Classification of Minimal Singularity Thresholds

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April 6, 2025

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- $\mathrm{lct}(I)$ measures whether one can hope to apply the MMP to $(\mathrm{Spec}\, R, \mathrm{Spec}\, R/I)$.

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[MTW05, Theorem 3.3]

For an ideal $I \subseteq \mathbb{Z}[x_1, \ldots, x_n]$, we have

$$\operatorname{lct}(I \otimes_{\mathbb{Z}} \mathbb{C}) = \lim_{p \to \infty} \operatorname{fpt}(I \otimes_{\mathbb{Z}} \mathbb{F}_p).$$

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- $h_1, \dots, h_n$ general linear forms
- For $0 \leq j \leq n$, the mixed multiplicity $e_j(I)$ is given by the following.

$$e_j(I) = e\left(\frac{I + (h_{j+1}, \dots, h_n)}{(h_{j+1}, \dots, h_n)}\right).$$

- Can also be computed from the function length$(R/I^r\mathfrak{m}^s)$ for $r, s \in \mathbb{Z}^+$.

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[DP14, Theorem 1.2], equivalent restatement

- If $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is $(\underline{x})$-primary, then $\mathrm{DP}(I) \leq \mathrm{lct}(I)$.
- If $J = (x_1^{d_1}, \ldots, x_n^{d_1})$, then $\mathrm{DP}(J) = \mathrm{lct}(J)$.

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Theorem (B. 2025):

If I is homogeneous and $\mathrm{lct}(I) = \mathrm{DP}(I)$ (or $\mathrm{fpt}(I) = \mathrm{DP}(I)$ in char $p > 0$), then

$$I = (x_1^{e_1(I)/e_0(I)}, \ldots, x_n^{e_n(I)/e_{n-1}(I)})$$

up to change of variables and integral closure.

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References

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