

EXTREMAL F-THRESHOLDS IN REGULAR LOCAL RINGS

BENJAMIN BAILY

ABSTRACT. Let (R, \mathfrak{m}) be a regular local ring of characteristic $p > 0$. Among all proper ideals $\mathfrak{a} \subseteq R$ with a fixed order of vanishing $\text{ord}_{\mathfrak{m}}(\mathfrak{a})$, we classify the ideals for which the F -threshold $\text{ft}^{\mathfrak{m}}(\mathfrak{a})$ is minimal.

1. INTRODUCTION

Let (R, \mathfrak{m}) be a regular local ring of characteristic $p > 0$ and let \mathfrak{a} denote a proper ideal of R . Assume that R is F -finite – that is, we assume that the Frobenius map $F : R \rightarrow R$ is finite – we will relax this assumption later. The F -pure threshold of (R, \mathfrak{a}) is a nonnegative rational number [4] measuring the singularities of the pair (R, \mathfrak{a}) at the point \mathfrak{m} in $\text{Spec } R$. In this setting, the F -pure threshold can be computed as

$$\text{fpt}(\mathfrak{a}) = \sup \left\{ \frac{t}{p^e} : \mathfrak{a}^t \not\subseteq \mathfrak{m}^{[p^e]} \right\}, \text{ where } I^{[p^e]} := \sum_{z \in I} z^{p^e} R.$$

The idea is that smaller values of the F -pure threshold correspond to “worse singularities” of the closed subscheme of $\text{Spec } R$ defined by \mathfrak{a} . For example, if $\mathfrak{a} = fR$ is principal, then $\text{fpt}(f) \leq 1$ with equality if and only if R/fR is Frobenius split; in this case, R/fR must be reduced.

If $d = \text{ord}_{\mathfrak{m}}(\mathfrak{a})$ denotes the greatest positive integer such that $\mathfrak{a} \subseteq \mathfrak{m}^d$ and n is the dimension of R , then $\text{fpt}(\mathfrak{a})$ is bounded below by $\frac{1}{d}$ and bounded above by $\min(\frac{n}{d}, 1)$ [28]. In this paper, we describe when the F -pure threshold achieves the lower end of this range.

Main Theorem (Theorem 3.1). *Let (R, \mathfrak{m}) be an F -finite regular local ring of characteristic $p > 0$ and $\mathfrak{a} \subseteq R$ a proper ideal. Let $d = \text{ord}_{\mathfrak{m}}(\mathfrak{a})$ and write $d = qs$ where q is a power of p and s is coprime to p . Then $\text{fpt}(\mathfrak{a}) = \frac{1}{d}$ if and only if \mathfrak{a} is a principal ideal and there exists g in $\mathfrak{m}^{[q]}$ such that $\mathfrak{a} = g^s R$.*

For a more general statement and a proof, see Section 3. In particular, we show that the F -finite hypothesis can be weakened to the assumption that the formal fiber $\widehat{R} \otimes_R \frac{R_{(\pi)}}{\pi R_{(\pi)}}$ is reduced for all prime elements π in R , which is satisfied by any excellent local ring. The hypothesis on the height-1 formal fibers of R cannot be weakened further; see Example 3.20.

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Other authors have considered a related problem for squarefree homogeneous polynomials over an algebraically closed field k of characteristic $p > 0$; see [20]. For f in $k[x_0, \dots, x_n]$ squarefree and homogeneous of degree d , one has $\text{fpt}(f) \geq \frac{1}{d-1}$ with equality if and only if $d = p^e + 1$ and $f \in \mathfrak{m}^{[p^e]}$. In the absence of homogeneity, however, we cannot improve on the lower bound $\text{fpt}(f) \geq \frac{1}{d}$ even if we assume that f is squarefree: the binomial $x_1^d + x_2^N$ is reduced whenever d, N are coprime and $\text{fpt}(x_1^d + x_2^N)$ converges to $\frac{1}{d}$ as $N \rightarrow \infty$ (Example 3.21).

The characteristic zero analog of the main theorem concerning the log canonical threshold (lct) is well-understood. If (R, \mathfrak{m}) is an excellent regular local ring of equal characteristic zero and $\mathfrak{a} \subseteq R$ an ideal, it is known to experts that $\text{lct}(\mathfrak{a}) \geq \frac{1}{\text{ord}_{\mathfrak{m}}(\mathfrak{a})}$ with equality if and only if $\mathfrak{a} = x^d R$ for some d in \mathbb{Z}^+ and some x in $\mathfrak{m} \setminus \mathfrak{m}^2$. More generally, for the germ at 0 of a plurisubharmonic function $u : \mathbb{C}^n \rightarrow \mathbb{C}$ with Lelong number $\nu(u) = 1$, one has $\text{lct}(u) \geq 1$ with equality if and only if $u = \log|z_1| + v$ with z_1 a local coordinate and $\nu(v) = 0$ [12].

A straightforward consequence of the main theorem is that the lower bound $\frac{1}{\text{ord}_{\mathfrak{m}}(\mathfrak{a})} \leq \text{fpt}(\mathfrak{a})$ is only attained by height-1 ideals. In a forthcoming paper [2], the author proves a stronger bound for higher-height ideals, analogous to a result of Demainay and Pham on log canonical thresholds [8]. Specifically, for an F -finite regular local ring (R, \mathfrak{m}) of characteristic $p > 0$ and an ideal $\mathfrak{a} \subseteq R$ of height l , there is a log convex sequence of positive integers $\sigma_0(\mathfrak{a}), \dots, \sigma_l(\mathfrak{a})$ with $\sigma_0(\mathfrak{a}) = 1, \sigma_1(\mathfrak{a}) = \text{ord}_{\mathfrak{m}}(\mathfrak{a})$, and

$$(1) \quad \frac{\sigma_0(\mathfrak{a})}{\sigma_1(\mathfrak{a})} + \cdots + \frac{\sigma_{l-1}(\mathfrak{a})}{\sigma_l(\mathfrak{a})} \leq \text{fpt}(\mathfrak{a}).$$

The integers $\sigma_j(\mathfrak{a})$ are constructed in [3], where they are written as $\sigma_j(\mathfrak{a}, \mathfrak{m})$. The main result of [2] is a description of the cases of equality in Equation (1) when R is a polynomial ring and $\mathfrak{a} \subseteq R$ is a homogeneous ideal.

Notation 1.1. *In this paper, all rings are commutative and Noetherian. The letter p always denotes a prime number and q a power of p .*

2. PRELIMINARIES

On several occasions, we need to work with monomial orders and leading terms in power series rings. With a few exceptions, Gröbner theory in power series rings is analogous to the theory in polynomial rings. For the unfamiliar reader, we recommend [7, Chapter 4].

Notation 2.1. *For a vector of ring elements $\mathbf{f} = f_1, \dots, f_r$ and a vector of nonnegative integers $\mathbf{a} = a_1, \dots, a_r$, we let $\mathbf{f}^\mathbf{a}$ denote the element $f_1^{a_1} \cdots f_r^{a_r}$. If \mathbf{a} is instead a tuple of nonnegative real numbers, we let $\mathbf{f}^\mathbf{a}$ denote the \mathbb{R} -divisor $a_1 \text{div}(f_1) + \cdots + a_r \text{div}(f_r)$.*

Definition 2.2. Let k be a field. Let $R = k[\![x_1, \dots, x_n]\!]$. A *local monomial order* on R is a partial ordering $>$ on the set of monomials of R such that

for all $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{Z}_{\geq 0}^n$, if $\mathbf{x}^\mathbf{a} \leq \mathbf{x}^\mathbf{b}$, then

$$\mathbf{x}^{\mathbf{a}+\mathbf{c}} \leq \mathbf{x}^{\mathbf{b}+\mathbf{c}} \leq \mathbf{x}^\mathbf{b}.$$

In particular, the greatest monomial in R is 1.

Let $f = \sum_{\mathbf{a} \in \mathbb{Z}_{\geq 0}^r} \gamma_{\mathbf{a}} \mathbf{x}^\mathbf{a}$ be an element of R . Among all $\mathbf{b} \in \mathbb{Z}_{\geq 0}^n$ such that $\gamma_{\mathbf{b}} \neq 0$, let S denote the set of vectors for which $\mathbf{x}^\mathbf{b}$ is maximal with respect to $>$. The *initial term* of f with respect to $>$, denoted $\text{in}_>(f)$, is the element $\sum_{\mathbf{b} \in S} \gamma_{\mathbf{b}} \mathbf{x}^\mathbf{b}$.

2.1. Background on F-Thresholds and Test Ideals. To begin, we define F -thresholds and test ideals and collect a few relevant properties. For further background, we refer the reader to [28, 22].

Definition-Proposition 2.3 ([22]). Let R be a regular local ring of characteristic $p > 0$. Let $\mathfrak{a} \subseteq R$ be an ideal and $\mathfrak{b} \subseteq R$ a proper ideal such that $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$. For $e \geq 0$, define $\nu_{\mathfrak{a}}^{\mathfrak{b}}(p^e)$ to be the greatest integer t such that $\mathfrak{a}^t \not\subseteq \mathfrak{b}^{[p^e]}$. Then the sequence $\frac{\nu_{\mathfrak{a}}^{\mathfrak{b}}(p^e)}{p^e}$ has a limit as $e \rightarrow \infty$, and we refer to this limit as the F -threshold $\text{ft}^{\mathfrak{b}}(\mathfrak{a})$. Additionally, we have

$$\text{ft}^{\mathfrak{b}}(f) = \inf \left\{ \frac{t}{p^e} : \mathfrak{a}^t \subseteq \mathfrak{b}^{[p^e]} \right\} = \sup \left\{ \frac{t}{p^e} : \mathfrak{a}^t \not\subseteq \mathfrak{b}^{[p^e]} \right\}.$$

For the majority of this paper, the above notion of the F -threshold suffices. In a few isolated instances, we need a more general notion of what it means for a pair (R, f^t) to be F -pure.

Definition 2.4 ([25], §1.1). Let R be a ring of characteristic $p > 0$. For any $e > 0$, the module $F_*^e R$ has underlying Abelian group isomorphic to R and an R -module action defined by restriction of scalars along the Frobenius map $F^e : R \rightarrow R$. Concretely, the elements of $F_*^e R$ are $\{F_*^e f : f \in R\}$, and $x F_*^e f = F_*^e(x^{p^e} f)$. We say that R is F -finite if $F_* R$ is a finite R -module, in which case $F_*^e R$ is finite for all $e > 0$.

Definition 2.5 ([28, 24]). Let R be an F -finite reduced ring. Let $\mathfrak{a} \subseteq R$ be an ideal and t a nonnegative real number. We say that the pair (R, \mathfrak{a}^t) is *sharply F -pure* if, for a single $e > 0$ (equivalently, all integers ne where $n \in \mathbb{Z}^+$), there exists $d \in \mathfrak{a}^{\lceil t(p^e - 1) \rceil}$ such that the map

$$R \rightarrow F_*^e R \quad 1 \mapsto F_*^e d$$

splits.

The F -pure threshold $\text{fpt}(\mathfrak{a})$ of the pair (R, \mathfrak{a}) is defined to be the supremum of all real $t \geq 0$ such that (R, \mathfrak{a}^t) is sharply F -pure.

By [22, Remark 1.5], if (R, \mathfrak{m}) is a regular local ring then $\text{fpt}(\mathfrak{a}) = \text{ft}^{\mathfrak{m}}(\mathfrak{a})$.

Definition 2.6 ([13, 4]). Let R be an F -finite regular ring, let $\mathfrak{a} \subseteq R$ be an ideal, and let t be a nonnegative real number. The *test ideal* of the pair (R, \mathfrak{a}^t) is equal to the image, for $e \gg 0$, of the map

$$\mathfrak{a}^{\lceil tp^e \rceil} \cdot \text{Hom}_R(F_*^e R, R) \rightarrow R \quad d \cdot \phi \mapsto \phi(F_*^e d).$$

Proposition 2.7. Let (R, \mathfrak{m}) be a regular local ring of dimension n and characteristic $p > 0$. Let f be an element of R and let $\mathfrak{b} \subseteq R$ such that $f \in \sqrt{\mathfrak{b}}$. Then the following hold.

- (1) If $\mathfrak{c} \subseteq \mathfrak{a}$, then $\text{ft}^{\mathfrak{b}}(\mathfrak{c}) \leq \text{ft}^{\mathfrak{b}}(\mathfrak{a})$.
- (2) For s in \mathbb{Z}^+ , we have $\text{ft}^{\mathfrak{b}}(\mathfrak{a}^s) = \frac{1}{s} \text{ft}^{\mathfrak{b}}(\mathfrak{a})$.
- (3) The inequality $\frac{1}{\text{ord}_{\mathfrak{m}}(f)} \leq \text{ft}^{\mathfrak{m}}(f) \leq \frac{n}{\text{fpt}(f)}$ holds.
- (4) For all $e > 0$, we have $\frac{\nu_{\mathfrak{a}}^{\mathfrak{b}}(p^e)}{p^e} < \text{ft}^{\mathfrak{b}}(\mathfrak{a})$.
- (5) For x in $\mathfrak{m} \setminus \mathfrak{m}^2$, if $\bar{f}, \bar{\mathfrak{b}}$ denote the images of f, \mathfrak{b} in R/xR , then $\text{ft}^{\mathfrak{b}}(f) \geq \text{ft}^{\bar{\mathfrak{b}}}(\bar{f})$.
- (6) $\text{ft}^{\mathfrak{b}}(f) = \inf\{t : \tau(f^t) \subseteq \mathfrak{b}\}$.
- (7) If R is a power series ring, \mathfrak{b} is a monomial ideal, and $>$ is a local monomial order, then $\text{ft}^{\mathfrak{b}}(\text{in}_>(f)) \leq \text{ft}^{\mathfrak{b}}(f)$.

Proof. For (1)-(4), see [22, Proposition 1.7]. The proof of claims (5)-(7) are enumerated below.

- (5) This standard fact follows from an observation in [27, Theorem 3.11]. For any positive integers a, e , if $f^a \in \mathfrak{b}^{[p^e]}$ then $(\bar{f})^a \in (\bar{\mathfrak{b}})^{[p^e]}$. The assumption that $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ is to ensure that the quotient ring R/xR is again regular, but is not necessary if one defines the F -threshold in greater generality.
- (6) This fact follows from [22, Proposition 2.7].
- (7) The argument of the claim in [28, Proof of Proposition 4.5] works verbatim.

□

2.2. Critical Points, after Hernández and Teixeira.

Notation 2.8. For a vector $\mathbf{a} = (a_1, \dots, a_n)$ in $\mathbb{R}_{\geq 0}^n$, the symbol $\|\mathbf{a}\|$ denotes the number $a_1 + \dots + a_n$.

A key point in the proof of Theorem 3.1 uses ideas of Hernández and Teixeira [16] in the 2-dimensional setting, generalized to fit our needs. In op. cit., the authors consider the following problem.

Question 2.9. Let k be a field of characteristic $p > 0$ and $\mathbf{h} = h_1, \dots, h_r$ be a tuple of distinct linear forms in $R = k[x, y]$. For which \mathbf{t} in $\mathbb{R}_{\geq 0}^r$ is $\tau(R, \mathbf{h}^{\mathbf{t}}) = R$? More generally, for a homogeneous system of parameters U, V and $\mathfrak{b} = (U, V)$, how can we describe the set of \mathbf{t} in $\mathbb{R}_{\geq 0}^r$ for which $\tau(R, \mathbf{h}^{\mathbf{t}}) \subseteq \mathfrak{b}$?

To address this question, the authors study *syzygy gaps*. For \mathbf{a} in $\mathbb{Z}_{\geq 0}^r$ and $q = p^e$, one can determine whether $\tau(R, \mathbf{h}^{\mathbf{a}/q}) \subseteq \mathfrak{b}$ by computing the graded free resolution

$$R(-m) \oplus R(-n) \rightarrow R(-q \deg U) \oplus R(-q \deg V) \oplus R(-\|\mathbf{a}\|) \rightarrow R/(U, V, \mathbf{h}^\mathbf{a}) \rightarrow 0.$$

The number $\Delta(\mathbf{a})$ is defined to be the “syzygy gap” $|m - n|$. Defining $\Delta(\mathbf{a}/q)$ to be $\frac{1}{q}\Delta(\mathbf{a})$ yields a well-defined map $\Delta : \mathbb{Z}[1/p]_{\geq 0}^r \rightarrow \mathbb{Z}[1/p]_{\geq 0}$. By Lemma 3.2 in op. cit., Δ extends uniquely to a continuous map from $\mathbb{Q}_{\geq 0}^r \rightarrow \mathbb{Q}_{\geq 0}$, and by Proposition 3.5 in op. cit., Δ determines the behavior of the test ideal: for \mathbf{t} in $\mathbb{Q}_{\geq 0}^r$, we have $\tau(R, \mathbf{h}^\mathbf{t}) \subseteq \mathfrak{b}$ if and only if $\Delta(\mathbf{t}) = |\|\mathbf{t}\| - \deg(UV)|$. By Corollary 3.11 in op. cit., we have $\tau(R, \mathbf{h}^\mathbf{t}) \subseteq \mathfrak{b}$ whenever $\|\mathbf{t}\| \geq \deg(UV)$, so the nontrivial behavior of the test ideal is confined to the region $\|\mathbf{t}\| < \deg(UV)$. By Theorem 5.9 in op. cit., this nontrivial behavior is completely determined by a family of distinguished points in $\mathbb{Z}[1/p]_{\geq 0}$ called *critical points*.

The question we consider is similar. Rather than arbitrary parameter ideals, we consider special parameter ideals \mathfrak{b} of the form (y, x^ℓ) and rather than tuples of linear forms, we consider tuples of polynomials $\mathbf{h} = h_1, \dots, h_r$, where $h_i = y - g_i$ and $g_i \in x^\ell k[x]$.

Question 2.10. *Let k be a field of characteristic $p > 0$ and $R = k[[x, y]]$. Let $\ell > 0$ and let $\mathfrak{b} := (y, x^\ell)$. Let $r \geq 2$ and for $1 \leq i \leq r$, let $h_i = y - g_i$, where $g_i \in x^\ell k[x]$. Let \mathbf{h} denote the vector h_1, \dots, h_r . For which \mathbf{t} in $\mathbb{R}_{\geq 0}^r$ is $\tau(R, \mathbf{h}^\mathbf{t}) \subseteq \mathfrak{b}$?*

Analogously to Question 2.9, we have $\tau(R, \mathbf{h}^\mathbf{t}) \not\subseteq \mathfrak{b}$ when $\|\mathbf{t}\| < 1$ and $\tau(R, \mathbf{h}^\mathbf{t}) \subseteq \mathfrak{b}$ when $\|\mathbf{t}\| \geq 2$, so the nontrivial behavior of $\tau(R, \mathbf{h}^\mathbf{t})$ is contained in the strip $1 \leq \|\mathbf{t}\| < 2$. If D is the largest degree of any of the polynomials g_i , then the behavior of $\tau(R, \mathbf{h}^\mathbf{t})$ is determined by critical points in the strip $1 \leq \|\mathbf{t}\| < 1 + \frac{\ell}{D}$ by Corollary 2.19, and may not be determined by critical points in the strip $1 + \frac{\ell}{D} \leq \|\mathbf{t}\| < 2$ (Example 2.21).

Definition 2.11. Consider the setup of Question 2.10. We define a partition of the set $(\mathbb{Z}[\frac{1}{p}]_{\geq 0})^r$ into the *upper and lower regions attached to \mathbf{h} and \mathfrak{b}* . For \mathbf{a} in $\mathbb{Z}_{\geq 0}^r$, $q = p^e$, we say that $\frac{\mathbf{a}}{q} \in \mathcal{U}$ (the upper region) if $\mathbf{h}^\mathbf{a} = h_1^{a_1} \dots h_r^{a_r} \in \mathfrak{b}^{[q]}$. Otherwise, $\frac{\mathbf{a}}{q} \in \mathcal{L}$ (the lower region).

Because R is F -split, it follows that $\mathbf{h}^\mathbf{a} \in \mathfrak{b}^{[q]}$ if and only if $\mathbf{h}^{p\mathbf{a}} \in \mathfrak{b}^{[pq]}$, so the regions \mathcal{U}, \mathcal{L} are well-defined.

Lemma 2.12. *For any $\frac{\mathbf{a}}{q} \leq \frac{\mathbf{a}'}{q'}$, if $\frac{\mathbf{a}}{q} \in \mathcal{U}$ then $\frac{\mathbf{a}'}{q'} \in \mathcal{U}$.*

Proof. By well-definedness of \mathcal{U} , we may set $q'' = \max(q, q')$ and rewrite $\frac{\mathbf{a}}{q} = \frac{\mathbf{b}}{q''}, \frac{\mathbf{a}'}{q'} = \frac{\mathbf{c}}{q''}$. Suppose $\frac{\mathbf{b}}{q''} \in \mathcal{U}$. As $\mathbf{b} \leq \mathbf{c}$, it follows that $\mathbf{h}^\mathbf{b} \mid \mathbf{h}^\mathbf{c}$, so $\mathbf{h}^\mathbf{c} \in \mathfrak{b}^{[q'']}$ and $\frac{\mathbf{c}}{q''} \in \mathcal{U}$. \square

Lemma 2.13. Suppose that $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and that q is a power of p . Let e_1, \dots, e_r denote the standard unit vectors in \mathbb{Z}^r . For any $1 \leq s \leq r$, we have $\frac{\mathbf{a}}{q} - \frac{e_s}{q} \in \mathcal{U}$ if and only if $\frac{\mathbf{a}}{q} - \frac{e_s}{pq} \in \mathcal{U}$.

Proof. The implication \Rightarrow follows from Lemma 2.12. For the reverse implication, start by considering the map $\phi : R \rightarrow R$ given by $\phi(x) = x$ and $\phi(y) = h_s$. Since ϕ is an isomorphism and $\phi(\mathbf{b}) = \mathbf{b}$, we may without loss of generality assume $h_s = y$.

Suppose $\frac{\mathbf{a}}{q} - \frac{e_s}{pq} \in \mathcal{U}$. Then $\mathbf{h}^{p(\mathbf{a}-e_s)}y^{p-1} = \mathbf{h}^{p\mathbf{a}-e_s} \in \mathbf{b}^{[pq]}$. Applying [20, Lemma 3.5], we obtain

$$\mathbf{h}^{p(\mathbf{a}-e_s)} \in (\mathbf{b}^{[pq]} : y^{p-1}) = (y^{pq-p+1}, x^{\ell pq})$$

The monomials $\{x^i y^j\}_{0 \leq i, j \leq p-1}$ form a free basis for F_*R over R ; write $\Phi : F_*R \rightarrow R$ for projection onto the $F_*(xy)^{p-1}$ factor. Then $\Phi \circ (\cdot F_*(xy)^{p-1})$ is a splitting of the Frobenius map $R \rightarrow F_*R$, so

$$\mathbf{h}^{\mathbf{a}-e_s} = \Phi(F_*(xy)^{p-1} \mathbf{h}^{p(\mathbf{a}-e_s)}) \in \Phi(F_*(y^{pq}, x^{\ell pq+p-1})) = \mathbf{b}^{[q]}.$$

□

Lemma 2.14. Let $\frac{\mathbf{a}}{q}$ be a vector in \mathcal{U} . If $\frac{\mathbf{a}-e_s}{q} \in \mathcal{L}$ for all $1 \leq s \leq r$ with $a_s > 0$, then $\frac{\mathbf{a}'}{q'} \in \mathcal{L}$ for all $\frac{\mathbf{a}'}{q'} < \frac{\mathbf{a}}{q}$.

Proof. The implication \Leftarrow is tautological. For the implication \Rightarrow , suppose that $\frac{\mathbf{a}-e_s}{q} \in \mathcal{L}$ for all $1 \leq s \leq r$ with $a_s > 0$. By Lemma 2.13, it follows that $\frac{p^n \mathbf{a} - e_s}{p^n q} \in \mathcal{L}$ for all $n > 0$ and all s with $a_s > 0$. For $\frac{\mathbf{a}'}{q'} < \frac{\mathbf{a}}{q}$, suppose without loss of generality that $\mathbf{a}'_1 < \mathbf{a}_1$. Then $\frac{\mathbf{a}'}{q'} \leq \frac{q' \mathbf{a} - e_1}{qq'}$, which implies $\frac{\mathbf{a}'}{q'} \in \mathcal{L}$ by Lemma 2.12. □

Definition 2.15 (c.f. [17], Definition 5.1). Let $\frac{\mathbf{c}}{q}$ be a vector in \mathcal{U} . If $\frac{\mathbf{c}}{q}$ satisfies either of the equivalent conditions of Lemma 2.14, we say that $\frac{\mathbf{c}}{q}$ is a *critical point* attached to \mathbf{h} and \mathbf{b} . We let \mathcal{C} denote the set of such critical points.

Lemma 2.16 (c.f. [16], Corollary 5.7). Let \mathbf{a} be in $\mathbb{Z}_{\geq 0}^r$ and q be a power of p . Then $\frac{\mathbf{a}}{q} \in \mathcal{U}$ if and only there exists a critical point $\frac{\mathbf{c}}{q}$ in \mathcal{C} with $\mathbf{c} \leq \mathbf{a}$.

Proof. Suppose $\frac{\mathbf{a}}{q} \in \mathcal{U}$. The set S of elements \mathbf{u} in $\mathbb{Z}_{\geq 0}^r$ such that $\frac{\mathbf{u}}{q} \in \mathcal{U}$, $\mathbf{u} \leq \mathbf{a}$ is finite and nonempty, so choose $\mathbf{c} \in S$ such that $\|\mathbf{c}\|$ is minimal. By construction, we have $\frac{\mathbf{c}-e_s}{q} \in \mathcal{L}$ for all $1 \leq s \leq r$, hence $\frac{\mathbf{c}}{q}$ is a critical point by Lemma 2.14. Conversely, if $\frac{\mathbf{a}}{q} \geq \frac{\mathbf{c}}{q} \in \mathcal{C}$, then $\frac{\mathbf{a}}{q} \in \mathcal{U}$ by Lemma 2.12. □

Lemma 2.17. If $\frac{\mathbf{c}}{q} \in \mathcal{U}$, then $\left\| \frac{\mathbf{c}}{q} \right\| \geq 1$.

Proof. First, we note that $h_i \in \mathbf{b}$ for all $1 \leq i \leq r$, so $\mathbf{h}^\mathbf{a} \in \mathbf{b}^{\|\mathbf{a}\|}$ for any \mathbf{a} in $\mathbb{Z}_{\geq 0}^r$. Conversely, the coefficient of $y^{\|\mathbf{a}\|}$ in $\mathbf{h}^\mathbf{a}$ is equal to 1, so $\text{ord}_\mathbf{b}(\mathbf{h}^\mathbf{a}) = \|\mathbf{a}\|$. If $\frac{\mathbf{c}}{q} \in \mathcal{U}$, then $\mathbf{h}^\mathbf{c} \in \mathbf{b}^{[q]} \subseteq \mathbf{b}^q$, so $\|\mathbf{c}\| = \text{ord}_\mathbf{b}(\mathbf{h}^\mathbf{c}) \geq q$. □

Lemma 2.18 (c.f. [16], Remark 5.8). *Let D denote the largest total degree of any of the polynomials g_1, \dots, g_r . If $\frac{\mathbf{b}}{q}, \frac{\mathbf{c}}{q}$ are distinct elements of \mathcal{C} , then $\left\| \max\left(\frac{\mathbf{b}}{q}, \frac{\mathbf{c}}{q}\right) \right\| \geq 1 + \frac{\ell}{D}$.*

Proof. Set $\mathbf{c}^{(0)} = \mathbf{c}$. To prove the claim, we produce a sequence of critical points $\mathbf{c}^{(0)}, \mathbf{c}^{(1)}, \dots, \mathbf{c}^{(s)}$ and possibly non-critical points $\mathbf{a}^{(0)}, \dots, \mathbf{a}^{(s)}$ such that $\mathbf{a}^{(s)} \leq \max(\mathbf{b}, \mathbf{c}^{(0)})$; we will show that $\|\mathbf{a}^{(s)}\| \geq 1 + \frac{\ell}{D}$.

Consider the following auxiliary conditions, which we will show hold for all $1 \leq m \leq s$:

$$(2) \quad \max(\mathbf{b}, \mathbf{c}^{(m)}) \leq \max(\mathbf{b}, \mathbf{a}^{(m-1)})$$

$$(3) \quad \max(\mathbf{b}, \mathbf{a}^{(m)}) < \max(\mathbf{b}, \mathbf{c}^{(m)})$$

Suppose we have constructed $\mathbf{c}^{(0)}, \dots, \mathbf{c}^{(n)}, \mathbf{a}^{(0)}, \dots, \mathbf{a}^{(n-1)}$ satisfying the conditions

- (i) $\mathbf{c}^{(n)} \neq \mathbf{b}$;
- (ii) $\frac{\mathbf{a}^{(0)}}{q}, \dots, \frac{\mathbf{a}^{(n-1)}}{q} \in \mathcal{U}$;
- (iii) Equation 2 holds for $1 \leq m \leq n$;
- (iv) Equation 3 holds for $0 \leq m \leq n-1$.

For the base case $n = 0$, the only non-vacuous hypothesis is that $\mathbf{c}^{(0)} \neq \mathbf{b}$, which holds by assumption. By Definition 2.15, the distinct critical points $\frac{\mathbf{b}}{q}, \frac{\mathbf{c}^{(n)}}{q}$ are incomparable, so let $1 \leq i_n, j_n \leq r$ such that $c_{i_n}^{(n)} > b_{i_n}$ and $c_{j_n}^{(n)} < b_{j_n}$. Set $\mathbf{a}^{(n)} = \max(\mathbf{b}, \mathbf{c}^{(n)}) - e_{i_n} - e_{j_n}$. We compute

$$(4) \quad \max(b_t, a_t^{(n)}) = \begin{cases} c_{i_n}^{(n)} - 1 & t = i_n \\ b_{j_n} & t = j_n \\ \max(c_t^{(n)}, b_t) & t \neq i_n, j_n, \end{cases}$$

so Equation 3 holds for $m = n$. If $\frac{\mathbf{a}^{(n)}}{q} \in \mathcal{L}$, we set $s = n$ and terminate the sequence. Otherwise, we apply Lemma 2.16 to produce $\mathbf{c}^{(n+1)} \leq \mathbf{a}^{(n)}$ where $\frac{\mathbf{c}^{(n+1)}}{q} \in \mathcal{C}$. By construction, Equation 2 holds for $m = n+1$. As $c_{j_n}^{(n+1)} \leq a_{j_n}^{(n)} = b_{j_n} - 1$, the vectors $\mathbf{c}^{(n+1)}, \mathbf{b}$ are again distinct.

By Equations 2,3, we have a strictly decreasing sequence of nonnegative integer vectors

$$\max(\mathbf{b}, \mathbf{c}^{(0)}) > \max(\mathbf{b}, \mathbf{c}^{(1)}) > \dots$$

which necessarily terminates. In particular, there exists some index $s \geq 0$ such that $\frac{\mathbf{a}^{(s)}}{q} \in \mathcal{L}$. Additionally, by construction of $\mathbf{a}^{(s)}$ we compute

$$\mathbf{a}^{(s)} = \max(\mathbf{b}, \mathbf{c}^{(s)}) - e_{i_s} - e_{j_s} \leq \max(\mathbf{b}, \mathbf{c}^{(0)}) - e_{i_s} - e_{j_s},$$

so $\|\max(\mathbf{b}, \mathbf{c}^{(0)})\| \geq \|\mathbf{a}^{(s)}\| + 2$. To prove that $\left\|\max\left(\frac{\mathbf{b}}{q}, \frac{\mathbf{c}^{(0)}}{q}\right)\right\| \geq 1 + \frac{\ell}{D}$, it therefore suffices to show that

$$(5) \quad \left\|\mathbf{a}^{(s)}\right\| \geq q + \frac{\ell q}{D} - 2$$

Without loss of generality, write $\mathbf{a} = \mathbf{a}^{(s)}, \mathbf{c} = \mathbf{c}^{(s)}, i_s = 1$ and $j_s = 2$. With $h_1 = y - g_1, h_2 = y - g_2$, consider the automorphism $\phi : R \rightarrow R$ given by $x \mapsto x$ and $y \mapsto y - g_1$. Then $\phi(h_1) = y, \phi(\mathbf{b}) = \mathbf{b}$, and $\phi(h_i) = y - g_i - g_1$, where $g_i - g_1$ is a polynomial in $k[x]$ of total degree at most D . We may therefore assume without loss of generality that $g_1 = 0$ so that $h_1 = y$.

Set $m = \text{ord}_x(g_2)$. As x^ℓ divides g_2 and $\deg(g_2) \leq D$, we necessarily have $\ell \leq m \leq D$. We compute $(h_1, h_2) = (y, x^m)$. As $\frac{\mathbf{a}}{q} \in \mathcal{L}$, we have $\mathbf{h}^\mathbf{a} \notin \mathbf{b}^{[q]}$. On the other hand, $\frac{\mathbf{a}+e_1}{q} \geq \frac{\mathbf{b}}{q}$ and $\frac{\mathbf{a}+e_2}{q} \geq \frac{\mathbf{c}}{q}$, so $\frac{\mathbf{a}+e_1}{q}, \frac{\mathbf{a}+e_2}{q} \in \mathcal{U}$ by Lemma 2.16; it follows that $\mathbf{h}^\mathbf{a}(h_1, h_2) \subseteq \mathbf{b}^{[q]}$. Rewriting (h_1, h_2) as (y, x^m) and applying [20, Lemma 3.5], we deduce that

$$\mathbf{h}^\mathbf{a} \in (\mathbf{b}^{[q]} : (h_1, h_2)) = (y^q, x^{\ell q}, x^{\ell q-m} y^{q-1}).$$

Write $\mathbf{h}^\mathbf{a} = \sum_{i \geq 0} z_i y^i$, where $z_i \in k[x]$. The quotient module

$$\frac{(y^q, x^{\ell q}, x^{\ell q-m} y^q - 1)}{(y^q, x^{\ell q})}$$

is spanned by the monomials $y^{q-1} x^{\ell q-m}, \dots, y^{q-1} x^{\ell q-1}$, so z_{q-1} is nonzero and has degree at least $\ell q - m$. On the other hand, z_{q-1} is a polynomial of degree $\|\mathbf{a}\| - (q - 1)$ in the inputs g_1, \dots, g_r , so we have

$$\ell q - m \leq \deg(z_{q-1}) \leq D(\|\mathbf{a}\| - (q - 1)).$$

As $m \leq D$, we conclude that $\|\mathbf{a}\| \geq (q - 1) + \frac{\ell q - D}{D}$, so Equation (5) holds. \square

Corollary 2.19 (c.f. [16], Theorem 5.9). *Let h_1, \dots, h_r be polynomials as in Question 2.10 where the g_i are distinct. Let D denote the largest degree of the polynomials g_1, \dots, g_r . Let f be an element of $k[[x, y]]$ such that $f = \mathbf{h}^\mathbf{t} = h_1^{t_1} \dots h_r^{t_r}$ for some positive integers t_1, \dots, t_r . Setting $\mathbf{b} = (y, x^\ell)$, either $\mathbf{ft}^\mathbf{b}(f) \geq \frac{1+\ell/D}{\|\mathbf{t}\|}$ or there exists a unique critical point $\mathbf{c} \leq \frac{1+\ell/D}{\|\mathbf{t}\|} \mathbf{t}$ which computes $\mathbf{ft}^\mathbf{b}(f)$: that is, $\mathbf{ft}^\mathbf{b}(f) = \max_{1 \leq i \leq r} \frac{c_i}{t_i}$.*

Proof. Let $\lambda = \frac{1+\ell/D}{\|\mathbf{t}\|}$. Write $\mu := \mathbf{ft}^\mathbf{b}(f)$ and suppose that $\mu < \lambda$. For all $e > 0$, Proposition 2.7 (4) implies that $f^{[p^e \mu]} \in \mathbf{b}^{[p^e]}$. By definition, $\frac{[p^e \mu]}{p^e} \mathbf{t} \in \mathcal{U}$, so by Lemma 2.16, there exists $\mathbf{c}^{(e)}$ in \mathcal{C} with $\mathbf{c}^{(e)} \leq \frac{[p^e \mu]}{p^e} \mathbf{t}$. Choose $e_0 \gg 0$ such that for all $e \geq e_0$ we have $\frac{[p^e \mu]}{p^e} < \lambda$. For any $e > e_0$ we have

$$\left\|\max(\mathbf{c}^{(e_0)}, \mathbf{c}^{(e)})\right\| \leq \left\|\frac{[p^e \mu]}{p^e} \mathbf{t}\right\| = \frac{[p^e \mu]}{p^e} \|\mathbf{t}\| < \lambda \|\mathbf{t}\| = 1 + \frac{\ell}{D}.$$

By Lemma 2.18, we must have $\mathbf{c}^{(e)} = \mathbf{c}^{(e_0)}$. Let $\mathbf{c} := \mathbf{c}^{(e_0)}$.

As $\mathbf{c} \leq \frac{\lceil p^e \rceil \mu}{p^e} \mathbf{t}$ for all $e \geq e_0$, it follows that $\mathbf{c} \leq \mu \mathbf{t}$. On the other hand, for any $\mu' < \mu$, we may choose $e \gg 0$ such that $\lceil p^e \mu' \rceil < \nu_f^\mathbf{b}(p^e)$ by Definition-Proposition 2.3. Consequently, $\frac{\lceil p^e \mu' \rceil}{p^e} \mathbf{t} \notin \mathcal{U}$, so $\mathbf{c}^{(e_0)} \not\leq \frac{\lceil p^e \mu' \rceil}{p^e} \mathbf{t}$ and hence $\mathbf{c} \not\leq \mu' \mathbf{t}$. We conclude that μ is the smallest real number for which $\mathbf{c} \leq \mu \mathbf{t}$, hence $\mu = \max_{1 \leq i \leq r} \frac{c_i}{t_i}$. \square

Remark 2.20. Putting $\ell = n = 1$, the above corollary gives an alternate proof of [16, Theorem 5.9] in the special case $\mathbf{b} = (x, y)$: given any homogeneous polynomial f , we may apply a linear change of coordinates so that $x \nmid f$, after which Corollary 2.19 applies.

The following example shows that the parameter D is necessary. Unlike the homogeneous case ([17, Theorem 5.9]), if $\text{ft}^\mathbf{b}(f) < \frac{2}{\|\mathbf{t}\|}$, there may not be a unique critical point $\frac{\mathbf{c}}{q} \leq \frac{2\mathbf{t}}{\|\mathbf{t}\|}$ computing $\text{ft}^\mathbf{b}(f)$.

Example 2.21. Let $R = \mathbb{F}_2[[x, y]]$ and $\ell = 1$ so that $\mathbf{b} = (x, y)$ and $\text{ft}^\mathbf{b}(-) = \text{fpt}(-)$. We define $h_1 = y + x, h_2 = y + x^2, h_3 = y + x^4$. We consider $\mathbf{t} = (1, 2, 1), f := \mathbf{h}^\mathbf{t} = h_1 h_2^2 h_3$. Then $f^7 \in \mathfrak{m}^{[16]}$, so $\text{fpt}(f) \leq \frac{7}{16} < \frac{2}{\|\mathbf{t}\|}$. There are many critical points below $\frac{2\mathbf{t}}{\|\mathbf{t}\|} = (\frac{1}{2}, 1, \frac{1}{2})$, however: for instance, the points $(\frac{1}{2}, 1, \frac{1}{2}), (\frac{3}{8}, \frac{13}{16}, \frac{7}{16}), (\frac{1}{4}, \frac{7}{8}, \frac{3}{8})$ are all critical. Moreover, none of these points compute the actual value of $\text{fpt}(f)$, which is equal to $\frac{3}{7}$ by a computation in Macaulay2 [11] using the FrobeniusThresholds [5] package.

3. PROOF OF THEOREM 3.1

We state a general version of the main theorem and outline a proof.

Theorem 3.1. *Let (R, \mathfrak{m}) be a regular local ring of characteristic $p > 0$ and $\mathfrak{a} \subseteq R$ a proper ideal. Let $d = \text{ord}_\mathfrak{m}(\mathfrak{a})$ and write $d = qs$ where q is a power of p and $\gcd(p, s) = 1$. Then*

$$(6) \quad \text{If } \text{ft}^\mathfrak{m}(\mathfrak{a}) = \frac{1}{d}, \text{ then there exists } g \text{ in } \mathfrak{m}^{[q]} \widehat{R} \text{ such that } g^s \widehat{R} = \mathfrak{a} \widehat{R}.$$

Suppose further that, for all prime elements π in R , the formal fiber $\widehat{R} \otimes_R R_{(\pi)}/\pi R_{(\pi)}$ is reduced. Then

$$(7) \quad \text{If } \text{ft}^\mathfrak{m}(\mathfrak{a}) = \frac{1}{d}, \text{ there exists } h \text{ in } \mathfrak{m}^{[q]} \text{ such that } h^s R = \mathfrak{a} R.$$

Since F -finite rings are excellent [10, Remark 13.6], our main theorem follows from this more precise version. A version for pairs $(R, \mathfrak{a}_1^{t_1} \dots \mathfrak{a}_r^{t_r})$ where $t_i \in \mathbb{R}$ will appear in the author's dissertation.

As a special case of Theorem 3.1, one can classify homogeneous polynomials f in $k[[x_1, \dots, x_n]]$ for which $\text{fpt}(f) = \frac{1}{\deg(f)}$. This case is given in

[20, Remark 3.2]; our proof follows a similar strategy. We reduce the claim to the case of a principal ideal in a complete local ring over an algebraically closed field, after which we apply a local Bertini theorem (Lemma 3.11) to reduce to the 2-dimensional case.

At this point, the two arguments diverge: unlike homogeneous polynomials, a power series in two variables may be irreducible, so we use the Weierstrass preparation theorem to write f as $u(y^d + a_1y^{d-1} + \dots + a_d)$ where $u \in k[[x, y]]^\times$ and $a_1, \dots, a_d \in k[[x]]$ (Lemma 3.5). Next, we pass to a finite flat extension $(R, \mathfrak{m}) \rightarrow (U, \mathfrak{n})$ over which f factors as $(y - \theta_1) \dots (y - \theta_d)$. This comes at the cost of having to consider the F -threshold of f at $\mathfrak{m}U$ instead of the F -pure threshold of f . Using [23, Proposition 5.1], we reduce to the case that the θ_i are polynomials in a finite extension $k[x^{1/u}]$ of $k[x]$. Finally, we use the critical point framework (Corollary 2.19) to deduce that $\theta_1 = \dots = \theta_d$, after which we apply Lemma 3.6 to deduce that f admits an s th root in R . As $d = qs$, if $f = g^s$, then $\text{fpt}(g) = s \text{fpt}(f) = \frac{1}{q} = \frac{1}{\text{ord}_\mathfrak{m}(g)}$, so the result follows from the degree- q case (Lemma 3.2). We divide the proof into five subsections.

- (1) Equation (6) holds when \mathfrak{a} is principal, $k = \bar{k}$, and $R = k[[x, y]]$.
- (2) Equation (6) holds when \mathfrak{a} is principal, $k = \bar{k}$, and $R = k[[x_1, \dots, x_n]]$.
- (3) Equation (6) holds when \mathfrak{a} is principal, k is any field, and $R = k[[x_1, \dots, x_n]]$.
- (4) Equation (6) holds when \mathfrak{a} is any ideal, k is any field, and $R = k[[x_1, \dots, x_n]]$.
- (5) Theorem 3.1 holds.

3.1. Step (1).

Lemma 3.2. *Let (R, \mathfrak{m}) be a regular local ring of characteristic $p > 0$ and let f be an element of R . If $q = p^e$ and $\text{ord}_\mathfrak{m}(f) = q$, then $\text{ft}^\mathfrak{m}(f) = \frac{1}{q}$ if and only if $f \in \mathfrak{m}^{[q]}$.*

Proof. As $f^1 \in \mathfrak{m}^{[q]}$, the inequality $\text{ft}^\mathfrak{m}(f) \leq \frac{1}{q}$ follows from Definition-Proposition 2.3. Conversely, $\text{ft}^\mathfrak{m}(f) = \frac{1}{q}$, then $\frac{\nu_f^\mathfrak{m}(q)}{q} < \frac{1}{q}$, so $\nu_f^\mathfrak{m}(q) = 0$ and $f \in \mathfrak{m}^{[q]}$. \square

Lemma 3.3 ([26]Tag 05CK). *Let $A \rightarrow B$ be a faithfully flat map of rings and $I \subseteq A$ an ideal. Then $IB \cap A = I$.*

Lemma 3.4. *Let $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local map of regular local rings. Suppose $\mathfrak{a}, \mathfrak{b} \subseteq R$ are ideals with $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$. Then $\text{ft}^\mathfrak{b}(\mathfrak{a}) = \text{ft}^{\mathfrak{b}S}(\mathfrak{a}S)$ and $\text{ord}_\mathfrak{b}(\mathfrak{a}) = \text{ord}_{\mathfrak{b}S}(\mathfrak{a}S)$.*

Proof. The first claim follows from [19, Proposition 2.2 (v)]. For the second, let $t > 0$. Lemma 3.3 implies

$$\mathfrak{a} \subseteq \mathfrak{b}^t \implies \mathfrak{a}S \subseteq \mathfrak{b}^t \implies \mathfrak{a}S \cap R \subseteq \mathfrak{b}^t \cap R \iff \mathfrak{a} \subseteq \mathfrak{b}^t,$$

so $\text{ord}_b(\mathfrak{a}) = \text{ord}_b S(\mathfrak{a}S)$. \square

Lemma 3.5. *Let k be an algebraically closed field of characteristic $p > 0$. Let $R = k[[z, w]]$ and set $\mathfrak{m} = (z, w)$. If $f \in \mathfrak{m}^d \setminus \mathfrak{m}^{d+1}$ and $\text{fpt}(f) = \frac{1}{d}$, then there exists:*

- A regular system of parameters x, y for R ;
- A degree- d monic polynomial P in $k[[x]][y]$;
- A unit u in R^\times

such that $f = uP$.

Proof. Write $f = f_d + f_{>d}$, where f_d is homogeneous of degree d and $f_{>d} \in \mathfrak{m}^{d+1}$. By Proposition 2.7 (7), we have $\text{fpt}(f_d) \leq \text{fpt}(f)$. As the lower bound $\frac{1}{d} = \frac{1}{\text{ord}_{\mathfrak{m}}(f_d)} \leq \text{fpt}(f_d)$ of Proposition 2.7 (3) still applies, we have $\text{fpt}(f_d) = \frac{1}{d}$. By [20, Remark 3.2], we have $f_d = \ell^d$ for some homogeneous linear form ℓ . Choosing a new regular system of parameters x, y for R such that $\ell = y$, we have $f = y^d + f_{>d}$.

Write $f = a_0 + a_1y + \dots$ as a power series in $k[[x]][y]$. As $f - y^d \in \mathfrak{m}^{d+1}$, it follows that $a_0, \dots, a_{d-1} \in \mathfrak{m}$ and $a_d \equiv 1 \pmod{\mathfrak{m}}$. The claim now follows from the Weierstrass preparation theorem [21, Theorems IV.9.1, IV.9.2]. \square

Lemma 3.6. *Let L be a field. Let f be an element of $L[[x]]$ with $x \nmid f$. Let u, s be positive integers such that s is not a multiple of $\text{char } k$. If $g \in L[[x^{1/u}]]$ such that $g^s = f$, then $g \in L[[x]]$.*

Proof. Write $g = a_0 + a_1x^{1/u} + \dots \in L[[x^{1/u}]]$. As $x \nmid f$, we have $a_0 \neq 0$. If $a_i = 0$ for all $u \nmid i$, then $g \in L[[x]]$. Otherwise, for the sake of contradiction, let i be minimal such that $u \nmid i$ and $a_i \neq 0$. By equating the coefficients of $x^{i/u}$ in g^s and in f , we obtain

$$0 = sa_0^{s-1}a_i.$$

We assumed that s is not a multiple of $\text{char } k$, so s is a unit in k , which contradicts the assumption that $a_0a_i \neq 0$. \square

Lemma 3.7. *Let k be a field and $T = k[[t]]$. Let $\theta_1, \dots, \theta_d$ be elements of T and consider $f = (y - \theta_1)\dots(y - \theta_d)$ as an element of $T[[y]]$. Let $\ell \geq 1$. If $f \in (y, t^\ell)^d$, then $\text{ord}_t(\theta_i) \geq \ell$ for all $1 \leq i \leq d$.*

Proof. Write $f = y^d + g_1y^{d-1} + \dots + g_d$ where $g_1, \dots, g_d \in T$. Re-order the roots $\theta_1, \dots, \theta_d$ such that

$$\text{ord}_t(\theta_1) = \dots = \text{ord}_t(\theta_r) < \text{ord}_t(\theta_{r+1}) \leq \dots \leq \text{ord}_t(\theta_d).$$

As $f \in (y, t^\ell)^d$, we have $y^{d-r}g_r \in (y, t^\ell)^d$ and hence $g_r \in (y, t^\ell)^{d-r} \cap T = t^{\ell r}T$. We have $g_r = \theta_1 \dots \theta_r + \text{higher order terms}$, so $\text{ord}_t(g_r) = \text{ord}_t(\theta_1 \dots \theta_r) = r \text{ord}_t(\theta_1)$, so for all $1 \leq i \leq d$ we have $\text{ord}_t(\theta_i) \geq \text{ord}_t(\theta_1) \geq \frac{\ell r}{r} = \ell$. \square

Lemma 3.8. *Let k be an algebraically closed field of characteristic $p > 0$. Let $R = k[[x, y]]$, $\mathfrak{m} = (x, y)$. Let d be a positive integer and f an element of*

R such that $\text{ord}_{\mathfrak{m}}(f) = d$, $\text{fpt}(f) = \frac{1}{d}$. Write $d = qs$, where q is a power of p and $\gcd(p, s) = 1$. Then there exists g in $\mathfrak{m}^{[q]}$ such that $g^s = f$.

Proof. By Lemma 3.5, we may assume f is a monic polynomial of degree d in $k[\![x]\!][y]$. Set $S = k[\![x]\!]$ and $K = \text{Frac}(S)$. Every monic polynomial in $S[y]$ factors completely over \overline{K} , so we may write

$$f = (y - \theta_1)^{e_1} \dots (y - \theta_r)^{e_r}$$

for some $\theta_1, \dots, \theta_r$ distinct roots in \overline{K} ; unlike the previous lemma, we will count roots with multiplicity.

Let T denote the integral closure of S in $K(\theta_1, \dots, \theta_r)$. By [18, Theorem 4.3.4], T is a complete local domain and $S \rightarrow T$ is finite. As T is a 1-dimensional complete normal ring of equal characteristic, the Cohen structure theorem gives $T \cong k'[\![t]\!]$ for some field extension k'/k . Furthermore, the restriction of $S \rightarrow T$ to $k \rightarrow k'$ is finite and k is algebraically closed, so $k = k'$.

We define $\ell := \text{ord}_t(xT)$, and note that by Lemma 3.7 that $t^\ell \mid \theta_i$ for all $1 \leq i \leq r$. Set $U := T[\![y]\!] = k[\![y, t]\!]$. As $R \hookrightarrow U$ is a finite local map between regular local rings of the same dimension, $R \rightarrow U$ is flat by the miracle flatness lemma [26, Tag 00R4]. Set $\mathfrak{b} = \mathfrak{m}U = (y, t^\ell)U$. As $R \rightarrow U$ is flat, we have $\text{fpt}(f) = \text{ft}^{\mathfrak{m}}(f) = \text{ft}^{\mathfrak{b}}(f) = \frac{1}{d}$ by Lemma 3.4. Write $d = qs$, where s is coprime to p . We aim to show that there exists g in U such that $g^s = f$. If $r = 1$ this is clear; we simply have $f = h_1^{qs}$ and $g = h_1^q$. Otherwise, suppose $r \geq 2$.

Let $\mathfrak{n} = (y, t) \subseteq U$. For all $1 \leq i \leq r$, we have $y - \theta_i \in \mathfrak{n} \setminus \mathfrak{n}^2$, so $h_i := y - \theta_i$ is irreducible. As argued in [23, Corollary 5.2], the pair $(U, (h_1 \dots h_r)^1)$ is sharply F -pure away from \mathfrak{n} , so by [23, Theorem 5.1] there exists D in \mathbb{Z}^+ such that for all t_1, \dots, t_r in $[0, 1)$ and $\tilde{h}_1, \dots, \tilde{h}_r$ such that $h_i \equiv \tilde{h}_i \pmod{\mathfrak{n}^D}$, we have

$$(8) \quad \tau(U, h_1^{t_1} \dots h_r^{t_r}) = \tau(U, \tilde{h}_1^{t_1} \dots \tilde{h}_r^{t_r}).$$

Without loss of generality, choose D to be larger than $\text{ord}_{\mathfrak{n}}(\theta_i - \theta_j)$ for all $1 \leq i < j \leq r$; in particular this value of D satisfies $D \geq \ell$. For $1 \leq i \leq r$, let $\tilde{\theta}_i$ in $k[t]$ denote the truncation of the power series θ_i at the D th term; we have $\theta_i \equiv \tilde{\theta}_i \pmod{\mathfrak{n}^D}$ and $t^\ell \mid \tilde{\theta}_i$. Set $\tilde{h}_i = y - \tilde{\theta}_i$ and $\tilde{f} = \tilde{h}_1^{e_1} \dots \tilde{h}_r^{e_r}$. By assumption that $r \geq 2$ we have $e_i \leq d - 1$ for all $1 \leq i \leq r$, hence $ce_i \in [0, 1)$ for all c in $[0, \frac{1}{d}]$. By Equation (8), for all $c \in [0, \frac{1}{d}]$, we have

$$\tau(R, f^c) = \tau(R, h_1^{ce_1} \dots h_r^{ce_r}) = \tau(R, \tilde{h}_1^{ce_1} \dots \tilde{h}_r^{ce_r}) = \tau(R, \tilde{f}^c).$$

As $\text{ft}^{\mathfrak{b}}(f) = \frac{1}{d}$, it follows that $\tau(R, \tilde{f}^c) \not\subseteq \mathfrak{b}$ for all $0 \leq c < \frac{1}{d}$ and $\tau(R, \tilde{f}^{1/d}) \subseteq \mathfrak{b}$, so $\text{ft}^{\mathfrak{b}}(\tilde{f}) = \frac{1}{d}$.

By our choice of D , the factors $\tilde{h}_1, \dots, \tilde{h}_r$ are distinct. In the notation of Definition 2.11, write $\tilde{f} = \tilde{\mathbf{h}}^{\mathbf{e}} = \tilde{h}_1^{e_1} \dots \tilde{h}_r^{e_r}$ with $e_i > 0$. As $\text{ft}^{\mathfrak{b}}(\tilde{f}) = \frac{1}{\|\mathbf{e}\|} < \frac{1+\ell/D}{\|\mathbf{e}\|}$, by Corollary 2.19, there exists a critical point $\mathbf{c} = \frac{\mathbf{a}}{q_0} \leq \frac{1+\ell/D}{\|\mathbf{e}\|} \mathbf{e}$ such

that $\text{ft}^{\mathfrak{b}}(\tilde{f}) = \max_{1 \leq i \leq r} \frac{c_i}{e_i}$. We then have

$$(9) \quad 1 = \|\mathbf{e}\| \cdot \text{ft}^{\mathfrak{b}}(\tilde{f}) = \sum_{i=1}^r e_i \max\left(\frac{c_1}{e_1}, \dots, \frac{c_r}{e_r}\right) \geq \sum_{i=1}^r c_i = \|\mathbf{c}\|.$$

By Lemma 2.17, we have $\|\mathbf{c}\| \geq 1$, so the inequality in Equation (9) is an equality. In particular, $c_i = e_i \max(\frac{c_1}{e_1}, \dots, \frac{c_r}{e_r})$ for all $1 \leq i \leq r$, so $\frac{c_i}{e_i} = \max(\frac{c_1}{e_1}, \dots, \frac{c_r}{e_r})$ for all $1 \leq i \leq r$, hence $\mathbf{c} = \text{ft}^{\mathfrak{b}}(\tilde{f})\mathbf{e}$. For all $1 \leq i \leq r$, we conclude that $\frac{e_i}{d} = \frac{a_i}{q_0}$. Recall that $d = qs$, where s is coprime to p . Then $e_i q_0 = a_i s q$, so we conclude that $s \mid e_i$. Consequently, f has an s th root g in U : similar to the $r = 1$ case, we take $g = h_1^{e_1/s} \dots h_r^{e_r/s}$.

Recall that $x = ut^\ell$ for some unit u in U^\times . As U is a power series ring over the algebraically closed field k , we have $u^{1/\ell} \in U$. Consequently, $x^{1/\ell} = tu^{1/\ell}$ is conjugate to t , hence $U = k[\![y, x^{1/\ell}]\!]$. Set $L = \text{Frac}(k[\![y]\!])$ and consider f as an element of $L[\![x]\!]$. As $g \in U \subseteq L[\![x^{1/\ell}]\!]$, it follows from Lemma 3.6 that $g \in L[\![x]\!] \subseteq \text{Frac}(R)$. As R is integrally closed and g is integral over R , it follows that $g \in R$.

As $f = g^s$, by Proposition 2.7 (2) we have $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{s} \text{ft}^{\mathfrak{m}}(g)$. Additionally, we have $\text{ord}_{\mathfrak{m}}(g) = \frac{1}{s} \text{ord}_{\mathfrak{m}}(f)$, so $\text{ft}^{\mathfrak{m}}(g) = \frac{1}{q} = \frac{1}{\text{ord}_{\mathfrak{m}}(g)}$. By Lemma 3.2, we conclude that $g \in \mathfrak{m}^{[q]}$, proving the claim. \square

Remark 3.9. Using Schwede's results on centers of F -purity, one can give a short proof of Lemma 3.8 in the special case that d is coprime to p . Assume that $\text{fpt}(f) = \frac{1}{d}$ and that d is coprime to p ; then $(p^e - 1)\frac{1}{d} \in \mathbb{Z}^+$ for some $e > 0$. By [14, Theorem 4.9], the pair $(R, f^{1/d})$ is sharply F -pure but not strongly F -regular. Let \mathfrak{p} be a center of sharp F -purity for $(R, f^{1/d})$ as in [24], which is a nonzero prime ideal by [24, Proposition 4.6]. We conclude $f^{\frac{p^e-1}{d}} \in (\mathfrak{p}^{[p^e]} : \mathfrak{p})$ by [24, Propositions 3.11 and 4.7].

If $\text{ht}(\mathfrak{p}) = 1$, then we have $\mathfrak{p} = (g)$ for some irreducible element g in R . In this case, we have $f^{\frac{p^e-1}{d}} \in g^{p^e-1}$. As $p^e - 1 = \text{ord}_{\mathfrak{m}}(f^{\frac{p^e-1}{d}}) \geq (p^e - 1) \text{ord}_{\mathfrak{m}}(g)$, we must have $\text{ord}_{\mathfrak{m}}(g) = 1$, so $q = 1$ and $g \in \mathfrak{m}^{[q]}$. By unique factorization of f we have that $f = ug^d$ for some unit u in R . As k is algebraically closed, u admits a d th root in R , proving the claim in this case. To finish the proof, we'll show that $\text{ht}(\mathfrak{p}) \neq 2$. Suppose for the sake of contradiction that $\mathfrak{p} = \mathfrak{m}$. Then $f^{\frac{p^e-1}{d}} \in (x_1^{p^e}, x_2^{p^e}, (x_1 x_2)^{p^e-1}) \subseteq \mathfrak{m}^{p^e}$. But this contradicts the fact that $\text{ord}_{\mathfrak{m}}(f^{\frac{p^e-1}{d}}) = p^e - 1$, proving the claim.

3.2. Step (2). For this step, we require a local Bertini theorem due to Flenner [9].

Lemma 3.10 ([9], Satz 2.1). *Let (A, \mathfrak{m}) be a complete local ring with coefficient field k . Let $\mathfrak{J} = (z_1, \dots, z_l) \subseteq A$ be a proper ideal. Suppose that $Q = \{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$ is a finite set of prime ideals of $D(\mathfrak{J})$. Then there exists x in \mathfrak{J} such that:*

- (i) $x \equiv z_1 \pmod{(z_2, \dots, z_l) + \mathfrak{m}\mathfrak{J}}$;
- (ii) $x \notin \mathfrak{q}_1 \cup \dots \cup \mathfrak{q}_r$;
- (iii) $x \notin \mathfrak{p}^{(2)}$ for all $x \in D(\mathfrak{J})$.

Proof. While Flenner does not explicitly note the congruence condition on x , we can specify one detail of the construction to ensure $x \equiv z_1 \pmod{(z_2, \dots, z_l) + \mathfrak{m}\mathfrak{J}}$. In Flenner's notation, we assume y_1, \dots, y_m is a generating set for \mathfrak{m} . We then let x_1, \dots, x_n denote the set of elements $z_1, \dots, z_l, z_1y_1, \dots, z_ly_m$ with $x_i = z_i$ for $1 \leq i \leq l$. With $S \subseteq A$ as in loc. cit., there exist a_2, \dots, a_n in S such that $x := x_1 + a_2x_2 + \dots + a_nx_n$ satisfies (ii) and (iii). Noting that $x \equiv z_1 + a_2z_2 + \dots + a_lz_l \pmod{\mathfrak{m}\mathfrak{J}}$, the claim (i) follows. \square

The following lemma is essentially an application of [9, Korollar 3.5], where we replace Satz 2.1 in op. cit. with our modification Lemma 3.10. Although Flenner's argument works for us *mutatis mutandis*, because the work op. cit. is written in German, we spell out the changes explicitly for the reader's convenience.

Lemma 3.11. *Let k be an infinite field. Let $n \geq 3$, $R = k[[x_1, \dots, x_n]]$, $\mathfrak{m} = (x_1, \dots, x_n)$. Let f be an element of R with $\text{ord}_{\mathfrak{m}}(f) = d > 0$. Let $s > 0$ and suppose that f does not have an s th root in R . There exists a in $\mathfrak{m} \setminus \mathfrak{m}^2$ such that the image \bar{f} of f in R/aR does not have an s th root in R/xR and $\text{ord}_{\mathfrak{m}/(x)}(\bar{f}) = d$.*

Proof. Recall that a regular local ring is a unique factorization domain (UFD) [1]. Factor f as $f = \pi_1^{e_1} \dots \pi_l^{e_l}$, where π_1, \dots, π_l are distinct irreducible factors of f . By assumption that f does not have an s th root in R , the multiplicities e_i are not all divisible by s . Set $g = \pi_1 \dots \pi_l$. Write $f = f_d + f_{>d}$, where f_d is a homogeneous polynomial of degree d and $f_{>d} \in \mathfrak{m}^{d+1}$. Let $Q \subseteq \text{Spec } R$ denote the set of primes \mathfrak{q} such that either \mathfrak{q} is minimal over (f_d, g) or such that $\text{ht}(\mathfrak{q}) < n$ and the image of \mathfrak{q} in R/gR is a minimal element of $\text{Sing}(R/gR)$. As $\dim R = n \geq 3$ and each of the minimal primes \mathfrak{q} over (f_d, g) has $\text{ht}(\mathfrak{q}) \leq 2$, we conclude that $\mathfrak{m} \notin Q$.

By assumption that k is infinite¹, there exists a homogeneous linear form in R which is not a factor of f_d . Choose homogeneous coordinates z_1, \dots, z_n for R such that $z_1 \nmid f_d$. Let $\mathfrak{J} = (z_1, z_2^{d+1}, \dots, z_n^{d+1})$. As $\sqrt{\mathfrak{J}} = \mathfrak{m}$, we have that $Q \subseteq D(\mathfrak{J})$. Apply Lemma 3.10 to R, \mathfrak{J} and Q to produce x in \mathfrak{J} such that $x \equiv z_1 \pmod{(z_2^{d+1}, \dots, z_n^{d+1}) + \mathfrak{m}\mathfrak{J}}$, $x \notin \bigcup_{\mathfrak{q} \in Q} \mathfrak{q}$, and $x \notin \mathfrak{p}^{(2)}$ for all $\mathfrak{p} \neq \mathfrak{m}$. Then $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ by construction.

We study the image \bar{f} of f in R/xR , which we show has $\text{ord}_{\mathfrak{m}/(x)}(\bar{f}) = d$. To see this, it suffices to show that $f \notin xR + \mathfrak{m}^{d+1}$. As $x \equiv z_1 \pmod{(z_2^{d+1}, \dots, z_n^{d+1}) + z_1\mathfrak{m}}$, there exists y in \mathfrak{m} such that $x \equiv z_1 + yz_1 \pmod{\mathfrak{m}^{d+1}}$. As z_1, f_d are homogeneous polynomials and $z_1 \nmid f_d$, we have $f_d \notin z_1R + \mathfrak{m}^{d+1} = xR + \mathfrak{m}^{d+1}$, so $f \equiv f_d \not\equiv 0 \pmod{xR + \mathfrak{m}^{d+1}}$ and $\text{ord}_{\mathfrak{m}/(x)}(f) = d$.

¹It suffices to have $\#\mathbb{P}^{n-1}(k) > d$.

Since R/gR is reduced, the argument of [9, Korollar 3.5] allows us to conclude that the ring $R/(g, x)R$ is reduced. In particular, the image of g is squarefree in R/xR , so we may factor the image of each π_i in R/xR as $\overline{\pi_i} = \rho_{i1} \dots \rho_{is_i}$ where the ρ_{ij} are irreducible and pairwise distinct. It follows that we may factor \overline{f} as $\prod_{i=1}^l \prod_{j=1}^{s_i} \rho_{ij}^{e_i}$. By assumption, the multiplicities e_i are not all divisible by s . As $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, the ring R/xR is regular, hence a UFD, so \overline{f} does not have an s th root in R/xR . \square

Lemma 3.12. *Let k be an algebraically closed field of characteristic $p > 0$. Let $n \geq 1$ and let $R = k[[x_1, \dots, x_n]]$. Let f be an element of R with $\text{ord}_{\mathfrak{m}}(f) = d$. Write $d = qs$, where $q = p^e$ and s is coprime to p . If $\text{fpt}(f) = \frac{1}{d}$, then there exists g in $\mathfrak{m}^{[q]}$ such that $f = g^s$.*

Proof. If $n = 1$, then $f = ux_1^d$ for some u in R^\times . As $k = \overline{k}$, every unit in R has a d th root in R , so we have $f = (u^{1/d}x_1)^d$ and $g = u^{q/d}x_1^q$. The $n = 2$ case is Lemma 3.8; we now suppose the claim holds in dimension $n - 1$ and let $n \geq 3$. Let f be an element of R with $\text{ord}_{\mathfrak{m}}(f) = d$, and write $d = qs$ where $q = p^e$ and $\gcd(s, p) = 1$. If f does not have an s th root in R , then by Lemma 3.11 there exists x in $\mathfrak{m} \setminus \mathfrak{m}^2$ such that, writing \overline{f} for the image of f in R/xR , the element \overline{f} does not have an s th root in R/xR and $\text{ord}_{\mathfrak{m}/(x)}(\overline{f}) = d$. By the classification in dimension $n - 1$, we have $\text{ft}^{\mathfrak{m}/(x)}(\overline{f}) > \frac{1}{d}$. By Proposition 2.7 (5), we have $\text{ft}^{\mathfrak{m}}(f) \geq \text{ft}^{\mathfrak{m}/(x)}(\overline{f}) > \frac{1}{d}$.

Suppose now that $\text{fpt}(f) = \frac{1}{d}$. By the previous paragraph, f admits an s th root g in R . Writing $f = g^s$, by Proposition 2.7 (2) and the fact that $\text{ord}_{\mathfrak{m}}$ is a valuation, we have

$$\text{ft}^{\mathfrak{m}}(g) = s \text{ft}^{\mathfrak{m}}(f) = \frac{1}{q} = \frac{1}{\text{ord}_{\mathfrak{m}}(g)}.$$

By Lemma 3.2, we conclude that $g \in \mathfrak{m}^{[q]}$, proving the claim. \square

3.3. Step (3). Over an arbitrary field k , an element f in $k[[x_1, \dots, x_n]]$ with $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{\text{ord}_{\mathfrak{m}}(f)}$ may not have an s th root at all.

Example 3.13. Let $R = k[[x]]$. Let $q = p^e$ and $s > 0$ such that $\gcd(s, p) = 1$ and write $d = qs$. If $a \in k^\times$ such that a does not have a s th root in k , then $\text{ft}^{\mathfrak{m}}(ax^d) = \text{ft}^{\mathfrak{m}}(x^d) = \frac{1}{d}$, but ax^d does not have an s th root in R .

The following lemma shows that Example 3.13 is as pathological as can possibly occur in a power series ring.

Lemma 3.14. *Let k be a field, $R = k[[x_1, \dots, x_n]]$, and set $S = R \widehat{\otimes}_k \overline{k}$. Let f be an element of R and s in \mathbb{Z}^+ such that $\text{char}(k)$ does not divide s . If there exists g in S such that $g^s = f$, then there exists a unit u in k^\times and an element h in R such that $f = uh^s$.*

Proof. Let Γ denote the set of monomials in S . Let \succ be a local monomial order on S such that (Γ, \succ) is order-isomorphic to $(\mathbb{N}, >)$; for example, we may take \succ such that $x^I \succ x^J \iff x^I <_{\text{deglex}} x^J$. Write

$g = \sum_{i=0}^N a_i x^{I_i}$ where the a_i are nonzero and $I_1 < I_2 < \dots$. We allow N to be a positive integer or infinity; the proof is identical in both cases. As $a_1^s(x_1^{I_1})^s = \text{in}_\succ(g^s) = \text{in}_\succ(f) \in R$, it follows that $a_1^s \in k$. Set $u = a_1^{-s}$ and $h = a_1^{-1}g$; we have $f = uh^s$ and $h^s \in R$. To prove the claim, we'll show that $h \in R$.

For ease of notation, write $b_i = \frac{a_i}{a_1}, i \geq 2$ so that $h = x^{I_1} + b_2 x^{I_2} + \dots$. Suppose $j \geq 2$ such that $b_2, \dots, b_{j-1} \in k$ and consider the coefficient c_j of $x^{(s-1)I_1+I_j}$ in h^s . A priori, we have

$$(10) \quad c_j = \sum_{\substack{\ell_1, \dots, \ell_s \\ I_{\ell_1} + \dots + I_{\ell_s} = (s-1)I_1 + I_j}} b_{\ell_1} \dots b_{\ell_s}.$$

If $\ell_1 \leq \dots \leq \ell_s \in \mathbb{N}$ such that $I_{\ell_1} + \dots + I_{\ell_s} = (s-1)I_1 + I_j$, then we have $(s-1)I_1 + I_{\ell_s} \leq (s-1)I_1 + I_j$, so $\ell_s \leq j$ with equality if and only if $\ell_1 = \dots = \ell_{s-1} = 1$. Consequently, we may refine Equation (10):

$$(11) \quad c_j = sb_j + \sum_{\substack{\ell_1, \dots, \ell_s < j \\ I_{\ell_1} + \dots + I_{\ell_s} = (s-1)I_1 + I_j}} b_{\ell_1} \dots b_{\ell_s}.$$

As $h^s \in R$, the coefficient c_j is in k . By assumption that $b_1, \dots, b_{j-1} \in k$, the second term of Equation (11) is an element of k , so $sb_j \in k$. As s is a unit in k , we conclude that $b_j \in k$. \square

Lemma 3.15. *Let k be a field of characteristic $p > 0$, $R = k[[x_1, \dots, x_n]]$, and $\mathfrak{m} = (x_1, \dots, x_n)$. Let f be an element of R such that $\text{ft}^\mathfrak{m}(f) = \frac{1}{d}$. If q is the largest power of p dividing d and $d = qs$, then there exists a unit u in R^\times and an element g in $\mathfrak{m}^{[q]}$ such that $f = ug^s$.*

Proof. Let $S = R \widehat{\otimes}_k \bar{k}$ and consider the map $R \rightarrow S$. We have that $R \rightarrow S$ is faithfully flat, $\mathfrak{n} = \mathfrak{m}S$ is the maximal ideal of S , and $S = \bar{k}[[x_1, \dots, x_n]]$. By Lemma 3.4, we have $\text{ft}^\mathfrak{m}(f) = \text{ft}^\mathfrak{n}(f)$ and $\text{ord}_\mathfrak{m}(f) = \text{ord}_\mathfrak{n}(f)$. As $\text{ft}^\mathfrak{n}(f) = \frac{1}{d} = \frac{1}{\text{ord}_\mathfrak{n}(f)}$, by Lemma 3.12 there exists g in $\mathfrak{m}^{[q]}$ such that $f = g^s$. By Lemma 3.14, there exist h in R , u in R^\times such that $uh^s = f$, and $h \in \mathfrak{m}^{[q]}$ by Lemma 3.2. \square

3.4. Step (4).

Lemma 3.16. *Let k be a field of characteristic $p > 0$, $R = k[[x_1, \dots, x_n]]$, and $\mathfrak{m} = (x_1, \dots, x_n)$. Let $\mathfrak{a} \subseteq R$ such that $\text{ft}^\mathfrak{m}(\mathfrak{a}) = \frac{1}{d}$. If q is the largest power of p dividing d and $d = qs$, then there exists g in $\mathfrak{m}^{[q]}$ such that $g^sR = \mathfrak{a}$.*

Proof. Let f be an element of \mathfrak{a} such that $\text{ord}_\mathfrak{m}(f) = d$. By Proposition 2.7 (1), (3) we have $\frac{1}{d} \leq \text{ft}^\mathfrak{m}(f) \leq \text{ft}^\mathfrak{m}(\mathfrak{a}) = \frac{1}{d}$. By Lemma 3.15, there exists u in R^\times , g in $\mathfrak{m}^{[q]}$ such that $f = ug^s$. We will show that g^s generates \mathfrak{a} .

Let $e > 0$ such that $s \mid (p^e - 1)$; write $t_e = \frac{p^e - 1}{s}$. By Proposition 2.7 (4), for all $e > 0$ we have $\nu_\mathfrak{a}^\mathfrak{m}(qp^e) < qp^e \text{ft}^\mathfrak{m}(\mathfrak{a}) \leq \lceil qp^e / qs \rceil = t_e + 1$, so $\mathfrak{a}^{t_e+1} \subseteq \mathfrak{m}^{[p^e]}$ for all $e > 0$. Let z be an arbitrary element of \mathfrak{a} and let r be a positive integer.

As $\mathfrak{a}^{t_e+1} \subseteq \mathfrak{m}^{[qp^e]}$, we in particular have $(g^s)^{t_e-r}z^{r+1} \in \mathfrak{m}^{[qp^e]}$. Write $g = a_1x_1^q + \dots + a_nx_n^q$. As $\text{ord}_{\mathfrak{m}}(g) = q$, there is some $1 \leq i \leq n$ such that $a_i \in R^\times$. As $g \equiv a_ix_i^q \pmod{(x_1^q, \dots, x_{i-1}^q, x_{i+1}^q, \dots, x_n^q)}$, it follows that $\mathfrak{m}^{[q]} = (x_1^q, \dots, x_{i-1}^q, g, x_{i+1}^q, \dots, x_n^q)$. In particular, $x_1, \dots, x_{i-1}, g, x_{i+1}, \dots, x_n$ is a system of parameters for R . By [20, Lemma 3.5], we have

$$(12) \quad \begin{aligned} z^{r+1} \in (\mathfrak{m}^{[qp^e]} : (g^s)^{t_e-r}) &= ((x_1^{qp^e}, \dots, g^{p^e}, \dots, x_n^{qp^e}) : g^{p^e-1-sr}) \\ &= (x_1^{qp^e}, \dots, g^{1+sr}, \dots, x_n^{qp^e}). \end{aligned}$$

Applying Equation (12) and letting $e \rightarrow \infty$, we obtain $z^{r+1} \in g^{1+sr}R \subseteq g^{sr}R$ for all $r > 0$. By [18, Corollary 6.8.11], z is contained in the integral closure of the ideal g^sR , which by [18, Proposition 1.5.2] is equal to g^sR itself. \square

3.5. Step (5).

Lemma 3.17. *Let (A, \mathfrak{m}) be a regular local ring and $I \subseteq A$ an ideal such that $I\widehat{A}$ is principal. Then I is principal.*

Proof. Let M be an A -module. Since $I\widehat{A}$ is principal, $I\widehat{A}$ is flat, so $\text{Tor}_1^{\widehat{A}}(I\widehat{A}, M \otimes_A \widehat{A}) = 0$. Consequently, by [26, Tag 00M8] we have

$$0 = \text{Tor}_1^{\widehat{A}}(I\widehat{A}, M \otimes_A \widehat{A}) = \text{Tor}_1^{\widehat{A}}(I \otimes_A \widehat{A}, M \otimes_A \widehat{A}) = \text{Tor}_1^A(I, M) \otimes_A \widehat{A},$$

so by faithful flatness of $A \rightarrow \widehat{A}$ we conclude $\text{Tor}_1^A(I, M) = 0$. As M was arbitrary, we deduce that I is a flat A -module, hence I is a principal ideal. \square

Lemma 3.18. *Let (A, \mathfrak{m}) be a regular local ring and $I \subseteq A$ an ideal such that $I\widehat{A}$ is principal and generated by an element g^s for some s in \mathbb{Z}^+ , g in \widehat{A} . Suppose that for all prime elements $\pi \in A$, the formal fiber $\widehat{A} \otimes_A A_{(\pi)} / \pi A_{(\pi)}$ is reduced. Then there exists h in A such that $I = h^sA$.*

Proof. By Lemma 3.17, choose f in A such that $I = fA$. As $I\widehat{A} = f\widehat{A} = g^s\widehat{A}$, it follows from Lemma 3.14 that $f = ug^s$ for some u in \widehat{A}^\times .

Recall again that A, \widehat{A} are UFDs [1]. Factor f in A as $f = v\pi_1^{d_1} \dots \pi_l^{d_l}$, with v in A^\times and π_1, \dots, π_l pairwise coprime and irreducible. In \widehat{A} , factor each π_i as $\rho_{i1}^{e_{i1}} \dots \rho_{is_i}^{e_{is_i}}$ where each ρ_{ij} is irreducible and $\rho_{ij}, \rho_{ij'}$ are coprime for $j \neq j'$. By assumption that the formal fiber $\widehat{A} \otimes_A A_{(\pi_i)} / \pi_i A_{(\pi_i)}$ is reduced for all $1 \leq i \leq l$, it follows that $e_{ij} = 1$ for all $1 \leq i \leq l, 1 \leq j \leq s_i$. Moreover, as $\rho_{ij}\widehat{A} \cap A = \pi_iA$, it follows that the $\rho_{ij}, \rho_{i'j'}$ are coprime for all $(i, j) \neq (i', j')$. Writing

$$ug^s = f = v\pi_1^{d_1} \dots \pi_l^{d_l} = \prod_{i=1}^l \prod_{j=1}^{s_i} \rho_{ij}^{d_i},$$

we conclude that $s \mid d_i$ for all $1 \leq i \leq l$. Setting $h = \pi_1^{d_1/s} \dots \pi_l^{d_l/s}$, we have $I = h^sA$. \square

Proof of Theorem 3.1. Let (A, \mathfrak{m}) be a regular local ring of characteristic $p > 0$ and $\mathfrak{a} \subseteq A$ an ideal with $\text{ord}_{\mathfrak{m}}(\mathfrak{a}) = d > 0$. Factor d as qs , where $q = p^e$ and $\gcd(p, s) = 1$. Suppose $\text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$. By Lemma 3.4 we have

$$\text{ft}^{\mathfrak{m}\widehat{A}}(\mathfrak{a}\widehat{A}) = \text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d} = \frac{1}{\text{ord}_{\mathfrak{m}}(\mathfrak{a})} = \frac{1}{\text{ord}_{\mathfrak{n}}(\mathfrak{a}\widehat{A})}.$$

By Lemma 3.16, there exists g in $\mathfrak{m}^{[q]}\widehat{A}$ such that $g^s = \mathfrak{a}\widehat{A}$. If we additionally assume that for all prime elements π of A that the formal fiber $\widehat{A} \otimes_A A_{(\pi)}/\pi A_{(\pi)}$ is reduced, then by Lemma 3.18 there exists h in $\mathfrak{m}^{[q]}$ such that $\mathfrak{a} = h^s A$.

Conversely, suppose that $\mathfrak{a}\widehat{A} = g^s\widehat{A}$ for some g in $\mathfrak{m}^{[q]}\widehat{A}$. By Lemmas 3.2 and 3.4 we have $\text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$. Similarly, if there exists h in $\mathfrak{m}^{[q]}$ such that $\mathfrak{a} = h^s A$, then $\text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$ by Lemma 3.2. \square

Our hypothesis on the formal fibers is necessary by Example 3.20, which relies on the following lemma.

Lemma 3.19. *Let (T, \mathfrak{m}) be a complete local domain of dimension at least 2 which satisfies Serre's condition S_2 . Suppose T has a coefficient field k . Let x be a nonzero element of \mathfrak{m} . Then there exists a Noetherian local subring $(A, \mathfrak{m} \cap A)$ such that $x \in A$, $\widehat{A} = T$, and x is a prime element of A .*

Proof. Let $C = \{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$ denote the set of minimal primes over x . Let Π denote the prime subring of T – that is, $\Pi = \mathbb{Q}$ if $\text{char } k = 0$ and $\Pi = \mathbb{F}_p$ if $\text{char } k = p > 0$. The claim follows once we check that this setup satisfies the hypotheses of [6, Theorem 1.1].

- (0) The primes $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ have height at most 1, and are thus nonmaximal. Moreover, we have $x \in \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r$.
- (1) As $\text{Ass}(T) = \{(0)\}$, we trivially have $\mathfrak{p} \cap \Pi[x] = (0)$ for all \mathfrak{p} in $\text{Ass}(T)$.
- (2) As T is S_2 and x is a regular element, it follows that T/xT is S_1 . Consequently, the associated primes of T/xT are precisely the minimal primes of T/xT , so $\text{Ass}(T/xT) = C$.
- (3) For all $1 \leq i \leq r$, we claim that $\text{Frac}(\Pi[x]) \cap \mathfrak{q}_i \subseteq xT$. To see this, suppose that there exist d, e in $\mathbb{Z}_{\geq 0}$, $a_1, \dots, a_d, b_1, \dots, b_e$ in Π , f in \mathfrak{q}_i such that

$$(13) \quad \frac{x^d + a_1x^{d-1} + \dots + a_d}{x^e + b_1x^{e-1} + \dots + b_e} = f.$$

Moreover, we suppose that the left-hand side of Equation (13) is written in reduced form. Then we have

$$a_d = x(x^{d-1} + a_1x^{d-2} + \dots + a_{d-1}) - f(x^e + b_1x^{e-1} + \dots + b_e) \in \mathfrak{m}.$$

As $a_d \in \Pi \cap \mathfrak{m}$, it follows that $a_d = 0$, so we have

$$b_e f = x(x^{d-1} + a_1x^{d-2} + \dots + a_{d-1} - f(x^{e-1} + b_1x^{e-2} + \dots + b_{e-1})) \in xT.$$

As the left-hand side of Equation (13) is written in reduced form and x divides the numerator $x^d + \cdots + xa_{d-1}$, it follows that $b_e \neq 0$, so $f \in xT$. \square

As an application of the above lemma, we show that for all $d > 0$, there exist many regular local rings (A, \mathfrak{m}) and prime elements f in A such that $\text{ord}_{\mathfrak{m}}(f) = d$, $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{d}$.

Example 3.20. Let k be a field of characteristic $p > 0$, $T = k[[x_1, \dots, x_n]]$ and $\mathfrak{m} = (x_1, \dots, x_n)$. Fix some $e \geq 0$, $q = p^e$, and $s > 0$ such that $\gcd(s, p) = 1$. Choose g in $\mathfrak{m}^{[q]}$ such that $\text{ord}_{\mathfrak{m}}(g) = q$ and set $f = g^s$. By Lemma 3.2, we have $\text{ft}^{\mathfrak{m}}(f) = \frac{1}{s} \text{ft}^{\mathfrak{m}}(g) = \frac{1}{qs} = \frac{1}{\text{ord}_{\mathfrak{m}}(f)}$.

By Lemma 3.19, there exists a Noetherian local subring $(A, \mathfrak{m} \cap A)$ such that $f \in A$, $\widehat{A} = T$, and such that f is a prime element of A . Let $\mathfrak{n} := \mathfrak{m} \cap A$. As argued in Lemma 3.15, we have $\text{ft}^{\mathfrak{n}}(f) = \text{ft}^{\mathfrak{m}}(f)$ and $\text{ord}_{\mathfrak{n}}(f) = \text{ord}_{\mathfrak{m}}(f)$, so $\text{ft}^{\mathfrak{n}}(f) = \frac{1}{\text{ord}_{\mathfrak{n}}(f)}$.

To conclude this article, we consider the effect on the main theorem of adding an additional reducedness hypothesis.

Example 3.21. If R is a regular local ring, x, y part of a regular system of parameters for R , and a, b coprime integers, then $\widehat{R}/(x^a - y^b)\widehat{R}$ is geometrically integral. To see this, writing $\widehat{R} = k[[x, y, z_1, \dots, z_n]]$, we have $\widehat{R}/(x^a - y^b)\widehat{R} \cong k[[t^a, t^b, z_1, \dots, z_n]]$.

Let (R, \mathfrak{m}) be a regular local ring of characteristic $p > 0$. Let $\mathfrak{a} \subseteq R$ such that $\text{ord}_{\mathfrak{m}}(\mathfrak{a}) = d$, $\text{ft}^{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{d}$, and $\widehat{R}/\mathfrak{a}\widehat{R}$ is reduced. Factor d as $d = qs$, where $q = p^e$ and $\gcd(s, p) = 1$. By Theorem 3.1, there exists g in $\mathfrak{m}^{[q]}\widehat{R}$ such that $\mathfrak{a}\widehat{R} = g^s\widehat{R}$, so reducedness of $\widehat{R}/g^s\widehat{R}$ forces $s = 1$. By Lemma 3.3 we also have $\mathfrak{a} \subseteq \mathfrak{m}^{[q]}$. For any $q = p^e$, if x, y is part of a regular system of parameters for R , then with $f = x^q - y^{q+1}$, we have $\text{ord}_{\mathfrak{m}}(f) = q$ by construction, $\text{ft}^{\mathfrak{m}}(f_q) = \frac{1}{q}$ by Lemma 3.2, and $\widehat{R}/f_q\widehat{R}$ is *geometrically* reduced.

To show that $\frac{1}{d}$ is an optimal lower bound on $\text{ft}^{\mathfrak{m}}(\mathfrak{a})$ when $\text{ord}_{\mathfrak{m}}(\mathfrak{a}) = d$ and $\widehat{R}/\mathfrak{a}\widehat{R}$ is reduced (or geometrically reduced), suppose $\dim R \geq 2$ and x, y is part of a regular system of parameters for R . Setting $g_{d,t} = x^d - y^{td+1}$, we have $\text{ft}^{\mathfrak{m}}(g_{d,t}) \leq \text{ft}^{\mathfrak{m}}((x^d, y^{td+1})) \leq \frac{1}{d} + \frac{1}{td+1} \searrow \frac{1}{d}$, where the inequality is by Proposition 2.7 (1) and the equality by [15, Proposition 36].

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