

# EXTREMAL $F$ -THRESHOLDS IN REGULAR LOCAL RINGS

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ABSTRACT. Let  $(R, \mathfrak{m})$  be a regular local ring of characteristic  $p > 0$ . Among all proper ideals  $\mathfrak{a} \subseteq R$  with a fixed order of vanishing  $\text{ord}_{\mathfrak{m}}(\mathfrak{a})$ , we classify the ideals for which the  $F$ -threshold  $\text{fpt}(\mathfrak{a})$  is minimal.

## 1. INTRODUCTION

Let  $(R, \mathfrak{m})$  be a regular local ring of characteristic  $p > 0$  and let  $\mathfrak{a}$  denote a proper ideal of  $R$ . Assume that  $R$  is  $F$ -finite – that is, we assume that the Frobenius map  $F : R \rightarrow R$  is finite – we will relax this assumption later. The  $F$ -pure threshold of  $(R, \mathfrak{a})$  is a nonnegative rational number [4] measuring the singularities of the pair  $(R, \mathfrak{a})$  at the point  $\mathfrak{m}$  in  $\text{Spec } R$ . In this setting, the  $F$ -pure threshold can be computed as

$$\text{fpt}(\mathfrak{a}) = \sup \left\{ \frac{t}{p^e} : \mathfrak{a}^t \not\subseteq \mathfrak{m}^{[p^e]} \right\}, \text{ where } I^{[p^e]} := \sum_{z \in I} z^{p^e} R.$$

The idea is that smaller values of the  $F$ -pure threshold correspond to “worse singularities” of the closed subscheme of  $\text{Spec } R$  defined by  $\mathfrak{a}$ . For example, if  $\mathfrak{a} = fR$  is principal, then  $\text{fpt}(f) \leq 1$  with equality if and only if  $R/fR$  is Frobenius split; in this case,  $R/fR$  must be reduced.

If  $d = \text{ord}_{\mathfrak{m}}(\mathfrak{a})$  denotes the greatest positive integer such that  $\mathfrak{a} \subseteq \mathfrak{m}^d$  and  $n$  is the dimension of  $R$ , then  $\text{fpt}(\mathfrak{a})$  is bounded below by  $\frac{1}{d}$  and bounded above by  $\min(\frac{n}{d}, 1)$  [28]. In this paper, we describe when the  $F$ -pure threshold achieves the lower end of this range.

**Main Theorem** (Theorem 3.1). *Let  $(R, \mathfrak{m})$  be an  $F$ -finite regular local ring of characteristic  $p > 0$  and  $\mathfrak{a} \subseteq R$  a proper ideal. Let  $d = \text{ord}_{\mathfrak{m}}(\mathfrak{a})$  and write  $d = qs$  where  $q$  is a power of  $p$  and  $s$  is coprime to  $p$ . Then  $\text{fpt}(\mathfrak{a}) = \frac{1}{d}$  if and only if  $\mathfrak{a}$  is a principal ideal and there exists  $g$  in  $\mathfrak{m}^{[q]}$  such that  $\mathfrak{a} = g^s R$ .*

For a more general statement and a proof, see Section 3. In particular, we show that the  $F$ -finite hypothesis can be weakened to the assumption that the formal fiber  $\widehat{R} \otimes_R \frac{R_{(\pi)}}{\pi R_{(\pi)}}$  is reduced for all prime elements  $\pi$  in  $R$ , which is satisfied by any excellent local ring. The hypothesis on the height-1 formal fibers of  $R$  cannot be weakened further; see Example 3.20.

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Other authors have considered a related problem for squarefree homogeneous polynomials over an algebraically closed field  $k$  of characteristic  $p > 0$ ; see [20]. For  $f$  in  $k[x_0, \dots, x_n]$  squarefree and homogeneous of degree  $d$ , one has  $\text{fpt}(f) \geq \frac{1}{d-1}$  with equality if and only if  $d = p^e + 1$  and  $f \in \mathfrak{m}^{[p^e]}$ . In the absence of homogeneity, however, we cannot improve on the lower bound  $\text{fpt}(f) \geq \frac{1}{d}$  even if we assume that  $f$  is squarefree: the binomial  $x_1^d + x_2^N$  is reduced whenever  $d, N$  are coprime and  $\text{fpt}(x_1^d + x_2^N)$  converges to  $\frac{1}{d}$  as  $N \rightarrow \infty$  (Example 3.21).

The characteristic zero analog of the main theorem concerning the log canonical threshold ( $\text{lct}$ ) is well-understood. If  $(R, \mathfrak{m})$  is an excellent regular local ring of equal characteristic zero and  $\mathfrak{a} \subseteq R$  an ideal, it is known to experts that  $\text{lct}(\mathfrak{a}) \geq \frac{1}{\text{ord}_{\mathfrak{m}}(\mathfrak{a})}$  with equality if and only if  $\mathfrak{a} = x^d R$  for some  $d$  in  $\mathbb{Z}^+$  and some  $x$  in  $\mathfrak{m} \setminus \mathfrak{m}^2$ . More generally, for the germ at 0 of a plurisubharmonic function  $u : \mathbb{C}^n \rightarrow \mathbb{C}$  with Lelong number  $\nu(u) = 1$ , one has  $\text{lct}(u) \geq 1$  with equality if and only if  $u = \log |z_1| + v$  with  $z_1$  a local coordinate and  $\nu(v) = 0$  [12].

A straightforward consequence of the main theorem is that the lower bound  $\frac{1}{\text{ord}_{\mathfrak{m}}(\mathfrak{a})} \leq \text{fpt}(\mathfrak{a})$  is only attained by height-1 ideals. In a forthcoming paper [2], the author proves a stronger bound for higher-height ideals, analogous to a result of Demailly and Pham on log canonical thresholds [8]. Specifically, for an  $F$ -finite regular local ring  $(R, \mathfrak{m})$  of characteristic  $p > 0$  and an ideal  $\mathfrak{a} \subseteq R$  of height  $l$ , there is a log convex sequence of positive integers  $\sigma_0(\mathfrak{a}), \dots, \sigma_l(\mathfrak{a})$  with  $\sigma_0(\mathfrak{a}) = 1, \sigma_1(\mathfrak{a}) = \text{ord}_{\mathfrak{m}}(\mathfrak{a})$ , and

$$(1) \quad \frac{\sigma_0(\mathfrak{a})}{\sigma_1(\mathfrak{a})} + \dots + \frac{\sigma_{l-1}(\mathfrak{a})}{\sigma_l(\mathfrak{a})} \leq \text{fpt}(\mathfrak{a}).$$

The integers  $\sigma_j(\mathfrak{a})$  are constructed in [3], where they are written as  $\sigma_j(\mathfrak{a}, \mathfrak{m})$ . The main result of [2] is a description of the cases of equality in Equation (1) when  $R$  is a polynomial ring and  $\mathfrak{a} \subseteq R$  is a homogeneous ideal.

**Notation 1.1.** *In this paper, all rings are commutative and Noetherian. The letter  $p$  always denotes a prime number and  $q$  a power of  $p$ .*

## 2. PRELIMINARIES

On several occasions, we need to work with monomial orders and leading terms in power series rings. With a few exceptions, Gröbner theory in power series rings is analogous to the theory in polynomial rings. For the unfamiliar reader, we recommend [7, Chapter 4].

**Notation 2.1.** *For a vector of ring elements  $\mathbf{f} = f_1, \dots, f_r$  and a vector of nonnegative integers  $\mathbf{a} = a_1, \dots, a_r$ , we let  $\mathbf{f}^{\mathbf{a}}$  denote the element  $f_1^{a_1} \dots f_r^{a_r}$ . If  $\mathbf{a}$  is instead a tuple of nonnegative real numbers, we let  $\mathbf{f}^{\mathbf{a}}$  denote the  $\mathbb{R}$ -divisor  $a_1 \text{div}(f_1) + \dots + a_r \text{div}(f_r)$ .*

**Definition 2.2.** Let  $k$  be a field. Let  $R = k[[x_1, \dots, x_n]]$ . A *local monomial order* on  $R$  is a partial ordering  $>$  on the set of monomials of  $R$  such that

for all  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  in  $\mathbb{Z}_{\geq 0}^n$ , if  $\mathbf{x}^{\mathbf{a}} \leq \mathbf{x}^{\mathbf{b}}$ , then

$$\mathbf{x}^{\mathbf{a}+\mathbf{c}} \leq \mathbf{x}^{\mathbf{b}+\mathbf{c}} \leq \mathbf{x}^{\mathbf{b}}.$$

In particular, the greatest monomial in  $R$  is 1.

Let  $f = \sum_{\mathbf{a} \in \mathbb{Z}_{\geq 0}^n} \gamma_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$  be an element of  $R$ . Among all  $\mathbf{b} \in \mathbb{Z}_{\geq 0}^n$  such that  $\gamma_{\mathbf{b}} \neq 0$ , let  $S$  denote the set of vectors for which  $\mathbf{x}^{\mathbf{b}}$  is maximal with respect to  $>$ . The *initial term* of  $f$  with respect to  $>$ , denoted  $\text{in}_{>}(f)$ , is the element  $\sum_{\mathbf{b} \in S} \gamma_{\mathbf{b}} \mathbf{x}^{\mathbf{b}}$ .

**2.1. Background on F-Thresholds and Test Ideals.** To begin, we define  $F$ -thresholds and test ideals and collect a few relevant properties. For further background, we refer the reader to [28, 22].

**Definition-Proposition 2.3** ([22]). Let  $R$  be a regular local ring of characteristic  $p > 0$ . Let  $\mathfrak{a} \subseteq R$  be an ideal and  $\mathfrak{b} \subseteq R$  a proper ideal such that  $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$ . For  $e \geq 0$ , define  $\nu_{\mathfrak{a}}^{\mathfrak{b}}(p^e)$  to be the greatest integer  $t$  such that  $\mathfrak{a}^t \not\subseteq \mathfrak{b}^{[p^e]}$ . Then the sequence  $\frac{\nu_{\mathfrak{a}}^{\mathfrak{b}}(p^e)}{p^e}$  has a limit as  $e \rightarrow \infty$ , and we refer to this limit as the  $F$ -threshold  $\text{ft}^{\mathfrak{b}}(\mathfrak{a})$ . Additionally, we have

$$\text{ft}^{\mathfrak{b}}(f) = \inf \left\{ \frac{t}{p^e} : \mathfrak{a}^t \subseteq \mathfrak{b}^{[p^e]} \right\} = \sup \left\{ \frac{t}{p^e} : \mathfrak{a}^t \not\subseteq \mathfrak{b}^{[p^e]} \right\}.$$

For the majority of this paper, the above notion of the  $F$ -threshold suffices. In a few isolated instances, we need a more general notion of what it means for a pair  $(R, f^t)$  to be  $F$ -pure.

**Definition 2.4** ([25], §1.1). Let  $R$  be a ring of characteristic  $p > 0$ . For any  $e > 0$ , the module  $F_*^e R$  has underlying Abelian group isomorphic to  $R$  and an  $R$ -module action defined by restriction of scalars along the Frobenius map  $F^e : R \rightarrow R$ . Concretely, the elements of  $F_*^e R$  are  $\{F_*^e f : f \in R\}$ , and  $x F_*^e f = F_*^e(x^{p^e} f)$ . We say that  $R$  is  $F$ -finite if  $F_*^e R$  is a finite  $R$ -module, in which case  $F_*^e R$  is finite for all  $e > 0$ .

**Definition 2.5** ([28, 24]). Let  $R$  be an  $F$ -finite reduced ring. Let  $\mathfrak{a} \subseteq R$  be an ideal and  $t$  a nonnegative real number. We say that the pair  $(R, \mathfrak{a}^t)$  is *sharply  $F$ -pure* if, for a single  $e > 0$  (equivalently, all integers  $ne$  where  $n \in \mathbb{Z}^+$ ), there exists  $d \in \mathfrak{a}^{\lceil t(p^e-1) \rceil}$  such that the map

$$R \rightarrow F_*^e R \quad 1 \mapsto F_*^e d$$

splits.

The  $F$ -pure threshold  $\text{fpt}(\mathfrak{a})$  of the pair  $(R, \mathfrak{a})$  is defined to be the supremum of all real  $t \geq 0$  such that  $(R, \mathfrak{a}^t)$  is sharply  $F$ -pure.

By [22, Remark 1.5], if  $(R, \mathfrak{m})$  is a regular local ring then  $\text{fpt}(\mathfrak{a}) = \text{ft}^{\mathfrak{m}}(\mathfrak{a})$ .

**Definition 2.6** ([13, 4]). Let  $R$  be an  $F$ -finite regular ring, let  $\mathfrak{a} \subseteq R$  be an ideal, and let  $t$  be a nonnegative real number. The *test ideal* of the pair  $(R, \mathfrak{a}^t)$  is equal to the image, for  $e \gg 0$ , of the map

$$\mathfrak{a}^{\lceil tp^e \rceil} \cdot \operatorname{Hom}_R(F_*^e R, R) \rightarrow R \quad d \cdot \phi \mapsto \phi(F_*^e d).$$

**Proposition 2.7.** *Let  $(R, \mathfrak{m})$  be a regular local ring of dimension  $n$  and characteristic  $p > 0$ . Let  $f$  be an element of  $R$  and let  $\mathfrak{b} \subseteq R$  such that  $f \in \sqrt{\mathfrak{b}}$ . Then the following hold.*

- (1) *If  $\mathfrak{c} \subseteq \mathfrak{a}$ , then  $\operatorname{ft}^{\mathfrak{b}}(\mathfrak{c}) \leq \operatorname{ft}^{\mathfrak{b}}(\mathfrak{a})$ .*
- (2) *For  $s$  in  $\mathbb{Z}^+$ , we have  $\operatorname{ft}^{\mathfrak{b}}(\mathfrak{a}^s) = \frac{1}{s} \operatorname{ft}^{\mathfrak{b}}(\mathfrak{a})$ .*
- (3) *The inequality  $\frac{1}{\operatorname{ord}_{\mathfrak{m}}(f)} \leq \operatorname{ft}^{\mathfrak{m}}(f) \leq \frac{n}{\operatorname{fpt}(f)}$  holds.*
- (4) *For all  $e > 0$ , we have  $\frac{\nu_{\mathfrak{a}}^{\mathfrak{b}}(p^e)}{p^e} < \operatorname{ft}^{\mathfrak{b}}(\mathfrak{a})$ .*
- (5) *For  $x$  in  $\mathfrak{m} \setminus \mathfrak{m}^2$ , if  $\bar{f}, \bar{\mathfrak{b}}$  denote the images of  $f, \mathfrak{b}$  in  $R/xR$ , then  $\operatorname{ft}^{\mathfrak{b}}(f) \geq \operatorname{ft}^{\bar{\mathfrak{b}}}(\bar{f})$ .*
- (6)  *$\operatorname{ft}^{\mathfrak{b}}(f) = \inf\{t : \tau(f^t) \subseteq \mathfrak{b}\}$ .*
- (7) *If  $R$  is a power series ring,  $\mathfrak{b}$  is a monomial ideal, and  $>$  is a local monomial order, then  $\operatorname{ft}^{\mathfrak{b}}(\operatorname{in}_{>}(f)) \leq \operatorname{ft}^{\mathfrak{b}}(f)$ .*

*Proof.* For (1)-(4), see [22, Proposition 1.7]. The proof of claims (5)-(7) are enumerated below.

- (5) This standard fact follows from an observation in [27, Theorem 3.11]. For any positive integers  $a, e$ , if  $f^a \in \mathfrak{b}^{[p^e]}$  then  $(\bar{f})^a \in (\bar{\mathfrak{b}})^{[p^e]}$ . The assumption that  $x \in \mathfrak{m} \setminus \mathfrak{m}^2$  is to ensure that the quotient ring  $R/xR$  is again regular, but is not necessary if one defines the  $F$ -threshold in greater generality.
- (6) This fact follows from [22, Proposition 2.7].
- (7) The argument of the claim in [28, Proof of Proposition 4.5] works verbatim.

□

## 2.2. Critical Points, after Hernández and Teixeira.

**Notation 2.8.** *For a vector  $\mathbf{a} = (a_1, \dots, a_n)$  in  $\mathbb{R}_{\geq 0}^n$ , the symbol  $\|\mathbf{a}\|$  denotes the number  $a_1 + \dots + a_n$ .*

A key point in the proof of Theorem 3.1 uses ideas of Hernández and Teixeira [16] in the 2-dimensional setting, generalized to fit our needs. In op. cit., the authors consider the following problem.

**Question 2.9.** *Let  $k$  be a field of characteristic  $p > 0$  and  $\mathbf{h} = h_1, \dots, h_r$  be a tuple of distinct linear forms in  $R = k[x, y]$ . For which  $\mathbf{t}$  in  $\mathbb{R}_{\geq 0}^r$  is  $\tau(R, \mathbf{h}^{\mathbf{t}}) = R$ ? More generally, for a homogeneous system of parameters  $U, V$  and  $\mathfrak{b} = (U, V)$ , how can we describe the set of  $\mathbf{t}$  in  $\mathbb{R}_{\geq 0}^r$  for which  $\tau(R, \mathbf{h}^{\mathbf{t}}) \subseteq \mathfrak{b}$ ?*

To address this question, the authors study *syzygy gaps*. For  $\mathbf{a}$  in  $\mathbb{Z}_{\geq 0}^r$  and  $q = p^e$ , one can determine whether  $\tau(R, \mathbf{h}^{\mathbf{a}/q}) \subseteq \mathfrak{b}$  by computing the graded free resolution

$$R(-m) \oplus R(-n) \rightarrow R(-q \deg U) \oplus R(-q \deg V) \oplus R(-\|\mathbf{a}\|) \rightarrow R/(U, V, \mathbf{h}^{\mathbf{a}}) \rightarrow 0.$$

The number  $\Delta(\mathbf{a})$  is defined to be the “syzygy gap”  $|m - n|$ . Defining  $\Delta(\mathbf{a}/q)$  to be  $\frac{1}{q}\Delta(\mathbf{a})$  yields a well-defined map  $\Delta : \mathbb{Z}[1/p]_{\geq 0}^r \rightarrow \mathbb{Z}[1/p]_{\geq 0}$ . By Lemma 3.2 in op. cit.,  $\Delta$  extends uniquely to a continuous map from  $\mathbb{Q}_{\geq 0}^r \rightarrow \mathbb{Q}_{\geq 0}$ , and by Proposition 3.5 in op. cit.,  $\Delta$  determines the behavior of the test ideal: for  $\mathbf{t}$  in  $\mathbb{Q}_{\geq 0}^r$ , we have  $\tau(R, \mathbf{h}^{\mathbf{t}}) \subseteq \mathfrak{b}$  if and only if  $\Delta(\mathbf{t}) = \|\mathbf{t}\| - \deg(UV)$ . By Corollary 3.11 in op. cit., we have  $\tau(R, \mathbf{h}^{\mathbf{t}}) \subseteq \mathfrak{b}$  whenever  $\|\mathbf{t}\| \geq \deg(UV)$ , so the nontrivial behavior of the test ideal is confined to the region  $\|\mathbf{t}\| < \deg(UV)$ . By Theorem 5.9 in op. cit., this nontrivial behavior is completely determined by a family of distinguished points in  $\mathbb{Z}[1/p]_{\geq 0}$  called *critical points*.

The question we consider is similar. Rather than arbitrary parameter ideals, we consider special parameter ideals  $\mathfrak{b}$  of the form  $(y, x^\ell)$  and rather than tuples of linear forms, we consider tuples of polynomials  $\mathbf{h} = h_1, \dots, h_r$ , where  $h_i = y - g_i$  and  $g_i \in x^\ell k[x]$ .

**Question 2.10.** *Let  $k$  be a field of characteristic  $p > 0$  and  $R = k[[x, y]]$ . Let  $\ell > 0$  and let  $\mathfrak{b} := (y, x^\ell)$ . Let  $r \geq 2$  and for  $1 \leq i \leq r$ , let  $h_i = y - g_i$ , where  $g_i \in x^\ell k[x]$ . Let  $\mathbf{h}$  denote the vector  $h_1, \dots, h_r$ . For which  $\mathbf{t}$  in  $\mathbb{R}_{\geq 0}^r$  is  $\tau(R, \mathbf{h}^{\mathbf{t}}) \subseteq \mathfrak{b}$ ?*

Analogously to Question 2.9, we have  $\tau(R, \mathbf{h}^{\mathbf{t}}) \not\subseteq \mathfrak{b}$  when  $\|\mathbf{t}\| < 1$  and  $\tau(R, \mathbf{h}^{\mathbf{t}}) \subseteq \mathfrak{b}$  when  $\|\mathbf{t}\| \geq 2$ , so the nontrivial behavior of  $\tau(R, \mathbf{h}^{\mathbf{t}})$  is contained in the strip  $1 \leq \|\mathbf{t}\| < 2$ . If  $D$  is the largest degree of any of the polynomials  $g_i$ , then the behavior of  $\tau(R, \mathbf{h}^{\mathbf{t}})$  is determined by critical points in the strip  $1 \leq \|\mathbf{t}\| < 1 + \frac{\ell}{D}$  by Corollary 2.19, and may not be determined by critical points in the strip  $1 + \frac{\ell}{D} \leq \|\mathbf{t}\| < 2$  (Example 2.21).

**Definition 2.11.** Consider the setup of Question 2.10. We define a partition of the set  $\left(\mathbb{Z}[\frac{1}{p}]_{\geq 0}\right)^r$  into the *upper and lower regions attached to  $\mathbf{h}$  and  $\mathfrak{b}$* . For  $\mathbf{a}$  in  $\mathbb{Z}_{\geq 0}^r$ ,  $q = p^e$ , we say that  $\frac{\mathbf{a}}{q} \in \mathcal{U}$  (the upper region) if  $\mathbf{h}^{\mathbf{a}} = h_1^{a_1} \dots h_r^{a_r} \in \mathfrak{b}^{[q]}$ . Otherwise,  $\frac{\mathbf{a}}{q} \in \mathcal{L}$  (the lower region).

Because  $R$  is  $F$ -split, it follows that  $\mathbf{h}^{\mathbf{a}} \in \mathfrak{b}^{[q]}$  if and only if  $\mathbf{h}^{p\mathbf{a}} \in \mathfrak{b}^{[pq]}$ , so the regions  $\mathcal{U}, \mathcal{L}$  are well-defined.

**Lemma 2.12.** *For any  $\frac{\mathbf{a}}{q} \leq \frac{\mathbf{a}'}{q'}$ , if  $\frac{\mathbf{a}}{q} \in \mathcal{U}$  then  $\frac{\mathbf{a}'}{q'} \in \mathcal{U}$ .*

*Proof.* By well-definedness of  $\mathcal{U}$ , we may set  $q'' = \max(q, q')$  and rewrite  $\frac{\mathbf{a}}{q} = \frac{\mathbf{b}}{q''}$ ,  $\frac{\mathbf{a}'}{q'} = \frac{\mathbf{c}}{q''}$ . Suppose  $\frac{\mathbf{b}}{q''} \in \mathcal{U}$ . As  $\mathbf{b} \leq \mathbf{c}$ , it follows that  $\mathbf{h}^{\mathbf{b}} \mid \mathbf{h}^{\mathbf{c}}$ , so  $\mathbf{h}^{\mathbf{c}} \in \mathfrak{b}^{[q'']}$  and  $\frac{\mathbf{c}}{q''} \in \mathcal{U}$ .  $\square$

**Lemma 2.13.** *Suppose that  $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$  and that  $q$  is a power of  $p$ . Let  $e_1, \dots, e_r$  denote the standard unit vectors in  $\mathbb{Z}^r$ . For any  $1 \leq s \leq r$ , we have  $\frac{\mathbf{a}}{q} - \frac{e_s}{q} \in \mathcal{U}$  if and only if  $\frac{\mathbf{a}}{q} - \frac{e_s}{pq} \in \mathcal{U}$ .*

*Proof.* The implication  $\Rightarrow$  follows from Lemma 2.12. For the reverse implication, start by considering the map  $\phi : R \rightarrow R$  given by  $\phi(x) = x$  and  $\phi(y) = h_s$ . Since  $\phi$  is an isomorphism and  $\phi(\mathbf{b}) = \mathbf{b}$ , we may without loss of generality assume  $h_s = y$ .

Suppose  $\frac{\mathbf{a}}{q} - \frac{e_s}{pq} \in \mathcal{U}$ . Then  $\mathbf{h}^{p(\mathbf{a}-e_s)}y^{p-1} = \mathbf{h}^{p\mathbf{a}-e_s} \in \mathbf{b}^{[pq]}$ . Applying [20, Lemma 3.5], we obtain

$$\mathbf{h}^{p(\mathbf{a}-e_s)} \in (\mathbf{b}^{[pq]} : y^{p-1}) = (y^{pq-p+1}, x^{\ell pq})$$

The monomials  $\{x^i y^j\}_{0 \leq i, j \leq p-1}$  form a free basis for  $F_* R$  over  $R$ ; write  $\Phi : F_* R \rightarrow R$  for projection onto the  $F_*(xy)^{p-1}$  factor. Then  $\Phi \circ (\cdot F_*(xy)^{p-1})$  is a splitting of the Frobenius map  $R \rightarrow F_* R$ , so

$$\mathbf{h}^{\mathbf{a}-e_s} = \Phi(F_*(xy)^{p-1} \mathbf{h}^{p(\mathbf{a}-e_s)}) \in \Phi(F_*(y^{pq}, x^{\ell pq+p-1})) = \mathbf{b}^{[q]}.$$

□

**Lemma 2.14.** *Let  $\frac{\mathbf{a}}{q}$  be a vector in  $\mathcal{U}$ . If  $\frac{\mathbf{a}-e_s}{q} \in \mathcal{L}$  for all  $1 \leq s \leq r$  with  $a_s > 0$ , then  $\frac{\mathbf{a}'}{q'} \in \mathcal{L}$  for all  $\frac{\mathbf{a}'}{q'} < \frac{\mathbf{a}}{q}$ .*

*Proof.* The implication  $\Leftarrow$  is tautological. For the implication  $\Rightarrow$ , suppose that  $\frac{\mathbf{a}-e_s}{q} \in \mathcal{L}$  for all  $1 \leq s \leq r$  with  $a_s > 0$ . By Lemma 2.13, it follows that  $\frac{p^n \mathbf{a} - e_s}{p^n q} \in \mathcal{L}$  for all  $n > 0$  and all  $s$  with  $a_s > 0$ . For  $\frac{\mathbf{a}'}{q'} < \frac{\mathbf{a}}{q}$ , suppose without loss of generality that  $\mathbf{a}'_1 < \mathbf{a}_1$ . Then  $\frac{\mathbf{a}'}{q'} \leq \frac{q' \mathbf{a} - e_1}{qq'}$ , which implies  $\frac{\mathbf{a}'}{q'} \in \mathcal{L}$  by Lemma 2.12. □

**Definition 2.15** (c.f. [17], Definition 5.1). Let  $\frac{\mathbf{c}}{q}$  be a vector in  $\mathcal{U}$ . If  $\frac{\mathbf{c}}{q}$  satisfies either of the equivalent conditions of Lemma 2.14, we say that  $\frac{\mathbf{c}}{q}$  is a *critical point* attached to  $\mathbf{h}$  and  $\mathbf{b}$ . We let  $\mathcal{C}$  denote the set of such critical points.

**Lemma 2.16** (c.f. [16], Corollary 5.7). *Let  $\mathbf{a}$  be in  $\mathbb{Z}_{\geq 0}^r$  and  $q$  be a power of  $p$ . Then  $\frac{\mathbf{a}}{q} \in \mathcal{U}$  if and only if there exists a critical point  $\frac{\mathbf{c}}{q}$  in  $\mathcal{C}$  with  $\mathbf{c} \leq \mathbf{a}$ .*

*Proof.* Suppose  $\frac{\mathbf{a}}{q} \in \mathcal{U}$ . The set  $S$  of elements  $\mathbf{u}$  in  $\mathbb{Z}_{\geq 0}^r$  such that  $\frac{\mathbf{u}}{q} \in \mathcal{U}$ ,  $\mathbf{u} \leq \mathbf{a}$  is finite and nonempty, so choose  $\mathbf{c} \in S$  such that  $\|\mathbf{c}\|$  is minimal. By construction, we have  $\frac{\mathbf{c}-e_s}{q} \in \mathcal{L}$  for all  $1 \leq s \leq r$ , hence  $\frac{\mathbf{c}}{q}$  is a critical point by Lemma 2.14. Conversely, if  $\frac{\mathbf{a}}{q} \geq \frac{\mathbf{c}}{q} \in \mathcal{C}$ , then  $\frac{\mathbf{a}}{q} \in \mathcal{U}$  by Lemma 2.12. □

**Lemma 2.17.** *If  $\frac{\mathbf{c}}{q} \in \mathcal{U}$ , then  $\left\| \frac{\mathbf{c}}{q} \right\| \geq 1$ .*

*Proof.* First, we note that  $h_i \in \mathbf{b}$  for all  $1 \leq i \leq r$ , so  $\mathbf{h}^{\mathbf{a}} \in \mathbf{b}^{[\mathbf{a}]}$  for any  $\mathbf{a}$  in  $\mathbb{Z}_{\geq 0}^r$ . Conversely, the coefficient of  $y^{[\mathbf{a}]}$  in  $\mathbf{h}^{\mathbf{a}}$  is equal to 1, so  $\text{ord}_{\mathbf{b}}(\mathbf{h}^{\mathbf{a}}) = [\mathbf{a}]$ . If  $\frac{\mathbf{c}}{q} \in \mathcal{U}$ , then  $\mathbf{h}^{\mathbf{c}} \in \mathbf{b}^{[q]} \subseteq \mathbf{b}^q$ , so  $\|\mathbf{c}\| = \text{ord}_{\mathbf{b}}(\mathbf{h}^{\mathbf{c}}) \geq q$ . □

**Lemma 2.18** (c.f. [16], Remark 5.8). *Let  $D$  denote the largest total degree of any of the polynomials  $g_1, \dots, g_r$ . If  $\frac{\mathbf{b}}{q}, \frac{\mathbf{c}}{q}$  are distinct elements of  $\mathcal{C}$ , then  $\left\| \max(\frac{\mathbf{b}}{q}, \frac{\mathbf{c}}{q}) \right\| \geq 1 + \frac{\ell}{D}$ .*

*Proof.* Set  $\mathbf{c}^{(0)} = \mathbf{c}$ . To prove the claim, we produce a sequence of critical points  $\mathbf{c}^{(0)}, \mathbf{c}^{(1)}, \dots, \mathbf{c}^{(s)}$  and possibly non-critical points  $\mathbf{a}^{(0)}, \dots, \mathbf{a}^{(s)}$  such that  $\mathbf{a}^{(s)} \leq \max(\mathbf{b}, \mathbf{c}^{(0)})$ ; we will show that  $\|\mathbf{a}^{(s)}\| \geq 1 + \frac{\ell}{D}$ .

Consider the following auxiliary conditions, which we will show hold for all  $1 \leq m \leq s$ :

$$\begin{aligned} (2) \quad & \max(\mathbf{b}, \mathbf{c}^{(m)}) \leq \max(\mathbf{b}, \mathbf{a}^{(m-1)}) \\ (3) \quad & \max(\mathbf{b}, \mathbf{a}^{(m)}) < \max(\mathbf{b}, \mathbf{c}^{(m)}) \end{aligned}$$

Suppose we have constructed  $\mathbf{c}^{(0)}, \dots, \mathbf{c}^{(n)}, \mathbf{a}^{(0)}, \dots, \mathbf{a}^{(n-1)}$  satisfying the conditions

- (i)  $\mathbf{c}^{(n)} \neq \mathbf{b}$ ;
- (ii)  $\frac{\mathbf{a}^{(0)}}{q}, \dots, \frac{\mathbf{a}^{(n-1)}}{q} \in \mathcal{U}$ ;
- (iii) Equation 2 holds for  $1 \leq m \leq n$ ;
- (iv) Equation 3 holds for  $0 \leq m \leq n-1$ .

For the base case  $n = 0$ , the only non-vacuous hypothesis is that  $\mathbf{c}^{(0)} \neq \mathbf{b}$ , which holds by assumption. By Definition 2.15, the distinct critical points  $\frac{\mathbf{b}}{q}, \frac{\mathbf{c}^{(n)}}{q}$  are incomparable, so let  $1 \leq i_n, j_n \leq r$  such that  $c_{i_n}^{(n)} > b_{i_n}$  and  $c_{j_n}^{(n)} < b_{j_n}$ . Set  $\mathbf{a}^{(n)} = \max(\mathbf{b}, \mathbf{c}^{(n)}) - e_{i_n} - e_{j_n}$ . We compute

$$(4) \quad \max(b_t, a_t^{(n)}) = \begin{cases} c_{i_n}^{(n)} - 1 & t = i_n \\ b_{j_n} & t = j_n \\ \max(c_t^{(n)}, b_t) & t \neq i_n, j_n, \end{cases}$$

so Equation 3 holds for  $m = n$ . If  $\frac{\mathbf{a}^{(n)}}{q} \in \mathcal{L}$ , we set  $s = n$  and terminate the sequence. Otherwise, we apply Lemma 2.16 to produce  $\mathbf{c}^{(n+1)} \leq \mathbf{a}^{(n)}$  where  $\frac{\mathbf{c}^{(n+1)}}{q} \in \mathcal{C}$ . By construction, Equation 2 holds for  $m = n+1$ . As  $c_{j_n}^{(n+1)} \leq a_{j_n}^{(n)} = b_{j_n} - 1$ , the vectors  $\mathbf{c}^{(n+1)}, \mathbf{b}$  are again distinct.

By Equations 2,3, we have a strictly decreasing sequence of nonnegative integer vectors

$$\max(\mathbf{b}, \mathbf{c}^{(0)}) > \max(\mathbf{b}, \mathbf{c}^{(1)}) > \dots$$

which necessarily terminates. In particular, there exists some index  $s \geq 0$  such that  $\frac{\mathbf{a}^{(s)}}{q} \in \mathcal{L}$ . Additionally, by construction of  $\mathbf{a}^{(s)}$  we compute

$$\mathbf{a}^{(s)} = \max(\mathbf{b}, \mathbf{c}^{(s)}) - e_{i_s} - e_{j_s} \leq \max(\mathbf{b}, \mathbf{c}^{(0)}) - e_{i_s} - e_{j_s},$$

so  $\|\max(\mathbf{b}, \mathbf{c}^{(0)})\| \geq \|\mathbf{a}^{(s)}\| + 2$ . To prove that  $\left\|\max\left(\frac{\mathbf{b}}{q}, \frac{\mathbf{c}^{(0)}}{q}\right)\right\| \geq 1 + \frac{\ell}{D}$ , it therefore suffices to show that

$$(5) \quad \left\|\mathbf{a}^{(s)}\right\| \geq q + \frac{\ell q}{D} - 2$$

Without loss of generality, write  $\mathbf{a} = \mathbf{a}^{(s)}$ ,  $\mathbf{c} = \mathbf{c}^{(s)}$ ,  $i_s = 1$  and  $j_s = 2$ . With  $h_1 = y - g_1$ ,  $h_2 = y - g_2$ , consider the automorphism  $\phi : R \rightarrow R$  given by  $x \mapsto x$  and  $y \mapsto y - g_1$ . Then  $\phi(h_1) = y$ ,  $\phi(\mathbf{b}) = \mathbf{b}$ , and  $\phi(h_i) = y - g_i - g_1$ , where  $g_i - g_1$  is a polynomial in  $k[x]$  of total degree at most  $D$ . We may therefore assume without loss of generality that  $g_1 = 0$  so that  $h_1 = y$ .

Set  $m = \text{ord}_x(g_2)$ . As  $x^\ell$  divides  $g_2$  and  $\deg(g_2) \leq D$ , we necessarily have  $\ell \leq m \leq D$ . We compute  $(h_1, h_2) = (y, x^m)$ . As  $\frac{\mathbf{a}}{q} \in \mathcal{L}$ , we have  $\mathbf{h}^{\mathbf{a}} \notin \mathbf{b}^{[q]}$ . On the other hand,  $\frac{\mathbf{a}+e_1}{q} \geq \frac{\mathbf{b}}{q}$  and  $\frac{\mathbf{a}+e_2}{q} \geq \frac{\mathbf{c}}{q}$ , so  $\frac{\mathbf{a}+e_1}{q}, \frac{\mathbf{a}+e_2}{q} \in \mathcal{U}$  by Lemma 2.16; it follows that  $\mathbf{h}^{\mathbf{a}}(h_1, h_2) \subseteq \mathbf{b}^{[q]}$ . Rewriting  $(h_1, h_2)$  as  $(y, x^m)$  and applying [20, Lemma 3.5], we deduce that

$$\mathbf{h}^{\mathbf{a}} \in (\mathbf{b}^{[q]} : (h_1, h_2)) = (y^q, x^{\ell q}, x^{\ell q - m} y^{q-1}).$$

Write  $\mathbf{h}^{\mathbf{a}} = \sum_{i \geq 0} z_i y^i$ , where  $z_i \in k[x]$ . The quotient module

$$\frac{(y^q, x^{\ell q}, x^{\ell q - m} y^{q-1} - 1)}{(y^q, x^{\ell q})}$$

is spanned by the monomials  $y^{q-1} x^{\ell q - m}, \dots, y^{q-1} x^{\ell q - 1}$ , so  $z_{q-1}$  is nonzero and has degree at least  $\ell q - m$ . On the other hand,  $z_{q-1}$  is a polynomial of degree  $\|\mathbf{a}\| - (q - 1)$  in the inputs  $g_1, \dots, g_r$ , so we have

$$\ell q - m \leq \deg(z_{q-1}) \leq D(\|\mathbf{a}\| - (q - 1)).$$

As  $m \leq D$ , we conclude that  $\|\mathbf{a}\| \geq (q - 1) + \frac{\ell q - D}{D}$ , so Equation (5) holds.  $\square$

**Corollary 2.19** (c.f. [16], Theorem 5.9). *Let  $h_1, \dots, h_r$  be polynomials as in Question 2.10 where the  $g_i$  are distinct. Let  $D$  denote the largest degree of the polynomials  $g_1, \dots, g_r$ . Let  $f$  be an element of  $k[[x, y]]$  such that  $f = \mathbf{h}^{\mathbf{t}} = h_1^{t_1} \dots h_r^{t_r}$  for some positive integers  $t_1, \dots, t_r$ . Setting  $\mathbf{b} = (y, x^\ell)$ , either  $\text{ft}^{\mathbf{b}}(f) \geq \frac{1+\ell/D}{\|\mathbf{t}\|}$  or there exists a unique critical point  $\mathbf{c} \leq \frac{1+\ell/D}{\|\mathbf{t}\|} \mathbf{t}$  which computes  $\text{ft}^{\mathbf{b}}(f)$ : that is,  $\text{ft}^{\mathbf{b}}(f) = \max_{1 \leq i \leq r} \frac{c_i}{t_i}$ .*

*Proof.* Let  $\lambda = \frac{1+\ell/D}{\|\mathbf{t}\|}$ . Write  $\mu := \text{ft}^{\mathbf{b}}(f)$  and suppose that  $\mu < \lambda$ . For all  $e > 0$ , Proposition 2.7 (4) implies that  $f^{[p^e \mu]} \in \mathbf{b}^{[p^e]}$ . By definition,  $\frac{[p^e \mu]}{p^e} \mathbf{t} \in \mathcal{U}$ , so by Lemma 2.16, there exists  $\mathbf{c}^{(e)}$  in  $\mathcal{C}$  with  $\mathbf{c}^{(e)} \leq \frac{[p^e \mu]}{p^e} \mathbf{t}$ . Choose  $e_0 \gg 0$  such that for all  $e \geq e_0$  we have  $\frac{[p^e \mu]}{p^e} < \lambda$ . For any  $e > e_0$  we have

$$\left\|\max(\mathbf{c}^{(e_0)}, \mathbf{c}^{(e)})\right\| \leq \left\|\frac{[p^e \mu]}{p^e} \mathbf{t}\right\| = \frac{[p^e \mu]}{p^e} \|\mathbf{t}\| < \lambda \|\mathbf{t}\| = 1 + \frac{\ell}{D}.$$

By Lemma 2.18, we must have  $\mathbf{c}^{(e)} = \mathbf{c}^{(e_0)}$ . Let  $\mathbf{c} := \mathbf{c}^{(e_0)}$ .

As  $\mathbf{c} \leq \frac{[p^e]\mu}{p^e} \mathbf{t}$  for all  $e \geq e_0$ , it follows that  $\mathbf{c} \leq \mu \mathbf{t}$ . On the other hand, for any  $\mu' < \mu$ , we may choose  $e \gg 0$  such that  $[p^e \mu'] < \nu_f^b(p^e)$  by Definition-Proposition 2.3. Consequently,  $\frac{[p^e \mu']}{p^e} \mathbf{t} \notin \mathcal{U}$ , so  $\mathbf{c}^{(e_0)} \not\leq \frac{[p^e \mu']}{p^e} \mathbf{t}$  and hence  $\mathbf{c} \not\leq \mu' \mathbf{t}$ . We conclude that  $\mu$  is the smallest real number for which  $\mathbf{c} \leq \mu \mathbf{t}$ , hence  $\mu = \max_{1 \leq i \leq r} \frac{c_i}{t_i}$ .  $\square$

**Remark 2.20.** Putting  $\ell = n = 1$ , the above corollary gives an alternate proof of [16, Theorem 5.9] in the special case  $\mathbf{b} = (x, y)$ : given any homogeneous polynomial  $f$ , we may apply a linear change of coordinates so that  $x \nmid f$ , after which Corollary 2.19 applies.

The following example shows that the parameter  $D$  is necessary. Unlike the homogeneous case ([17, Theorem 5.9]), if  $\text{ft}^b(f) < \frac{2}{\|\mathbf{t}\|}$ , there may not be a unique critical point  $\frac{\mathbf{c}}{q} \leq \frac{2\mathbf{t}}{\|\mathbf{t}\|}$  computing  $\text{ft}^b(f)$ .

**Example 2.21.** Let  $R = \mathbb{F}_2[[x, y]]$  and  $\ell = 1$  so that  $\mathbf{b} = (x, y)$  and  $\text{ft}^b(-) = \text{fpt}(-)$ . We define  $h_1 = y + x, h_2 = y + x^2, h_3 = y + x^4$ . We consider  $\mathbf{t} = (1, 2, 1), f := \mathbf{h}^{\mathbf{t}} = h_1 h_2^2 h_3$ . Then  $f^7 \in \mathfrak{m}^{[16]}$ , so  $\text{fpt}(f) \leq \frac{7}{16} < \frac{2}{\|\mathbf{t}\|}$ . There are many critical points below  $\frac{2\mathbf{t}}{\|\mathbf{t}\|} = (\frac{1}{2}, 1, \frac{1}{2})$ , however: for instance, the points  $(\frac{1}{2}, 1, \frac{1}{2}), (\frac{3}{8}, \frac{13}{16}, \frac{7}{16}), (\frac{1}{4}, \frac{7}{8}, \frac{3}{8})$  are all critical. Moreover, none of these points compute the actual value of  $\text{fpt}(f)$ , which is equal to  $\frac{3}{7}$  by a computation in Macaulay2 [11] using the FrobeniusThresholds [5] package.

### 3. PROOF OF THEOREM 3.1

We state a general version of the main theorem and outline a proof.

**Theorem 3.1.** *Let  $(R, \mathfrak{m})$  be a regular local ring of characteristic  $p > 0$  and  $\mathfrak{a} \subseteq R$  a proper ideal. Let  $d = \text{ord}_{\mathfrak{m}}(\mathfrak{a})$  and write  $d = qs$  where  $q$  is a power of  $p$  and  $\gcd(p, s) = 1$ . Then*

$$(6) \quad \text{If } \text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}, \text{ then there exists } g \text{ in } \mathfrak{m}^{[q]} \widehat{R} \text{ such that } g^s \widehat{R} = \mathfrak{a} \widehat{R}.$$

*Suppose further that, for all prime elements  $\pi$  in  $R$ , the formal fiber  $\widehat{R} \otimes_R R_{(\pi)}/\pi R_{(\pi)}$  is reduced. Then*

$$(7) \quad \text{If } \text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}, \text{ there exists } h \text{ in } \mathfrak{m}^{[q]} \text{ such that } h^s R = \mathfrak{a} R.$$

Since  $F$ -finite rings are excellent [10, Remark 13.6], our main theorem follows from this more precise version. A version for pairs  $(R, \mathfrak{a}_1^{t_1} \dots \mathfrak{a}_r^{t_r})$  where  $t_i \in \mathbb{R}$  will appear in the author's dissertation.

As a special case of Theorem 3.1, one can classify homogeneous polynomials  $f$  in  $k[[x_1, \dots, x_n]]$  for which  $\text{fpt}(f) = \frac{1}{\deg(f)}$ . This case is given in

[20, Remark 3.2]; our proof follows a similar strategy. We reduce the claim to the case of a principal ideal in a complete local ring over an algebraically closed field, after which we apply a local Bertini theorem (Lemma 3.11) to reduce to the 2-dimensional case.

At this point, the two arguments diverge: unlike homogeneous polynomials, a power series in two variables may be irreducible, so we use the Weierstrass preparation theorem to write  $f$  as  $u(y^d + a_1y^{d-1} + \cdots + a_d)$  where  $u \in k[[x, y]]^\times$  and  $a_1, \dots, a_d \in k[[x]]$  (Lemma 3.5). Next, we pass to a finite flat extension  $(R, \mathfrak{m}) \rightarrow (U, \mathfrak{n})$  over which  $f$  factors as  $(y - \theta_1) \cdots (y - \theta_d)$ . This comes at the cost of having to consider the  $F$ -threshold of  $f$  at  $\mathfrak{m}U$  instead of the  $F$ -pure threshold of  $f$ . Using [23, Proposition 5.1], we reduce to the case that the  $\theta_i$  are polynomials in a finite extension  $k[x^{1/u}]$  of  $k[x]$ . Finally, we use the critical point framework (Corollary 2.19) to deduce that  $\theta_1 = \cdots = \theta_d$ , after which we apply Lemma 3.6 to deduce that  $f$  admits an  $s$ th root in  $R$ . As  $d = qs$ , if  $f = g^s$ , then  $\text{fpt}(g) = s \text{fpt}(f) = \frac{1}{q} = \frac{1}{\text{ord}_{\mathfrak{m}}(g)}$ , so the result follows from the degree- $q$  case (Lemma 3.2). We divide the proof into five subsections.

- (1) Equation (6) holds when  $\mathfrak{a}$  is principal,  $k = \bar{k}$ , and  $R = k[[x, y]]$ .
- (2) Equation (6) holds when  $\mathfrak{a}$  is principal,  $k = \bar{k}$ , and  $R = k[[x_1, \dots, x_n]]$ .
- (3) Equation (6) holds when  $\mathfrak{a}$  is principal,  $k$  is any field, and  $R = k[[x_1, \dots, x_n]]$ .
- (4) Equation (6) holds when  $\mathfrak{a}$  is any ideal,  $k$  is any field, and  $R = k[[x_1, \dots, x_n]]$ .
- (5) Theorem 3.1 holds.

### 3.1. Step (1).

**Lemma 3.2.** *Let  $(R, \mathfrak{m})$  be a regular local ring of characteristic  $p > 0$  and let  $f$  be an element of  $R$ . If  $q = p^e$  and  $\text{ord}_{\mathfrak{m}}(f) = q$ , then  $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{q}$  if and only if  $f \in \mathfrak{m}^{[q]}$ .*

*Proof.* As  $f^1 \in \mathfrak{m}^{[q]}$ , the inequality  $\text{ft}^{\mathfrak{m}}(f) \leq \frac{1}{q}$  follows from Definition-Proposition 2.3. Conversely,  $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{q}$ , then  $\frac{\nu_f^{\mathfrak{m}}(q)}{q} < \frac{1}{q}$ , so  $\nu_f^{\mathfrak{m}}(q) = 0$  and  $f \in \mathfrak{m}^{[q]}$ .  $\square$

**Lemma 3.3** ([26]Tag 05CK). *Let  $A \rightarrow B$  be a faithfully flat map of rings and  $I \subseteq A$  an ideal. Then  $IB \cap A = I$ .*

**Lemma 3.4.** *Let  $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be a flat local map of regular local rings. Suppose  $\mathfrak{a}, \mathfrak{b} \subseteq R$  are ideals with  $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$ . Then  $\text{ft}^{\mathfrak{b}}(\mathfrak{a}) = \text{ft}^{\mathfrak{b}S}(\mathfrak{a}S)$  and  $\text{ord}_{\mathfrak{b}}(\mathfrak{a}) = \text{ord}_{\mathfrak{b}S}(\mathfrak{a}S)$ .*

*Proof.* The first claim follows from [19, Proposition 2.2 (v)]. For the second, let  $t > 0$ . Lemma 3.3 implies

$$\mathfrak{a} \subseteq \mathfrak{b}^t \implies \mathfrak{a}S \subseteq \mathfrak{b}^t \implies \mathfrak{a}S \cap R \subseteq \mathfrak{b}^t \cap R \iff \mathfrak{a} \subseteq \mathfrak{b}^t,$$

so  $\text{ord}_{\mathfrak{b}}(\mathfrak{a}) = \text{ord}_{\mathfrak{b}S}(\mathfrak{a}S)$ .  $\square$

**Lemma 3.5.** *Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $R = k[[z, w]]$  and set  $\mathfrak{m} = (z, w)$ . If  $f \in \mathfrak{m}^d \setminus \mathfrak{m}^{d+1}$  and  $\text{fpt}(f) = \frac{1}{d}$ , then there exists:*

- A regular system of parameters  $x, y$  for  $R$ ;
- A degree- $d$  monic polynomial  $P$  in  $k[[x]][y]$ ;
- A unit  $u$  in  $R^\times$

such that  $f = uP$ .

*Proof.* Write  $f = f_d + f_{>d}$ , where  $f_d$  is homogeneous of degree  $d$  and  $f_{>d} \in \mathfrak{m}^{d+1}$ . By Proposition 2.7 (7), we have  $\text{fpt}(f_d) \leq \text{fpt}(f)$ . As the lower bound  $\frac{1}{d} = \frac{1}{\text{ord}_{\mathfrak{m}}(f_d)} \leq \text{fpt}(f_d)$  of Proposition 2.7 (3) still applies, we have  $\text{fpt}(f_d) = \frac{1}{d}$ . By [20, Remark 3.2], we have  $f_d = \ell^d$  for some homogeneous linear form  $\ell$ . Choosing a new regular system of parameters  $x, y$  for  $R$  such that  $\ell = y$ , we have  $f = y^d + f_{>d}$ .

Write  $f = a_0 + a_1y + \dots$  as a power series in  $k[[x]][[y]]$ . As  $f - y^d \in \mathfrak{m}^{d+1}$ , it follows that  $a_0, \dots, a_{d-1} \in \mathfrak{m}$  and  $a_d \equiv 1 \pmod{\mathfrak{m}}$ . The claim now follows from the Weierstrass preparation theorem [21, Theorems IV.9.1, IV.9.2].  $\square$

**Lemma 3.6.** *Let  $L$  be a field. Let  $f$  be an element of  $L[[x]]$  with  $x \nmid f$ . Let  $u, s$  be positive integers such that  $s$  is not a multiple of  $\text{char } k$ . If  $g \in L[[x^{1/u}]]$  such that  $g^s = f$ , then  $g \in L[[x]]$ .*

*Proof.* Write  $g = a_0 + a_1x^{1/u} + \dots \in L[[x^{1/u}]]$ . As  $x \nmid f$ , we have  $a_0 \neq 0$ . If  $a_i = 0$  for all  $u \nmid i$ , then  $g \in L[[x]]$ . Otherwise, for the sake of contradiction, let  $i$  be minimal such that  $u \nmid i$  and  $a_i \neq 0$ . By equating the coefficients of  $x^{i/u}$  in  $g^s$  and in  $f$ , we obtain

$$0 = sa_0^{s-1}a_i.$$

We assumed that  $s$  is not a multiple of  $\text{char } k$ , so  $s$  is a unit in  $k$ , which contradicts the assumption that  $a_0a_i \neq 0$ .  $\square$

**Lemma 3.7.** *Let  $k$  be a field and  $T = k[[t]]$ . Let  $\theta_1, \dots, \theta_d$  be elements of  $T$  and consider  $f = (y - \theta_1) \dots (y - \theta_d)$  as an element of  $T[y]$ . Let  $\ell \geq 1$ . If  $f \in (y, t^\ell)^d$ , then  $\text{ord}_t(\theta_i) \geq \ell$  for all  $1 \leq i \leq d$ .*

*Proof.* Write  $f = y^d + g_1y^{d-1} + \dots + g_d$  where  $g_1, \dots, g_d \in T$ . Re-order the roots  $\theta_1, \dots, \theta_d$  such that

$$\text{ord}_t(\theta_1) = \dots = \text{ord}_t(\theta_r) < \text{ord}_t(\theta_{r+1}) \leq \dots \leq \text{ord}_t(\theta_d).$$

As  $f \in (y, t^\ell)^d$ , we have  $y^{d-r}g_r \in (y, t^\ell)^d$  and hence  $g_r \in (y, t^\ell)^{d-r} \cap T = t^{\ell r}T$ . We have  $g_r = \theta_1 \dots \theta_r + \text{higher order terms}$ , so  $\text{ord}_t(g_r) = \text{ord}_t(\theta_1 \dots \theta_r) = r \text{ord}_t(\theta_1)$ , so for all  $1 \leq i \leq d$  we have  $\text{ord}_t(\theta_i) \geq \text{ord}_t(\theta_1) \geq \frac{\ell r}{r} = \ell$ .  $\square$

**Lemma 3.8.** *Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $R = k[[x, y]]$ ,  $\mathfrak{m} = (x, y)$ . Let  $d$  be a positive integer and  $f$  an element of*

$R$  such that  $\text{ord}_{\mathfrak{m}}(f) = d, \text{fpt}(f) = \frac{1}{d}$ . Write  $d = qs$ , where  $q$  is a power of  $p$  and  $\gcd(p, s) = 1$ . Then there exists  $g$  in  $\mathfrak{m}^{[q]}$  such that  $g^s = f$ .

*Proof.* By Lemma 3.5, we may assume  $f$  is a monic polynomial of degree  $d$  in  $k[[x]][y]$ . Set  $S = k[[x]]$  and  $K = \text{Frac}(S)$ . Every monic polynomial in  $S[y]$  factors completely over  $\overline{K}$ , so we may write

$$f = (y - \theta_1)^{e_1} \dots (y - \theta_r)^{e_r}$$

for some  $\theta_1, \dots, \theta_r$  distinct roots in  $\overline{K}$ ; unlike the previous lemma, we will count roots with multiplicity.

Let  $T$  denote the integral closure of  $S$  in  $K(\theta_1, \dots, \theta_r)$ . By [18, Theorem 4.3.4],  $T$  is a complete local domain and  $S \rightarrow T$  is finite. As  $T$  is a 1-dimensional complete normal ring of equal characteristic, the Cohen structure theorem gives  $T \cong k'[[t]]$  for some field extension  $k'/k$ . Furthermore, the restriction of  $S \rightarrow T$  to  $k \rightarrow k'$  is finite and  $k$  is algebraically closed, so  $k = k'$ .

We define  $\ell := \text{ord}_t(xT)$ , and note that by Lemma 3.7 that  $t^\ell \mid \theta_i$  for all  $1 \leq i \leq r$ . Set  $U := T[[y]] = k[[y, t]]$ . As  $R \hookrightarrow U$  is a finite local map between regular local rings of the same dimension,  $R \rightarrow U$  is flat by the miracle flatness lemma [26, Tag 00R4]. Set  $\mathfrak{b} = \mathfrak{m}U = (y, t^\ell)U$ . As  $R \rightarrow U$  is flat, we have  $\text{fpt}(f) = \text{ft}^{\mathfrak{m}}(f) = \text{ft}^{\mathfrak{b}}(f) = \frac{1}{d}$  by Lemma 3.4. Write  $d = qs$ , where  $s$  is coprime to  $p$ . We aim to show that there exists  $g$  in  $U$  such that  $g^s = f$ . If  $r = 1$  this is clear; we simply have  $f = h_1^{qs}$  and  $g = h_1^q$ . Otherwise, suppose  $r \geq 2$ .

Let  $\mathfrak{n} = (y, t) \subseteq U$ . For all  $1 \leq i \leq r$ , we have  $y - \theta_i \in \mathfrak{n} \setminus \mathfrak{n}^2$ , so  $h_i := y - \theta_i$  is irreducible. As argued in [23, Corollary 5.2], the pair  $(U, (h_1 \dots h_r)^1)$  is sharply  $F$ -pure away from  $\mathfrak{n}$ , so by [23, Theorem 5.1] there exists  $D$  in  $\mathbb{Z}^+$  such that for all  $t_1, \dots, t_r$  in  $[0, 1)$  and  $\tilde{h}, \dots, \tilde{h}_r$  such that  $h_i \equiv \tilde{h}_i \pmod{\mathfrak{n}^D}$ , we have

$$(8) \quad \tau(U, h_1^{t_1} \dots h_r^{t_r}) = \tau(U, \tilde{h}^{t_1} \dots \tilde{h}_r^{t_r}).$$

Without loss of generality, choose  $D$  to be larger than  $\text{ord}_{\mathfrak{n}}(\theta_i - \theta_j)$  for all  $1 \leq i < j \leq r$ ; in particular this value of  $D$  satisfies  $D \geq \ell$ . For  $1 \leq i \leq r$ , let  $\tilde{\theta}_i$  in  $k[[t]]$  denote the truncation of the power series  $\theta_i$  at the  $D$ th term; we have  $\theta_i \equiv \tilde{\theta}_i \pmod{\mathfrak{n}^D}$  and  $t^\ell \mid \tilde{\theta}_i$ . Set  $\tilde{h}_i = y - \tilde{\theta}_i$  and  $\tilde{f} = \tilde{h}_1^{e_1} \dots \tilde{h}_r^{e_r}$ . By assumption that  $r \geq 2$  we have  $e_i \leq d - 1$  for all  $1 \leq i \leq r$ , hence  $ce_i \in [0, 1)$  for all  $c$  in  $[0, \frac{1}{d}]$ . By Equation (8), for all  $c \in [0, \frac{1}{d}]$ , we have

$$\tau(R, f^c) = \tau(R, h_1^{ce_1} \dots h_r^{ce_r}) = \tau(R, \tilde{h}_1^{ce_1} \dots \tilde{h}_r^{ce_r}) = \tau(R, \tilde{f}^c).$$

As  $\text{ft}^{\mathfrak{b}}(f) = \frac{1}{d}$ , it follows that  $\tau(R, \tilde{f}^c) \not\subseteq \mathfrak{b}$  for all  $0 \leq c < \frac{1}{d}$  and  $\tau(R, \tilde{f}^{1/d}) \subseteq \mathfrak{b}$ , so  $\text{ft}^{\mathfrak{b}}(\tilde{f}) = \frac{1}{d}$ .

By our choice of  $D$ , the factors  $\tilde{h}_1, \dots, \tilde{h}_r$  are distinct. In the notation of Definition 2.11, write  $\tilde{f} = \tilde{\mathbf{h}}^{\mathbf{e}} = \tilde{h}_1^{e_1} \dots \tilde{h}_r^{e_r}$  with  $e_i > 0$ . As  $\text{ft}^{\mathfrak{b}}(\tilde{f}) = \frac{1}{\|\mathbf{e}\|} < \frac{1+\ell/D}{\|\mathbf{e}\|}$ , by Corollary 2.19, there exists a critical point  $\mathbf{c} = \frac{\mathbf{a}}{q_0} \leq \frac{1+\ell/D}{\|\mathbf{e}\|} \mathbf{e}$  such

that  $\text{ft}^b(\tilde{f}) = \max_{1 \leq i \leq r} \frac{c_i}{e_i}$ . We then have

$$(9) \quad 1 = \|\mathbf{e}\| \cdot \text{ft}^b(\tilde{f}) = \sum_{i=1}^r e_i \max\left(\frac{c_1}{e_1}, \dots, \frac{c_r}{e_r}\right) \geq \sum_{i=1}^r c_i = \|\mathbf{c}\|.$$

By Lemma 2.17, we have  $\|\mathbf{c}\| \geq 1$ , so the inequality in Equation (9) is an equality. In particular,  $c_i = e_i \max(\frac{c_1}{e_1}, \dots, \frac{c_r}{e_r})$  for all  $1 \leq i \leq r$ , so  $\frac{c_i}{e_i} = \max(\frac{c_1}{e_1}, \dots, \frac{c_r}{e_r})$  for all  $1 \leq i \leq r$ , hence  $\mathbf{c} = \text{ft}^b(\tilde{f})\mathbf{e}$ . For all  $1 \leq i \leq r$ , we conclude that  $\frac{c_i}{d} = \frac{a_i}{q_0}$ . Recall that  $d = qs$ , where  $s$  is coprime to  $p$ . Then  $e_i q_0 = a_i s q$ , so we conclude that  $s \mid e_i$ . Consequently,  $f$  has an  $s$ th root  $g$  in  $U$ : similar to the  $r = 1$  case, we take  $g = h_1^{e_1/s} \dots h_r^{e_r/s}$ .

Recall that  $x = ut^\ell$  for some unit  $u$  in  $U^\times$ . As  $U$  is a power series ring over the algebraically closed field  $k$ , we have  $u^{1/\ell} \in U$ . Consequently,  $x^{1/\ell} = tu^{1/\ell}$  is conjugate to  $t$ , hence  $U = k[[y, x^{1/\ell}]]$ . Set  $L = \text{Frac}(k[[y]])$  and consider  $f$  as an element of  $L[[x]]$ . As  $g \in U \subseteq L[[x^{1/\ell}]]$ , it follows from Lemma 3.6 that  $g \in L[[x]] \subseteq \text{Frac}(R)$ . As  $R$  is integrally closed and  $g$  is integral over  $R$ , it follows that  $g \in R$ .

As  $f = g^s$ , by Proposition 2.7 (2) we have  $\text{ft}^m(f) = \frac{1}{s} \text{ft}^m(g)$ . Additionally, we have  $\text{ord}_m(g) = \frac{1}{s} \text{ord}_m(f)$ , so  $\text{ft}^m(g) = \frac{1}{q} = \frac{1}{\text{ord}_m(g)}$ . By Lemma 3.2, we conclude that  $g \in \mathfrak{m}^{[q]}$ , proving the claim.  $\square$

**Remark 3.9.** Using Schwede's results on centers of  $F$ -purity, one can give a short proof of Lemma 3.8 in the special case that  $d$  is coprime to  $p$ . Assume that  $\text{fpt}(f) = \frac{1}{d}$  and that  $d$  is coprime to  $p$ ; then  $(p^e - 1)\frac{1}{d} \in \mathbb{Z}^+$  for some  $e > 0$ . By [14, Theorem 4.9], the pair  $(R, f^{1/d})$  is sharply  $F$ -pure but not strongly  $F$ -regular. Let  $\mathfrak{p}$  be a center of sharp  $F$ -purity for  $(R, f^{1/d})$  as in [24], which is a nonzero prime ideal by [24, Proposition 4.6]. We conclude  $f^{\frac{p^e-1}{d}} \in (\mathfrak{p}^{[p^e]} : \mathfrak{p})$  by [24, Propositions 3.11 and 4.7].

If  $\text{ht}(\mathfrak{p}) = 1$ , then we have  $\mathfrak{p} = (g)$  for some irreducible element  $g$  in  $R$ . In this case, we have  $f^{\frac{p^e-1}{d}} \in g^{p^e-1}$ . As  $p^e - 1 = \text{ord}_m(f^{\frac{p^e-1}{d}}) \geq (p^e - 1) \text{ord}_m(g)$ , we must have  $\text{ord}_m(g) = 1$ , so  $q = 1$  and  $g \in \mathfrak{m}^{[q]}$ . By unique factorization of  $f$  we have that  $f = ug^d$  for some unit  $u$  in  $R$ . As  $k$  is algebraically closed,  $u$  admits a  $d$ th root in  $R$ , proving the claim in this case. To finish the proof, we'll show that  $\text{ht}(\mathfrak{p}) \neq 2$ . Suppose for the sake of contradiction that  $\mathfrak{p} = \mathfrak{m}$ . Then  $f^{\frac{p^e-1}{d}} \in (x_1^{p^e}, x_2^{p^e}, (x_1 x_2)^{p^e-1}) \subseteq \mathfrak{m}^{p^e}$ . But this contradicts the fact that  $\text{ord}_m(f^{\frac{p^e-1}{d}}) = p^e - 1$ , proving the claim.

**3.2. Step (2).** For this step, we require a local Bertini theorem due to Flenner [9].

**Lemma 3.10** ([9], Satz 2.1). *Let  $(A, \mathfrak{m})$  be a complete local ring with coefficient field  $k$ . Let  $\mathfrak{J} = (z_1, \dots, z_l) \subseteq A$  is a proper ideal. Suppose that  $Q = \{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$  is a finite set of prime ideals of  $D(\mathfrak{J})$ . Then there exists  $x$  in  $\mathfrak{J}$  such that:*

- (i)  $x \equiv z_1 \pmod{(z_2, \dots, z_l) + \mathfrak{m}\mathfrak{J}}$ ;
- (ii)  $x \notin \mathfrak{q}_1 \cup \dots \cup \mathfrak{q}_r$ ;
- (iii)  $x \notin \mathfrak{p}^{(2)}$  for all  $x \in D(\mathfrak{J})$ .

*Proof.* While Flenner does not explicitly note the congruence condition on  $x$ , we can specify one detail of the construction to ensure  $x \equiv z_1 \pmod{(z_2, \dots, z_l) + \mathfrak{m}\mathfrak{J}}$ . In Flenner's notation, we assume  $y_1, \dots, y_m$  is a generating set for  $\mathfrak{m}$ . We then let  $x_1, \dots, x_n$  denote the set of elements  $z_1, \dots, z_l, z_1 y_1, \dots, z_l y_m$  with  $x_i = z_i$  for  $1 \leq i \leq l$ . With  $S \subseteq A$  as in loc. cit., there exist  $a_2, \dots, a_n$  in  $S$  such that  $x := x_1 + a_2 x_2 + \dots + a_n x_n$  satisfies (ii) and (iii). Noting that  $x \equiv z_1 + a_2 z_2 + \dots + a_l z_l \pmod{\mathfrak{m}\mathfrak{J}}$ , the claim (i) follows.  $\square$

The following lemma is essentially an application of [9, Korollar 3.5], where we replace Satz 2.1 in op. cit. with our modification Lemma 3.10. Although Flenner's argument works for us *mutatis mutandis*, because the work op. cit. is written in German, we spell out the changes explicitly for the reader's convenience.

**Lemma 3.11.** *Let  $k$  be an infinite field. Let  $n \geq 3$ ,  $R = k[[x_1, \dots, x_n]]$ ,  $\mathfrak{m} = (x_1, \dots, x_n)$ . Let  $f$  be an element of  $R$  with  $\text{ord}_{\mathfrak{m}}(f) = d > 0$ . Let  $s > 0$  and suppose that  $f$  does not have an  $s$ th root in  $R$ . There exists a  $\mathfrak{m} \setminus \mathfrak{m}^2$  such that the image  $\bar{f}$  of  $f$  in  $R/aR$  does not have an  $s$ th root in  $R/aR$  and  $\text{ord}_{\mathfrak{m}/(x)}(\bar{f}) = d$ .*

*Proof.* Recall that a regular local ring is a unique factorization domain (UFD) [1]. Factor  $f$  as  $f = \pi_1^{e_1} \dots \pi_l^{e_l}$ , where  $\pi_1, \dots, \pi_l$  are distinct irreducible factors of  $f$ . By assumption that  $f$  does not have an  $s$ th root in  $R$ , the multiplicities  $e_i$  are not all divisible by  $s$ . Set  $g = \pi_1 \dots \pi_l$ . Write  $f = f_d + f_{>d}$ , where  $f_d$  is a homogeneous polynomial of degree  $d$  and  $f_{>d} \in \mathfrak{m}^{d+1}$ . Let  $Q \subseteq \text{Spec } R$  denote the set of primes  $\mathfrak{q}$  such that either  $\mathfrak{q}$  is minimal over  $(f_d, g)$  or such that  $\text{ht}(\mathfrak{q}) < n$  and the image of  $\mathfrak{q}$  in  $R/gR$  is a minimal element of  $\text{Sing}(R/gR)$ . As  $\dim R = n \geq 3$  and each of the minimal primes  $\mathfrak{q}$  over  $(f_d, g)$  has  $\text{ht}(\mathfrak{q}) \leq 2$ , we conclude that  $\mathfrak{m} \notin Q$ .

By assumption that  $k$  is infinite<sup>1</sup>, there exists a homogeneous linear form in  $R$  which is not a factor of  $f_d$ . Choose homogeneous coordinates  $z_1, \dots, z_n$  for  $R$  such that  $z_1 \nmid f_d$ . Let  $\mathfrak{J} = (z_1, z_2^{d+1}, \dots, z_n^{d+1})$ . As  $\sqrt{\mathfrak{J}} = \mathfrak{m}$ , we have that  $Q \subseteq D(\mathfrak{J})$ . Apply Lemma 3.10 to  $R, \mathfrak{J}$  and  $Q$  to produce  $x$  in  $\mathfrak{J}$  such that  $x \equiv z_1 \pmod{(z_2^{d+1}, \dots, z_n^{d+1}) + \mathfrak{m}\mathfrak{J}}$ ,  $x \notin \bigcup_{\mathfrak{q} \in Q} \mathfrak{q}$ , and  $x \notin \mathfrak{p}^{(2)}$  for all  $\mathfrak{p} \neq \mathfrak{m}$ . Then  $x \in \mathfrak{m} \setminus \mathfrak{m}^2$  by construction.

We study the image  $\bar{f}$  of  $f$  in  $R/xR$ , which we show has  $\text{ord}_{\mathfrak{m}/(x)}(\bar{f}) = d$ . To see this, it suffices to show that  $f \notin xR + \mathfrak{m}^{d+1}$ . As  $x \equiv z_1 \pmod{(z_2^{d+1}, \dots, z_n^{d+1}) + z_1 \mathfrak{m}}$ , there exists  $y$  in  $\mathfrak{m}$  such that  $x \equiv z_1 + y z_1 \pmod{\mathfrak{m}^{d+1}}$ . As  $z_1, f_d$  are homogeneous polynomials and  $z_1 \nmid f_d$ , we have  $f_d \notin z_1 R + \mathfrak{m}^{d+1} = xR + \mathfrak{m}^{d+1}$ , so  $f \equiv f_d \not\equiv 0 \pmod{xR + \mathfrak{m}^{d+1}}$  and  $\text{ord}_{\mathfrak{m}/(x)}(f) = d$ .

<sup>1</sup>It suffices to have  $\#\mathbb{P}^{n-1}(k) > d$ .

Since  $R/gR$  is reduced, the argument of [9, Korollar 3.5] allows us to conclude that the ring  $R/(g, x)R$  is reduced. In particular, the image of  $g$  is squarefree in  $R/xR$ , so we may factor the image of each  $\pi_i$  in  $R/xR$  as  $\bar{\pi}_i = \rho_{i1} \dots \rho_{is_i}$  where the  $\rho_{ij}$  are irreducible and pairwise distinct. It follows that we may factor  $\bar{f}$  as  $\prod_{i=1}^l \prod_{j=1}^{s_i} \rho_{ij}^{e_i}$ . By assumption, the multiplicities  $e_i$  are not all divisible by  $s$ . As  $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ , the ring  $R/xR$  is regular, hence a UFD, so  $\bar{f}$  does not have an  $s$ th root in  $R/xR$ .  $\square$

**Lemma 3.12.** *Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $n \geq 1$  and let  $R = k[[x_1, \dots, x_n]]$ . Let  $f$  be an element of  $R$  with  $\text{ord}_{\mathfrak{m}}(f) = d$ . Write  $d = qs$ , where  $q = p^e$  and  $s$  is coprime to  $p$ . If  $\text{fpt}(f) = \frac{1}{d}$ , then there exists  $g$  in  $\mathfrak{m}^{[q]}$  such that  $f = g^s$ .*

*Proof.* If  $n = 1$ , then  $f = ux_1^d$  for some  $u$  in  $R^\times$ . As  $k = \bar{k}$ , every unit in  $R$  has a  $d$ th root in  $R$ , so we have  $f = (u^{1/d}x_1)^d$  and  $g = u^{q/d}x_1^q$ . The  $n = 2$  case is Lemma 3.8; we now suppose the claim holds in dimension  $n - 1$  and let  $n \geq 3$ . Let  $f$  be an element of  $R$  with  $\text{ord}_{\mathfrak{m}}(f) = d$ , and write  $d = qs$  where  $q = p^e$  and  $\gcd(s, p) = 1$ . If  $f$  does not have an  $s$ th root in  $R$ , then by Lemma 3.11 there exists  $x$  in  $\mathfrak{m} \setminus \mathfrak{m}^2$  such that, writing  $\bar{f}$  for the image of  $f$  in  $R/xR$ , the element  $\bar{f}$  does not have an  $s$ th root in  $R/xR$  and  $\text{ord}_{\mathfrak{m}/(x)}(\bar{f}) = d$ . By the classification in dimension  $n - 1$ , we have  $\text{ft}^{\mathfrak{m}/(x)}(\bar{f}) > \frac{1}{d}$ . By Proposition 2.7 (5), we have  $\text{ft}^{\mathfrak{m}}(f) \geq \text{ft}^{\mathfrak{m}/(x)}(\bar{f}) > \frac{1}{d}$ .

Suppose now that  $\text{fpt}(f) = \frac{1}{d}$ . By the previous paragraph,  $f$  admits an  $s$ th root  $g$  in  $R$ . Writing  $f = g^s$ , by Proposition 2.7 (2) and the fact that  $\text{ord}_{\mathfrak{m}}$  is a valuation, we have

$$\text{ft}^{\mathfrak{m}}(g) = s \text{ft}^{\mathfrak{m}}(f) = \frac{1}{q} = \frac{1}{\text{ord}_{\mathfrak{m}}(g)}.$$

By Lemma 3.2, we conclude that  $g \in \mathfrak{m}^{[q]}$ , proving the claim.  $\square$

**3.3. Step (3).** Over an arbitrary field  $k$ , an element  $f$  in  $k[[x_1, \dots, x_n]]$  with  $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{\text{ord}_{\mathfrak{m}}(f)}$  may not have an  $s$ th root at all.

**Example 3.13.** Let  $R = k[[x]]$ . Let  $q = p^e$  and  $s > 0$  such that  $\gcd(s, p) = 1$  and write  $d = qs$ . If  $a \in k^\times$  such that  $a$  does not have a  $s$ th root in  $k$ , then  $\text{ft}^{\mathfrak{m}}(ax^d) = \text{ft}^{\mathfrak{m}}(x^d) = \frac{1}{d}$ , but  $ax^d$  does not have an  $s$ th root in  $R$ .

The following lemma shows that Example 3.13 is as pathological as can possibly occur in a power series ring.

**Lemma 3.14.** *Let  $k$  be a field,  $R = k[[x_1, \dots, x_n]]$ , and set  $S = R \hat{\otimes}_k \bar{k}$ . Let  $f$  be an element of  $R$  and  $s$  in  $\mathbb{Z}^+$  such that  $\text{char}(k)$  does not divide  $s$ . If there exists  $g$  in  $S$  such that  $g^s = f$ , then there exists a unit  $u$  in  $k^\times$  and an element  $h$  in  $R$  such that  $f = uh^s$ .*

*Proof.* Let  $\Gamma$  denote the set of monomials in  $S$ . Let  $\succ$  be a local monomial order on  $S$  such that  $(\Gamma, \succ)$  is order-isomorphic to  $(\mathbb{N}, >)$ ; for example, we may take  $\succ$  such that  $x^I \succ x^J \iff x^I <_{\text{deglex}} x^J$ . Write

$g = \sum_{i=0}^N a_i x^{I_i}$  where the  $a_i$  are nonzero and  $I_1 < I_2 < \dots$ . We allow  $N$  to be a positive integer or infinity; the proof is identical in both cases. As  $a_1^s (x_1^{I_1})^s = \text{in}_>(g^s) = \text{in}_>(f) \in R$ , it follows that  $a_1^s \in k$ . Set  $u = a_1^{-s}$  and  $h = a_1^{-1}g$ ; we have  $f = uh^s$  and  $h^s \in R$ . To prove the claim, we'll show that  $h \in R$ .

For ease of notation, write  $b_i = \frac{a_i}{a_1}, i \geq 2$  so that  $h = x^{I_1} + b_2 x^{I_2} + \dots$ . Suppose  $j \geq 2$  such that  $b_2, \dots, b_{j-1} \in k$  and consider the coefficient  $c_j$  of  $x^{(s-1)I_1 + I_j}$  in  $h^s$ . A priori, we have

$$(10) \quad c_j = \sum_{\substack{\ell_1, \dots, \ell_s \\ I_{\ell_1} + \dots + I_{\ell_s} = (s-1)I_1 + I_j}} b_{\ell_1} \dots b_{\ell_s}.$$

If  $\ell_1 \leq \dots \leq \ell_s \in \mathbb{N}$  such that  $I_{\ell_1} + \dots + I_{\ell_s} = (s-1)I_1 + I_j$ , then we have  $(s-1)I_1 + I_{\ell_s} \leq (s-1)I_1 + I_j$ , so  $\ell_s \leq j$  with equality if and only if  $\ell_1 = \dots = \ell_{s-1} = 1$ . Consequently, we may refine Equation (10):

$$(11) \quad c_j = sb_j + \sum_{\substack{\ell_1, \dots, \ell_s < j \\ I_{\ell_1} + \dots + I_{\ell_s} = (s-1)I_1 + I_j}} b_{\ell_1} \dots b_{\ell_s}.$$

As  $h^s \in R$ , the coefficient  $c_j$  is in  $k$ . By assumption that  $b_1, \dots, b_{j-1} \in k$ , the second term of Equation (11) is an element of  $k$ , so  $sb_j \in k$ . As  $s$  is a unit in  $k$ , we conclude that  $b_j \in k$ .  $\square$

**Lemma 3.15.** *Let  $k$  be a field of characteristic  $p > 0$ ,  $R = k[[x_1, \dots, x_n]]$ , and  $\mathfrak{m} = (x_1, \dots, x_n)$ . Let  $f$  be an element of  $R$  such that  $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{d}$ . If  $q$  is the largest power of  $p$  dividing  $d$  and  $d = qs$ , then there exists a unit  $u$  in  $R^\times$  and an element  $g$  in  $\mathfrak{m}^{[q]}$  such that  $f = ug^s$ .*

*Proof.* Let  $S = R \widehat{\otimes}_k \bar{k}$  and consider the map  $R \rightarrow S$ . We have that  $R \rightarrow S$  is faithfully flat,  $\mathfrak{n} = \mathfrak{m}S$  is the maximal ideal of  $S$ , and  $S = \bar{k}[[x_1, \dots, x_n]]$ . By Lemma 3.4, we have  $\text{ft}^{\mathfrak{m}}(f) = \text{ft}^{\mathfrak{n}}(f)$  and  $\text{ord}_{\mathfrak{m}}(f) = \text{ord}_{\mathfrak{n}}(f)$ . As  $\text{ft}^{\mathfrak{n}}(f) = \frac{1}{d} = \frac{1}{\text{ord}_{\mathfrak{n}}(f)}$ , by Lemma 3.12 there exists  $g$  in  $\mathfrak{m}^{[q]}$  such that  $f = g^s$ . By Lemma 3.14, there exist  $h$  in  $R$ ,  $u$  in  $R^\times$  such that  $uh^s = f$ , and  $h \in \mathfrak{m}^{[q]}$  by Lemma 3.2.  $\square$

#### 3.4. Step (4).

**Lemma 3.16.** *Let  $k$  be a field of characteristic  $p > 0$ ,  $R = k[[x_1, \dots, x_n]]$ , and  $\mathfrak{m} = (x_1, \dots, x_n)$ . Let  $\mathfrak{a} \subseteq R$  such that  $\text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$ . If  $q$  is the largest power of  $p$  dividing  $d$  and  $d = qs$ , then there exists  $g$  in  $\mathfrak{m}^{[q]}$  such that  $g^s R = \mathfrak{a}$ .*

*Proof.* Let  $f$  be an element of  $\mathfrak{a}$  such that  $\text{ord}_{\mathfrak{m}}(f) = d$ . By Proposition 2.7 (1), (3) we have  $\frac{1}{d} \leq \text{ft}^{\mathfrak{m}}(f) \leq \text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$ . By Lemma 3.15, there exists  $u$  in  $R^\times$ ,  $g$  in  $\mathfrak{m}^{[q]}$  such that  $f = ug^s$ . We will show that  $g^s$  generates  $\mathfrak{a}$ .

Let  $e > 0$  such that  $s \mid (p^e - 1)$ ; write  $t_e = \frac{p^e - 1}{s}$ . By Proposition 2.7 (4), for all  $e > 0$  we have  $\nu_{\mathfrak{a}}^{\mathfrak{m}}(qp^e) < qp^e \text{ft}^{\mathfrak{m}}(\mathfrak{a}) \leq \lceil qp^e / qs \rceil = t_e + 1$ , so  $\mathfrak{a}^{t_e + 1} \subseteq \mathfrak{m}^{[p^e]}$  for all  $e > 0$ . Let  $z$  be an arbitrary element of  $\mathfrak{a}$  and let  $r$  be a positive integer.

As  $\mathfrak{a}^{t_e+1} \subseteq \mathfrak{m}^{[qp^e]}$ , we in particular have  $(g^s)^{t_e-r} z^{r+1} \in \mathfrak{m}^{[qp^e]}$ . Write  $g = a_1 x_1^q + \dots + a_n x_n^q$ . As  $\text{ord}_{\mathfrak{m}}(g) = q$ , there is some  $1 \leq i \leq n$  such that  $a_i \in R^\times$ . As  $g \equiv a_i x_i^q \pmod{(x_1^q, \dots, x_{i-1}^q, x_{i+1}^q, \dots, x_n^q)}$ , it follows that  $\mathfrak{m}^{[q]} = (x_1^q, \dots, x_{i-1}^q, g, x_{i+1}^q, \dots, x_n^q)$ . In particular,  $x_1, \dots, x_{i-1}, g, x_{i+1}, \dots, x_n$  is a system of parameters for  $R$ . By [20, Lemma 3.5], we have

$$(12) \quad \begin{aligned} z^{r+1} \in (\mathfrak{m}^{[qp^e]} : (g^s)^{t_e-r}) &= \left( (x_1^{qp^e}, \dots, g^{p^e}, \dots, x_n^{qp^e}) : g^{p^e-1-sr} \right) \\ &= (x_1^{qp^e}, \dots, g^{1+sr}, \dots, x_n^{qp^e}). \end{aligned}$$

Applying Equation (12) and letting  $e \rightarrow \infty$ , we obtain  $z^{r+1} \in g^{1+sr} R \subseteq g^{sr} R$  for all  $r > 0$ . By [18, Corollary 6.8.11],  $z$  is contained in the integral closure of the ideal  $g^s R$ , which by [18, Proposition 1.5.2] is equal to  $g^s R$  itself.  $\square$

### 3.5. Step (5).

**Lemma 3.17.** *Let  $(A, \mathfrak{m})$  be a regular local ring and  $I \subseteq A$  an ideal such that  $I\hat{A}$  is principal. Then  $I$  is principal.*

*Proof.* Let  $M$  be an  $A$ -module. Since  $I\hat{A}$  is principal,  $I\hat{A}$  is flat, so  $\text{Tor}_1^{\hat{A}}(I\hat{A}, M \otimes_A \hat{A}) = 0$ . Consequently, by [26, Tag 00M8] we have

$$0 = \text{Tor}_1^{\hat{A}}(I\hat{A}, M \otimes_A \hat{A}) = \text{Tor}_1^{\hat{A}}(I \otimes_A \hat{A}, M \otimes_A \hat{A}) = \text{Tor}_1^A(I, M) \otimes_A \hat{A},$$

so by faithful flatness of  $A \rightarrow \hat{A}$  we conclude  $\text{Tor}_1^A(I, M) = 0$ . As  $M$  was arbitrary, we deduce that  $I$  is a flat  $A$ -module, hence  $I$  is a principal ideal.  $\square$

**Lemma 3.18.** *Let  $(A, \mathfrak{m})$  be a regular local ring and  $I \subseteq A$  an ideal such that  $I\hat{A}$  is principal and generated by an element  $g^s$  for some  $s$  in  $\mathbb{Z}^+$ ,  $g$  in  $\hat{A}$ . Suppose that for all prime elements  $\pi \in A$ , the formal fiber  $\hat{A} \otimes_A A_{(\pi)} / \pi A_{(\pi)}$  is reduced. Then there exists  $h$  in  $A$  such that  $I = h^s A$ .*

*Proof.* By Lemma 3.17, choose  $f$  in  $A$  such that  $I = fA$ . As  $I\hat{A} = f\hat{A} = g^s \hat{A}$ , it follows from Lemma 3.14 that  $f = ug^s$  for some  $u$  in  $\hat{A}^\times$ .

Recall again that  $A, \hat{A}$  are UFDs [1]. Factor  $f$  in  $A$  as  $f = v\pi_1^{d_1} \dots \pi_l^{d_l}$ , with  $v$  in  $A^\times$  and  $\pi_1, \dots, \pi_l$  pairwise coprime and irreducible. In  $\hat{A}$ , factor each  $\pi_i$  as  $\rho_{i1}^{e_{i1}} \dots \rho_{is_i}^{e_{is_i}}$  where each  $\rho_{ij}$  is irreducible and  $\rho_{ij}, \rho_{ij'}$  are coprime for  $j \neq j'$ . By assumption that the formal fiber  $\hat{A} \otimes_A A_{(\pi_i)} / \pi_i A_{(\pi_i)}$  is reduced for all  $1 \leq i \leq l$ , it follows that  $e_{ij} = 1$  for all  $1 \leq i \leq l, 1 \leq j \leq s_i$ . Moreover, as  $\rho_{ij}\hat{A} \cap A = \pi_i A$ , it follows that the  $\rho_{ij}, \rho_{i'j'}$  are coprime for all  $(i, j) \neq (i', j')$ . Writing

$$ug^s = f = v\pi_1^{d_1} \dots \pi_l^{d_l} = \prod_{i=1}^l \prod_{j=1}^{s_i} \rho_{ij}^{d_i},$$

we conclude that  $s \mid d_i$  for all  $1 \leq i \leq l$ . Setting  $h = \pi_1^{d_1/s} \dots \pi_l^{d_l/s}$ , we have  $I = h^s A$ .  $\square$

*Proof of Theorem 3.1.* Let  $(A, \mathfrak{m})$  be a regular local ring of characteristic  $p > 0$  and  $\mathfrak{a} \subseteq A$  an ideal with  $\text{ord}_{\mathfrak{m}}(\mathfrak{a}) = d > 0$ . Factor  $d$  as  $qs$ , where  $q = p^e$  and  $\gcd(p, s) = 1$ . Suppose  $\text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$ . By Lemma 3.4 we have

$$\text{ft}^{\mathfrak{m}\hat{A}}(\mathfrak{a}\hat{A}) = \text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d} = \frac{1}{\text{ord}_{\mathfrak{m}}(\mathfrak{a})} = \frac{1}{\text{ord}_{\mathfrak{n}}(\mathfrak{a}\hat{A})}.$$

By Lemma 3.16, there exists  $g$  in  $\mathfrak{m}^{[q]}\hat{A}$  such that  $g^s = \mathfrak{a}\hat{A}$ . If we additionally assume that for all prime elements  $\pi$  of  $A$  that the formal fiber  $\hat{A} \otimes_A A_{(\pi)}/\pi A_{(\pi)}$  is reduced, then by Lemma 3.18 there exists  $h$  in  $\mathfrak{m}^{[q]}$  such that  $\mathfrak{a} = h^s A$ .

Conversely, suppose that  $\mathfrak{a}\hat{A} = g^s \hat{A}$  for some  $g$  in  $\mathfrak{m}^{[q]}\hat{A}$ . By Lemmas 3.2 and 3.4 we have  $\text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$ . Similarly, if there exists  $h$  in  $\mathfrak{m}^{[q]}$  such that  $\mathfrak{a} = h^s A$ , then  $\text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$  by Lemma 3.2.  $\square$

Our hypothesis on the formal fibers is necessary by Example 3.20, which relies on the following lemma.

**Lemma 3.19.** *Let  $(T, \mathfrak{m})$  be a complete local domain of dimension at least 2 which satisfies Serre's condition  $S_2$ . Suppose  $T$  has a coefficient field  $k$ . Let  $x$  be a nonzero element of  $\mathfrak{m}$ . Then there exists a Noetherian local subring  $(A, \mathfrak{m} \cap A)$  such that  $x \in A$ ,  $\hat{A} = T$ , and  $x$  is a prime element of  $A$ .*

*Proof.* Let  $C = \{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$  denote the set of minimal primes over  $x$ . Let  $\Pi$  denote the prime subring of  $T$  – that is,  $\Pi = \mathbb{Q}$  if  $\text{char } k = 0$  and  $\Pi = \mathbb{F}_p$  if  $\text{char } k = p > 0$ . The claim follows once we check that this setup satisfies the hypotheses of [6, Theorem 1.1].

- (0) The primes  $\mathfrak{q}_1, \dots, \mathfrak{q}_r$  have height at most 1, and are thus nonmaximal. Moreover, we have  $x \in \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r$ .
- (1) As  $\text{Ass}(T) = \{(0)\}$ , we trivially have  $\mathfrak{p} \cap \Pi[x] = (0)$  for all  $\mathfrak{p}$  in  $\text{Ass}(T)$ .
- (2) As  $T$  is  $S_2$  and  $x$  is a regular element, it follows that  $T/xT$  is  $S_1$ . Consequently, the associated primes of  $T/xT$  are precisely the minimal primes of  $T/xT$ , so  $\text{Ass}(T/xT) = C$ .
- (3) For all  $1 \leq i \leq r$ , we claim that  $\text{Frac}(\Pi[x]) \cap \mathfrak{q}_i \subseteq xT$ . To see this, suppose that there exist  $d, e$  in  $\mathbb{Z}_{\geq 0}$ ,  $a_1, \dots, a_d, b_1, \dots, b_e$  in  $\Pi$ ,  $f$  in  $\mathfrak{q}_i$  such that

$$(13) \quad \frac{x^d + a_1 x^{d-1} + \dots + a_d}{x^e + b_1 x^{e-1} + \dots + b_e} = f.$$

Moreover, we suppose that the left-hand side of Equation (13) is written in reduced form. Then we have

$$a_d = x(x^{d-1} + a_1 x^{d-2} + \dots + a_{d-1}) - f(x^e + b_1 x^{e-1} + \dots + b_e) \in \mathfrak{m}.$$

As  $a_d \in \Pi \cap \mathfrak{m}$ , it follows that  $a_d = 0$ , so we have

$$b_e f = x(x^{d-1} + a_1 x^{d-2} + \dots + a_{d-1}) - f(x^{e-1} + b_1 x^{e-2} + \dots + b_{e-1}) \in xT.$$

As the left-hand side of Equation (13) is written in reduced form and  $x$  divides the numerator  $x^d + \cdots + xa_{d-1}$ , it follows that  $b_e \neq 0$ , so  $f \in xT$ .

□

As an application of the above lemma, we show that for all  $d > 0$ , there exist many regular local rings  $(A, \mathfrak{m})$  and prime elements  $f$  in  $A$  such that  $\text{ord}_{\mathfrak{m}}(f) = d$ ,  $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{d}$ .

**Example 3.20.** Let  $k$  be a field of characteristic  $p > 0$ ,  $T = k[[x_1, \dots, x_n]]$  and  $\mathfrak{m} = (x_1, \dots, x_n)$ . Fix some  $e \geq 0$ ,  $q = p^e$ , and  $s > 0$  such that  $\gcd(s, p) = 1$ . Choose  $g$  in  $\mathfrak{m}^{[q]}$  such that  $\text{ord}_{\mathfrak{m}}(g) = q$  and set  $f = g^s$ . By Lemma 3.2, we have  $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{s} \text{ft}^{\mathfrak{m}}(g) = \frac{1}{qs} = \frac{1}{\text{ord}_{\mathfrak{m}}(f)}$ .

By Lemma 3.19, there exists a Noetherian local subring  $(A, \mathfrak{m} \cap A)$  such that  $f \in A$ ,  $\widehat{A} = T$ , and such that  $f$  is a prime element of  $A$ . Let  $\mathfrak{n} := \mathfrak{m} \cap A$ . As argued in Lemma 3.15, we have  $\text{ft}^{\mathfrak{n}}(f) = \text{ft}^{\mathfrak{m}}(f)$  and  $\text{ord}_{\mathfrak{n}}(f) = \text{ord}_{\mathfrak{m}}(f)$ , so  $\text{ft}^{\mathfrak{n}}(f) = \frac{1}{\text{ord}_{\mathfrak{n}}(f)}$ .

To conclude this article, we consider the effect on the main theorem of adding an additional reducedness hypothesis.

**Example 3.21.** If  $R$  is a regular local ring,  $x, y$  part of a regular system of parameters for  $R$ , and  $a, b$  coprime integers, then  $\widehat{R}/(x^a - y^b)\widehat{R}$  is geometrically integral. To see this, writing  $\widehat{R} = k[[x, y, z_1, \dots, z_n]]$ , we have  $\widehat{R}/(x^a - y^b)\widehat{R} \cong k[[t^a, t^b, z_1, \dots, z_n]]$ .

Let  $(R, \mathfrak{m})$  be a regular local ring of characteristic  $p > 0$ . Let  $\mathfrak{a} \subseteq R$  such that  $\text{ord}_{\mathfrak{m}}(\mathfrak{a}) = d$ ,  $\text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$ , and  $\widehat{R}/\mathfrak{a}\widehat{R}$  is reduced. Factor  $d$  as  $d = qs$ , where  $q = p^e$  and  $\gcd(s, p) = 1$ . By Theorem 3.1, there exists  $g$  in  $\mathfrak{m}^{[q]}\widehat{R}$  such that  $\mathfrak{a}\widehat{R} = g^s\widehat{R}$ , so reducedness of  $\widehat{R}/g^s\widehat{R}$  forces  $s = 1$ . By Lemma 3.3 we also have  $\mathfrak{a} \subseteq \mathfrak{m}^{[q]}$ . For any  $q = p^e$ , if  $x, y$  is part of a regular system of parameters for  $R$ , then with  $f = x^q - y^{q+1}$ , we have  $\text{ord}_{\mathfrak{m}}(f) = q$  by construction,  $\text{ft}^{\mathfrak{m}}(f_q) = \frac{1}{q}$  by Lemma 3.2, and  $\widehat{R}/f_q\widehat{R}$  is *geometrically* reduced.

To show that  $\frac{1}{d}$  is an optimal lower bound on  $\text{ft}^{\mathfrak{m}}(\mathfrak{a})$  when  $\text{ord}_{\mathfrak{m}}(\mathfrak{a}) = d$  and  $\widehat{R}/\mathfrak{a}\widehat{R}$  is reduced (or geometrically reduced), suppose  $\dim R \geq 2$  and  $x, y$  is part of a regular system of parameters for  $R$ . Setting  $g_{d,t} = x^d - y^{td+1}$ , we have  $\text{ft}^{\mathfrak{m}}(g_{d,t}) \leq \text{ft}^{\mathfrak{m}}((x^d, y^{td+1})) \leq \frac{1}{d} + \frac{1}{td+1} \searrow \frac{1}{d}$ , where the inequality is by Proposition 2.7 (1) and the equality by [15, Proposition 36].

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