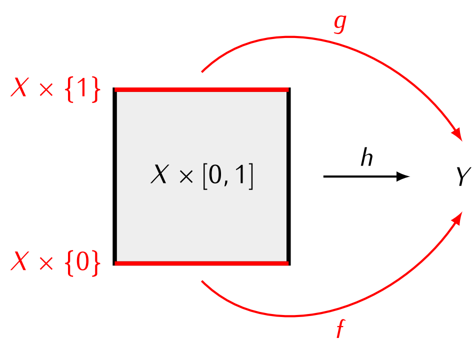
Contents

1 | Review: Homotopies

1.1 Definition. Let $f, g\colon X \to Y$ be continuous functions. A *homotopy* between f and g is a continuous function $h\colon X \times [0,1] \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$:



If such homotopy exists then we say that the functions f and g are *homotopic* and we write $f \simeq g$. We will also write $h\colon f \simeq g$ to indicate that h is a homotopy between f and g.

1.2 Note. Given a homotopy $h\colon X \times [0,1] \to Y$ it will be convenient denote by $h_t\colon X \to Y$ the function defined by $h_t(x) = h(x,t)$. If $h\colon f \simeq g$ then $h_0 = f$ and $h_1 = g$.

1.3 Definition. Let X be a space and let $A \subseteq X$. If $f, g\colon X \to Y$ are functions such that $f|_A = g|_A$ then we say that f and g are *homotopic relative to* A if there exists a homotopy $h\colon X \times [0,1] \to Y$ such that $h_0 = f$, $h_1 = g$ and $h_t|_A = f|_A = g|_A$ for all $t \in [0,1]$. In such case we write $f \simeq g$ (rel A).

1.4 Definition. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. If such maps exist we say that the spaces X and Y are *homotopy equivalent* and we write $X \simeq Y$.

1.5 Note. If f and g are maps as in Definition 1.4 then we say that g is a *homotopy inverse* of f.

1.6 Definition. If X is a space such that $X \simeq *$ then we say that X is a *contractible space*.

1.7 Definition. A subspace $A \subseteq X$ is a *deformation retract* of a space X if there exists a homotopy $h\colon X \times [0,1] \to X$ such that

1) $h_0 = \mathrm{id}_X$
2) $h_t|_A = \mathrm{id}_A$ for all $t \in [0,1]$
3) $h_1(x) \in A$ for all $x \in X$

In such case we say that h is a *deformation retraction* of X onto A.

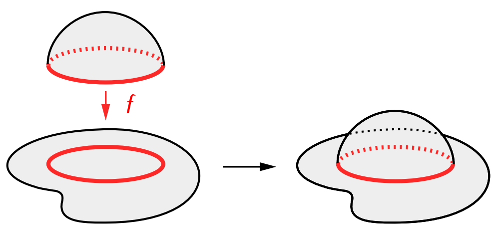
1.8 Proposition. *If $A \subseteq X$ is a deformation retract of X then $A \simeq X$.*

1.9 Note. Let X, X' be spaces, $A \subseteq X$, $A' \subseteq X'$. By a map $f\colon (X, A) \to (X', A')$ we will understand a function $f\colon X \to X'$ such that $f(A) \subseteq A'$. A homotopy of such maps is a homotopy $h\colon X \times [0,1] \to X'$ such that $h_t(A) \subseteq A'$ for each $t \in [0,1]$.

We will also use a variant of this for triples of spaces. If $B \subseteq A \subseteq X$ and $B' \subseteq A' \subseteq X'$ then a map $f\colon (X, A, B) \to (X', A', B')$ is a function $f\colon X \to X'$ such that $f(A) \subseteq A'$ and $f(B) \subseteq B'$. A homotopy h of such maps satisfies the same conditions on every level h_t.

2 | Review: CW Complexes

2.1 Definition. Let X be a space and let $f\colon S^{n-1} \to X$ be a continuous function. We say that a space Y is obtained by *attaching an n-cell* to X if $Y = X \sqcup D^n/{\sim}$ where $\sim$ is the equivalence relation given by $x \sim f(x)$ for all $x \in S^{n-1} \subseteq D^n$. We write $Y = X \cup_f e^n$.



2.2 Some terminology:

- The map $f\colon S^{n-1} \to X$ is called the *attaching map* of the cell e^n.
- The map $\bar{f}\colon D^n \to X \sqcup D^n \to X \cup_f e^n$ is called the *characteristic map* of the cell e^n.
- The subspace $e^n = \bar{f}(D^n \smallsetminus S^{n-1}) \subseteq X \cup_f e^n$ is called the *open cell*.
- The subspace $\bar{e}^n = \bar{f}(D^n) \subseteq X \cup_f e^n$ is called the *closed cell*.

2.3 Proposition. *If $f, g\colon S^{n-1} \to X$ are maps such that $f \simeq g$ then $X \cup_f e^n \simeq X \cup_g e^n$.*

2.4 Definition. Let X be topological space and let $A \subseteq X$. The pair (X, A) is a *relative CW complex* if $X = \bigcup_{n=-1}^{\infty} X^{(n)}$ where

1) $X^{(-1)} = A$;
2) for $n \geq 0$ the space $X^{(n)}$ is obtained by attaching n-cells to $X^{(n-1)}$;
3) the topology on X is defined so that a set $U \subseteq X$ is open if and only if $U \cap X^{(n)}$ is open in $X^{(n)}$ for all n.

2.5 Note. If (X, A) is a relative CW complex then the space $X^{(n)}$ is called the *n-skeleton* of X.

2.6 Note. By part 3) of Definition 2.4 if (X, A) is a relative CW complex then a function $f\colon X \to Z$ is continuous if and only if $f|_{X^{(n)}}\colon X^{(n)} \to Z$ is continuous for all $n \geq -1$.

2.7 Note. Assume that (X, A) is a relative CW complex and that we are given a map $g\colon A \to Z$. In such situation, we will often want to construct a map $\bar{g}\colon X \to Z$ such that $\bar{g}|A = g$. Usually, this construction will proceed inductively with respect to the skeleta of X. We will assume that we have already constructed a map $\bar{g}_{n-1}\colon X^{(n-1)} \to Z$ such that $\bar{g}_{n-1}|_A = g$, and we will attempt to extend $\bar{g}_{n-1}$ to $\bar{g}_n\colon X^{(n)} \to Z$. The space $X^{(n)}$ is the quotient space of $X^{(n-1)} \sqcup \bigsqcup_i D^n$ with the equivalence relation defined by the attaching maps of n-cells. Therefore, to define $\bar{g}_n$ it will suffice, for each n-cell e^n with the attaching map $f\colon S^{n-1} \to Z$, to give a map $\varphi\colon D^n \to Z$ such that $\varphi|_{S^{n-1}} = \bar{g}_{n-1}f$.

Once we have maps $\bar{g}_n$ for all n, we can define $\bar{g}\colon X \to Z$ by setting $\bar{g}|_{X^{(n)}} = \bar{g}_n$. The map $\bar{g}$ is continuous by (2.6).

2.8 Definition. A *CW complex* is a space X such that $(X, \varnothing)$ is a relative CW complex.

2.9 Definition. 1) A CW complex X is *finite* if it consists of finitely many cells.

2) A CW complex X is *finite dimensional* if $X = X^{(n)}$ for some n.

3) The *dimension* of a CW complex X is defined by

$$\dim X = \begin{cases} \min\{n \mid X = X^{(n)}\} & \text{if } X \text{ is finite dimensional} \\ \infty & \text{otherwise} \end{cases}$$

2.10 Definition. Let X, Y be relative CW complexes. A map $f\colon X \to Y$ is *cellular* if $f(X^{(n)}) \subseteq Y^{(n)}$ for all $n \geq 0$.

2.11 Cellular Approximation Theorem. *Let X, Y be relative CW complexes. For any map $f\colon X \to Y$ there exists a cellular map $g\colon X \to Y$ such that $f \simeq g$. Moreover, if $A \subseteq X$ is a subcomplex and $f|_A\colon A \to Y$ is a cellular map then g can be selected so that $f|_A = g|_A$ and $f \simeq g$* (rel A).

2.12 Corollary. *If $n > m$ then every map $f\colon S^m \to S^n$ is homotopic to a constant map.*

Proof. Consider S^n with the structure of a CW complex with one 0-cell and one n-cell. By Theorem 2.11 any map $f\colon S^m \to S^n$ is homotopic to a cellular map. Since the m-skeleton of S^n consists of a single point, such a cellular map is constant. □

2.13 Definition. Let X be a topological space, and let $A \subseteq X$. The pair (X, A) has the *homotopy extension property* if any map

$$h\colon X \times \{0\} \cup A \times [0, 1] \to Y$$

can be extended to a map $\bar{h}\colon X \times [0, 1] \to Y$.

2.14 Theorem. *Any relative CW complex (X, A) has the homotopy extension property.*

2.15 Proposition. *If (X, A) has the homotopy extension property and A is a contractible space, then the quotient map $q\colon X \to X/A$ is a homotopy equivalence.*

2.16 Inductive Homotopy Lemma. *Let (X, A) be a relative CW complex and let $A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ be subcomplexes of X such that $\bigcup_n X_n = X$. Assume that for $n \geq -1$ we have maps $f_n\colon X \to Y$ such that*

1) $f_n|_{X_{n-1}} = f_{n-1}|_{X_{n-1}}$ *for all* $n \geq 0$
2) $f_n \simeq f_{n-1}$ $(\mathrm{rel}\ X_{n-1})$ *for all* $n \geq 0$

Let $g\colon X \to Y$ be given by $g(x) = f_n(x)$ if $x \in X_n$. Then g is a continuous function and $f_{-1} \simeq g$ $(\mathrm{rel}\ A)$.

2.17 Example. Take

$$S^n = \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

Denote also $S^{-1} = \varnothing$. For each n we have an embedding $j\colon S^n \hookrightarrow S^{n+1}$ given by $j(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n+1}, 0)$. Define $S^\infty = \bigcup_n S^n$. A set $U \subseteq S^\infty$ is open if for each $n \geq 0$ the set $U \cap S^n$ is open in S^n.

The space S^∞ has a CW complex structure where S^n is the n-skeleton of S^∞.

2.18 Proposition. *S^∞ is a contractible space.*

Proof. Let $x_0 \in S^0 \subseteq S^\infty$. We can assume that S^∞ has a CW complex structure such that x_0 is a 0-cell. By Lemma 2.16 it will suffice to construct functions $f_n\colon S^\infty \to S^\infty$ for $n \geq 0$ such that

1) $f_{-1} = \mathrm{id}_{S^\infty}$
2) $f_n|_{S^n} = x_0$ for all $n \geq 0$
3) $f_n \simeq f_{n-1}$ $(\mathrm{rel}\ S^{n-1})$ for all $n \geq 0$

We will construct functions f_n by induction with respect to n. Assume that we already have a function f_n satisfying the above properties. This, in particular, means that $f_n|_{S^n} = x_0$. We want to get a function f_{n+1} such that $f_{n+1}|_{S^{n+1}} = x_0$ and $f_n \simeq f_{n+1}$ $(\mathrm{rel}\ S^n)$. By Theorem 2.11, the function f_n is homotopic $(\mathrm{rel}\ S^n)$ to a cellular function $g\colon S^\infty \to S^\infty$. The function g restricts to a map $g|_{S^{n+1}}\colon S^{n+1} \to S^{n+2} \subseteq S^\infty$. Using Corollary 2.12 we obtain that there exists a homotopy $h\colon S^{n+1} \times [0, 1] \to S^\infty$ between $g|_{S^{n+1}}$ and the constant map to x_0. We can choose this homotopy so that it is relative to S^n. By Theorem 2.14 we can extend h to a homotopy $\bar{h}\colon S^\infty \times [0, 1] \to S^\infty$. Take $f_{n+1} = \bar{h}_1$.

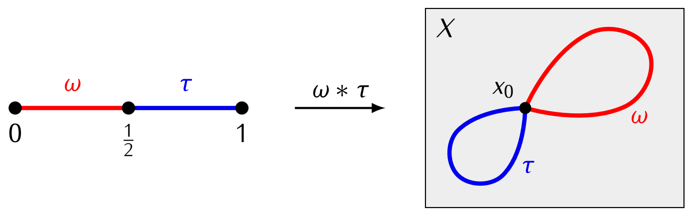
$\square$

3 | Higher Homotopy Groups

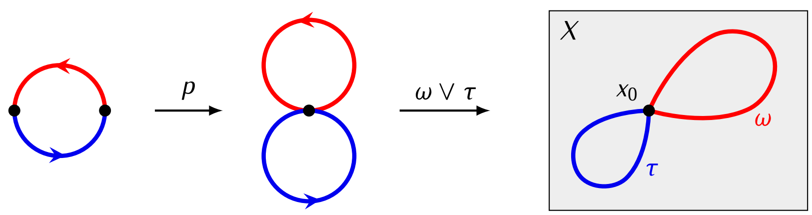
3.1 Notation.

$$I^n = \{(s_1, \dots, s_n) \mid s_i \in [0,1], i = 1, \dots, n\}$$
$$\partial I^n = \{(s_1, \dots, s_n) \in I^n \mid s_i \in \{0,1\} \text{ for some } i\}$$
$$D^n = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 \leq 1\}$$
$$S^{n-1} = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 = 1\}$$

Recall that the fundamental group $\pi_1(X, x_0)$ of a pointed space (X, x_0) is the group whose elements are homotopy classes of maps $\omega : (I, \partial I) \to (X, x_0)$. Multiplication is given by concatenation of such maps.



Alternatively, $\pi_1(X, x_0)$ can be described as a group whose elements are homotopy classes of maps $\omega\colon (S^1, s_0) \to (X, x_0)$. In this setting, the multiplication in $\pi_1(X, x_0)$ is defined using the pinch map $p\colon S^1 \to S^1 \vee S^1$:



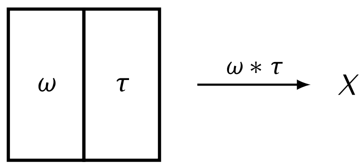
This construction can be generalized to define higher homotopy groups.

3.2 Definition/Proposition. Let (X, x_0) be a pointed space. For $n \geq 1$ the *n-th homotopy group* of (X, x_0) is the group $\pi_n(X, x_0)$ whose elements are homotopy classes of maps $\omega\colon (I^n, \partial I^n) \to (X, x_0)$. Multiplication in $\pi_n(X, x_0)$ is defined as follows. If $\omega, \tau\colon (I^n, \partial I^n) \to (X, x_0)$ then

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

where $\omega * \tau\colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$(\omega * \tau)(s_1, s_2, \ldots, s_n) = \begin{cases} \omega(2s_1, s_2, \ldots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \ldots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$



The trivial element of $\pi_n(X, x_0)$ is the homotopy class of the constant map $c_{x_0}\colon I^n \to X$. Also, for $[\omega] \in \pi_1(X, x_0)$ we have $[\omega]^{-1} = [\overline{\omega}]$ where $\overline{\omega}\colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$\overline{\omega}(s_1, s_2, \ldots, s_n) = \omega(1 - s_1, s_2, \ldots, s_n)$$

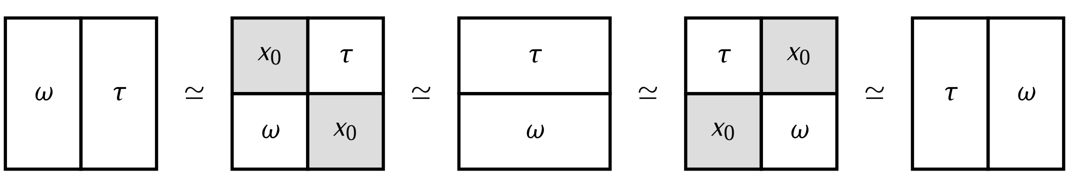
3.3 Note. A part of Definition 3.2 makes sense also for $n = 0$. In this case we have $I^0 = \{*\}$ and $\partial I^0 = \varnothing$. We define $\pi_0(X, x_0)$ as the set of homotopy classes of maps $\omega\colon (I^0, \partial I^0) \to (X, x_0)$. Giving such a map is the same as selecting a point $\omega(*) = x_\omega \in X$. Giving a homotopy of such maps is equivalent to giving a path between the corresponding points. Thus two points x_ω and x_τ represent the same element of $\pi_0(X, x_0)$ if they belong to the same path connected component. In other words, we get

$$\pi_0(X, x_0) \;\cong\; \begin{pmatrix} \text{path connected} \\ \text{components of } X \end{pmatrix}$$

The trivial element of $\pi_0(X, x_0)$ is given by the map $c_{x_0}\colon I^0 \to X$ such that $c_{x_0}(*) = x_0$. This corresponds to the path connected component of x_0 in X. In this way $\pi_0(X, x_0)$ becomes a pointed set. There is no multiplication defined in $\pi_0(X, x_0)$.

3.4 Theorem. *For $n \geq 2$ then the group $\pi_n(X, x_0)$ is abelian for any pointed space (X, x_0).*

Pictorial proof. A homotopy $\omega * \tau \simeq \tau * \omega$ can be depicted as follows:



The shaded squares in the pictures are mapped to the basepoint $x_0 \in X$. $\square$

A more rigorous proof can be obtained using the following fact.

3.5 Eckmann–Hilton Theorem. *Let M be a set equipped with two binary operations*

$$\circ\colon M \times M \to M, \quad \bullet\colon M \times M \to M$$

Assume that there exist elements $1_\circ, 1_\bullet \in M$ such that $m \circ 1_\circ = 1_\circ \circ m = m$ and $m \bullet 1_\bullet = 1_\bullet \bullet m = m$ for all $m \in M$. Assume also, that for any $m_1, m_2, n_1, n_2 \in M$ we have

$$(m_1 \circ m_2) \bullet (n_1 \circ n_2) = (m_1 \bullet n_1) \circ (m_2 \bullet n_2)$$

Then for any $m, n \in M$ we have $m \circ n = m \bullet n$, and $m \circ n = n \circ m$.

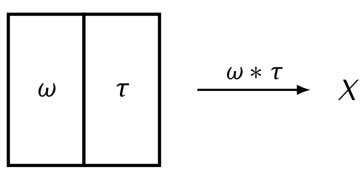
Proof. Exercise. $\square$

Proof of Theorem 3.4. Recall that multiplication in $\pi_n(X, x_0)$ is defined by If $\omega, \tau\colon (I^n, \partial I^n) \to (X, x_0)$ then

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

where $\omega * \tau\colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$

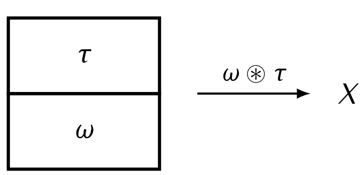


Since $n \geq 2$, we can also define a multiplication in $\pi_n(X, x_0)$ by

$$[\omega] \odot [\tau] = [\omega \circledast \tau]$$

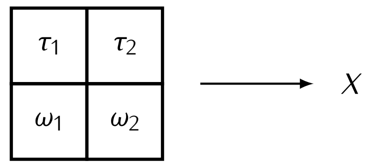
where

$$(\omega \circledast \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(s_1, 2s_2, \dots, s_n) & \text{for } s_2 \in [0, \frac{1}{2}] \\ \tau(s_1, 2s_2 - 1, \dots, s_n) & \text{for } s_2 \in [\frac{1}{2}, 1] \end{cases}$$



Notice that for any $\omega_1, \omega_2, \tau_1, \tau_2 \colon (I^n, \partial I^n) \to (X, x_0)$ we have

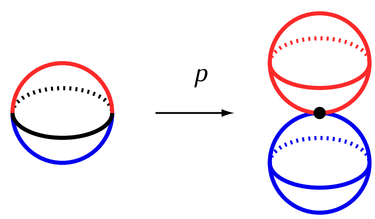
$$(\omega_1 * \omega_2) \circledast (\tau_1 * \tau_2) = (\omega_1 \circledast \tau_1) * (\omega_2 \circledast \tau_2)$$



The result follows from Theorem 3.5. $\square$

3.6 Alternative construction. Just as for the fundamental group, higher homotopy groups can be also described using maps from spheres. Since $I^n/\partial I^n \cong S^n$, giving a map $(I^n, \partial I^n) \to (X, x_0)$ is equivalent to giving a map $(S^n, s_0) \to (X, x_0)$ for some basepoint $s_0 \in S^n$. Thus elements of $\pi_n(X, x_0)$ can be described as homotopy classes of such maps.

To describe multiplication in $\pi_n(X, x_0)$ in this setting, consider the pinch map $p \colon S^n \to S^n \vee S^n$ that maps the upper hemisphere of S^n onto one copy of $S^n \subseteq S^n \vee S^n$, the lower hemisphere onto the second copy, and the equator of S^n to the basepoint of $S^n \vee S^n$:



Given two basepoint preserving maps $\omega, \tau \colon (S^n, s_0) \to (X, x_0)$, let $\omega \vee \tau \colon S^n \vee S^n \to X$ be the function that maps the first copy of S^n using ω and the second copy using τ. Then we have

$$[\omega] \cdot [\tau] = [(\omega \vee \tau) \circ p]$$

The following fact is often useful:

3.7 Proposition. *A map $\omega \colon (S^n, s_0) \to (X, x_0)$ represents the trivial element of $\pi_n(X, x_0)$ if and only if there exists a map $\omega' \colon D^{n+1} \to X$ such that $\omega'|_{S^n} = \omega$.*

Proof. Exercise. $\square$

3.8 Functoriality. Let $f \colon (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. For any $\omega \colon (I^n, \partial I^n) \to (X, x_0)$, composition with f gives a map $f \circ \omega \colon (I^n, \partial I^n) \to (Y, y_0)$. If $\omega, \tau \colon (I^n, \partial I^n) \to (X, x_0)$ and $\omega \simeq \tau$, then $f \circ \omega \simeq f \circ \tau$. Therefore we get a well defined function

$$f_* \colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

given by $f_*([\omega]) = [f \circ \omega]$. If $n \geq 0$ then f_* is a homomorphism of groups. For maps $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ we have $(gf)_* = g_* f_*$. Also, if $\mathrm{id}_X\colon (X, x_0) \to (X, x_0)$ is the identity map, then $\mathrm{id}_{X*}\colon \pi_n(X, x_0) \to \pi_n(X, x_0)$ is the identity homomorphism. This shows that the assignments $(X, x_0) \to \pi_n(X, x_0)$ define functors:

$$\begin{aligned}
\pi_0 &\colon \mathbf{Top}_* \to \mathbf{Set}_* \\
\pi_1 &\colon \mathbf{Top}_* \to \mathbf{Gr} \\
\pi_n &\colon \mathbf{Top}_* \to \mathbf{Ab}
\end{aligned}$$

for $n \geq 2$, where $\mathbf{Top}_*$ is the category of pointed topological spaces, and $\mathbf{Set}_*$, $\mathbf{Gr}$, $\mathbf{Ab}$ are the categories of pointed sets, groups, and abelian groups, respectively.

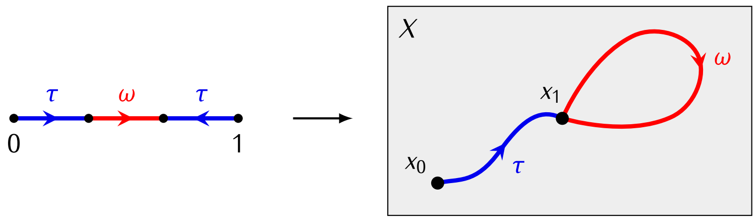
As a consequence, if $f\colon (X, x_0) \to (Y, y_0)$ is a homeomorphism, then $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism for $n \geq 0$.

4 | Dependence on The Basepoint

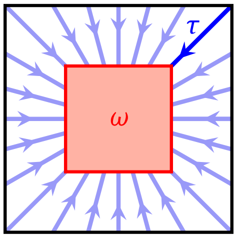
Let X be a space, and let $x_0, x_1 \in X$. Recall that any path $\tau\colon [0,1] \to X$ such that $\tau(0) = x_0$ and $\tau(1) = x_1$ defines a isomorphism of fundamental groups

$$s_\tau\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$$

given by $s_\tau([\omega]) = [\tau * \omega * \overline{\tau}]$, where $\overline{\tau}$ is obtained from τ by reverting orientation.



In a similar way, given a path $\tau\colon [0,1] \to X$ with $\tau(0) = x_0$ and $\tau(1) = x_1$ we can define a map $s_\tau\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$. To do this, given a map $\omega\colon (I^n, \partial I^n) \to (X, x_1)$, define a map $\omega_\tau\colon (I^n, \partial I^n) \to (X, x_0)$ as follows:



The smaller cube is mapped by ω and each radial ray joining the boundaries of the larger and smaller cube is mapped by the path τ.

Let $\pi_1(X, x_0, x_1)$ denote the set of homotopy classes of paths $\tau\colon [0,1] \to X$ such that $\tau(0) = x_0$ and $\tau(1) = x_1$, with homotopies preserving the endpoints.

4.1 Lemma. *Let $\omega, \omega'\colon (I^n, \partial I^n) \to (X, x_1)$ be maps such that $\omega \simeq \omega'$ (rel ∂I^n), and let $\tau, \tau'\colon [0,1] \to X$ be paths such that $\tau(0) = \tau'(0) = x_0$, $\tau(1) = \tau'(1) = x_1$ and $\tau \simeq \tau'$ (rel $\{0,1\}$). Then $\omega_\tau \simeq \omega'_{\tau'}$ (rel ∂I^n).*

Equivalently, if $[\omega] = [\omega'] \in \pi_n(X, x_1)$ and $[\tau] = [\tau'] \in \pi_1(X, x_0, x_1)$ then $[\omega_\tau] = [\omega'_{\tau'}] \in \pi_n(X, x_0)$

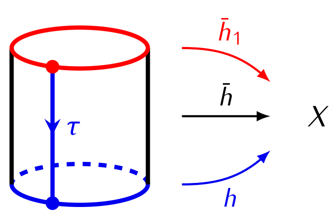
Proof. Exercise. □

4.2 Note. The homotopy class $[\omega_\tau]$ can be also described as follows. Consider the homotopy $h\colon \partial I^n \times [0,1] \to X$ given by $h(x,t) = \tau(1-t)$. Since the pair $(I^n, \partial I^n)$ has the homotopy extension property, we can extend h to a homotopy $\bar{h}\colon I^n \times [0,1] \to X$ such that $\bar{h}_0 = \omega$. The map $\bar{h}_1$ defines an element $[\bar{h}_1] \in \pi_n(X, x_0)$. This element does not depend on the choice of the extension $\bar{h}$ (exercise), and we have $[\bar{h}_1] = [\omega_\tau]$.

4.3 Note. Recall that elements of $\pi_n(X, x_1)$ can be alternatively defined as pointed homotopy classes of maps $\omega\colon (S^n, s_0) \to (X, x_1)$. In this setting, for $[\tau] \in \pi_1(X, x_0, x_1)$ the element $[\omega_\tau] \in \pi_n(X, x_0)$ can be described using a similar approach as in (4.2). Given such ω and τ we can define a function

$$h\colon (S^n \times \{0\}) \cup (\{s_0\} \times [0,1]) \to X$$

so that $h(s, 0) = \omega(s)$ and $h(s_0, t) = \tau(1-t)$. Since the pair (S^n, s_0) has the homotopy extension property, thus h can be extended to a homotopy $\bar{h}\colon S^n \times [0,1] \to X$. One can check that the pointed homotopy class of the map $\bar{h}_1\colon (S^n, s_0) \to (X, x_0)$ does not depend on the choice of the extension $\bar{h}$. We set: $[\omega_\tau] = [h_1] \in \pi_n(X, x_0)$.



4.4 Definition. Given $[\tau] \in \pi_1(X, x_0, x_1)$ let

$$s_{[\tau]}\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$$

denote the function given by $s_{[\tau]}([\omega]) = [\omega_\tau]$.

4.5 Proposition. *1) For any $[\tau] \in \pi_1(X, x_0, x_1)$ the function $s_{[\tau]}$ is a group homomorphism.*

2) If $[\tau] \in \pi_1(X, x_0, x_1)$ and $[\sigma] \in \pi_1(X, x_1, x_2)$ then

$$s_{[\tau*\sigma]} = s_{[\tau]} \circ s_{[\sigma]} \colon \pi_n(X, x_2) \to \pi_n(X, x_0)$$

3) If $c_{x_0} \colon [0,1] \to X$ is the constant path, $c_{x_0}(t) = x_0$ for all $t \in [0,1]$, then $s_{[c_{x_0}]} \colon \pi_n(X, x_0) \to \pi_n(X, x_0)$ is the identity homomorphism.

Proof. Exercise. □

4.6 Corollary. *Let X be a space and let $x_0, x_1 \in X$. For any path $\tau \colon [0,1] \to X$ be a path such that $\tau(0) = x_0$, $\tau(1) = x_1$ the homomorphism $s_{[\tau]} \colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ is an isomorphism.*

Proof. Let $\bar{\tau}$ be the inverse of τ. This defines homomorphisms

$$s_{[\tau]} \colon \pi_n(X, x_1) \leftrightarrows \pi_n(X, x_0) \colon s_{[\overline{\tau}]}$$

We will show that $s_{[\overline{\tau}]} = s_{[\tau]}^{-1}$. Indeed, by Proposition 4.5 we have

$$s_{[\overline{\tau}]} \circ s_{[\tau]} = s_{[\overline{\tau}*\tau]} = s_{[c_{x_0}]} = \mathrm{id}_{\pi_n(X, x_1)}$$

Analogously, $s_{[\tau]} \circ s_{[\overline{\tau}]} = \mathrm{id}_{\pi_n(X, x_0)}$. □

Corollary 4.6 implies that if x_0, x_1 are in the same path connected component of X then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$. On the other hand, if points $x_0, x_1 \in X$ belong to different path connected components of X, then in general there is no relationship between $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$.

4.7 Proposition. *Let X be a space, $x_0 \in X$, and let X_0 be the path connected component of X such that $x_0 \in X_0$. Then the inclusion map $i \colon X_0 \hookrightarrow X$ induces an isomorphism*

$$i_* \colon \pi_n(X_0, x_0) \xrightarrow{\cong} \pi_n(X, x_0)$$

Proof. Since I^n is path connected, for any map $\omega \colon (I^n, \partial I^n) \to (X, x_0)$ we have $\omega(I^n) \subseteq X_0$. This shows that i_* is onto. Also, if $h \colon I^n \times [0,1] \to X$ is a homotopy $h \colon \omega \simeq \omega'$ where $\omega, \omega' \colon I^n \to X_0$ then, since $I^n \times [0,1]$ is path connected, we have $h(I^n \times [0,1]) \subseteq X_0$. It implies that i_* is 1-1. □

4.8 Note. Given a path connected space X we will sometimes write $\pi_n(X)$ to denote the n-th homotopy group of X taken with respect to some unspecified basepoint of X. By Corollary 4.6 this will not create problems as long as we are interested in the isomorphism type of the fundamental group only.

Similarly as for the fundamental group we have:

4.9 Proposition. *Let $f, g\colon X \to Y$ be homotopic maps and let $h\colon f \simeq g$. For $x_0 \in X$ let τ be the path in Y given by $\tau(t) = h(x_0, t)$. The following diagram commutes:*

$$\begin{tikzcd} & \pi_n(Y, g(x_0)) \arrow[d, "s_{[\tau]}", "\cong"'] \\ \pi_n(X, x_0) \arrow[ur, "g_*"] \arrow[dr, "f_*"'] & \\ & \pi_n(Y, f(x_0)) \end{tikzcd}$$

Proof. Exercise. □

4.10 Note. Proposition 4.9 implies, in particular, that if $f, g\colon (X, x_0) \to (Y, y_0)$ are maps of pointed spaces and $f \simeq g$ (rel $\{x_0\}$) then $f_* = g_*$.

4.11 Corollary. *If $f, g\colon X \to Y$ are maps such that $f \simeq g$ then the homomorphism $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism (or it is trivial or 1-1 or onto) if and only if the homomorphism $g_*\colon \pi_n(X, x_0) \to \pi_n(Y, g(x_0))$ has the same property.*

4.12 Proposition. *If $f\colon X \to Y$ is a homotopy equivalence then for any $x_0 \in X$ the homomorphism $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. Consider the sequence of homomorphisms

$$\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0)) \xrightarrow{g_*} \pi_n(X, gf(x_0)) \xrightarrow{f_*} \pi_n(Y, fgf(x_0))$$

Composition of the first two homomorphisms satisfies $g_*f_* = (gf)_*$. Since $gf \simeq \mathrm{id}_X$ and id_{X*} is an isomorphism, by Corollary 4.11 we obtain that g_*f_* is an isomorphism. This implies in particular that g_* is onto. Similarly, composing the last two homomorphisms we obtain $f_*g_* = (fg)_*$ and since $fg \simeq \mathrm{id}_Y$ we get that f_*g_* is an isomorphism. This means that g_* is 1-1. As a consequence g_* is an isomorphism. It follows that the first homomorphism f_* is a composition of two isomorphisms: $f_* = g_*^{-1}(g_*f_*)$, and so f_* is an isomorphism. □

4.13 Corollary. *If X, Y are path connected spaces and $X \simeq Y$ then $\pi_n(X, x_0) \overset{\cong}{=} \pi_n(Y, y_0)$ for any $x_0 \in X$, $y_0 \in Y$.*

4.14 The action of π_1. If $[\tau] \in \pi_1(X, x_0)$ then $s_{[\tau]}$ is an isomorphism

$$s_{[\tau]}\colon \pi_n(X, x_0) \to \pi_n(X, x_0)$$

Denote $[\tau] \odot [\omega] := s_{[\tau]}(\omega)$.

4.15 Definition. For $n \geq 0$ the *action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$* is the map

$$\begin{aligned} \pi_1(X, x_0) \times \pi_n(X, x_0) &\to \pi_n(X, x_0) \\ ([\tau]\,,\,[\omega]) &\mapsto [\tau] \odot [\omega] \end{aligned}$$

4.16 Note. By Proposition 4.5, for any $[\tau], [\tau'] \in \pi_1(X, x_0)$ and $\omega, \omega' \in \pi_n(X, x_0)$ we have:

- $[\tau] \odot ([\omega] \cdot [\omega']) = ([\tau] \odot [\omega]) \cdot ([\tau \odot [\omega'])$
- $([\tau] \cdot [\tau']) \odot [\omega] = [\tau] \odot ([\tau'] \odot [\omega])$
- $[c_{x_0}] \odot [\omega] = [\omega]$ where $[c_{x_0}] \in \pi_1(X, x_0)$ is the trivial element.
- $[\tau] \odot [c_{x_0}] = [c_{x_0}]$ where $[c_{x_0}] \in \pi_n(X, x_0)$ is the trivial element.

4.17 Proposition. *For any map $f\colon (X, x_0) \to (Y, y_0)$ the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X, x_0) \times \pi_n(X, x_0) \arrow[r, "\odot"] \arrow[d, "f_* \times f_*"'] & \pi_n(X, x_0) \arrow[d, "f_*"] \\ \pi_1(Y, y_0) \times \pi_n(Y, y_0) \arrow[r, "\odot"'] & \pi_n(Y, y_0) \end{tikzcd}$$

Proof. Exercise. □

4.18 Example. The action of $\pi_1(X, x_0)$ on $\pi_1(X, x_0)$ if given by conjugation:

$$[\tau] \odot [\omega] = [\tau] \cdot [\omega] \cdot [\tau]^{-1}$$

4.19 Definition. A path connected space X is *n-simple* if for some $x_0 \in X$ the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ is trivial: $[\tau] \odot [\omega] = [\omega]$ for all $[\tau] \in \pi_1(X, x_0)$ and $[\omega] \in \pi_n(X, x_0)$. A path connected space is *simple* if it is n-simple for all $n \geq 1$.

The following fact implies that n-simplicity of a space X does not depend on the choice of a basepoint $x_0 \in X$:

4.20 Proposition. *Let X be a space, let $x_0, x_1 \in X$, and let $\tau\colon [0, 1] \to X$ be a path such that $\tau(0) = x_0$ and $\tau(1) = x_1$. Then the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X, x_1) \times \pi_n(X, x_1) \arrow[r, "\odot"] \arrow[d, "s_{[\tau]} \times s_{[\tau]}"'] & \pi_n(X, x_1) \arrow[d, "s_{[\tau]}"] \\ \pi_1(X, x_0) \times \pi_n(X, x_0) \arrow[r, "\odot"'] & \pi_n(X, x_0) \end{tikzcd}$$

Proof. Exercise. □

For spaces X, Y let $[X, Y]$ denote the set of homotopy classes of maps $X \to Y$. Notice that for any space X and any n we have a map of sets

$$\phi\colon \pi_n(X, x_0) \to [S^n, X]$$

which maps the pointed homotopy class of map $\omega\colon (S^n, s_0) \to (X, x_0)$ to the unpointed homotopy class of the same map.

4.21 Proposition. *Let X be a path connected space, and let $n \geq 1$. The following conditions are equivalent:*

1) *X is n-simple.*
2) *For any $x_0, x_1 \in X$, $[\tau], [\sigma] \in \pi_1(X, x_0, x_1)$ and $[\omega] \in \pi_n(X, x_1)$ we have $s_{[\tau]}([\omega]) = s_{[\sigma]}([\omega])$. Thus there is a canonical isomorphism $\pi_n(X, x_1) \xrightarrow{\cong} \pi_n(X, x_0)$.*
3) *For any $x_0 \in X$ the map $\phi\colon \pi_n(X, x_0) \to [S^n, X]$ is a bijection. Therefore any (unpointed) map $f\colon S^n \to X$ defines a unique element of $\pi_n(X, x_0)$.*

Proof. 1) ⇒ 2) Let $[\tau], [\sigma] \in \pi_1(X, x_0, x_1)$ and $[\omega] \in \pi_n(X, x_1)$. Since $[\overline{\tau} * \sigma] \in \pi_1(X, x_1)$, by 1) we obtain

$$s_{[\overline{\tau}]} s_{[\sigma]}([\omega]) = s_{[\overline{\tau}*\sigma]}([\omega]) = [\omega]$$

Also, since $s_{[\overline{\tau}]}$ is the inverse isomorphism of $s_{[\tau]}$ we get

$$s_{[\sigma]}([\omega]) = s_{[\tau]} s_{[\overline{\tau}]} s_{[\sigma]}([\omega]) = s_{[\tau]}([\omega])$$

2) ⇒ 1) Let $[\tau], [c_{x_0}] \in \pi_1(X, x_0)$, where $[c_{x_0}]$ is the trivial elemment. By 2) we have

$$s_{[\tau]}([\omega]) = s_{[c_{x_0}]}([\omega]) = [\omega]$$

for any $[\omega] \in \pi_n)(X, x_0)$. Therefore X is n-simple.

1) ⇒ 3) The map ϕ is always onto. Indeed, take any map $\omega\colon S^n \to X$. Since X is path connected, there exists a path $\tau\colon [0, 1] \to X$ such that $\tau(0) = x_0$ and $\tau(1) = \omega(s_0)$. Consider the map

$$h\colon (S^n \times \{0\}) \cup (\{s_0\} \times [0, 1]) \to X$$

so that $h(s, 0) = \omega(s)$ and $h(s_0, t) = \tau(1 - t)$. The pair (S^n, s_0) has the homotopy extension property, so h can be extended to a homotopy $\bar{h}\colon S^n \times [0, 1] \to X$. The for the map h_1 we have $h_1(s_0) = x_0$, so $[h_1] \in \pi_n(X, x_0)$. Also, h is homotopic to $h_0 = \omega$. Therefore we have $\phi([h_1]) = [\omega]$.

To show that ϕ is 1-1, we will use the description of $s_{[\tau]}$ in terms of maps from spheres given in Note 4.2. Given two elements $[\omega_0], [\omega_1] \in \pi_n(X, x_0)$ assume that $\phi([\omega_0]) = \phi[\omega_1]$. This means that there exists a homotopy $h\colon S^n \times [0,1] \to X$ such that $h_0 = \omega_0$ and $h_1 = \omega_1$. Let $\tau\colon [0,1] \to X$ be a path given by $\tau(t) = h(s_0, t)$. Then $[\tau] \in \pi_1(X, x_0)$, and by (4.2) we have

$$[\omega_1] = [\bar{\tau}] \odot [\omega_0] = s_{[\bar{\tau}]}([\omega_0])$$

By 1) we have $s_{[\bar{\tau}]}([\omega_0]) = [\omega_0]$. Thus $[\omega_1] = [\omega_0] \in \pi_n(X, x_0)$.

3) $\Rightarrow$ 1) Let $[\tau] \in \pi_1(X, x_0)$, $[\omega] \in \pi_n(X, x_0)$. Let $\omega_\tau\colon (S^n, s_0) \to (X, x_0)$ be some map such that $[\omega_\tau] = s_{[\tau]}([\omega])$. By (4.2) the maps ω_τ and ω are freely homotopic, i.e. $\phi([\omega_\tau]) = \phi([\omega])$. By assumption ϕ is 1-1, thus we obtain

$$[\omega] = [\omega_\tau] = s_{[\tau]}([\omega]) = [\tau] \odot [\omega]$$

in $\pi_n(X, x_0)$.

$\square$

5 | Some Computations

5.1 Proposition. *If X is a contractible space then $\pi_n(X) = 0$ for all $n \geq 0$.*

Proof. Since $X \simeq *$ thus $\pi_n(X) \overset{\simeq}{=} \pi_n(*) = 0$. $\square$

5.2 Proposition. *If X is a relative CW-complex, $X^{(n)}$ is the n-skeleton of X, and $x_0 \in X^{(n)}$, then the homomorphism $i_*\colon \pi_k(X^{(n)}, x_0) \to \pi_k(X, x_0)$ induced by the inclusion map $i\colon X^{(n)} \hookrightarrow X$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

Proof. We can assume that $x_0 \in X^{(0)}$. Consider S^k as a CW complex with a 0-cell $s_0 \in S^k$. By the Cellular Approximation Theorem 2.11 any map $\omega\colon (S^k, s_0) \to (X, x_0)$ is homotopic (relative to the basepoint) to cellular map $\omega'\colon (S^k, s_0) \to (X, x_0)$. If $k \leq n$ then $\omega'(S^k) \subseteq X^{(n)}$, so ω' represents an element of $\pi_k(X^{(n)}, x_0)$ such that $i_*([\omega']) = [\omega]$. This shows that i_* is an epimorphism for $k \leq n$.

Next, take $[\omega_0], [\omega_1] \in \pi_k(X^{(n)}, x_0)$. We can assume that the maps $\omega_0, \omega_1\colon (S^k, s_0) \to (X^{(n)}, x_0)$ are cellular. If $i_*([\omega_0]) = i_*([\omega_1])$ then there is a homotopy $h\colon S^k \times [0,1] \to X$. Using the Cellular Approximation Theorem 2.11 again, we can assume that this homotopy is a cellular map. Since $\dim S^k \times [0,1] = k+1$, we obtain that if $k < n$ then $h(S^k \times [0,1]) \to X^{(n)}$. Thus h gives a homotopy between ω_0 and ω_1 in $X^{(n)}$. Therefore $[\omega_0] = [\omega_1] \in \pi_k(X^{(n)}, x_0)$. This shows that i_* is a monomorphism for $k < n$. $\square$

5.3 Corollary. *If $k < n$ then $\pi_k(S^n) = 0$*

Proof. A sphere S^n can be given a CW-complex structure with one 0-cell and one n-cell. Then by Proposition 5.2 for $k < n$ we have an epimorphism

$$\pi_k((S^n)^{(k)}) \to \pi_k(S^n)$$

Since $(S^n)^{(k)} = *$, thus $\pi_k((S^n)^{(k)}) = 0$ and so $\pi_k(S^n) = 0$. $\square$

5.4 Definition. A space X is *n-connected* if $\pi_k(X) = 0$ for all $k \leq n$.

Corollary 5.3 can be restated by saying that the sphere S^n is $(n-1)$-connected.

5.5 Proposition. *For any space X and $n \geq 0$ the following conditions are equivalent:*

1) *X is n-connected.*
2) *For any $k \leq n$ and any map $\varphi\colon S^k \to X$ there exists a map $\overline{\varphi}\colon D^{k+1} \to X$ such that $\overline{\varphi}|_{S^k} = \varphi$.*

Proof. Follows from Proposition 3.7. □

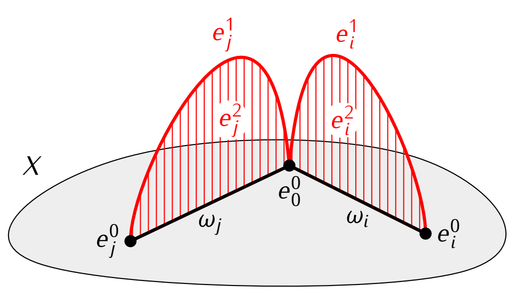
By Proposition 5.2 if X is a CW complex that has only one 0-cell and no k-cells for $k \leq n$ (i.e. $X^{(n)} = *$) then X is n-connected. One can show that the opposite is also true, up to a homotopy equivalence:

5.6 Proposition. *If X is an n-connected CW complex, then there exists a CW complex Y such that such that $X \simeq Y$ and $Y^{(n)} = *$.*

Proof. We will show inductively that for any $k = 0, \dots, n$ there exists a CW complex Y_k such that $X \simeq Y_k$ and $Y_k^{(k)} = *$.

Choose a 0-cell $e_0^0 \in X$. Since $\pi_0(X) = 0$, the space X is path connected. Thus for any 0-cell e_i^0 we can select a path $\omega_i\colon [0,1] \to X$ such that $\omega(0) = e_0$ and $\omega(1) = e_i^0$. By the Cellular Approximation Theorem 2.11, we can assume that ω_i is a path in $X^{(1)}$. We construct a new CW complex Y_0'' by attaching cells to X as follows.

1) First, for each 0-cell e_i^0 we attach to X a 1-cell e_i^1 using the attaching map $\varphi_i\colon S^0 = \{-1, 1\} \to X$ such that $\varphi_i(-1) = e_0^0$ and $\varphi_i(1) = e_i^0$. Let $Y_0' = X \cup \bigcup_i e_i^1$ be the CW complex obtained in this way.
2) In Y_0' each 0-cell e_i^0 is connected to e_0^0 by two different paths: ω_i, and a path τ_i that traverses the new cell e_i^1. For each i we attach a 2-cell e_i^2 using an attaching map $\psi_i\colon S^1 \to Y_0'$ that send the lower half circle to ω_i and the upper half circle to τ_i. Let $Y_0'' = Y_0' \cup \bigcup_i e_i^2$.



Notice that X is a deformation retract of Y_0'', so the inclusion map $j\colon X \hookrightarrow Y_0''$ is a homotopy equivalence. Also $A = X^{(0)} \cup \bigcup_i e_i^1$ is a contractible subcomplex of Y_0''. By Proposition 2.15, the quotient map

$q\colon Y_0'' \to Y_0''/A$ is a homotopy equivalence. Since Y_0''/A has a CW complex structure with only one 0-cell we can take $Y_0 = Y_0''/A$.

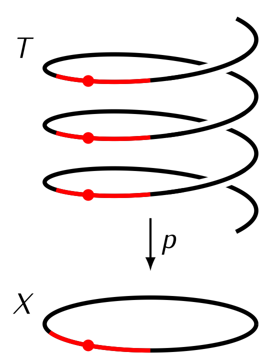
Next, assume that for some $k \leq n$ we have already constructed a CW complex Y_{k-1} such that $X \simeq Y_{k-1}$ and $Y_{k-1}^{(k-1)} = *$. This means that the k-skeleton of Y_{k-1} is given by $Y_{k-1}^{(k)} = \bigvee_i S^k$, with one copy of S^k for each k-cell e_i^k in Y_{k-1}. Let $\varphi_j\colon S^k \hookrightarrow \bigvee_i S^k \subseteq Y_{k-1}$ be the inclusion of the j-th copy of S^k. Since $\pi_k(Y_{k-1}) \cong \pi_k(X) = 0$, each map φ_i extends to a map $\omega_i\colon D^{k+1} \to Y_{k-1}$. We construct a new CW complex Y_k'' by attaching cells to Y_{k-1} as follows.

1) First, for each i we attach a $(k+1)$-cell e_i^{k+1} using $\varphi_i\colon S^k \to Y_{k-1}$ as the attaching map. Let $Y_k' = Y_{k-1} \cup \bigcup_i e_i^{k+1}$ be the CW complex obtained in this way.
2) For each i we have now two maps $D^{k+1} \to Y_k'$: the map ω_i, and the characteristic map τ_i of the cell e_i^{k+1}. Using these maps we attach, for each i, a $(k+2)$-cell e_i^{k+2}, using an attaching map $\psi_i\colon S^{k+1} \to Y_k'$ that sends the lower hemisphere of S^{k+1} to ω_i and the upper hemisphere to τ_i. Let $Y_k'' = Y_k' \cup \bigcup_i e_i^2$.

As before, we observe that Y_{k-1} is a deformation retract of Y_k'', and that $A = Y_{k-1}^{(k)} \cup \bigcup_i e_i^k$ is a contractible subcomplex of Y_k''. Therefore we obtain a $X \simeq Y_{k-1} \simeq Y_k'' \simeq Y_k''/A$. It remains to notice that the space $Y_k = Y_k''/A$ has a CW-complex structure such that $Y_k^{(k)} = *$.

□

5.7 Homotopy groups and coverings. Recall that covering of a space X is a map $p\colon T \to X$ which is locally homeomorphic to the projection map $\mathrm{pr}_1\colon U \times D \to U$ for some discrete space D.



Recall also, that one of the main properties of coverings is the following fact:

5.8 Theorem (Lifting Criterion). *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Assume that Y is a connected and locally path connected space and let $y_0 \in Y$. A map $f\colon (Y, y_0) \to (X, x_0)$*

has a lift $\tilde{f}\colon (Y, y_0) \to (T, \tilde{x}_0)$ *if and only if* $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$.

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Moreover, if a lift $\tilde{f}$ *exists, then it is unique.*

Recall that for any covering $p\colon (T, \tilde{x}_0) \to (X, x_0)$ the induced homomorphism $p_*\colon \pi_1(T, \tilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism. Using Theorem 5.8 we can generalize this as follows:

5.9 Proposition. *If* $p\colon T \to X$ *is a covering,* $x_0 \in X$ *and* $\tilde{x}_0 \in p^{-1}(x_0)$*, then the induced homomorphism*

$$p_*\colon \pi_n(T, \tilde{x}_0) \to \pi_n(X, x_0)$$

is an isomorphism for all $n > 1$.

Proof. Let $n > 1$ and $\omega\colon (S^n, s_0) \to (X, x_0)$ represents an element of $\pi_n(X, x_0)$. Since $\pi_1(S^n) = 0$, by Theorem 5.8 there exists a map $\widetilde{\omega}\colon (S^n, s_0) \to (T, \tilde{x}_0)$ such that $p\widetilde{\omega} = \omega$. This shows that $p_*\colon \pi_n(T, \tilde{x}_0) \to \pi_n(X, x_0)$ is onto.

Next, assume that $\omega_0, \omega_1\colon (S^n, s_0) \to (T, \tilde{x}_0)$ are maps such that $p_*([\omega_0]) = p_*([\omega_1])$. This means that there exists a basepoint preserving homotopy $h\colon S^n \times [0,1] \to X$, such that $h_0 = p\omega_0$, $h_1 = p\omega_1$. Since $S^n \times [0,1] \simeq S^n$ we have $\pi_1(S^n \times [0,1]) \cong \pi_1(S^n) = 0$. Thus by Theorem 5.8, there exists a homotopy $\widetilde{h}\colon S^n \times [0,1] \to T$ such that $p\widetilde{h} = h$ and $\widetilde{h}(s_0, 0) = \tilde{x}_0$. Using the uniqueness of lifts, one can check that $\widetilde{h}_0 = \omega_0$ and $\widetilde{h}_1 = \omega_1$, and that the homotopy $\widetilde{h}$ preserves the basepoint (exercise). It follows that $[\omega_0] = [\omega_1]$ in $\pi_1(T, \tilde{x}_0)$. Therefore p_* is a monomorphism.

□

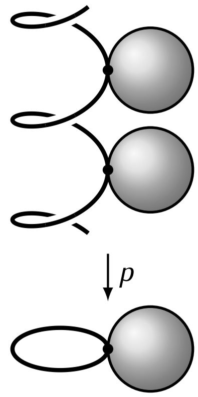
5.10 Example. $\pi_n(S^1) = 0$ for all $n > 1$.

Indeed, universal covering of S^1 is given by a map $p\colon \mathbb{R} \to S^1$. Since $\mathbb{R}$ is a contractible space, by Proposition 5.9 for $n > 1$ we obtain

$$\pi_n(S^1) \cong \pi_n(\mathbb{R}) = 0$$

5.11 Example. If $m > 1$ then $\pi_n(S^1 \vee S^m) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^m)$ for all $n > 1$.

To see this, notice that the universal covering of $S^1 \vee S^m$ is the space $\widetilde{X}$ obtained by attaching copies of S^m at all integer points of the real line:



The space $\widetilde{X}$ can be given the structure of a CW complex, such that the real line $\mathbb{R}$ is its subcomplex. Since $\mathbb{R} \simeq \{*\}$, by Theorem 2.14 we have $\widetilde{X} \simeq \widetilde{X}/\mathbb{R} \cong \bigvee_{i\in\mathbb{Z}} S^m$. Therefore for $n > 1$ we obtain

$$\pi_n(S^1 \vee S^m) \cong \pi_n(\widetilde{X}) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^m)$$

5.12 Note. Example 5.11 can be used to show that if X, Y are spaces such that $\pi_n(X) \cong \pi_n(Y)$ for all $n \geq 0$, then this does not imply that $X \simeq Y$.

Take, for example, $X = S^1 \vee S^m$ for some $m > 1$, and let $Y = S^1 \vee S^m \vee S^m$. These spaces are not homotopy equivalent, since they have different homology groups: $H_m(X;\mathbb{Z}) \cong \mathbb{Z}$ and $H_m(Y;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$.

On the other hand, since both these spaces are path connected, we have $\pi_0(X) \cong \pi_0(Y) \cong \{*\}$. Also, since $\pi_1(S^m) = 0$, thus by van Kampen's theorem we get $\pi_1(X) \cong \pi_1(S^1) \cong \pi_1(Y)$.

The universal covering space $\widetilde{Y}$ of Y is the space obtained by attaching $S^m \vee S^m$ at all integer points of $\mathbb{R}$. Using the same argument as in Example 5.11, we obtain $\widetilde{Y} \simeq \bigvee_{i\in\mathbb{Z}}(S^m \vee S^m) \cong \bigvee_{i\in\mathbb{Z}} S^m$. Therefore for $n \geq 2$ we have

$$\pi_n(X) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^m) \cong \pi_n(Y)$$

5.13 Theorem. *For a family $(X_i, \bar{x}_i)_{i\in I}$ be a family of pointed spaces there is an isomorphism*

$$\pi_n\left(\prod_{i\in I} X_i,\ (\bar{x}_i)_{i\in I}\right) \cong \prod_{i\in I} \pi_n(X_i, \bar{x}_i)$$

Proof. For $j \in I$ let $p_j\colon \prod_{i\in I} X_i \to X_j$ denote the projection onto the j-th factor. The induced homomorphisms p_{j*} define a homomorphism:

$$\prod_{i\in I} p_{i*}\colon \pi_n\left(\prod_{i\in I} X_i,\ (\bar{x}_i)_{i\in I}\right) \to \prod_{i\in I} \pi_n(X_i, \bar{x}_i)$$

To obtain a homomorphism going in the opposite direction, let $([\omega_i])_{i\in I}$ be an element of $\prod_{i\in I}\pi_n(X_i, \bar{x}_i)$. Then each ω_i is a map $\omega_i\colon (S^n, s_0) \to (X_i, \bar{x}_i)$. Take the product map

$$\prod_{i\in I}\omega_i\colon (S^n, s_0) \to (\prod_i X_i, \bar{x}_i)$$

One can check that the assignment $([\omega_i])_{i\in I} \mapsto [\prod_{i\in I}\omega_i]$ gives a well-defined homomorphism

$$g\colon \prod_{i\in I}\pi_n(X_i, \bar{x}_i) \to \pi_n\left(\prod_{i\in I} X_i,\ (\bar{x}_i)_{i\in I}\right)$$

and that the compositions $g \circ \prod_{i\in I} p_{i*}$ and $\prod_{i\in I} p_{i*} \circ g$ are identity homomorphisms (exercise). $\square$

5.14 Example. Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_n(S^1) = 0$ for $n > 1$, thus for any set I we have

$$\pi_n\left(\prod_{i\in I} S^1\right) \cong \begin{cases} \prod_{i\in I}\mathbb{Z} & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{cases}$$

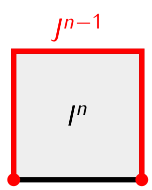
6 | Relative Homotopy Groups

6.1 Notation. Let $X \subseteq A_1 \subseteq A_2$ and $Y \subseteq B_1 \subseteq B_2$. By a map $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ we will understand a map $f\colon X \to Y$ such that $f(A_i) \subseteq B_i$ for $i = 1, 2$. A homotopy of such maps is a homotopy $h\colon X \times [0, 1] \to Y$ such that $h_t(A_i) \subseteq B_i$ for $i = 1, 2$ and all $t \in [0, 1]$.

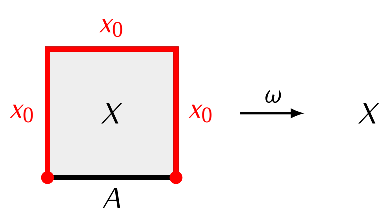
6.2 Notation. For $n \geq 1$ let J^{n-1} denote the subspace of $I^n = I^{n-1} \times I$ given by

$$J^{n-1} = I^{n-1} \times \{1\} \cup \partial I^{n-1} \times I$$

We have: $I^n \subseteq \partial I^n \subseteq J^{n-1}$.



6.3 Definition/Proposition. Let $x_0 \in A \subseteq X$. For $n \geq 2$, the *n-th relative homotopy group* of (X, A, x_0) is the group $\pi_n(X, A, x_0)$ whose elements are homotopy classes of maps $\omega\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$.

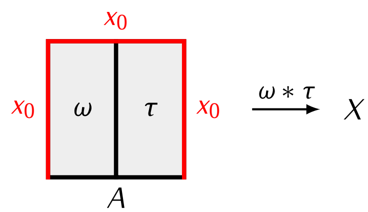


Multiplication in $\pi_n(X, A, x_0)$ is defined as follows. If $\omega, \tau\colon (I^n, \partial I^n) \to (X, x_0)$ then

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

where $\omega * \tau \colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$



The trivial element of $\pi_n(X, x_0)$ is the homotopy class of the constant map $c_{x_0} \colon I^n \to X$. Also, for $[\omega] \in \pi_1(X, x_0)$ we have $[\omega]^{-1} = [\overline{\omega}]$ where $\overline{\omega} \colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$\overline{\omega}(s_1, s_2, \dots, s_n) = (1 - s_1, s_2, \dots, s_n)$$

By a similar argument as in the case of absolute homotopy groups (3.4) we obtain:

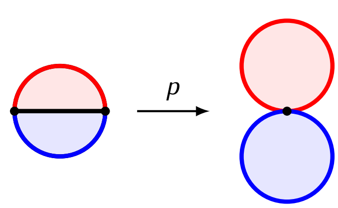
6.4 Theorem. *If $n \geq 3$ then the group $\pi_n(X, A, x_0)$ is abelian for any pointed pair (X, A, x_0).*

6.5 Note. A part of Definition 6.3 makes sense also for $n = 1$. In this case we have $\partial I^1 = \{0, 1\}$ and $J^0 = \{1\}$. Giving map $(I^1, \partial I_1, J^0) \to (X, A, x_0)$ is the same a defining a path in X that starts at x_0 and ends in A. Homotopy classes of such paths form the set $\pi_1(X, A, x_0)$. In general, this set does not have a group structure, but it has a basepoint defined by the constant path $c_{x_0} \colon I^1 \to X$ such that $c_{x_0}(I^1) = x_0$.

6.6 Proposition. *For any space X we have $\pi_n(X, x_0, x_0) \cong \pi_n(X, x_0)$.*

6.7 Proposition. *For any space X we have $\pi_n(X, X, x_0) = 0$.*

6.8 Alternative construction. Just as absolute homotopy groups we can described in terms of maps from spheres, relative homotopy groups can be constructed using maps from discs. Let $s_0 \in S^{n-1} \subseteq D^n$. Elements of $\pi_n(X, A, x_0)$ can be identified with homotopy classes of maps $\omega \colon (D^n, S^{n-1}, s_0) \to (X, A, x_0)$. For $n \geq 2$, multiplication in $\pi_n(X, A, x_0)$ is induced by the pinch map $p \colon D^n \to D^n \vee D^n$, which collapses the equatorial subdisc $D^{n-1} \subseteq D^n$ into a point.



6.9 For any $n \geq 1$, a map $f\colon (X, A, x_0) \to (Y, B, y_0)$ induces a map

$$f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$$

given by $f_*([\omega]) = [f \circ \omega]$. For $n \geq 2$, the map f_* is a homomorphism of groups. In this way we obtain functors

$$\begin{aligned} \pi_1\colon \mathbf{Top}^2_* &\to \mathbf{Set}_* \\ \pi_2\colon \mathbf{Top}^2_* &\to \mathbf{Gr} \\ \pi_n\colon \mathbf{Top}^2_* &\to \mathbf{Ab} \end{aligned}$$

for $n \geq 3$, where $\mathbf{Top}^2_*$ is the category of pointed pairs (X, A, x_0) as objects and maps of such pairs as morphisms.

6.10 Proposition. *If $f, g\colon (X, A, x_0) \to (Y, B, y_0)$ are maps such that $f \simeq g$ (as maps of pointed pairs) then $f_* = g_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all $n \geq 1$.*

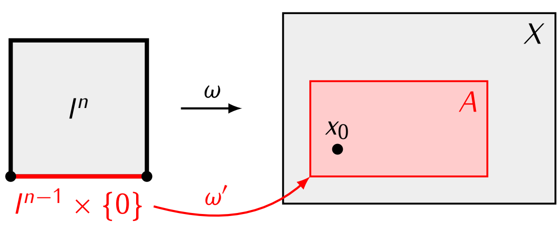
6.11 Long exact sequence of a pair. Consider a pointed pair (X, A, x_0). The inclusion $i\colon (A, x_0) \hookrightarrow (X, x_0)$ induces homomorphisms $i_*\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ for $n \geq 0$. Also, the map of pointed pairs $j\colon (X, x_0, x_0) \to (X, A, x_0)$ induces homomorphisms

$$j_*\colon \pi_n(X, x_0) = \pi_n(X, x_0, x_0) \to \pi_n(X, A, x_0)$$

for $n \geq 1$. We also have homomorphisms

$$\partial\colon \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$$

defined as follows. For $\omega\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, let $\omega'\colon (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$ be the restriction of ω to $I^{n-1} \times \{0\}$. Then $\partial([\omega]) = [\omega']$.



6.12 Theorem. *For any pointed pair (X, A, x_0) the following sequence is exact:*

$$\ldots \longrightarrow \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \ldots$$
$$\ldots \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)$$

Proof. Exercise. □

6.13 Note. The end of the exact sequence in Theorem 6.12 consists of maps of pointed sets. Given such maps

$$(S_2, s_2) \xrightarrow{f} (S_1, s_1) \xrightarrow{g} (S_0, s_0)$$

exactness at S_1 means that $f(S_2) = g^{-1}(s_0)$.

7 | Fibrations

7.1 Definition. A map $p\colon E \to B$ has the *homotopy lifting property* for a space X if for any commutative diagram of the form

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\bar{f}"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B \end{tikzcd}$$

there exists a map $\bar{h}\colon X \times [0,1] \to E$ such that $\bar{h}i = \bar{f}$ and $p\bar{h} = h$.

In the setting of Definition 7.1 we will say that $\bar{h}$ is a lift of h beginning at $\bar{f}$.

7.2 Definition. A map $p\colon E \to B$ is

- a *Hurewicz fibration* if it has the homotopy lifting property for any space X.
- a *Serre fibration* if it has the homotopy lifting property for any CW complex X.

7.3 Note. Every Hurewicz fibration is a Serre fibration.

7.4 Example. For any spaces B, F the projection map $\mathrm{pr}_B\colon B \times F \to B$ is a Hurewicz fibration. Indeed, assume that we have a commutative diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\bar{f}"] \arrow[d, hook, "i"'] & B \times F \arrow[d, "\mathrm{pr}_B"] \\ X \times [0,1] \arrow[r, "h"'] & B \end{tikzcd}$$

Let $\mathrm{pr}_F : B \times F \to F$ be the projection onto F. We can define $\bar{h} : X \times [0, 1] \to B \times F$ by

$$\bar{h}(x, t) = (h(x, t),\ \mathrm{pr}_F \bar{f}(x, 0))$$

7.5 Example. Every covering map $p : E \to B$ is a Hurewicz fibration.

7.6 Definition. Let $A \subseteq X$. A map $p : E \to B$ has the *relative homotopy lifting property* for the pair (X, A) if for any commutative diagram of the form

$$\begin{tikzcd} X \times \{0\} \cup A \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B \end{tikzcd}$$

there exists a map $\bar{h} : X \times [0, 1] \to E$ such that $\bar{h}i = \bar{f}$ and $p\bar{h} = h$.

7.7 Theorem. *Let $p : E \to B$ be a map. The following conditions are equivalent:*

1) *p is a Serre fibration;*
2) *p has the homotopy lifting property for D^n for all $n \geq 0$;*
3) *p has the relative homotopy lifting property for (D^n, S^{n-1}) for all $n \geq 0$;*
4) *p has the relative homotopy lifting property for all relative CW-complexes (X, A).*

Proof. 1) $\Rightarrow$ 2) Obvious.

2) $\Rightarrow$ 3) Assume that we have a diagram

$$\begin{tikzcd} D^n \times \{0\} \cup S^{n-1} \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B \end{tikzcd}$$

We want to show that the map $\bar{h}$ exists.

We can construct a homeomorphism $\varphi : D^n \times [0, 1] \to D^n \times [0, 1]$ such that $\varphi(D^n \times \{0\}) = D^n \times \{0\} \cup S^{n-1} \times [0, 1]$. This gives a commutative diagram

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "\varphi"] \arrow[d, hook] & D^n \times \{0\} \cup S^{n-1} \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r, "\varphi"'] \arrow[urr, dashed, "h'"] & D^n \times [0,1] \arrow[r, "h"'] & B \end{tikzcd}$$

The map $h'\colon D^n \times [0,1] \to E$ exists by 2). Then we can take $\bar{h} = h'\varphi^{-1}$.

3) $\Rightarrow$ 4) Let (X, A) be a relative complex, and assume that we have a commutative diagram

$$\begin{tikzcd}
X \times \{0\} \cup A \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B
\end{tikzcd}$$

We want to show that the map $\bar{h}$ exists.

Assume that X is obtained by attaching a single n-dimensional cell e^n to A using an attaching map $\varphi\colon S^{n-1} \to A$, i.e. $X = A \cup_\varphi e^n$. Let $\overline{\varphi}\colon D^n \to X$ be the characteristic map of e^n (2.2). Then the above diagram can be extended as follow:

$$\begin{tikzcd}
D^n \times \{0\} \cup S^{n-1} \times [0,1] \arrow[r, "\overline{\varphi} \times \{0\} \cup \varphi \times [0,1]"] \arrow[d, hook] & X \times \{0\} \cup A \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\
D^n \times [0,1] \arrow[r, "\overline{\varphi} \times [0,1]"'] \arrow[urr, dashed, "h'"] & X \times [0,1] \arrow[r, "h"'] & B
\end{tikzcd}$$

The map h' exists by 3). Since $X \times [0,1]$ is a quotient space of $A \times [0,1] \sqcup D^n \times [0,1]$, the map

$$\bar{f} \sqcup h'\colon A \times [0,1] \sqcup D^n \times [0,1] \to E$$

defines the desired map $\bar{h}\colon X \times [0,1] \to E$. The general statement can be obtained from here by induction with respect to cell attachments.

4) $\Rightarrow$ 1) Let X be a CW complex and $A = \varnothing$. Then the relative lifting property for (X, A) is the same as the lifting property for X. $\square$

7.8 Note. Property 3) in Theorem 7.7 can be equivalently stated as follows. Given a cube I^n, let K be a subset of ∂I^n consisting of all but one face of I^n. Then for any commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B
\end{tikzcd}$$

there exists a map $\bar{h}\colon I^n \to E$ such that this diagram commutes.

7.9 Lemma. *Let $p\colon E \to B$ be a Serre fibration. Let $e_0 \in E$ and $b_0 \in B$ be points such that $p(e_0) = b_0$, and let $F = p^{-1}(b_0)$. For any $n \geq 1$ the map $p\colon (E, F, e_0) \to (B, b_0, b_0)$ induces an isomorphism of homotopy groups*

$$p_*\colon \pi_n(E, F, e_0) \longrightarrow \pi_n(B, b_0, b_0) = \pi_n(B, b_0)$$

Proof. To check that $p_*\colon \pi_n(E, F, e_0) \to \pi_n(B, b_0)$ is onto, take a map $\omega\colon (I^n, \partial I^n) \to (B, b_0)$. By the relative homotopy lifting property for $(I^{n-1}, \partial I^{n-1})$, we can find a map $\overline{\omega}\colon I^n \to E$ such that $\overline{\omega}(J^{n-1}) = e_0$ and $p\overline{\omega} = \omega$.

$$\begin{tikzcd}
J^{n-1} = I^{n-1} \times \{1\} \cup \partial I^{n-1} \times [0,1] \arrow[r, "c_{e_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n = I^{n-1} \times [0,1] \arrow[r, "\omega"'] \arrow[ur, dashed, "\overline{\omega}"] & B
\end{tikzcd}$$

Then $\overline{\omega}$ represents and element of $\pi_n(E, F, e_0)$, and $p_*([\overline{\omega}]) = [\omega]$.

It remains to verify that p_* is 1-1. Assume that $\omega_0, \omega_1\colon (I^n, \partial I^n, J^{n-1}) \to (E, F, e_0)$ be maps such that $p_*([\omega_0]) = p_*([\omega_1])$. Then there exists a homotopy $h\colon I^n \times I \to B$ with $h_0 = p\omega_0$ and $h_1 = p\omega_1$, and such that $h(\partial I^n \times [0,1]) = b_0$. Take the subset $K \subseteq I^n \times I$ given by

$$K = I^n \times \{0, 1\} \cup J^{n-1} \times [0,1]$$

Notice that K consists of all faces of the cube $I^{n+1} = I^n \times [0,1]$, except for the face $I^{n-1} \times \{0\} \times [0,1]$. Define $\overline{f}\colon X \to E$ by

$$\overline{f}(x) = \begin{cases} \omega_0(x) & \text{for } x \in I^n \times \{0\} \\ \omega_1(x) & \text{for } x \in I^n \times \{1\} \\ e_0 & \text{for } x \in J^{n-1} \times [0,1] \end{cases}$$

By (7.8) we can find a map $\overline{h}\colon I^{n+1} \to E$ such that $\overline{h}|_K = \overline{f}$ and $p\overline{h} = h$. Such map $\overline{h}$ gives a homotopy between ω_0 and ω_1. Therefore $[\omega_0] = [\omega_1]$ in $\pi_n(E, F, e_0)$. ☐

7.10 Theorem. *Let $p\colon E \to B$ be a Serre fibration. Let $e_0 \in E$ and $b_0 \in B$ be such that $p(e_0) = b_0$, and let $F = p^{-1}(b_0)$. Let $i\colon F \to E$ be the inclusion map. For any $n \geq 1$ define a homomorphism $\partial\colon \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ given by*

$$\partial\colon \pi_n(B, b_0) \xrightarrow{p_*^{-1}} \pi_n(E, F, e_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0)$$

Then the following sequence is exact:

$$\begin{aligned} \dots \longrightarrow \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \dots \\ \dots \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0) \end{aligned}$$

Proof. Exactness in almost all places follows from the exactness of the long exact sequence of the triple (E, F, e_0), and the commutativity of the following diagram:

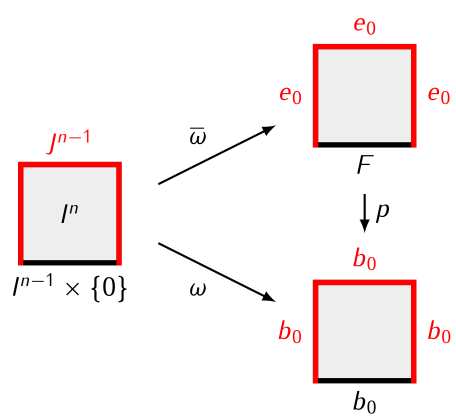
$$\begin{tikzcd}
\dots \arrow[r] & \pi_n(F,e_0) \arrow[r, "i_*"] & \pi_n(E,e_0) \arrow[r, "p_*"] & \pi_n(B,b_0) \arrow[r, "\partial"] & \pi_{n-1}(F,e_0) \arrow[r, "i_*"] & \pi_{n-1}(E,e_0) \arrow[r] & \dots \\
\dots \arrow[r] & \pi_n(F,e_0) \arrow[r, "i_*"'] \arrow[u, "="] & \pi_n(E,e_0) \arrow[r, "j_*"'] \arrow[u, "="] & \pi_n(E,F,e_0) \arrow[r, "\partial"'] \arrow[u, "p_*", "\cong"'] & \pi_{n-1}(F,e_0) \arrow[r, "i_*"'] \arrow[u, "="] & \pi_{n-1}(E,e_0) \arrow[r] \arrow[u, "="] & \dots
\end{tikzcd}$$

Since the long exact sequence of (E, F, x_0) ends at $\pi_0(E, e_0)$, exactness of the sequence

$$\pi_0(F,e_0) \xrightarrow{i_*} \pi_0(E,e_0) \xrightarrow{p_*} \pi_0(B,b_0)$$

needs to be checked separately (exercise). □

7.11 Note. The map $\partial\colon \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ can be described directly as follows. Take a map $\omega\colon (I^n, \partial I^n) \to (B, x_0)$. Since $p\colon E \to B$ is a Serre fibration, by (7.8) we can find $\overline{\omega}\colon I^n \to E$ such that $p\overline{\omega} = \omega$, and $\overline{\omega}(J^{n-1}) = e_0$. Then $\partial([\omega]) = [\overline{\omega}|_{I^{n-1}\times\{0\}}]$.



7.12 Example. Consider the product fibration $\mathrm{pr}_B\colon B \times F \to B$. For $b_0 \in B$ we have $\mathrm{pr}_B^{-1}(b_0) = \{b_0\} \times F \cong F$. This for $f_0 \in F$ the exact sequence looks as follows:

$$\dots \to \pi_{n+1}(B,b_0) \xrightarrow{\partial} \pi_n(F,f_0) \xrightarrow{i_*} \pi_n(B\times F,(b_0,f_0)) \xrightarrow{p_*} \pi_n(B,b_0) \xrightarrow{\partial} \pi_{n-1}(F,f_0) \xrightarrow{i_*} \dots$$

The projection map $\mathrm{pr}_F\colon B \times F \to F$ induces homomorphisms $\mathrm{pr}_{F*}\colon \pi_n(B \times F, (b_0, f_0)) \to \pi_n(F, f_0)$ such that $\mathrm{pr}_{F*}\, i_* = \mathrm{id}_{\pi_n(F,f_0)}$. This means that $\operatorname{Im}\partial = \operatorname{Ker} i_* = 0$. Therefore for each $n \geq 1$ we obtain a split short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \pi_n(F,f_0) \arrow[r, shift left, "i_*"] & \pi_n(B\times F,(b_0,f_0)) \arrow[l, shift left, "\mathrm{pr}_{F*}"] \arrow[r, "p_*"] & \pi_n(B,b_0) \arrow[r] & 0
\end{tikzcd}$$

This shows that $\pi_n(B \times F, (b_0, f_0)) \cong \pi_n(B, b_0) \times \pi_n(F, f_0)$, which is a special case of the product formula (5.13).

7.13 Example. Let $p\colon E \to B$ be a covering, let $b_0 \in B$ and let $e_0 \in p^{-1}(b_0)$. The space $F = p^{-1}(b_0)$ is discrete, so $\pi_n(F) = 0$ for all $n \geq 1$. Therefore the exact sequence of the fibration becomes

$$\begin{aligned} \ldots \longrightarrow\ 0 \longrightarrow \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \longrightarrow 0\ \longrightarrow \ldots \\ \ldots \longrightarrow 0 \longrightarrow \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0) \end{aligned}$$

This shows that $p_*\colon \pi_n(E, e_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 2$, and it is a monomorphism for $n = 1$. This recovers the statement of Proposition 5.9.

The image of $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ coincides with $\mathrm{Ker}(\partial\colon \pi_1(B, b_0) \to \pi_0(F, e_0))$. By the definition of the map ∂, an element $[\omega] \in \pi_1(B, b_0)$ is in $\mathrm{Ker}\,\partial$ if $\omega\colon [0, 1] \to B$ has a lift $\overline{\omega}\colon [0, 1] \to E$ such that $\overline{\omega}(1) = e_0$ and $\omega(1)$ is in the same path connected component of F as e_0. Since F is discrete, it means that $\overline{\omega}(1) = e_0 = \overline{\omega}(0)$. As a consequence, we obtain that $\mathrm{Im}(p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ consists of elements $[\omega] \in \pi_1(B, b_0)$ such that the lift of ω ending at e_0 is a loop.

7.14 Theorem. *Let $p\colon E \to B$ be map and let $\{U_i\}_{i\in I}$ be an open cover of B. Assume that for each $i \in I$ the map $p_i\colon p^{-1}(U_i) \to U_i$, which is the restriction of p is a Serre fibration. Then p is a Serre fibration.*

7.15 Note. An analogous fact is true for Hurewicz fibrations, under the assumption that B is a paracompact space.

Proof of Theorem 7.14. See e.g. Hatcher *Algebraic Topology*, Proposition 4.48 p. 379. □

7.16 Definition. A map $p\colon E \to B$ is a *fiber bundle* with fiber F if for every point $b \in B$ there exists an open neighborhood $b \in U \subseteq B$ and a homeomorphism $h_U\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

Here $\mathrm{pr}_1\colon U \times F \to U$ is the projection map $\mathrm{pr}_1(x, y) = x$.

7.17 Proposition. *Every fiber bundle is a Serre fibration.*

Proof. This follows from Theorem 7.14 and Example 7.4. □

7.18 Example. Every covering space $p\colon E \to B$ is a fiber bundle whose fiber is a discrete space.

7.19 Example. Mobius band

7.20 Example. Klein bottle

7.21 Example. Consider S^{2n+1} as a subspace of the complex space $\mathbb{C}^n$:

$$S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} ||z_i||^2 = 1\}$$

In particular, $S^1 = \{z \in \mathbb{C} \mid ||z|| = 1\}$. The *$n$-dimensional complex projective space* is the quotient space

$$\mathbb{CP}^n = S^{2n+1}/\sim$$

where $(z_1, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$ for all $\lambda \in S^1$. We will denote by $[z_0, \dots, z_n] \in \mathbb{CP}^n$ the equivalence class of $(z_0, \dots, z_n)$. Let $p\colon S^{2n+1} \to \mathbb{CP}^n$ be the quotient map $p(z_0, \dots, z_n) = [z_0, \dots, z_n]$.

We will show that $p\colon S^{2n+1} \to \mathbb{CP}^n$ is a fiber bundle with fiber S^1. Let $b = [z_0, \dots, z_n] \in \mathbb{CP}^n$. Choose $0 \le i \le n$ such that $z_i \neq 0$, and take $U_i = \{[w_0, \dots, w_n] \in \mathbb{CP}^n \mid w_i \neq 0\}$. This set is an open neighborhood of b in $\mathbb{CP}^n$. We have

$$p^{-1}(U_i) = \{(w_0, \dots, w_n) \in S^{2n+1} \mid w_i \neq 0\}$$

Define a map $h_i\colon p^{-1}(U_i) \to U_i \times S^1$ by $h_i(w_0, \dots, w_n) = ([w_0, \dots, w_n], w_i/||w_i||)$. This is a homeomorphism, with the inverse given by

$$h_i^{-1}([v_0, \dots, v_n], \lambda) = \frac{||v_i||}{v_i} \cdot \lambda \cdot (v_0, \dots, v_n)$$

Let $n \ge 1$. The long exact sequence of the bundle $p\colon S^{2n+1} \to \mathbb{CP}^n$ has the form

$$\dots \longrightarrow \pi_m(S^1) \xrightarrow{i_*} \pi_m(S^{2n+1}) \xrightarrow{p_*} \pi_m(\mathbb{CP}^n) \xrightarrow{\partial} \pi_{m-1}(S^1) \longrightarrow \dots$$
$$\dots \longrightarrow \pi_2(S^{2n+1}) \xrightarrow{p_*} \pi_2(\mathbb{CP}^n) \xrightarrow{\partial} \pi_1(S^1) \xrightarrow{i_*} \pi_1(S^{2n+1}) \xrightarrow{p_*} \pi_1(\mathbb{CP}^n, b_0) \xrightarrow{\partial} \pi_0(S^1) = 0$$

Since $\pi_m(S^1) = 0$ for $m > 1$, we obtain that $\pi_m(\mathbb{CP}^n) \cong \pi_m(S^{2n+1})$ for $m \ge 3$. Also, since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_m(S^{2n+1}) = 0$ for $m < 2n+1$, thus $\pi_2(\mathbb{CP}^n) \cong \pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(\mathbb{CP}^n) = 0$.

7.22 Example. As a special case of Example 7.21, take $n = 1$. In this case, we have a homeomorphism $\mathbb{CP}^1 \cong S^2$. To see this, define a map $h\colon \mathbb{CP}^1 \smallsetminus \{[1, 0]\} \to \mathbb{C}$ by $h([z_0, z_1]) = \frac{z_0}{z_1}$. This is a homeomorphism with the inverse given by $h^{-1}(z) = \frac{1}{1+||z||} \cdot [z, 1]$. Since S^2 is homeomorphic to the one-point compactification of $\mathbb{C}$, i.e. $S^2 \cong \mathbb{C} \cup \{\infty\}$, the map h can be extended to a homeomomrphism $h\colon \mathbb{CP}^1 \to S^2$ by setting $h([1, 0]) = \infty$.

Under the identification $\mathbb{CP}^1 \cong S^2$ the bundle $S^1 \to S^3 \xrightarrow{p} \mathbb{CP}^1$ becomes $S^1 \to S^3 \xrightarrow{p} S^2$. This bundle is called the *Hopf bundle* (or the *Hopf fibration*).

Using the long exact sequence of the Hopf fibration we obtain:

7.23 Theorem. $\pi_2(S^2) \cong \mathbb{Z}$.

8 | From Maps to Fibrations

As we have seen any fibration $F \to E \xrightarrow{p} B$ has the associated long exact sequence

$$\cdots \to \pi_n(F) \longrightarrow \pi_n(E) \xrightarrow{p_*} \pi_n(B) \to \dots$$

that relates the homotopy groups of the spaces B, E, and F. The main goal of this chapter is to show that this approach to computing homotopy groups can be used with an arbitrary map $f\colon X \to Y$ taken in place of a fibration p. We will show that the following holds:

8.1 Theorem. *Given any map $f\colon X \to Y$ there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "g_f", "\simeq"'] \arrow[dr, "f"'] & & E_f \arrow[dl, "p_f"] \\ & Y & \end{tikzcd}$$

such that $p_f\colon E_f \to Y$ is a Hurewicz fibration and $g\colon E_f \to X$ is a homotopy equivalence.

For $x_0 \in X$ and $e_0 = g_f(x_0) \in E_f$ we will get $\pi_n(X, x_0) \cong \pi_n(E_f, e_0)$ for all $n \geq 0$. In this way, the long exact sequence of a fibration gives an exact sequence

$$\cdots \to \pi_n(F, e_0) \longrightarrow \pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, y_0) \to \dots$$

where $F = p_f^{-1}(y_0)$.

8.2 Mapping spaces. For spaces X, Y, let $\mathrm{Map}(X, Y)$ denote the set of all continuous functions $X \to Y$. For $A \subseteq X$ and $U \subseteq Y$ let $P(A, U) \subseteq \mathrm{Map}(X, Y)$ be the set

$$P(A, U) = \{f \in \mathrm{Map}(X, Y) \mid f(A) \subseteq U\}$$

8.3 Definition. The compact-open topology on $\mathrm{Map}(X, Y)$ is the topology with subbasis given by all sets of the form $P(A, U)$ where $A \subseteq X$ is compact and $U \subseteq Y$ is open.

Let X, Y, Z be spaces. For a function $\varphi\colon Z \to \mathrm{Map}(X, Y)$ denote by $\varphi^\sharp\colon Z \times X \to Y$ the function given by $\varphi^\sharp(z, x) = \varphi(z)(x)$. We will say that $\varphi^\sharp$ is the *adjoint* of φ.

8.4 Theorem. *If X is a locally compact Hausdorff space, then the compact-open topology on* $\mathrm{Map}(X, Y)$ *is the unique topology with the property that a map $\varphi\colon Z \to \mathrm{Map}(X, Y)$ is continuous if and only if $\varphi^\sharp\colon Z \times X \to Y$ is continuous.*

8.5 All mapping spaces below are equipped with the compact-open topology. The following properties hold:

1) The evaluation map $\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$ given by $\mathrm{ev}(f, x) = f(x)$ is continuous.
2) In particular for every $x_0 \in X$ the map $\mathrm{ev}_{x_0}\colon \mathrm{Map}(X, Y) \to Y$, $\mathrm{ev}_{x_0}(f) = f(x_0)$ is continuous.
3) If $\{*\}$ is a one point space, then the map $\mathrm{ev}_*\colon \mathrm{Map}(\{*\}, Y) \to Y$ is a homeomorphism.
4) For any continuous function $f\colon X \to Y$ and any space Z the induced function $f_*\colon \mathrm{Map}(Z, X) \to \mathrm{Map}(Z, Y)$ given by $f_*(g) = f \circ g$ is continuous.
5) For any continuous function $f\colon X \to Y$ and any space Z the induced function $f^*\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(Y, X)$ given by $f^*(g) = g \circ f$ is continuous.
6) If Y is a locally compact Hausdorff space, then for any spaces X and Y the map $F\colon \mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ given by $F(f, g) = g \circ f$ is continuous.
7) If Y is a locally compact Hausdorff space and X is a Hausdorff space then for any space Z the map $\mathrm{adj}\colon \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \to \mathrm{Map}(X \times Y, Z)$ given by $\mathrm{adj}(\varphi) = \varphi^\sharp$ is a homeomorphism.

From now on, all mapping spaces will taken with the compact-open topology.

8.6 Example. Let X be a locally compact space and let $f, g\colon X \to Y$. Giving a map $\omega\colon [0, 1] \to \mathrm{Map}(X, Y)$ such that $\omega(0) = f$ and $\omega(1) = g$ is equivalent to giving a homotopy $\omega^\sharp\colon X \times [0, 1] \to Y$ between f and g. In effect, homotopy classes of maps $X \to Y$ correspond to path connected components of the space $\mathrm{Map}(X, Y)$.

8.7 Example. Let X be a space. The *path space* of X is the space $PX = \mathrm{Map}([0, 1], X)$.

For $x_0 \in X$ consider the subspace of PX given by

$$\Omega_{x_0} X = \{\omega \in PX \mid \omega(0) = \omega(1) = x_0\}$$

This space is called the *loop space* of X based at x_0. Denote by $c_{x_0} \in \Omega_{x_0}X$ the constant loop $c_{x_0}(t) = x_0$ for all $t \in [0, 1]$.

Notice that every element $\omega \in \Omega_{x_0}X$ represents an element of $\pi_1(X, x_0)$. Similarly as in Example 8.6 we also obtain that path connected components of $\Omega_{x_0}X$ correspond to homotopy classes of loops in X. In this way, the assignment $[\omega] \mapsto [\omega^\sharp]$ gives a bijection

$$\pi_0(\Omega_{x_0}X, c_{x_0}) \xrightarrow{\cong} \pi_1(X, x_0)$$

Concatenation of loops defines a map $\Omega_{x_0}X \times \Omega_{x_0}X \to \Omega_{x_0}X$ which, in turn, induces a map

$$\pi_0(\Omega_{x_0}X, c_{x_0}) \times \pi_0(\Omega_{x_0}X, c_{x_0}) \to \pi_0(\Omega_{x_0}X, c_{x_0})$$

This defines a group structure on $\pi_0(\Omega_{x_0}X, c_{x_0})$ such that the bijection $\pi_0(\Omega_{x_0}X, c_{x_0}) \cong \pi_1(X, x_0)$ becomes an isomorphism of groups.

Generalizing this, any element of $\pi_n(\Omega_{x_0}, c_{x_0})$ is represented by a map $\omega\colon (I^n, \partial I^n) \to (\Omega_{x_0}X, c_{x_0})$. The adjoint of ω is a map $\omega^\sharp\colon I^n \times [0, 1] = I^{n+1} \to X$ such that $\omega^\sharp(\partial I^{n+1}) = x_0$. In other words, we obtain a map $\omega^\sharp\colon (I^{n+1}, \partial I^{n+1}) \to (X, x_0)$. It is easy to verify that maps $\omega_1, \omega_2\colon (I^n, \partial I^n) \to (\Omega_{x_0}X, c_{x_0})$ are homotopic if and only of their adjoints $\omega_1^\sharp$, $\omega_2^\sharp$ are homotopic. Thus the correspondence $[\omega] \mapsto [\omega^\sharp]$ defines a bijection

$$\pi_n(\Omega_{x_0}X, c_{x_0}) \xrightarrow{\cong} \pi_{n+1}(X, x_0)$$

One can check that this is an isomorphism of groups.

8.8 Note. A map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ induces a map of loop spaces $\Omega f\colon \Omega_{x_0}X \to \Omega_{y_0}Y$. In this way we obtain a functor

$$\Omega\colon \mathbf{Top}_* \to \mathbf{Top}_*$$

8.9 Example. Let $x_0 \in A \subseteq X$. Denote

$$P(X, A, x_0) = \{\omega\colon [0, 1] \to X \mid \omega(0) \in A,\ \omega(1) = x_0\}$$

Similarly as in Example 8.7, one can check that for any map $\omega\colon (I^n, \partial I^n) \to (P(X, A, x_0), c_{x_0})$ the adjoint $\omega^\sharp\colon I^{n+1} \to X$ represents an element $[\omega^\sharp] \in \pi_{n+1}(X, A, x_0)$. The assignment $[\omega] \to [\omega^\sharp]$ gives an isomorphism

$$\pi_n(P(X, A, x_0)) \xrightarrow{\cong} \pi_{n+1}(X, A, x_0)$$

for any $n \geq 1$.

Let $f\colon X \to Y$ be a map, and let PY be the path space of Y. Define

$$E_f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0)\} \subseteq X \times PY$$

We have a map $r_f\colon PX \to E_f$ give by $r_f(\omega) = (\omega(0), f\omega))$

8.10 Proposition. *For a map $f: X \to Y$ the following conditions are equivalent:*

1) *The map f is a Hurewicz fibration.*
2) *The map f has the homotopy lifting property for the space E_f*
3) *There exists a map $s_f: E_f \to PX$ such that $r_f s_f = \mathrm{id}_{E_f}$*

Proof. 1) $\Rightarrow$ 2) Obvious.

2) $\Rightarrow$ 3) Consider the following commutative diagram:

$$\begin{tikzcd} E_f \times \{0\} \arrow[r, "\bar{k}"] \arrow[d, hook] & X \arrow[d, "f"] \\ E_f \times [0,1] \arrow[r, "g"'] \arrow[ur, dashed, "\bar{g}"] & Y \end{tikzcd} \tag{$*$}$$

Here $\bar{k}((x, \omega), 0) = x$ and $g((x, \omega), t) = \omega(t)$. By 2) there exists a homotopy $\bar{g}$ that commutes this the rest of the diagram. Take $s_f = \bar{g}^\sharp$, the adjoint of $\bar{g}$.

3) $\Rightarrow$ 1) Assume that we have the following commutative diagram and we want to show that a homotopy lift $\bar{h}$ exists:

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "\bar{d}"] \arrow[d, hook] & X \arrow[d, "f"] \\ Z \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Y \end{tikzcd}$$

For $z \in Z$ let $\omega_z: [0,1] \to Y$ be the path given by $\omega_z(t) = h(z, t)$. Define a map $u: Z \to E_f$ by $u(z) = (\bar{k}(z, 0), \omega_z)$. Notice that, in the notation of diagram ($*$) we have $\bar{d} = \bar{k}(u \times \mathrm{id}_{\{0\}})$ and $h = g(u \times \mathrm{id}_{[0,1]})$. As a consequence, we can take $\bar{h} = \bar{g}(u \times \mathrm{id}_{[0,1]})$. $\square$

Proof of Theorem 8.1. Let $f: X \to Y$ be a map. As before, define

$$E_f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0)\}$$

Let $g_f: X \to E_f$ be given by $g_f(x) = (x, c_{f(x)})$ where $c_{f(x)}: [0,1] \to Y$ is the constant path at $f(x)$. Also, let $p_f: E_f \to Y$ be given by $p_f(x, \omega) = \omega(1)$. We have $f = p_f g_f$.

We will show that g_f is a homotopy equivalence with the homotopy inverse given by the projection map $\mathrm{pr}: E_f \to X$, $\mathrm{pr}(x, \omega) = x$. We have $\mathrm{pr}\, g_f = \mathrm{id}_X$. The composition $g_f \mathrm{pr}: E_f \to E_f$ is given by $g_f \mathrm{pr}(x, \omega) = (x, c_x)$. A homotopy $h: g_f \mathrm{pr} \simeq \mathrm{id}_{E_f}$ is defined by $h((x, \omega), t) = (x, \omega_t(x))$, where $\omega_t: [0,1] \to Y$, $\omega_t(s) = \omega(ts)$.

It remains to show that $p_f\colon E_f \to Y$ is a Hurewicz fibration. To see this, assume that we have a commutative diagram

$$\begin{tikzcd} Z\times\{0\} \arrow[r, "\bar{k}"] \arrow[d, hook] & E_f \arrow[d, "p_f"] \\ Z\times[0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Y \end{tikzcd}$$

Denote $\bar{k}(z) = (x_z, \omega_z)$. By commutativity of the diagram we have $\omega_z(1) = h_0(z)$. Then the lift $\bar{h}$ can be defined by

$$\bar{h}(z,t) = (x_z, \tau_{z,t} * \omega_z)$$

where $\tau_{z,t} * \omega_z$ is the concatenation of ω_z with the path $\tau_{z,t}\colon [0,1] \to Y$ given by $\tau_{z,t}(s) = h(z, st)$. □

8.11 Definition. Let $f\colon X \to Y$ be a map, and let $p_f\colon E_f \to Y$ be the Hurewicz fibration associated to f, as in Theorem 8.1. The *homotopy fiber* of f over a point $y_0 \in Y$ is the space

$$\mathrm{hofib}_{y_0} f = p_f^{-1}(y_0)$$

Explicitly:

$$\mathrm{hofib}_{y_0} f = \{(x,\omega) \in X \times PY \mid f(x) = \omega(0),\ \omega(1) = y_0\}$$

8.12 Corollary. *Let $f\colon (X, x_0) \to (Y, y_0)$ be map of pointed spaces. Denote $v_0 = (x_0, c_{y_0}) \in \mathrm{hofib}_{y_0} f$. We have a long exact sequence of homotopy groups*

$$\ldots \longrightarrow \pi_{n+1}(Y,y_0) \longrightarrow \pi_n(\mathrm{hofib}_{x_0} f, v_0) \xrightarrow{i(f)_*} \pi_n(X,x_0) \xrightarrow{f_*} \pi_n(Y,y_0) \longrightarrow \ldots$$
$$\ldots \xrightarrow{f_*} \pi_1(Y,y_0) \longrightarrow \pi_0(\mathrm{hofib}_{x_0} f, v_0) \xrightarrow{i(f)_*} \pi_0(X,x_0) \xrightarrow{f_*} \pi_0(Y,y_0)$$

Here the map

$$i(f)\colon\ \mathrm{hofib}_{y_0} f = \{(x,\omega) \in X \times PY \mid f(x) = \omega(0),\ \omega(1) = y_0\} \longrightarrow Y$$

is given by $i(f)(x,\omega) = x$.

8.13 Example. Given a space X and $x_0 \in X$, consider a map $f\colon \{*\} \to X$, $f(*) = x_0$. Then

$$\begin{aligned} \mathrm{hofib}_{x_0} f &= \{(*,\omega) \in \{*\} \times PX \mid \omega(0) = x_0 = \omega(1)\} \\ &\cong \{\omega \in PX \mid \omega(0) = x_0 = \omega(1)\} \\ &= \Omega_{x_0} X \end{aligned}$$

Since $\pi_n(\{*\}) = 0$ for all $n \geq 0$ the exact sequence becomes

$$\ldots \longrightarrow 0 \longrightarrow \pi_n(X,x_0) \longrightarrow \pi_{n-1}(\Omega_{x_0}X, c_{x_0}) \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow \pi_1(X,x_0) \longrightarrow \pi_0(\Omega_{x_0}X, c_{x_0}) \longrightarrow 0$$

where $c_{x_0} \in \Omega_{x_0} X$ is the constant loop at x_0. This recovers the isomorophisms $\pi_n(\Omega_{x_0} X, c_{x_0}) \cong \pi_{n+1}(X, x_0)$, which we obtained in Example 8.7

Notice that if x_1 belongs to a different path connected component of X than x_0, then $\operatorname{hofib}_{x_1} f = \varnothing$.

8.14 Example. The map $f\colon * \to X$ in Example 8.7 can be interpreted as an inclusion $\{x_0\} \hookrightarrow X$. Generalizing it, for $A \subseteq X$, consider the inclusion map $j\colon A \hookrightarrow X$. In this case we have

$$\begin{aligned} E_j &= \{(a, \omega) \in A \times PX \mid j(a) = \omega(0)\} \\ &\cong \{\omega \in P(X) \mid \omega(0) \in A\} \end{aligned}$$

For $x_0 \in X$ we get:

$$\begin{aligned} \operatorname{hofib}_{x_0} j &= \{\omega \in PX \mid \omega(0) \in A,\ \omega(1) = x_0\} \\ &= P(X, A, x_0) \end{aligned}$$

Recall (8.9) that we have isomorphisms $\pi_n(P(X, A, x_0), c_{x_0}) \xrightarrow{\cong} \pi_{n+1}(X, A, x_0)$. They fit into a commutative diagram of exact sequences:

$$\begin{tikzcd}
\dots \arrow[r] & \pi_{n+1}(X, A, x_0) \arrow[r, "\partial"] & \pi_n(A, x_0) \arrow[r, "j_*"] & \pi_n(X, x_0) \arrow[r] & \pi_n(X, A, x_0) \arrow[r] & \dots \\
\dots \arrow[r] & \pi_n(P(X, A, x_0), c_{x_0}) \arrow[r] \arrow[u, "\cong"] & \pi_n(E_j, c_{x_0}) \arrow[r, "p_{j*}"'] \arrow[u, "\cong", "g_{j*}"'] & \pi_n(X, x_0) \arrow[r, "\partial"'] \arrow[u, "="] & \pi_{n-1}(P(X, A, x_0), c_{x_0}) \arrow[r, "i_*"'] \arrow[u, "\cong"] & \dots
\end{tikzcd}$$

Here $g_j\colon E_j \to A$ and $p_j\colon E_j \to X$ are given by $g_j(\omega) = \omega(0)$ and $p_j(\omega) = \omega(1)$.

8.15 Definition. Consider a commutative diagram

$$\begin{tikzcd}
E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}$$

where p_1, p_2 are Hurewicz fibrations. The map f is a *fibrewise homotopy equivalence* if there exists a map $g\colon E_2 \to E_1$ such that $p_1 g = p_2$ and homotopies $h\colon gf \simeq \mathrm{id}_{E_1}$, $h'\colon fg \simeq \mathrm{id}_{E_2}$ such that $p_1 h_t = p_1$ and $p_2 h'_t = p_2$ for all $t \in [0, 1]$

8.16 Note. In the notation of Definition 8.15, if $f\colon E_1 \to E_2$ is a fibrewise homotopy equivalence then for any subspace $A \subseteq B$ the map $f|_{p_1^{-1}(A)}\colon p_1^{-1}(A) \to p_2^{-1}(A)$ is a homotopy equivalence. In particular, for any $b_0 \in B$ the map of fibers $f|_{p_1^{-1}(b_0)}\colon p^{-1}(b_0) \to p_2^{-1}(b_0)$ is a homotopy equivalence.

8.17 Proposition. *For a map $f\colon X \to Y$ consider the commutative diagram as in Theorem 8.1:*

$$\begin{tikzcd} X \arrow[rr, "g_f", "\simeq"'] \arrow[dr, "f"'] & & E_f \arrow[dl, "p_f"] \\ & Y & \end{tikzcd}$$

where $E_f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0)\}$, $p_f(x, \omega) = \omega(1)$ and $g_f(x) = (x, c_{f(x)})$.

If f is a Hurewicz fibration then g_f is a fiberwise homotopy equivalence.

Proof. Exercise. $\square$

8.18 Corollary. *Let $f\colon X \to Y$ is a Hurewicz fibration and let $g_f\colon X \to E_f$ be given as in Proposition 8.17. Then for $y_0 \in Y$ the map*

$$g_f|_{f^{-1}(y_0)}\colon f^{-1}(y_0) \to p_f^{-1}(y_0) = \mathrm{hofib}_{y_0} f$$

is a homotopy equivalence.

Proof. It follows from Proposition 8.17 and Note 8.16. $\square$

9 | Exact Puppe Sequence

Recall that a sequence of pointed sets

$$(S_2, s_2) \xrightarrow{f} (S_1, s_1) \xrightarrow{g} (S_0, s_0)$$

is *exact at* S_1 if $f(S_2) = g^{-1}(s_0)$.

For pointed spaces (X, x_0) and (Y, y_0) let $[X, Y]_*$ denote the set of pointed homotopy classes of maps $X \to Y$. This is a pointed set, with the basepoint represented by the constant function $c_{y_0} \colon X \to Y$, $c_{y_0}(x) = y_0$ for all $x \in X$.

9.1 Definition. A pointed space (X, x_0) is well-pointed if the pair (X, x_0) has the homotopy extension property.

9.2 Definition. A sequence of maps of spaces

$$(X_0, x_0) \xrightarrow{f_0} (X_1, x_1) \xrightarrow{f_1} (X_2, x_2)$$

is *exact at* X_1 is for any well-pointed space (Y, y_0) the sequence pointed sets

$$[Y, X_0]_* \xrightarrow{f_{0*}} [Y, X_1]_* \xrightarrow{f_{1*}} [Y, X_2]_*$$

is exact at $[Y, X_1]_*$.

9.3 Proposition. *If $p \colon E \to B$ is a Hurewicz fibration, $e_0 \in E$, $b_0 = p(e_0) \in B$ $F = p^{-1}(b_0)$, and $i \colon F \to E$ is the inclusion map then the sequence $(F, e_0) \xrightarrow{i} (E, e_0) \xrightarrow{p} (B, b_0)$ is exact at E.*

Proof. Exercise. $\square$

Let $f \colon (X, x_0) \to (Y, y_0)$ be any pointed map. Consider the sequence

$$\operatorname{hofib} f \xrightarrow{i(f)} X \xrightarrow{f} Y$$

where

$$i(f)\colon \operatorname{hofib}_{y_0} f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0),\ \omega(1) = y_0\} \longrightarrow X$$

is given by $i(f)(x, \omega) = x$. Since this sequence is homotopy equivalent to a sequence given by a Hurewicz fibration, it is exact at X. We can continue this construction inductively, by taking consecutive homotopy fibers:

$$\ldots \longrightarrow \operatorname{hofib} i^3(f) \xrightarrow{i^4(f)} \operatorname{hofib} i^2(f) \xrightarrow{i^3(f)} \operatorname{hofib} i(f) \xrightarrow{i^2(f)} \operatorname{hofib} f \xrightarrow{i(f)} X \xrightarrow{f} Y \tag{$*$}$$

In this way we obtain a sequence which is exact at all spaces. As it turns out, this sequence has a more convenient description. The starting point for it is the following fact:

9.4 Proposition. *Let $f\colon X \to Y$ be a map and $y_0 \in Y$. Then the map $i(f)\colon \operatorname{hofib}_{y_0} f \longrightarrow X$ is a Hurewicz fibration.*

Proof. Exercise. □

9.5 Corollary. *For any map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ we have a commutative diagram*

$$\begin{tikzcd}
\operatorname{hofib} i(f) \arrow[r, "i^2(f)"] & \operatorname{hofib} f \arrow[r, "i(f)"] & X \arrow[r, "f"] & Y \\
\Omega Y \arrow[u, "g", "\simeq"'] \arrow[ur, "j"'] & & &
\end{tikzcd}$$

Proof. We have

$$i(f)^{-1}(x_0) = \{(x_0, \omega) \in X \times PY \mid \omega(0) = f(x_0) = y_0,\ \omega(1) = y_0\} \cong \Omega Y$$

Thus ΩY can be identified with the fiber of $i(f)$ over y_0, and the map $j\colon \Omega Y \to \operatorname{hofib} f$, $j(\omega) = (x_0, \omega)$ with the inclusion of the fiber. By Proposition 9.4 and Corollary 8.18 we obtain a homotopy equivalence $g\colon \Omega Y \to \operatorname{hofib} i(f)$ such that the above diagram commutes. □

9.6 Note. The homotopy equivalence in Corollary 9.5 can be explicitly described as follows. Up to a homeomorphism we have

$$\operatorname{hofib} i(f) = \{(\omega, \tau) \in PX \times PY \mid f\omega(0) = \tau(0),\ \omega(1) = y_0,\ \tau(1) = x_0\}$$

Then $i^2(f)\colon \operatorname{hofib} i(f) \to \operatorname{hofib} f$ is given by $(\omega, \tau) = (\omega(0), \tau)$ and $g(\tau) = (c_{x_0}, \tau)$.

Applying Corollary 9.5 iteratively to the sequence (∗) we get homotopy equivalences

$$\begin{aligned}
&\operatorname{hofib} i(f) \xleftarrow{\simeq} \Omega Y\\
&\operatorname{hofib} i^2(f) \xleftarrow{\simeq} \Omega X\\
&\operatorname{hofib} i^3(f) \xleftarrow{\simeq} \Omega \operatorname{hofib} f\\
&\operatorname{hofib} i^4(f) \xleftarrow{\simeq} \Omega \operatorname{hofib} i(f) \simeq \Omega^2 Y\\
&\operatorname{hofib} i^5(f) \xleftarrow{\simeq} \Omega \operatorname{hofib} i^2(f) \simeq \Omega^2 X\\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots
\end{aligned}$$

Moreover, one can check that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\dots \arrow[r] & \operatorname{hofib} i^4(f) \arrow[r, "i^5(f)"] & \operatorname{hofib} i^3(f) \arrow[r, "i^4(f)"] & \operatorname{hofib} i^2(f) \arrow[r, "i^3(f)"] & \operatorname{hofib} i(f) \arrow[r, "i^2(f)"] & \operatorname{hofib} f \arrow[r, "i(f)"] & X \arrow[r, "f"] & Y \\
\dots \arrow[r] & \Omega^2 Y \arrow[r, "\Omega j"'] \arrow[u, "\simeq"] & \Omega \operatorname{hofib} f \arrow[r, "\Omega i(f)"'] \arrow[u, "\simeq"] & \Omega X \arrow[r, "\Omega f"'] \arrow[u, "\simeq"] & \Omega Y \arrow[r, "j"'] \arrow[u, "\simeq"] & \operatorname{hofib} f \arrow[r, "i(f)"'] \arrow[u, equal] & X \arrow[r, "f"'] \arrow[u, equal] & Y \arrow[u, equal]
\end{tikzcd} \qquad (**)$$

Since the upper row of this diagram is exact, the same is true for the lower row.

9.7 Definition. The sequence in the lower row of the diagram (∗∗) is called the *Puppe exact sequence* associated to the map f.

As a consequence, for any map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ and any well-pointed space (Z, z_0) we obtain a long exact sequence of sets:

$$\begin{aligned}
\dots \xrightarrow{\Omega^2 f_*} [Z, \Omega^2 Y]_* \xrightarrow{\Omega j_*} [Z, \Omega \operatorname{hofib} f]_* \xrightarrow{\Omega i(f)_*} [Z, \Omega X]_* \xrightarrow{\Omega f_*} [Z, \Omega Y]_* \\
\xrightarrow{j_*} [Z, \operatorname{hofib} f]_* \xrightarrow{i(f)_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_* \quad (\maltese)
\end{aligned}$$

9.8 Note. For any pointed space (X, x_0) and $n \geq 1$ the loop space $\Omega^n X$ is quipped with a multiplication map $\mu\colon \Omega^n X \times \Omega^n X \to \Omega^n X$ given by concatenation of loops. For any pointed space (Z, z_0) this defines a multiplication

$$\mu_*\colon [Z, \Omega^n X]_* \times [Z, \Omega^n X]_* \to [Z, \Omega^n X]_*$$

given by $\mu_*([\varphi], [\psi]) = [\mu \circ (\varphi \times \psi)]$. This equips the set $[Z, \Omega^n X]_*$ with a group structure. Moreover, for $n \geq 2$ the multiplication μ commutes up to homotopy, and in effect $[Z, \Omega^n X]_*$ becomes an abelian group.

As a result the exact sequence (✠) becomes an exact sequence of groups starting at $[Z, \Omega Y]_*$ and its groups are abelian starting with $[Z, \Omega^2 Y]_*$

9.9 Loop spaces and suspensions. There is a different way of interpreting group structures appearing in the sequence (✠), which uses suspensions of a space in place of loop spaces.

9.10 Definition. Let X be a space. The *unreduced suspension* of X if the space

$$SX = X \times [0,1]/(X \times \{0,1\})$$

9.11 Note. Any map $f\colon X \to Y$ defines a map $Sf\colon SX \to SY$ given by $Sf([x,t]) = [f(x), t]$. This map is called the suspension of f. In this way we obtain the suspension functor

$$S\colon \mathbf{Top} \to \mathbf{Top}$$

This functor preserves homotopy classes of maps: if $f, g\colon X \to Y$ and $f \simeq g$ then $Sf \simeq Sg$.

9.12 Example. For a sphere S^n we have $SS^n \cong S^{n+1}$.

9.13 Definition. Let (X, x_0) be a pointed space. The *reduced suspension* of X is the pointed space

$$\Sigma X = X \times [0,1]/(X \times \{0,1\} \cup \{x_0\} \times [0,1])$$

or equivalently $\Sigma X = SX/\{[x_0, t] \mid t \in [0,1]\}$. The basepoint in ΣX is given by $[x_0, 0] \in \Sigma X$.

9.14 Note. If (X, x_0) is a well-pointed space, then Proposition 2.15 implies that the quotient map $SX \to \Sigma X$ is a homotopy equivalence. In particular, for any basepoint $x_0 \in S^n$ we have $\Sigma S^n \simeq SS^n \cong S^{n+1}$. One can show that actually there is a homeomorphism $\Sigma S^n \cong S^{n+1}$

9.15 Note. Any map $f\colon (X, x_0) \to (Y, y_0)$ of pointed spaces, defines a map $\Sigma f\colon \Sigma X \to \Sigma Y$ given by $\Sigma f([x,t]) = [f(x), t]$. This defines the suspension functor

$$\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$$

Similarly as for the unreduced suspension, the reduced suspension preserves homotopy classes: if $f, g\colon (X, x_0) \to (Y, y_0)$ are maps of pointed spaces and $f \simeq g$ then $\Sigma f \simeq \Sigma g$.

Let X be a Hausdorff space. By properties of mapping spaces (8.5) the adjunction map $\mathrm{adj}(\omega) = \omega^\sharp$ defines a homeomorphism $\mathrm{adj}\colon \mathrm{Map}(X \times [0,1], Y) \to \mathrm{Map}(X, PY)$. Let (X, x_0) and (Y, y_0) be pointed spaces. Consider $\Omega_{y_0} Y$ as a subspace of PY and let $\mathrm{Map}_*(X, \Omega_{y_0} Y)$ be the subspace of $\mathrm{Map}(X, PY)$ consisting of basepoint preserving maps. Then adj restricts to a homeomorphism between this subspace and the subspace of $\mathrm{Map}(X \times [0,1], Y)$ consisting of all maps $f\colon X \times [0,1] \to Y$ such that $f(X \times \{0,1\} \cup \{x_0\} \times [0,1]) = y_0$. Such maps are in a bijective correspondence with basepoint preserving maps $\Sigma X \to Y$. In this way we obtain a homeomorphism

$$\mathrm{adj}\colon \mathrm{Map}_*(\Sigma X, Y) \xrightarrow{\cong} \mathrm{Map}_*(X, \Omega Y)$$

On the level of homotopy classes of maps this gives a bijection

$$\mathrm{adj}\colon [\Sigma X, Y]_* \xrightarrow{\cong} [X, \Omega Y]_*$$

The set of the right hand side has a group structure induced by concatenation of loops. A group structure on the left hand side can be defined using the pinch map $\Sigma X \to \Sigma X \vee \Sigma X$. In this way the above bijection becomes an isomorphism of groups.

As a result, the exact sequence (✠) can be equivalently written as

$$\ldots \xrightarrow{f_*} [\Sigma^2 Z, Y]_* \xrightarrow{j_* \,\mathrm{adj}} [\Sigma Z, \mathrm{hofib}\, f]_* \xrightarrow{i(f)_*} [\Sigma Z, X]_* \xrightarrow{f_*} [\Sigma Z, Y]_* \xrightarrow{j_* \,\mathrm{adj}} [Z, \mathrm{hofib}\, f]_* \xrightarrow{i(f)_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_*$$

Consider this sequence with $Z = S^0$. Since $\Sigma^n S^0 \cong S^n$ we obtain

$$\ldots \xrightarrow{f_*} [S^2, Y]_* \xrightarrow{j_* \,\mathrm{adj}} [S^1, \mathrm{hofib}\, f]_* \xrightarrow{i(f)_*} [S^1, X]_* \xrightarrow{f_*} [S^1, Y]_* \xrightarrow{j_* \,\mathrm{adj}} [S^0, \mathrm{hofib}\, f]_* \xrightarrow{i(f)_*} [S^0, X]_* \xrightarrow{f_*} [S^0, Y]_*$$

Since $[S^n, Y]_* = \pi_n(Y)$ we recover the long exact sequence from Corollary 8.12.

10 | Cofibrations

10.1 Definition. A map $i\colon A \to X$ has the *homotopy extension property* for a space Y if for any commutative diagram of the form

$$\begin{tikzcd} Y & X \arrow[l, "\bar{f}"'] \arrow[dl, dashed, "\bar{h}"'] \\ PY \arrow[u, "\mathrm{ev}_0"] & A \arrow[l, "h"] \arrow[u, "i"'] \end{tikzcd}$$

there exists a map $\bar{h}\colon X \times [0,1] \to E$ such that $\bar{h}i = \bar{f}$ and $p\bar{h} = h$. Here PY is the path space of Y and $\mathrm{ev}_0\colon PY \to Y$ is the evaluation at 0 map: $\mathrm{ev}_0(\omega) = \omega(0)$.

Equivalently, $i\colon A \to X$ has the homotopy extension property for Y if given any map $\bar{f}\colon X \to Y$ and a homotopy $h^\sharp\colon A \times [0,1] \to Y$ such that $h_0^\sharp = \bar{f}i$ we can find a homotopy $\bar{h}^\sharp\colon X \times [0,1]$, such that $\bar{h}_0^\sharp = \bar{f}$ and $\bar{h}^\sharp(i(a), t) = h^\sharp(a, t)$ for all $(a, t) \in A \times [0,1]$.

In this setting we will say that $\bar{h}^\sharp$ is an extension of $h^\sharp$ beginning at $\bar{f}$.

10.2 Definition. A map $i\colon A \to X$ is a *cofibration* if it has the homotopy extension property for any space Y. In such case we also say that the space $X/i(A)$ is the *cofiber* of i.

10.3 Example. By Theorem 2.14 if (X, A) is a relative CW complex then the inclusion $i\colon A \hookrightarrow X$ is a cofibration.

Recall that the mapping cylinder of a map $f\colon X \to Y$ is the quotient space

$$M_f = (X \times [0,1] \sqcup Y)/\!\sim$$

where $(x, 0) \sim f(x)$ for all $x \in X$. We have a map $s_f\colon M_f \to Y \times [0,1]$ such that $s_f(x, t) = (f(x), t)$ for $(x, t) \in X \times [0,1]$ and $f(y) = (y, 0)$ for $y \in Y$.

10.4 Proposition. *For a map $i\colon A \to X$ the following conditions are equivalent:*

1) *The map i is a cofibration.*
2) *The map i has the homotopy extension property for the space M_i*
3) *There exists a map $r_f\colon X \times [0,1] \to M_i$ such that $r_f s_f = \mathrm{id}_{M_i}$*

Proof. Exercise. □

10.5 Corollary. *If $i\colon A \to X$ is a cofibration then i is an embedding.*

Proof. Exercise. Use condition 3) in Proposition 10.4. □

10.6 Proposition. *Given any map $f\colon X \to Y$ the map $i_f\colon X \to M_f$ given by $i_f(x) = (x, 1)$ is a cofibration.*

Proof. Exercise. □

10.7 Note. Given a map $f\colon X \to Y$, let $d_f\colon M_f \to Y$ be the strong deformation retraction. As a consequence of Proposition 10.6, we have a commutative diagram

$$\begin{tikzcd} Y & & M_f \arrow[ll, "d_f"', "\simeq"] \\ & X \arrow[ul, "f"] \arrow[ur, "i_f"'] & \end{tikzcd}$$

where i_f is a cofibration. A homotopy inverse of d_f is given by the inclusion map $j_f\colon Y \to M_f$.

10.8 Note. Recall that the mapping cone of a map $f\colon X \to Y$ is the space $C_f = M_f/X \times \{1\}$. The space C_f is the cofiber of the cofibration $i_f\colon X \to M_f$.

10.9 Coexact Puppe sequence. The construction of the coexact Puppe sequence of a map is dual to the construction of the exact Puppe sequence given in Chapter 9.

As in Chapter 9 we will be interested here in pointed spaces and homotopy classes of maps that preserve basepoints. In this case we will use a slightly weakened version of a cofibration: a map of pointed spaces $i\colon (A, a_0) \to (X, x_0)$ is a cofibration if has the homotopy extension property for all pointed maps $(X, x_0) \to (Y, y_0)$ and pointed homotopies $A \times [0,1] \to Y$. In this context we modify the constructions of the mapping mapping cylinder and the mapping cone as follows:

10.10 Definition. For a map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ the *reduced mapping cylinder* of f is the space $\overline{M}_f = M_f/\{x_0\} \times [0,1]$. The *reduced mapping cone* is the space $\overline{C}_f = \overline{M}_f/X \times \{1\}$.

The reduced mapping cylinder and mapping cone come with a natural choice of basepoints. As in (10.7) for any map $f\colon (X, x_0) \to (Y, y_0)$ we have a commutative diagram

$$\begin{tikzcd} Y & & \overline{M}_f \arrow[ll, "d_f"', "\simeq"] \\ & X \arrow[ul, "f"] \arrow[ur, "i_f"'] & \end{tikzcd}$$

where i_f is a pointed cofibration and d_f is a pointed homotopy equivalence. Also, $\overline{C}_f$ is the cofiber of i_f.

10.11 Definition. A sequence of maps of spaces

$$(X_0, x_0) \xrightarrow{f_0} (X_1, x_1) \xrightarrow{f_1} (X_2, x_2)$$

is *coexact at* X_1 is for any pointed space (Y, y_0) the sequence pointed sets

$$[X_2, Y]_* \xrightarrow{f_1^*} [X_1, Y]_* \xrightarrow{f_0^*} [X_0, Y]_*$$

is exact at $[X_1, Y]_*$.

10.12 Proposition. *If $i\colon A \to X$ is a cofibration, $q\colon X \to X/i(A)$ is the quotient map, $x_0 \in A$ then the sequence $(A, x_0) \xrightarrow{i} (X, i(x_0)) \xrightarrow{q} (X/A, qi(x_0))$ is coexact at X.*

For any map $f\colon (X, x_0) \to (Y, y_0)$ consider the sequence

$$X \xrightarrow{f} Y \xrightarrow{q(f)} \overline{C}_f$$

where $q(f)(y) = (y, 0)$. Since this sequence is homotopy equivalent to the cofibration sequence $X \xrightarrow{i_f} \overline{M}_f \longrightarrow \overline{C}_f$, is is coexact at Y. Continuing this construction inductively we obtain a coexact sequence

$$X \xrightarrow{f} Y \xrightarrow{q(f)} \overline{C}_f \xrightarrow{q^2(f)} \overline{C}_{q(f)} \xrightarrow{q^3(f)} \overline{C}_{q(f)} \xrightarrow{q^4(f)} \overline{C}_{q^3(f)} \longrightarrow \dots \tag{$*$}$$

As in Chapter 9 our goal will be to show that this sequence admits a more convenient description. This will depend on two facts that dualize Propositon 9.4 and Corollary 8.18

10.13 Proposition. *For any map $f\colon (X, x_0) \to (Y, y_0)$ the map $q(f)\colon X \to \overline{C}_f$ is a cofibration.*

Proof. Exercise. □

10.14 Proposition. *If $f\colon (X, x_0) \to (Y, y_0)$ is a cofibration then the quotient map*

$$\overline{C}_f \to Y/f(X)$$

is a homotopy equivalence.

Proof. Exercise. □

Notice that $\overline{C}_f/q(f) \cong \Sigma X$, where ΣX is the reduced suspension of X. In this way we obtain:

10.15 Proposition. *For any map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "q(f)"] & \overline{C}_f \arrow[r, "q^2(f)"] \arrow[dr, "g"'] & \overline{C}_{q(f)} \arrow[d, "\simeq"] \\ & & & \Sigma X \end{tikzcd}$$

Applying Proposition 10.14 iteratively to the sequence (∗) we get homotopy equivalences

$$\begin{aligned} &\overline{C}_{q(f)} \xrightarrow{\simeq} \Sigma X \\ &\overline{C}_{q^2(f)} \xrightarrow{\simeq} \Sigma Y \\ &\overline{C}_{q^3(f)} \xrightarrow{\simeq} \Sigma \overline{C}_f \\ &\overline{C}_{q^4(f)} \xrightarrow{\simeq} \Sigma \overline{C}_{q(f)} \simeq \Sigma^2 X \\ &\overline{C}_{q^5(f)} \xrightarrow{\simeq} \Sigma \overline{C}_{q^2(f)} \simeq \Sigma^2 Y \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

Moreover, one can check that the following diagram commutes up to homotopy:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "="'] & Y \arrow[r, "q(f)"] \arrow[d, "="'] & \overline{C}_f \arrow[r, "q^2(f)"] \arrow[d, "="'] & \overline{C}_{q(f)} \arrow[r, "q^3(f)"] \arrow[d, "\simeq"'] & \overline{C}_{q^2(f)} \arrow[r, "q^4(f)"] \arrow[d, "\simeq"'] & \overline{C}_{q^3(f)} \arrow[r, "q^5(f)"] \arrow[d, "\simeq"'] & \overline{C}_{q^4(f)} \arrow[r] \arrow[d, "\simeq"'] & \dots \\ X \arrow[r, "f"'] & Y \arrow[r, "q(f)"'] & \overline{C}_f \arrow[r, "g"'] & \Sigma X \arrow[r, "\Sigma f"'] & \Sigma Y \arrow[r, "\Sigma q(f)"'] & \Sigma \overline{C}_f \arrow[r, "\Sigma g"'] & \Sigma^2 X \arrow[r] & \dots \end{tikzcd} \tag{$**$}$$

10.16 Definition. The sequence in the lower row of the diagram (∗∗) is called the *Puppe coexact sequence* associated to the map f.

As a consequence, for any map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ and any pointed space (Z, z_0) we obtain a long exact sequence of sets:

$$[X, Z]_* \xleftarrow{f^*} [Y, Z]_* \xleftarrow{q(f)^*} [\overline{C}_f, Z]_* \xleftarrow{g^*} [\Sigma X, Z]_* \xleftarrow{\Sigma f^*} [\Sigma Y, Z]_* \xleftarrow{\Sigma q(f)^*} [\Sigma \overline{C}_f, Z]_* \xleftarrow{\Sigma g^*} [\Sigma^2 X, Z]_* \longleftarrow \dots \tag{✠}$$

Starting with $[\Sigma X, Z]_*$ the sets in this sequence have a group structure defined by the suspension, and all maps are homomorphisms of groups. Starting with $[\Sigma^2, Z]_*$ all groups are abelian.

10.17 Note. 1) Using the adjunction $\mathrm{adj}\colon [\Sigma X, Y]_* \xrightarrow{\cong} [X, \Omega Y]_*$ as in (9.15) we can rewrite the sequence (✠) in the form

$$[X, Z]_* \xleftarrow{f^*} [Y, Z]_* \xleftarrow{q(f)^*} [\overline{C}_f, Z]_* \xleftarrow{g^*} [X, \Omega Z]_* \xleftarrow{f^*} [Y, \Omega Z]_* \xleftarrow{q(f)^*} [\overline{C}_f, \Omega Z]_* \xleftarrow{g^*} [X, \Omega^2 Z]_* \longleftarrow \dots$$

In this setting, groups structures are induced the multiplication in loop spaces.

2) Assume that the map $f\colon (X, x_0) \to (Y, y_0)$ is a cofibration. Using Corollary 10.5 we can then assume that X is a subspace of Y and that f is the inclusion map. By Proposition 10.14 we have $\overline{C}_f \simeq Y/X$, so the above sequence can be written as

$$[X, Z]_* \xleftarrow{f^*} [Y, Z]_* \xleftarrow{q^*} [Y/X, Z]_* \xleftarrow{g^*} [X, \Omega Z]_* \xleftarrow{f^*} [Y, \Omega Z]_* \xleftarrow{q^*} [Y/X, \Omega Z]_* \xleftarrow{g^*} [X, \Omega^2 Z]_* \longleftarrow \dots$$

where $q\colon Y \to Y/X$ is the quotient map.

11 | Excision

One of the main properties of homology groups is excision. It can stated as follows:

11.1 Theorem. *Let X be a space and $X_1, X_2 \subseteq X$ be open sets such that $X = X_1 \cup X_2$. Then the map of pairs $i\colon (X_1, X_1 \cap X_2) \to (X, X_2)$ induces an isomorphism*

$$i_*\colon H_n(X_1, X_1 \cap X_2) \xrightarrow{\cong} H_n(X, X_2)$$

for all $n \geq 0$.

The same property does not holds in general for homotopy groups. However, it does hold under some extra assumptions. In order to make this precise we will need a definition.

11.2 Definition. Let $A \subseteq X$ and let $0 \leq n \leq \infty$. The pair (X, A) is *n-connected* if the map $\pi_0(A) \to \pi_0(X)$ is onto and $\pi_k(X, A, x_0) = \{1\}$ for all $x_0 \in A$ and all $1 \leq k \leq n$.

11.3 Proposition. *Let $A \subseteq X$. The following conditions are equivalent.*

1) *(X, A) is n-connected.*
2) *The homomorphism $i_*\colon \pi_k(A, x_0) \to \pi_k(X, x_0)$ induced by the inclusion map $i\colon A \hookrightarrow X$ is an isomorphism for all $x_0 \in A$ and all $k < n$ and it is an epimorphism for $k = n$.*
3) *For $k \leq n$, any map $(I^k, \partial I^k) \to (X, A)$ is homotopic relative to ∂I^k to a map $I^k \to A$.*
4) *For $k \leq n$, any map $h\colon I^k \cup (\partial I^k \times I) \to X$ such that $h(\partial I^k \times \{1\}) \subseteq A$ can be extended to a map $\bar{h}\colon I^k \times I \to X$ such that $h(I^k \times \{1\}) \subseteq A$.*

Proof. Exercise. □

11.4 Excision Theorem. *Let X be a space and $X_1, X_2 \subseteq X$ be open such that $X = X_1 \cup X_2$. Assume that*

- *$(X_1, X_1 \cap X_2)$ is m-connected*

- *$(X_2, X_1 \cap X_2)$ is n-connected*

for some $m, n \geq 0$. Then for any $x_0 \in X_1 \cap X_2$ the homomorphism

$$i_* \colon \pi_k(X_1, X_1 \cap X_2, x_0) \to \pi_k(X, X_2, x_0)$$

induced by the inclusion map, is an isomorphism for $1 \leq k < m + n$ and it is onto for $k = m + n$.

In this chapter we will explore some consequences Theorem 11.4, an we will return to its proof in Chapter 13.

11.5 Proposition. *Let (X, A) be a pair with the homotopy extension property and let $q \colon X \to X/A$ be the quotient map. Let $x_0 \in A$ and $* = q(A) \in X/A$. If (X, A) is m-connected and the space A is n-connected for some $m, n \geq 0$ then the homomorphism*

$$q_* \colon \pi_k(X, A, x_0) \to \pi_k(X/A, *, *) = \pi_k(X/A, *)$$

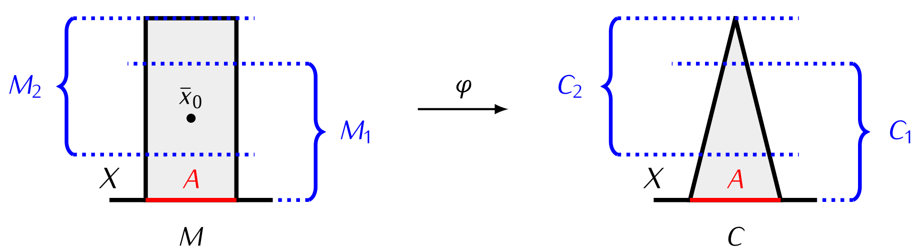
is an isomorphism for $k \leq m + n$ and it is an epimorphism for $k = m + n + 1$.

Proof. Let $j \colon A \hookrightarrow X$ be the inclusion map. Let M denote the the mapping cylinder of j:

$$M = (A \times [0, 1] \sqcup X)/{\sim}$$

where $(x, 0) \sim x$ for all $x \in A$. Also, let $C = M/(A \times \{1\})$ be the mapping cone of j. In other words, C is obtained by attaching the cone $CA = A \times [0, 1]/(A \times \{1\})$ to X.

Take the quotient map $\varphi \colon M \to C$. Denote by $M_1, M_2 \subseteq M$ the subspaces of M given by $M_1 = X \cup A \times [0, \frac{3}{4}]$ and $M_2 = A \times [\frac{1}{4}, 1]$, and let $C_i = \varphi(M_i)$ for $i = 1, 2$. Also, let $\bar{x}_0 = (x_0, \frac{1}{2}) \in M_1 \cap M_2$.



Let $r \colon M \to X$ be the retraction map, and let $s \colon C \to X/A$ be the map that sends the cone $CA \subseteq C$ to the point $* \in X/A$. Both r and s are homotopy equivalences. For s this follows from Proposition 2.15 using the fact that since (X, A) has the homotopy extension property, then (C, CA) also has this property.

For any $k \geq 1$ the following diagram commutes:

$$\begin{tikzcd}
\pi_k(X, A, x_0) \arrow[r, "q_*"] & \pi_k(X/A, *, *) \\
\pi_k(M, M_2, \bar{x}_0) \arrow[r, "\varphi_*"] \arrow[u, "r_*", "\cong"'] & \pi_k(C, C_2, \varphi(\bar{x}_0)) \arrow[u, "\cong", "s_*"'] \\
\pi_k(M_1, M_1 \cap M_2, \bar{x}_0) \arrow[r, "\varphi|_{M_1 *}", "\cong"'] \arrow[u, "i_*", "\cong"'] & \pi_k(C_1, C_1 \cap C_2, \varphi(\bar{x}_0)) \arrow[u, "i'_*"']
\end{tikzcd}$$

Here $\varphi|_{M_1}$ is the restriction of φ and the homomorphisms i_*, i'_* are induced by inclusions. Since $i\colon (M_1, M_1 \cap M_2) \to (M_j, M_1 \cap M_2)$ is a homotopy equivalence and $\varphi|_{M_1}\colon (M_1, M_1 \cap M_2) \to (C_1, C_1 \cap C_2)$ is a homeomorphism, i_* and $\varphi|_{M_1 *}$ are isomorphisms. It follows that q_* is an isomorphism or epimorphism if and only if i'_* has the same property.

From the above diagram we also obtain that $\pi_k(C_1, C_1 \cap C_2, \varphi(\bar{x}_0)) \cong \pi_k(X, A, x_0)$ for all k, so $(C_1, C_1 \cap C_2)$ is m-connected. Also, since C_2 is a contractible space, from the long exact sequence of the pair $(C_2, C_1 \cap C_2)$ we get

$$\pi_k(C_2, C_1 \cap C_2, \varphi(\bar{x}_0)) \cong \pi_{k-1}(C_1 \cap C_2, \varphi(\bar{x}_0)) \cong \pi_{k-1}(A, x_0)$$

Since by assumption A is n-connected, thus $(C_2, C_1 \cap C_2)$ is $(n+1)$-connected. By the Excision Theorem 11.4 we obtain that i'_* (and thus also q_*) is an isomorphism for $k \leq m + n$ and an epimorphism for $k = m + n + 1$. □

Let (X, x_0) be a pointed space and let $\omega\colon (I^n, \partial I^n) \to (X, x_0)$ represent an element $[\omega] \in \pi_n(X, x_0)$. Let ΣX be the reduced suspension of X. Consider the map $\Sigma'\omega\colon I^{n+1} \to \Sigma X$ obtained the composition

$$\Sigma'\omega\colon I^{n+1} = I^n \times [0, 1] \xrightarrow{q} \Sigma I^n \xrightarrow{\Sigma\omega} \Sigma X$$

where q is the quotient map. One can check that $\Sigma'\omega$ represents an element of $\pi_{n+1}(\Sigma X, \bar{x}_0)$.

11.6 Definition/Proposition. The assignment $[\omega] \mapsto [\Sigma'\omega]$ defines a homomorphism of groups

$$\Sigma_*\colon \pi_n(X, x_0) \to \pi_{n+1}(\Sigma X, \bar{x}_0)$$

which is called the *suspension homomorphism*.

Proof. The function Σ_* is well defined since the suspension functor preserves homotopy classes of maps. It remains to check that Σ_* is a group homomorphism (exercise). □

11.7 Freudenthal Suspension Theorem. *Let (X, x_0) be a well-pointed, n-connected space. Let $\bar{x}_0$ denote the basepoint in the reduced suspension ΣX. The suspension homomorphism*

$$\Sigma_*\colon \pi_k(X, x_0) \to \pi_{k+1}(\Sigma X, \bar{x}_0)$$

is an isomorphism for $k \leq 2n$ and it is an epimorphism for $k = 2n + 1$.

Proof. First, let $CX = X \times [0,1]/X \times \{1\}$ be the cone on X. Identifying X with $X \times \{0\}$ we can consider it as a subspace of CX. Since CX is a contractible space, in the long exact sequence of the pair (CX, X) the homomorphism $\partial\colon \pi_{k+1}(CX, X, x_0) \to \pi_k(X, x_0)$ is an isomorphism for all $k \geq 0$.

One can check (exercise) that if (X, x_0) is a well-pointed space, then for any $k \geq 0$ the following diagram commutes:

$$
\begin{tikzcd}
\pi_k(X, x_0) \arrow[r, "\Sigma_*"] & \pi_{k+1}(\Sigma X, \bar{x}_0) \\
\pi_{k+1}(CX, X, x_0) \arrow[u, "\partial"', "\cong"] \arrow[r, "q_*"'] & \pi_{k+1}(CX/X, \bar{x}_0) \arrow[u, "\cong"', "q'_*"]
\end{tikzcd}
$$

Here q_* and q'_* are induced by the quotient maps $q\colon CX \to CX/X$ and $q'\colon CX/X = SX \to \Sigma X$.

Since (X, x_0) is well-pointed, the map q' is a homotopy equivalence, and thus q'_* is an isomorphism. It follows that Σ_* is an isomorphism or epimorphism if and only if this holds for q_*. Since X is n-connected and CX is contractible, the pair (CX, X) is $n+1$-connected. Therefore, by Proposition 11.5, q_* is an isomorphism for $k + 1 \leq 2n + 1$ (or $k \leq 2n$) and an epimorphism for $k + 1 = 2n + 2$ (i.e. $k = 2n + 1$) $\square$

Since the sphere S^n is $(n-1)$-connected, by Theorem 11.7 we obtain:

11.8 Corollary. *The suspension homomorphism*

$$\Sigma_*\colon \pi_k(S^n) \to \pi_{k+1}(\Sigma S^n) \cong \pi_{k+1}(S^{n+1})$$

is an isomorphism for $k \leq 2n - 2$ and an epimorphism for $k = 2n - 1$.

11.9 Corollary. *For any $n \geq 1$ we have $\pi_n(S^n) \cong \mathbb{Z}$.*

Proof. We argue by induction with respect to n. We already know that $\pi_1(S^1) \cong \mathbb{Z}$. Also, by Theorem 7.23 we have $\pi_2(S^2) \cong \mathbb{Z}$.

Next, assume that $\pi_n(S^n) \cong \mathbb{Z}$ for some $n \geq 2$. In such case $2n - 2 \geq n$, so by Corollary 11.8 we obtain $\mathbb{Z} \cong \pi_n(S^n) \cong \pi_{n+1}(S^{n+1})$. $\square$

11.10 Note. 1) By Corollary 11.8 the suspension homomorphism $\Sigma_*\colon \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism for all $n \geq 2$. By the same corollary $\Sigma_*\colon \pi_1(S^1) \to \pi_2(S^2)$ is onto, and since every epimorphism $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism, it follows that this is an isomorphism as well.

2) The generator of the group $\pi_n(S^n)$ is represented by the identity map $\mathrm{id}\colon S^n \to S^n$. For $n = 1$ it follows from the direct computation of $\pi_1(S^1)$, and for $n > 1$ it holds since the suspension isomorphism maps the homotopy class of $\mathrm{id}_{S^{n-1}}$ to the homotopy class of id_{S^n}

11.11 Corollary. *$\pi_3(S^2) \cong \mathbb{Z}$ and the generator of $\pi_3(S^2)$ is given by the homotopy class of the Hopf bundle map (7.22).*

Proof. The long exact sequence of the Hopf fibration $S^1 \to S^3 \xrightarrow{p} S^2$ gives an exact sequence:

$$0 = \pi_3(S^1) \longrightarrow \pi_3(S^3) \xrightarrow{p_*} \pi_3(S^2) \xrightarrow{\partial} \pi_2(S^1) = 0$$

Therefore p_* is an isomorphism and so $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$. Also, since $[\mathrm{id}_{S^3}]$ is a generator of $\pi_3(S^3)$, thus $p_*([\mathrm{id}_{S^3}]) = [p]$ is a generator of $\pi_3(S^2)$. □

11.12 Note. Notice that since $\pi_2(S^1) = 0$, the suspension homomorphism $\Sigma_*\colon \pi_2(S^1) \to \pi_3(S^2)$ is not an isomorphism.

11.13 Corollary. *For $n \geq 1$ the group $\pi_{n+1}(S^n)$ is cyclic.*

Proof. We have $\pi_2(S^1) = 0$ and $\pi_3(S^2) \cong \mathbb{Z}$. By Corollary 11.8 the suspension homomorphism $\mathbb{Z} \cong \pi_3(S^2) \to \pi_4(S^3)$ is onto, so $\pi_4(S^3)$ is a cyclic group. By the same corollary we have $\pi_{n+1}(S^n) \cong \pi_{n+2}(S^{n+1})$ for all $n \geq 3$. □

12 | All Groups Are Homotopy Groups

Recall that van Kampen's Theorem implies that for any group G we can find a space X such that $\pi_1(X) \cong G$. The goal of this chapter is to extend this result to higher homotopy groups. Since all groups $\pi_n(X)$ with $n \geq 2$ are abelian (3.4), we will show that the following holds:

12.1 Theorem. *For any abelian group G and any $n \geq 2$ there exists a space X such that $\pi_n(X) \cong G$. Moreover, such space X can be constructed in such way, that X is a CW complex and $X^{(n-1)} = *$.*

For every abelian group G there exists an epimorphism $\varphi\colon \bigoplus_{i\in I} \mathbb{Z} \to G$ for some set I. Indeed, it is enough to take $I = G$, the set of elements of the group G. Then we can define φ by $\varphi(e_g) = g$, where e_g is the generator of the copy of $\mathbb{Z} \subseteq \bigoplus_{h\in G} \mathbb{Z}$ indexed by g. Given such a homomorphism φ we get $G \cong \bigoplus_{i\in I} \mathbb{Z}/\ker(\varphi)$.

Based on this, in order to prove Theorem 12.1 it will suffice to show that:

1) for any set I and $n \geq 2$ there exists a space X such that $\pi_n(X) \cong \bigoplus_{i\in I} \mathbb{Z}$.
2) for any subgoup $H \subseteq \bigoplus_{i\in I} \mathbb{Z}$ and any $n \geq 2$ there exists a space X such that $\pi_n(X) \cong \bigoplus_{i\in I} \mathbb{Z}/H$.

12.2 Lemma. *Let $\{(X_i, \bar{x}_i)\}_{i\in I}$ be a family of pointed Hasdorff spaces. Let $X = \bigvee_{i\in I} X_i$, and let $* \in X$ denote the basepoint. For $k \in I$ let $r_k\colon X \to X_k$ be the retraction map. Then for any $n \geq 2$ we an epimorphism $\varphi\colon \pi_n(X, *) \to \bigoplus_{i\in I} \pi_n(X_i, \bar{x}_i)$ given by $\varphi([\omega]) = \sum_{i\in I} r_{i*}([\omega])$.*

Proof. For each $k \in I$ let $j_k\colon X_k \to X$ be the inclusion map. We have a homomorphism

$$\psi := \bigoplus_{i\in I} j_{i*}\colon \bigoplus_{i\in I} \pi_n(X_i, \bar{x}_i) \to \pi_n(X, *)$$

The retractions r_i define a map

$$\varphi := \prod_{i\in I} r_{i*}\colon \pi_n(X, *) \to \prod_{i\in I} \pi_n(X_i, \bar{x}_i)$$

We claim that $\mathrm{Im}(\varphi) \subseteq \bigoplus_{i\in I} \pi_n(X_i, \bar{x}_i) \subseteq \prod_{i\in I} \pi_n(X_i, \bar{x}_i)$. Indeed, if $\omega\colon (I^n, \partial I^n) \to (X, *)$ is a map representing an element $[\omega] \in \pi_n(X, *)$, then, by compactness of I^n, we have $\omega(I^n) \cap X_i \neq *$ for finitely many $i \in I$ only, and so $r_{i*}([\omega]) \neq 0$ for finitely many $i \in I$. Thus $\varphi([\omega]) \subseteq \bigoplus_{i\in I} \pi_n(X_i, \bar{x}_i)$. It follows that we can consider φ as a homomorphism $\pi_n(X, *) \to \bigoplus_{i\in I} \pi_n(X_i, \bar{x}_i)$.

Since $r_i j_i = \mathrm{id}_{X_i}$ for all $i \in I$, and $r_{i'} j_i$ is the constant map for all $i \neq i'$, it follows that $\varphi\psi$ is the identity homomorphism, and so φ is onto. □

12.3 Note. In general, the epimorphism φ in Lemma 12.2 is not an isomorphism. For example, recall (5.11) that for $n \geq 2$ we have $\pi_n(S^1 \vee S^n) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^n)$. By Lemma 12.2 we get an epimorphism

$$\pi_n(S^1 \vee S^n) \cong \pi_n(\textstyle\bigvee_{i\in\mathbb{Z}} S^n) \to \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$$

which shows that the group $\pi_n(S^1 \vee S^n)$ is not finitely generated. Therefore $\pi_n(S^1 \vee S^n) \ncong \pi_n(S^1) \oplus \pi_n(S^n) \cong \mathbb{Z}$.

12.4 Proposition. *Let $\{(X_i, \bar{x}_i)\}_{i\in I}$ be a family of pointed CW-complexes. Given $n \geq 1$, assume that each complex X_i is n-connected. Then the homomorphism $\varphi\colon \pi_m(\bigvee_{i\in I} X_i, *) \to \bigoplus_{i\in I} \pi_m(X_i, \bar{x}_i)$ is an isomorphism for $m \leq 2n$.*

Proof. For each CW complex X_i we can assume that $\bar{x}_i$ is a 0-cell of X_i. Also, by Proposition 5.6 we can assume that X_i has no other 0-cells, and no k-cells for $k \leq n$ [1].

By Proposition 12.4 φ is onto. It will suffice to show that $\ker\varphi = 0$ for $m \leq 2n$.

Assume first, that the set I is finite, so $\bigvee_{i\in I} X_i = X_1 \vee \dots \vee X_k$ for some $k \geq 0$. Take the product $X_1 \times \ldots \times X_k$. The inclusion maps $\psi_j\colon X_j \to X_1 \times \ldots \times X_k$ given by $\psi_j(x) = (\bar{x}_1, \ldots, \bar{x}_{j-1}, x, \bar{x}_{j+1}, \ldots \bar{x}_k)$ define an embedding $\psi\colon X_1 \vee \ldots \vee X_k \to X_1 \times \ldots \times X_k$. This gives a commutative diagram:

$$\begin{tikzcd}
\pi_m(X_1 \vee \ldots \vee X_k) \arrow[r, "\psi_*"] \arrow[d, "\varphi"'] & \pi_m(X_1 \times \ldots \times X_k) \arrow[d, "\cong"] \\
\bigoplus_{j=1}^k \pi_{k+1}(X_i) \arrow[r, "="] & \prod_{j=1}^k \pi_{k+1}(X_i)
\end{tikzcd}$$

This shows that φ is a monomorphism if any only if ψ_* is one. If $X_1, \ldots, X_k$ are finite CW complexes, then the space $X_1 \times \cdots \times X_k$ also has the structure of a CW complex, with cells given by products $e_1 \times \cdots \times e_k$ where e_i is a cell in X_i. All cells of $X_1 \times \cdots \times X_k$ that are not contained in $X_1 \vee \cdots \vee X_k$ have dimension $2n+2$ or higher, so $X_1 \vee \cdots \vee X_k$ is the $(2n+1)$-skeleton of $X_1 \times \cdots \times X_k$. Thus, by Proposition 5.2, ψ_* is an isomorphism for all $m \leq 2n$.

Next, assume that the set I is infinite, and let $\omega\colon (I^m, \partial I^m) \to (\bigvee_{i\in I} X_i, *)$ be a map such that $\varphi([\omega]) = 0$. By compactness of I^m we have $\omega(I^m) \cap X_i \neq *$ for finitely many $i \in I$ only. Thus we can consider ω as

[1] This uses the fact that if $X_i \simeq X_i'$ for all $i \in I$ then $\bigvee_{i\in I} X_i \simeq \bigvee X_{i\in I}'$. This holds for well-pointed, path connected spaces.

a map $\omega\colon (I^m, \partial I^m) \to (X_{i_1} \vee \ldots \vee X_{i_k}, *)$ for some $i_1, \ldots, i_k \in I$. Since $\varphi([\omega]) = 0$, the homomorphism $\pi_m(X_{i_1} \vee \ldots \vee X_{i_k}) \to \bigoplus_{j=1}^{k} \pi_{k+1}(X_{i_j})$ also maps $[\omega]$ to 0. By the finite case this means that $[\omega] = 0$. $\square$

12.5 Note. The proof of Proposition 12.4 uses the fact that if X and Y are CW complexes, then $X \times Y$ has the structure of a CW complex with cells given by products of cells in X and Y. An issue with this statement is that the topology induced on $X \times Y$ by this cell structure (where a set $U \subseteq X \times Y$ is open if and only if its intersection with each cell is an open subset of the cell) need not be the same as the product topology on $X \times Y$. The topology induced by the cell structure on $X \times Y$ is called the compactly generated topology. Let $X \times_{cg} Y$ denote the product taken with this topology, and let $X \times Y$ denote the product taken with the product topology. Every open set in $X \times Y$ is also open in $X \times_{cg} Y$, so the identity map $\mathrm{id}\colon X \times_{cg} Y \to X \times Y$ is continuous. Moreover, this map induces an isomorphism of homotopy groups $\pi_n(X \times_{cg} Y) \xrightarrow{\cong} \pi_n(X \times Y)$ for all n. For this reason this change of topology does not affect the proof of Proposition 12.4.

12.6 Corollary. *For any set I and any $n \geq 2$ we have an isomorphism*

$$\pi_n(\textstyle\bigvee_{i\in I} S^n) \cong \bigoplus_{i \in I} \mathbb{Z}$$

Moreover, the group $\pi_n(\bigvee_{i\in I} S^n)$ is generated by elements $[j_k]$ for $k \in I$ where $j_k\colon S^n \hookrightarrow \bigvee_{i\in I} S^n$ is the inclusion of the k-th copy of S^n.

12.7 Proposition. *Let (X, x_0) be a simply connected space, and let $\varphi_i\colon (S^n, s_0) \to (X, x_0)$ be maps representing elements of $\pi_n(X, x_0)$ for some $n \geq 2$. Consider the space $Y = X \cup \bigcup_i e_i^{n+1}$ obtained by attaching $(n+1)$-cells to X using φ_i as the attaching maps. If $j\colon X \hookrightarrow Y$ is the inclusion map, then the induced homomorphism*

$$j_*\colon \pi_k(X, x_0) \to \pi_k(Y, x_0)$$

is an isomorphism for $k < n$ and an epimorphism for $k = n$. Moreover, $\ker(j_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0))$ *is the subgroup of $\pi_n(X, x_0)$ generated by the elements $[\varphi_i]$.*

Proof. We can consider the pair (Y, X) as a relative CW complex with the n-skeleton given by X. Then j_* is an isomorphism for $k < n$ and epimorphism for $k = n$ by Proposition 5.2. Notice that by Proposition 11.3 this is equivalent to saying that the pair (X, Y) is n-connected.

It remains to verify the statement about the kernel of j_* for $k = n$. Consider the exact sequence of the pair (Y, X):

$$\cdots \to \pi_{n+1}(Y, X) \xrightarrow{\partial} \pi_n(X) \xrightarrow{j_*} \pi_n(Y) \to \pi_n(Y, X) \to \ldots$$

We have $\ker j_* = \operatorname{Im} \partial$. Since the pair (X, Y) is n-connected and, by assumption, the space X is 1-connected, from Theorem 11.5 we obtain that the quotient map $q\colon Y \to Y/X$ induces an isomorphism

$$q_*\colon \pi_{n+1}(Y, X) \xrightarrow{\cong} \pi_{n+1}(Y/X) \cong \pi_{n+1}(\textstyle\bigvee_i S^{n+1}) \cong \oplus_i \mathbb{Z}$$

This implies that $\pi_{n+1}(Y, X)$ is generated by homotopy classes of maps $\overline{\varphi}_i\colon D^{n+1} \to Y$ which are the characteristic maps of the cells e_i^{n+1}. The boundary homorphism is given by $\partial[\overline{\varphi}_i] = [\varphi_i]$. Therefore $\operatorname{Im} \partial = \ker j_*$ is the subgroup of $\pi_n(X)$ generated by the elements $[\varphi_i]$. $\square$

Proposition 12.7 can be generalized to non-simply connected spaces as follows. Recall (4.14) that higher homotopy groups admit the action of the fundamental group:

$$\begin{array}{rcl} \pi_1(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([\tau]\ ,\ [\omega]) & \mapsto & [\tau] \odot [\omega] \end{array}$$

We have:

12.8 Proposition. *Let (X, x_0) be a space which is connected, locally path connected, and semi-locally simply connected. Let $\varphi_i \colon (S^n, s_0) \to (X, x_0)$ be maps representing elements of $\pi_n(X, x_0)$ for some $n \geq 2$. Consider the space $Y = X \cup \bigcup_i e_i^{n+1}$ obtained by attaching $(n+1)$-cells to X using φ_i as the attaching maps. If $j \colon X \hookrightarrow Y$ is the inclusion map, then the induced homomorphism*

$$j_* \colon \pi_k(X, x_0) \to \pi_k(Y, x_0)$$

is an isomorphism for $k \leq n$ and an epimorphism for $k = n$. Moreover, $\ker(j_* \colon \pi_n(X, x_0) \to \pi_n(Y, x_0))$ *is the subgroup of $\pi_n(X, x_0)$ generated by the elements $[\omega] \odot [\varphi_i]$ for all $[\omega] \in \pi_1(X, x_0)$.*

Proof. The only non-trivial part is the statement about $\ker j_*$. The conditions on the space X guarantee that it has a universal covering $p_X \colon \widetilde{X} \to X$. Let $p_X^{-1}(x_0) = \{\widetilde{x}_k\}_{k \in K}$ and let $\widetilde{\varphi}_{i,k} \colon S^n \to \widetilde{X}$ denote the lift of φ_i such that $\widetilde{\varphi}_{i,k}(s_0) = \widetilde{x}_k$. Let $\widetilde{Y} = \widetilde{X} \cup \bigcup_{i,j} e_{i,k}^{n+1}$ be the space obtained by attaching $(n+1)$-cells to $\widetilde{X}$ using $\varphi_{i,k}$ as attaching maps. The natural map $p_Y \colon \widetilde{Y} \to Y$ is a universal covering of Y. We get a commutative diagram:

$$\begin{tikzcd} \pi_n(\widetilde{X}, \widetilde{x}_0) \arrow[r, "\tilde{j}_*"] \arrow[d, "p_{X*}"', "\cong"] & \pi_n(\widetilde{Y}, \widetilde{x}_0) \arrow[d, "\cong"', "p_{Y*}"] \\ \pi_n(X, x_0) \arrow[r, "j_*"'] & \pi_n(Y, x_0) \end{tikzcd}$$

where $\tilde{j} \colon \widetilde{X} \to \widetilde{Y}$ is the inclusion and $\widetilde{x}_0 \in p_X^{-1}(x_0)$. Since p_{X*} and p_{Y*} are isomorphisms (5.9), we obtain that $\ker j_* = p_{X*}(\ker \tilde{j}_*)$.

For each $\widetilde{x}_k \in p^{-1}(x_0)$ let $\widetilde{\omega}_k$ be a path in $\widetilde{X}$ such that $\widetilde{\omega}_k(0) = \widetilde{x}_0$ and $\widetilde{\omega}_k(1) = \widetilde{x}_k$. Then for each $[\omega] \in \pi_1(X, x_0)$ we have $[\omega] = [p_X \widetilde{\omega}_k]$ for some k. Let $s_k \colon \pi_n(\widetilde{X}, \widetilde{x}_k) \to \pi_n(\widetilde{X}, \widetilde{x}_0)$ be the change of the basepoint isomorphism defined by $\widetilde{\omega}_k$ (4.4). Since $\widetilde{X}$ is simply connected, using Proposition 12.7 we obtain that $\ker \tilde{j}_*$ is generated by the elements $s_k[\widetilde{\varphi}_{i,k}]$ for all i, k. Thus $\ker j_*$ is generated by elements $p_{X*} s_k[\widetilde{\varphi}_{i,k}]$. It remains to notice that $p_{X*} s_k[\widetilde{\varphi}_{i,k}] = [p_X \omega_k] \odot [p_X \widetilde{\varphi}_{i,k}] = [p_X \omega_k] \odot [\varphi_i]$ (exercise).

□

Proof of Theorem 12.1. Given an abelian group G and $n \geq 2$, we can find a set I and an epimorphism

$$\Phi \colon \pi_n(\bigvee_{i \in I} S^n) \cong \bigoplus_{i \in I} \mathbb{Z} \to G$$

Let $\ker\Phi = \{[\varphi_k\colon S^n \to \bigvee_{i\in I} S^n]\}_{k\in K}$, and let X be the space obtained by attaching $(n+1)$-cells to $\bigvee_{i\in} S^n$ using the maps φ_i. By Proposition 12.7 we obtain $\pi_n(X) \cong \pi_n(\bigvee_{i\in I} S^n)/\ker\Phi \cong G$. □

12.9 Definition. Given a group G and an integer $n \geq 1$, an *Eilenberg-MacLane space* of the type $K(G, n)$ is a path connnected CW complex X such that

$$\pi_i(X) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

12.10 Note. Eilenberg-MacLane spaces are not uniquely defined, but as we will see later (14.10), they are unique up to homotopy equivalence. By abuse of notation we will write $X = K(G, n)$ to indicate that X has the type of $K(G, n)$.

12.11 Example. $S^1 = K(\mathbb{Z}, 1)$.

12.12 Example. Recall that the n-dimensional real projective space $\mathbb{RP}^n$ is the quotient space of S^n obtained by identifying antipodal points: $\mathbb{RP}^n = S^n/\sim$ where $x \sim -x$ for all $x \in S^n$. The quotient map $q\colon S^n \to \mathbb{RP}^n$ is the 2-fold universal cover of $\mathbb{RP}^n$. It follows that

$$\pi_i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}/2 & \text{if } i = 1 \\ \pi_i(S^n) & i \geq 2 \end{cases}$$

Embeddings of spheres $S^1 \hookrightarrow S^2 \hookrightarrow \dots$ induce embeddings of projective spaces $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2 \hookrightarrow \dots$. Take $S^\infty = \bigcup_{n=1}^\infty S^n$ and $\mathbb{RP}^\infty = \bigcup_{n=1}^\infty \mathbb{RP}^n$. The quotient map $q\colon S^\infty \to \mathbb{RP}^\infty$ is a 2-fold universal covering of $\mathbb{RP}^\infty$. Since S^∞ is a contractible space (2.18), we obtain

$$\pi_i(\mathbb{RP}^\infty) \cong \begin{cases} \mathbb{Z}/2 & \text{if } i = 1 \\ 0 & i\text{if } \geq 2 \end{cases}$$

Therefore $\mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$.

12.13 Example. Recall (7.21) that for a complex projective space the quotient map $p\colon S^{2n+1} \to \mathbb{CP}^n$ is a Serre fibration with the fiber S^1. The long exact sequence of this fibration gives

$$\pi_i(\mathbb{CP}^n) \cong \begin{cases} 0 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ \pi_i(S^{2n+1}) & \text{if } i \geq 3 \end{cases}$$

The embedding maps $S^3 \hookrightarrow S^5 \hookrightarrow \dots$ induce embeddings $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \hookrightarrow \dots$. We again have $S^\infty = \bigcup_{n=1}^\infty S^{2n+1}$. Also, define $\mathbb{CP}^\infty = \bigcup_{n=1}^\infty \mathbb{CP}^n$. The map $p\colon S^\infty \to \mathbb{CP}^\infty$ is again a Serre fibration with fiber S^1. Since S^∞ is contractible, the long exact sequence of this fibration gives

$$\pi_i(\mathbb{CP}^\infty) \cong \begin{cases} 0 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i \geq 3 \end{cases}$$

Thus $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

12.14 Proposition. *For any $n \geq 1$ and any group G (abelian if $n \geq 2$) there exists an Eilenberg-MacLane space $K(G, n)$. Moreover, it is possible to construct such space so that $K(G, n)^{(n-1)} = *$.*

Proof. By Theorem 12.1, if $n \geq 2$ then we can find a path connected CW complex (X_n, x_0) such that $X_n^{(n-1)} = *$ and

$$\pi_i(X_n, x_0) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

For $n = 1$ such CW complex can be constructed using van Kampen's theorem. Let X_{n+1} be the space obtained by attaching an $(n + 2)$-cells to X_n using all possible maps $(S^{n+1}, s_0) \to (X_n, x_0)$. Then $X_n \subseteq X_{n+1}$, and using Proposition 5.2 we obtain

$$\pi_i(X_{n+1}, x_0) \cong \begin{cases} 0 & \text{if } i = n+1 \\ G & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

In the same way, for any $m > n + 1$ we can can inductively construct a space X_m such that X_m is obtained by attaching $(m + 1)$-cells to X_{m-1} and

$$\pi_i(X_m, x_0) \cong \begin{cases} 0 & \text{if } n < i \leq m \\ G & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

Then we can take $K(G, n) = \bigcup_{m=n}^\infty X_m$. □

12.15 Corollary. *For any sequence of groups $G_1, G_2, \ldots$ such that G_i is abelian for $i \geq 2$, there exists a path connected CW complex X such that $\pi_i(X) \cong G_i$ for all $i \geq 1$.*

Proof. Take $X = \prod_{i=1}^\infty K(G_i, i)$. □

13 | Proof of the Excision Theorem

Based on Tammo tom Dieck, *Algebraic Topology* sec. 6.9.

The goal of this section is to give a proof a the Excision Theorem. For reference, we bring up again its statement:

11.4 **Excision Theorem.** *Let X be a space and $X_1, X_2 \subseteq X$ be open such that $X = X_1 \cup X_2$. Assume that*

- *$(X_1, X_1 \cap X_2)$ is m-connected*
- *$(X_2, X_1 \cap X_2)$ is n-connected*

for some $m, n \geq 0$. Then for any $x_0 \in X_1 \cap X_2$ the homomorphism

$$i_*\colon \pi_k(X_1, X_1 \cap X_2, x_0) \to \pi_k(X, X_2, x_0)$$

induced by the inclusion map, is an isomorphism for $1 \leq k < m + n$ and it is onto for $k = m + n$.

13.1 Cubical subdivisions. The proof of Theorem 11.4 will involve working with certain subdivisions of cubes I^n. Here we set some terminology and notation related to such subdivisions.

Let $N \geq 1$ be some fixed integer. For $j = 0, \dots, N$ denote $c_j = \frac{j}{N}$. Also, let $\delta = \frac{1}{N}$. The numbers c_j define a subdivision of the interval $I = [0, 1]$ into subintervals $[c_j, c_{j+1}] = [c_j, c_j + \delta]$. More generally, an n-dimensional cube I^n has a subdivision into subcubes of the form

$$\begin{aligned} C_{j_1,\dots,j_n} &= [c_{j_1}, c_{j_1+1}] \times [c_{j_2}, c_{j_j+1}] \\ &= [c_{j_1}, c_{j_1} + \delta] \times [c_{j_2}, c_{j_2} + \delta] \times \ldots \times [c_{j_n}, c_{j_n} + \delta] \end{aligned}$$

for some $0 \leq j_1, \dots, j_n \leq N - 1$. We will call this the N-cubical subdivision of I^n. This subdivision defines a CW complex structure on I^n. An m-dimensional cell in I^n is an m-dimensional subcube

$$C = [c_{j_1}, c_{j_1} + \epsilon_1] \times [c_{j_2}, c_{j_2} + \epsilon_2] \times \ldots \times [c_{j_n}, c_{j_n} + \epsilon_n]$$

where $\epsilon_i = \delta$ for m values of the index i and $\epsilon_i = 0$ otherwise. We will denote by $I^n(m)$ the m-skeleton of I^n with this cell structure.

Let $C_{j_1,\ldots,j_n}$ be an n-dimensional subcube:

$$C_{j_1,\ldots,j_n} = \{(t_1, \ldots, t_n) \in I^n \mid c_{j_i} \leq t_i \leq c_{j_i} + \delta\}$$

For $0 \leq p \leq N$ we will denote by $S_p C_{j_1,\ldots,j_n}$ and $L_p C_{j_1,\ldots,j_n}$ the subspaces of $C_{j_1,\ldots,j_n}$ given by

$$S_p C_{j_1,\ldots,j_n} = \{(t_1, \ldots, t_n) \in C_{j_1,\ldots,j_n} \mid c_{j_i} < t_i < c_{j_i} + \tfrac{\delta}{2} \text{ for at least } p \text{ coordinates } t_i\}$$
$$L_p C_{j_1,\ldots,j_n} = \{(t_1, \ldots, t_n) \in C_{j_1,\ldots,j_n} \mid c_{j_i} + \tfrac{\delta}{2} < t_i < c_k + \delta \text{ for at least } p \text{ coordinates } t_i\}$$

Also, denote

$$S_p = \bigcup_{j_1,\ldots,j_n} S_p C_{j_1,\ldots,j_n} \qquad L_p = \bigcup_{j_1,\ldots,j_n} L_p C_{j_1,\ldots,j_n}$$

13.2 Lemma. *Consider I^n with the N-cubical subdivision for some $N > 0$. Assume that $A, B \subseteq I^n$ are closed, disjoint sets, such that $A \cap I^n(p) = \varnothing$ for some $p \leq n$. There exists a homotopy $\Phi\colon I^n \times [0,1] \to I^n$ satisfying the following conditions:*

(i) $\Phi(C \times [0,1]) \subseteq C$ *for each subcube (of any dimension) in* I^n.

(ii) $\Phi_0 = \mathrm{id}_{I^n}$.

(iii) $\Phi_1^{-1}(A) \subseteq S_{p+1}$ *and* $\Phi_1^{-1}(B) = B$.

Also, there exists a homotopy $\Psi\colon I^n \times [0,1] \to I^n$ that satisfies (i) and (ii) and

(iii') $\Psi_1^{-1}(A) \subseteq L_{p+1}$ *and* $\Psi_1^{-1}(B) = B$.

Proof. Let $\varphi\colon [0,1] \times [0,1] \to [0,1]$ be a homotopy defined as follows:

$$\varphi(t, s) = (1 - s)t + s \cdot \min(c_j + \delta, 2t - c_j)$$

for $t \in [c_j, c_j + \delta]$. This is a homotopy between the identity map on $[0,1]$ and a map that on each subinterval $[c_j, c_j + \delta]$ sends $[c_j + \frac{\delta}{2}, c_j + \delta]$ to the point $c_j + \delta$ and stretches $[c_j, c_j + \frac{\delta}{2}]$ linearly to $[c_j, c_j + \delta]$. Define $\widetilde{\Phi}\colon I^n \times [0,1] \to I^n$ by

$$\widetilde{\Phi}((t_1, \ldots, t_n), s) = (\varphi(t_1, s), \ldots, \varphi(t_n, s))$$

The homotopy $\widetilde{\Phi}$ satisfies conditions (i) and (ii). Moreover, $\widetilde{\Phi}_1(t_1, \ldots, t_n) \notin I^n(p)$ if and only if $(t_1, \ldots, t_n) \in S_{p+1}$. Since $A \cap I^n(p) = \varnothing$ this gives $\widetilde{\Phi}_1^{-1}(A) \subseteq S_{p+1}$. Let $\varrho\colon I^n \to [0,1]$ be a function such that $\varrho(A) = 1$ and $\varrho(B) = 0$. Define $\Phi\colon I^n \times [0,1] \to I^n$ by

$$\Phi(x, s) = \widetilde{\Phi}(x, s\varrho(x))$$

Then $\Phi_1^{-1}(A) = \widetilde{\Phi}_1^{-1}(A) \subseteq S_p$ and $\Phi_1^{-1}(B) = \widetilde{\Phi}_0^{-1}(B) = B$

The homotopy Ψ can be obtained analogously.

□

13.3 Corollary. *Consider the cube I^n with the N-cubical subdivision for some $N \geq 1$. Assume that $A, B \subseteq I^n$ are closed, disjoint sets, such that $A \cap I^n(p) = \varnothing$ and $B \cap I^n(q) = \varnothing$ for some $p, q \leq n$. There exists a homotopy $\Lambda\colon I^n \times [0,1] \to I^n$ satisfying the following conditions:*

(i) $\Lambda(C \times [0,1]) \subseteq C$ *for each subcube (of any dimension) in* I^n.

(ii) $\Lambda_0 = \mathrm{id}_{I^n}$.

(iii) $\Lambda_1^{-1}(A) \subseteq S_{p+1}$ *and* $\Lambda_1^{-1}(B) \subseteq L_{q+1}$.

Proof. Take a homotopy Φ as in Lemma 13.2. Using the same lemma with $A = \Phi_1^{-1}(A)$ and $B = \Phi_1^{-1}(B) = B$ we obtain a homotopy Ψ that satisfies (i), (ii) and $\Psi_1^{-1}(\Phi_1^{-1}(A)) = \Phi_1^{-1}(A) \subseteq S_{p+1}$ and $\Psi_1^{-1}(\Phi_1^{-1}(B)) = \Psi_1^{-1}(B) \subseteq L_{q+1}$. The homotopy Λ can be then defined by

$$\Lambda(x,s) = \begin{cases} \Phi(x, 2s) & \text{for } s \leq \frac{1}{2} \\ \Psi(\Phi(x,1), 2s) & \text{for } s \geq \frac{1}{2} \end{cases}$$

□

Proof of Theorem 11.4. Denote $X_0 = X_1 \cap X_2$. We will first show that the homomorphism

$$i_*\colon \pi_k(X_1, X_0, x_0) \to \pi_k(X, X_2, x_0)$$

is onto for $k \leq m + n$.

Assume then $k \leq m + n$ and let $\omega\colon I^k \to X$ be a map representing an element of $\pi_k(X, X_2, x_0)$. We have $\omega(I^{k-1} \times \{0\}) \subseteq X_2$ and $\omega((\partial I^k \times I) \cup (I^{k-1} \times \{1\})) = x_0$. We need to show that ω is homotopic through such maps to $\tau\colon I^k \to X$ such that $\tau(I^k) \subseteq X_1$ and $\tau(I^{k-1} \times \{0\}) \subseteq X_0$.

Consider I^k with a N-cubical subdivision such that for each subcube $C \subseteq I^k$ we have either $\omega(C) \subseteq X_1$ or $\omega(C) \subseteq X_2$. We claim that there exists a homotopy $h\colon \omega \simeq \omega_1$ such that

1) if $\omega(C) \subseteq X_0$ then $h(x,t) = \omega(x)$ for $(x,t) \in C \times [0,1]$
2) if $\omega(C) \subseteq X_i$ for $i = 1, 2$ then $h(C \times [0,1]) \subseteq X_i$.
3) $\omega_1^{-1}(X_1 \smallsetminus X_0) \cap I^k(m) = \varnothing$
4) $\omega_1^{-1}(X_2 \smallsetminus X_0) \cap I^k(n) = \varnothing$.

The homotopy h can be constructed by induction with respect to skeleta of I^k. Let C^0 be a 0-dimensional subcube of I^k. If $\omega(C^0) \in X_0$ take $h|_{C^0 \times [0,1]}$ to be the constant map to the point $\omega(C^0)$ if $C^0 \in X_i \smallsetminus X_0$ for $i = 1, 2$ take $h|_{C^0 \times [0,1]}$ to be a path in X_i that joins $\omega(C^0)$ with a point in X_0. Such path exists by the connectivity assumption on the pair (X_i, X_0). In effect we obtain a homotopy

$h\colon I^k(0)\times[0,1]\to X$ satisfying 1)–4). For the inductive step, assume that we already constructed a homotopy $h\colon I^k(r)\times[0,1]\to X$ for some $r\geq 0$, and let C^{r+1} be an $(r+1)$-dimensional cube. The homotopy h is already defined on ∂C^{r+1}. If $\omega(C^{r+1})\subseteq X_0$, we extend h to C^{r+1} using condition 1). If $\omega(C^{r+1})\subseteq X_1$ and $r+1\leq m$ then we can extend h to a homotopy $h\colon C^{r+1}\times[0,1]\to X_1$ such that $h_1(C^{r+1})\subseteq X^0$ by Proposition 11.3. We proceed analogously if $\omega(C^{r+1})\subseteq X_2$ and $r+1\leq k$. In all other cases we extend h to C^{r+1} in an arbitary way that satisfies condition 2).

To check that the resulting map $\omega_1=h_1\colon I^k\to X$ satisfies condition 3), let $C^r\subseteq I^k(m)$ be an r-dimensional subcube for some $r\leq m$. If $\omega(C^r)\subseteq X_1$ then $\omega_1(C^r)\subseteq X_0$ and if $\omega(C^r)\subseteq X_2$ then $\omega_1(C^r)\subseteq X_2$. Thus $\omega_1(C^r)\cap(X_1\smallsetminus X_0)=\varnothing$. Condition 4) is satisfied by the same argument.

Next, consider the homotopy Λ as in Corollary 13.3 for the sets $A=\omega_1^{-1}(X_1\smallsetminus X_0)$ and $B=\omega_1^{-1}(X_2\smallsetminus X_0)$. The composition $\omega_1\Lambda\colon I^k\times I\to X$ gives a homotopy between ω_1 and a map ω_2 satisfying $\omega_2^{-1}(X_1\smallsetminus X_0)\subseteq S_{m+1}$ and $\omega_2^{-1}(X_2\smallsetminus X_0)\subseteq L_{n+1}$. Take the projection map $\mathrm{pr}\colon I^k\to I^{k-1}$, $\mathrm{pr}(t_1,\dots,t_{k-1},t_k)=(t_1,\dots,t_{k-1})$. We claim that the sets $\mathrm{pr}(S_{m+1})$ and $\mathrm{pr}(L_{n+1})$ are disjoint. Indeed, if $(t_1,t_2,\dots,t_{k-1})\in\mathrm{pr}(S_{m+1})\cap\mathrm{pr}(L_{n+1})$ then there are numbers $c_{j_1},\dots,c_{j_{k-1}}\in\{0,\frac{1}{N},\dots,\frac{N-1}{N}\}$ such that $c_{j_i}<t_i<c_{j_i}+\frac{\delta}{2}$ for at least m coordinates t_i and $c_{j_i}+\frac{\delta}{2}<t_i<c_{j_i}+\delta$ for at least n coordinates t_i. However, by assumption $k-1<m+n$, so this is impossible. As a consequence, the sets $\mathrm{pr}(\omega_2^{-1}(X_1\smallsetminus X_0))$ and $\mathrm{pr}(\omega_2^{-1}(X_2\smallsetminus X_0))$ are disjoint. We also have $\partial I^{k-1}\cap\mathrm{pr}(\omega_2^{-1}(X_2\smallsetminus X_0))=\varnothing$ so $\mathrm{pr}(\omega_2^{-1}(X_1\smallsetminus X_0))\cup\partial I^{k-1}$ and $\mathrm{pr}(\omega_2^{-1}(X_2\smallsetminus X_0))$ are disjoint, closed subsets of I^{k-1}. Take a function $\varrho\colon I^{k-1}\to[0,1]$ such that $\varrho(\mathrm{pr}(\omega_2^{-1}(X_1\smallsetminus X_0))\cup\partial I^{k-1})=1$ and $\varrho(\mathrm{pr}(\omega_2^{-1}(X_2\smallsetminus X_0)))=0$. Define a homotopy $h\colon I^k\times[0,1]\to X$ by

$$h((t_1,\dots,t_{k-1},t_k),s)=\omega_2(t_1,\dots,t_{k-1},(1-s)t_k+s\varrho(t_1,\dots,t_{k-1})t_k)$$

Then $h_0=\omega_2\simeq\omega$ while h_1 gives the desired map τ.

The argument that $i_*\colon\pi_k(X_1,X_0,x_0)\to\pi_k(X,X_2,x_0)$ is 1-1 for $k<m+n$ is analogous. In such case we start with two maps $\omega_0,\omega_1\colon I^k\to X_1$ representing two elements of $\pi_k(X_1,X_0,x_0)$. If these maps represent the same element in $\pi_k(X,X_2,x_0)$ then there exists $h\colon I^{k+1}=I^k\times I\to X$ such that $h|_{I^k\times\{i\}}=\omega_i$ for $i=0,1$ and that satisfies the appropriate conditions on the other faces of I^{k+1}. We want to show that h is homotopic to a map $h'\colon I^{k+1}\to X_1$. Since $k+1\leq m+n$ this can be done in the same way as above. $\square$

14 | Weak Equivalences

14.1 Definition. Let $0 \leq n \leq \infty$. A map $f\colon X \to Y$ is an *n-equivalence* if the induced homomorphism $f_*\colon \pi_i(X, x_0) \to \pi_i(Y, f(x_0))$ is an isomorphism for $0 \leq i < n$ and it is an epimorphism for $i = n$ for all $x_0 \in X$. A map f is a *weak (homotopy) equivalence* if it is an ∞-equivalence.

Recall that for a map $f\colon X \to Y$ the mapping cylinder of f is the space

$$M_f = (X \times [0,1] \sqcup Y)/{\sim}$$

where $(x, 0) \sim f(x)$ for all $x \in X$. We will consider X as a subspace of M_f by identifying it with $X \times \{1\}$.

14.2 Proposition. *Given a map $f\colon X \to Y$ the following conditions are equivalent:*

1) *f is an n-equivalence.*
2) *For $k \leq n$, given any commutative diagram*

$$\begin{tikzcd} S^{k-1} \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^k \arrow[r, "\psi"'] \arrow[ur, dashed, "\overline{\psi}"] & Y \end{tikzcd}$$

   there exists a map $\overline{\psi}\colon D^k \to Y$ such that $\overline{\psi}|_{S^{k-1}} = \varphi$ and $f\overline{\psi} \simeq \psi$ (rel S^{k-1}).
3) *For $k \leq n$, given any diagram*

$$\begin{tikzcd} S^{k-1} \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^k \arrow[r, "\psi"'] \arrow[ur, dashed, "\overline{\psi}"] & Y \end{tikzcd}$$

and a homotopy $\Phi\colon f\varphi \simeq \psi|_{S^{k-1}}$ *there exists a map* $\overline{\psi}\colon D^k \to Y$ *and a homotopy* $\overline{\Phi}\colon f\overline{\psi} \simeq \psi$ *such that* $\overline{\psi}|_{S^{n-1}} = \varphi$ *and* $\overline{\Phi}|_{S^{k-1}\times[0,1]} = \Phi$.

4) *The pair* (M_f, X) *is* n*-connected.*

Proof. Exercise. □

14.3 Proposition. *1) If* $f, g\colon X \to Y$ *are maps such that* $f \simeq g$ *and* f *is an* n*-equivalence then so is* g.

2) If $f\colon X \to Y$, $g\colon Y \to Z$, *and any two of the maps* f, g, gf *are weak equivalences, then so is the third map.*

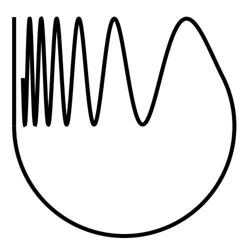
3) Every homotopy equivalence is a weak equivalence.

Proof. Exercise. □

One of the main goals of this chapter will be the proof of the following fact:

14.4 Theorem. *If* X, Y *are CW complexes then any weak equivalence* $f\colon X \to Y$ *is a homotopy equivalence.*

14.5 Note. Theorem 14.4 does not hold in general for spaces that are not CW complexes. For example, let W be the Warsaw circle (shown below). Since $\pi_i(W) = 0$ for all i, the constant map $W \to *$ is a weak equivalence. However, it is not a homotopy equivalence.



The proof Theorem 14.4 will use the following fact:

14.6 Proposition. *Assume that we have a diagram*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "f"] \\
X \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Z
\end{tikzcd}$$

where (X, A) is a relative CW complex such that $\dim(X \smallsetminus A) \leq n$ *for some* $n \leq \infty$*, and* $f\colon Y \to Z$ *is an* n*-equivalence. Assume also that* $\Phi\colon A \times [0,1] \to Z$ *is a homotopy such that* $\Phi\colon h|_A \simeq gf$*. Then there exists a map* $\bar{h}\colon X \to Y$ *and a homotopy* $\bar{\Phi}\colon X \times [0,1] \to Z$ *such that* $\bar{h}|_A = g$*,* $\bar{\Phi}\colon h \simeq f\bar{h}$ *and* $\bar{\Phi}|_{A\times[0,1]} = \Phi$*.*

Proof. By induction on skeleta of (X, A), using Proposition 14.2. □

As a special case of Proposition 14.6 we obtain:

14.7 Corollary. *Assume that we have a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "f"] \\ X \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Z \end{tikzcd}$$

where (X, A) be a relative CW complex such that $\dim(X \smallsetminus A) \leq n$ *for some* $n \leq \infty$*, and* $f\colon Y \to Z$ *is an* n*-equivalence. Then there exists a map* $\bar{h}\colon X \to Y$ *such that* $\bar{h}|_A = g$ *and* $f\bar{h} \simeq h$ (rel A)*.*

Recall that by $[X, Y]$ we denote the set of homotopy classes of maps $X \to Y$. A map $f\colon Y \to Z$ induces a map of sets $f_*\colon [X, Y] \to [X, Z]$ given by $f_*[\varphi] = [f\varphi]$.

14.8 Corollary. *Let* $f\colon Y \to Z$ *be an* n*-equivalence for some* $n \leq \infty$*. For any CW complex* X *the map*

$$f_*\colon [X, Y] \to [X, Z]$$

is a bijection if $\dim X \leq n - 1$ *and it is onto if* $\dim X \leq n$*.*

Proof. The onto part follows from Corollary 14.7 with $A = \varnothing$. It remains to show that f_* is 1-1 if $\dim X \leq n - 1$. Assume then that for some $\varphi_0, \varphi_1\colon X \to Y$ there is a homotopy $h\colon X \times [0,1] \to Z$ such that $h_0 = f\varphi_0$ and $h_1 = f\varphi_1$. This gives a commutative diagram

$$\begin{tikzcd} X \times \{0,1\} \arrow[r, "\varphi_0 \sqcup \varphi_1"] \arrow[d, hook, "i"'] & Y \arrow[d, "f"] \\ X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Z \end{tikzcd}$$

Consider the relative CW complex $(X \times [0,1], X \times \{0,1\})$. Since $\dim X \times [0,1] \leq n$, using Corollary 14.7 again we obtain that there exists $\bar{h}\colon X \times [0,1] \to Y$ which is homotopy between φ_0 and φ_1. □

Proof of Theorem 14.4. Let $f\colon X \to Y$ be a weak equivalence of CW complexes. By Corollary 14.8, the map

$$f_*\colon [Y, X] \to [Y, Y]$$

is a bijection. Therefore, there exists $g\colon Y \to X$ such that $f_*[g] = [\mathrm{id}_Y]$. Equivalently, $fg \simeq \mathrm{id}_Y$. Next, consider the bijection

$$f_*\colon [X, X] \to [X, Y]$$

We have $f_*[gf] = [fgf] = [f] = f_*[\mathrm{id}_X]$, which gives $[gf] = [\mathrm{id}_X]$, or equivalently $gf \simeq \mathrm{id}_X$. Therefore f is a homotopy equivalence with a homotopy inverse g. □

We have seen before (5.12) that two CW complexes X, Y that have isomorphic homotopy groups need not be homotopy equivalent. The issue is, that even if $\pi_i(X) \cong \pi_i(Y)$ for all $i \geq 0$, there may be no map $X \to Y$ which induces such isomorphisms. However, in two cases homotopy groups alone are enough to determine the homotopy type of a CW complex: for contractible spaces and for Eilenberg-MacLane spaces.

14.9 Proposition. *If X is a CW complex such that $\pi_i(X) = 0$ for all $i \geq 0$ then $X \simeq *$.*

Proof. The constant map $X \to *$ is weak equivalence, so by Theorem 14.4 it is a homotopy equivalence. □

14.10 Proposition. *Let X_1, X_2 be Eilenberg-MacLane spaces of type $K(G, n)$. That is, X_1, X_2 are path connected CW complexes such that*

$$\pi_i(X_k) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2$. Then $X_1 \simeq X_2$.

Proof. Recall (12.14) that we can construct an Eilenberg-MacLane space X_0 of the type $K(G, n)$ such that $X_0^{(n-1)} = *$. It will be enough to show that for any other Eileberg-MacLane space Y of the same type there exists a weak equivalence $X_0 \to Y$. Indeed, by Theorem 14.4 this will give $X_0 \simeq Y$, and applying it to the spaces X_1 and X_2 we will obtain $X_1 \simeq X_0 \simeq X_2$.

Let then X_0, Y be Eilenberg-MacLane spaces of type $K(G, n)$ such that $X_0^{(n-1)} = *$. We can assume that the 0-cell $* \in X_0$ is the basepoint of X_0, and let $y_0 \in Y$ be a basepoint in Y. Let $\varphi\colon \pi_n(X_0, *) \to \pi_n(Y, y_0)$ be an isomorphism of groups. We will construct a map $f\colon (X_0, *) \to (Y, y_0)$ such that $f_* = \varphi$. To do this, notice that $X_0^{(n)} = \bigvee_{i \in I} S^n$. For $k \in I$ let $j_k\colon S^n \hookrightarrow X_0^n$ be the inclusion of the k-th copy of S^n. Let $[ij_k] \in \pi_n(X_0, *)$ be the element represented by $S^n \overset{j_k}{\hookrightarrow} X_0^{(n)} \overset{i}{\hookrightarrow} X_0$, and let $\omega_k\colon S^n \to Y$ be a map such that $[\omega_k] = \varphi([ij_k])$. Define $f_n\colon X_0^{(n)} \to Y$ by $f_n = \bigvee_{k \in I} \omega_k$.

Assume that we can extend f_n to some map $f\colon X_0 \to Y$. Then f induces a homomorphism $f_*\colon \pi_n(X_0, *) \to \pi_n(Y, y_0)$ such that

$$f_*([ij_k]) = [\omega_k] = \varphi([ij_k]) \tag{$*$}$$

for all $k \in I$. By Corollary 12.6 the elements $[j_k]$ generate the group $\pi_n(X_0^{(n)}, *)$, and by Proposition 5.2 the homomorphism $i_*\colon \pi_n(X_0^{(n)}, *) \to \pi_n(X_0, *)$ is onto. Therefore elements $[ij_k]$ generate $\pi_n(X_0, *)$. As a consequence, the equation $(*)$ implies that $f_*([\tau]) = \varphi([\tau])$ for all $[\tau] \in \pi_n(X_0, *)$. It follows that $f_*\colon \pi_i(X_0, *) \to \pi_i(Y, y_0)$ is an isomorphism for $i = n$ and since all other homotopy groups of X_0 and Y are trivial, f_* is an isomorphism for all $i \neq n$ as well. Therefore f is a weak equivalence.

An extension of $f_0\colon X_0^{(n)} \to Y$ to $f\colon X_0 \to Y$ can be constructed by induction with respect to skeleta of X_0. Assume that for some $m \geq n$ we have a map $f_m\colon X_0^{(m)} \to Y$ that extends f_n. Then $X_0^{(m+1)} = X_0^{(m)} \cup \bigcup_{j\in J} e_j^{m+1}$ for some $(m+1)$-cells e_j. Let $\varphi_j\colon S^m \to X^{(m)}$ be the attaching map of e_j^{m+1}, and let $\overline{\varphi}_j\colon D^{m+1} \to X^{(m)}$ be the characteristic map. Since $\pi_m(Y) = 0$, the map $f_m\varphi_j$ extends to $\psi_j\colon D^{m+1} \to Y$ We define $f_{m+1}\colon X_0^{(m+1)} \to Y$ by

$$f_{m+1}(x) = \begin{cases} f_m(x) & \text{if } x \in X^{(m)} \\ \psi_j(\overline{\varphi}_j^{-1}(x)) & \text{if } x \in e_j \end{cases}$$

$\square$

Using similar arguments as in the proof of Proposition 14.10 we can obtain:

14.11 Proposition. *Let $K(G, n)$, $K(H, n)$ be Eilenberg-MacLane spaces for some groups G, H and $n \geq 1$. For any homomorphism of groups $\varphi\colon \pi_n(K(G, n), x_0) \to \pi_n(K(H, n), y_0)$ there exists a map $f\colon (K(G, n), x_0) \to (K(H, n), y_0)$ such that $f_* = \varphi$.*

Proof. Exercise. $\square$

15 | Weak Homotopy Type

A complication with studying weak equivalences is that two spaces can be related via a chain of weak equivalences even when there is no direct weak equivalence between them. For example, take $X, Y \subseteq \mathbb{R}$ where X consist of all rational numbers and $Y = \{\frac{1}{n} \mid n = 1, 2, \dots\} \cup \{0\}$. Since every path connected component of X and Y consists of a single point, $\pi_0(X)$ and $\pi_0(Y)$ are countable sets and all higher homotopy groups are trivial. A weak equivalence $X \to Y$ would need to be a continuous bijection in order to induce a bijection $\pi_0(X) \to \pi_0(Y)$. However, one can check that there is no such continuous bijection. By the same argument, there is no weak equivalence $Y \to X$. On the other hand, if we take the set of integers $\mathbb{Z}$ with the discrete topology, then any bijections $\mathbb{Z} \to X$ and $\mathbb{Z} \to Y$ are continuous functions and they are weak equivalences. Thus the spaces X and Y are related by a chain of weak equivalences:

$$X \leftarrow \mathbb{Z} \rightarrow Y$$

This motivates the following definition:

15.1 Definition. Spaces X and Y are *weakly equivalent* (or have the same *weak homotopy type*) if they can be connected by a zigzag of weak equivalences

$$X = Z_0 \to Z_1 \leftarrow Z_2 \to \dots \leftarrow Z_{n-1} \to Z_n = Y$$

15.2 Proposition. *If X, Y are CW complexes then they are weakly equivalent if and only if they are homotopy equivalent.*

Proof. Assume that X, Y are connected by a zigzag of n weak equivalences:

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \to \dots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y \tag{$*$}$$

We will show that $X \simeq Y$ by induction with respect to n. If $n = 1$, then we have a weak equivalence $X = Z_0 \to Z_1 = Y$, which by Theorem 14.4 is a homotopy equivalence.

Assume that the statement is true for any zigzag consisting of $n-1$ or fewer weak equivalences and that X, Y are connected by a sequence $(*)$. By Corollary 14.8 the map $f_{2*}\colon [X, Z_2] \to [X, Z_1]$ is a bijection. This means that there exists a map $g\colon X \to Z_2$ such that $f_2 g \simeq f_1$. By Proposition 14.3 the map g is a weak equivalence. Thus we obtain a zigzag of weak equivalences of the form:

$$X \xrightarrow{g} Z_2 \xrightarrow{f_3} Z_3 \leftarrow \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

By the inductive assumption $X \simeq Y$. □

For spaces that are not CW complexes, the study of their weak homotopy type can be simplified using the notion of a CW approximation.

15.3 Definition. A *CW approximation* of a space X is a CW complex Y together with a weak equivalence $f\colon Y \to X$.

More generally, a *CW approximation* of a pair (X, A) is a relative CW complex (Y, A) together with a weak equivalence $f\colon Y \to X$ such that $f|_A = \mathrm{id}_A$.

Notice that a CW approximation of a space X is the same as a CW approximation of the pair $(X, \varnothing)$.

We will show that the following holds:

15.4 Theorem. *Any pair (X, A) has a CW approximation. Moreover, any two CW approximations for such a pair are homotopy equivalent.*

15.5 Corollary. *Spaces X, Y are weakly eqivalent if and only if there exists a space Z and weak equivalences $X \leftarrow Z \to Y$.*

Proof. If such a space Z exists, then by definition X and Y are weakly equivalent. Conversely, assume that we have a zigzag of weak equivalences connecting X and Y:

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \to \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

We can extend it to

$$X' \xrightarrow{g_X} X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \to \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y \xleftarrow{g_Y} Y'$$

where $g_X\colon X' \to X$ and $g_Y\colon Y' \to Y$ are CW approximations of X and Y, respectively. By Proposition 15.2 there exists a homotopy equivalence $h\colon X' \to Y'$. Thus we obtain a diagram of weak equivalences: $X \xleftarrow{g_X} X' \xrightarrow{g_Y h} Y$. □

Proof of Theorem 15.4. Assume first that X is a path connected space. For $n = 0, 1, \ldots$ we will construct relative CW complexes $(Y^{(n)}, A)$ and maps $f^{(n)}\colon Y^{(n)} \to X$ such that

1) $Y^{(n)}$ is obtained from $Y^{(n-1)}$ by attaching n-cells.
2) $f^{(0)}|_A = \mathrm{id}_A$ and $f^{(n)}|_{Y^{(n-1)}} = f^{(n-1)}$
3) $f^{(n)}_*\colon \pi_i(Y^{(n)}) \to \pi_i(X)$ is an isomorphism for $i < n$ and epimorphism for $i = n$.

Then the map $\bigcup_n f^{(n)}\colon \bigcup_n Y^{(n)} \to X$ will give a CW approximation of (X, A).

Let $\{A_i\}_{i\in I}$ be path connected components of A. Also, let $x_0 \in X$. For each $i \in I$ choose a point $a_i \in A_i$. Let $(Y^{(1)}, A)$ be a 1-dimensional relative CW complex obtained by:

- adding to A a single 0-cell e^0;
- for each $i \in I$ adding to $A \cup e^0$ a 1-cell e^1_i attached to the points e_0 and a_i.
- for each element $[\tau\colon (S^1, s_0) \to (X, x_0)] \in \pi_1(X, x_0)$ attaching to the resulting space a circle S^1_τ, by identifying s_0 with e^0.

Since X is path connected, for each $i \in I$ there is a path $\omega_i\colon [0, 1] \to X$ such that $\omega_i(0) = x_0$ and $\omega_i(1) = a_i$. Take a map $f^{(1)}\colon Y^{(1)} \to X$ such that $f^{(1)}(x) = x$ for all $x \in A$, $f^{(1)}(e^0) = x_0$. Also, $f^{(1)}$ maps each cell e^1_i using the path ω_i, and each circle S^1_τ using the map τ. Notice that $f^{(1)}_*\colon \pi_i(Y^{(1)}, e_0) \to \pi_i(X, x_0)$ is a bijection for $i = 0$ and it is onto for $i = 1$.

Next, assume that for $i = 1, \dots, n$ we already constructed spaces $Y^{(i)}$ and maps $f^{(i)}\colon Y^{(i)} \to X$ satisfying conditions 1)–3). Take the epimorphism $f^{(n)}_*\colon \pi_n(Y^{(n)}, e^0) \to \pi_n(X, x_0)$. Let $\overline{Y}^{(n+1)}$ denote the space obtained by attaching to $Y^{(n)}$ an $(n+1)$-cell e^{n+1}_ω for each element $[\omega\colon (S^n, s_0) \to (Y^{(n)}, e^0)] \in \ker f^{(n)}_*$, using ω as the attaching map. Since $[f^{(n)}\omega] = 0$ in $\pi_n(X, x_0)$, the map $f^{(n)}\omega\colon S^n \to X$ can be extended to a map $D^{n+1} \to X$. We can use this to extend $f^{(n)}$ to a map $\bar{f}^{(n+1)}\colon \overline{Y}^{(n+1)} \to X$. Subsequently, take $Y^{(n+1)}$ to be the space obtained by attaching to $\overline{Y}^{(n+1)}$ a sphere $S^{(n+1)}_\tau$ for each $[\tau\colon (S^{n+1}, s_0) \to (X, x_0)] \in \pi_{n+1}(X, x_0)$, by identifying s_0 with e^0. Extend $\bar{f}^{(n+1)}$ to $f^{(n+1)}\colon Y^{(n+1)} \to X$, mapping $S^{(n+1)}_\tau$ using τ.

We have a commutative diagram

$$\begin{tikzcd}
\pi_n(Y^{(n)}, e^0) \arrow[rr, "i_*"] \arrow[dr, "f^{(n)}_*"'] & & \pi_n(Y^{(n+1)}, e^0) \arrow[dl, "f^{(n+1)}_*"] \\
& \pi_n(X, x_0) &
\end{tikzcd}$$

where $i\colon Y^{(n)} \hookrightarrow Y^{(n+1)}$ is the inclusion map. Since $f^{(n)}_*$ is onto, thus so is $f^{(n+1)}_*$. Also, by construction $\ker f^{(n+1)} = 0$. Therefore $f^{(n+1)}_*\colon \pi_i(Y^{(n+1)}, e^0) \to \pi_i(X, x_0)$ is an isomorphism for $i \leq n$ and it is an epimorphism for $i = n + 1$.

Next, assume that X is not path connected and let $\{X_i\}_{i\in I}$ be path connected components of X. Construct a CW approximation Y_i for each pair $(X_i, A \cap X_i)$, using the procedure described above. Then a CW approximation of (X, A) can be obtained by taking the quotient space $A \sqcup \bigsqcup_{i\in I} Y_i/\sim$, where the relation $\sim$ identifies points of $X_i \cap A \subseteq Y_i$ with the corresponding points of A.

Finally, assume that for $i = 1, 2$ a map $f_i\colon (Y_i, A) \to (X, A)$ is a CW approximation of (X, A). This gives a commutative diagram

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, hook] & Y_2 \arrow[d, "f_2"] \\
Y_1 \arrow[r, "f_1"'] \arrow[ur, dashed, "g"] & X
\end{tikzcd}
$$

By Corollary 14.7 there exists $g\colon Y_1 \to Y_2$ such that $g(x) = x$ for all $x \in A$ and $f_2 g \simeq f_1$ (rel A) By the same argument, there exists $h\colon Y_2 \to Y_1$ such that $h(x) = x$ for all $x \in A$ and $f_1 h \simeq f_2$ (rel A). This shows that there exists a map $\varphi\colon Y_1 \times [0,1] \to X$ which gives a homotopy $f_1 \simeq f_1 hg$ (rel A).

Consider the space $Z_1 = Y_1 \times \{0,1\} \cup A \times [0,1] \subseteq Y_1 \times [0,1]$. Then $(Y_1 \times [0,1], Z_1)$ is a relative CW complex. We have a commutative diagram

$$
\begin{tikzcd}
Z_1 \arrow[r, "\psi"] \arrow[d, hook] & Y_1 \arrow[d, "f_1"] \\
Y_1 \times [0,1] \arrow[r, "\varphi"'] \arrow[ur, dashed, "\overline{\varphi}"] & X
\end{tikzcd}
$$

where

$$\psi(y,t) = \begin{cases} y & \text{if } t < 1 \\ hg(y) & \text{if } t = 1 \end{cases}$$

Using Corollary 14.7 again, we obtain that there exists $\overline{\varphi}\colon Y_1 \times [0,1] \to X$, which gives a homotopy $\mathrm{id}_{Y_1} \simeq hg$ (rel A). Analogously, we obtain that $\mathrm{id}_{Y_2} \simeq gh$ (rel A). □

16 | Weak Equivalences and Homology

The main goal of this chapter is to show that the following holds:

16.1 Theorem. *If $f\colon X \to Y$ is a weak equivalence then the induced homomorphisms $f_*\colon H_i(X) \to H_i(Y)$ and $f^*\colon H^i(X) \to H^i(Y)$ are isomorphisms for all $i \geq 0$.*

16.2 Brief review of homological algebra.

- A *chain complex* C_* consists of a sequence of abelian groups and group homomorphisms
$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots$$
such that $\partial_n \partial_{n+1} = 0$ for all n. The homomorphisms ∂_n are called *differentials*.
- The *n-th homology group* of a chain complex C_* is the group $H_n(C_*) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$.
- A *chain map* $f_*\colon C_* \to D_*$ is a sequence of homomophisms $f_n\colon C_n \to D_n$ such that $\partial_n f_n = f_{n-1}\partial_n$.
- A chain map $f_*\colon C_* \to D_*$ induces homomorphisms of homology groups $f_*\colon H_n(C_*) \to H_n(D_*)$.
- If $f_*, g_*\colon C_* \to D_*$ are chain maps, then a *chain homotopy* $s_*\colon C_* \to D_*$ from f_* to g_* is a sequence of homomorphisms $s_n\colon C_n \to D_{n+1}$ such that $f_n - g_n = \partial_{n+1}s_n + s_{n-1}\partial_n$
- If there exists a chain homotopy between chain maps $f_*, g_*\colon C_* \to D_*$ then f_* and g_* induce the same homorphism between homology groups $H_*(C_*) \to H_*(D_*)$.

16.3 Brief review of singular homology.

- A *singular chain complex* of a topological space X is a chain complex $C_*(X)$ such that $C_n(X)$ is the free abelian group generated by all singular simplices $\sigma\colon \Delta^n \to X$.
- Differentials in $C_*(X)$ are defined using face maps $d^i_n\colon \Delta^{n-1} \to \Delta^n$ for $i = 0, \ldots, n$: $\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma d^i_n$.

- *Singular homology groups* of a space X are homology groups of $C_*(X)$: $H_n(X) := H_n(C_*(X))$.
- Any map of spaces $f\colon X \to Y$ defines a chain map of singular chain complexes $f_*\colon C_*(X) \to C_*(Y)$ given by $f_*(\sigma) = f\sigma$ for a singular simplex $\sigma\colon \Delta^n \to X$. This induces a homomorphism of homology groups $f_*\colon H_*(X) \to H_*(Y)$.
- For a space X let $i_0, i_1\colon X \to X \times [0,1]$ denote the inclusions $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$. There exists a chain homotopy $s^X_*\colon C_*(X) \to C_{*+1}(X \times [0,1])$ from i_{0*} to i_{1*}.
- The chain homotopy s^X_* can be used to show that if $f, g\colon X \to Y$ are homotopic maps then they induce that same homomorphism of homology groups $H_*(X) \to H_*(Y)$.

Proof of Theorem 16.1. Assume first that $X \subseteq Y$ and that $f\colon X \hookrightarrow Y$ is the inclusion map. Notice that in this case $C_*(X)$ is a subcomplex of $C_*(Y)$ and the chain map $f_*\colon C_*(X) \to C_*(Y)$ is an inclusion.

We will associate to each singular simplex $\sigma\colon \Delta^n \to Y$ a homotopy $h^\sigma\colon \Delta^n \times [0,1] \to Y$ such that:

1) $h^\sigma_0 = \sigma$ and $h^\sigma_1(\Delta^n) \subseteq X$.
2) If $\sigma(\Delta^n) \subseteq X$ then $h^\sigma_t = \sigma$ for all $t \in [0,1]$.
3) $h^{\sigma d^i_n} = h^\sigma(d^i_n \times \mathrm{id}_{[0,1]})$

The homotopies h^σ will be constructed by induction with respect to n. For $n = 0$ giving a simplex $\sigma\colon \Delta^0 = \{*\} \to X$ is the same a giving a point $\sigma(*) = y \in Y$. Since f is a weak equivalence, there exist a path $h^\sigma\colon \Delta^0 \times [0,1] \to Y$ such that $h^\sigma(*, 0) = y$ and $h^\sigma(*, 1) \in X$. If $y \in X$ take h^σ to be the constant path.

Assume that we have already constructed homotopies h^τ satisfying 1)–3) for all $\tau\colon \Delta^k \to Y$ with $k < n$, and let $\sigma\colon \Delta^n \to Y$. If $\sigma(\Delta^n) \subseteq X$, define h^σ using condition 2). Otherwise, let $\partial\Delta^n := \bigcup_{i=0}^n d^i_n(\Delta^{n-1}) \subseteq \Delta^n$. Since homotopies $h^{\sigma d^i_n}$ are already defined, condition 3) determines a map $h\colon \Delta^n \times \{0\} \cup \partial\Delta^n \times [0,1] \to Y$ such that $h_0 = \sigma$ and $h_1(\partial\Delta^n) \subseteq X$. The pair $(\Delta^n, \partial\Delta^n)$ is a relative CW complex, so by Proposition 14.6 we can extend h to a homotopy $h^\sigma\colon \Delta^n \times [0,1] \to Y$ such that $h^\sigma_0 = \sigma$ and $h^\sigma_1(\Delta^n) \subseteq X$.

Define a map $\varphi_*\colon C_*(Y) \to C_*(X)$ by $\varphi(\sigma) = h^\sigma_1$. Condition 3) implies that φ_* is a chain map. Also, by condition 2) we obtain $\varphi_* f_* = \mathrm{id}_{C_*(X)}$. Finally, a chain homotopy $\Phi_*\colon C_*(Y) \to C_*(Y)$ between $f_*\varphi_*$ and $\mathrm{id}_{C_*(Y)}$ can be obtained as follows. Given a singular simplex $\sigma\colon \Delta^n \to Y$ the homotopy h^σ induces a chain map $h^\sigma_*\colon C_*(\Delta^n \times [0,1]) \to C_*(Y)$. We also have a chain homotopy $s^{\Delta^n}_*\colon C_*(\Delta^n) \to C_*(\Delta^n \times [0,1])$. The identity map $\mathrm{id}_{\Delta^n}\colon \Delta^n \to \Delta^n$ is a singular simplex in $C_n(\Delta^n)$. We set $\varphi(\sigma) = h^\sigma_* s^{\Delta^n}_*(\mathrm{id}_{\Delta^n})$.

For a general weak equivalence $f\colon X \to Y$ consider the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\ & M_f \arrow[ur, "r"'] & \end{tikzcd}$$

where M_f is the mapping cylinder of f, i is the inclusion map and r is the retraction. Since f is a weak equivalence and r is a homotopy equivalence, thus i is a weak equivalence. Therefore, by the argument above, i induces an isomorphism on homology groups $i_*\colon H_*(X) \xrightarrow{\cong} H_*(M_f)$. Also, since every homotopy equivalence induces an isomorphism on homology, thus we get an isomorphism $r_*\colon H_*(M_f) \xrightarrow{\cong} H_*(Y)$. Therefore $f_* = r_* i_*\colon H_*(X) \to H_*(Y)$ is an isomorphism.

The statement that a weak equivalence induces an isomorphism of cohomology groups follows from the Universal Coefficients Theorem, which implies that if a map $f\colon X \to Y$ gives an isomorphism on homology, then it also induces an isomorphism on cohomology. □

Using the same arguments as in the proof of Theorem 16.1, this theorem can be generalized as follows:

16.4 Theorem. *If $f\colon X \to Y$ is an n-equivalence for some $n \geq 1$ then then $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for all $i < n$ and it is an epimorphism for $i = n$.*

17 | Hurewicz Theorem

Hurewicz homomorphism is a map that connects homotopy and homology groups. Recall that $H_n(S^n) \cong \mathbb{Z}$ We will denote by γ_n a chosen generator of $H_n(S^n)$. Given an element $[\varphi\colon (S^n, s_0) \to (X, x_0)] \in \pi_n(X, x_0)$ consider the homomorphism $\varphi_*\colon H_*(S^n) \to H_n(X)$. This homomorphism depends only on the homotopy class of φ.

17.1 Definition. The *Hurewicz homomorphism* is a function

$$h\colon \pi_n(X, x_0) \to H_n(X)$$

given by $h([\varphi]) = \varphi_*(\gamma_n)$.

17.2 Proposition. *For any function $f\colon X \to Y$ the following diagram commutes:*

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, "f_*"] \arrow[d, "h"'] & \pi_n(Y, f(x_0)) \arrow[d, "h"] \\
H_n(X) \arrow[r, "f_*"'] & H_n(Y)
\end{tikzcd}$$

Proof. For $[\varphi] \in \pi_n(X, x_0)$ we have

$$hf_*([\varphi]) = h([f\varphi]) = (f\varphi)_*(\gamma_n) = f_*\varphi_*(\gamma_n) = f_*h([\varphi])$$

□

17.3 Proposition. *The Hurewicz homorphism is a group homomorphism.*

Proof. Let $\varphi, \psi\colon (S^n, s_0) \to (X, x_0)$ where $n \geq 1$. Recall that the element $[\varphi] \cdot [\psi] \in \pi_n(X, x_0)$ is the homotopy class of the map

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\varphi \vee \psi} X$$

where $p\colon S^n \to S^n \vee S^n$ is the pinch map. Let $r_1, r_2\colon S^n \vee S^n \to S^n$ be the retractions of $S^n \vee S^n$ onto the first and, respectively, the second copy of S^n. We have a commutative diagram

$$\begin{tikzcd} H_n(S^n) \arrow[r, "p_*"] \arrow[dr, "\mathrm{id}_* \oplus \mathrm{id}_*"'] & H_n(S^n \vee S^n) \arrow[r, "(\varphi \vee \psi)_*"] \arrow[d, "r_{1*} \oplus r_{2*}", "\cong"'] & H_n(X) \\ & H_n(S^n) \oplus H_n(S^n) \arrow[ur, "\varphi_* + \psi_*"'] & \end{tikzcd}$$

This gives:

$$h([\varphi]\cdot[\psi]) = ((\varphi \vee \psi)p)_*(\gamma_n) = (\varphi_* + \psi_*)(\mathrm{id}_* \oplus \mathrm{id}_*)(\gamma_n) = \varphi_*(\gamma_n) + \psi_*(\gamma_n) = h([\varphi]) + h([\psi])$$

□

17.4 Hurewicz Isomorphism Theorem. *Let X be a path connected space such that for some $n \geq 2$ we have $\pi_i(X) = 0$ for $i < n$. Then $H_i(X) = 0$ for $0 < i < n$ and the Hurewicz homomorphism*

$$h\colon \pi_n(X, x_0) \to H_n(X)$$

is an isomorphism.

Proof. Assume first that $X = S^n$. We have $H_i(S^n) = 0$ for $0 < i < n$. In degree n the Hurewicz homomorphism is a map $h\colon \mathbb{Z} \cong \pi_n(S^n) \to H_n(S^n) \cong \mathbb{Z}$. The group $\pi_n(S^n)$ is generated by the homotopy class of the identity map $\mathrm{id}_{S^n}\colon S^n \to S^n$ (11.10). We have $h([\mathrm{id}_{S^n}]) = \mathrm{id}_{S^n *}(\gamma_n) = \gamma_n$. Therefore h maps a generator of $\pi_n(S^n)$ to a generator of $H^n(S^n)$, and so it is an isomorphism.

Next, assume that $X = \bigvee_{i\in I} S^n$. Again, in this case $H_i(\bigvee_{i\in I} S^n) = 0$ for $0 < i < n$. Also, the retraction maps $r_i\colon \bigvee_{i\in I} S^n \to S^n$ give a commutative diagram

$$\begin{tikzcd} \pi_n(\bigvee_{i\in I} S^n) \arrow[r, "\bigoplus r_{i*}", "\cong"'] \arrow[d, "h"] & \bigoplus_{i\in I} \pi_n(S^n) \arrow[d, "\bigoplus_{i\in I} h", "\cong"'] \\ H_n(\bigvee_{i\in I} S^n) \arrow[r, "\cong", "\bigoplus r_{i*}"'] & \bigoplus_{i\in I} H_n(S^n) \end{tikzcd}$$

The the map $\bigoplus_{i\in I} h$ is an isomorphism by the previous case, so the left vertical map h is also an isomorphism.

For the next step, assume that X is an arbitrary CW complex with $\pi_i(X) = 0$ for $i < n$. By Proposition 5.6 we can assume that $X^{(n-1)} = *$, which gives $H_i(X) = 0$ for $0 < i < n$.

Let $j\colon X^{(n+1)} \hookrightarrow X$ be the inclusion of the $(n+1)$-skeleton of X. By Proposition 17.2 we have a commutative diagram

$$\begin{tikzcd}
\pi_n(X^{(n+1)}) \arrow[r, "j_*", "\cong"'] \arrow[d, "h"'] & \pi_n(X) \arrow[d, "h"] \\
H_n(X^{(n+1)}) \arrow[r, "\cong", "j_*"'] & H_n(X)
\end{tikzcd}$$

The upper homomorphism j_* is an isomorphism by Proposition 5.2, and the lower j_* is an isomorphism by properties of homology groups. As a consequence, it is enough to show that $h\colon \pi_n(X^{(n+1)}) \to H_n(X^{(n+1)})$ is an isomorphism.

Since $X^{(n-1)} = *$, it follows that $X^{(n)} = \bigvee_{i\in I} S^n$ and $X^{(n+1)} = X^{(n)} \cup \bigcup_{k\in K} e_k^{(n+1)}$ where $\{e_k^{(n+1)}\}_{k\in K}$ are $(n+1)$-cells of X. Let $\varphi_k\colon S^n \to X^{(n)}$ be the attaching map of the cell e_k^{n+1}, and let $i\colon X^{(n)} \hookrightarrow X^{(n+1)}$ denote the inclusion map. We have a commutative diagram

$$\begin{tikzcd}
\pi_n(\bigvee_{k\in K} S^n) \arrow[r, "(\bigvee_{k\in K}\varphi_k)_*"] \arrow[d, "h"', "\cong"] & \pi_n(X^{(n)}) \arrow[r, "i_*"] \arrow[d, "h"', "\cong"] & \pi_n(X^{(n+1)}) \arrow[r] \arrow[d, "h"] & 0 \arrow[d, "\cong"] \\
H_n(\bigvee_{k\in K} S^n) \arrow[r, "(\bigvee_{k\in K}\varphi_k)_*"'] & H_n(X^{(n)}) \arrow[r, "i_*"'] & H_n(X^{(n+1)}) \arrow[r] & H_{n-1}(\bigvee_{k\in K} S^n)
\end{tikzcd}$$

The upper row of this diagram is exact by Proposition 12.7, and the lower row is exact by the long homology sequence associated to the map $\bigvee_{k\in K}\varphi_k$. By the Five Lemma we obtain that $h\colon \pi_n(X^{(n+1)}) \to H_n(X^{(n+1)})$ is an isomorphism.

Finally, let X be an arbitary space with $\pi_i(X) = 0$ for $i < n$. Let $f\colon Y \to X$ be a CW approximation of X (15.3). Using Theorem 16.1 and the previous case we get $H_i(X) \cong H_i(Y) = 0$ for $0 < i < n$.

By Proposition 17.2 we have a commutative diagram

$$\begin{tikzcd}
\pi_n(Y) \arrow[r, "f_*", "\cong"'] \arrow[d, "h"', "\cong"] & \pi_n(X) \arrow[d, "h"] \\
H_n(Y) \arrow[r, "\cong", "f_*"'] & H_n(X)
\end{tikzcd}$$

Since f is a weak equivalence, the upper homomorphism f_* is an isomorphism by definition, and the lower f_* is an isomorphism by Theorem 16.1. Also, since Y is a CW complex the left vertical map is an isomorphism by the previous case. Therefore $h\colon \pi_n(X) \to H_n(X)$ is an isomorphism. $\square$

17.5 Inverse Hurewicz Isomorphism Theorem. *Let X be a simply connected space, and let $H_i(X) = 0$ for $1 \leq i < n$ for some $n \geq 2$. Then $\pi_i(X) = 0$ for $i < n$ and the Hurewicz homomorphism $h\colon \pi_n(X) \to H_n(X)$ is an isomorphism.*

Proof. Exercise. □

Since all homology groups $H_i(X)$ are abelian but the fundamental group $\pi_1(X)$ need not be abelian, in general the Hurewicz homomorphism $h\colon \pi_1(X) \to H_1(X)$ is not an isomorphism. However, a version of Theorem 17.4 still holds with the following modification. Recall that if G is a group then the commutator of G is the subgroup $[G, G] \subseteq G$ generated by all elements of the form $ghg^{-1}h^{-1}$ for $g, h \in G$. The commutator is a normal subgroup of G, and the quotient group $G^{\mathrm{ab}} := G/[G, G]$ is an abelian group. The group G^{ab} is called the abelianization of G.

If H is an abelian group then any homomorphism $\varphi\colon G \to H$ defines a unique homomorphism $\overline{\varphi}\colon G^{\mathrm{ab}} \to H$ such that $\varphi = \overline{\varphi}\eta$ where $\eta\colon G \to G^{\mathrm{ab}}$ is the quotient homomorphism. Also, if $\psi\colon G \to H$ is a homomorphism of arbitary groups, then $\psi([G, G]) \subseteq [H, H]$, and so ψ induces a homomorphism of abelianizations $\psi^{\mathrm{ab}}\colon G^{\mathrm{ab}} \to H^{\mathrm{ab}}$.

17.6 Theorem. *Let X be a path connected space and let $h\colon \pi_1(X, x_0) \to H_1(X)$ be the Hurewicz homomorphism. Then the induced homomorphism $\overline{h}\colon \pi_1(X, x_0)^{ab} \to H_1(X)$ is an isomorphism.*

The proof will use the following algebraic fact.

17.7 Lemma. *Consider a sequence of group homomorphisms*

$$G \xrightarrow{\varphi} H \xrightarrow{\psi} K$$

such that ψ is onto and $\ker\psi = N_H(\operatorname{Im}\varphi)$ where $N_H(\operatorname{Im}\varphi)$ is the normalizer of $\operatorname{Im}\varphi$ in H. Then the induced sequence

$$G^{ab} \xrightarrow{\varphi^{ab}} H^{ab} \xrightarrow{\psi^{ab}} K^{ab} \longrightarrow 0$$

is exact.

Proof. Exercise. □

Proof of Theorem 17.6. Take $X = S^1$. As in the proof of Theorem 17.4 we obtain that $h\colon \pi_1(S^1) \to H_1(S^1)$ is an isomorphism. Also, since $\pi_1(S^1) \cong \mathbb{Z}$ is an abelian group, thus $\pi_1(S^1) \cong \pi_1(S^1)^{\mathrm{ab}}$ and, up to this isomorphism, $\overline{h}$ coincides with h.

Next, take $X = \bigvee_{i\in I} S^1$ and let $r_i\colon \bigvee_{i\in I} S^1 \to S^1$ be retraction maps. We have a commutative diagram

$$\begin{tikzcd}
\pi_1(\bigvee_{i\in I} S^1) \arrow[r, "\bigoplus r_{i*}"] \arrow[d, "h"'] & \bigoplus_{i\in I}\pi_1(S^1) \arrow[d, "\cong"', "\bigoplus_{i\in I} h"] \\
H_n(\bigvee_{i\in I} S^1) \arrow[r, "\cong", "\bigoplus r_{i*}"'] & \bigoplus_{i\in I} H_n(S^1)
\end{tikzcd}$$

The upper map $\bigoplus r_{i*}$ essentially conincides with the abelianization of $\pi_1(\bigvee_{i\in I} S^1)$, and the map $\bigoplus_{i\in I} h$ coincides, up to an isomorphism, with $\overline{h}\colon \pi_1(\bigvee_{i\in I} S^1)^{\mathrm{ab}} \to H_1(\bigvee_{i\in I})$. It remains to notice that $\bigoplus_{i\in I} h$ is an isomorphism by the previous case.

As in the proof of Theorem 17.4, it remains to consider the case where X is a 2-dimensional CW complex of the form $X = \bigvee_{i\in I} S^1 \cup \bigcup_{k\in K} e^2_k$. Let $\varphi_k\colon S^1 \to \bigvee_{i\in I} S^1$ be the attaching map of the cell e^2_k. Denote $\psi := \bigvee_{k\in K} \varphi_k\colon \bigvee_{k\in K} S^1 \to \bigvee_{i\in I} S^1$ Also, let $j\colon \bigvee_{i\in I} S^1 \hookrightarrow X$ be the inclusion of the 1-skeleton of X. We have a sequence of group homomorphisms

$$\pi_1(\bigvee_{k\in K} S^1) \xrightarrow{\psi_*} \pi_1(\bigvee_{i\in I} S^1) \xrightarrow{j_*} \pi_1(X)$$

By van Kampen's Theorem the hoomorphism j_* is onto and $\ker j_* = N_{\pi_1(\bigvee_{i\in I} S^1)}(\operatorname{Im}\psi_*)$. Cosider the commutative diagram

$$\begin{tikzcd}
\pi_1(\bigvee_{k\in K} S^1)^{\mathrm{ab}} \arrow[r, "\psi_*"] \arrow[d, "\overline{h}"', "\cong"] & \pi_1(\bigvee_{i\in I} S^1)^{\mathrm{ab}} \arrow[r, "i_*"] \arrow[d, "\overline{h}"', "\cong"] & \pi_1(X)^{\mathrm{ab}} \arrow[r] \arrow[d, "\overline{h}"] & 0 \arrow[d, "\cong"] \\
H_1(\bigvee_{i\in I} S^1) \arrow[r, "\psi_*"'] & H_1(\bigvee_{i\in I} S^1) \arrow[r, "i_*"'] & H_1(X) \arrow[r] & 0
\end{tikzcd}$$

The upper row is exact by Lemma 17.7 and the lower row is exact by the long exact homology sequence associated to ψ_*. Using the Five Lemma we obtain that $\overline{h}\colon \pi_1(X)^{\mathrm{ab}} \to H_1(X)$ is an isomorphism. $\square$

17.8 Relative Hurewicz Homomorphism. Recall that $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$. Let $\bar{\gamma}_n$ denote a chosen generator of $H_n(D^n, S^{n-1})$. Given a pointed pair (X, A, x_0), and an element $[\varphi\colon (D^n, S^{n-1}, s_0) \to (X, A, x_0)] \in \pi_n(X, A, x_0)$ consider the function $\varphi_*\colon H_n(D^n, S^{n-1}) \to H_n(X, A)$.

17.9 Definition. The *relative Hurewicz homomorphism* is a function

$$h\colon \pi_n(X, A, x_0) \to H_n(X, A)$$

given by $h([\varphi]) = \varphi_*(\bar{\gamma}_n)$.

17.10 Proposition. *The relative Hurewicz homomorphism is a group homomorphism for $n \geq 2$.*

17.11 Relative Inverse Hurewicz Isomorphism Theorem. *Let (X, A) be a pair of simply connected CW complexes. If $H_i(X, A) = 0$ for all $0 < i < n$ for some $n \geq 2$ then $\pi_i(X, A) = 0$ for all $i < n$ and the Hurewicz homomorphism $h\colon \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.*

Proof. See tom Dieck, Theorem 20.1.3 p. 497. Uses commutativity of the diagram

$$\begin{tikzcd} \pi_i(X,A) \arrow[r] \arrow[d, "h"'] & \pi_i(X/A) \arrow[d, "h"] \\ H_i(X,A) \arrow[r] & H_i(X/A) \end{tikzcd}$$

and the Inverse Hurewicz Theorem 17.5 applied to the space X/A. □

17.12 Theorem. *Let X, Y be simply connected CW complexes and let $f\colon X \to Y$ be a map such that for some $n \geq 2$ the homorphism $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for $i < n$ and epimorphism for $i = n$. Then $f_*\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$ and epimorphism for $i = n$.*

Proof. Let M_f be the mapping cylinder of f. The assumption about f is equivalent to the condition that $H_i(M_f, X) = 0$ for $i < n$. By Theorem 17.11 this gives $\pi_i(M_f, X) = 0$ for $i < n$. The statement then follows from the long exact sequence of homotopy groups of the pair (M_f, X). □

17.13 Corollary. *Let $f\colon X \to Y$ be a map of simply connected CW complexes such that $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for all $i \geq 0$. Then f is a homotopy equivalence.*

Let $p_X\colon \widetilde{X} \to X$ denote the universal cover of a space X. Given a map $f\colon X \to Y$ we can find a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "p_X"'] & \widetilde{Y} \arrow[d, "p_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

17.14 Theorem. *Let $f\colon X \to Y$ be a map of path connected CW complexes. If the homomorphisms $f_*\colon \pi_1(X) \to \pi_1(Y)$ and $\widetilde{f}_*\colon H_i(\widetilde{X}) \to H_i(\widetilde{Y})$ for all $i \geq 0$ are isomorphisms then f is a homotopy equivalence.*

Proof. By Theorem 17.12 the map $\widetilde{f}_*\colon \pi_i(\widetilde{X}) \to \pi_i(\widetilde{Y})$ is an isomorphism for all $i \geq 0$. Since $p_{X*}\colon \pi_i(\widetilde{X}) \to \pi_i(X)$ and $p_{Y*}\colon \pi_i(\widetilde{Y}) \cong \pi_i(Y)$ are isomorphisms for $i \geq 2$, this gives that $f_*\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \geq 2$. By assumption, $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism as well, so f is a weak equivalence and thus a homotopy equivalence. □

18 | Spectral Sequences

18.1 Motivation. Hurewicz Isomorphism Theorem 17.4 lets us compute the first non-trivial homotopy group of a space X using homological methods. In order to extend this to to higher homotopy groups of X, one can attempt the following approach. Assume that $\pi_k(X) = 0$ for $k < n$ and that we know homology groups of X. This in particular gives us $\pi_n(X) \cong H_n(X)$. We can construct a map $f_n \colon X \to K(\pi_n(X), n)$ which induces an isomorphism on n-th homotopy groups. Let X_n denote the homotopy fiber of f_n. The long exact sequence of a fibration shows that $\pi_k(X_n) = 0$ for $k < n+1$ and $\pi_k(X) \cong \pi_k(X_n)$ for $k \geq n+1$. In particular using the Hurewicz Isomorphism Theorem we obtain

$$\pi_{n+1}(X) \cong \pi_{n+1}(X_n) \cong H_{n+1}(X_n)$$

Thus computations of $\pi_{n+1}(X)$ are reduced to computing a homology group of the space X_n.

This procedure can be repeated: once we know $\pi_{n+1}(X_n)$, we can construct a map $f_{n+1} \colon X_n \to K(\pi_{n+1}(X_n), n+1)$ that induces an isomorphism on $(n+1)$-st homotopy groups. Taking X_{n+1} to be the homotopy fiber of this map we obtain isomorphisms

$$\pi_{n+2}(X) \cong \pi_{n+2}(X_n) \cong \pi_{n+2}(X_{n+1}) \cong H_{n+2}(X_{n+1})$$

Proceeding recursively, we obtain that in order to compute homotopy groups of X it suffices to compute homology groups of spaces X_k for $k \geq n-1$ such that $X_{n-1} = X$ and which are connected by fibration sequences

$$X_{k+1} \to X_k \to K(\pi_{k+1}(X), k+1)$$

In order to carry out this program we would need to;

- calculate homology groups of Eilenberg-MacLane spaces $K(G, k)$;
- given a fibration sequence $F \to E \to B$ find a relationship between homology groups of the spaces F, E, and B.

Spectral sequences provide a tool for achieving the second of these objectives. They are helpful with the first one as well.

In this chapter we give the definition of a spectral sequence and some examples how spectral sequences are used. Explanation in which circumstances spectral sequences occur is left for later.

18.2 Definition. A *bigraded* abelian group G_{**} is a collection of abelian groups $G_{p,q}$ for $p, q \in \mathbb{Z}$.

18.3 Definition. A *(first quadrant, homological) spectral sequence* (E^r_{**}, d^r) is a sequence of bigraded abelian groups E^r_{**} for $r = 1, 2, \ldots$ such that:

1) $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$.
2) Each E^r_{**} is equipped with homomorphisms (*differentials*)

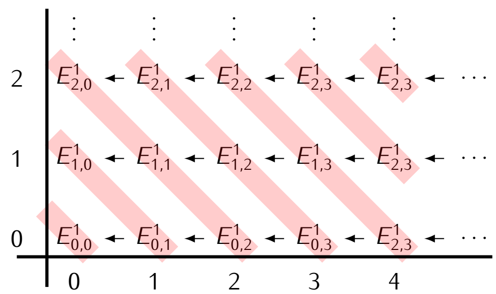
$$d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

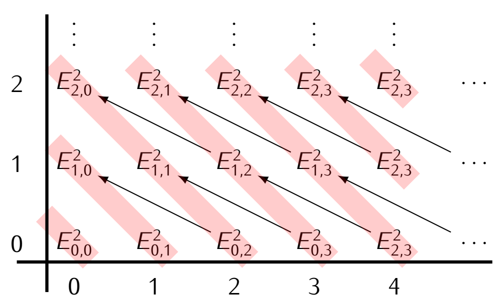
   satisfying $d^r d^r = 0$.
3) For each $r \geq 0$ we have $E^{r+1}_{p,q} \cong H_{p,q}(E^r_{**})$ where

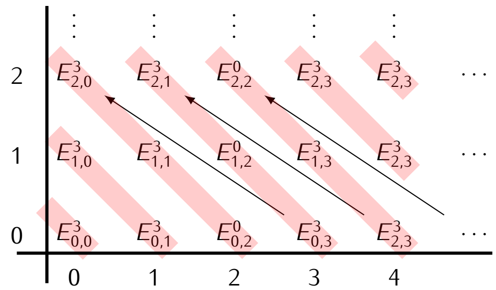
$$H_{p,q}(E^r_{**}) = \frac{\mathrm{Ker}(d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathrm{Im}(d^r \colon E^r_{p+r,q-r+1} \to E^r_{p,q})}$$

18.4 Note. The bigraded group E^r_{**} is called the *r-th page* of the spectral sequence.

Below are pictures of the first three pages of a spectral sequence. Notice that the differentials d^r always go between groups $E^r_{p,q}$ where $p + q = n$ for some n and groups where $p + q = n - 1$.







Since all groups $E^r_{p,q}$ with negative p or q are trivial, the differentials d^r originating at $E^r_{p,q}$ are trivial for $r > p$. Likewise, the differentials d^r terminating at $E^r_{p,q}$ are trivial if $r > q + 1$. As a consequence, for $r \leq$ max(p+1, q+2) we get

$$E^r_{p,q} = E^{r+1}_{p,q} = E^{r+2}_{p,q} = \dots$$

For each p, q, let $E^\infty_{p,q}$ denote this recurring group. These groups form a bigraded group E^∞_{**}.

In typical applications of spectral sequences, E^∞_{**} is related to some object of interest, e.g. homology groups of some space. This is done as follows. We start with a graded abelian group H_* i.e. a collection of abelian groups H_n for $n \in \mathbb{Z}$. A *filtration* of H_* is a sequence of graded subgrops:

$$0 = F_{-1}H_* \subseteq F_0H_* \subseteq F_1H_* \subseteq \dots \subseteq H_*$$

such that $\bigcup_{p=0}^\infty F_pH_* = H_*$.

18.5 Definition. We say that a spectral sequence (E^r_{**}, d^r) *converges* to a graded group H_* if there exists a filtration of H_* such that

$$E^\infty_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$$

for all p, q.

Results on existence spectral sequences usually say that there exists a spectral sequence for which we can say describe in some useful way groups $E^r_{p,q}$ for some fixed r, and that this sequence converges to some interesting graded group H_*. Here is one example of such a statement:

18.6 Theorem. *Let $p\colon E \to B$ be a Serre fibration and let $F = p^{-1}(b_0)$ for some $b_0 \in B$. If the space B is simply connected then there exists a spectral sequence (E^r_{**}, d^r) such that*

$$E^2_{p,q} \cong H_p(B, H_q(F))$$

for all p, q, and which converges to $H_*(E)$.

The spectral sequence described in this theorem is called the *Serre spectral sequence* of the fibration p.

The next result provides an example how spectral sequences are used in computations.

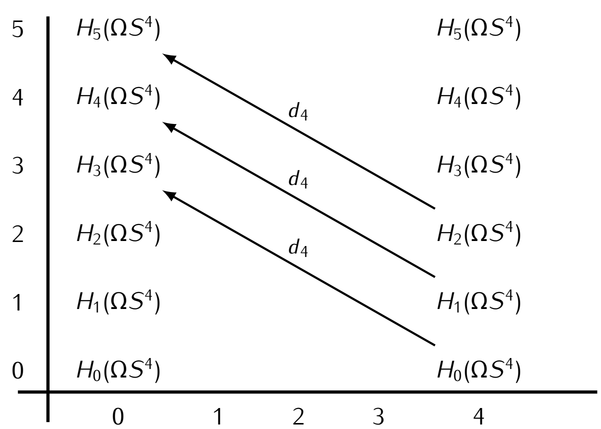
18.7 Theorem. *If* $n \geq 2$ *then*

$$H_m(\Omega S^n) \cong \begin{cases} \mathbb{Z} & \text{if } (n-1)|m \\ 0 & \text{otherwise} \end{cases}$$

Proof. The space ΩS^n is the fiber of a Serre fibration $p\colon P \to S^n$ with a contractible space P. Consider the Serre spectral sequence of this fibration. We have

$$E^2_{p,q} \cong H_p(S^n, H_q(\Omega S^n)) \cong \begin{cases} H_q(\Omega S^n) & \text{if } p = 0, n \\ 0 & \text{otherwise} \end{cases}$$

For example, for $n = 4$ the second page of this spectral sequence looks as follows:



All differentials in the spectral sequence are trivial, except, possibly $d^n\colon E^n_{p,q} \to E^n_{0,q+n-1}$. It follows that $E^2_{**} = E^n_{**}$ and $E^{n+1}_{**} = E^\infty_{**}$. The total space P of the fibration is contractible, so $H_0(P) = \mathbb{Z}$ and $H_p(P) = 0$ for $p > 0$. By Theorem 18.6 we have $E^\infty_{p,q} \cong F_pH_{p+q}(P)/F_{p-1}H_{p+q}(P)$ for some filtration $\{F_pH_*(P)\}$ of $H_*(P)$. It follows that

$$E^{n+1}_{p,q} = E^\infty_{p,q} \cong \begin{cases} \mathbb{Z} & \text{if } (p,q) = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

Since $E^{n+1}_{p,q} \cong H_{p,q}(E^2_{**})$ we obtain that $H_0(\Omega S^n) \cong \mathbb{Z}$ and $H_p(\Omega S^n) = 0$ for $0 < p \leq n-2$. Also, all differentials d^n must be isomorphisms. This gives:

$$H_p(\Omega S^n) \cong H_{p+(n-1)}(\Omega S^n) \cong H_{p+2(n-1)}(\Omega S^n) \cong H_{p+3(n-1)}(\Omega S^n) \cong \dots$$

Taking $p = 0$ we obtain that $H_m(\Omega S^n) \cong \mathbb{Z}$ if $(n-1)|m$. In all other cases $H_m(\Omega S^n) \cong H_p(\Omega S^n)$ for some $0 < p \leq n-2$, and so it is a trivial group. □

18.8 Note. The proof of Theorem 18.7 used the observation that all differentials d^r in the Serre spectral sequence of the fibration $p\colon P \to S^n$ were trivial for $r \geq n+1$. A situation like this appears frequently in computations involving spectral sequences, which motivates the next definition.

18.9 Definition. We say that a spectral sequence *collapses* at the page r_0 if all differentials d^r are trivial for $r \geq r_0$.

If a spectral sequence collapses at the page r_0 then we have $E^{r_0}_{p,q} = E^{r_0+1}_{p,q} = \ldots = E^{\infty}_{p,q}$.

19 | Spectral Sequence From a Filtration

The goal of this chapter is to describe a construction of a spectral sequence associated to a filtration of a chain complex. By a chain complex we will mean here a non-negatively graded chain complex, i.e. a chain complex of abelian group

$$\ldots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots$$

such that $C_n = 0$ for $n < 0$.

19.1 Definition. Let C_* be a chain complex. A *filtration* of C_* is a sequence of subcomplexes

$$0 = F_{-1}C_* \subseteq F_0C_* \subseteq \ldots \subseteq C_*$$

such that $\bigcup_p F_pC_* = C_*$. The filtration is *first quadrant* if $H_k(F_pC_*/F_{p-1}C_*) = 0$ for $k < p$.

19.2 Example. Let X be a CW complex. The filtration of X with respect to the skeleta

$$\varnothing = X^{(-1)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X$$

defines a filtration of the singular chain complex of X:

$$0 = C_*(X^{(-1)}) \subseteq C_*(X^{(1)}) \subseteq C_*(X^{(2)}) \subseteq \ldots \subseteq C_*(X)$$

Since $H_p(C_*(X^{(q)}), C_*(X^{(q-1)})) \cong H_p(X^{(q)}, X^{(q-1)}) = 0$ for $p < q$, so this is a first quadrant filtration.

19.3 Note. A filtration $\{F_pC_*\}$ of a chain complex C_* induces a filtration of homology groups of C_*

$$0 = F_{-1}H_n(C_*) \subseteq F_1H_n(C_*) \subseteq \ldots \subseteq H_n(C_*)$$

where $F_pH_n(C_*) := \mathrm{Im}(H_n(F_pC_*)) \to H_n(C_*))$. Since $\bigcup F_pC_* = C_*$ we have $\bigcup_p F_pH_n(C_*) = H_n(C_*)$.

Assume that we are given a chain complex C_* with differentials $\partial\colon C_n \to C_{n-1}$, and that $\{F_pC_*\}$ is a filtration of C_*. Denote $E^0_{p,q} := F_pC_{p+q}/F_{p-1}C_{p+q}$. We will consider subgroups $B^\infty_{p,q}, Z^\infty_{p,q} \subseteq E^0_{p,q}$ defined as follows:

$$\begin{aligned} Z^\infty_{p,q} &= \{[x] \in E^0_{p,q} \mid \partial z = 0 \in C_{p+q-1} \text{ for some } z \in [x]\} \\ B^\infty_{p,q} &= \{[x] \in E^0_{p,q} \mid \partial b \in [x] \text{ for some } b \in C_{p+q+1}\} \end{aligned}$$

We have $B^\infty_{p,q} \subseteq Z^\infty_{p,q}$. Define $E^\infty_{p,q} := Z^\infty_{p,q}/B^\infty_{p,q}$.

19.4 Proposition. *$E^\infty_{p,q} \cong F_pH_{p+q}(C_*)/F_{p-1}H_{p+q}(C_*)$.*

Proof. Exercise. □

The spectral sequence we are constructing will introduce intermediate stages $E^0_{p,q}$ between $E^0_{p,q}$ and $E^\infty_{p,q}$ such that each stage is closer approximation of $E^\infty_{p,q}$. More precisely, for $r = 1, 2, \dots$ define:

$$\begin{aligned} Z^r_{p,q} &= \{[x] \in E^0_{p,q} \mid \partial z \in F_{p-r}C_{p+q-1} \text{ for some } z \in [x]\} \\ B^r_{p,q} &= \{[x] \in E^0_{p,q} \mid \partial b \in [x] \text{ for some } b \in F_{p+r-1}C_{p+q+1}\} \end{aligned}$$

We have inclusions

$$B^1_{p,q} \subseteq B^2_{p,1} \subseteq \dots \subseteq B^\infty_{p,q} \subseteq Z^\infty_{p,q} \subseteq \dots \subseteq Z^2_{p,q} \subseteq Z^1_{p,q}$$

Define: $E^r_{p,q} := Z^r_{p,q}/B^r_{p,q}$.

19.5 Proposition. *In the setting described above we have*

1) *$B^\infty_{p,q} = \bigcup_r B^r_{p,q}$ and $Z^\infty_{p,q} = \bigcap_r Z^r_{p,q}$.*
2) *$E^1_{p,q} \cong H_{p+q}(F_pC_*/F_{p-1}C_*)$.*

Proof. Exercise. □

19.6 Note. Since $F_pC_* = 0$ if $p < 0$, we get that $E^1_{p,q} = 0$ for $p < 0$. If F_pC_* is a first quadrant filtration, then we also get $E^1_{p,q} = 0$ for $q < 0$.

The groups $E^r_{p,q}$ will form pages of our spectral sequence. In order to finish the construction we still need to specify differentials $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$. This can be done as follow. By definition, every element of $E^r_{p,q} = Z^r_{p,q}/B^r_{p,q}$ is represented by $z \in F_pC_{p+q}$ such that $\partial z \in F_{p-r}C_{p+q-1}$. We set $d^r([z]) = [\partial z]$.

19.7 Proposition. *The function $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ is a well-defined homomorphism. Moreover, $d^rd^r = 0$ and $H_{p,q}(E^r_{**}, d^r) \cong E^{r+1}_{p,q}$.*

Proof. Exercise. □

Here is a result summarizing the above constructions:

19.8 Theorem. *Let C_* be a chain complex with a first quadrant filtration*

$$0 = F_{-1}C_* \subseteq F_0 C_* \subseteq \ldots \subseteq C_*$$

such that $\bigcup_p F_p C_* = C_*$. Then there exists a first quadrant spectral sequence E^r_{**} such that

- $E^1_{p,q} = H_{p+q}(F_p C_* / F_{p-1}(C_*))$;
- *the sequence converges to $H_*(C_*)$.*

Applying this to the singular chain complex of a topological space we obtain:

19.9 Theorem. *Let X be a space with a filtration*

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X$$

such that for every compact subset $A \subseteq X$ we have $A \subseteq X_p$ for some $p \geq 0$. Assume also that $H_k(X_p, X_{p-1}) = 0$ for $k < p$. Then there exists a first quadrant spectral sequence E^r_{**} such that

- $E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$
- *The sequence converges to $H_*(X)$. More precisely,*

  $$E^\infty_{p,q} = F_p H_{p+q}(X)/F_{p-1}H_{p+q}(X)$$

  where $F_p H_n(X) = \mathrm{Im}(H_n(X_p) \to H_n(X))$.

Proof. The filtration of the space X induces a filtration of the singular chain complex of X:

$$0 = C_*(X_{-1}) \subseteq C_*(X_0) \subseteq C_*(X_1) \subseteq \ldots \subseteq C_*(X)$$

The condition on the compact sets in X implies that $\bigcup_p C_*(X_p) = C_*(X)$. Thus the statement follows from Theorem 19.8. □

19.10 Note. The differentials $d^1 \colon E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1}, X_{p-2}) = E^1_{p-1,q}$ can be more explicitly described as compositions

$$H_{p+q}(X_p, X_{p-1}) \xrightarrow{\delta} H_{p+q-1}(X_{p-1}) \to H_{p+q-1}(X_{p-1}, X_{p-2})$$

where δ is the boundary map from the homology long exact sequence of the pair (X_p, X_{p-1}).

19.11 Example. For a CW complex X consider the filtration of X by its skeleta:

$$\varnothing = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X$$

In the spectral sequence associated to this filtration we have

$$E^1_{p,q} = H_{p+q}(X^{(p)}, X^{(p-1)}) = \begin{cases} H_p(X^{(p)}, X^{(p-1)}) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

As a consequence the first page of the spectral sequence looks as follows:

$$\begin{array}{c|cccccc}
2 & \vdots & & \vdots & & \vdots & \\
1 & 0 & & 0 & & 0 & \\
0 & H_0(X^{(0)}, X^{(-1)}) & \xleftarrow{d^1} & H_1(X^{(1)}, X^{(0)}) & \xleftarrow{d^1} & H_2(X^{(2)}, X^{(1)}) & \xleftarrow{d^1} \cdots \\
\hline
 & 0 & & 1 & & 2 &
\end{array}$$

The spectral sequence collapses at the second page, so $E^2_{p,q} \cong E^\infty_{p,q}$. We also have

$$E^\infty_{p,q} = \frac{\operatorname{Im}(H_{p+q}(X^{(p)}) \to H_{p+q}(X))}{\operatorname{Im}(H_{p+q}(X^{(p-1)}) \to H_{p+q}(X))} \cong \begin{cases} H_p(X) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

As a consequence, singular homology groups of X are isomorphic to the homology groups of the chain complex given by the first row of E^1. This chain complex is the cellular chain complex of X.

20 | Serre Spectral Sequence

The Serre spectral sequence is a special case of the spectral sequence associated to a filtration described in Theorem 19.9.

20.1 Definition. Let $p\colon E \to B$ is a Serre fibration where B is a connected CW complex. Let

$$\varnothing = B^{(-1)} \subseteq B^{(0)} \subseteq \dots B$$

be the filtration of B by skeleta. Taking $E^p := p^{-1}(B^{(k)})$ we obtain a filtration of the space E:

$$\varnothing = E^{-1} \subseteq E^0 \subseteq \dots \subseteq E$$

The *Serre spectral sequence* of the fibration p is the spectral sequence associated to this filtration.

By Theorem 19.9 we get that $E^1_{p,q} = H_{p+q}(E^p, E^{p-1})$ and that E^r_{**} converges to $H_*(E)$. The advantage of the Serre spectral sequence is that we can explicitly describe its second page:

20.2 Theorem. *Let E^r_{**} be the Serre spectral sequence of a fibration $p\colon E \to B$. Let $F = p^{-1}(b_0)$ for some $b_0 \in B$. If the space B is simply connected then $E^2_{p,q} \cong H_p(B, H_q(F))$.*

While we will skip the proof of this result, it is useful to point out that the assumption that B is simply connected is needed in order to obtain a canonical identification between fibers of p taken over different points. Assume for a moment p is a Hurewicz fibration and that $b_0, b_1 \in B$, Let $F_i = p^{-1}(b_i)$

for $i = 0, 1$. Given a path $\omega\colon [0,1] \to B$ such that $\omega(0) = b_0$ $\omega(1) = b_1$, consider the diagram

$$\begin{tikzcd}
F_0 \times \{0\} \arrow[r, "i_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
F_0 \times [0,1] \arrow[r, "\omega\,\mathrm{pr}"'] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

where $\mathrm{pr}\colon F_0 \times [0,1] \to [0,1]$ is the projection map and $i_0\colon F_0 \to E$ is the inclusion. A lift h of $\omega\,\mathrm{pr}$ gives a homotopy in E between the map i_0 and a certain map $h_1\colon F_0 \to F_1$. One can show that this map h_1 is a homotopy equivalence and that its homotopy class depends only on the homotopy class of the path ω (relative its endpoints). If the space B is simply connected, all paths joining b_0 and b_1 are homotopic, so the homotopy class of h_1 is uniquely defined. In particular, we obtain canonical isomorphisms of homology groups $h_{1*}\colon H_q(F_0) \xrightarrow{\cong} H_q(F_1)$. If p is a Serre fibration, we can use the same argument, but in order to get the lift h we replace F_0 by its CW approximation.

We have seen already one application of the Serre spectral sequence in Theorem 18.7. Here is another one:

20.3 Proposition. *Let $S^k \to S^m \xrightarrow{p} S^n$ be a homotopy fibration sequence with $n \geq 1$. Then $k = n - 1$ and $m = 2n - 1$.*

Proof. If $n = 1$ then the long exact sequence of homotopy groups shows that we must have $m = 1$ and $k = 0$. Assume then that $n \geq 2$. Consider the Serre spectral sequence of this fibration. Its second page $E^2_{p,q} \cong H_p(S^n, H_q(S^k))$ has only four non-zero terms, all isomorphic to $\mathbb{Z}$:

$$\begin{array}{c|cc}
k & E^2_{0,k} & E^2_{n,k} \\
 & & \\
0 & E^2_{0,0} & E^2_{n,0} \\
\hline
 & 0 & n
\end{array}$$

All differentials originating and terminating at $E^r_{0,0}$ and $E^r_{k,n}$ are trivial, so $E^2_{,0} = E^\infty_{0,0}$ and $E^2_{n,k} = E^\infty_{n,k}$. The page E^∞_{**} can have non-zero terms $E^\infty_{p,q}$ only if $(p,q) = (0,0)$ or $p + q = m$. It follows that $k + n = m$. The terms $E^2_{0,k}$ and $E^2_{n,0}$ must kill each other, so they must be connected by a differential. This is possible only if $k = n - 1$. Taken together these observations imply that p is a fibration sequence of the form $S^{n-1} \to S^{2n-1} \xrightarrow{p} S^n$. $\square$

Hopf bundles give examples of fibration sequences $S^{n-1} \to S^{2n-1} \to S^n$ for $n = 1, 2, 4, 8$. A theorem of Adams implies that these are the only fibration sequences where all three spaces are spheres.

21 | Serre classes

The motivation for this chapter is to show that the following holds.

21.1 Theorem. *The homotopy groups $\pi_n(S^m)$ are finitely generated for all $n, m \geq 1$.*

This will follow from a more general result that will be stated in terms of Serre classes.

21.2 Definition. A *Serre class* is a non-empty collection $\mathcal{C}$ of abelian groups satisfying the property that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of abelian groups then $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$

We will say that a Serre class $\mathcal{C}$ is a *Serre ring* if in addition it satisfies that if $A, B \in \mathcal{C}$ then $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}(A, B) \in \mathcal{C}$.

We will also say that a Serre class is *acyclic* if for every group $A \in \mathcal{C}$ we have $H_q(K(A, 1)) \in \mathcal{C}$ for all $q > 0$.

21.3 Proposition. *Let $\mathcal{C}$ is a Serre class. The following hold:*

1) $0 \in \mathcal{C}$.
2) *If $A \in \mathcal{C}$ and $A' \cong A$ then $A' \in \mathcal{C}$.*
3) *If $B \subseteq A$ then $A \in \mathcal{C}$ if and only if $B, A/B \in \mathcal{C}$.*
4) *If $A \to B \to C$ is an exact sequence and $A, C \in \mathcal{C}$ then $B \in \mathcal{C}$.*
5) *If $0 = A_{-1} \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$ then $A_n \in \mathcal{C}$ if and only if $A_i/A_{i-1} \in \mathcal{C}$ for all i.*

Proof. Exercise. $\square$

21.4 Proposition. *Let $\mathcal{C}$ is a Serre ring. If X is a path connected space such that $H_q(X) \in \mathcal{C}$ for all $0 < q < p$ then $H_p(X; G) \in \mathcal{C}$ for any group $G \in \mathcal{C}$.*

Proof. By the Universal Coefficient Theorem we have

$$H_p(X;G) \cong (H_p(X) \otimes G) \oplus \mathrm{Tor}(H_{p-1}(X), G)$$

This immediately gives that $H_p(X;G) \in \mathcal{C}$ for $p \geq 2$ For $p = 0$ we gave $H_0(X;G) \cong G \in \mathcal{C}$ while for $p = 1$ we obtain $H_1(X;G) \cong H_1(X) \otimes G \in \mathcal{C}$. □

21.5 Proposition. *All of the following are acyclic Serre rings:*

- $\mathcal{C}_{fin}$ *= the class of all finite abelian groups.*
- $\mathcal{C}_{fg}$ *= the class of all finitely generated abelian groups.*
- $\mathcal{C}_{tor}$ *= the class of all torsion abelian groups.*
- $\mathcal{C}_p$ *= the class of all p-torsion abelian groups for a given prime p.*

21.6 Theorem. *Let* $F \to E \xrightarrow{p} B$ *be a Serre fibration with a simply connected space* B*, and let* $\mathcal{C}$ *be a Serre ring. If for two of the spaces* F*,* E*,* B *the homology groups* $H_q(-)$ *are in* $\mathcal{C}$ *for all* $q > 0$ *then the same holds for the third space.*

Proof. There are three cases to consider.

Case 1: $H_q(F), H_q(B) \in \mathcal{C}$ for all $q > 0$.

Consider the Serre spectral sequence of the fibration p. We have $E^2_{p,q} = H_p(B, H_q(F))$ so by Proposition 21.4 we get that $E^2_{p,q} \in \mathcal{C}$ for all $(p,q) \neq (0,0)$. Next, since groups $E^3_{p,q}$ are obtained by taking quotients of subgroups of the groups $E^2_{p,q}$, we get that $E^3_{p,q} \in \mathcal{C}$ for all $(p,q) \neq (0,0)$. Inductively, we obtain that $E^r_{p,q} \in \mathcal{C}$ for all $r \geq 2$ and $(p,q) \neq (0,0)$, and so also $E^\infty_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$. For $q > 0$ the groups $H_q(E)$ admit a finite filtration such filtration quotients are isomorphic to groups $E^\infty_{p,q}$ with $(p,q) \neq 0$. This implies that $H_q(E) \in \mathcal{C}$.

Case 2: $H_q(F), H_q(E) \in \mathcal{C}$ for all $q > 0$.

Since all groups $E^\infty_{p,q}$ are quotients of subgroups of $H_{p+q}(E)$, we have $E^\infty_{p,q} \in \mathcal{C}$ for all $(p,q) \neq (0,0)$. We will show that $E^2_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$ by induction with respect to p. For $p = 0$ this holds since $E^2_{0,q} \cong H_q(F)$. Assume that it also holds for $E^2_{i,q}$ for all $i < p$. It follows that $E^r_{i,q} \in \mathcal{C}$ for all $i < p$ and all $r \geq 2$.

Since all differentials terminating at $E^r_{p,0}$ are trivial, for each r we have an exact sequence

$$E^{r+1}_{p,0} \to E^r_{p,0} \xrightarrow{d^r} E^r_{p-r,r-1}$$

By assumption $E^r_{p-r,r-1} \in \mathcal{C}$, so if $E^{r+1}_{p,0} \in \mathcal{C}$ then the same is true for $E^r_{p,0}$. Since $E^{p+1}_{p,q} = E^\infty_{p,q} \in \mathcal{C}$, arguing inductively over decreasing values of r we obtain that $E^r_{p,0} \in \mathcal{C}$ for all $r \geq 2$. In particular, $H_p(B) = E^2_{p,0} \in \mathcal{C}$. Using Proposition 21.4 we obtain that $E^2_{p,q} = H_p(B, H_q(F)) \in \mathcal{C}$ for all $q \geq 0$.

Case 3: $H_q(B), H_q(E) \in \mathcal{C}$ for all $q > 0$.

This is similar to case 2. □

21.7 Proposition. *If $\mathcal{C}$ is an acyclic Serre ring then for every $A \in \mathcal{C}$ and $n \geq 1$ we have $H_q(K(A, n)) \in \mathcal{C}$.*

Proof. We argue by induction with respect to n. The case $n = 1$ holds by definition of acyclicity of a Serre class. Assume that the statement is true for some $n \geq 1$. For $A \in \mathcal{C}$ consider the homotopy fibration sequence $K(A, n) = \Omega K(A, n+1) \to * \to K(A, n+1)$. Since $H_q(K(A, n)), H_q(*) \in \mathcal{C}$ for all $q > 0$, by Theorem 21.6 we obtain that $H_q(K(A, n+1)) \in \mathcal{C}$. □

21.8 Theorem. *Let $\mathcal{C}$ be an acyclic Serre ring. If X is a simply connected space then the following conditions are equivalent:*

1) $\pi_n(X) \in \mathcal{C}$ for all $n \geq 1$
2) $H_n(X) \in \mathcal{C}$ for all $n \geq 1$

The proof of Theorem 21.8 will make use of the notion of Postnikov sections:

21.9 Definition. Let X be a path connected space. The n-th Postnikov section of X is a space X_n together with a map $f: X \to X_n$ such that

1) $f_*: \pi_q(X) \to \pi_q(X_n)$ is an isomorphism for $q \leq n$
2) $\pi_q(X) = 0$ for $q > n$.

The n-th Postnikov section of a space X can be constructed glueing to X cells in dimensions $n+1$ and higher to kill all homotopy groups above $\pi_n(X)$. The map $f: X \to X_n$ is then given by the inclusion.

Proof of Theorem 21.8.

1) ⇒ 2) Let X_n denote the n-th Postnikov section of X. By Theorem 16.4 we have $H_q(X) \cong H_q(X_n)$ for all $q < n$, so it will be enough to show that $H_q(X_n) \in \mathcal{C}$ for all $n, q > 0$. We will prove this by induction with respect to n. For $n = 2$ we have $X_2 = K(\pi_2(X), 2)$, so the statement holds by Proposition 21.7. Assume that it also holds for some $n \geq 2$. Notice that we have a fibration sequence

$$K(\pi_{n+1}(X), n+1) \to X_{n+1} \to X_n$$

Using Proposition 21.7 again we get that $H_q(K(\pi_{n+1}(X), n+1)) \in \mathcal{C}$ for $q > 0$, so using Theorem 21.6 we obtain that $H_q(X_{n+1}) \in \mathcal{C}$ for $q > 0$.

2) ⇒ 1) We will show that $\pi_n(X) \in \mathcal{C}$ by induction with respect to n. Since X is simply connected, for $n = 2$ by the Hurewicz Isomorphism Theorem we get $\pi_2(X) \cong H_2(X) \in \mathcal{C}$. Next, assume that $\pi_q(X) \in \mathcal{C}$ for all $q \leq n$ and consider the fibration sequence

$$\mathrm{hofib}\, f \to X \to X_n$$

where X_n is the n-th Postnikov section of X. Notice that

$$\pi_q(\mathrm{hofib}\, f) = \begin{cases} 0 & if q \leq n \\ \pi_q(X) & if q > n \end{cases}$$

Since $\pi_q(X_n) \in \mathcal{C}$ for all q, thus by part 1) $\Rightarrow$ 2) we get that $H_q(X_n) \in \mathcal{C}$ for all $q > 0$. By assumption $H_q(X) \in \mathcal{C}$ for $q > 0$. Therefore, using Theorem 21.6 we obtain that $H_q(\mathrm{hofib}\, f) \in \mathcal{C}$ for $q > 0$. Since $\mathrm{hofib}\, f$ is n-connected, by the Hurewicz Isomorphism Theorem we get $H_{n+1}(\mathrm{hofib}\, f) \cong \pi_{n+1}(\mathrm{hofib}\, f) \cong \pi_{n+1}(X)$. This gives $\pi_{n+1}(X) \in \mathcal{C}$.

□

22 | Cohomology via Homotopy

Recall (12.9) that for an abelian group G by $K(G, n)$ we denote the Eilenberg-MacLane space such that $\pi_n(K(G, n)) \cong G$. We will also denote by $K(G, 0)$ the discrete space consisting of elements of the group G. Notice that for every n we have a weak equivalence

$$K(G, n) \xrightarrow{\simeq} \Omega K(G, n+1) \xrightarrow{\simeq} \Omega^2 K(G, n+2)$$

For any pointed CW complex X this induces a bijection of sets of pointed homotopy classes

$$[X, K(G, n)]_* \xrightarrow{\cong} [X, \Omega^2 K(G, n+2)]_*$$

Since $[X, \Omega^2 K(G, n+2)]_*$ has a natural structure of an abelian group (9.8), we obtain in this way an abelian group structure on $[X, K(G, n)]_*$.

The main goal of this chapter is to show that the following holds:

22.1 Theorem. *Let G be an abelian group.*

1) For any pointed CW complex X and $n \geq 0$ there exists an isomorphism

$$T_X : [X, K(G, n)]_* \xrightarrow{\cong} \widetilde{H}^n(X; G)$$

where $\widetilde{H}^n(X; G)$ is the n-th reduced singular cohomology group of X with coefficients in G.

2) These isomorphisms are natural. That is, if $f : X \to Y$ is a map of pointed CW complexes then the following diagram commutes:

$$\begin{tikzcd}
{[X, K(G, n)]_*} \arrow[d, "T_X"', "\cong"] & {[Y, K(G, n)]_*} \arrow[l, "f^*"'] \arrow[d, "\cong"', "T_Y"] \\
\widetilde{H}^n(X; G) & \widetilde{H}^n(Y; G) \arrow[l, "f^*"]
\end{tikzcd}$$

22.2 Note. Let $\varphi\colon X \to K(G,n)$ be a pointed map. By part 2) of Theorem 22.1 we obtain a commutative diagram

$$\begin{tikzcd}
{[X, K(G,n)]_*} \arrow[d, "T_X"', "\cong"] & {[K(G,n), K(G,n)]_*} \arrow[l, "\varphi^*"'] \arrow[d, "\cong"', "T_{K(G,n)}"] \\
\widetilde{H}^n(X; G) & \widetilde{H}^n(K(G,n); G) \arrow[l, "\varphi^*"]
\end{tikzcd}$$

This gives:

$$T_X([\varphi]) = T_X(\varphi^*([\mathrm{id}_{K(G,n)}])) = \varphi^* T_{K(G,n)}([\mathrm{id}_{K(G,n)}])$$

This implies that for any pointed CW complex X the bijection T_X is determined by the cohomology class $\alpha_n = T_{K(G,n)}([\mathrm{id}_{K(G,n)}]) \in \widetilde{H}^n(K(g,n); G)$. This class is called the *fundamental class*.

22.3 Note. An unpointed version of Theorem 22.1 also holds: for any CW complex X there exists a natural isomorphism $T_X\colon [X, K(G,n)] \xrightarrow{\cong} H^n(X; G)$. This can be derived from Theorem 22.1 as follows. For a CW complex X let X_+ denote the space obtained by adding one 0-cell to X: $X_+ = X \sqcup \{+\}$. We consider $+$ as the basepoint of X_+. We have bijections $[X, K(G,n)] \cong [X_+, K(G,n)]_*$ and $H^n(X; G) \cong \widetilde{H}^n(X_+; G)$. Thus if $[X_+, K(G,n)]_* \cong \widetilde{H}^n(X_+; G)$ then $[X, K(G,n)] \cong H^n(X; G)$.

22.4 Example. For any CW complex X we have:

- $[X, S^1] \cong H^1(X, \mathbb{Z})$
- $[X, \mathbb{CP}^\infty] \cong H^2(X, \mathbb{Z})$
- $[X, \mathbb{RP}^\infty] \cong H^2(X, \mathbb{Z}/2)$

The proof of Theorem 22.1 will proceed as follows. First, we will define a general notion of a cohomology theory, which consists of a sequence of functors $\{h^n\}_{n\in\mathbb{Z}}$ from the category of pointed CW complexes to the category of abelian groups satisfying certain axioms. We will show that both assignments $X \mapsto \widetilde{H}^n(X; G)$ and $X \mapsto [X, K(G,n)]_*$ satisfy this definition. Then we will prove that if $\{h^n\}$ is any generalized cohomology theory such that $h^n(S^0) \cong \widetilde{H}^n(S^0; G)$ for all n, then for every pointed CW complex X and every n there is a natural isomorphism $h^n(X) \to H^n(X; G)$. Since the cohomology theory defined by Eilenberg-MacLane space satisfies this property, Theorem 22.1 will follow.

22.5 Definition. Let $\mathbf{CW}_*$ denote the category of pointed CW complexes and basepoint preserving maps. A *(reduced) cohomology theory* consists of:

- A sequence contravariant functors $\{h^n\colon \mathbf{CW}_* \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$.
- For every $X \in \mathbf{CW}_*$ and every $n \in \mathbb{Z}$ a natural isomorphism $\Sigma\colon h^n(X) \to h^{n+1}(\Sigma X)$. Naturality

means that for any map $f\colon X \to Y$ we have a commutative diagram

$$\begin{tikzcd} h^n(Y) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "f^*"'] & h^{n+1}(\Sigma Y) \arrow[d, "\Sigma f^*"] \\ h^n(X) \arrow[r, "\cong", "\Sigma"'] & h^{n+1}(\Sigma X) \end{tikzcd}$$

Moreover, the following axioms are satisfied:

- (**Homotopy axiom**) If $f, g\colon X \to Y$ are maps such that $f \simeq g$ then $f^* = g^*\colon h^n(Y) \to h^n(X)$ for all n.
- (**Exactness axiom**) For any pair (X, A) where $A \subseteq X$ is a subcomplex, $i\colon A \hookrightarrow X$ is the inclusion and $q\colon X \to X/A$ is the quotient map, the following sequence is exact:

$$h^n(A) \xleftarrow{i^*} h^n(X) \xleftarrow{q^*} h^n(X/A)$$

- (**Wedge axiom**) For any family of pointed CW complexes $\{X_i\}_{i\in I}$ the inclusion maps $X_j \hookrightarrow \bigvee_{i\in I} X_i$ induce isomorphisms $h^n\left(\bigvee_{i\in I} X_i\right) \xrightarrow{\cong} \prod_{i\in I} h^n(X_i)$ for all n.

22.6 Some consequences of the axioms.

- $h^n(*) = 0$ for all n.
- For any pair (X, A) where $A \subseteq X$ is a subcomplex, there is a long exact sequence

$$\dots \longleftarrow h^n(A) \xleftarrow{i^*} h^n(X) \xleftarrow{q^*} h^n(X/A) \xleftarrow{\delta} h^{n-1}(A) \longleftarrow \dots$$

The map $\delta\colon h^{n-1}(A) \to h^n(X/A)$ is the composition of the suspension isomorphism $\Sigma\colon h^{n-1}(A) \to h^n(\Sigma A)$, the homomorphism induced by the quotient map $C_i \to C_i/X \cong \Sigma A$, where C_i is the cone of the inclusion $i\colon A \hookrightarrow X$, and the isomorphism induced by the homotopy equivalence $X/A \xrightarrow{\simeq} C_i$.

22.7 Example. Given an abelian group G, consider the reduced singular cohomology functors $X \mapsto \widetilde{H}^n(X; G)$. For $n < 0$ set $\widetilde{H}^n(X; G) = 0$ for all X. Then the functors $\{\widetilde{H}^n(-; G)\}$ define a cohomology theory.

22.8 Example. For an abelian group G, let $h^n_G(X) = [X, K(G, n)]_*$. For $n < 0$ we set $K(G, n) = *$. Then the functors $\{h^n_G\}$ form a cohomology theory. To define the suspension isomorphism

$$\Sigma\colon h^n_G(X) = [X, K(G, n)]_* \longrightarrow [\Sigma X, K(G, n+1)]_* = h^{n+1}_G(\Sigma X)$$

choose a week equivalence $\varphi_n\colon K(G, n) \to \Omega K(G, n{+}1)$. This induces an isomorphism $\varphi_n^*\colon [X, K(G, n)]_* \to [X, \Omega K(G+1, n)]_*$. Then we compose it with the adjunction isomorphism $[X, \Omega K(G+1, n)]_* \xrightarrow{\cong} [\Sigma X, K(G+1, n)]_*$

It is obvious that $\{h^n_G\}$ satisfies the homotopy axiom. The exactness axiom is also satisfied by Proposition 10.12. The wedge axiom holds since for any family of well-pointed spaces $\{X_i\}_{i\in I}$ and any pointed space Z, inclusion maps induce a bijection $[\bigvee_{i\in I} X_i, Z]_* \to \prod_{i\in I}[X_i, Z]_*$.

Notice that the only property of the spaces $K(G,n)$ used in Example 22.8 is that for each n there exist a weak homotopy equivalence $\varphi_n\colon K(G,n) \to \Omega K(G,n+1)$. This motivates the following definition.

22.9 Definition. An *Ω-spectrum* $(K_n, \varphi_n)_{n\in\mathbb{Z}}$ is a sequence of pointed spaces K_n and weak homotopy equivalences $\varphi_n\colon K_n \xrightarrow{\cong} \Omega K_{n+1}$.

By the same argument as in Example 22.8 we obtain:

22.10 Proposition. *Every Ω-spectrum $(K_n, \varphi_n)_{n\in\mathbb{Z}}$ defines a cohomology theory $\{h^n\}_{n\in\mathbb{Z}}$ given by $h^n(X) = [X, K_n]_*$.*

22.11 Definition. A cohomology theory $\{h^n\}$ satisfies the *dimension axiom* if $h^n(S^0) = 0$ for $n \neq 0$.

22.12 Theorem. *Let $\{h^n_1\}_{n\in\mathbb{Z}}$ and $\{h^n_2\}_{n\in\mathbb{Z}}$ be cohomology theories that satisfy the dimension axiom and such that $h^0_1(S^0) \cong h^0_2(S^0)$. Then for each pointed CW complex there exists natural isomorphism $T_X\colon h^n_1(X) \xrightarrow{\cong} h^n_2(X)$. Naturality means that each pointed map $f\colon X \to Y$ gives a commutative diagram*

$$\begin{tikzcd}
h^*_1(Y) \arrow[r, "f^*"] \arrow[d, "T_Y"', "\cong"] & h^*_1(X) \arrow[d, "\cong"', "T_X"] \\
h^*_2(Y) \arrow[r, "f^*"'] & h^*_2(X)
\end{tikzcd}$$

Proof of Theorem 22.1. For the reduced singular cohomology theory we have

$$\widetilde{H}^n(S^0; G) \;\cong\; \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Also,

$$[S^0, K(G,n)]_* \;\cong\; \pi_0(K(G,n)) \;\cong\; \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore we can apply Theorem 22.12. $\square$

The proof of Theorem 22.12 will require some preparation.

22.13 Lemma. *Let $\{h^n\}$ be a cohomology theory satisfying the dimension axiom. Then:*

1) $h^q(\bigvee_{i\in I} S^n) = 0$ *for* $q \neq n$.

2) For any CW complex X *the inclusion of the* n*-th skeleton* $j\colon X^{(n)} \hookrightarrow X$ *induces an isomorphism* $j^*\colon h^q(X) \xrightarrow{\cong} h^q(X^{(n)})$ *for all* $q < n$ *Also,* $h^q(X^{(n)}) = 0$ *for* $q > n$.

Proof. 1) This follows from the isomorphisms

$$h^q(\bigvee_{i\in I} S^n) \cong \prod_{i\in I} h^q(S^n) \cong \prod_{i\in I} h^q(\Sigma^n S^0) \cong \prod_{i\in I} h^{q-n}(S^0)$$

2) For finite-dimensional CW complexes this can be proved by induction on skeleta of X, using cofibration sequences $X^{(k-1)} \hookrightarrow X^{(k)} \to \bigvee S^k$. This can be generalized to the case $\dim X = \infty$ using the infinite telescope construction (see e.g. Hatcher, *Algebraic Topology* pp. 138–139), which gives a cofibration sequence $\bigvee_k X^{(k)} \to X \to \bigvee_k \Sigma X^{(k)}$. □

For abelian groups G, H let $\mathrm{Hom}(G, H)$ denote the set of homomorphisms $G \to H$. This set has a group structure with addition defined by $(\varphi + \psi)(g) = \varphi(g) + \psi(g)$ for $\varphi, \psi \in \mathrm{Hom}(G, H)$.

22.14 Proposition. *Let* $\{h^n\}$ *be a cohomology theory. For a pointed CW complex* X *and* $n \geq 1$ *consider the map*

$$\Phi\colon \pi_n(X) \to \mathrm{Hom}(h^q(X), h^q(S^n))$$

that sends an element $[\varphi\colon S^n \to X] \in \pi_n(X)$ *to the induced homomorphism* $\varphi^*\colon h^q(X) \to h^q(S^n)$*. Then the map* Φ *is a homomorphism of groups.*

Proof. The constant map $S^n \to X$ induces the trivial homomorphism $h^q(X) \to h^q(S^n)$, so Φ preserves trivial elements. Let $[\varphi], [\psi] \in \pi_n(X)$. The element $[\varphi]\cdot[\psi] \in \pi_n(X)$ is the homotopy class of the map

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\varphi\vee\psi} S^n$$

where p is the pinch map. We need to show that $p^*(\varphi \vee \psi)^* = \varphi^* + \psi^*\colon h^q(X) \to h^q(S^n)$. This follows from commutativity of the following diagram:

$$\begin{tikzcd}
h^q(X) \arrow[r, "(\varphi\vee\psi)^*"] \arrow[dr, bend right, "\varphi^*\times\psi^*"'] & h^q(S^n\vee S^n) \arrow[r, "p^*"] \arrow[d, "\cong"] & h^q(S^n) \\
 & h^q(S^n)\times h^q(S^n) \arrow[ur, bend right, "\mu"'] &
\end{tikzcd}$$

Here the isomorphism $h^q(S^n \vee S^n) \to h^q(S^n) \times h^q(S^n)$ is induced by the inclusion maps and μ is given by $\mu(x, y) = x + y$. □

22.15 Corollary. *Let $\{h_1^n\}$, $\{h_2^n\}$ be cohomology theories and let $T\colon h_1^q(S^n) \to h_2^q(S^n)$ be an arbitrary homomorphism. Then for any map $f\colon S^n \to S^n$ the following diagram commutes:*

$$\begin{tikzcd} h_1^q(S^n) \arrow[r, "f^*"] \arrow[d, "T"'] & h_1^q(S^n) \arrow[d, "T"] \\ h_2^q(S^n) \arrow[r, "f^*"'] & h_2^q(S^n) \end{tikzcd} \tag{$*$}$$

Proof. Using Proposition 22.14 we obtain that homotopy classes of maps f for which the diagram $(*)$ commutes form a subgroup of $\pi_n(S^n)$. Since the homotopy class of the identity map $\mathrm{id}_{S^n}\colon S^n \to S^n$ belongs to this subgroup, the subgroup contains all elements of $\pi_n(S^n)$. □

Let $\{h^n\}$ be a cohomology theory and let X be a CW complex. For $n \geq 0$ consider the map

$$\varphi_n\colon X^{(n+1)}/X^{(n)} \to \Sigma X^{(n)} \to \Sigma(X^{(n)}/X^{(n-1)})$$

Let $d^n\colon h^n(X^{(n)}/X^{(n-1)}) \to h^{n+1}(X^{(n+1)}/X^{(n)})$ be a homomorphism given by the composition

$$d^n\colon h^n(X^{(n)}/X^{(n-1)}) \xrightarrow{\Sigma} h^{n+1}(\Sigma(X^{(n)}/X^{(n-1)})) \xrightarrow{\varphi_n^*} h^{n+1}(X^{(n+1)}/X^{(n)})$$

22.16 Proposition. *Let $\{h^n\}$ be a cohomology theory. For a CW complex X consider the maps*

$$h^{n-1}(X^{(n-1)}/X^{(n-2)}) \xrightarrow{d^{n-1}} h^n(X^{(n)}/X^{(n-1)}) \xrightarrow{d^n} h^{n+1}(X^{(n+1)}/X^{(n)})$$

Then $\mathrm{Im}(d^{n-1}) \subseteq \mathrm{Ker}(d^n)$. Moreover, of $\{h^n\}$ satisfies the dimension axiom then $h^n(X) \cong h^n(X^{(n+1)}) \cong \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$.

Proof. Exercise. Use Lemma 22.13 and long exact sequences for the pairs $(X^{(n+1)}, X^{(n)})$, $(X^{(n)}, X^{(n-1)})$ and $(X^{(n-1)}, X^{(n-2)})$. □

Proof of Theorem 22.12. We will construct natural isomorphisms $T_X\colon h_1^*(X) \to h_2^*(X)$ in a few steps.

1) We define $T_{S^n}\colon h_1^*(S^n) \to h_2^*(S^n)$ by induction with respect to n. By assumption we have isomorphisms $T_{S^0}\colon h_1^*(S^0) \to h_2^*(S^0)$. Assume that T_{S^n} is already defined for some n. Choose a homeomorphism $f_{n+1}\colon S^{n+1} \to \Sigma S^n$ and define $T_{S^{n+1}}$ so that the following diagram commutes:

$$\begin{tikzcd} h_1^*(S^n) \arrow[r, "f_n^*\Sigma", "\cong"'] \arrow[d, "T_{S^n}"', "\cong"] & h_1^*(S^{n+1}) \arrow[d, "T_{S^{n+1}}"] \\ h_2^*(S^n) \arrow[r, "\cong", "f_n^*\Sigma"'] & h_2^*(S^{n+1}) \end{tikzcd}$$

2) By the definition of a cohomology theory, for any set J inclusion maps $S^n \to \bigvee_{j\in J} S^n$ induce isomorphisms $h_i^*(\bigvee_{j\in J} S^n) \xrightarrow{\cong} \prod_{j\in J} h_i^*(S^n)$ Choose isomorphisms $T_{\bigvee_{j\in J} S^n}$ so that the following diagram commutes:

$$\begin{tikzcd} h_1^*(\bigvee_{j\in J} S^n) \arrow[r, "\cong"'] \arrow[d, "T_{\bigvee_{j\in J} S^n}"', "\cong"] & \prod_{j\in J} h_1^*(S^n) \arrow[d, "\cong"', "\prod_{j\in J} T_{S^n}"] \\ h_2^*(\bigvee_{j\in J} S^n) \arrow[r, "\cong"] & \prod_{j\in J} h_2^*(S^n) \end{tikzcd}$$

We claim that isomorphisms $T_{\bigvee_{j\in J} S^n}$ defined above are natural with respect to all maps $f\colon \bigvee_{j\in J} S^n \to \bigvee_{k\in K} S^n$. That is, for any such map the following diagram commutes:

$$\begin{tikzcd} h_1^*(\bigvee_{k\in K} S^n) \arrow[r, "f^*"] \arrow[d, "T_{\bigvee_{k\in K} S^n}"', "\cong"] & h_1^*(\bigvee_{j\in J} S^n) \arrow[d, "\cong"', "T_{\bigvee_{j\in J} S^n}"] \\ h_2^*(\bigvee_{k\in K} S^n) \arrow[r, "f^*"'] & h_2^*(\bigvee_{j\in J} S^n) \end{tikzcd}$$

Using the isomorphisms $h_i^*(\bigvee_{j\in J} S^n) \cong \prod_{j\in J} h_i^*(S^n)$ and compactness of spheres, this can be reduced (exercise) to checking that for any pointed map $f\colon S^n \to S^n$ we have $f^* T_{S^n} = T_{S^n} f_*$. This, however, follows from Corollary 22.15.

3) There are now two possible ways of obtaining an isomorphism $h_1^*(\Sigma \bigvee_{i\in I} S^n) \to h_2^*(\Sigma \bigvee_{i\in I} S^n)$. One is to use the suspension isomorphisms $\Sigma\colon h_k^*(\bigvee_{i\in I} S^n) \to h_k^{*+1}(\Sigma \bigvee_{i\in I} S^n)$ and the already defined isomorphism $T_{\bigvee_{i\in I} S^n}$. Another is to use the homeomorphism

$$\bigvee_{i\in I} S^{n+1} \xrightarrow{\bigvee f_{n+1}} \bigvee_{i\in I} \Sigma S^n \xrightarrow{\bigvee \Sigma j_i} \Sigma \bigvee_{i\in I} S^n$$

and the isomorphism $T_{\bigvee_{i\in I} S^{n+1}}$. Here $f_{n+1}\colon S^{n+1} \to \Sigma S^n$ is a homeomorphism and Σj_i is the suspension of the inclusion map $j_i\colon S^n \to \bigvee_{i\in I} S^n$. One can check that both these methods give the same isomorphism $T_{\Sigma \bigvee S^n}\colon h_1^*(\Sigma \bigvee_{i\in I} S^n) \to h_2^*(\Sigma \bigvee_{i\in I} S^n)$. Using naturality of isomorphisms $T_{\bigvee S^{n+1}}$ established in 2), we obtain that for any map $f\colon \bigvee_{j\in J} S^{n+1} \to \Sigma \bigvee_{k\in K} S^n$ we have a commutative diagram

$$\begin{tikzcd} h_1^{*-1}(\bigvee_{k\in K} S^n) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "T_{\bigvee_{k\in K} S^n}"', "\cong"] & h_1^*(\Sigma \bigvee_{k\in K} S^n) \arrow[r, "f^*"] \arrow[d, "T_{\Sigma \bigvee_{k\in K} S^n}"', "\cong"] & h_1^*(\bigvee_{j\in J} S^{n+1}) \arrow[d, "\cong"', "T_{\bigvee_{j\in J} S^{n+1}}"] \\ h_2^{*-1}(\bigvee_{k\in K} S^n) \arrow[r, "\cong", "\Sigma"'] & h_2^*(\Sigma \bigvee_{k\in K} S^n) \arrow[r, "f^*"'] & h_2^*(\bigvee_{j\in J} S^{n+1}) \end{tikzcd}$$

4) Let now X be an arbitrary pointed CW complex. By Proposition 22.16 for $k = 1, 2$ we have isomorphisms $h_k^n(X) \cong \mathrm{Ker}(d_k^n)/\mathrm{Im}(d_k^{n-1})$ where $d_k^n\colon h_k^n(X^{(n)}/X^{(n-1)}) \to h_k^{n+1}(X^{(n+1)}/X^{(n)})$. If we could

find isomorphisms $T_{X,n}\colon h_1^n(X^{(n)}/X^{(n-1)}) \to h_2^n(X^{(n)}/X^{(n-1)})$ such that $d_2^n T_{X,n} = T_{X,n+1} d_1^n$, then they would induce isomorphisms

$$T_X\colon h_1^n(X) \cong \operatorname{Ker}(d_1^n)/\operatorname{Im}(d_1^{n-1}) \longrightarrow \operatorname{Ker}(d_2^n)/\operatorname{Im}(d_2^{n-1}) \cong h_2^n(X)$$

Such isomorphisms $T_{X,n}$ can be constructed as follows. For each n choose a homeomorphism $f_n\colon \bigvee_{i\in I_n} S^n \to X^{(n)}/X^{(n-1)}$. Then define $T_{X,n}$ so that the following diagram commutes:

$$\begin{tikzcd}
h_1^n(X^{(n)}/X^{(n-1)}) \arrow[r, "f_n^*", "\cong"'] \arrow[d, "T_{X,n}"'] & h_1^n(\bigvee_{i\in I_n} S^n) \arrow[d, "\cong"', "T_{\bigvee_{i\in I_n} S^n}"] \\
h_2^n(X^{(n)}/X^{(n-1)}) \arrow[r, "\cong", "f_n^*"'] & h_2^n(\bigvee_{i\in I_n} S^n)
\end{tikzcd}$$

Commutativity of isomorphisms $T_{X,n}$ with the maps d_k^n follows from commutativity of the following diagram:

$$\begin{tikzcd}
h_1^n(X^{(n)}/X^{(n-1)}) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "f_n^*"', "\cong"] \arrow[rr, bend left, "d_1^n"] \arrow[ddd, bend right, "T_{X,n}"'] & h_1^{n+1}(\Sigma(X^{(n)}/X^{(n-1)})) \arrow[r, "\varphi_n^*"] \arrow[d, "\cong"', "(\Sigma f_n)^*"] & h_1^{n+1}(X^{(n+1)}/X^{(n)}) \arrow[d, "\cong"', "f_{n+1}^*"] \arrow[ddd, bend left, "T_{X,n+1}"] \\
h_1^n(\bigvee_{i\in I_n} S^n) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "T_{\bigvee_{i\in I^n} S^n}"', "\cong"] & h_1^{n+1}(\Sigma\bigvee_{i\in I_n} S^n) \arrow[r, "((\Sigma f_n)^{-1}\varphi_n f_{n+1})^*"] \arrow[d, "T_{\Sigma\bigvee_{i\in I^n} S^n}"', "\cong"] & h_1^{n+1}(\bigvee_{i\in I_{n+1}} S^n) \arrow[d, "\cong"', "T_{\bigvee_{i\in I^{n+1}} S^{n+1}}"] \\
h_2^n(\bigvee_{i\in I_n} S^n) \arrow[r, "\Sigma", "\cong"'] & h_2^{n+1}(\Sigma\bigvee_{i\in I_n} S^n) \arrow[r, "((\Sigma f_n)^{-1}\varphi_n f_{n+1})^*"] & h_2^{n+1}(\bigvee_{i\in I_{n+1}} S^n) \\
h_2^n(X^{(n)}/X^{(n-1)}) \arrow[r, "\Sigma", "\cong"'] \arrow[u, "f_n^*", "\cong"'] \arrow[rr, bend right, "d_2^n"'] & h_2^{n+1}(\Sigma(X^{(n)}/X^{(n-1)})) \arrow[r, "\varphi_n^*"] \arrow[u, "\cong", "(\Sigma f_n)^*"'] & h_2^{n+1}(X^{(n+1)}/X^{(n)}) \arrow[u, "\cong", "f_{n+1}^*"']
\end{tikzcd}$$

The maps φ_n are defined as in Proposition 22.16. The middle squares commute by 3).

To check that the isomorphisms T_X are natural with respect to maps $f\colon X \to Y$, notice that we can assume that f is cellular and so it induces homomorphisms $f^*\colon h_k^n(Y^{(n)}/Y^{(n-1)}) \to h_k^n(X^{(n)}/X^{(n-1)})$ which commute with the maps d_k^n. Then it remains check that $f^* T_{Y,n} = T_{X,n} f^*$. This can be verified using naturality of the isomorphisms $T_{\bigvee S^n}$ established in 2).

□

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