf{PCov}(X)$ denote the category of all such covering of X. Also, for a group G let $\mathbf{TSet}_G$ denote the category of all G-sets with a transitive action of G. By Proposition 22.5 the functor Λ restricts to a functor

$$\Lambda\colon \mathbf{PCov}(X) \to \mathbf{TSet}_{\pi_1(X,x_0)}$$

Our next goal is to show that the following holds:

22.10 Theorem. *Let X be a connected, locally path connected, and semi-locally simply connected space, and let $x_0 \in X$. The functor*

$$\Lambda\colon \mathbf{PCov}(X) \to \mathbf{TSet}_{\pi_1(X,x_0)}$$

is an equivalence of categories.

By Proposition 21.3 the proof of Theorem 22.10 can be split into two parts:

1) We need to show that any set with a transitive action of the group $\pi_1(X, x_0)$ is isomorphic to a $\pi_1(X, x_0)$-set $\Lambda(p\colon T \to X) = p^{-1}(x_0)$ for some path connected covering p.

2) We also need to show that maps of path connected coverings of X are in a bijective correspondence with $\pi_1(X, x_0)$-equivariant maps of their fibers.

Part 1) will follow immediately from the following result:

22.11 Proposition. *Let X be a connected, locally path connected, and semi-locally simply connected space and let $x_0 \in X$. The map*

$$\Lambda\colon \begin{pmatrix} \textit{isomorphism classes} \\ \textit{of path connected} \\ \textit{coverings of } X \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{isomorphism classes} \\ \textit{of sets with transitive} \\ \textit{action of } \pi_1(X, x_0) \end{pmatrix}$$

given by $\Lambda(p\colon T \to X) = p^{-1}(x_0)$ is a bijection.

The proof of Proposition 22.11 will use some properties of transitive G-sets that we develop below.

22.12 Definition. Let G be a group, and S be a G-set. The *stabilizer* of en element $s \in S$ is the subgroup $G_s \subseteq G$ given by:

$$G_s = \{g \in G \mid sg = s\}$$

22.13 Proposition. *Let $p\colon T \to X$ be a covering, and let $x_0 \in X$. The stabilizer of an element $\tilde{x} \in p^{-1}(x_0)$ under the monodromy action is the subgroup $p_*(\pi_1(T, \tilde{x})) \subseteq \pi_1(X, x_0)$.*

Proof. Exercise. □

22.14 Lemma. *Let G be a group.*

1) *If G acts transitively on a set S and $s, s' \in S$ then the stabilizers G_s and $G_{s'}$ are conjugate subgroups of the group G.*
2) *Let S be a set with an action of G and let $s \in S$. The assignment $S \mapsto G_s$ defines a bijective correspondence:*

$$\Phi\colon \begin{pmatrix} \textit{isomorphism classes} \\ \textit{of sets with a transitive} \\ \textit{action of } G \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{conjugacy classes} \\ \textit{of subgroups} \\ \textit{of } G \end{pmatrix}$$

Proof. 1) Since G acts transitively we have $s' = sh$ for some $h \in G$. We will show that $G_{s'} = h^{-1}G_s h$.

For $g \in G_s$ we have:

$$s'(h^{-1}gh) = sgh = sh = s'$$

Therefore $h^{-1}G_sh \subseteq G_{s'}$. Conversely, if $g \in G_{s'}$ then

$$sh = s' = s'g = shg$$

This implies that $s = s(hgh)^{-1}$, so $hgh^{-1} \in G_s$, or equivalently $g \in h^{-1}G_sh$. Thus $G_{s'} \subseteq h^{-1}G_sh$.

2) We will construct a function

$$\Psi\colon \begin{pmatrix} \text{conjugacy classes} \\ \text{of subgroups} \\ \text{of } G \end{pmatrix} \longrightarrow \begin{pmatrix} \text{isomorphism classes} \\ \text{of sets with transitive} \\ \text{action of } G \end{pmatrix}$$

which is the inverse of Φ. For a subgroup $H \subseteq G$ let $H\backslash G$ denote the set of right cosets of H in G. Define an action of G on $H\backslash G$ by $(Hg)g' = H(gg')$. This action is transitive since for any $Hg, Hg' \in H\backslash G$ we have $Hg' = (Hg)(g^{-1}g')$. Let $\Psi(H) = H\backslash G$. In order to show that Ψ is well defined on conjugacy classes we need to check that if $H' \subseteq G$ is a subgroup conjugate to H then the G-sets $H\backslash G$ and $H'\backslash G$ are isomorphic. Assume then that $H' = kHk^{-1}$ for some $k \in G$. Define $f\colon H\backslash G \to H'\backslash G$ by $f(Hg) = H'kg$. One can check that this is a well defined isomorphism of G-sets (exercise). Let $e \in G$ be the trivial element. Since the stabilizer of $He \in H\backslash G$ is the subgroup H, we obtain that $\Phi\Psi([H]) = [H]$ where $[H]$ denotes the conjugacy class of H, and so $\Phi\Psi$ is an identity function. One can check that the composition $\Psi\Phi$ also is an identity (exercise).

□

Proof of Proposition 22.11. Consider the diagram:

$$\begin{tikzcd}
\begin{pmatrix} \text{isomorphism classes} \\ \text{of path connected} \\ \text{coverings of } X \end{pmatrix} \arrow[rr, "\Lambda"] \arrow[dr, "\Omega"'] & & \begin{pmatrix} \text{isomorphism classes} \\ \text{of sets with transitive} \\ \text{action of } \pi_1(X, x_0) \end{pmatrix} \arrow[dl, "\Phi"] \\
& \begin{pmatrix} \text{conjugacy classes} \\ \text{of subgroups} \\ \text{of } \pi_1(X, x_0) \end{pmatrix} &
\end{tikzcd}$$

The map Φ is defined as in proposition 22.14, and and Ω is defined as in Theorem 21.1. By Proposition 22.13 this diagram commutes. Since Φ is a bijection by Proposition 22.14, and Ω is a bijection by Theorem 21.1 we obtain that Λ is a bijection. □

Next, we turn to properties of the functor Λ related to maps of coverings. We will show that the following holds.

22.15 Proposition. *Let X be a connected and locally path connected space, and let $x_0 \in X$. For any path connected coverings $p_i\colon T_i \to X$, $i = 1, 2$ the assignment*

$$\Lambda\colon \left(\begin{array}{c} \textit{maps of coverings} \\ T_1 \to T_2 \end{array}\right) \longrightarrow \left(\begin{array}{c} \pi_1(X, x_0)\textit{-equivariant maps} \\ p_1^{-1}(x_0) \to p_2^{-1}(x_0) \end{array}\right)$$

is a bijection.

The proof of Proposition 22.15 will use the following fact. Recall that for a G-set S by G_s we denote the stabilizer of an element $s \in S$

22.16 Lemma. *Let S, T be sets with a transitive action of a group G, and let $s_0 \in S$, $t_0 \in T$. A G-equivariant map $f\colon S \to T$ such that $f(s_0) = t_0$ exists if and only if $G_{s_0} \subseteq G_{t_0}$. Moreover, if such map exists then it is unique.*

Proof. Exercise. □

Proof of Proposition 22.15. We will prove first that Λ is onto. Let $f\colon p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ be a $\pi_1(X, x_0)$-equivariant map. We need to show that there exists a map of coverings $\bar{f}\colon T_1 \to T_2$ such that $\Lambda(\bar{f}) = f$. Let $\tilde{x}_1 \in p_1^{-1}(x_0)$, and let $\tilde{x}_2 = f(\tilde{x}_1)$. Combining Proposition 22.13 and Lemma 22.16 we obtrain

$$p_{1*}(\pi_1(T_1, \tilde{x}_1)) \subseteq p_{2*}(\pi_1(T_2, \tilde{x}_2))$$

Therefore, by the lifting criterion (19.5) there exists a map of coverings $\bar{f}\colon T_1 \to T_2$ such that $f(\tilde{x}_1) = \tilde{x_2}$. Since the map $\Lambda(\bar{f})\colon p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ satisfies $\Lambda(\bar{f})(\tilde{x}_1) = \tilde{x}_2$ the uniqueness part of Lemma 22.16 gives $\Lambda(\bar{f}) = f$.

Next, assume that $f, f'\colon T_1 \to T_2$ are maps of coverings such that $\Lambda(f) = \Lambda(f')$. This implies that for $\tilde{x} \in p^{-1}(x_0)$ we have

$$f(\tilde{x}) = \Lambda(f)(\tilde{x}) = \Lambda(f')(\tilde{x}) = f'(\tilde{x})$$

By Lemma 17.11 this gives $f = f'$.

□

Proof of Theorem 22.10. Follows directly from Propositions 21.3, 22.11 and 22.15. □

Exercises to Chapter 22

E22.1 Exercise. Prove Proposition 22.5.

E22.2 Exercise. Prove Proposition 22.8.

E22.3 Exercise. Prove Proposition 22.13.

E22.4 Exercise. Prove Lemma 22.16.

23 | Deck Transformations

23.1 Definition. Let $p\colon T \to X$ be a covering. A *deck transformation* of p is an isomorphism of coverings

$$\begin{tikzcd} T \arrow[rr, "f", "\cong"'] \arrow[dr, "p"'] & & T \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Deck transformations form a group under composition of isomorphisms. We will denote this group by $D(p)$. In this chapter we will compute the group $D(p)$ for a path connected covering p in terms of fundamental groups of X and T. Recall that in Chapter 22 we constructed a functor

$$\Lambda\colon \mathbf{PCov}(X) \to \mathbf{TSet}_{\pi_1(X,x_0)}$$

from the category of path connected coverings of a space X to the category of transitive $\pi_1(X, x_0)$-sets. We also showed (22.15) that if X is a connected and locally path connected space then this functor is a bijection of sets of morphisms. Since any functor preserves isomorphism, if $f : T_1 \to T_2$ is an isomorphism of coverings of X, then $\Lambda(f)$ is an isomorphism of $\pi_1(X, x_0)$-sets. The following fact implies that the converse is also true:

23.2 Lemma. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor such that for any $c, c' \in \mathbf{C}$ the map the map $\mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$ given by $f \mapsto F(f)$ is a bijection. A morphism $f\colon c \to c'$ i in $\mathbf{C}$ is an isomorphism if and only if $F(f)\colon F(c) \to F(c')$ is an isomorphism.*

Proof. Exercise. □

As a consequence we obtain:

23.3 Corollary. *Let X be a connected and locally path connected space, $x_0 \in X$, and let $p\colon T \to X$ be a path connected covering. The group of deck transformations $D(p)$ is isomorphic to the group of $\pi_1(X, x_0)$-equivariant isomorphisms $p^{-1}(x_0) \to p^{-1}(x_0)$.*

Proof. Exercise. □

In view of Corollary 23.3 the problem of computing the group of deck transformations reduces to the problem of computing the group of G-equivariant isomorphisms of a G-set S. Denote this group by $\mathrm{Iso}_G(S)$.

23.4 Definition. Let G be a group, and let $H \subseteq G$ be a subgroup. The *normalizer* of H in G is the subgroup $N_G(H) \subseteq G$ defined by

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

23.5 Note. $N_G(H)$ is the largest subgroup of G that contains H as its normal subgroup. In particular H is a normal subgroup of G if and only if $N_G(H) = G$.

Recall that if S is a G-set and $s \in S$ then by G_s we denote the stabilizer of s.

23.6 Proposition. *Let G be a group, and let S is a transitive G-set. For any $s \in S$ there exists an isomorphism of groups*

$$\mathrm{Iso}_G(S) \cong N_G(G_s)/G_s$$

Proof. Let $f\colon S \to S$ be a G-equivariant isomorphism. Since the action of G on S is transitive we we have $f(s) = sg_f$ for some $g_f \in G$ (depending on s). We claim that $g_f \in N_G(G_s)$. Indeed, for any $h \in G_s$ we have

$$s(g_f h g_f^{-1}) = f(s)(hg_f^{-1}) = f(sh)g_f^{-1} = f(s)g_f^{-1} = s(g_f g_f^{-1}) = s$$

which shows that $g_f h g_f^{-1} \in G_s$.

Define a map

$$\varphi\colon \mathrm{Iso}_G(S) \to N_G(G_s)/G_s$$

by $\varphi(f) := g_f G_s$. To verify that φ is well defined we need to check that if $\bar{g}_f \in G$ is another element such that $f(s) = s\bar{g}_f$ then $g_f G_s = \bar{g}_f G_s$. Since $sg_f = f(s) = s\bar{g}_f$ we get $s = s\bar{g}_f g_f^{-1}$ which gives $\bar{g}_f g_f^{-1} \in G_s$. By the observation above $g_f \in N_G(G_s)$, so $(\bar{g}_f g_f^{-1})g_f = g_f h$ for some $h \in G_s$. This gives:

$$\bar{g}_f G_s = \bar{g}_f g_f^{-1} g_f G_s = g_f h G_s = g_f G_s$$

Next, we claim that φ is a group homomorphism. Indeed, if $f, f' \in \mathrm{Iso}_G(S)$, $f(s) = sg_f$, $f'(s) = sg_{f'}$ then

$$f' \circ f(s) = f'(sg_f) = f'(s)g_f = sg_{f'}g_f$$

and so $\varphi(f' \circ f) = (g_{f'}g_f)G_s = \varphi(f') \cdot \varphi(f)$. It remains to show that φ is an isomorphism (exercise). □

23.7 Proposition. *Let X be a connected and locally path connected space, and let $x_0 \in X$. For a path connected covering $p\colon T \to X$ and $\tilde{x} \in p^{-1}(x_0)$ there exists an isomorphism of groups:*

$$D(p) \cong N_{\pi_1(X,x_0)}(p_*(\pi_1(T,\tilde{x})))/p_*(\pi_1(T,\tilde{x}))$$

23.8 Note. Recall that a covering $p\colon T \to X$ is regular if $p_*(\pi_1(T,\tilde{x}))$ is a normal subgroup of $\pi_1(X,x_0)$. In such case the isomorphism in Proposition 23.7 gives

$$D(p) \cong \pi_1(X,x_0)/p_*(\pi_1(T,\tilde{x}))$$

In particular, for the universal covering $\tilde{p}\colon \widetilde{X} \to X$ we obtain $D(\tilde{p}) \cong \pi_1(X,x_0)$.

Exercises to Chapter 23

E23.1 Exercise. For a function $f\colon X \to X$ by $\mathrm{Fix}(f)$ we will denote the set of fixed points of f:

$$\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$$

Let X be a connected and locally path connected space, let $\widetilde{p}\colon \widetilde{X} \to X$ be the universal covering of X, and let $f\colon X \to X$ be a map. We will say that a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ is a lift of f if the following diagram commutes:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{p}"'] & \widetilde{X} \arrow[d, "\widetilde{p}"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

Let S denote the set of all lifts of f.

a) Show that $\mathrm{Fix}(f) = \bigcup_{\widetilde{f}\in S} \widetilde{p}(\mathrm{Fix}(\widetilde{f}))$.

b) Let $\widetilde{f}_1, \widetilde{f}_2 \in S$. Show that the following conditions are equivalent:

(i) $\widetilde{p}(\mathrm{Fix}(\widetilde{f}_1)) \cap \widetilde{p}(\mathrm{Fix}(\widetilde{f}_2)) \neq \varnothing$

(ii) There exists a deck transformation $g\colon \widetilde{X} \to \widetilde{X}$ such that $\widetilde{f}_2 = g\widetilde{f}_1 g^{-1}$

(iii) $\widetilde{p}(\mathrm{Fix}(\widetilde{f}_1)) = \widetilde{p}(\mathrm{Fix}(\widetilde{f}_2))$

c) Let $f\colon (S^1,x_0) \to (S^1,x_0)$ be a map such that the homomorphism $f_*\colon \pi_1(S^1,x_0) \to \pi_1(S^1,x_0)$ is given by $f_*([\omega]) = n \cdot [\omega]$ for some $n \in \mathbb{Z}$. Show that $\mathrm{Fix}(f)$ consists of at least $|n-1|$ points.