HOMOGENEOUS IDEALS WITH MINIMAL SINGULARITY THRESHOLDS

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ABSTRACT. Let $(\mathcal{O}_n, \mathfrak{m})$ denote the ring of germs of holomorphic functions $\mathbb{C}^n \to \mathbb{C}$, and let $I \subseteq \mathcal{O}_n$ be an $\mathfrak{m}$-primary ideal. Demailly and Pham showed that $\operatorname{lct}(I) \geq \frac{1}{e_1(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)}$, where $e_j(I)$ is the mixed multiplicity $e(I, \ldots, I, \mathfrak{m}, \ldots, \mathfrak{m})$, with I repeated j times and $\mathfrak{m}$ repeated $n-j$ times.

We generalize the lower bound to the case of an arbitrary ideal of an excellent regular local (or standard-graded) ring of equal characteristic, with $\operatorname{lct}(I)$ replaced by the F-threshold $c^{\mathfrak{m}}(I)$ in positive characteristic. Our main result is a classification of homogeneous ideals in polynomial rings for which the lower bound is attained.

1. Introduction

Let X be a smooth scheme, $Y \subseteq X$ a proper closed subscheme, and $y \in Y$ a point. We study numerical invariants measuring the singularities of the pair (X, Y) at the point y: in characteristic zero, we consider the log canonical threshold; in positive characteristic, we consider the F-pure threshold.

The log canonical threshold (lct) has attracted considerable attention in algebraic geometry due to its connections with the Minimal Model Program and singularity theory. The F-pure threshold (fpt) is an analog of the lct in positive characteristic, related via reduction to characteristic $p \gg 0$ [21]. The fpt has provided insights into positive characteristic singularities [tk] and the connection to the lct has been leveraged to better understand singularities in characteristic zero [tk].

Both the lct and fpt are numerical invariants associated to a pair (X, Y) which measure the singularities of the embedding $Y \subseteq X$. In recent years, many authors have found lower bounds on the lct (and fpt) in terms of more classical invariants of the embedding [35, 16, 15, 40, 29, 13, 28]. The first of these bounds is due to Skoda [38], who showed that

$$\frac{1}{\operatorname{mult}_y(Y)} \leq \operatorname{lct}_y(X, Y). \tag{1}$$

The case of equality in Equation (1) has long been known to experts: $\frac{1}{\operatorname{mult}_y(Y)} = \operatorname{lct}_y(X, Y)$ if and only if Y is a smooth divisor at y. Skoda's original formulation of Equation (1) concerned germs of plurisubharmonic functions, and the

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equality case in the analytic formulation was settled in ambient dimension 2 by Favre and Jonsson [14] and in full generality by Guan and Zhou [19]. We consider the equality case in a generalization of Skoda's lower bound.

Let I denote the defining ideal of Y in $\mathcal{O}_{X,y}$ and $n = \dim \mathcal{O}_{X,y}$. In the case that $\mathcal{O}_{X,y}/I$ is zero-dimensional, Demailly and Pham strenghthened Equation (1) by incorporating higher-codimension information about the embedding, namely the mixed multiplicities $e_j(I) := e(I, \dots, I, \mathfrak{m}_y, \dots, \mathfrak{m}_y)$.

$$\frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)} \leq \mathrm{lct}_y(Y, X). \tag{2}$$

Suppose instead $l < n$ and $\mathrm{ht}(I) \geq l$. As in [36, Theorem 1.4], we may let $L = V(h_{l+1}, \dots, h_n)$ denote the vanishing locus of $n-l$ general linear forms in $\mathcal{O}_n$ and apply the bound Equation (2) and [30, Example 9.5.4] to $I|_L$. This yields

$$\frac{1}{e_1(I|_L)} + \cdots + \frac{e_{l-1}(I|_L)}{e_l(I|_L)} \leq \mathrm{lct}(I|_L) \leq \mathrm{lct}(I). \tag{3}$$

We then define $E_l(I) := \frac{1}{e_1(I|_L)} + \cdots + \frac{e_{l-1}(I|_H)}{e_l(I|_L)}$, which is well-defined for sufficiently general L. The content of Equation (3) is that $E_l(I) := E_l(I|_H) \leq \mathrm{lct}(I)$. If instead $(R, \mathfrak{m})$ is an arbitrary regular local ring and $I \subseteq R$ with $\mathrm{ht}(I) \geq l$, $E_l(I)$ is defined similarly – in particular, when $R/\mathfrak{m}$ is infinite, we let H denote the vanishing set of $n-l$ elements of $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are sufficiently general and set $E_l(I) := E_l(I|_H)$. See Definition 2.35 for a precise definition.

Our first contribution is to generalize Equation (3) to arbitrary characteristic.

Theorem A (Theorem 3.9)**.** *Let $(R, \mathfrak{m})$ be a regular local ring or a polynomial ring over a perfect field. When $\mathrm{char}\, R = 0$, assume that R is excellent. Let $I \subseteq R$ be an ideal of height at least l, which is assumed to be homogeneous when R is graded. Let $c(I)$ denote the lct (resp. fpt) of I. Then $E_l(I) \geq c(I)$.*

In contrast to Skoda's bound, little is known about the case of equality in Equation (2). The best result in this direction is due to Bivià-Ausina [5], who proved an analog of Rees's theorem for the invariant E_n and classified cases of equality under the assumption that the integral closure $\overline{I}$ of I is a monomial ideal. Our main result is a classification of the cases of equality in the graded setting.

Theorem B (Theorem 5.11)**.** *Let k be an algebraically-closed field of characteristic zero (resp $p > 0$). Let $R = k[x_1, \dots, x_n]$ and let $I \subseteq R$ be a homogeneous ideal of height at least l. If $E_l(I) = \mathrm{lct}(I)$ (resp. $E_l(I) = \mathrm{fpt}(I)$), then there exist integers $d_1, \dots, d_l$ such that, in suitable coordinates, we have*

$$\overline{I} = \overline{\left(x_1^{d_1}, \dots, x_l^{d_l}\right)}.$$

We briefly outline the proof of Theorem 5.11, which reduces to the case $l = n$; we focus on this case. Assume $E_n(I) = c(I)$. Write $I = I_1 + \cdots + I_r$, where I_j is generated by d_j-forms and $d_1 < \cdots < d_r$.

(1) Using techniques from [5, 32], we control the generic initial ideals $\{\mathrm{gin}(I^n)\}_{n\geq 1}$.
(2) Using (1), we obtain a formula for $e_j(I)$ in terms of the numbers $\{d_j, \mathrm{ht}(I_1 + \cdots + I_j)\}$, which allows us to reduce to the case of a complete intersection.
(3) We prove the result by induction on the number of distinct degrees $d_1, \ldots, d_r$.

In the case $r = 1$, any $\mathfrak{m}$-primary ideal I generated by d-forms automatically satisfies $\overline{I} = \mathfrak{m}^d$, so there is no way to use $r = 1$ as a useful base case for our induction. Instead, we use $r = 2$. In this case, we show (Lemma 4.6) that $\mathrm{fpt}_0(I) = E_n(I)$ if and only if $\mathrm{fpt}_0(I_1) = \mathrm{ht}(I_1)/d_1$. As $\overline{I} = \overline{\overline{I_1} + \mathfrak{m}^{d_2}}$, it suffices to show that $\overline{I_1} = (x_1, \ldots, x_{\mathrm{ht}(I_1)})^{d_1}$ in suitable coordinates. In characteristic zero, this follows from [15, Theorem 3.5]. In positive characteristic, this fact is [2, Theorem 3.17].

We conjecture that Theorem 5.11 holds more generally; see Conjectures 6.2 and 6.4.

2. Preliminaries

2.1. ***F*-Pure and Log Canonical Thresholds.** We begin with a formal definition of the lct. For a detailed introduction, see [33].

Definition 2.1 (Log Resolution)**.** Let X be a smooth variety over a characteristic zero field and $Y \subseteq X$ a proper closed subvariety with defining ideal $\mathfrak{a}$. Let W be a smooth variety. A projective morphism $\pi : W \to X$ is a *log resolution* of (X, Y) if π is an isomorphism over $X \setminus Y$ and the inverse image $\mathfrak{a} \cdot \mathcal{O}_W$ is the ideal of a Cartier divisor D such that $D + K_{W/X}$ has simple normal crossings.

Definition 2.2 (Log canonical threshold, [33] Theorem 1.1)**.** Let X be a smooth variety and $Y \subseteq X$ a closed subvariety with defining ideal $\mathfrak{a}$. By Hironaka's theorem on resolution of singularities in characteristic zero, there exists a log resolution $\pi : W \to X$ of the pair (X, Y). If $E_1, \ldots, E_N$ are the exceptional divisors of π, then we can write

$$D = \sum_{i=1}^{N} a_i E_i \qquad \text{and} \qquad K_{W/X} = \sum_{i=1}^{N} k_i E_i.$$

For $y \in Y$, the quantity $\min_{i: y \in \pi(E_i)} \frac{k_i+1}{a_i}$ does not depend on the log resolution π and is denoted $\mathrm{lct}_y(X, Y)$, the **log canonical threshold** of (X, Y) at y. If a point y is not specified, we define

$$\mathrm{lct}(X, Y) = \min_{y \in Y} \mathrm{lct}_y(X, Y) = \min_i \frac{k_i + 1}{a_i}.$$

We make the convention that $\mathrm{lct}_y(X, Y) = \infty$ when $y \notin Y$. Additionally, when k is a characteristic zero field, $R = k[x_1, \ldots, x_n]$, and $I \subseteq R$ is an ideal, we set $\mathrm{lct}(R, I) := \mathrm{lct}(\mathbb{A}^n, V(I))$ and $\mathrm{lct}_0(R, I) := \mathrm{lct}_0(\mathbb{A}^n, V(I))$.

In fact, one can define the log canonical threshold of (X, Y) whenever X is an integral *excellent* scheme; see [17, §2].

For background on the F-pure threshold, we direct the reader to [40]. In this subsection, we summarize several key definitions and results.

Definition 2.3. Let R be a ring of characteristic $p > 0$. We let F_*R denote the R-module structure on R given by restriction of scalars along the Frobenius map $F : R \to R$. We say R is F-finite if F_*R is module-finite over R.

Definition 2.4 ([40]). Let R be an F-finite ring, $I \subseteq R$ an ideal, and $t \in \mathbb{R}^+$. The pair (R, I^t) is *F-split* if for $e \gg 0$, the map

$$I^{\lfloor t(p^e-1)\rfloor} \cdot \mathrm{Hom}(F^e_* R, R) \to R$$

is surjective. The *F-pure threshold* of the pair (R, I) is the supremum of all t such that (R, I^t) is F-split. We denote this quantity by $\mathrm{fpt}(R, I)$, or $\mathrm{fpt}(I)$ when the ambient ring is clear.

Let k be an F-finite field of characteristic $p > 0, R = k[x_1, \ldots, x_n]$, and $I \subseteq R$. Let $\mathfrak{m} := (x_1, \ldots, x_n)$ denote the homogeneous maximal ideal of R. We then let $\mathrm{fpt}_0(I)$ denote the quantity $\mathrm{fpt}(R_\mathfrak{m}, IR_\mathfrak{m})$.

In practice, we do not use the above definitions. Instead, we use the characterization of the F-pure threshold as the F-threshold at the maximal ideal and the lct as a limit of fpt.

Definition 2.5. Let R be a ring of characteristic $p > 0$. Let $\mathfrak{a}, J$ be ideals of R such that $\mathfrak{a} \subseteq \sqrt{J}$. For each nonnegative integer e, we define

$$\nu^J_\mathfrak{a}(p^e) := \max\{t \in \mathbb{Z}^+ : \mathfrak{a}^t \not\subseteq J^{[p^e]}\}.$$

By [8], the sequence $\dfrac{\nu^J_\mathfrak{a}(p^e)}{p^e}$ has a limit as $e \to \infty$; we denote this limit by $c^J(\mathfrak{a})$ and refer to it as the *F-threshold of $\mathfrak{a}$ at J.*

Proposition 2.6. *Let $(R, \mathfrak{m})$ be an F-finite regular local ring. Then the F-pure threshold of the pair (R, I) is equal to $c^\mathfrak{m}(I)$. If instead R is a polynomial ring over an F-finite field and $I \subseteq R$ is a homogeneous ideal, then the same holds when we let $\mathfrak{m}$ denote the homogeneous maximal ideal of R.*

Proof. The local case is [34, Remark 1.5] and the graded case is [7, Proposition 3.10]. □

Proposition 2.7 ([22], Theorem 6.8)**.** *Let A be a finite-type $\mathbb{Z}$-algebra and $\mathfrak{a} \subseteq A[x_1, \ldots, x_n]$ an ideal. Set $k = Frac(A)$. Then we have*

$$\mathrm{lct}_0(\mathfrak{a} \otimes_A k) = \lim_{\mu \in \max \mathrm{Spec}\, A, |A/\mu| \to \infty} \mathrm{fpt}_0(A/\mu[x_1, \ldots, x_n], \mathfrak{a} \otimes_A A/\mu).$$

Many of our results make sense for both fpt and lct, so we introduce the following notation to avoid stating the same results once each for characteristic zero and positive characteristic.

Notation 2.8. *Let k be a field, $R = k[x_1, \dots, x_n], \mathfrak{m} = (x_1, \dots, x_n)$, and $I \subseteq R$ an ideal. We define the quantity $c(R, I)$ as follows:*

$$c(R, I) = \begin{cases} c^{\mathfrak{m}}(I) & \text{char } R = p > 0 \\ \mathrm{lct}_0(R, I) & \text{char } R = 0 \end{cases}.$$

Similarly, let $(R, \mathfrak{m})$ be a regular local ring and $I \subseteq R$ an ideal. We define

$$c(R, I) = \begin{cases} c^{\mathfrak{m}}(I) & \text{char } R = p > 0 \\ \mathrm{lct}(R, I) & \text{char } R = 0 \textit{ and } R \textit{ is excellent} \end{cases}$$

If the ambient ring is clear, we will use $c(I)$ as shorthand.

Proposition 2.9 (Properties of the singularity threshold)**.** *Assume either setting of Notation 2.8, and let I, J be nonzero ideals of R. The following properties hold for $c(-)$ and $c_0(-)$:*

(i) If $I \subseteq J$, then $c(I) \leq c(J)$.
(ii) For all $m > 0$, we have $c(I^m) = m^{-1}c(I)$.
(iii) We have $c(I) = c(\overline{I})$, where $\overline{I}$ denotes the integral closure of I.
(iv) If $J \subseteq R$ is an ideal such that R/J is regular, then $c(R, I) \geq c(R/J, (I + J)/J)$.
(v) If $J \subseteq R$ is another nonzero ideal, then $c(I + J) \leq c(I) + c(J)$.
(vi) Suppose $R \subseteq S$ is a ring extension. If $\text{char } R = 0$, assume S is excellent. In both of the following cases, $c(I) = c(IS)$.
- *$(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are local and $\mathfrak{m}S = \mathfrak{n}$;*
- *$R = k[x_1, \dots, x_n]$, L/k is a field extension, and $S = L[x_1, \dots, x_n]$.*

Proof. First, we verify (i)-(v). For characteristic zero, see [33, Properties 1.12, 1.13, 1.15, 1.17, 1.20]. For characteristic $p > 0$, see [40, Proposition 2.2 (1), (2), (6), Proposition 4.4].

For (vi), note that both cases are regular and faithfully flat extensions. In characteristic zero, see [27, Proposition 1.9]. In characteristic $p > 0$, see [26, Proposition 2.2 (v)]. □

2.2. Monomial Ideals.

Notation 2.10. *For a vector of ring elements $\mathbf{f} = f_1, \dots, f_r$ and a vector of nonnegative integers $\mathbf{a} = a_1, \dots, a_r$, we let $\mathbf{f}^{\mathbf{a}}$ denote the element $f_1^{a_1} \dots f_r^{a_r}$. A boldface, lowercase letter always refers to a tuple of integers or ring elements.*

Definition 2.11. Let k be a field and let $R = k[x_1, \dots, x_n]$ or $k[\![x_1, \dots, x_n]\!]$. For a tuple $\mathbf{x} = x_{i_1}, \dots, x_{i_r}$ with $1 \leq i_1 < \dots < i_r \leq n$, we let $\mathrm{Mon}(\mathbf{x})$ denote the monoid of monomials in the variables $\mathbf{x}$. In particular, $\mathrm{Mon}(x_1, \dots, x_n)$ denotes the set of all monomials in R.

When working with monomial ideals, one often identifies a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with the point $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$. For future reference, it will help to give a name to this identification.

Definition 2.12. Let k be a field. We define the map

$$\log : \mathrm{Mon}(x_{i_1}, \ldots, x_{i_r}) \to \mathbb{Z}^r_{\geq 0}, \qquad \log(x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}) = (a_1, \ldots, a_r).$$

Definition 2.13. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal. Then the *Newton Polytope* of $\mathfrak{a}$, denoted $\Gamma(\mathfrak{a})$, is the convex hull in $\mathbb{R}^n_{\geq 0}$ of $\log(\mathfrak{a})$. We let $\mathrm{conv}(-)$ denote the convex hull of a set.

Remark 2.14. We record several properties of $\Gamma(\mathfrak{a})$.

(i) $\Gamma(\mathfrak{a})$ is a closed, convex, unbounded subset of the first orthant of $\mathbb{R}^n$.
(ii) When $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, the complement of $\Gamma(\mathfrak{a})$ inside the first orthant is an open, bounded polyhedron.
(iii) For two ideals $\mathfrak{a}, \mathfrak{b}$, the Minkowski sum of $\Gamma(\mathfrak{a})$ and $\Gamma(\mathfrak{b})$ is equal to $\Gamma(\mathfrak{a}\mathfrak{b})$. In particular, $\Gamma(\mathfrak{a}^n) = n\Gamma(\mathfrak{a})$.

The following proposition shows that Newton polytope of a monomial ideal determines the singularity threshold.

Proposition 2.15. *Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal. Then*

$$c(\mathfrak{a}) = \frac{1}{\mu}, \text{ where } \mu = \inf\{t : t\vec{1} \in \Gamma(\mathfrak{a})\}.$$

Proof. See [24], Example 5 for characteristic zero and [23], Proposition 36 for prime characteristic. □

Following the proof of [16], Theorem 1.4 and the terminology of [32], we also define the *limiting polytope* of a graded system of monomial ideals.

Definition 2.16. Let $\mathfrak{a}_\bullet$ be a graded system of monomial ideals in $k[x_1, \ldots, x_n]$ – that is, suppose $\mathfrak{a}_r\mathfrak{a}_s \subseteq \mathfrak{a}_{r+s}$ for all $r, s \in \mathbb{Z}^+$. We define $\Gamma(\mathfrak{a}_\bullet)$ as the closure in $\mathbb{R}^{n+1}$ of the ascending union $\{\frac{1}{2^m}\Gamma(\mathfrak{a}_{2^m})\}_{m>0}$.

Definition 2.17. Let $>$ be a monomial order on R. We set $\Gamma_>(I) = \Gamma(\mathfrak{a}_\bullet)$, where $\mathfrak{a}_n = \mathrm{in}_>(I^n)$.

We fix our conventions for working with monomial orders and initial terms in polynomial rings. For further background, see [11, Chapter 15].

Definition 2.18. Let k be a field and let $R = k[x_1, \ldots, x_n]$ or $k[\![x_1, \ldots, x_n]\!]$. A *monomial order* $>$ on R is a partial order on $\mathrm{Mon}(x_1, \ldots, x_n)$ compatible with the divisibility relations: that is, for monomials $\mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b}, \mathbf{x}^\mathbf{c}$, if $\mathbf{x}^\mathbf{a} > \mathbf{x}^\mathbf{b}$ then $\mathbf{x}^{\mathbf{a}+\mathbf{c}} > \mathbf{x}^{\mathbf{b}+\mathbf{c}}$.

For an element $f \in R$, write $f = \sum_{\mathbf{a} \in \mathbb{Z}^n_{\geq 0}} \beta_\mathbf{a} \mathbf{x}^\mathbf{a}$ where $\beta_\mathbf{a} \in k$. The set $\{\mathbf{x}^\mathbf{a} : \beta_\mathbf{a} \neq 0\}$ is called the *support* of f and is denoted $\mathrm{supp}(f)$. The *initial term* of f with respect to $>$, denoted $\mathrm{in}_>(f)$, is given by

$$\sum_{\substack{\mathbf{a} : \mathbf{x}^\mathbf{a} \in \mathrm{supp}(f) \\ \mathbf{x}^\mathbf{a} \text{ maximal with respect to } >}} \beta_\mathbf{a} \mathbf{x}^\mathbf{a}.$$

For an ideal $I \subseteq R$, the *initial ideal* $\mathrm{in}_>(I)$ is the ideal generated by the elements $\{\mathrm{in}_>(f) : f \in I\}$.

A critical property of initial ideals is semicontinuity with respect to the singularity threshold.

Proposition 2.19. *Let $R = k[x_1, \ldots, x_n]$ and let $>$ be a monomial order. If $I \subseteq R$ is an ideal, then $c_0(\mathrm{in}_>(I)) \leq c_0(I)$.*

Proof. For characteristic zero, see [9] for the semicontinuity of the lct. For positive characteristic, see the claim preceding Remark 4.6 in [40]. $\square$

When I is homogeneous, k is infinite, and $>$ is a total order compatible with the partial order by degree, there is a well-defined notion of the initial ideal in "generic coordinates."

Definition-Proposition 2.20 ([11], Theorem 15.18)**.** Let k be an infinite field, $R = k[x_1, \ldots, x_n]$, and $I \subseteq R$ a homogeneous ideal. Suppose that $>$ is a monomial total order on R such that $x_1 > \cdots > x_n$ and $\mathbf{x}^{\mathbf{a}} > \mathbf{x}^{\mathbf{b}}$ whenever $\deg(\mathbf{x}^{\mathbf{a}}) > \deg(\mathbf{x}^{\mathbf{b}})$. Then there exists a nonempty open subset $U \subseteq \mathrm{GL}_n(k)$ and a monomial ideal J such that:

- For all $\gamma \in U$, we have $\mathrm{in}_>(\gamma I) = J$;
- U is *Borel-fixed*: if $B \subseteq \mathrm{GL}_n(k)$ denotes the subgroup of lower triangular matrices, then $BU = U$.

The ideal J is called the *generic initial ideal* of I with respect to $>$, and is denoted $\mathrm{gin}_>(I)$.

2.3. **Integral Closure of Ideals.**

Definition 2.21. Let I be an ideal in a ring R. An element $r \in R$ is integral over I if there exists an integer n and elements $a_1, \ldots, a_n, a_i \in I^i$ such that

$$r^n + a_1 r^{n-1} + \cdots + a_n.$$

We then define the integral closure $\overline{I}$ of I as the set of elements $r \in R$ which are integral over I.

Those hoping for an exhaustive discussion of the integral closure of ideals should consult [25]. For now, we will list some basic properties of $\overline{I}$.

Proposition 2.22 (Properties of the Integral Closure, [25] Chapter 1)**.** *Let R be a ring and $I \subseteq R$ an ideal. Let $\varphi : R \to S$. Then we have*

(i): $\overline{I}$ *is an ideal.*
(ii): $\overline{(\overline{I})} = \overline{I}$.
(iii): $\overline{I}S \subseteq \overline{IS}$.
(iv): If $J \subseteq S$ *is an ideal, then* $\varphi^{-1}(\overline{J}) = \overline{\varphi^{-1}(J)}$.
(v): For any multiplicatively-closed subset $W \subseteq R$*, we have* $W^{-1}\overline{I} = \overline{W^{-1}I}$.
(vi): The integral closure of a monomial ideal $\mathfrak{a}$ *in a polynomial ring* $k[x_1, \ldots, x_n]$ *is generated by the set* $\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \Gamma(\mathfrak{a})$.

(vii): If φ is faithfully flat or an integral extension, then $\overline{I}S \cap R = \overline{I}$.

Integral closure is an operation which respects many numerical invariants we are interested in this paper.

Theorem 2.23 ([25], Proposition 11.2.1, Theorem 11.3.1)**.** *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and $I \subseteq J$ two $\mathfrak{m}$-primary ideals. Then $e(I) = e(J)$ if and only if $\overline{I} = \overline{J}$.*

The same result holds in the case that $(R, \mathfrak{m})$ is instead standard-graded.

Proposition 2.24. *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then $c(I) = c(\overline{I})$.*

Proof. For characteristic zero, see [33], Property 1.15. For positive characteristic, see [40], Proposition 2.2 (6). □

To conclude this subsection, we recall a version of the Briançon-Skoda theorem, due to Lipman and Sathaye in the level of generality we need.

Theorem 2.25. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d and $I \subseteq R$ an ideal. Then for all integers $t > 0$, we have*

$$\overline{I}^{t+d} \subseteq \overline{I^{t+d}} \subseteq I^{t+1}.$$

Proof. The first containment is [25, Proposition 5.3.1]. For the second, we reduce to the case of an infinite residue field; we let $R(X) := R[X]_{\mathfrak{m}R[X]}$ as in op. cit. By Proposition 2.22 we have $\overline{I} = \overline{IR(X)} \cap R$, so it suffices to prove the claim for $R = R(X)$.

By [25, Proposition 5.1.6 and Theorem 8.6.6] there exists an ideal $J \subseteq I$ with $\overline{J^t} = \overline{I^t}$ for all $t > 0$ and such that J can be generated by at most d elements. By [31, Theorem 1"] we deduce

$$\overline{I^{t+d}} = \overline{J^{t+d}} \subseteq J^{t+1} \subseteq I^{t+1}.$$

□

2.4. **Mixed Multiplicities and the Demailly-Pham Invariant.**

We recall the definition of the mixed multiplicity symbol $e(I_1, \ldots, I_d; M)$.

Definition 2.26. Let M be a finite-length R-module. We let $\lambda_R(M)$ denote the length of M as an R-module.

Theorem 2.27 ([25], Theorem 17.4.2)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I_1, \ldots, I_k$ ideals of R primary to $\mathfrak{m}$, and M a finitely-generated R-module. Then there exists a polynomial $P(n_1, \ldots, n_k)$ with rational coefficients and total degree at most $\dim R$ such that for all $n_1, \ldots, n_k \gg 0$, we have*

$$P(n_1, \ldots, n_k) = \lambda_R\left(\frac{M}{I_1^{n_1} \ldots I_k^{n_k} M}\right).$$

Remark 2.28. Suppose instead that S is a Noetherian ring, not necessarily local, and $\mathfrak{n}$ is any maximal ideal of S. If $I_1, \dots, I_k$ are $\mathfrak{n}$-primary ideals in S, then $I_1^{n_1} \cdots I_k^{n_k}$ is $\mathfrak{n}$-primary for all $n_1, \dots, n_k > 0$. Consequently, we have

$$\lambda_S\left(\frac{S}{I_1^{n_1} \cdots I_k^{n_k} S}\right) = \lambda_{S_\mathfrak{n}}\left(\frac{S_\mathfrak{n}}{I_1^{n_1} \cdots I_k^{n_k} S_\mathfrak{n}}\right)$$

for all $n_1, \dots, n_k$, so Theorem 2.27 holds for $I_1, \dots, I_k$ without assuming that S is local.

Definition 2.29 (Mixed Multiplicity)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $I_1, \dots, I_k$ be $\mathfrak{m}$-primary ideals of R. Let $Q(n_1, \dots, n_k)$ denote the degree-d part of $P(n_1, \dots, n_k)$. The coefficients of Q define the mixed multiplicities $e(I_1^{\langle d_1 \rangle}, \dots, I_k^{\langle d_k \rangle}; M)$:

$$Q(n_1, \dots, n_k) = \sum_{d_1 + \cdots + d_k = d} \binom{d}{d_1, \dots, d_k}^{-1} e(I_1^{\langle d_1 \rangle}, \dots, I_k^{\langle d_k \rangle}; M) \tag{4}$$

The expression $e(I_1^{\langle d_1 \rangle}, \dots, I_k^{\langle d_k \rangle}; M)$ is shorthand for the expression

$$e(I_1, \dots, I_1, \dots, I_k, \dots, I_d; M),$$

where I_j is repeated d_j times.

Remark 2.30. Other authors [25] have used the notation $e(I_1^{[d_1]}, \dots, I_k^{[d_k]}; M)$ To avoid confusion with the bracket powers $I_j^{[p^e]}$ of the ideals I_j, we use angle brackets in the exponent.

We now define the mixed multiplicities $e_j(I)$.

Definition 2.31. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and let I denote an $\mathfrak{m}$-primary ideal. We define

$$e_j(I) = e(I^{\langle j \rangle}, \mathfrak{m}^{\langle d-j \rangle}; R).$$

We record a few basic properties of the numbers $e_j(I)$.

Proposition 2.32. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n with infinite residue field. Let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal.*

(i) We have $e_0(I) = 1, e_1(I) = \mathrm{ord}_\mathfrak{m}(I)$, and $e_n(I) = e(I)$.
(ii) If $h_1, \dots, h_n$ are elements of $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are sufficiently general, then for all $0 \le j \le n$ we have

$$e_j(I) = e\left(\frac{I + (h_1, \dots, h_{n-j})}{(h_1, \dots, h_{n-j})}\right) = e\left(I + (h_1, \dots, h_{n-j})\right),$$

where $e(-)$ denotes the usual Hilbert-Samuel multiplicity.
(iii) Minkowski inequality: for $j \ge 1$, we have $e_j(I)^2 \le e_{j-1}(I) e_{j+1}(I)$.

Proof.

(i) Follows from (ii).

(ii) The first equality is [25, Corollary 17.4.7]l; the second is [25, Proposition 11.1.19].
(iii) Follows from [25, Theorem 17.7.2].

□

To state Theorem 3.9, we must extend Definition 2.29 to the case of an ideal $I \subseteq R$ which is not necessarily $\mathfrak{m}$-primary.

Definition 2.33 ([4], Definition 2.4)**.** Let $(R, \mathfrak{m})$ be an n-dimensional local ring and $I_1, \ldots, I_n$ ideals of R. We define

$$\sigma(I_1, \ldots, I_n) = \sup_{t>0} \; e(I_1 + \mathfrak{m}^t, \ldots, I_n + \mathfrak{m}^t). \tag{5}$$

As a special case, for $1 \leq j \leq n$ and $I \subseteq R$, we define $\sigma_j(I) := \sigma(I^{\langle j \rangle}, \mathfrak{m}^{\langle n-j \rangle})$ as in Definition 2.31.

The quantity (5) may be infinite. The following proposition summarizes basic properties of $\sigma(I_1, \ldots, I_n)$, among which is a characterization of when the quantity (5) is finite.

Proposition 2.34 ([4])**.** *Let $(R, \mathfrak{m})$ be an n-dimensional regular local ring and $I_1, \ldots, I_n$ ideals of R.*

(i) If $I_1, \ldots, I_n$ have height n, then $\sigma(I_1, \ldots, I_n) = e(I_1, \ldots, I_n)$.
(ii) We have $\sigma(I_1, \ldots, I_n) < \infty$ if and only if there exist $g_1 \in I_1, \ldots, g_n \in I_n$ such that $(g_1, \ldots, g_n)$ is $\mathfrak{m}$-primary. In this case, $\sigma(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)$ for elements $g_i \in I_i$ whose images in $I_i/\mathfrak{m}I_i$ are sufficiently general.
(iii) In particular, for an ideal $I \subseteq R$, we have $\sigma_j(I) < \infty$ if and only if $\operatorname{ht}(I) \geq j$. In this case, if $h_1, \ldots, h_{n-j} \in \mathfrak{m}$ are elements whose images in $\mathfrak{m}/\mathfrak{m}^2$ are sufficiently general, we have

$$\sigma_j(I) = e\left(\frac{I + (h_1, \ldots, h_{n-j})}{(h_1, \ldots, h_{n-j})}\right) = e\left(I + (h_1, \ldots, h_{n-j})\right). \tag{6}$$

(iv) The Minkowski inequalities hold: if $\sigma_{j-1}(I), \sigma_j(I)$ are finite then $\sigma_j(I)^2 \leq \sigma_{j-1}(I)\sigma_{j+1}(I)$.

Proof.

(i) For $t \gg 0$, we have $\mathfrak{m}^t \subseteq I_i$ for all $1 \leq i \leq n$. See the remark after Definition 2.4 in op. cit.
(ii) This is Proposition 2.4 in op. cit.
(iii) When $j \leq \operatorname{ht} I$, use prime avoidance to choose $g_1, \ldots, g_j \in I$ such that $\operatorname{ht}((g_1, \ldots, g_j)) = j$ and $g_{j+1}, \ldots, g_n \in \mathfrak{m}$ such that $\operatorname{ht}((g_1, \ldots, g_n)) = n$. When $j \geq \operatorname{ht}(I) + 1$, for any $g_1, \ldots, g_j \in I, g_{j+1}, \ldots, g_n \in \mathfrak{m}$ we have

$$\operatorname{ht}((g_1, \ldots, g_n)) \leq \operatorname{ht}(I + (g_{j+1}, \ldots, g_n)) \leq \operatorname{ht}(I) + n - j \leq n - 1.$$

To see that the first equality in Equation (6) holds, pick pick $t \gg 0$ such that $\sigma_j(I) = e_j(I + \mathfrak{m}^t)$. Use (ii) to pick $g_1, \ldots, g_j \in I, h_1, \ldots, h_{n-j} \in$

$\mathfrak{m}$ generally such that $\sigma_j(I) = e((f_1+g_1,\dots,f_j+g_j,h_1,\dots,h_{n-j}))$. Use Proposition 2.32(iii) to pick the $h_1,\dots,h_{n-j}$ even more generally so that $e_j(I+\mathfrak{m}^t) = e(I+\mathfrak{m}^t+(h_1,\dots,h_{n-j}))$. Using monotonicity of Hilbert-Samuel multiplicity, we compute

$$\begin{aligned}\sigma_j(I) = e_j(I+\mathfrak{m}^t) &= e(I+\mathfrak{m}^t+(h_1,\dots,h_{n-j})) \le e(I+(h_1,\dots,h_{n-j}))\\ &\le e(g_1,\dots,g_j,h_1,\dots,h_{n-j}) = \sigma_j(I).\end{aligned}$$

(iv) Follows immediately from Proposition 2.32 (iv).

□

We will now define the Demailly-Pham invariant [10].

Definition 2.35. Let $(R,\mathfrak{m})$ be a regular local ring, $l\in\mathbb{Z}^+$, and I an ideal with $\operatorname{ht}(I)\ge l$. Then we set

$$E_l(I) := \frac{1}{\sigma_1(I)} + \cdots + \frac{\sigma_{l-1}(I)}{\sigma_l(I)}.$$

Suppose instead $R = k[x_1,\dots,x_n]$ and $\mathfrak{m} = (x_1,\dots,x_n)$. Let $I\subseteq R$ be an $\mathfrak{m}$-primary ideal. For all $r,s>0$, we have

$$\lambda_R\left(\frac{R}{I^r\mathfrak{m}^s}\right) = \lambda_{R_\mathfrak{m}}\left(\frac{R_\mathfrak{m}}{I^r\mathfrak{m}^sR_\mathfrak{m}}\right),$$

so Theorem 2.27 holds for $\lambda_R\left(\frac{R}{I^r\mathfrak{m}^s}\right)$ without assuming that R is local. In particular, for $0\le j\le n$, the numbers $e(I^{\langle j\rangle},\mathfrak{m}^{\langle n-j\rangle})$ can be defined intrinsically from the polynomial $P(r,s)$ and agree with the quantities $e_j(IR_\mathfrak{m})$. Additionally, in this setting we have an analog of Theorem 2.23.

Proposition 2.36. *Let R be a regular ring and $\mathfrak{m}\subseteq R$ a maximal ideal such that $\dim R_\mathfrak{m} = n$. Let $I_1,I_2\subseteq R$ be $\mathfrak{m}$-primary ideals. Then $E_n(I_1R_\mathfrak{m})\le E_n(I_2R_\mathfrak{m})$ with equality if and only if $\overline{I_1}=\overline{I_2}$.*

Proof. The case that $(R,\mathfrak{m})$ is local is [5, Corollary 11]. In the non-local case, we have $\overline{I_1R_\mathfrak{m}} = \overline{I_2R_\mathfrak{m}}$, so $I_1R_\mathfrak{m}$ is a reduction of $I_2R_\mathfrak{m}$ by [25, Corollary 1.2.5]. By definition of reduction, there exists $t>0$ such that $I_1I_2^tR_\mathfrak{m} = I_2^{t+1}R_\mathfrak{m}$. The ideals $I_1I_2^t, I_2^{t+1}$ are $\mathfrak{m}$-primary, so we have

$$I_1I_2^t = I_1I_2^tR_\mathfrak{m}\cap R = I_2^{t+1}R_\mathfrak{m}\cap R = I_2^{t+1},$$

so I_1 is a reduction of I_2. By [25, Corollary 1.2.5] again, we deduce that $\overline{I_1}=\overline{I_2}$. □

We extend Definition 2.35 to the ring $k[x_1,\dots,x_n]$.

Definition 2.37. Let $R = k[x_1,\dots,x_n], \mathfrak{m} = (x_1,\dots,x_n)$ and let $I\subseteq\mathfrak{m}$ be an ideal. For $1\le j\le\operatorname{ht}(IR_\mathfrak{m})$ we define $\sigma_j(I) := \sigma_j(IR_\mathfrak{m})$ and $E_j(I) := E_j(IR_\mathfrak{m})$.

Remark 2.38. Even though Definition 2.37 defines a local invariant, we suppress the reference to the maximal ideal $\mathfrak{m}$, as we will only ever need to consider singularities at the origin.

Proposition 2.39. *Let $R = k[x_1, \ldots, x_n], \mathfrak{m} = (x_1, \ldots, x_n)$, and let $I \subseteq R$ be an ideal with $\operatorname{ht}(I) \geq l$.*

(a) If I is $\mathfrak{m}$-primary, then Proposition 2.32 holds for I.
(b) If I is homogeneous, then Proposition 2.34 (i), (iii), (iv) hold, provided we assume the elements $h_1, \ldots, h_{n-j}$ in (iii) are homogeneous.

Proof.

(a) As in Proposition 2.32, (i) follows from (ii). For (ii), for any $h_1, \ldots, h_{n-j} \in \mathfrak{m}$, the ideal $(I + (h_1, \ldots, h_{n-j}))$ is $\mathfrak{m}$-primary as well. As $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}R_{\mathfrak{m}}/\mathfrak{m}^2R_{\mathfrak{m}}$, choosing $h_1, \ldots, h_{n-j} \in \mathfrak{m}$ whose images in $\mathfrak{m}R_{\mathfrak{m}}/\mathfrak{m}^2R_{\mathfrak{m}}$ are sufficiently general, we have

$$e_j(I) := e_j(IR_{\mathfrak{m}}) = e\left((I + (h_1, \ldots, h_{n-j})R_{\mathfrak{m}}\right) = e\left(I + (h_1, \ldots, h_{n-j})\right).$$

The other equality in (ii) is similar, and (iv) is immediate:

$$e_{j-1}(I)e_{j+1}(I) := e_{j-1}(IR_{\mathfrak{m}})e_{j+1}(IR_{\mathfrak{m}}) \leq e_j(IR_{\mathfrak{m}})^2 =: e_j(I)^2.$$

(b) For Proposition 2.32 (i), we note that a homogeneous ideal of height n is automatically primary to $\mathfrak{m}$. For (iii), we note that when $h_1, \ldots, h_{n-j}$ are homogeneous and sufficiently general, $\operatorname{ht}(I+(h_1, \ldots, h_{n-j}))$ is homogeneous of height n and thus $\mathfrak{m}$-primary. Additionally, we note that $I + \mathfrak{m}^t$ is $\mathfrak{m}$-primary for all $t > 0$, after which we appeal to part (a) (ii) of this proposition. Similarly to part (a), Proposition 2.34 (iv) follows from the local case.

$\square$

Lemma 2.40. *Suppose we are in one of the following situations:*

(1) L/k is a field extension, $R = k[x_1, \ldots, x_n], \mathfrak{m} = (x_1, \ldots, x_n), S = L[x_1, \ldots, x_n]$, and $I \subseteq R$ is $\mathfrak{m}$-primary or homogeneous.
(2) $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is an extension of n-dimensional regular local rings, $\mathfrak{m}S = \mathfrak{n}$, and $I \subseteq R$ is an ideal.

Then for all $1 \leq j \leq n$ such that $\sigma_j(I)$ is defined, we have $\sigma_j(I) = \sigma_j(IS)$.

Proof. The assumptions imply that $\lambda_R(J) = \lambda_S(JS)$ for every $\mathfrak{m}$-primary ideal $J \subseteq R$. In particular, this applies to the ideals $(I + \mathfrak{m}^t)^a\mathfrak{m}^b$ for all $a, b, t > 0$; the result follows from Definitions 2.29 and 2.33. $\square$

3. Proof of Theorem A

3.1. ***F*-Pure Thresholds and the Demailly-Pham Invariant.** In this subsection, we require an asymptotic version of [5, Theorem 13] in arbitrary characteristic. Our argument follows from Bivià-Ausina's work with the necessary changes being made.

Lemma 3.1. *Let k be a field, $R = k[x_1, \ldots, x_n]$, and $J \subseteq R$ an ideal. Let $1 \leq j \leq n$ and define $\pi_j : R \to R/(x_{j+1}, \ldots, x_n) \cong k[x_1, \ldots, x_j]$. Let $>_{lex}$ denote the usual lexicographic order. Let $>$ denote the negative lexicographic* *order. That is, $>$ is the monomial order given by $x_1^{a_1}\dots x_r^{a_r} < x_1^{b_r}\dots x_m^{b_r}$ if and only if $x_1^{a_r}\dots x_r^{a_1} >_{lex} x_1^{b_r}\dots x_r^{b_1}$. Then*

$$\operatorname{in}_> \pi_j(J) = \pi_j(\operatorname{in}_>(J)).$$

Proof. Let $f \in J$. Write $f = g + h$, where $h \in (x_{j+1}, \dots, x_n)$ and $g \in k[x_1, \dots, x_j]$. If $g = 0$, then $\pi_j(f) = 0$. If $g \neq 0$, then $\operatorname{in}_>(f) = \operatorname{in}_>(g)$. In both cases, we have $\pi_j(\operatorname{in}_>(f)) = \operatorname{in}_>(\pi_j(f))$. □

Remark 3.2. In other words, $>$ is obtained from $>_{\text{lex}}$ by reversing the order of variables and then reversing the entire monomial order. Consequently, $>$ agrees with the reverse lexicographic order for monomials of the same degree.

The following is a general lemma which will be used repeatedly throughout the rest of this article.

Lemma 3.3. *Let L be a field, $S = L[x_1, \dots, x_n]$, and $J \subseteq S$ an $\mathfrak{m}$-primary homogeneous ideal generated by forms of degree $\leq d$. Then $\mathfrak{m}^d \subseteq \overline{J}$.*

Proof. We first prove the result in the case that L is infinite. First, choose forms $f_1, \dots, f_n$ from among the generators of J such that $(f_1, \dots, f_n)$ is $\mathfrak{m}$-primary. If $h_1, \dots, h_n$ are general linear forms, then

$$J' := (h_1^{d-\deg(f_1)} f_1, \dots, h_n^{d-\deg(f_n)} f_n)$$

is an $\mathfrak{m}$-primary $(d, \dots, d)$-complete intersection contained in J. As $J' \subseteq \mathfrak{m}^d$ and $e(J)' = d^n = e(\mathfrak{m}^d)$, we have $\mathfrak{m}^d = \overline{\mathfrak{m}^d} \subseteq \overline{J'} \subseteq \overline{J}$ by Theorem 2.23.

Now, let L be an arbitrary field, and set $S' = \overline{L}[x_1, \dots, x_n]$. We then have $\overline{J} = \overline{JS'} \cap S \supseteq \mathfrak{m}^d S' \cap S = \mathfrak{m}^d$ by Proposition 2.22 (vii) and the infinite field case. □

Lemma 3.4 ([5], Theorem 4)**.** *Let k be an infinite field, $R = k[x_1, \dots, x_n]$, and I an $\mathfrak{m}$-primary ideal. If $>$ denotes the monomial order of Lemma 3.1 and $\gamma \in \mathrm{GL}_n(k)$ is general, then for all $1 \leq j \leq n$ we have*

$$\lim_{t\to\infty} \frac{e_j(\operatorname{in}_>(\gamma^{-1} I^t))}{t^j} = e_j(I). \tag{7}$$

To reassure the reader that Bivià-Ausina's arguments hold in arbitrary characteristic, we summarize the main argument below.

Proof of Lemma 3.4. By Proposition 2.32 (ii), choose $\gamma \in \mathrm{GL}_n(k)$ so that $e_j(I) = e_j\left(\frac{I+\gamma(x_n,\dots,x_{n-j+1})}{\gamma(x_n,\dots,x_{n-j+1})}\right)$. Set $J = \gamma^{-1} I$. By Proposition 2.32 (ii), let $\gamma_t \in \mathrm{GL}_n(k)$ such that $e_j(\operatorname{in}_>(J^t)) = e(\pi_j(\gamma_t^{-1} \operatorname{in}_>(J^t)))$. By [39, Theorem 3.4 (ii)], Hilbert-Samuel multiplicity is upper semicontinuous in finite families. Applying this to the specializations $\pi_j(\gamma_t^{-1} \operatorname{in}_>(J^t)) \rightsquigarrow \pi_j(\operatorname{in}_>(J^t))$ and $J^t \rightsquigarrow \operatorname{in}_>(J^t)$ as in [5, Proposition 2 and Theorem 4], we deduce

$$e(\pi_j(\operatorname{in}_>(J^t))) \geq e(\pi_j(\gamma_t^{-1} \operatorname{in}_>(J^t)) = e_j(\operatorname{in}_>(J^t)) \geq e_j(J^t). \tag{8}$$

By [35, Corollary 1.13], Equation (8), and Lemma 3.1 we have

$$\begin{aligned} e_j(I) = e_j(J) = e(\pi_j(J)) &= \lim_{t\to\infty} \frac{e(\pi_j(J)^t)}{t^j} \\ &= \lim_{t\to\infty} \frac{e(\mathrm{in}_>(\pi_j(J^t))}{t^j} = \lim_{t\to\infty} \frac{e(\pi_j(\mathrm{in}_>(J^t)))}{t^j} \\ &\geq \lim_{t\to\infty} \frac{e_j(\mathrm{in}_>(J^t))}{t^j} \geq \lim_{t\to\infty} \frac{e_j(J^t)}{t^j} = e_j(J). \end{aligned} \tag{9}$$

It follows that the inequalities in Equation (9) must be sharp, proving the claim. □

Remark 3.5. The generic change of coordinates in Lemma 3.4 is necessary when $\dim(R) \geq 3$. Consider $R = \mathbb{Q}[x, y, z]$ and $I = (xy, x^2 + z^2, x^4, y^4, z^4)$. Let $\succ$ denote the degrevlex order with $x \succ y \succ z$. Using the ReesAlgebra package in Macaulay2 [18, 12], we compute $e(\mathrm{in}_\succ(I^5)) = 2000 = 5^3 \cdot e(I) = e(I^5)$. For all $t > 0$, we have $\mathrm{in}_\succ(I^{5t}) \subseteq \mathrm{in}_\succ(I^5)^t$, hence

$$t^3 e(I^5) = e(I^{5t}) \leq e(\mathrm{in}_\succ(I^{5t})) \leq e(\mathrm{in}_\succ(I^5))^t = t^3 e(I^5).$$

Set $\mathfrak{a} = \mathrm{in}_\succ(I^5)$. By Theorem 2.23, we deduce that $\overline{\mathrm{in}_\succ(I^{5t})} = \overline{\mathfrak{a}^t}$ for all $t > 0$. It follows from Proposition 2.36 that

$$\lim_{t\to\infty} \frac{e_2(\mathrm{in}_\succ(I^t))}{t^2} = \lim_{t\to\infty} \frac{e_2(\mathrm{in}_\succ(I^{5t}))}{(5t)^3} = \frac{e_2(\mathfrak{a}^t)}{(5t)^3} = \frac{e_2(\mathfrak{a})}{5^3}.$$

To compute $e_2(\mathfrak{a})$, we use Proposition 2.32 (ii). Suppose $h = \alpha x + \beta y + \gamma z$ is such that $e_2(\mathfrak{a}) = e\left(\frac{\mathfrak{a}+(h)}{(h)}\right)$. Let $\gamma \in \mathrm{GL}_n(k)$ denote the map $(x, y, z) \mapsto (\alpha x, \beta y, \gamma z)$. As $\mathfrak{a}$ is a monomial ideal, we have $\gamma^{-1}\mathfrak{a} = \mathfrak{a}$, hence

$$e\left(\frac{\mathfrak{a} + \gamma(x + y + z)}{\gamma(x + y + z)}\right) = e\left(\frac{\mathfrak{a} + (x + y + z)}{(x + y + z)}\right). \tag{10}$$

The quantity in Equation (10) can be computed directly using the ReesAlgebra package, giving us

$$\lim_{t\to\infty} \frac{e_2(\mathrm{in}_\succ(I^t))}{t^2} = \frac{\mathrm{in}_\succ(I^5)}{5^2} = \frac{22}{5}.$$

Using Lemmas 3.7 and 3.8, we compute $e_2(I) = 4 < \frac{22}{5}$.

Definition 3.6. Let k be a field, $R = k[x_1, \dots, x_n], \mathfrak{m} = (x_1, \dots, x_n)$, and let $\mathfrak{a}_\bullet$ be a graded system of $\mathfrak{m}$-primary ideals. We define:

- The asymptotic mixed multiplicities: $e_j(\mathfrak{a}_\bullet) = \liminf_m \frac{e_j(\mathfrak{a}_m)}{m^j}$.
- The asymptotic Demailly-Pham invariant: $E_n(\mathfrak{a}_\bullet) = \frac{1}{e_1(\mathfrak{a}_\bullet)} + \cdots + \frac{e_{n-1}(\mathfrak{a}_\bullet)}{e_n(\mathfrak{a}_\bullet)}$
- The asymptotic singularity threshold: $c(\mathfrak{a}_\bullet) = \liminf_m mc(\mathfrak{a}_m)$.

Before we prove Theorem 3.9 and our asymptotic version of [5, Theorem 13], we require the following standard facts.

Lemma 3.7. *Let L be a field and $S = L[x_1, \dots, x_n]$. Let I be a homogeneous ideal of S and $J \subseteq I$ denote the ideal of S generated by the homogeneous forms in I of degree $\leq d$. If $\operatorname{ht}(J) = n$, then $\overline{J} = \overline{I}$.*

Proof. It is clear that $\overline{J} \subseteq \overline{I}$. Let $\mathfrak{m} := (x_1, \dots, x_n)$. For the reverse containment, note that $I \subseteq J + \mathfrak{m}^{d+1}$. By Lemma 3.3 we have

$$\overline{I} \subseteq \overline{J + \mathfrak{m}^{d+1}} \subseteq \overline{J + \mathfrak{m}^{d}} \subseteq \overline{J}.$$

□

Lemma 3.8. *Let L be a field and $S = L[x_1, \dots, x_n]$. Let $J = (f_1, \dots, f_n)$ be a complete intersection where $\deg f_i = d_i$ and $d_1 \leq \cdots \leq d_n$. Then we have the following:*

(i) If L is infinite, then for a general hyperplane section $H \subseteq \operatorname{Spec} R$, we have $e(I|_H) = d_1 \cdots d_{n-1}$.
(ii) With no assumption on $|L|$, we have $e_j(I) = d_1 \dots d_j$, and hence $E_n(I) = \frac{1}{d_1} + \cdots + \frac{1}{d_n}$.

Proof. As the $f_1, \dots, f_n$ form a homogeneous system of parameters for S, it is well-known that

$$e(J) = d_1 \dots d_n. \tag{11}$$

Choosing H generally so that $\operatorname{ht}(f_1, \dots, f_{n-1})|_H = n - 1$, Lemma 3.7 gives $e(I|_H) = e((f_1|_H, \dots, f_{n-1}|_H))$, so (i) follows from Equation (11). For (ii), Lemma 2.40 shows that $e_j(J)$ is invariant under extension of the base field, so it suffices to consider the case of an infinite field. The result follows from (i) and Proposition 2.32 (ii). □

Theorem 3.9. *Let n be a positive integer and $1 \leq l \leq n$. Suppose either:*

(1) $(R, \mathfrak{m})$ is a regular local ring of dimension n, excellent if $\operatorname{char} R = 0$, and that $I \subseteq R$ is an ideal of height at least l;
(2) k is a field, $R = k[x_1, \dots, x_n], \mathfrak{m} = (x_1, \dots, x_n)$, and $I \subseteq R$ is an ideal which is $\mathfrak{m}$-primary or homogeneous of height at least l.

Then $E_l(I) \leq c(I)$.

Proof. We reduce to the case of an $\mathfrak{m}$-primary ideal in a polynomial ring over an infinite field. In the case (ii), Proposition 2.9 (vi) and Lemma 2.40 show that E_l, c are invariant under field extension. If J is an ideal generated by $n - l$ general linear forms, then $E_l(\frac{I+J}{J}) = E_l(I)$ and $c(\frac{I+J}{J}) \leq c(I)$ by Proposition 2.9 (iv) and Proposition 2.39, so we may assume $l = n$.

Suppose we are in the case (i). By Proposition 2.9 (vi) and Lemma 2.40, the quantities c, E_l are invariant under completion and field extension, so we may without loss of generality assume R is complete local with infinite residue field k. As $c(I) \geq c((I + J)/J)$ for any ideal $J \subseteq R$, we let J be an ideal generated by a $n-l$ elements of $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are sufficiently general; our goal is now to show that $c((I + J)/J) \geq E_l((I + J)/J))$, where $(I + J)/J$ is an ideal of height l in a complete regular local ring of dimension

l. Therefore, without loss of generality, we may assume $l = n$, and hence I is $\mathfrak{m}$-primary. Set $R' = k[x_1, \dots, x_n], \mathfrak{m}' = (x_1, \dots, x_n)$. Since I is $\mathfrak{m}$-primary, there exists $I' \subseteq R'$ with $I'R = I$. Another application of Proposition 2.9 (vi) and Lemma 2.40 gives $c(I) = c(I'R'_{\mathfrak{m}'})$ and $E_n(I) = E_n(I'R'_{\mathfrak{m}'})$, so Notation 2.8 and Definition 2.37 we have $c(I) = c(I'), E_n(I) = E_n(I')$.

Let $\gamma \in \mathrm{GL}_n(k)$ such that Equation (7) holds. Set $\mathfrak{a}_m = \mathrm{in}_>(\gamma^{-1} I^m)$ for $m > 0$. By Lemma 3.4 we have $E_n(\mathfrak{a}_\bullet) = E_n(I)$ and by Proposition 2.19 we have $c(\mathfrak{a}_\bullet) \leq c(I)$. Let $\mu = \inf_t : (t, \dots, t) \in \Gamma$ and set $\vec{\mu} = (\mu, \dots, \mu)$. Since Γ is convex and $\mu \in \partial\Gamma$, by [37, Corollary 11.6.1] there exists a half-space $H^+ \subseteq \mathbb{R}^n$ such that $\Gamma \subseteq H^+$ and that $\mu \in \partial H^+$. Since Γ is closed under translation by elements of $R^n_{\geq 0}$ and the complement of Γ in $\mathbb{R}^n_{\geq 0}$ is bounded, the same is true for H^-. Consequently, the complement of H^- in $\mathbb{R}^n_{\geq 0}$ is a simplex which we denote by $\mathrm{conv}(0, (b_1, 0, \dots, 0), \dots, (0, \dots, 0, b_n))$.

Define a graded system of monomial ideals $\mathfrak{b}_\bullet$ by $\mathfrak{b}_m = \{x^\alpha : \alpha \in mH^+\}$. By assumption that $\Gamma \subseteq H^+$, we have $\mathfrak{a}_m \subseteq \mathfrak{b}_m$ for all m. Consequently, we have $E_n(\mathfrak{a}_\bullet) \leq E_n(\mathfrak{b}_\bullet)$ by Proposition 2.36. By Proposition 2.15, we also have $c(\mathfrak{b}_\bullet) = c(\mathfrak{a}_\bullet)$. Altogether, we have

$$E_n(I) = E_n(\mathfrak{a}_\bullet) \leq E_n(\mathfrak{b}_\bullet) = \frac{1}{b_1} + \dots + \frac{1}{b_r} = c(\mathfrak{b}_\bullet) = c(\mathfrak{a}_\bullet) \leq c(I).$$

□

Corollary 3.10. *Let k be an uncountably infinite field, $R = k[x_1, \dots, x_n]$, and I an $\mathfrak{m}$-primary homogeneous ideal. Let $>$ denote the degree-reverse lexicographic order and let $\mathfrak{a}_m := \mathrm{gin}(I^m)$. Lastly, let $\vec{b_1}, \dots, \vec{b_n}$ denote the standard unit vectors in $\mathbb{R}^n$. If $E_n(I) = c(I)$, then*

$$\overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)} = \mathrm{conv}\left(\vec{0}, e_1(I)\vec{b_1}, \frac{e_1(I)}{e_2(I)}\vec{b_2}, \dots, \frac{e_{n-1}(I)}{e_n(I)}\vec{b_n}\right). \tag{12}$$

Proof. By Remark 3.2, $>$ agrees with the degree reverse lexicographic order on homogeneous ideals. Consequently, we may choose $\gamma \in \mathrm{GL}_n(k)$ very generally so that Equation (7) holds and $\mathrm{in}_>(\gamma^{-1} I^m) = \mathrm{gin}_>(I^m)$ for all $m > 0$. With this choice of γ, let $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet$ be as in Theorem 3.9.

Suppose $E_n(I) = c(I)$. Then we also have $E_n(\mathfrak{a}_\bullet) = E_n(\mathfrak{b}_\bullet)$. By [5, Proposition 10], we further have that $e_j(I) = e_j(\mathfrak{a}_\bullet) = e_j(\mathfrak{b}_\bullet)$ for all $1 \leq j \leq n$. In particular, $e_n(\mathfrak{a}_\bullet) = e_n(\mathfrak{b}_\bullet)$, so by [35, Theorem 2.12 and Lemma 2.13], we have $\mathrm{vol}(\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)) = \mathrm{vol}(\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{b}_\bullet))$. Since $\Gamma(\mathfrak{a}_\bullet), \Gamma(\mathfrak{b}_\bullet)$ are closed and convex with positive volume, it follows that $\Gamma(\mathfrak{a}_\bullet) = \Gamma(\mathfrak{b}_\bullet)$.

Since the generic initial ideal is Borel-fixed, we have $b_1 \leq \dots \leq b_n$. Consequently, we can compute $e_j(\mathfrak{b}_\bullet)$ in terms of the numbers b_j: we have

$$\overline{\left(x_1^{\lfloor mb_1 \rfloor}, \dots, x_n^{\lfloor mb_n \rfloor}\right)} \subseteq \mathfrak{b}_m \subseteq \overline{\left(x_1^{\lceil mb_1 \rceil}, \dots, x_n^{\lceil mb_n \rceil}\right)}.$$

It follows that $e_j(\mathfrak{b}_\bullet) = b_1 \cdots b_j$. As $e_j(I) = e_j(\mathfrak{a}_\bullet) = e_j(\mathfrak{b}_\bullet)$, the result follows. □

Remark 3.11. The condition Equation (12) is necessary to have $c(I) = E_n(I)$, but not sufficient. By [32] in characteristic zero, Equation (12) holds for any homogeneous complete intersection $J = (f_1, \dots, f_n)$.

In fact, Equation (12) holds for any homogeneous complete intersection in positive characteristic, too. Mayes's argument in [32] mostly transfers over, but a small modification must be made. In particular, lemma 3.6 in op. cit. does not hold in positive characteristic: in the ring $\overline{\mathbb{F}_p}[x, y]$ we have $\operatorname{gin}(x^p, y^p) = (x^p, y^p)$. The details will appear in the author's thesis.

4. Theorem B in the Complete Intersection Case

4.1. **Behavior of the Singularity Threshold Under Modification.** Fix the following notation throughout this subsection.

Assumption 4.1. *Let k be a field, $R = k[x_1, \dots, x_n]$, and let $\mathfrak{m}$ denote the homogeneous maximal ideal. Let $I \subseteq R$ be an $\mathfrak{m}$-primary homogeneous ideal. Write $I = I_1 + \cdots + I_r$, where I_j is generated by forms of degree d_j and $d_1 < \cdots < d_j$.*

Lemma 4.2 ([3], Lemma 3.2)**.** *Let $R = k[x_1, \dots, x_n]$ and let $\mathfrak{m}$ denote the homogeneous maximal ideal. For any $e, t \in \mathbb{Z}^+$, we have*

$$(\mathfrak{m}^{[p^e]} : \mathfrak{m}^t) = \begin{cases} R & t \geq np^e - n + 1 \\ \mathfrak{m}^{[p^e]} + \mathfrak{m}^{np^e - n + 1 - t} & t < np^e - n + 1 \end{cases}$$

More generally, we have the following.

Lemma 4.3. *Let $R = k[x_1, \dots, x_n]$. Let v be a monomial valuation on R with $v(x_i) \geq 0$ for all $1 \leq i \leq n$. For $\lambda \in \mathbb{R}^+$, let $\mathfrak{a}_\lambda$ denote the ideal $\{f \in R : v(f) \geq \lambda\}$ and $\mathfrak{a}_\lambda^+ = \{f \in R : v(f) > \lambda\}$. Let $q \in \mathbb{Z}^+, \lambda \in \mathbb{R}^+$. Then we have*

$$((x_1^q, \dots, x_n^q) : \mathfrak{a}_\lambda) = (x_1^q, \dots, x_n^q) + \mathfrak{a}^+_{(q-1)v(x_1 \dots x_n) - \lambda}. \tag{13}$$

Proof. The argument is the same as Lemma 4.2. Let $m \notin (x_1^q, \dots, x_n^q)$ be a monomial. Then $m \mid (x_1 \cdots x_n)^{q-1}$, so $\mathfrak{a}_\lambda m \not\subseteq (x_1^q, \dots, x_n^q)$ if and only if $\frac{(x_1 \dots x_n)^{q-1}}{m} \in \mathfrak{a}_\lambda$, which holds if and only if $v((x_1 \cdots x_n)^{q-1}) - v(m) \leq \lambda$. We've shown that the two sides of Equation (13) contain the same monomials; both sides are monomial ideals, so the result follows. □

Lemma 4.4 ([2], Lemma 4.2)**.** *Let k be a field of characteristic $p > 0$, let $R = k[x_1, \dots, x_n]$, and $I \subseteq R$ a homogeneous ideal. For a hyperplane H cut out by a linear form ℓ, we let $I|_H$ denote the image of I in $R/\ell R$. In this case, we have*

$$\nu_{I|_H}(p^e) \leq \max\{r : I^r \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{(n-1)(p^e-1)+1}\}, \tag{14}$$

Corollary 4.5. *Assume the setup of Assumption 4.1 and let $H \subseteq \operatorname{Spec} R$ be a hyperplane cut out by a linear form ℓ. Then $c(I) - c(I|_H) \geq 1/d_r$.*

Proof. We first prove the claim in characteristic $p > 0$. Combining Lemma 4.2 and Lemma 4.4, we have

$$\nu_{I|_H}(p^e) \leq \max\{s : \mathfrak{m}^{p^e} I^s \not\subseteq \mathfrak{m}^{[p^e]}\}.$$

By Lemma 3.3, we have $\mathfrak{m}^{d_r} \subseteq \overline{I}$, so $\max\{s : \mathfrak{m}^{p^e} I^s \not\subseteq \mathfrak{m}^{[p^e]}\} \leq \nu_I(p^e) - \left\lfloor \frac{p^e}{d_r} \right\rfloor$, so we have $\nu_{I|_H}(p^e) \leq \nu_I(p^e) - \left\lfloor \frac{p^e}{d} \right\rfloor$. Dividing by p^e and taking the limit as $e \to \infty$ gives the result.

We now prove the claim in characteristic 0. The result follows from Proposition 2.7, the details of which we will spell out explicitly in this case. Let $A \subseteq k$ be a finitely-generated $\mathbb{Z}$-algebra and $J \subseteq A[x_1, \ldots, x_n]$ an ideal such that $JR = I$. Such a subring A can always be constructed by adjoining to $\mathbb{Z}$ the field coefficients appearing in a generating set for I; we'll also choose A to contain the coefficients of the linear form ℓ. If μ is a maximal ideal of A, we let I_μ denote the image of J in $(A/\mu)[x_1, \ldots, x_n]$. Applying Proposition 2.7 to both I and $I|_H$, we obtain

$$c(I) - c(I|_H) = \lim_{\substack{\mu \in \operatorname{Spec} A \\ \operatorname{char} A/\mu \to \infty}} c(I|_\mu) - c(I_\mu|_{H_\mu}) \geq 1/d_r.$$

□

Lemma 4.6. *Assume the setup of Assumption 4.1. Suppose $r = 2$. Then we have $c(I) = \frac{n}{d_2} + c(I_1)\frac{d_2 - d_1}{d_2}$. In particular, for any $0 \leq s \leq n$, we have $c(I) = \frac{s}{d_1} + \frac{n-s}{d_2}$ if and only if $c(I_1) = \frac{s}{d_1}$.*

Proof. We prove the claim first in positive characteristic. By Lemma 3.3, we have $\mathfrak{m}^{d_2} \subseteq \overline{I}$, so $I \subseteq I_1 + \mathfrak{m}^{d_2} \subseteq \overline{I}$, so $c(I) = c(I_1 + \mathfrak{m}^{d_2})$. Consequently, we have

$$\nu_{\overline{I}}(p^e) = \max\left\{a + b : I_1^a \mathfrak{m}_\mu^{bd_n} \not\subseteq \mathfrak{m}^{[p^e]}\right\} = \max\{a + b : I_1^a \not\subseteq (\mathfrak{m}_p^{[p^e]} : \mathfrak{m}_p^{bd_n})\}.$$

By Lemma 4.2, we obtain

$$\nu_{\overline{I}}(p^e) = \max\{a + b : I_1^a \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{np^e - n + 1 - bd_2}\}. \tag{15}$$

The ideal I_1^a is generated in degree ad_1, so $I_1^a \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^t$ if and only $I_1^a \not\subseteq \mathfrak{m}^{[p^e]}$ and $ad_1 < t$. With this insight, we refine Equation (15):

$$\nu_I(p^e) = \max_{0 \leq a \leq \nu_{I_1}(p^e)} a + \frac{np^e - n - ad_1}{d_2}. \tag{16}$$

The quantity being maximized in Equation (16) is an increasing function of a, so the maximum occurs at $a = \nu_{I_1}(p^e)$ and

$$\nu_{\overline{I}}(p^e) = \frac{np^e - n}{d_2} + \nu_{I_1}(p^e)\frac{d_2 - d_1}{d_2}.$$

Dividing by p^e and letting $e \to \infty$, we obtain

$$c(I) = \frac{n}{d_2} + c(I_1)\frac{d_2 - d_1}{d_2}.$$

For characteristic zero, we can spread out to positive characteristic as in Corollary 4.5 and compute

$$c(I) = \lim_{\mu} c(I_\mu) = \lim_{\mu} \left(\frac{n}{d_2} + c(I_{1,\mu}) \frac{d_2 - d_1}{d_2} \right) = \frac{n}{d_2} + c(I) \frac{d_2 - d_1}{d_2}.$$

□

4.2. **Induction Setup.**

Assumption 4.7. *We assume a setup similar to Assumption 4.1. Let k be an algebraically closed field. Let $a_1, \dots, a_r, d_1, \dots, d_r \in \mathbb{Z}^+$. For $1 \le i \le r$, let $\mathbf{x}_i$ denote the tuple of variables $x_{i,1}, \dots, x_{i,a_i}$, and let $R = k[\mathbf{x}_1, \dots, \mathbf{x}_r]$. Let $n = a_1 + \cdots + a_r = \dim R$. Let $I \subseteq R$ be a complete intersection of the form $(f_{1,1}, \dots, f_{r,a_r})$ such that $f_{i,j}$ is a d_i-form. For $1 \le j \le r$, write $I_j = (f_{j,1}, \dots, f_{j,a_j})$. Let w_d denote the monomial valuation with $w_d(x_{i,j}) = 1/d_i$. We let $\mathfrak{D}$ denote the ideal $\overline{(\mathbf{x}_1^{d_1}, \dots, \mathbf{x}_r^{d_r})}$, which coincides with the set of elements z with $w_d(z) \ge 1$.*

Assumption 4.8. *Assume the setup of Assumption 4.7. We define the following condition on the ideal I:*

$$I_1 \text{ is extended from } k[\mathbf{x}_1] \text{ and } I \subseteq \mathfrak{D} + (\mathbf{x}_1). \tag{17}$$

Definition 4.9. For $r \in \mathbb{Z}^+$, we define the statements A_r, B_r.

(A_r) For all I as in Assumption 4.7, if $DP(I) = c(I)$ then there exists $\gamma \in \mathrm{GL}_n(k)$such that $\gamma I \subseteq \mathfrak{D}$.

(B_r) For all I as in Assumption 4.7, if $DP(I) = c(I)$ then there exists $\gamma \in \mathrm{GL}_n(k)$ such that γI satisfies Equation (17).

The goal of this section is to prove A_r for all r. We accomplish this via the following steps:

1. A_1, A_2 hold
2. For $r \ge 3$, $A_2, A_{r-1} \implies B_r$
3. For $r \ge 3$, B_r implies A_r.

The proof of step (1) is short, but steps (2) and (3) will each have a dedicated subsection.

Proof of A_1. If I is an $\mathfrak{m}$-primary complete intersection generated by forms of degree d for some d, by Lemma 3.3 we have $\overline{I} = \mathfrak{m}^d$. □

Remark 4.10. When $r = 1$, the above argument shows that $c(I) = E_n(I)$ is satisfied for all choices of I. Unfortunately, this means that the proof of our induction step $A_2, A_{r-1} \implies A_r$ can't be modified to show $A_1 \implies A_2$. We instead expand the scope of our base case.

To prove A_2, we introduce the notion of *essential dimension.*

Definition 4.11 (Essential Dimension)**.** Let $J \subseteq R = k[x_1, \dots, x_d]$ be a homogeneous ideal. The essential dimension $\operatorname{ess}(J)$ is equal to the minimal r for which there exist linear forms $\ell_1, \dots, \ell_r$ such that J is extended from $I \subseteq k[\ell_1, \dots, \ell_r]$.

We have the following result.

Proposition 4.12 ([2], Proposition 3.3)**.** *Let k be an algebraically-closed field, $R = k[x_0, \dots, x_n]$, and $J \subseteq R$ a homogeneous ideal. Set $r = \operatorname{ht}(J)$. Let $L = (\ell_{r+1}, \dots, \ell_n)$, where the ℓ_i are chosen generally. For $r \leq t \leq n$, set $L_t = (\ell_{t+1}, \dots, \ell_n)$ and $J_t = \frac{J+L_t}{L_t}$. Then for all $r \leq t \leq n$, we have $ess(J_t) = \max(t+1, ess(J))$.*

Proof of A_2. By Lemma 4.6, if $c(I) = E_n(I)$ then $c(I_1) = a_1/d_1$. By [15, Theorem 3.5] in characteristic zero and [2, Theorem 3.17] in positive characteristic, it follows that $\operatorname{ess}(I_1) = a_1$. By [2, Lemma 3.18], there exists $\gamma \in \mathrm{GL}_n(k)$ such that $\overline{\gamma(I_1)} = (\mathbf{x}_1)^{d_1}$, hence $\gamma I \subseteq (\mathbf{x}_1)^{d_1} + \mathfrak{m}^{d_2} \subseteq \mathfrak{D}$. $\square$

4.3. **Step (2): For $r \geq 3$, $A_2, A_{r-1} \implies B_r$.**

Lemma 4.13. *Assume the setup of Assumption 4.7 and suppose $c(I) = E_n(I)$. Let $\ell \in R$ be a general linear form and let H denote the zero locus of ℓ. Then $c(H, I|_H) = E_n(I|_H)$.*

Proof. By Lemma 3.8, Theorem 3.9, and Corollary 4.5, we have
$$\frac{a_1}{d_1} + \cdots + \frac{a_r - 1}{d_r} = E_{n-1}(I|_H) \leq c(I|_H) \leq \frac{a_1}{d_1} + \cdots + \frac{a_r - 1}{d_r}.$$
$\square$

Lemma 4.14. *Assume the setup of Assumption 4.7 and suppose $r \geq 3, c(I) = E_n(I)$. If A_{r-1} holds, there exists $\gamma \in \mathrm{GL}_n(k)$ such that $\gamma \overline{I_1} = (\mathbf{x}_1)^{d_1}$.*

Proof. Note that $E_n(I) = a_1/d_1 + \cdots + a_r/d_r$ by Lemma 3.8. Let L be an ideal of R generated by $a_3 + \cdots + a_d$ general linear forms. Since I is a complete intersection, $I_1 + I_2 + L$ is $\frac{\mathfrak{m}}{L}$-primary, so by Lemma 3.7 we have $c(R/L, \frac{I+L}{L}) = c(R/L, \frac{I_1+I_2+L}{L})$. Consequently, by repeated application of Corollary 4.5, we have
$$c(I) \geq c\left(R/L, \frac{I_1 + I_2 + L}{L}\right) + \frac{a_3}{d_3} + \cdots + \frac{a_r}{d_r}. \tag{18}$$
Assuming $c(I) = E_n(I)$, we have
$$\frac{a_1}{d_1} + \frac{a_2}{d_2} \leq c\left(R/L, \frac{I_1 + I_2 + L}{L}\right) \leq \frac{a_1}{d_1} + \frac{a_2}{d_2},$$
where the left-hand side is by Theorem 3.9 and the right-hand side is by Equation (18). Both inequalities are therefore equalities, so by the argument of A_2, we have $\operatorname{ess}\left(\frac{I_1+L}{L}\right) = a_1$. By Proposition 4.12, it follows that $\operatorname{ess}(I_1) = a_1$. The result then follows from [2, Lemma 3.18]. $\square$

Lemma 4.15. *Assume the setup of Assumption 4.7. Suppose $c(I) = E_n(I)$. Then there exists $\gamma \in \mathrm{GL}_n(k)$ such that γI satisfies Equation* (17).

Proof. By Lemma 4.14, we may assume I_1 is extended from $k[\mathbf{x}_1]$. Let $\succ$ denote the monomial partial order induced by the monomial valuation $w(x_{1,i}) = 0$ and $w(x_{i,j}) = 1$ for $i \geq 2$. For $2 \leq i \leq r, 1 \leq j \leq a_i$, let $g_{i,j} := \mathrm{in}_\succ(f_{i,j})$. Since I is a complete intersection, we have $f_{i,j} \notin \sqrt{I_1} = (\mathbf{x}_1)$, hence $g_{i,j} \notin (\mathbf{x}_1)$ and moreover $f_{i,j} - g_{i,j} \in (\mathbf{x}_1)$. Observe that

$$\mathrm{in}_\succ(I) \supseteq I_1 + \mathrm{in}_\succ(I_2 + \cdots + I_r) \supseteq I_1 + (g_{2,1}, \ldots, g_{r,a_r}). \tag{19}$$

Let I' denote the right-hand side of Equation (19). Because $g_{i,j}$ and $f_{i,j}$ have the same image modulo $(\mathbf{x}) = \sqrt{I_1}$, the ideal I' is a complete intersection. In particular, I' is a complete intersection of type $(\underbrace{d_1, \ldots, d_1}_{a_1}, \ldots, \underbrace{d_r, \ldots, d_r}_{a_r})$.

By Lemma 3.8 and Proposition 2.19, we have

$$E_n(I) = E_n(I') \leq c(I') \leq c(\mathrm{in}_\succ(I) \leq c(I) = E_n(I). \tag{20}$$

As $\overline{I_1} = (\mathbf{x}_1)^{d_1}$, we have $c(I_1) = a_1/d_1$. Since I_1 and $J := (g_{2,1}, \ldots, g_{r,a_r})$ are defined in terms of disjoint sets of variables, we have by [41], Theorem 2.4 (1) that

$$c(R, I') = c(k[\mathbf{x}_1], I_1) + c(k[\mathbf{x}_2, \ldots, \mathbf{x}_r], J) = \frac{a_1}{d_1} + c(k[\mathbf{x}_2, \ldots, \mathbf{x}_r], J). \tag{21}$$

It follows from Equations (20) and (21) that J, which is a complete intersection in $k[\mathbf{x}_2, \ldots, \mathbf{x}_r]$, satisfies $E_{a_2+\cdots+a_r}(J) = c(J)$. By A_{r-1}, there exists $\gamma \in \mathrm{GL}_{n-a_1}(k)$ such that $\gamma J \subseteq \overline{(\mathbf{x}_2^{d_2}, \ldots, \mathbf{x}_r^{d_r})}$.

We define $\gamma' := \begin{bmatrix} \mathrm{id}_{a_1} & 0 \\ 0 & \gamma \end{bmatrix}$ and we claim that $\gamma' I$ satisfies Equation (17). By construction, $\gamma' I'$ satisfies Equation (17). Since $g_{i,j} - f_{i,j} \in (\mathbf{x}_1)$ for all $2 \leq i \leq r, 1 \leq j \leq a_i$, we have

$$\gamma'(I_2 + \cdots + I_r) \subseteq \gamma'(g_{1,1}, \ldots, g_{r,a_r}) + (\mathbf{x}_1),$$

which proves that $\gamma' I$ also satisfies Equation (17). $\square$

Remark 4.16. In the proof of Lemma 4.15, we used the assumption A_{r-1} to show that $I|_H \subseteq \mathfrak{D}|_H$. This condition is necessary, but not sufficient, to have $I \subseteq \mathfrak{D}$ – even if I also satisfies Equation (17). For example, consider the ideal $I = (x^2, y^3 + xyz, z^7)$ and $\mathfrak{D} = \overline{(x^2, y^3, z^7)}$.

4.4. Step (3): $B_r \implies A_r$.

4.4.1. *Proof Sketch.* By B_r, any ideal I as in Assumption 4.7 with $c(I) = E_n(I)$ can be written in the form

$$I = (f_{1,1}, \ldots, f_{1,a_1}) + ((f'_{2,1} + f''_{2,1}) + \cdots + (f'_{r,a_r} + f''_{r,a_r}))$$

where $f'_{i,j} \in k[\mathbf{x}_2, \ldots, \mathbf{x}_r]$ and $f''_{i,j} \in (\mathbf{x}_1)$, and $f_{1,1}, \ldots, f_{1,a_1}, f'_{2,1}, \ldots, f'_{r,a_r}$ are all contained in $\mathfrak{D}$ and form a regular sequence. If $f''_{i,j} \in \mathfrak{D}$ for all i, j, then $I \subseteq \mathfrak{D}$ and Proposition 2.36 implies that $\overline{I} = \mathfrak{D}$ as desired.

To handle the case that $I \not\subseteq \mathfrak{D}$, we suppose that I is as close to $\mathfrak{D}$ as possible without having $I \subseteq \mathfrak{D}$. Specifically, we consider an ideal I with a unique index $2 \leq m \leq r-1$ such that $f''_{i,j} = 0$ for all $i \neq m$ and $f'_{i,j} = x_{i,j}^{d_i}$. We also suppose that for this fixed index m, the $f''_{m,j}$ are not all zero and no monomial summand of $f''_{m,j}$ is contained in $\mathfrak{D}$. Because of the (relatively) uncomplicated structure of I, we are able to use positive-characteristic arguments to directly show $c(I) > E_n(I)$ (Lemma 4.28).

In the general case (Lemma 4.27), we prove that any complete intersection satisfying Equation (17) but with $I \not\subseteq \mathfrak{D}$ can be degenerated to an ideal I' which is as described in the previous paragraph: close to $\mathfrak{D}$ while satisfying $I' \not\subseteq \mathfrak{D}$. By Proposition 2.19, we have $c(I) \geq c(I') > E_n(I') = E_n(I)$.

4.4.2. *Constructing the Partial Degeneration.*

Definition 4.17. For $\vec{u} = (u_1, \dots, u_n) \in (\mathbb{Z}^+)^n$, set $|u| = u_1 + \cdots + u_n$. For $d = (d_1, \dots, d_n) \in \mathbb{Z}^n$, we also define $\nu_d : \mathbb{Z}^n \to \mathbb{Q}$ by $\nu_d(\vec{u}) = \frac{u_1}{d_1} + \cdots + \frac{u_n}{d_n}$.

Lemma 4.18. *Let $0 < d_1 < \cdots < d_n \in (\mathbb{Z}^+)^n$ and set $d = (d_1, \dots, d_n)$. Let $S \subseteq (\mathbb{Z}_{\geq 0})^n$. Write $S = S_2 \sqcup \cdots \sqcup S_{n-1}$. Assume that S satisfies the following:*

(1) For all $\vec{u} \in S_i$, we have $|u| = d_i$.
(2) For all $\vec{u} \in S$, either $u_1 \geq 1$ or $\nu_d(\vec{u}) \geq 1$.
(3) For all $2 \leq i \leq n-1$, we have $d_i\vec{e_i} \notin S_i$, where $\vec{e}_i$ denotes the ith unit vector in $\mathbb{Z}^n$.
(4) There exists $\vec{u} \in S$ with $\nu_d(\vec{u}) < 1$ (and therefore $u_1 \geq 1$).

Then there exists a linear map $\lambda : \mathbb{Z}^n \to \mathbb{Z}$ and a unique index $2 \leq m \leq n-1$ such that

(A.i) *For all $2 \leq i \leq n-1, i \neq m, \vec{u} \in S_i$ we have $\lambda(\vec{u}) < \lambda(d_m\vec{e_m})$.*
(A.ii) *We have*

$$\max_{\vec{u} \in S_m} \lambda(\vec{u}) = \lambda(d_m\vec{e}_m). \tag{22}$$

(A.iii) *For any $\vec{u}$ achieving the maximum in Equation (22) we have $\nu_d(\vec{u}) < 1$ and $u_1 \geq 1$.*

To prove Lemma 4.18, we construct a sequence of auxilliary linear maps $\lambda_0, \dots, \lambda_s : \mathbb{Z}^n \to \mathbb{Q}$.

Lemma 4.19. *Assume the setup of Lemma 4.18. There exists a linear map $\lambda_0 : \mathbb{Z}^n \to \mathbb{Q}$ such that:*

(B.i) *For all $2 \leq i \leq n-1, \vec{u} \in S_i$ we have $\lambda_0(\vec{u}) \leq \lambda_0(d_i\vec{e_i})$.*
(B.ii) *For $\vec{u} \in S_i$, if $\lambda_0(\vec{u}) = \lambda_0(d_i\vec{e_i})$ then $\nu_d(\vec{u}) \leq 1$.*
(B.iii) *For $\vec{u} \in S_i$, if $\lambda_0(\vec{u}) = \lambda_0(d_i\vec{e_i})$ and $u_1 \geq 1$, then $\nu_d(\vec{u}) < 1$.*
(B.iv) *There exists an index $2 \leq j \leq n-1$ and some $\vec{u} \in S_j$ such that $u_1 \geq 1$ and $\lambda_0(\vec{u}) = d_j\lambda_0(\vec{e}_j)$.*

Proof. To start, we set

$$t_0 := \min_{\vec{u}\in S \text{ with } u_1\geq 1} \frac{\nu_d(\vec{u})-1}{u_1}. \tag{23}$$

We define $w_0 : \mathbb{Z}^n \to \mathbb{Q}$ by

$$\lambda_0(\vec{u}) = t_0 u_1 - \nu_d(\vec{u}).$$

We claim that λ_0 satisfies (B.i)-(B.iv).

(B.i) Let $2 \leq i \leq n-1, \vec{u} \in S_i$. Our proof that $\lambda_0(\vec{u}) \leq \lambda_0(d_i\vec{e}_i)$ splits into two cases.

$u_1 = 0$: By assumption (2) of Lemma 4.18 we have $w_0(\vec{u}) = -\nu_d(\vec{u}) \leq -1$.

$u_1 \geq 1$: We have $w_0(\vec{u}) \leq \left(\frac{\nu_d(\vec{u})-1}{u_1}\right) u_1 - \nu_d(\vec{u}) = -1$. The property (B.i) then follows from the fact that $\lambda_0(d_i\vec{e}_i) = \nu_d(d_i\vec{e}_i) = -1$ for all $2 \leq i \leq n-1$.

(B.ii) First, we observe that if $\lambda_0(\vec{u}) = \lambda_0(d_i\vec{e}_i) = -1$, then

$$\nu_d(\vec{u}) = t_0u_1 - \lambda_0(\vec{u}) = 1 + t_0u_1. \tag{24}$$

By assumption (4) of Lemma 4.18, there exists $\vec{u} \in S$ with $u_1 \geq 1$ and $\nu_d(\vec{u}) < 1$. It follows that the minimum in Equation (23) is negative, proving (B.ii).

(B.iii) The claim (B.iii) follows from Equation (24) together with the fact that $t_0u_1 < 0$.

(B.iv) If $\vec{u} \in S_j$ realizes the minimum in Equation (23), then

$$\lambda_0(\vec{u}) = \left(\frac{\nu_d(\vec{u})-1}{u_1}\right) u_1 + \nu_d(\vec{u}) = -1 = \lambda_0(d_j\vec{e}_j).$$

□

Notation 4.20. *Assume the notation of Lemma 4.18, and let λ_0 be as in Lemma 4.19. For $2 \leq i \leq n-1$, we define*

$$S_i^{(0)} = \{\vec{u} \in S_i : \lambda_0(\vec{u}) = \lambda_0(d_i\vec{e}_i)\}$$

and we set $S^{(0)} = S_2^{(0)} \sqcup \cdots \sqcup S_{n-1}^{(0)}$.

By Lemma 4.19, we have some control over $S^{(0)}$.

- By definition, we have $S^{(0)} \subseteq S$.
- By (B.ii) and (B.iii), for every $\vec{u} \in S^{(0)}$ we have $\nu_d(\vec{u}) \leq 1$ with equality only if $u_1 = 0$.
- By (B.iv), $S^{(0)}$ is nonempty. In particular, $S^{(0)}$ contains an element $\vec{u}$ with $u_1 \geq 1$.

Lemma 4.21. *Assume the notation of Notation 4.20. There exists a linear map $\lambda_1 : \mathbb{Z}^n \to \mathbb{Q}$ such that:*

(C.i) *For all $2 \leq i \leq n-1, \vec{u} \in S_i^{(0)}$, if $u_1 = 0$ then $\lambda_1(\vec{u}) < \lambda_1(d_i\vec{e}_i)$.*

(C.ii) *For all* $2 \le i \le n-1, \vec{u} \in S_i^{(0)}$*, if* $u_1 \ge 1$ *then we have* $\lambda_1(\vec{u}) \le \lambda_1(d_i\vec{e}_i)$.

(C.iii) *There exists an index* $2 \le j \le n-1$ *and some* $\vec{u} \in S_j^{(0)}$ *such that* $u_1 \ge 1$ *and* $\lambda_1(\vec{u}) = \lambda_1(d_j\vec{e}_j)$.

Proof. We set

(25)
$$t_1 := \min_{2\le i\le n-1} \min_{\vec{u}\in S_i^{(0)}} \frac{d_n^2u_2 + \cdots + d_n^nu_n - d_id_n^i}{u_1}, \quad \lambda_1(\vec{u}) := t_1u_1 - d_n^2u_2 - \cdots - d_n^nu_n.$$

We claim that λ_1 satisfies properties (C.i)-(C.iii).

(C.i) By assumption (2) and (B.ii), we have $\nu_d(\vec{u}) \le 1$. Consequently, we deduce

$$1 \ge \nu_d(\vec{u}) = \frac{u_1}{d_1} + \cdots + \frac{u_i}{d_i} \ge \frac{u_1 + \cdots + u_i}{d_i}. \tag{26}$$

If $u_1 + \cdots + u_i = d_i$, then each inequality in Equation (26) is an equality and hence $\vec{u} = d_i\vec{e}_i$. This would violate assumption (3), so there must instead be some index $i < j \le n$ such that $u_j \ge 1$. We then have

$$\lambda_1(\vec{u}) = -d_n^2u_2 - \cdots - d_n^nu_n \le -d_n^j < -d_id_n^i = \lambda_1(d_i\vec{e}_i).$$

(C.ii) We compute

(27)
$$\lambda_1(\vec{u}) \le \left(\frac{d_n^2u_2 + \cdots + d_n^nu_n - d_id_n^i}{u_i}\right)u_i - d_n^2u_2 - \cdots - d_n^nu_n = -d_id_n^i = \lambda_1(d_i\vec{e}_i).$$

(C.iii) By (B.iv), the set $\{\vec{u} \in S^{(0)} : u_1 \ge 1\}$ is nonempty. If $\vec{u}$ achieves the minimum in Equation (25), then the inequality in Equation (27) is an equality.

□

Notation 4.22. *Let* $k \ge 0$*. Suppose that* $\lambda_0, \ldots, \lambda_k$ *have been defined. For* $1 \le \ell \le k, 2 \le i \le n-1$*, we define*

$$S_i^{(\ell)} = \{\vec{u} \in S_i^{\ell-1} : \lambda_\ell(\vec{u}) = d_i\lambda_\ell(\vec{e}_i)\}$$

and set $S^{(\ell)} = S_2^{(\ell)} \sqcup \cdots \sqcup S_{n-1}^{(\ell)}$*. Finally, we set* $\Lambda_\ell = \{2 \le i \le n-1 : S_i^{(\ell)} \ne \varnothing\}$*.*

Similarly to the case of $S^{(0)}$, Lemmas 4.21 and 4.23 give us control over $S^{(\ell)}$ for $\ell \ge 1$. Assume we have constructed λ_{k+1} and proven Lemma 4.23 for $k \le \ell - 1$.

- By definition, we have $S^{(\ell)} \subseteq S^{(\ell-1)}$.
- By (B.iii), (C.i), and (C.ii), for all $\vec{u} \in S^{(\ell)}$ we have $u_1 \ge 1$ and $\nu_d(\vec{u}) < 1$.
- By (C.iii) if $\ell = 1$ and (D.iii) if $\ell \ge 2$, there exists an element $\vec{u} \in S^{(\ell)}$ with $u_1 \ge 1$.

Lemma 4.23. *Let $k \geq 0$ and suppose that $\lambda_0, \dots, \lambda_k$ have been defined. Assume the notation of Notation 4.22 and suppose that $|\Lambda_k| \geq 2$. Set $i_k = \min \Lambda_k$. There exists $\lambda_{k+1} : \mathbb{Z}^n \to \mathbb{Q}$ such that:*

(D.i) *For $\vec{u} \in S_{i_k}^{(k)}$, we have $\lambda_{k+1}(\vec{u}) < \lambda_{k+1}(d_i\vec{e}_i)$.*

(D.ii) *For $j > i_k, \vec{u} \in S_j^{(k)}$, we have $\lambda_{k+1}(\vec{u}) \leq \lambda_{k+1}(d_i\vec{e}_i)$.*

(D.iii) *There exists some index $j > i_k$ and some $\vec{u} \in S_j^{(k)}$ such that $\lambda_{k+1}(\vec{u}) = \lambda_{k+1}(d_j\vec{e}_j)$.*

Proof. We define

$$t_{k+1} := \max_{j > i_k, \vec{u} \in S_j^{(k)}} \frac{u_{i_k}}{u_1}, \quad \lambda_{k+1}(\vec{u}) := -t_{k+1}u_1 + u_{i_k}. \tag{28}$$

We claim that λ_{k+1} satisfies (D.i)-(D.iii).

(D.i) For $\vec{u} \in S_{i_k}^{(k)}$, since $\vec{u} \neq d_{i_k}\vec{e}_{i_k}$, we have

$$\lambda_{k+1}(\vec{u}) \leq u_{i_k} < d_{i_k} = d_{i_k}\lambda_{k+1}(\vec{e}_{i_k}).$$

(D.ii) For $j > k, \vec{u} \in S_j^{(k)}$ we have

$$\lambda_{k+1}(\vec{u}) \leq -\left(\frac{u_{i_k}}{u_1}\right)u_1 + u_{i_k} = 0 = \lambda_{k+1}(d_j\vec{e}_j). \tag{29}$$

(D.iii) If $\vec{u}$ attains the maximum in Equation (28), then the inequality in Equation (29) is sharp.

□

Lemma 4.24 (c.f. [11], Exercise 15.12)**.** *Let $\mu_0, \dots, \mu_s : \mathbb{Z}^n \to \mathbb{Q}$ be linear maps and $U \subseteq \mathbb{Z}^n$ a finite set. There exists a map $\mu : \mathbb{Z}^n \to \mathbb{Z}$ such that for all $\vec{u}, \vec{v} \in U$, we have $\mu(\vec{u}) < \mu(\vec{v})$ if and only if*

$$\begin{aligned} &\textit{there exists } 0 \leq k \leq s \textit{ such that } \mu_k(\vec{u}) < \mu_k(\vec{v}) \\ &\textit{and } \mu_\ell(\vec{u}) = \mu_\ell(\vec{v}) \textit{ for all } 0 \leq \ell \leq k-1 \end{aligned} \tag{30}$$

We may now prove Lemma 4.18.

Proof. First, we construct the auxilliary maps $\lambda_0, \dots, \lambda_s$. Apply Lemmas 4.19 and 4.21 to S to produce linear maps λ_0, λ_1. By (B.iv) and (C.iii), we have $|\Lambda_1| \geq 1$. We continue inductively: suppose we have constructed $\lambda_0, \dots, \lambda_k$. If $|\Lambda_k| = 1$, we set $s = k$ and proceed to the start of the following paragraph for the next step of the argument. Otherwise, apply Lemma 4.23 to produce a map λ_{k+1}. By (D.i) we have $\Lambda_{k+1} \subseteq \Lambda_k \setminus \{i_k\}$ and by (D.iii) we have $|\Lambda_{k+1}| \geq 1$. As $n - 2 \geq |\Lambda_1| > |\Lambda_2| > \cdots \geq 1$, we eventually arrive at an index s such that $|\Lambda_s| = 1$.

Apply Lemma 4.24 to the sequence $\lambda_0, \dots, \lambda_s$ to produce a map $\lambda : \mathbb{Z}^n \to \mathbb{Z}$. We show that λ satisfies (A.i) and (A.ii), where the index m is the unique element of Λ_s. By Lemmas 4.19, 4.21 and 4.23, for all $2 \leq i \leq n-1, \vec{u} \in S_i, 0 \leq k \leq s$ we have $\lambda_k(\vec{u}) \leq \lambda_k(d_i\vec{e}_i)$. It follows that $\lambda(\vec{u}) \leq \lambda(d_i\vec{e}_i)$. If

$\lambda(\vec{u}) = \lambda(d_i\vec{e}_i)$, then we must have $\lambda_k(\vec{u}) = \lambda_k(d_i\vec{e}_i)$ for all $0 \le k \le s$. Each successive equality $\lambda_k(\vec{u}) = \lambda_k(d_i\vec{e}_i)$ creates further restrictions on $\vec{u}$:

(a) As $\lambda_0(\vec{u}) = \lambda_0(d_i\vec{e}_i)$, by Lemma 4.19 (B.ii) and (B.iii) we have either $u_0 = 0$ and $\nu_d(\vec{u}) \le 1$ or $u_1 \ge 1$ and $\nu_d(\vec{u}) < 1$.

(b) As $\lambda_1(\vec{u}) = \lambda_1(d_i\vec{e}_i)$, by Lemma 4.21 (C.i), we must in fact have $u_1 \ge 1$ and $\nu_d(\vec{u}) < 1$.

(c) As $\lambda_k(\vec{u}) = \lambda_k(d_i\vec{e}_i)$ for all $1 \le i \le s$, we have $i \in \Lambda_1 \cap \cdots \cap \Lambda_s = \{m\}$.

Point (c) in the above list implies (A.i) and points (a), (b) imply (A.iii). For (A.ii), we have by construction that $\Lambda_s = \{m\}$ is nonempty, so there exists $\vec{u^*} \in S_m^{(s)} \subseteq S_m$. By Notation 4.22, the set $S_m^{(s)}$ is inductively defined as the set of $\vec{u} \in S_m$ such that $\lambda_k(\vec{u}) = \lambda_k(d_m\vec{e}_m)$ for all $0 \le k \le s$, so we have $\lambda(\vec{u^*}) = \lambda(d_m\vec{e}_m)$, hence Equation (22) holds. □

4.4.3. *Using the Partial Degeneration Order.*

Lemma 4.25. *Let R be as in Assumption 4.7. Suppose that $f_1, \ldots, f_{a_i}$ are d_i-forms comprising a regular sequence in R, and suppose that $f_j \in k[\mathbf{x}_i]$ for all $1 \le j \le a_i$. Then the integral closure J of $(f_1, \ldots, f_j)$ in R is equal to $(\mathbf{x}_i)^d$.*

Proof. Since $k[\mathbf{x}_i] \to R$ is faithfully flat, $f_1, \ldots, f_{a_i}$ form a regular sequence in $k[\mathbf{x}_i]$. By Theorem 2.23, the integral closure of $(f_1, \ldots, f_{a_i})$ in $k[\mathbf{x}_i]$ is $(\mathbf{x}_i)^d$. By [25, Proposition 1.6.2], we have $(\mathbf{x}_i)^{d_i} \subseteq J$. On the other hand, we have $(f_1, \ldots, f_{a_i}) \subseteq (\mathbf{x}_i)^{d_i}$. By [25, Proposition 1.4.6], $(\mathbf{x}_i)^d$ is integrally closed in R, so $J = (\mathbf{x}_i)^{d_i}$. □

Before proceeding, recall our conventions from Definition 2.18.

Lemma 4.26. *Assume the setting of Assumption 4.7. Suppose I satisfies Equation* (17)*. Then we have the following:*

(a) For all $1 \le i \le r, 1 \le j \le a_i$, there exists $f'_{i,j} \in k[\mathbf{x}_i]$ and $f''_{i,j} \in (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1})$ such that $f_{i,j} = f'_{i,j} + f''_{i,j}$.

(b) For all $1 \le i \le r$, we have $\sqrt{I_1 + \cdots + I_i} = (\mathbf{x}_1, \ldots, \mathbf{x}_i)$.

(c) For all $1 \le i \le r$, the elements $f'_{i,1}, \ldots, f'_{i,a_i}$ form a regular sequence.

Proof. The key to this proof is the following simple observation. Let $g = x_{1,1}^{e_{1,1}} \ldots x_{r,a_r}^{e_{r,a_r}}$ be a monomial of degree d_i such that $w_d(g) \ge 1$. If $g \notin (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1})$, then we have

$$1 \le w_d(g) \le \frac{e_{i,1} + \cdots + e_{i,a_i}}{d_i} + \frac{e_{i+1,1}, \cdots + e_{r,a_r}}{d_{i+1}} = \frac{e_{i,1}, \cdots + e_{r,a_r}}{d_i} + (d_{i+1} - d_i)\left(\frac{e_{i+1,1}, \cdots + e_{r,a_r}}{d_i d_{i+1}}\right) \tag{31}$$

As $e_{i,1} + \cdots + e_{r,a_r} = d_i$ and $\deg(g) = d_i$ we have $e_{k,\ell} = 0$ for all $k < i$. At the same time, Equation (31) implies that $e_{i+1,1} + \cdots + e_{r,a_r} = 0$, so $g \in k[\mathbf{x}_i]$.

For any $1 \leq i \leq r, 1 \leq j \leq a_i$, by Equation (17) we have $\operatorname{supp}(f_{i,j}) \subseteq \mathfrak{D} \sqcup (\mathbf{x}_1)$. By the previous paragraph, we have $\operatorname{supp}(f_{i,j}) \cap \mathfrak{D} \subseteq k[\mathbf{x}_i] \sqcup (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1})$. Writing $f_{i,j} = \sum_{y \in \operatorname{supp}(f_{i,j})} \beta_y^{(i,j)} y$, we may therefore define

$$f'_{i,j} := \sum_{y \in \operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_i]} \beta_y^{(i,j)} y, \qquad f''_{i,j} = \sum_{y \in \operatorname{supp}(f_{i,j}) \cap (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1})} \beta_y^{(i,j)} y,$$

which proves (a).

We prove (b) and (c) simultaneously by induction. By Equation (17), I_1 is a homogeneous ideal extended from $k[\mathbf{x}_1]$ and $\operatorname{ht}(I_1) = a_1$, so (b) holds for $i = 1$. As $f_{1,j} = f'_{1,j}$ for $1 \leq j \leq a_1$, (c) also holds for $i = 1$. Suppose that (b), (c) hold for $i - 1$. Then $\sqrt{I_1 + \cdots + I_{i-1}} = (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1})$, so we have

$$\begin{aligned} \sqrt{I_1 + \cdots + I_i} &= \sqrt{(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}) + (f_{i,1}, \ldots, f_{i,a_i})} \\ &= \sqrt{(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}) + (f'_{i,1}, \ldots, f'_{i,a_i})}. \end{aligned} \tag{32}$$

Taking the heights of all terms in Equation (32), we observe that the elements $f_{1,1}, \ldots, f_{i-1,a_{i-1}}, f'_{i,1}, \ldots, f'_{i,a_i}$ form a regular sequence. In particular, $f'_{i,1}, \ldots, f'_{i,a_i}$ is a regular sequence, proving claim (c). As $\operatorname{ht}((f'_{i,1}, \ldots, f'_{i,a_i})) = a_i$ and $(f'_{i,1}, \ldots, f'_{i,a_i})$ is a homogeneous ideal extended from $k[\mathbf{x}_i]$, we have $\sqrt{(f'_{i,1}, \ldots, f'_{i,a_i})} = (\mathbf{x}_i)$. Applying Equation (32) again, we deduce claim (b). $\square$

Lemma 4.27. *Assume the setting of Assumption 4.7. Suppose I satisfies Equation (17) and $I \not\subseteq \mathfrak{D}$. Then there exists an integer $2 \leq m \leq r - 1$ and d_m-forms $h_{m,1}, \ldots, h_{m,a_m}$ such that, setting*

$$J := (\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_{m-1})^{d_{m-1}} + (h_{m,1}, \ldots, h_{m,a_m}) + (\mathbf{x}_{m+1})^{d_{m+1}} + \cdots + (\mathbf{x}_r)^{d_r},$$

we have

(i) For all $1 \leq j \leq a_m$, we have $h_{m,j} = h'_{m,j} + h''_{m,j}$, where $h'_{m,j} \in k[\mathbf{x}_m], h''_{m,j} \in (\mathbf{x}_1)$, and $w_d(y) < 1$ for all $y \in \operatorname{supp}(h''_{m,j})$;
(ii) There exists some j such that $h''_{m,j} \neq 0$;
(iii) J is $\mathfrak{m}$-primary with $E_n(J) = E_n(I)$;
(iv) $c(J) \leq c(I)$.

Proof. Let $f'_{i,j}, f''_{i,j}$ be as in Lemma 4.26. Define a semigroup homomorphism $\rho : \operatorname{Mon}(\mathbf{x}_1, \ldots, \mathbf{x}_r) \to \mathbb{Z}^r_{\geq 0}$ by $\rho(x_{i,j}) = \vec{e}_i$. For $2 \leq i \leq r$, we define

$$S_i := \bigcup_{j=1}^{a_i} \rho\left(\operatorname{supp}(f''_{i,j})\right).$$

Set $S = S_2 \sqcup \cdots \sqcup S_{r-1}$. We claim that S satisfies the hypotheses (1)-(4) of Lemma 4.18.

(1) The claim follows from the fact that $f''_{i,j}$ is homogeneous of degree d_i and $\deg(y) = |\rho(y)|$ for any monomial $y \in R$.

(2) Let $y \in \operatorname{supp}(f''_{i,j})$ be a monomial. If $y \in (\mathbf{x}_1)$, then $\rho(y)_1 \geq 1$. Otherwise, as I satisfies Equation (17), we have $1 \leq w_d(y) = \nu_d(\rho(y))$.
(3) By construction, if $y \in \operatorname{supp}(f''_{i,j})$ then $y \in (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1})$, so $\rho(y)_1 + \cdots + \rho(y)_{i-1} \geq 1$, hence $\rho(y) \neq d_i\vec{e}_i$.
(4) By assumption, we have $I \not\subseteq \mathfrak{D}$, thus we have $f_{i,j} \notin \mathfrak{D}$ for some $1 \leq i \leq r, 1 \leq j \leq a_i$. By Equation (17), I_1 is extended from $k[\mathbf{x}_1]$, hence $I_1 \subseteq (\mathbf{x}_1)^{d_1} \subseteq \mathfrak{D}$. Moreover, by Lemma 3.3 we have $\mathfrak{m}^{d_r} \subseteq \mathfrak{D}$, hence $I_r \subseteq \mathfrak{D}$. We conclude that there exists some $2 \leq i \leq n-1$ and some $1 \leq j \leq a_i$ such that $f_{i,j} \notin \mathfrak{D}$. As $f'_{i,j} \in \mathfrak{m}^{d_i} \cap k[\mathbf{x}_i] \subseteq \mathfrak{D}$ and $f'_{i,j} + f''_{i,j} = f_{i,j} \notin \mathfrak{D}$, it must be the case that $f''_{i,j} \notin \mathfrak{D}$. In particular, there exists a monomial y in the support of $f''_{i,j}$ such that $y \notin \mathfrak{D}$. Recalling that $\mathfrak{D}$ consists precisely of the elements $f \in R$ with $w_d(f) \geq 1$, for this monomial y we have $\rho(y) \in S$ and $\nu_d(\rho(y)) = w_d(y) < 1$.

We now apply Lemma 4.18 to the set S to produce a map $\lambda : \mathbb{Z}^r \to \mathbb{Z}$ and an index $2 \leq m \leq r-1$ such that

(A.i) For all $2 \leq i \leq r-1, i \neq m, \vec{u} \in S_i$ we have $\lambda(\vec{u}) < \lambda(d_m\vec{e_m})$.
(A.ii) We have

$$\max_{\vec{u} \in S_m} \lambda(\vec{u}) = \lambda(d_m\vec{e}_m). \tag{8}$$

(A.iii) For any $\vec{u}$ achieving the maximum in Equation (22) we have $\nu_d(\vec{u}) < 1$ and $u_1 \geq 1$.

Let in_λ denote the monomial partial order given by $y <_\lambda z$ if $\lambda(\rho(y)) < \lambda(\rho(z))$. We consider the leading terms $\mathrm{in}_\lambda(f_{i,j})$ for $2 \leq i \leq r-1, 1 \leq j \leq a_i$. For any $y \in \operatorname{supp}(f'_{i,j})$, we have $\rho(y) = d_i\vec{e}_i$, and in particular $\lambda(\rho(y)) \leq \lambda(d_i\vec{e}_i)$. By (A.i) and (A.ii), we have $\lambda(\rho(y)) \leq \lambda(d_i\vec{e}_i)$ for all $y \in \operatorname{supp}(f''_{i,j})$. It follows that

$$\max_{y \in \operatorname{supp}(f_{i,j})} \lambda(\rho(y)) \leq \lambda(d_i\vec{e}_i).$$

By Lemma 4.26, the elements $f'_{i,j}$ form a regular sequence. In particular, $f'_{i,j} \neq 0$, so $\operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_i] = \operatorname{supp}(f'_{i,j}) \neq \varnothing$. If $y_{i,j} \in \operatorname{supp}(f'_{i,j})$, then we have

$$\max_{y \in \operatorname{supp}(f_{i,j})} \lambda(\rho(y)) \geq \lambda(\rho(y_{i,j})) = \lambda(d_i\vec{e}_i).$$

We now have an expression for the leading term $\mathrm{in}_\lambda(f_{i,j})$. If we write $f_{i,j} = \sum_{y \in \operatorname{supp}(f_{i,j})} \beta_y^{(i,j)} y$, then $\mathrm{in}_\lambda(f_{i,j})$ is the sum over all $\gamma_y^{(i,j)} y$ such that $\lambda(\rho(y))$ is as large as possible. To be precise, we have

$$\mathrm{in}_\lambda(f_{i,j}) = \sum_{y \in \operatorname{supp}(f_{i,j}) : \lambda(\rho(y)) = \lambda(d_i\vec{e}_i)} \beta_y^{(i,j)} y = f'_{i,j} + \sum_{y \in \operatorname{supp}(f''_{i,j}) : \lambda(\rho(y)) = \lambda(d_i\vec{e}_i)} \beta_y^{(i,j)} y. \tag{33}$$

For $1 \leq i \leq a_m$, set $h_{m,j} := \mathrm{in}_\lambda(f_{m,j})$. With this choice of $h_{m,1}, \ldots, h_{m,a_m}$, we verify that J satisfies conclusions (i)-(iv) of the lemma.

(i) We set $h'_{m,j} := f'_{m,j}$ and $h''_{m,j} := \sum_{y\in\operatorname{supp}(f''_{i,j}):\lambda(\rho(y))=\lambda(d_i\vec{e}_i)} \gamma_y^{(i,j)} y$. By Equation (33), we have $h'_{m,j} + h''_{m,j} = \operatorname{in}_\lambda(f_{m,j}) = h_{m,j}$. By the definition of $f'_{m,j}$ in Lemma 4.26, we have $\operatorname{supp}(h'_{m,j}) \subseteq k[\mathbf{x}_m]$. For any $y \in \operatorname{supp} h''_{m,j}$, we have $y \in \operatorname{supp}(f''_{m,j})$, so $\rho(y) \in S_m$. We also have $\lambda(\rho(y)) = \lambda(d_m\vec{e}_m)$, hence by (A.iii) we have $\rho(y)_1 \geq 1$ and $\nu_d(\rho(y)) < 1$, hence $y \in (\mathbf{x}_1)$ and $w_d(y) < 1$.

(ii) By (A.ii), there exists $\vec{u} \in S_m$ such that $\lambda(\vec{u}) = \lambda(d_m\vec{e}_m)$. By the definition of S_m, there exists some index $1 \leq j \leq a_m$ and some $y \in \operatorname{supp}(f''_{m,j})$ such that $\rho(y) = \vec{u}$. By Equation (33), we have $y \in \operatorname{supp}(h''_{m,j})$.

(iii) We argue similarly to Equation (32). By Lemma 4.25, we have $\sqrt{(f'_{m,1},\ldots,f'_{m,a_m})} = (\mathbf{x}_m)$. By Equation (33), we have $h_{m,j} \equiv f'_{m,j} \mod (\mathbf{x}_1)$, so it follows that

$$\begin{aligned}\sqrt{J} &= \sqrt{(\mathbf{x}_1)+\cdots+(\mathbf{x}_{m-1})+(h_{m,1},\ldots,h_{m,a_m})+(\mathbf{x}_{m+1})+\cdots+(\mathbf{x}_r)}\\ &= \sqrt{(\mathbf{x}_1)+\cdots+(\mathbf{x}_{m-1})+(f'_{m,1},\ldots,f'_{m,a_m})+(\mathbf{x}_{m+1})+\cdots+(\mathbf{x}_r)} = \mathfrak{m}.\end{aligned}$$

To see that $E_n(J) = E_n(I)$, we note that J has the same integral closure as the ideal

$$J' = (x_{1,1}^{d_1},\ldots,x_{m-1,a_{m-1}}^{d_{m-1}},h_{m,1},\ldots,h_{m,a_m},x_{m+1,1}^{d_{m+1}},\ldots,x_{r,a_r}^{d_r}).$$

As J' is a complete intersection, by Lemma 3.8 we have

$$E_n(J) = E_n(J') = \frac{a_1}{d_1}+\cdots+\frac{a_r}{d_r} = E_n(I).$$

(iv) Let $I' = (\operatorname{in}_\lambda(f_{1,1}),\ldots,\operatorname{in}_\lambda(f_{r,a_r}))$. We claim that $J \subseteq \overline{I'}$. By Equation (33) and (A.iii), we have $\operatorname{in}_\lambda(f_{i,j}) = f'_{i,j}$ for $i \neq m$. For $i \neq m$, it follows from Lemmas 4.25 and 4.26 that $(\mathbf{x}_i)^{d_i} = \overline{(f'_{i,1},\ldots,f'_{i,a_i})} \subseteq \overline{I'}$. The claim that $J \subseteq \overline{I'}$ follows once we note that $h_{m,j} = \operatorname{in}_\lambda(f_{i,j}) \in I'$. To see that $c(J) \leq c(I)$, by Propositions 2.9 and 2.19 we have

$$c(J) = c(J') \leq c(\overline{I'}) = c(I') \leq c(\operatorname{in}_\lambda(I)) \leq c(I).$$

□

Lemma 4.28. *Assume the setting of Assumption 4.7. Let $J \subseteq R$ be an ideal satisfying conditions (i)-(iii) of Lemma 4.27. Then $c(J) > E_n(J)$.*

Proof. Set $J_m = (h_{m,1},\ldots,h_{m,a_m})$. We first prove this result in characteristic $p > 0$.

By assumption, $J_m \not\subseteq \mathfrak{D}$. Define

$$\sigma := \max_{\substack{1\leq j\leq a_m\\ y\in\operatorname{supp}(h_{m,j})\setminus(\mathbf{x}_m)^{d_m}}} w_d(y),$$

which satisfies $\sigma < 1$ by conditions (ii) and (iv). By [2, Theorem 3.17] we have $c(J_m) > \frac{a_m}{d_m}$. Let $f = h_{1,m}^{t_1} \dots h_{m,a_m}^{t_{a_m}}$ be a homogeneous generator of $(J_m)^{\nu_{J_m}(p^e)}$ such that $f \notin \mathfrak{m}^{[p^e]}$. Write

$$f = \sum_{y \in \operatorname{supp}(f)} \beta_y y, \quad f' := \sum_{y \in \operatorname{supp}(f) \setminus \mathfrak{m}^{[p^e]}} \beta_y y.$$

As $f \equiv f' \mod \mathfrak{m}^{[p^e]}$, we have $f' J_m \subseteq \mathfrak{m}^{[p^e]}$.

Applying Theorem 2.25 and Lemma 4.25 to the ideal $\frac{J_m + (\mathbf{x}_1)}{(\mathbf{x}_1)}$, we have

$$\frac{(\mathbf{x}_m)^{a_m d_m} + (\mathbf{x}_1)}{(\mathbf{x}_1)} \subseteq \frac{J_m + (\mathbf{x}_1)}{(\mathbf{x}_1)}.$$

Let $\mu_1, \dots, \mu_M$ be a minimal set of monomial generators for $(\mathbf{x}_m)^{a_m d_m}$. Subsequently, let $\widetilde{\mu_1}, \dots, \widetilde{\mu_M} \in J_m$ be homogeneous elements such that $\mu_i - \widetilde{\mu_i} \in (\mathbf{x}_1)$ for all $1 \le i \le M$. Let $\succ$ denote the reverse lexicographic order after putting the variables in the order $\mathbf{x}_1 \prec \dots \prec \mathbf{x}_r$. By definition of the reverse lexicographic order, we have $y \prec z$ for any $y \in (\mathbf{x}_1), z \in k[\mathbf{x}_2, \dots, \mathbf{x}_r]$. In particular, we have $\operatorname{in}_\succ(\widetilde{\mu_i}) = \operatorname{in}_\succ(\mu_i + (\widetilde{\mu_i} - \mu_i)) = \mu_i$. It follows that $(\mathbf{x}_m)^{a_m d_m} \subseteq \operatorname{in}_\succ(J_m)$.

By construction, we have $f \notin \mathfrak{m}^{[p^e]}$ and $f J_m \subseteq J_m^{\nu_{J_m}(p^e)+1} \subseteq \mathfrak{m}^{[p^e]}$. As $f' \equiv f \mod \mathfrak{m}^{[p^e]}$, we have $f' \notin \mathfrak{m}^{[p^e]}$ and $f' J_m \subseteq \mathfrak{m}^{[p^e]} + J_m^{\nu_{J_m}(p^e)+1} \subseteq \mathfrak{m}^{[p^e]}$.

Let $y := \operatorname{in}_\succ(f')$. As no element of $\operatorname{supp}(f')$ is in $\mathfrak{m}^{[p^e]}$, we have $y \notin \mathfrak{m}^{[p^e]}$. Additionally, we have

$$\operatorname{in}_\succ(f') \operatorname{in}_\succ(J_m) \subseteq \operatorname{in}_\succ(f' J_m) \subseteq \operatorname{in}_\succ(\mathfrak{m}^{[p^e]}) = \mathfrak{m}^{[p^e]}.$$

By Lemma 4.3, we have

$$(\mathfrak{m}^{[p^e]} : J_m) \subseteq (\mathfrak{m}^{[p^e]} : (\mathbf{x}_m)^{a_m d_m}) = \mathfrak{m}^{[p^e]} + (\mathbf{x}_m)^{a_m(p^e-1) - a_m d_m + 1}.$$

As $y \notin \mathfrak{m}^{[p^e]}$, we have

$$\operatorname{ord}_{(\mathbf{x}_m)}(y) \ge a_m(p^e - 1) - a_m d_m + 1. \tag{34}$$

Additionally, we write

$$f = \prod_{j=1}^{a_m} (h'_{m,j} + h''_{m,j})^{t_j} = \prod_{j=1}^{a_m} \sum_{\substack{0 \le t'_j, t''_j \le t_j \\ t'_j + t''_j = t_j}} \binom{t_j}{t'_j} (h'_{m,j})^{t_j} (h''_{m,j})^{t''_j}.$$

As $\operatorname{supp}(f') \subseteq \operatorname{supp}(f)$, there exist $(t'_1, t''_1), \dots, (t'_{a_m}, t''_{a_m})$ such that $t'_j + t''_j = t_j$ for all $1 \le j \le a_m$ and

$$y \in \operatorname{supp}\left((h'_{m,1})^{t'_1} (h''_{m,1})^{t''_1} \dots (h'_{m,a_m})^{t'_{a_m}} (h''_{m,a_m})^{t''_{a_m}} \right).$$

Set $J'_m = (h'_{m,1}, \dots, h'_{m,a_m})$. As J_m is extended from $k[\mathbf{x}_m]$, by Lemma 4.25 we have $J_m \subseteq (\mathbf{x}_m)^{d_m}$, so

$$t'_1 + \dots + t'_{a_m} \le \nu_{J_m}(p^e) \le \nu_{(\mathbf{x}_m)^{d_m}}(p^e) \le \left\lfloor \frac{a_m(p^e - 1)}{d_m} \right\rfloor.$$

Consequently, we have

$$w_d(y) \leq \sum_{j=1}^{a_m} t'_j w_d(h'_{m,j}) + \sum_{j=1}^{a_m} t''_j \left(\max_{z \in \operatorname{supp}(h''_{m,j})} w_d(z) \right) \tag{35}$$

$$= \sum_{j=1}^{a_m} t'_j + \sum_{j=1}^{a_m} t''_j \left(\max_{z \in \operatorname{supp}(h''_{m,j})} w_d(z) \right) \leq \sum_{j=1}^{a_m} t'_j + \sigma \sum_{j=1}^{a_m} t''_j \tag{36}$$

$$\leq \left\lfloor \frac{a_m(p^e-1)}{d_m} \right\rfloor + \sigma \left(\nu_{J_m}(p^e) - \left\lfloor \frac{a_m(p^e-1)}{d_m} \right\rfloor \right). \tag{37}$$

As in Lemma 4.3, let $\mathfrak{a}_\beta, \mathfrak{a}_\beta^+$ denote the ideals $\{f \in R : w_d(f) \geq \beta\}, \{f \in R : w_d(f) > \beta\}$ respectively. Let t_e denote the quantity in Equation (37); as $w_d(y) \leq t_e$ we have $y \notin \mathfrak{a}_{t_e}^+$. By assumption, $y \notin \mathfrak{m}^{[p^e]}$, and as y is a monomial we have $y \notin \mathfrak{m}^{[p^e]} + \mathfrak{a}_{t_e}^+$. Set $u_e := (p^e-1)(\frac{a_1}{d_1} + \cdots + \frac{a_r}{d_r})$. It follows from Lemma 4.3 that

$$y \notin \mathfrak{m}^{[p^e]} + \mathfrak{a}_{t_e}^+ = (\mathfrak{m}^{[p^e]} : \mathfrak{a}_{u_e - t_e}).$$

Let $z \in \mathfrak{a}_{u_e-t_e}$ such that $yz \notin \mathfrak{m}^{[p^e]}$. Write $z = z'w$ where $w \in k[\mathbf{x}_m]$ and $z' \in k[\mathbf{x}_1, \ldots, \widehat{\mathbf{x}_m}, \ldots, \mathbf{x}_r]$. As $yz \notin \mathfrak{m}^{[p^e]}$, by Equation (34) we have

$$\operatorname{ord}_{(\mathbf{x}_m)}(w) \leq (p^e-1)a_m - \operatorname{ord}_{(\mathbf{x}_m)}(y) \leq a_m d_m,$$

hence $w_d(z') = w_d(z) - w_d(w) \geq u_e - t_e - a_m$. As $\mathfrak{D} = \overline{(\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_r)^{d_r}}$, by Theorem 2.25 we have

$$z' \in \mathfrak{D}^{\lfloor u_e - t_e \rfloor - a_m} \subseteq ((\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_r)^{d_r})^{\lfloor u_e - t_e \rfloor - a_m - n}.$$

Since $z' \notin (\mathbf{x}_m)$, we in fact have

$$z' \in ((\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_{m-1})^{d_{m-1}} + (\mathbf{x}_{m+1})^{d_{m+1}} (\mathbf{x}_r)^{d_r})^{\lfloor u_e - t_e \rfloor - a_m - n} \subseteq (J)^{\lfloor u_e - t_e \rfloor - a_m - n}.$$

It follows that $\nu_J(p^e) \geq \nu_{J_m}(p^e) + \lfloor u_e - t_e \rfloor - a_m - n$. Dividing by p^e and letting $e \to \infty$, we obtain

$$\begin{aligned} c(J) &\geq c(J_m) + \lim_{e\to\infty} \frac{u_e}{p^e} - \lim_{e\to\infty} \frac{t_e}{p^e} \\ &= c(J_m) + \left(\frac{a_1}{d_1} + \cdots + \frac{a_r}{d_r} \right) - \left(\frac{a_m}{d_m}(1-\sigma) + \sigma c(J_m) \right) \\ &= (1-\sigma)\left(c(J_m) - \frac{a_m}{d_m} \right) + \left(\frac{a_1}{d_1} + \cdots + \frac{a_r}{d_r} \right) \end{aligned}$$

Since $\sigma < 1$ and $c(J_m) > \frac{a_m}{d_m}$, it follows that the above quantity exceeds $DP(J)$.

In characteristic zero, one notes that for any ideal J satisfying conditions (i)-(iii), the reduction of the pair (R, J) to characteristic $p \gg 0$ satisfies conditions (i)-(iii). Moreover, the quantity σ is constant for $p \gg 0$. Assuming the reduction notation of Assumption 4.1, we have

$$c(J) = \lim_{\substack{\mu \in \operatorname{Spec} A \\ \operatorname{char} A/\mu \to \infty}} c(J_\mu) \geq (1-\sigma)\left(\lim_{\substack{\mu \in \operatorname{Spec} A \\ \operatorname{char} A/\mu \to \infty}} c(J_{m,\mu}) - \frac{a_m}{d_m}\right) + E_n(J)$$
$$= (1-\sigma)\left(c(J_m) - \frac{a_m}{d_m}\right) + E_n(J) > E_n(J).$$

□

Lemmas 4.15, 4.27 and 4.28 combine to give us a proof of Theorem 5.11 in the case of an $\mathfrak{m}$-primary complete intersection.

Proposition 4.29. *Assume the setup of Assumption 4.7 and suppose $c(I) = E_n(I)$. Then there exists $\gamma \in \mathrm{GL}_n(k)$ such that $\overline{\gamma I} = \mathfrak{D}$.*

Proof. Using Lemma 4.15, we produce $\gamma \in \mathrm{GL}_n(k)$ such that γI satisfies Equation (17). By Lemmas 4.27 and 4.28, we have $\overline{\gamma I} = \mathfrak{D}$. □

5. Proof of Theorem B

In this section, fix the following setup.

Assumption 5.1. *Let L be an uncountable, algebraically closed field. Let $S = L[x_1, \dots, x_n]$ and let $I \subseteq S$ be a homogeneous. As in Lemma 3.1, for $1 \leq j \leq n$, let $S_j := L[x_1, \dots, x_j], \pi_j : S \to S_j$ the natural projection map and $\iota_j : S_j \to S$ the natural embedding. Let $>$ denote the reverse lexicographic order refining the partial order by degree.*

Definition 5.2. Let k be a field, $R = k[x_1, \dots, x_n]$, $I \subseteq R$ a homogeneous ideal, and $t \in \mathbb{Z}^+$. We let $[I]_t$ denote the vector space of t-forms in I and we let $[I]_{\leq t}$ denote the direct sum of the $[I]_s$ for $s \leq t$.

Definition 5.3. Let k be a field, $R = k[x_1, \dots, x_n]$, and $I \subseteq R$ a homogeneous ideal of height $l > 0$. For $1 \leq i \leq l$, let $d_i(I) = \min\{j : \operatorname{ht}[I]_{\leq j} R \geq i\}$.

Lemma 5.4. *Assume the setting of Assumption 5.1. Let $\gamma \in \mathrm{GL}_n(L)$ be very general: for now, we impose the condition that for all $m > 0$, we have $\mathrm{in}_>(\gamma I^m) = \mathrm{gin}_>(I^m)$; we will impose countably many additional conditions in Lemma 5.9. For $1 \leq j \leq n, m > 0$, set $\mathfrak{a}_{j,m} := \mathrm{in}_>(\pi_j(\gamma I^m)) \subseteq S_j$. For $j > 0, 1 \leq i \leq j$, let $\vec{b}_i^{(j)}$ denote the ith unit vector of $\mathbb{R}^j$. For $j \geq i$, let $p_j(i) := \inf\{t : t\vec{b}_i^{(j)} \in \Gamma(\mathfrak{a}_{j,\bullet})\}$. Then for all j, we have $p_j(j) = p_n(j)$.*

Proof. By Lemma 3.1 we have $\iota_j(\mathfrak{a}_{j,\bullet}) \subseteq \mathfrak{a}_{n,\bullet}$ for all $1 \leq j \leq n$, so $p_n(j) \leq p_j(j)$. For the reverse direction, set $t = p_n(j)$. Since $t\vec{b}_j^{(n)} \in \overline{\bigcup_{m>0} \frac{1}{2^m}\Gamma(\mathfrak{a}_{n,2^m})}$, there exists a sequence $\{\mathbf{a}_m = (a_{m,1}, \dots, a_{m,n})\}_{m>0}$ such that $\mathbf{a}_m \in \Gamma(\mathfrak{a}_{n,2^m})$ for all m and $\lim_{m\to\infty} 2^{-m}\mathbf{a}_m = t\vec{b}_j^{(n)}$. For any choice of $\{(a_{m,1}, \dots, a_{m,n})\}_{m>0}$, we also have $(\lceil a_{m,1}\rceil, \dots, \lceil a_{m,n}\rceil) \in \Gamma(\mathfrak{a}_{n,2^m})$ and $\lim_{m\to\infty} \frac{(\lceil a_{m,1}\rceil, \dots, \lceil a_{m,n}\rceil)}{2^m} =$

$tb_{j,n}$. We may therefore assume without loss of generality that $a_m \in \mathbb{Z}^n_{\geq 0}$ for all $m > 0, 1 \leq i \leq n$, hence for all $m > 0$, we have $x^{a_m} \in \overline{\mathfrak{a}_{n,2^m}}$.

By [20, Theorem 2.1], $\overline{\mathfrak{a}_{n,2^m}}$ is Borel-fixed, so

$$x_1^{a_{m,1}} \cdots x_{j-1}^{a_{m,j-1}} x_j^{a_{m,j}+\cdots+a_{m,n}} \in \overline{\mathfrak{a}_{n,2^m}}.$$

Further note that $\mathfrak{a}_{j,m} = \pi_j(\mathfrak{a}_{n,m})$ by Lemma 3.1. By Proposition 2.22(iii), we conclude

$$\pi_j(x_1^{a_{m,1}} \cdots x_{j-1}^{a_{m,j-1}} x_j^{a_{m,j}+\cdots+a_{m,n}}) \in \pi_j(\overline{\mathfrak{a}_{n,2^m}}) \subseteq \overline{\pi_j(\mathfrak{a}_{n,2^m})} = \overline{\mathfrak{a}_{j,2^m}}. \tag{38}$$

It follows that

$$tb_{j,j} = \lim_{m\to\infty} \frac{(a_{m,1}, \ldots, a_{j-1}, a_j + \cdots + a_n)}{m} \in \Gamma(\mathfrak{a}_{j,\bullet}),$$

which proves $p_j(j) \leq t = p_n(j)$. □

Remark 5.5. Assume the setup of Lemma 5.4. A priori, there exists a sequence

$$\{\mathbf{a}_m = (a_{m,1}, \ldots, a_{m,j})\}_{m>0} \subseteq \mathbb{R}^j_{\geq 0}$$

such that for all $m > 0$ we have $\lim_{m\to\infty} 2^{-m}\mathbf{a}_m = t\vec{b}_j^{(j)}$ and $\mathbf{a}_m \in \Gamma(\mathfrak{a}_{2^m})$. Lemma 5.6 is a minor refinement, allowing us to assume that $\mathbf{x}^{\mathbf{a}_m} \in \mathfrak{a}_{2^m}$.

Lemma 5.6. *Assume the setup of Lemma 5.4. There exists a sequence* $\{\mathbf{a}'_m\}_{m>0} \subseteq \mathbb{Z}^j$ *such that* $\mathbf{x}^{\mathbf{a}'_m} \in \mathfrak{a}_{j,2^m}$ *for all* $m > 0$ *and* $\lim_{m\to\infty} 2^{-m}\mathbf{a}'_m = p_j(j)b_{j,j}$.

Proof. Let $\{\mathbf{a}_m\}_m$ be as in Remark 5.5. First, we recall that $\Gamma(\mathfrak{a}_{2^m}) = \operatorname{conv}(\log(\mathbf{x}^{\mathbf{u}}) : \mathbf{x}^{\mathbf{u}} \in \mathfrak{a}_{2^m})$. We may choose $\{\mathbf{u}_m^{(i)} = (u_{m,1}^{(i)}, \ldots, u_{m,j}^{(i)})\}_{i=0}^j$ such that $\mathbf{x}^{\mathbf{u}_m^{(i)}} \in \mathfrak{a}_{2^m}$ and $\mathbf{a}_m \in \operatorname{conv}(\mathbf{u}_m^{(0)}, \ldots, \mathbf{u}_m^{(j)})$. Reorder the $\mathbf{u}_m^{(i)}$ so that $u_{m,j}^{(0)} \leq \cdots \leq u_{m,j}^{(j)}$. Since $a_{m,j}$ is the average of the $u_{m,j}^{(i)}$, we have $u_{m,j}^{(0)} \leq a_{m,j}$. For $i < j$, we similarly have $\frac{u_{m,i}^{(0)}}{j+1} \leq \frac{u_{m,i}^{(0)}+\cdots+u_{m,i}^{(j)}}{j+1} = a_{m,i}$, so $u_{m,i}^{(0)} \leq (j+1)a_{m,i}$.

For all $m > 0$, set $\mathbf{a}'_m = \mathbf{u}_m^{(0)}$. Then we have $\lim_{m\to\infty} a'_{m,i} \leq (j+1)\lim_{m\to\infty} a_{m,i} = 0$ for all $i < j$, and

$$p_j(j) \leq \liminf_{m\to\infty} 2^{-m}a'_{m,j} \leq \lim_{m\to\infty} 2^{-m}a_{m,j} = p_j(j).$$

It follows that $\lim_{m\to\infty} 2^{-m}\mathbf{a}'_m = p_j(j)b_{j,j}$ and for all $m > 0, \mathbf{x}^{\mathbf{a}'_m} \in \mathfrak{a}_{2^m}$. □

Lemma 5.7. *Let k be an algebraically closed field and $R = k[x_1, \ldots, x_j]$. Let $\mathfrak{q}$ be a homogeneous prime ideal of height $j-1$ with $x_j \notin \mathfrak{q}$. If $>$ denotes the reverse lexicographic order, then for all $m > 0$ we have* $\operatorname{in}_>(\mathfrak{q}^m) = (x_1, \ldots, x_{j-1})^m$.

Proof. Since k is algebraically closed, there exist linear forms $\ell_1, \ldots, \ell_{j-1} \in R_1$ such that $q = (\ell_1, \ldots, \ell_{j-1})$. By [11, Theorem 15.17], $\mathfrak{q}$ and $\operatorname{in}_>(\mathfrak{q})$ have the same Hilbert series. Moreover, $\operatorname{in}_>(\mathfrak{q})$ is a monomial ideal not containing

x_j, so we must have $\mathrm{in}_>(\mathfrak{q}) = (x_1, \ldots, x_{j-1})$. For $m > 1$, a similar analysis applies. We have the standard containment $(x_1, \ldots, x_{j-1})^m = \mathrm{in}_>(\mathfrak{q})^m \subseteq \mathrm{in}_>(\mathfrak{q}^m)$. As $(x_1, \ldots, x_{j-1})^m$ has the same Hilbert series as $\mathfrak{q}^m$, the result follows. □

Lemma 5.8. *Let k be a field, $R = k[x_1, \ldots, x_j]$, and $\mathfrak{n} = (x_1, \ldots, x_j)$. If $I \subseteq R$ is a homogeneous ideal, then*

$$[I\mathfrak{n}^t]_u = [[I]_{u-t}R]_u.$$

Proof. First, we verify the following: if f is homogeneous of degree $u - t$, then $[f\mathfrak{n}^t]_u = [fR]_u$. The containment $\subseteq$ is clear. For $\supseteq$, if g is homogeneous and $gf \in [fR]_u$, then $g \in \mathfrak{n}^t$, so $gf \in [f\mathfrak{n}^t]_u$.

Now let $f_1, \ldots, f_r$ be homogeneous elements spanning $[I]_{u-t}$, and let $g_1, \ldots, g_s$ be homogeneous elements such that $f_1, \ldots, f_r, g_1, \ldots, g_s$ generate I and $\deg(g_i) > u - t$. By the first paragraph, we compute

$$[I\mathfrak{n}^t]_u = [(f_1, \ldots, f_r)\mathfrak{n}^t]_u + [(g_1, \ldots, g_s)\mathfrak{n}^t]_u = \sum_{i=1}^{r} [f_i\mathfrak{n}^t]_u = \sum_{i=1}^{r} [f_i]_u = [[I_{u-t}]R]_u.$$

□

Lemma 5.9. *Assume the setup of Lemmas 5.4 and 5.6. Then for all $1 \leq j \leq n$, we have $p_n(j) = d_j(I)$.*

Proof. Set $d_j := d_j(I)$. Before we begin the proof, we first state the additional generality conditions on γ. For all $m > 0$, assume that $\mathrm{in}_>(\pi_j(\gamma I^m)) = \mathrm{gin}_>(\pi_j(\gamma I^m))$. Since $\mathrm{ht}\, \pi_j(\gamma[I]_{\leq d_j-1}R) \leq \mathrm{ht}\, \gamma\left[I]_{\leq d_j-1}R\right) < j$, there is some $1 \leq i \leq j$ such that $x_i \notin \sqrt{\pi_j(\gamma[I]_{\leq d_j-1})R}$; we choose γ such that $x_j \notin \sqrt{\pi_j(\gamma[I]_{\leq d_j-1}R)}$. Each of these conditions is satisfied by a general choice of γ, so they may be realized simultaneously by a very general choice of γ.

Set $J = \pi_j(\gamma I)$. Write $J_1 = [J]_{\leq d_j-1}R$, and let J_2 be the ideal generated by the minimal homogeneous generators of J with degree at least d_j so that $J = J_1 + J_2$ and $J_2 \subseteq \mathfrak{m}^{d_j}$. By construction of γ, in the language of Lemma 5.4 we have $\mathfrak{a}_{j,m} = \mathrm{in}_>(J^m)$. Also by construction, we have $x_j \notin \sqrt{[J]_{\leq d_j-1}}$, so we may choose a minimal prime $\mathfrak{p}$ over J_1 such that $x_j \notin \mathfrak{p}$. As $\mathrm{ht}\, \mathfrak{p} \leq j - 1$, we may choose a homogeneous prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$ such that $\mathrm{ht}\, \mathfrak{q} = j - 1$ and $x_j \notin \mathfrak{q}$. By Lemma 5.7, we have $\mathrm{in}_>(\mathfrak{q}^m) = (x_1, \ldots, x_{j-1})^m$ for all $m > 0$.

By Lemma 5.6, choose a sequence $\{\mathbf{a}_m\}_{m>0}$ such that $\mathbf{x}^{\mathbf{a}_m} \in \mathfrak{a}_{2^m}$ for all $m > 0$ and $\lim_{m\to\infty} 2^{-m}\mathbf{a}_m = p_j(j)\vec{b}_j^{(j)}$. Let $e_m := a_{m,1} + \cdots + a_{m,j}$. For all $m > 0$, we have

$$[J^{2^m}]_{e_m} = \left[\sum_{\substack{a+b=2^m \\ a+bd_j \le e_m}} J_1 + J_2 \right]_{e_m} \tag{39}$$

We have shown that $J_1 \subseteq \mathfrak{q}$ and $J_2 \subseteq \mathfrak{m}^{d_j}$. By Equation (39) and Lemma 5.8, we have

$$\begin{aligned} \mathbf{x}^{\mathbf{a}_m} \in [J^{2^m}]_{e_m} \subseteq \left[\sum_{\substack{a+b=2^m \\ a+bd_j \le e_m}} \mathfrak{q}^a \mathfrak{m}^{bd_j} \right]_{e_m} &= \sum_{\substack{a+b=2^m \\ a+bd_j \le e_m}} \left[\mathfrak{q}^a \mathfrak{m}^{bd_j} \right]_{e_m} \\ &= \sum_{\substack{a+b=2^m \\ a+bd_j \le e_m}} = \left[\mathfrak{q}^{2^m - \left\lfloor \frac{e_m}{d_j} \right\rfloor} \right]_{e_m}. \end{aligned}$$

Taking initial terms of both sides, we have $\mathbf{x}^{\mathbf{a}_m} \in (x_1, \dots, x_{j-1})^{2^m - \left\lfloor \frac{e_m}{d_j} \right\rfloor}$. Consequently, we have $a_{m,1} + \cdots + a_{m,j-1} \ge 2^m - \left\lfloor \frac{e_m}{d_j} \right\rfloor$. As $\lim_{m\to\infty} 2^{-m}(a_{m,1} + \cdots + a_{m,j-1}) = 0$, this yields

$$0 \ge \liminf_{m\to\infty} 2^{-m} \left(2^m - \left\lfloor \frac{e_m}{d_j} \right\rfloor \right) = 1 - \frac{1}{d_j} \limsup_{m\to\infty} \frac{a_{m,j}}{2^m} = 1 - \frac{p_j(j)}{d_j}. \tag{40}$$

From the above equation, we have $p_j(j) \ge d_j$. For the reverse containment, we have by Lemma 3.3 that $\mathfrak{m}^{d_j} \subseteq \overline{J}$. It follows from Theorem 2.25 that $x_j^{(m+j-1)d_j} \in J^m$ for all $m > 0$, hence $p_j(j) \le d_j$. $\square$

Lemma 5.10. *Let $R = k[x_1, \dots, x_n]$ and let $\mathfrak{m} = (x_1, \dots, x_n)$. Let $I \subseteq R$ be a homogeneous ideal and $J \subseteq \mathfrak{m}$ any ideal. Then we have*

$$\bigcap_{m>0} \overline{I + J^m} = \overline{I}.$$

Proof. By [25, Corollary 6.8.5], we have

$$\overline{IR_\mathfrak{m}} \subseteq \bigcap_{m>0} \overline{IR_\mathfrak{m} + J^m R_\mathfrak{m}} \subseteq \bigcap_{m>0} \overline{IR_\mathfrak{m} + \mathfrak{m}^m R_\mathfrak{m}} = \overline{IR_\mathfrak{m}}.$$

By Proposition 2.22 (v), we have $\overline{I}R_\mathfrak{m} = \overline{IR_\mathfrak{m}}$. By [25, Corollary 5.2.3], $\overline{I}$ is homogeneous, hence $\overline{I}R_\mathfrak{m} \cap R = \overline{I}$. We have the following, from which the claim follows.

$$\overline{I} \subseteq \bigcap_{m>0} \left(\overline{IR_\mathfrak{m} + J^m R_\mathfrak{m}} \cap R \right) \subseteq \overline{IR_\mathfrak{m}} \cap R = \overline{I}.$$

$\square$

We are now able to prove Theorem 5.11.

Theorem 5.11. *Let k be an algebraically-closed field. Let $R = k[x_1, \ldots, x_n]$ and let $I \subseteq R$ be a homogeneous ideal with $\operatorname{ht}(I) \geq l$. If $E_l(I) = c(I)$, then there exist integers $d_1, \ldots, d_l$ and $\gamma \in \mathrm{GL}_n(k)$ such that*

$$\gamma \overline{I} = \overline{\left(x_1^{d_1}, \ldots, x_l^{d_l}\right)}.$$

Proof. If $\operatorname{ht}(I) \geq l+1$, then $c(I) \geq E_{l+1}(I) > E_l(I)$, so we must have $\operatorname{ht}(I) = l$. Let $\ell_{l+1}, \ldots, \ell_n$ be general linear forms and set $L = V(\ell_{l+1}, \ldots, \ell_n)$. By Proposition 2.39, we have $E_l(I) = E_l(I|_L) \leq c(I|_L) \leq c(I)$, so we again have $E_l(I) = c(I|_L)$. We'll compute $d_1(I|_L), \ldots, d_l(I|_L)$. Field extensions are faithfully flat, so $\operatorname{ht}([I]_{\leq t}|_L)$ is preserved under field extension, hence so are the quantities $d_1(I|_L), \ldots, d_l(I|_L)$. We may without loss of generality assume k is uncountable and algebraically-closed.

Let $\gamma \in \mathrm{GL}_n(k)$ be as in Lemmas 5.4 and 5.9 with respect to the ideal $I|_L$, and set $\mathfrak{a}_m = \operatorname{in}_>(\gamma(I|_L)^m) = \operatorname{gin}_>((I|_L)^m)$. As $c(I|_L) = E_l(I)$, by Corollary 3.10 we have

$$\overline{\mathbb{R}^l_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)} = \operatorname{conv}\left(\vec{0}, e_1(I)\vec{b_1}^{(l)}, \frac{e_2(I)}{e_1(I|_L)}\vec{b_2}^{(l)}, \ldots, \frac{e_{l-1}(I)}{e_l(I)}\vec{b_l}^{(l)}\right). \tag{41}$$

The quantities $p_n(1), \ldots, p_l(l)$ can be read off from Equation (41), so we have $\frac{e_j(I|_L)}{e_{j-1}(I|_L)} = d_j(I|_L)$ for all $1 \leq i \leq l$.

For $1 \leq j \leq l$, let f_i be an element of $[I]_{d_j}$ whose image mod $(\ell_{l+1}, \ldots, \ell_n)$ is general. As $\operatorname{ht}[I|_L]_{d_j} \geq j$, it follows that $f_1, \ldots, f_l, \ell_{l+1}, \ldots, \ell_n$ is a regular sequence. Let $J = (f_1, \ldots, f_l)$. By Lemma 3.8 we have $E_l(J) = E_l(I)$. For $t > 0$, set $J_t = J + (\ell_{l+1}^t, \ldots, \ell_n^t)$ and set $I_t = I + (\ell_{l+1}^t, \ldots, \ell_n^t)$. Since $J_t \subseteq I_t$ is a complete intersection, Lemma 3.8 and Propositions 2.9 and 2.36 give

$$E_l(J) + \frac{n-l}{t} = E_l(J_t) \leq E_l(I_t). \tag{42}$$

Combining Equation (42) with Theorem 3.9 and Proposition 2.9 (v) gives

$$E_l(J) + \frac{n-l}{t} \leq E_l(I_t) \leq c(I_t) \leq c(I) + c((\ell_{l+1}^t, \ldots, \ell_n^t)) = E_l(I) + \frac{n-l}{t}.$$

In particular, we have $E_l(I_t) = E_l(J_t) = c(J_t)$. Set

$$\mathfrak{D}_t := \overline{(x_1^{d_1(I)}, \ldots, x_l^{d_l(I)}, x_{l+1}^t, \ldots, x_l^t)}.$$

By Proposition 4.29, there exists $\gamma \in \mathrm{GL}_n(k)$ such that $\gamma_t \overline{J_t} = \mathfrak{D}_t$. Consequently, as $\mathfrak{D}_t \subseteq \gamma_t \overline{I_t}$ and $E_l(I_t) = E_l(\mathfrak{D}_t)$, by Proposition 2.36 we have $\overline{\mathfrak{D}_t} = \overline{I_t}$.

Set $\mathfrak{D} = \overline{(x_1^{d_1(I)}, \ldots, x_l^{d_l(I)})}$. Let d be the maximum degree of an irredundant generator of $\mathfrak{D}$ or $\overline{I}$. The k-vector space of forms of degree $\leq d$ in R is finite-dimensional, so the infinite descending chain $[\mathfrak{D}_1]_{\leq d} \supseteq [\mathfrak{D}_2]_{\leq d} \supseteq \ldots$, the limit of which is $[\mathfrak{D}]_{\leq d}$ by Lemma 5.10, must stabilize, so there exists $t_0 \gg 0$ such that $[\mathfrak{D}_t]_{\leq d} = [\mathfrak{D}]_{\leq d}$ for all $t \geq t_0$. Similarly, there exists $t_1 \gg 0$

such that $[\overline{I_t}]_{\leq d} = [\overline{I}]_{\leq d}$ for all $t \geq t_1$. For $t = \max(t_1, t_2)$, we have

$$\gamma_t[\overline{I}]_{\leq d} = \gamma_t[\overline{I_t}]_{\leq d} = [\mathfrak{D}_t]_{\leq d} = [\mathfrak{D}]_{\leq d}.$$

As $\gamma_t\overline{I}, \mathfrak{D}$ are generated in degree $\leq d$, it follows that $\gamma_t\overline{I} = \mathfrak{D}$. □

6. Future Work

The first natural question to ask is whether One can ask whether Theorem 5.11 can be generalized to the local case. We note that the answer is "no" in positive characteristic, and we pose a a conjecture generalizing Theorem 5.11 over the complex numbers.

Example 6.1. Let $R = \overline{\mathbb{F}_p}[\![x, y]\!]$, and let $I = (x^p + y^{p+1}) \subseteq R$. Then $E_1(I) = \frac{1}{p} = \mathrm{fpt}(I)$, but $I = \overline{I}$ and there are no coordinates for R in which I is a monomial ideal.

For a complete description of when $E_1(I) = c(I)$ in positive characteristic, see [1]. Whereas Theorem 5.11 fails to generalize to the local case in char $p > 0$, we are optimistic that a stronger result is possible in characteristic zero.

Conjecture 6.2. *Let $R = \mathbb{C}[\![x_1, \ldots, x_n]\!]$. Let $I \subseteq R$ be an ideal with $\mathrm{ht}(I) \geq l$. Then $\mathrm{lct}(I) = E_l(I)$ if and only if there exists a regular system of parameters $y_1, \ldots, y_n$ for R and positive integers $d_1, \ldots, d_l$ such that $\overline{I} = \overline{(y_1^{d_1}, \ldots, y_l^{d_l})}$.*

Compare Conjecture 6.2 with the discussion in [5, p. 1915]. Bivià-Ausina asks whether one can expect the statement of Conjecture 6.2 to hold with the additional assumption that $x_i \mapsto y_i$ is a *linear* change of coordinates. This does not hold in general.

Example 6.3. Let $R = \mathbb{C}[\![x, y]\!]$ and $I = (x+y^2, y^3)$. Then $c(I) = \frac{4}{3} = e_2(I)$. On the other hand, $I = \overline{I}$ does not contain any homogeneous linear form, so for any $\gamma \in \mathrm{GL}_n(\mathbb{C})$ we have $x \notin \gamma I$, so $\gamma\overline{I} \neq \overline{(x, y^3)}$.

We note that Conjecture 6.2 is strong enough to give an alternate proof of Theorem 5.11 over $\mathbb{C}$.

Proof. Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Let I be a homogeneous ideal with $\mathrm{ht}(I) \geq l$ and $E_l(I) = \mathrm{lct}(I)$. Let $S = \mathbb{C}[\![x_1, \ldots, x_n]\!]$. Notation 2.8 and Definition 2.37 imply that $E_l(IS) = E_l(I)$ and $\mathrm{lct}(IS) = \mathrm{lct}(I)$, so by Conjecture 6.2 there exists a regular system of parameters $y_1, \ldots, y_n$ for S such that $\overline{IS} = \overline{(y_1^{d_1}, \ldots, y_l^{d_l})}$.

Write each y_1 as a power series in $x_1, \ldots, x_n$ and let $z_1, \ldots, z_n$ be the homogeneous degree-1 terms of $y_1, \ldots, y_n$. As $y_i \equiv z_i \mod \mathfrak{m}^2$, the forms $z_1, \ldots, z_n$ in R generate the homogeneous maximal ideal of R.

As I is homogeneous, so is $\overline{I}$, hence $\overline{I} = \overline{IS} \cap R$. In particular, we have $y_1^{d_1}, \ldots, y_l^{d_l} \in \overline{I}$. By considering the ideals $\overline{I + (y_{l+1}^t, \ldots, y_n^t)}$ for $t > 0$

(which evidently satisfy $E_n = \mathrm{lct}$) and running an argument similar to Theorem 5.11, we deduce that $\overline{I} = \overline{(y_1^{d_1}, \dots, y_l^{d_l})}$ □

We propose a stronger conjecture in terms of valuations. Let $R = \mathbb{C}[\![x_1, \dots, x_n]\!]$. Let $v : R \to [0, \infty]$ be a valuation and let $A_R(v)$ be the log discrepancy as in [27]. For each positive integer n, define $\mathfrak{a}_n(v) = \{f \in R : v(f) \geq n$. The *height* of v is the number $\max_n \mathrm{ht}(\mathfrak{a}_n(v))$. By [27, Corollary 6.9] and [6, Lemma 3.5], we have

$$\mathrm{lct}(\mathfrak{a}_\bullet(v)) \leq A_R(v) \tag{43}$$

Conjecture 6.4. *Let $R, v, \mathfrak{a}_n(v)$ be as above and assume v has height at least l. If $E_l(\mathfrak{a}_\bullet(v)) = A_R(v)$ then v is a monomial valuation.*

We conclude this article by demonstrating that Conjecture 6.4 implies Conjecture 6.2. First, a few lemmas.

Lemma 6.5. *Let $(R, \mathfrak{m})$ be a regular local ring. Let $I \subseteq R$ be an ideal of height at least l and such that $l < \dim(R)$. If $z \in \mathfrak{m}$ such that the image of z in $\mathfrak{m}/\mathfrak{m}^2$ is general, then $\mathrm{ht}(I + z^t) \geq l + 1$ and $\sigma_{l+1}(I + z^t) \leq t\sigma_l(I)$ for all $t > 0$. If the analytic spread of I satisfies $\ell(I) = l$, then $\sigma_{l+1}(I + z^t) = t\sigma_l(I)$ for all $t > 0$ and $\ell(I + z^t) = l + 1$.*

Proof. By replacing R with $R[\![X]\!]_{\mathfrak{m}R[\![X]\!]}$ along the lines of [25, Lemma 8.4.2], we may without loss of generality assume that $R/\mathfrak{m}$ is uncountably infinite.

For $r > 0$, set

$$V_r := \underbrace{\frac{I + \mathfrak{m}^r}{\mathfrak{m}(I + \mathfrak{m}^r)} \oplus \cdots \oplus \frac{I + \mathfrak{m}^r}{\mathfrak{m}(I + \mathfrak{m}^r)}}_{l} \oplus \underbrace{\frac{\mathfrak{m}}{\mathfrak{m}^2} \oplus \cdots \oplus \frac{\mathfrak{m}}{\mathfrak{m}^2}}_{n-l}.$$

Choose $r > 0$ such that $\sigma_l(I) = e_l(I + \mathfrak{m}^r)$. By [25, Proposition 17.5.1], there exists a nonempty Zariski-open set $U_r \subseteq V_r$ such that for all $g_1, \dots, g_l \in I + \mathfrak{m}^r, z_{l+1}, \dots, z_n \in \mathfrak{m}$ whose projection to V_r is contained in U_r, we have for all $t > 0$

$$e(\underbrace{I + \mathfrak{m}^r, \dots, I + \mathfrak{m}^r}_{l}, \underbrace{\mathfrak{m}, \dots, \mathfrak{m}}_{n-l-1}, \mathfrak{m}^t) = e(g_1, \dots, g_l, z_{l+1}, \dots, z_{l+1}, \dots, z_{n-1}, z_n^t).$$

Choose a *fixed* $g_1, \dots, g_l \in I, h_1, \dots, h_l \in \mathfrak{m}^r, z \in \mathfrak{m}$ for which the set of tuples $(z_{l+1}, \dots, z_{n-1})$ such that

$$(\overline{g}_1, \dots, \overline{g}_l, \overline{z}_{l+1}, \dots, \overline{z}_{n-1}, \overline{z}) \in U_1 \qquad (\overline{g}_1 + \overline{h}_1, \dots, \overline{g}_l + \overline{h}_l, \overline{z}_{l+1}, \dots, \overline{z}_{n-1}, \overline{z}) \in U_r$$

is a nonempty open set in $\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right)^{\oplus l-1}$.

If $\mathrm{ht}(I) < \dim R$, we can additionally choose z to avoid the minimal primes over I, which guarantees that $\mathrm{ht}(I + z^t) = \min(\mathrm{ht}(I) + 1, \dim(R)) \geq l + 1$. By Proposition 2.34 we may choose $g_1, \dots, g_l \in I + \mathfrak{m}^t, z_{l+1}, \dots, z_{n-1} \in \mathfrak{m}$

very generally such that for all $t > 0$, with $J = (z_{l+1}, \dots, z_{n-1})$ we have $\sigma_{l+1}(I + z^t) = e\left(\frac{I+z^t+J}{J}\right)$. Then

$$\sigma_{l+1}(I + z^t) = e\left(\frac{I + z^t + J}{J}\right) \le e\left(\frac{(g_1 + h_1, \dots, g_l + h_l, z^t) + J}{J}\right) \tag{44}$$

$$= e(g_1 + h_1, \dots, g_l + h_l, z_{l+1}, \dots, z_{n-1}, z^t) = te_{l+1}(I + \mathfrak{m}^r) = t\sigma_{l+1}(I). \tag{45}$$

Let's now additionally assume that $\ell(I) = l$ and, by [25, Theorem 8.6.6] let $I' = (g_1, \dots, g_l) \subseteq I$ generate an ideal I' such that $\overline{I'} = \overline{I}$. Then for any z^t we have $\overline{I' + z^t} = \overline{I + z^t}$, so we may without loss of generality suppose $I = I'$. By Proposition 2.34, choose $z_{l+1}, \dots, z_{n-1}, z$ such that, with $J = (z_{l+1}, \dots, z_{n-1})$ we have $\sigma_l(I) = e(g_1, \dots, g_l, z_{l+1}, \dots, z_{n-1}, z)$. As in the first case, we may actually choose $z_{l+1}, \dots, z_{n-1}$ very generally such that $\sigma_{l+1}(I + z^t) = e(\frac{I+z^t+J}{J})$ for all $t > 0$.

By [25, Proposition 11.2.9], we have

$$t\sigma_l(I) = te(g_1, \dots, g_l, z_{l+1}, \dots, z_{n-1}, z) \tag{46}$$

$$= e(g_1, \dots, g_l, z^t, z_{l+1}, \dots, z_{n-1}) = \sigma_{l+1}(I). \tag{47}$$

Moreover, by [25, Proposition 8.3.7] we have

$$\ell(I + z^t) = \ell(\overline{I + z^t}) \le l + 1 \le \operatorname{ht}(I + z^t) \le \ell(I + z^t).$$

□

Lemma 6.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I, J \subseteq \mathfrak{m}$ proper ideals. Then $\bigcap_{t>0} \overline{I + J^t} = \overline{I}$.*

Proof. The case $J = \mathfrak{m}$ is [25, Corollary 6.8.5]. More generally, we have

$$\overline{I} \subseteq \bigcap_{t>0} \overline{I + J^t} \subseteq \bigcap_{t>0} \overline{I + \mathfrak{m}^t} = \overline{I}.$$

□

Lemma 6.7. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n. Let $1 \le l \le n$ and let $I \subseteq J$ be ideals of height l. If $E_l(I) = E_l(J)$ and $\ell(J) = l$, then $\overline{I} = \overline{J}$.*

Proof. We prove the claim by induction on $\dim R - l$; if $l = \dim R$ then the result is Proposition 2.36. If $l = 0$ then $I = J = (0)$, so we assume $1 \le l < \dim R$.

As $\operatorname{ht}(I) = \operatorname{ht}(J) = l$, by Definition 2.33 there exists $t_0 > 0$ such that $\sigma_j(I) = e_j(I + \mathfrak{m}^t)$ for all $1 \le j \le l$ and all $t \ge t_0$. Using Lemma 6.5, we may choose z very generally such that for all $t \ge t_0$ we have $\ell(J + z^t) = l + 1$ and

$$\sigma_{l+1}(I + z^t) \le t\sigma_l(I) \qquad \sigma_{l+1}(J + z^t) = t\sigma_{l+1}(J).$$

By [5, Proposition 10], the fact that $E_l(I) = E_l(J)$ implies $\sigma_j(I) = \sigma_j(J)$ for $j = 1, \dots, l$. As $I + z^t \subseteq J + z^t$, we have

$$\sigma_{l+1}(I + z^t) \le t\sigma_l(I) = t\sigma_l(J) = \sigma_{l+1}(J + z^t) \le \sigma_{l+1}(I), \tag{48}$$

hence $\sigma_{l+1}(I+z^t) = \sigma_{l+1}(J+z^t)$. Moreover, by the fact that $t \geq t_0$, for $K \in \{I, J\}$ and all $1 \leq j \leq l$ we have

$$\sigma_j(K) \geq \sigma_j(K+z^t) \geq e_j(K+z^t+\mathfrak{m}^{t_0}) = e_j(K+\mathfrak{m}^{t_0}) = \sigma_j(K), \tag{49}$$

so $\sigma_j(I+z^t) = \sigma_j(I) = \sigma_j(J) = \sigma_j(I+z^t)$. By Equations (48) and (49), we conclude that $E_{l+1}(I+z^t) = E_{l+1}(J+z^t)$. By induction, we deduce that $\overline{I+z^t} = \overline{J+z^t}$. By Lemma 6.6 we deduce

$$\overline{I} = \bigcap_{t>0} \overline{I+z^t} = \bigcap_{t>0} \overline{J+z^t} = \overline{J}.$$

□

We now prove the implication.

Proof of Conjecture 6.2 assuming Conjecture 6.4. Let $I \subseteq R$ be an ideal of height at least l such that $E_l(I) = \operatorname{lct}(I)$. By Definition 2.2, choose a divisorial valuation $v : R \to [0, \infty]$ such that $\operatorname{lct}(I) = \frac{A_R(I)}{v(I)}$. Letting $w = \frac{v}{v(I)}$, we have $w(I) = 1$ and $A_R(w) = \operatorname{lct}(I)$. As $w(I) = 1$, we have $I^n \subseteq \mathfrak{a}_n(w)$ for all $n > 0$, so $\operatorname{ht}(w) \geq l$ and $E_l(\mathfrak{a}_\bullet(w)) \geq E_l(I)$.

It follows from Theorem 3.9 and eq. (43) that

$$\operatorname{lct}(I) = E_l(I) \leq E_l(\mathfrak{a}_\bullet(w)) \leq \operatorname{lct}(\mathfrak{a}_\bullet(w)) \leq A_R(w) = \operatorname{lct}(I),$$

so in particular $E_l(\mathfrak{a}_\bullet(w)) = A_R(w)$. By Conjecture 6.4, w is a monomial valuation in some local coordinates $z_1, \ldots, z_n$ for R with $w(z_1) \geq w(z_2) \geq \cdots \geq w(z_n)$. Since $\sigma_j(I) \leq \sigma_j(\mathfrak{a}_\bullet(w))$ for $1 \leq j \leq l$ and $E_l(I) = E_l(\mathfrak{a}_\bullet(w))$, we conclude $\sigma_j(I) = \sigma_j(\mathfrak{a}_\bullet(w))$ by [5, Proposition 10]. We can read off the invariants $\sigma_1(\mathfrak{a}_\bullet(w)), \ldots, \sigma_l(\mathfrak{a}_\bullet(w))$ from the weights $w(z_i)$. In particular, we have $\sigma_j(\mathfrak{a}_\bullet(w)) = \frac{1}{w(z_1)\ldots w(z_j)}$, hence $w(z_j) = \frac{\sigma_{j-1}(I)}{\sigma_j(I)}$ for all $1 \leq j \leq l$. Moreover, by definition of the log discrepancy for quasi-monomial valuations we have $A_R(w) = w(z_1) + \cdots + w(z_n)$, so we must have $w(z_{l+1}) = \cdots = w(z_n) = 0$.

By construction, $I \subseteq \mathfrak{a}_1(w)$. If the numbers $\frac{1}{w(z_1)}, \ldots, \frac{1}{w(z_l)}$ are not all *integers*, then $\Gamma(\mathfrak{a}_1(w)) \subsetneq \Gamma(\mathfrak{a}_\bullet(w))$ – but this would imply $\operatorname{lct}(I) \leq \operatorname{lct}(\mathfrak{a}_1(w)) < \operatorname{lct}(\mathfrak{a}_\bullet(w))$, which is impossible. Write $d_i = \frac{1}{w(z_i)}$ for $1 \leq i \leq l$. We are now in the situation where $I \subseteq \overline{(z_1^{d_1}, \ldots, z_l^{d_l})} =: \mathfrak{D}$ and $E_l(I) = E_l(\mathfrak{D})$. Since $\mathfrak{D}$ is the integral closure of an ideal generated by l elements, we have $\ell(\mathfrak{D}) = l$. By Lemma 6.7 we deduce that $\overline{I} = \overline{\mathfrak{D}}$. □

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