CONVOLUTION
&
LTI Systems

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Prologue

Convolution & LTI Systems are the heart and soul of Digital Signal Processing (DSP). Yet the CONVOLUTION operation has neither been defined nor explained clearly. This pedagogical gem redeems this deficiency. The flow is smooth. The presentation is lucid, precise and concise.

This monogram is also a tribute to the Management and Faculty of NMAMIT. True to the ***alma mater***, this work is par excellence. The authors (all students) are to be congratulated.

Nitte | Stephen Vadakkan

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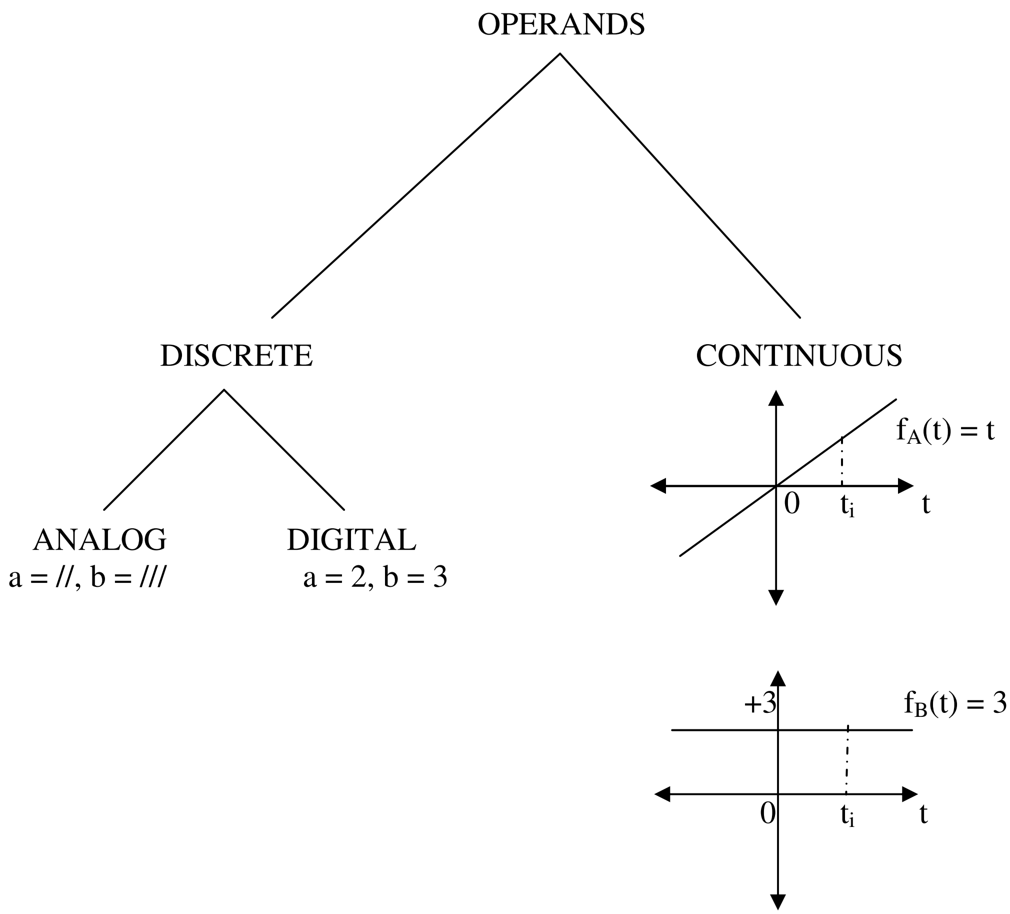
INTRODUCTION

Digital Signal Processing (DSP) is a vast and rich subject. One sees the convergence of many beautiful ideas from Physics, Mathematics and Electronics. While it can be difficult in some areas, the basic concepts and calculations can be understood with a few good examples based on the knowledge of just Higher Secondary level Physics and Mathematics.

This modest module is intended as a preview to material in standard text-books. It is neither rigorous nor exhaustive. It is meant to be intuitive – to give the beginner a feel for the subject. After going through this module it is strongly recommended that the student reads one of the best books on this subject:

The Scientist and Engineer's Guide to
Digital Signal Processing
by Steven W. Smith.

In high school we learn several types of operations on ***discrete*** operands: + and – , • and /, exponentiation as in x^n and its two inverses: taking the n^{th} root of x denoted by $\sqrt[n]{x}$ and taking the log of x to some base b denoted by $\log_b x$. We also learned two operations: differentiation and integration on well-behaved functions. We shall call these functions ***continuous*** operands. Let us briefly review the + and • binary operations as applied to both types of operands.



There is also a third type of operand known as ***digitized.*** We may digitize discrete operands and continuous operands. Digitizing continuous operands is a three step process: discretize, quantize and then encode.

Continuous operands may also be ***digitalized***. We may digitalize continuous operands using natural or what is called analog frequencies or we may digitalize continuous operands using (discrete) digital frequencies. Digitalized operands are also discrete.

Very often when books say digital, as in Analog to Digital Converter (ADC), they actually mean digitized. In digitized operands there is no frequency information. Where as in digitalized operands there is embedded/hidden somewhere the frequency information.

Generally, after we digitize a continuous operand we cannot recover the original continuous operand from the discrete digitized operand. But we can recover the original continuous operand from a discrete digitalized operand.

ADDITION OPERATION: +

ANALOG

$a + b = c$

// + /// = /////

concatenation

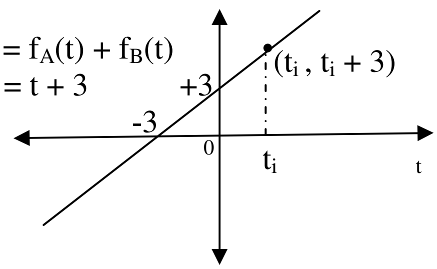
DIGITAL

$2 + 3 = 5$

addition

CONTINUOUS

$f_C(t) = f_A(t) + f_B(t)$
$= t + 3$



Notice that the addition is at ***corresponding*** instants

MULTIPLICATION OPERATION: •

ANALOG

$a \bullet b = c$

Repeated concatenation where the number of strokes are the same

a • b = // // // = //////
b • a = /// /// = //////

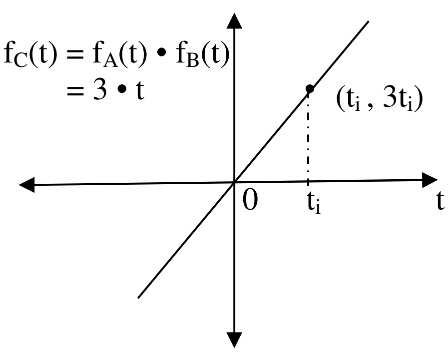
DIGITAL

Repeated addition where the addends are the same

$a \bullet b = 2+2+2 = 6$
$b \bullet a = 3 + 3 = 6$

CONTINUOUS

$f_C(t) = f_A(t) \bullet f_B(t)$
$= 3 \bullet t$



Notice that the multiplication is at ***corresponding*** instants

We are now going to learn a new binary operation called CONVOLUTION denoted by *. We may apply this to both discrete operands and continuous operands. CONVOLUTION is the heart of the DSP. It is **the** operation in Linear Time Invariant (LTI) Systems.

TWO GUITARS EXAMPLE

Imagine you have two guitars; g and G. Guitar g has only one string $\{s_3\}$ and guitar G has five strings $\{S_1, S_2, S_3, S_4, S_5\}$. Guitar g is with you and guitar G is a few meters away from you. You strum guitar g. What happens?

The sound waves or vibrations emanating from the sole string s_3 of guitar g reach and impact on all or EACH of the 5 strings of guitar G. What is the EFFECT of the impact of s_3 on EACH of the strings of G? $s_3 \bullet S_1$, $s_3 \bullet S_2$, $s_3 \bullet S_3$, $s_3 \bullet S_4$, $s_3 \bullet S_5$.

What is the sum total of all these EFFECTS: $s_3 \bullet S_1 + s_3 \bullet S_2 + s_3 \bullet S_3 + s_3 \bullet S_4 + s_3 \bullet S_5$. This we call the TOTAL EFFECT SUM and can be written as $(\sum s_3 \bullet S_j)$ for j = 1, 2, 3, 4, 5.

Only the ***corresponding*** string S_3 of guitar G resonates or RESPONDS. So the TOTAL EFFECT SUM is $s_3 \bullet S_3$. This is known as the ***frequency response.***

Now reverse the scenario. Keep guitar G with you and place guitar g a few meters away. Strum all five strings of G. What happens?

The sound waves or vibrations emanating from EACH of the five strings $\{S_1, S_2, S_3, S_4, S_5\}$ of G reach and impact the sole string s_3 of guitar g. What is the EFFECT of EACH of the five different sound waves from G on g?

$$S_1 \bullet s_3,\ S_2 \bullet s_3,\ S_3 \bullet s_3,\ S_4 \bullet s_3,\ S_5 \bullet s_3$$

What is the sum total of all these effects: $S_1 \bullet s_3 + S_2 \bullet s_3 + S_3 \bullet s_3 + S_4 \bullet s_3 + S_5 \bullet s_3$. This we call the TOTAL EFFECT SUM and can be written as $(\sum S_j \bullet s_3)$ for j = 1, 2, 3, 4, 5.

The sole string s_3 of guitar g RESPONDED only to the ***corresponding*** string S_3 of guitar G. So the TOTAL EFFECT SUM is $S_3 \bullet s_3$. The other waves had no EFFECT or produced zero ***frequency response***:

$$S_i \bullet s_j = 0 \text{ when } i \neq j.$$

Now let us upgrade guitar g to have five strings $\{s_1, s_2, s_3, s_4, s_5\}$. Keep g with you. Place G a few meters away from you. Now strum all five strings of g. What happens?

We know that the sound waves or vibrations of EACH of the five strings of g reach and impact EACH of the five strings of G. An EFFECT is $s_i \bullet S_j$. There are 5 x 5 = 25 such

EFFECTS. An EFFECT SUM i = ($\sum s_i \bullet S_j$) for j = 1, 2, 3, 4, 5. We know this EFFECT SUM i is equal to $s_i \bullet S_i$. There are 5 such EFFECT SUM i for i = 1, 2, 3, 4, 5.

We know from the previous two scenarios that EACH of the five strings of guitar G respond to EACH of the ***corresponding*** strings of guitar g. Also, we know from experience, that we do not hear the individual responses separately. What we hear is a TOTAL EFFECT SUM or sum of the 5 EFFECT SUM i = $\{s_1 \bullet S_1 + s_2 \bullet S_2 + s_3 \bullet S_3 + s_4 \bullet S_4 + s_5 \bullet S_5\}$.

How do we mathematically represent the computation process of the TOTAL EFFECT SUM of EACH string of g on EACH string of G?

Let:

$$g = \{s_1, s_2, s_3, s_4, s_5\} \qquad \text{sequence of strings}$$

$$G = \{S_1, S_2, S_3, S_4, S_5\} \qquad \text{sequence of strings}$$

Note that in a sequence, as opposed to a set, the order of terms in the sequence is important.

One way to represent the computation process is :

$$\begin{array}{ccccccccc}
\{S_1, S_2,S_3,S_4,S_5\} & & \{S_1, S_2, S_3, S_4, S_5\} & & \{\ldots\} & & \{\ldots\} & & \{S_1, S_2, S_3, S_4, S_5\} \\
\bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
\underline{\quad s_1 \quad} & & \underline{\quad s_2 \quad} & & \underline{\quad s_3 \quad} & & \underline{\quad s_4 \quad} & & \underline{\quad s_5 \quad} \\
s_1 \bullet S_1 & + & s_2 \bullet S_2 & + & s_3 \bullet S_3 & + & s_4 \bullet S_4 & + & s_5 \bullet S_5 \\
\text{EFFECT SUM} & & \text{EFFECT SUM} & & \text{EFFECT SUM} & & \text{EFFECT SUM} & & \text{EFFECT SUM}
\end{array}$$

This is also referred to as a ***weighted sum*** where the S_j are "weighted" by s_i and then summed up.

The MULTIPLICATION or SUPERPOSITION computes the EFFECT of the string s_i on the string S_j. The EFFECT is:

$$\begin{array}{lll} & s_{i.} \bullet S_j \neq 0 & \text{for } i = j \\ \text{and} & s_i \bullet S_j = 0 & \text{for } i \neq j \end{array}$$

This yields :{ $s_1 \bullet S_1$, $s_2 \bullet S_2$, $s_3 \bullet S_3$, $s_4 \bullet S_4$, $s_5 \bullet S_5$}, the EFFECT SUM sequence.

The summation of the EFFECT SUM sequence is the TOTAL EFFECT SUM of EACH on EACH. This is: $\{s_1 \bullet S_1 + s_2 \bullet S_2 + s_3 \bullet S_3 + s_4 \bullet S_4 + s_5 \bullet S_5\}$, which is what we hear.

We may formularize this computation process:

$$\left\{ \sum_{i=1}^{5} s_i \bullet \left(\sum_{j=1}^{5} s_j \right) \right\}$$

ASYMMETRIC versus SYMMETRIC

What happens if we REVERSE strum the strings of guitar g? That is to say , instead of strumming downwards $\{s_1,s_2,s_3,s_4,s_5\}$ we strum upwards $\{s_5,s_4,s_3,s_2,s_1\}$.

$$\left\{\sum_{i=5}^{1} s_i \bullet \left(\sum_{j=1}^{5} s_j\right)\right\}$$

This yields :$\{ s_5 \bullet S_5, s_4 \bullet S_4, s_3 \bullet S_3, s_2 \bullet S_2, s_1 \bullet S_1\}$, the EFFECT SUM sequence.

The EFFECT SUM sequences are not the same. They are ASYMMETRIC. In both cases the computation takes about (25+4) additions and about 25 multiplications.

The TOTAL EFFECT SUM is the same in both cases. What we hear is the same. But there is a difference in the order in the EFFECT SUM sequence. And in some situations, knowing the individual terms of a sequence and when they are computed or occur is of importance.

Can we think of a computational process where the EFFECT SUM sequences or order of the individual terms, are the same?

Let us begin by REVERSING the strings of guitar g. So now g = $\{s_5, s_4, s_3, s_2, s_1\}$.

This process of REVERSING or REFLECTING is known as CONVOLVE. We may write down the computation process as:

Step 1: CONVOLVE g.

$$\begin{array}{lccccccccccc}
\text{G:} & S_1 & S_2 & S_3 & S_4 & S_5 & * & & & & \\
\text{g convolved:} & & & & & & s_5 & s_4 & s_3 & s_2 & s_1
\end{array}$$

Step 2: SHIFT

$$\begin{array}{ccccccccc}
S_1 & S_2 & S_3 & S_4 & S_5 & * & & & \\
 & & & & s_5 & s_4 & s_3 & s_2 & s_1
\end{array}$$

Step 3: MULTIPLY

$$\begin{array}{ccccccccc}
S_1 & S_2 & S_3 & S_4 & S_5 & * & & & \\
 & & & & s_5 & s_4 & s_3 & s_2 & s_1 \\
\hline
 & & & & s_5 \bullet S_5 & & & &
\end{array}$$

Step 4: SHIFT AND MULTIPLY

$$\begin{array}{cccccccc}
S_1 & S_2 & S_3 & S_4 & S_5 & * & & \\
 & & & s_5 & s_4 & s_3 & s_2 & s_1 \\
\hline
 & & & & s_5 \bullet S_5 & & & \\
 & & & s_5 \bullet S_4 & & & & \\
 & & & s_4 \bullet S_5 & & & &
\end{array}$$

We may continue with this step by step “SHIFT and MULTIPLY” process until EACH of the strings of g has SHIFTED across and EFFECTED all or EACH of the strings of G to yield:

					S_1	S_2	S_3	S_4	S_5	*
s_5	s_4	s_3	s_2	s_1						
	$s_1 \bullet S_1$	$s_2 \bullet S_1$ $s_1 \bullet S_2$	$s_3 \bullet S_1$ $s_2 \bullet S_2$ $s_1 \bullet S_3$	$s_4 \bullet S_1$ $s_3 \bullet S_2$ $s_2 \bullet S_3$ $s_1 \bullet S_4$	$s_5 \bullet S_1$ $s_4 \bullet S_2$ $s_3 \bullet S_3$ $s_2 \bullet S_4$ $s_1 \bullet S_5$	$s_5 \bullet S_2$ $s_4 \bullet S_3$ $s_3 \bullet S_4$ $s_2 \bullet S_5$	$s_5 \bullet S_3$ $s_4 \bullet S_4$ $s_3 \bullet S_5$	$s_5 \bullet S_4$ $s_4 \bullet S_5$	$s_5 \bullet S_5$	
	$s_1 \bullet S_1$	0	$s_2 \bullet S_2$	0	$s_3 \bullet S_3$	0	$s_4 \bullet S_4$	0	$s_5 \bullet S_5$	

Finally, when there is no more SHIFTING and MULTIPLICATION (superposition) to be done, we can add up each individual column to get what we now call the CONVOLUTION SUM.
The CONVOLUTION SUMS form a sequence: $s_1 \bullet S_1$, 0, $s_2 \bullet S_2$, 0, $s_3 \bullet S_3$, 0, $s_4 \bullet S_4$, 0, $s_5 \bullet S_5$.
We may sum up these CONVOLUTION SUMS to get the TOTAL CONVOLUTION SUM.

How would we formularize this computation process?

We re-number the N = 5 strings with: i = 0 to 2N-1. Here 2N-1 = 9. Only strings 0 to 4 exist. The extra strings s_5 through s_9 and S_5 through S_9 do NOT exist and hence are zero. This is known as PADDING WITH ZEROS. And now we may write out the SYMMETRIC formula for the TOTAL CONVOLUTION SUM as:

$$\sum_{i=0}^{2N-1=9} \left\{ \sum_{k=0}^{i} s_k \bullet \ s_{i-k} \right\} = \sum_{i=0}^{2N-1=9} \left\{ \sum_{k=0}^{i} s_{i-k} \bullet s_k \right\}$$

We get nine CONVOLUTION SUMS which we number from 0 to 2N–2 = 8.
This process or operation of finding the symmetric sequence of CONVOLUTION SUMS is called CONVOLUTION. We may denote this process of computation by g*G.

The formula for CONVOLUTION SUM i is:

$$\left\{ \sum_{k=0}^{i} s_k \bullet \ s_{i-k} \right\} = \left\{ \sum_{k=0}^{i} s_{i-k} \bullet \ s_k \right\}$$

We note that the computation process is SYMMETRIC with respect to the sequence of CONVOLUTION SUMS:

The top two rows of G * g computation process is:

$$S_0\ S_1\ S_2\ S_3\ S_4\ S_5\ S_6\ S_7\ S_8\ S_9 *$$
$$s_9\ s_8\ s_7\ s_6\ s_5\ s_4\ s_3\ s_2\ s_1\ s_0$$

The top two rows of g * G computation process is:

$$s_0\ s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8\ s_9 *$$
$$S_9\ S_8\ S_7\ S_6\ S_5\ S_4\ S_3\ S_2\ S_1\ S_0$$

Despite this difference, the CONVOLUTION SUMS appear in the same order.

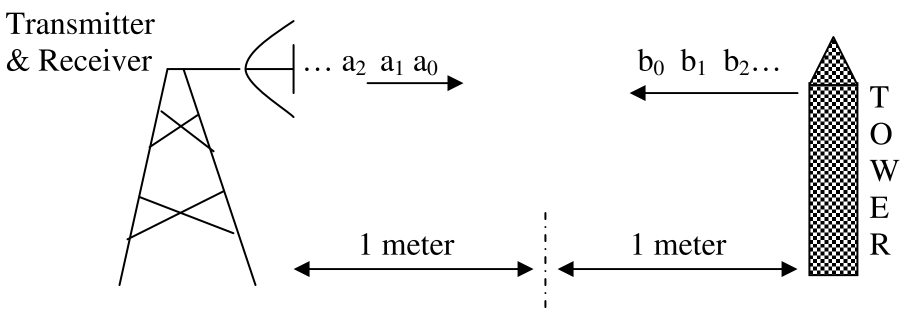
$$G * g \quad = \quad g * G$$

The convolution operation * is ***commutative.***

Now it seems that we did some more work (computation) using the above SYMMETRIC formulas to get the CONVOLUTION SUMS in the same order. It looks like we did some 45 additions and 45 multiplications. As an exercise, you may write out the computation tables for (k = 0 to i) for each CONVOLUTION SUM i (i = 0 to 9).We notice that the CONVOLUTION SUMS with even index i (i = 0, 2, 4, 6 and 8) are NON-ZERO. The odd index CONVOLUTION SUMS are zero. With such prior knowledge, we compute only the even index terms and we are down to (25 + 4) or so additions and 25 multiplications. It looks like we got better information with the same computation effort.

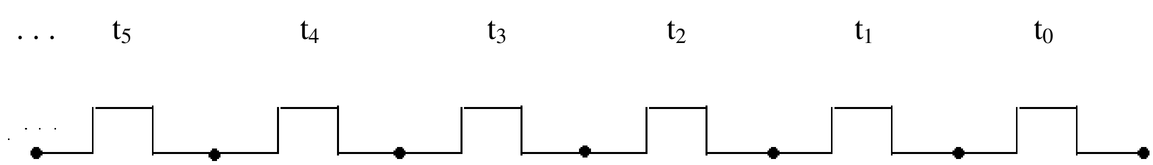
DISTANCE MEASURMENT USING CONVOLUTION

Let us now see a simple example where the order of the CONVOLUTION SUMS is important.



SPEED OF LIGHT=3 x 10^8 m/sec CLOCK PULSE: 3 x 10^8 cycles/sec [Hz]

BELOW IS THE TIMING DIAGRAM



a_5	a_4	a_3	a_2	a_1	a_0
0	0	0	0	1	1
0	1	1	0	0	0
b_5	b_4	b_3	b_2	b_1	b_0

Assuming we know the TRANSMITTED pulse sequence a[n] and the distance 2 meters, we may determine the RECEIVED pulse sequences b[n] shown above.

Now assume we do NOT know the distance. But we know the TRANSMITTED and RECEIVED pulse sequences a[n] and b[n].

How can we find the distance?

Let us now CONVOLVE the transmitted signal a[n] and the received signal b[n] to get

$$c[n] = a[n] * b[n]$$

• • • a_6	a_5	a_4	a_3	a_2	a_1	a_0	*
• • • 0	0	0	0	0	1	1	0 0 0 1 1 0…
							b_0 b_1 b_2 b_3 b_4 b_5 b_6
						$a_0 \bullet b_0$ 0	
					$a_1 \bullet b_0$ + $a_0 \bullet b_1$ 0+0		
				$a_2 \bullet b_0$+ $a_1 \bullet b_1$+ $a_0 \bullet b_2$ 0+0+0			
			$a_3 \bullet b_0$+ $a_2 \bullet b_1$+ $a_1 \bullet b_2$ + $a_0 \bullet b_3$ 0+0+0+1				
		$a_4 \bullet b_0$+ $a_3 \bullet b_1$+ $a_2 \bullet b_2$+ $a_1 \bullet b_3$+ $a_0 \bullet b_4$ 0+0+0 +1+1					
	$a_5 \bullet b_0$+ $a_4 \bullet b_1$+ $a_3 \bullet b_2$+ $a_2 \bullet b_3$+ $a_1 \bullet b_4$+ $a_0 \bullet b_5$ 0+0+0+ 0+0+1						
	1	2	1	0	0	0	c_i CONVOLUTION SUM
	5	4	3	2	1	0	i CONVOLUTION INDEX

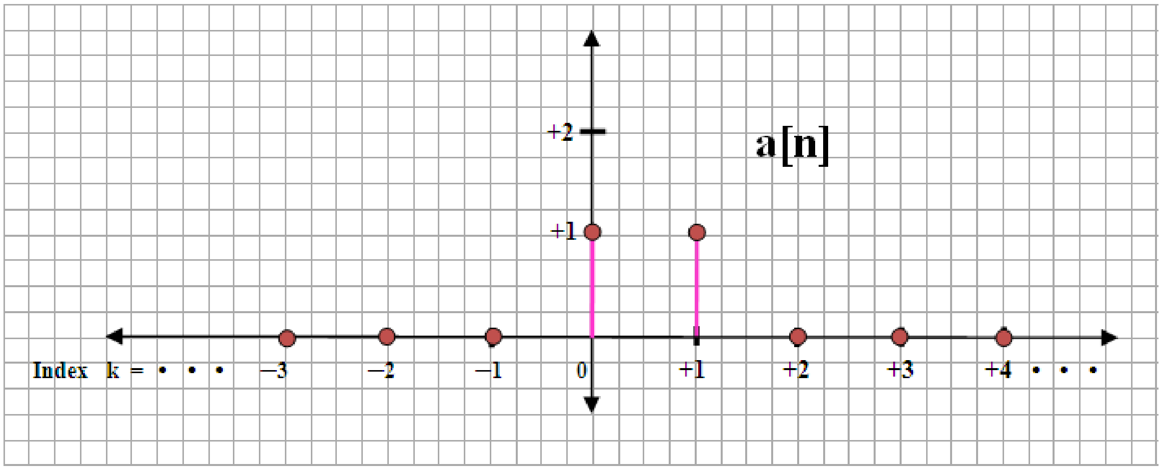
Peak CONVOLUTION SUM $c_i = 2$ is at CONVOLUTION INDEX $i = 4$, that is to say 4 clock pulses or 2 clock pulses each way.

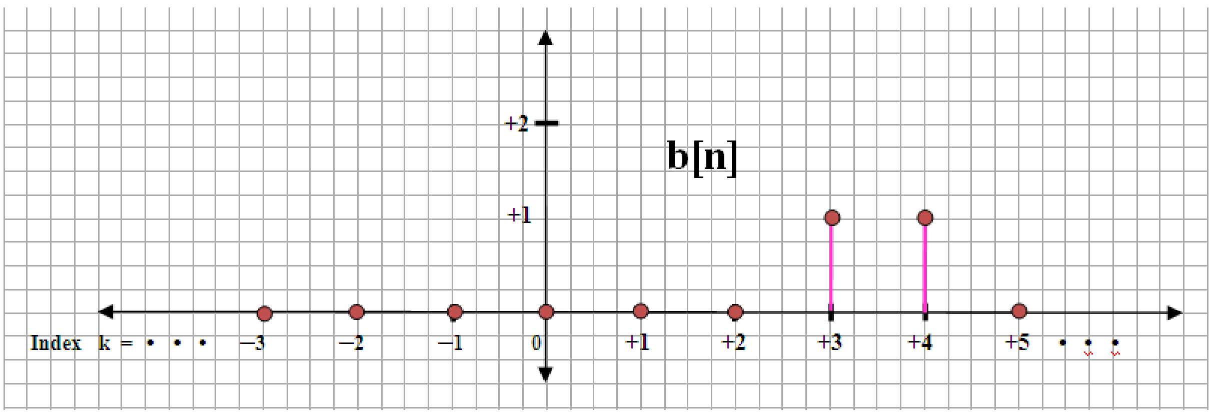
Distance = speed x time

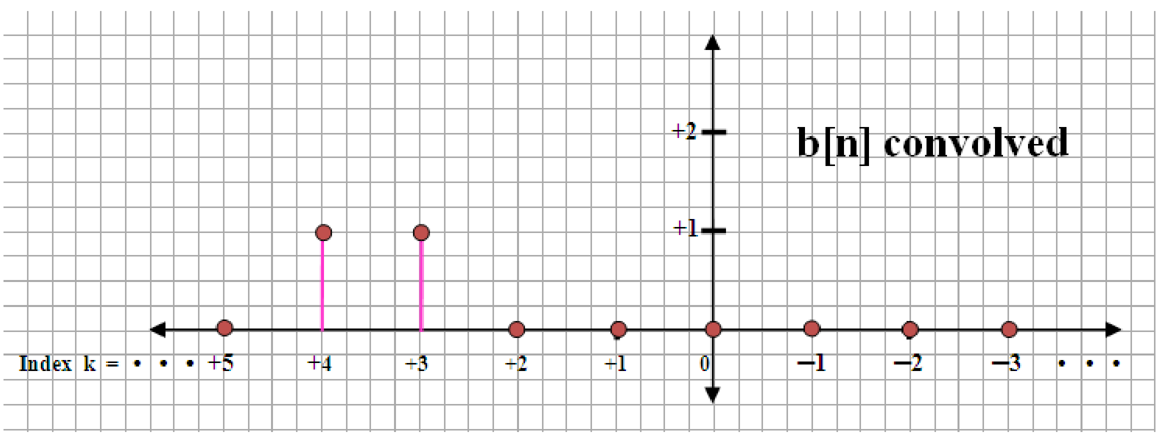
$= (3 \times 10^8 \text{ [m/sec]}) \times 2/(3 \times 10^8 \text{ [sec]})$

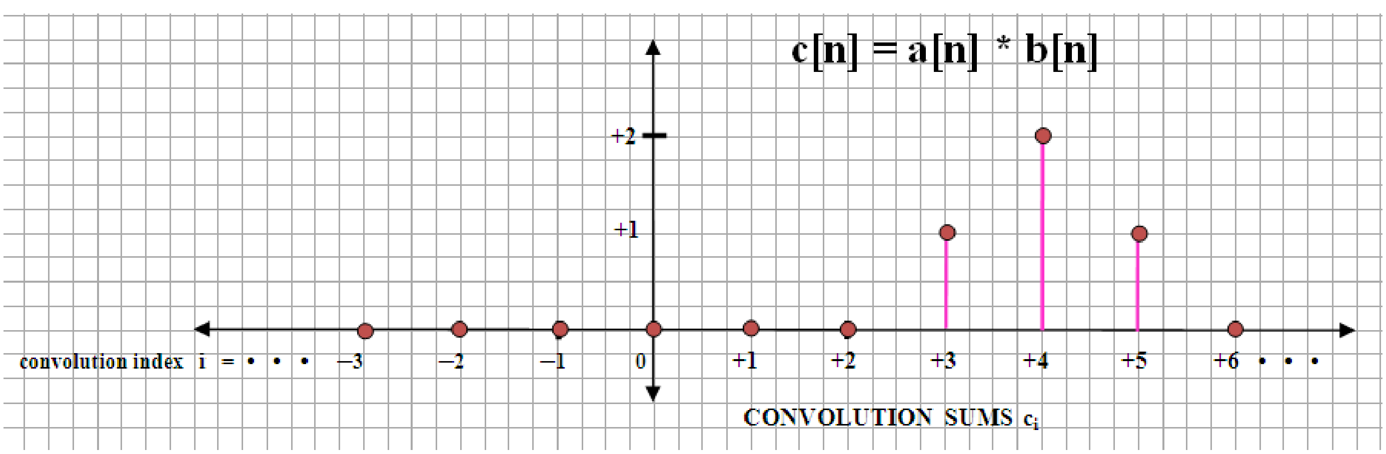
= 2 meters

We may represent the ***pure sequences*** a[n], b[n] and c[n] with lollipop diagrams.









So:

1. We need to know the PEAK value. In this case it is 2. In general, we need to know each CONVOLUTION SUM c_i .
2. We need to know where / when (in time) the PEAK occurs. In this case it is at index i = 4. In general, we need to know the CONVOLUTION INDEX.

This is the principle behind a pulse radar. Does this look like our familiar LONG MULTIPLICATION?

If we perform the LONG MULTIPLICATION of 11 and 11000, we get:

	1	2	1	0	0	0
index	5	4	3	2	1	0

LONG MULTIPLICATION WITHOUT CARRY

There is nothing new or mysterious about this method of computation. We encountered this method in fourth grade under the name LONG MULTIPLICATION. To compute 632 X 521, we wrote down the process.

			6	3	2	X
			5	2	1	
	30	15	10	0	0	
		12	6	4	0	
			6	3	2	
	30	27	22	7	2	
10^5	10^4	10^3	10^2	10^1	10^0	

This is what the result looks like without the CARRY. In fourth grade we also did the CARRY to the next higher place.

With the CARRY the result is: 329272.What is the advantage of the CARRY?

It is easy to compare two numbers in decimal digital form by inspection. For example, it is easier to see that,

10^1	10^0		10^1	10^0		10^1	10^0		10^1	10^0
3	2	<	3	3	rather than	3	2	?	2	13

More precisely and formally we may write this procedure of 632 X 521 as:

			6	3	2	* 1 2 5
					1•2	
				1•3 2•2		
			1•6 2•3 5•2			
		2•6 5•3				
	5•6					
	30	27	17	7	2	: sequence of CONVOLUTION SUMS
10^5	10^4	10^3	10^2	10^1	10^0	
5	4	3	2	1	0	: i CONVOLUTION INDEX

SHIFT
MULTIPLY
ADD

We may index the CONVOLUTION SUMS in the order in which they appear. Notice that it is not necessary to know the most significant digits in advance. We may initiate the CONVOLUTION process given (• • • 632) * (• • • 521). This is useful in real time applications where we may not know all the input in advance, or in the design of Infinite Impulse Response (IIR) filters.

Notice that: 632 * 521 = 521 * 632. The sequence of the CONVOLUTION SUMS is the same.

This method which we call LONG MULTIPLICATION (without carry) is actually CONVOLUTION. It was invented by Vedic Mathematicians after they invented zero and the decimal digital system. This method is simple and fast when compared to the usual method of MULTIPLICATION: repeated addition where the addends are the same.

Take a close look at the column of 10^2 for example. Will the products (1•6, 2•3, 5•2) in that column appear in the same order in 521 * 632? Will it be inverted?

If we were to plot these products or ***super-position*** in this column on a graph in the order in which they are appear, will the graphs of this column in 632 * 521 and 521 * 632 be the same?

Later we shall actually plot the products in a column i and call it the ***super-position curve*** with CONVOLUTION INDEX i.

CONVOLUTION SUM

Instead of particular numbers like 632 and 521, let us represent them in an abstract way and see if there is a pattern in the summation of each column which can be made in to a formula.

particular:	• • • 0 0 0 6 3 2	• • • 0 0 0 5 2 1
abstract:	a_5 a_4 a_3 a_2 a_1 a_0	b_5 b_4 b_3 b_2 b_1 b_0

So now we have two ***pure sequences***

$a[n]$ where $a_k = 0$ for $k > 2$
and $b[n]$ where $b_k = 0$ for $k > 2$.

This is known as PADDING WITH ZEROS. We are required to find the ***pure sequence***

$$c[n] = a[n] * b[n]$$

		a_2	a_1	a_0	* b_0 b_1 b_2
a_2a_2	a_2b_1	a_2b_0	a_1b_0	a_0b_0	
	a_1b_2	a_1b_1	a_0b_1		SHIFT MULTIPLY
		a_0b_2			ADD
c_4	c_3	c_2	c_1	c_0	

Pay careful attention to the subscripts of the products in each column. Do you see a pattern?

$c_0 = a_0b_0$

$c_1 = a_0b_1 + a_1b_0$

$c_2 = a_0b_2 + a_1b_1 + a_2b_1$

In each case the formula is

$$\text{CONVOLUTION SUM } c_i = \sum_{k=0}^{i} a_k b_{i-k}$$

Now $c_3 = a_1b_2 + a_2b_1$
$= a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0$

Again:

$$c_i = \sum_{k=0}^{i} a_k b_{i-k}$$

because $b_3 = 0$ and $a_3 = 0$
Now $c_4 = a_2b_2 = a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0$

Again:

$$c_i = \sum_{k=0}^{i} a_k b_{i-k}$$

because $b_4 = b_3 = a_3 = a_4 = 0$

So due to the PADDING WITH ZEROS we can use the same formula for c_i for the summation of the products in each column i.

c_i is known as the CONVOLUTION SUM.

$$\text{CONVOLUTION SUM } c_i = \sum_{k=0}^{i} a_k b_{i-k}$$

If we PAD WITH ZEROS all the way to -∞ to +∞, we may write the general form of the CONVOLUTION SUM as:

$$c_i = \sum_{k=-\infty}^{\infty} a_k b_{i-k}$$

Exercise: To gain more insight, compute the following and write out the sequence of CONVOLUTION TERMS with the appropriate CONVOLUTION INDEX below each term.

a) 632 * 0.00521

b) 0.00632 * 521

c) 632000 * 521

d) 632 * 521000

e) 632000 * 0.00521

f) 0.00632 *521000

Notice how the sequence of CONVOLUTION SUMS 30, 27, 22, 7, 2 is SHIFTED right or left. Despite this SHIFTING, the sequence of CONVOLUTION SUMS between the first non-zero term and last non-zero term is the same. We may call this property SHIFT INVARIANCE.

It is not necessary that we stick to only decimal digital values in the operands. We could use any values, as we would expect if we were to discretize two real value continuous functions over the time axis.

Exercise: CONVOLVE the two sequences below and write the appropriate CONVOLUTION INDEX below each CONVOLUTION TERM.

		(1,	√2,	-3)	and	(√2,	-5,	1/2)
Index	k:	-1	0	1		2	3	4

Exercise: You know that with decimal digital numbers:

$$(632 * 521) * 489 = 632 * (521 * 489)$$

This property is known as ***associativity***. Verify that convolution is an ***associative*** operation: (a[n] * b[n]) * c[n] = a[n] * (b[n] * c[n]).

We know that * is ***commutative***: a[n] * b[n] = b[n] * a[n] .

Commutativity allows us to change the order of the operands.

Associativity allows us to change the order of the evaluation.

CATALAN numbers tells us how many different ways of associating n applications of a binary operator. So, we have a lot of flexibility in the way we may perform the calculations involving convolution. This directly translates into the flexibility of hardware design of LTI Systems.

DEFINITION OF TERMS

Lets us state the terminology for the various formulae before we proceed further. We make a distinction between ***pure sequences*** and sequences $f_A[t_k]$ and $f_B[t_k]$ got from $f_A(t)$ and $f_B(t)$, ***continuous*** functions over time, made ***discrete*** using equal spacing T_s. We shall call these ***discrete functions.***

DISCRETE

pure sequences

ASYMMETRIC

1. EFFECT
2. EFFECT SUM
3. TOTAL EFFECT SUM

SYMMETRIC

1. EFFECT
2. CONVOLUTION SUM
3. TOTAL CONVOLUTION SUM

discrete functions

$f_A(t) \longrightarrow f_A[t_k]$

$[T_A^{-}, T_A^{+}]$ $[N_A^{-}, N_A^{+}]$

$f_B(t) \longrightarrow f_B[t_k]$

$[T_B^{-}, T_B^{+}]$ $[M_B^{-}, M_B^{+}]$

SYMMETRIC

1. COMBINED EFFECT
2. COMBINED EFFECT SUM
3. TOTAL COMBINED EFFECT SUM

We have 3 kinds of intervals to deal with in the CONVOLUTION process:

1. Equal sub-intervals T_s
2. Overlapping interval = contiguous overlapping sub-intervals.

   We plot the ***super-position curve*** over the overlapping interval.

   $\sum$ of the area under the ***super-position curve*** = COMBINED EFFECT SUM

   $\int$ of the area under the ***super-position curve*** = CONVOLUTION INTEGRAL
3. Convolution interval = the entire interval over which the CONVOLUTION is taking place.

   continuous case: $[T_A^{-} + T_B^{-}, T_A^{+} + T_B^{+}]$ is the convolution interval.

   discrete case: $[N_A^{-} + M_B^{-}, N_A^{+} + M_B^{+}]$ is the convolution interval.

   We plot the convolution curve over the convolution interval.

   A point on the convolution curve is:
   discrete function case: COMBINED EFFECT SUM
   continuous function case: CONVOLUTION INTEGRAL

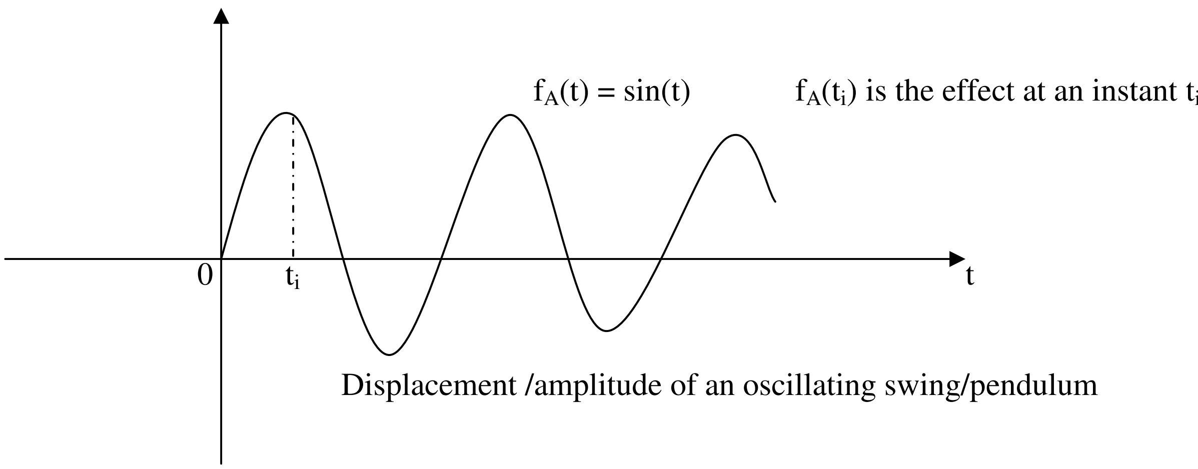
We have two special INSTANTS:

$$\text{super} - \text{position instant } \tau = \lim_{T_s \to 0} (k \,.\, T_s) \text{ in the overlapping interval.}$$

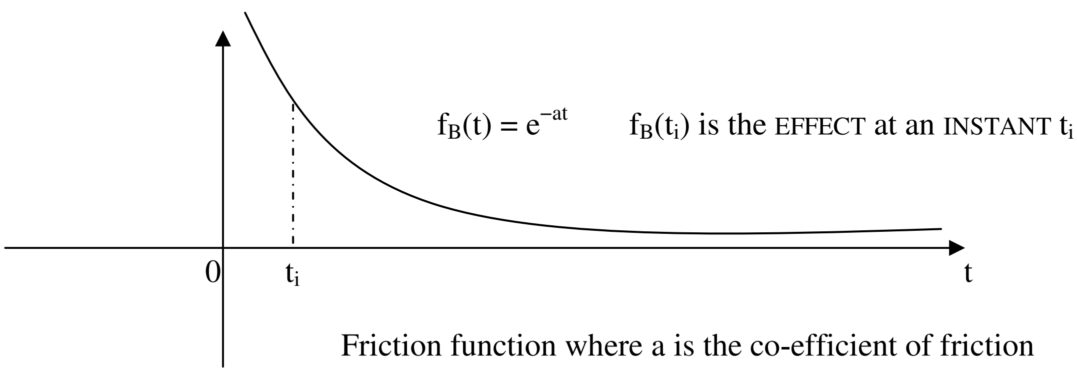
$$\text{convolution instant } t = \lim_{T_s \to 0} (i \,.\, T_s) \text{ in the convolution interval.}$$

Guitar strings form ***pure sequences.*** We want to be able to work with ***continuous functions*** and ***discrete functions***. We also need to go back and forth between both versions. So let us proceed to define more formally the CONVOLUTION SUM for discrete functions and then derive the CONVOLUTION INTEGRAL for ***continuous functions.***

1. EFFECT AT AN INSTANT

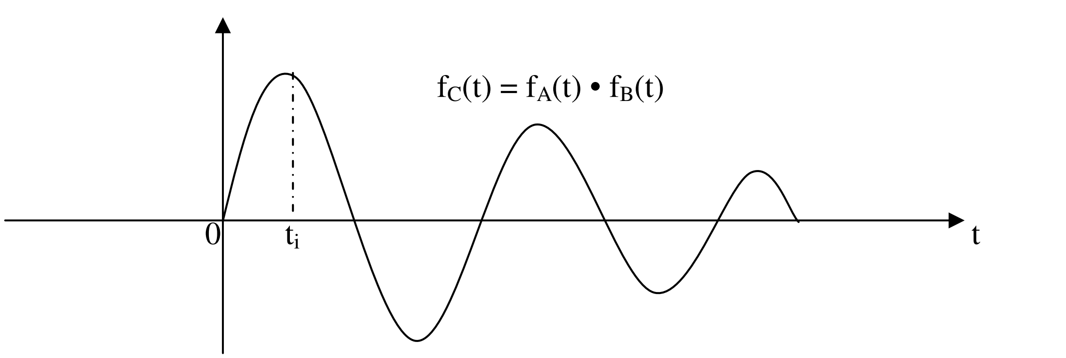


Displacement /amplitude of an oscillating swing/pendulum



Friction function where a is the co-efficient of friction

2. EFFECT AT CORRESPONDING INSTANTS



$f_A(t_i) \bullet f_B(t_i)$ is the EFFECT of $f_A(t)$ and $f_B(t)$ at CORESSPONDING instant t_i.

Another way to view this is: $f_C(t_i)$ is the EFFECT at INSTANT t_i.

3. COMBINED EFFECT OVER A SUB-INTERVAL (δt, Δt, T_s)

In high school Calculus, assuming $f(t) = \dfrac{dF(x)}{dx}$

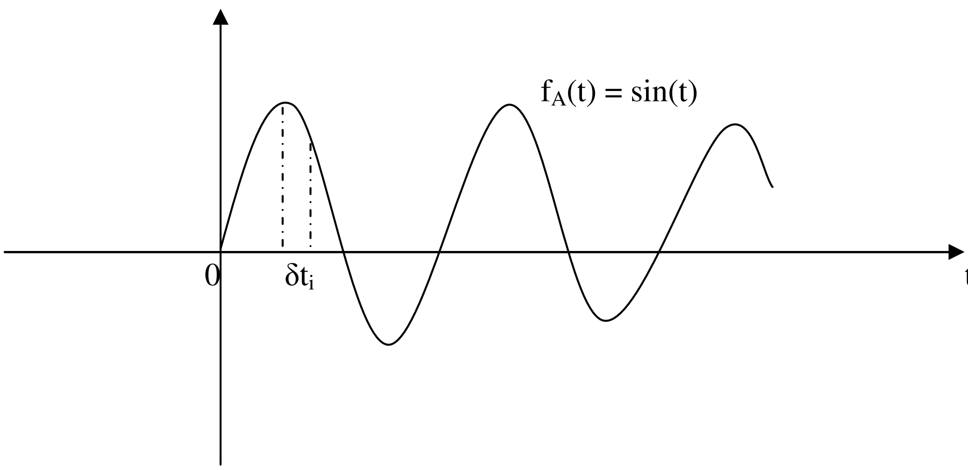
$$\begin{array}{lcl} \text{CHANGE in } F(t) & = & \text{Area under } f(t) & = & \displaystyle\int_a^b f(t)dt \\ \text{over interval } [a,b] & & \text{over interval}[a,b] & & \end{array}$$

So: CHANGE in F(t) over δt $= f(t) \bullet \delta t$

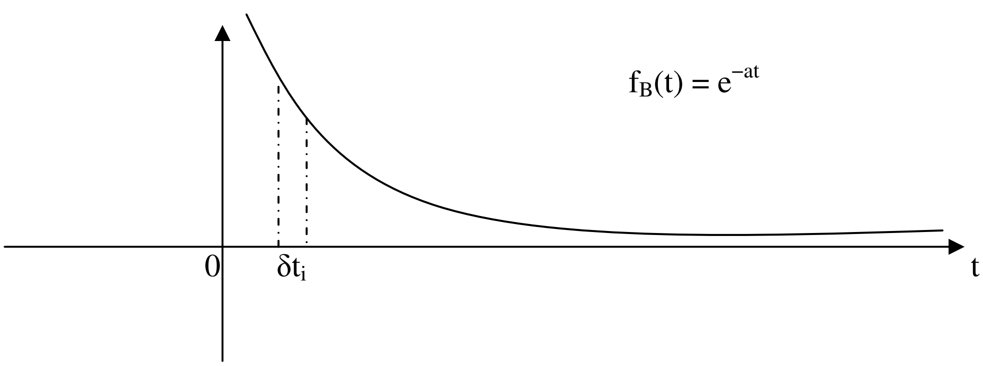
Now: CHANGE $\Longleftrightarrow$ EFFECT

COMBINED EFFECT of $f(t)$ = EFFECT of $f(t)$ over sub-interval δt $= f(t) \bullet \delta t$

Hence:

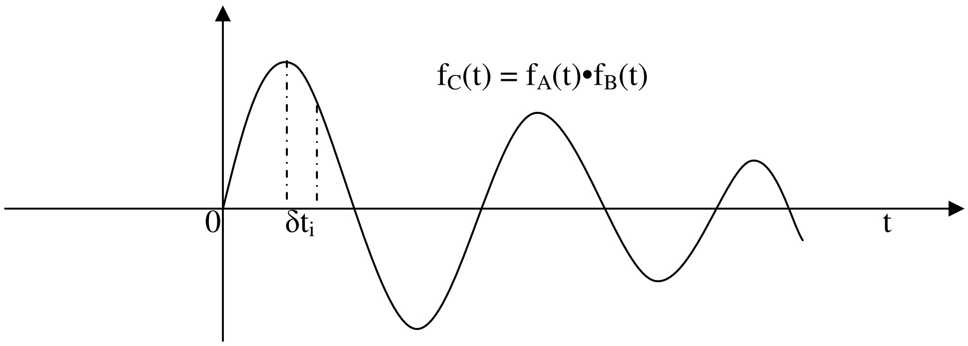


COMBINED EFFECT of $f_A(t_i)$ = EFFECT of $f_A(t_i)$ over sub-interval δt_i $= f_A(t_i) \bullet \delta t_i$



COMBINED EFFECT of $f_B(t)$ = EFFECT of $f_B(t)$ over sub-interval δt_i $= f_B(t_i) \bullet \delta t_i$

4. COMBINED EFFECT of $f_A(t)$ and $f_B(t)$ over CORRESPONDING SUB – INTERVALS of the equal size δt_i.

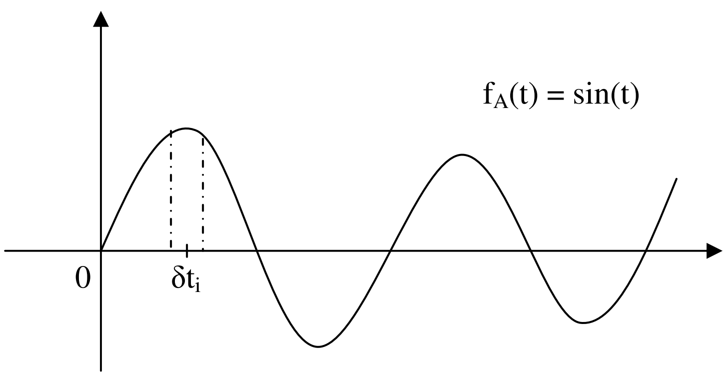


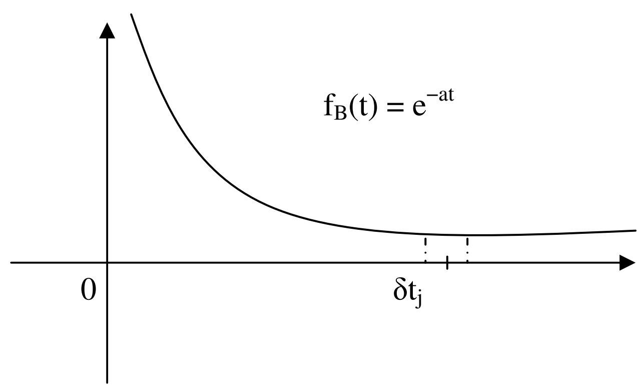
COMBINED EFFECT of $f_A(t)$ and $f_B(t)$ over CORESSPONDING SUB-INTERVAL δt_i is:

$$\text{COMBINED EFFECT of } f_C(t_i) = f_C(t_i) \bullet \delta t_i = f_A(t_i) \bullet f_B(t_i) \bullet \delta\, t_i$$

We can do this because the sub-intervals for both functions $f_A(t)$ and $f_B(t)$ are of the same size.

5. COMBINED EFFECT of $f_A(t)$ and $f_B(t)$ over NON-CORRESPONDING SUB-INTERVALS of the equal size $\delta t = \delta t_i = \delta t_j$:





$$f_A(t_i) \bullet f_B(t_j) \bullet \delta t$$

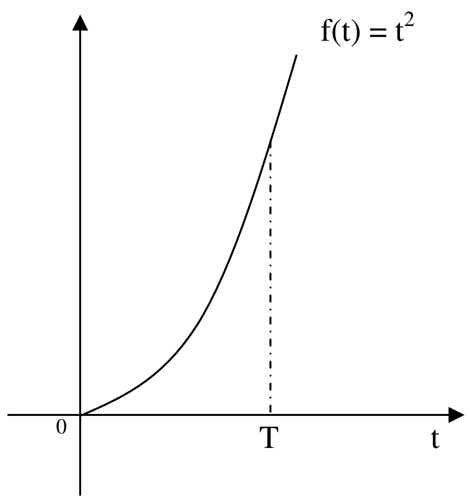
This step is crucial. We saw that in the CONVOLUTION process we do a SHIFT and MULTIPLY. To find the COMBINED EFFECT the SYMMETRIC way, the sub-intervals for both functions should be of the same size. Later, we shall refer to this important aspect in a different way and say: equally spaced points, implying sub-intervals of the same size. Hence we will say: ***without equal spacing there is no convolution.***

6. DISCRETISING a CONTINUOUS function with equal spacing T_s .

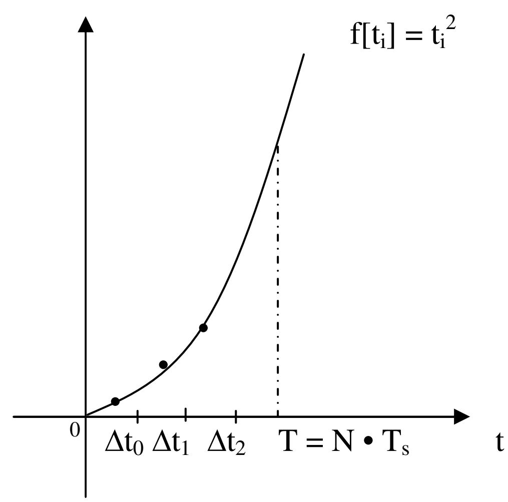
Now we can slowly move away from the language and notation of Calculus that deals with CONTINUOUS functions and get back to DISCRETE notation and calculations. A continuous function over an interval [0,T] can be made discrete by choosing a sub-interval of size T_s and dividing the interval [0,T] into $N = T/T_s$ equal parts. The subscript s in T_s usually refers to the word sample.

So, instead of an infinitesimally small sub-interval δt, we have a bigger (measurable) sub-interval Δt. We may denote a sub-interval by Δt_i. They are of the same size T_s. Now we take the discrete values $f(t_i)$, where the t_i are the mid-points of the sub-intervals Δt_i.

CONTINUOUS



DISCRETE



COMBINED EFFECT of $f[t_i]$ = EFFECT of $f[t_i]$ over a sub-interval $\Delta t_i = f[t_i] \bullet \Delta t_i$ or $f[t_i] \bullet T_s$.

With this terminology and notation in mind we may write the COMBINED EFFECT of $f_A[t_i]$ and $f_B[t_j]$ over NON-CORRESPONDING sub-interval of EQUAL size is:

$$f_A[t_i] \bullet f_B[t_j] \bullet T_s$$

7. OVERLAPPING INTERVAL

Due to the SHIFT the two discrete functions $f_A[t_k]$ and $f_B[t_{i-k}]$ will overlap over more than one sub-interval T_s. In fact, they will overlap over a contiguous set of sub-intervals. This common set of contiguous sub-intervals is called the overlapping interval.

8. COMBINED EFFECTS AND SUPER-POSITION CURVE

For each SHIFT identified by the CONVOLUTION INDEX i we have the products (super-positions) in column i. These products (super-positions) or COMBINED EFFECTS

$f_A[t_k] \bullet f_B[t_{i-k}] \bullet T_s$ are over the range of k in the overlapping interval. We may plot these COMBINED EFFECTS to get the ***super-position curve*** over this overlapping interval.

9. COMBINED EFFECT SUM

For each i we may sum up these COMBINED EFFECTS (in column i) to get the area under the ***super-position curve.*** The area under the ***super-position curve*** yields a single value which in the ***discrete case*** is:

$$(\text{COMBINED EFFECT SUM})i = \sum_{\text{column i}} (\text{COMBINED EFFECT}) = \sum_{k=-\infty}^{+\infty} f_A[t_k] \bullet f_B[t_{i-k}] \bullet T_s$$

and in the ***continuous case*** is the CONVOLUTION INTEGRAL.

10. CONVOLUTION INTERVAL AND CONVOLUTION CURVE

Convolution interval = the entire interval over which the CONVOLUTION is taking place.

continuous case: $[T_A^- + T_B^-, T_A^+ + T_B^+]$ is the convolution interval.

discrete case: $[N_A^- + M_B^-, N_A^+ + M_B^+]$ is the convolution interval.

For each CONVOLUTION INDEX i in the CONVOLUTION INTERVAL, we plot the COMBINED EFFECT SUM i for $i \bullet T_s$ on the time axis to get the CONVOLUTION CURVE.

What is the area under the CONVOLUTION CURVE? TOTAL COMBINED EFFECT SUM:

$$\sum_{i=-\infty}^{+\infty} (\text{COMBINED EFFECT SUM})i = \sum_{i=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} f_A[t_k] \bullet f_B[t_{i-k}] \bullet T_s$$

In the ***continuous case,*** we have the CONVOLUTION INSTANT $\mathbf{t} = \lim_{T_s \to 0} (i \,.\, T_s)$.

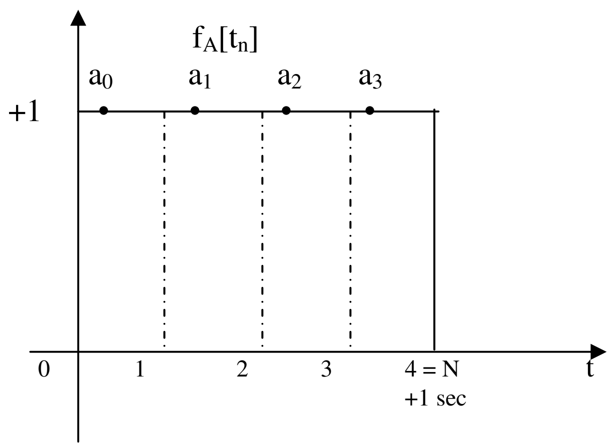
For each CONVOLUTION INSTANT t in the CONVOLUTION INTERVAL, we plot the CONVOLUTION INTEGRAL for INSTANT t on the time axis to get the CONVOLUTION CURVE.

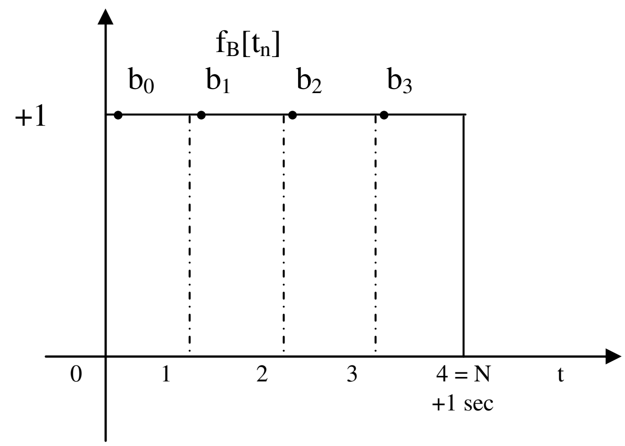
What is the area under the CONVOLUTION CURVE? TOTAL COMBINED EFFECT INTEGRAL:

$$\int_{-\infty}^{+\infty} (\text{CONVOLUTION INTEGRAL}) \,.\, dt = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} f_A(\tau) \,.\, f_B(t-\tau) \,.\, d\tau \right\} .\, dt$$

A SIMPLE EXAMPLE

Let us now take two very simple continuous functions and make them discrete over interval [0, 1] with T_s = ¼ sec implying N = 4. So the interval [0, 1] is divided into N=4 EQUAL sub-intervals. Index k of a_k and b_k runs from 0 to N-1 = 3.

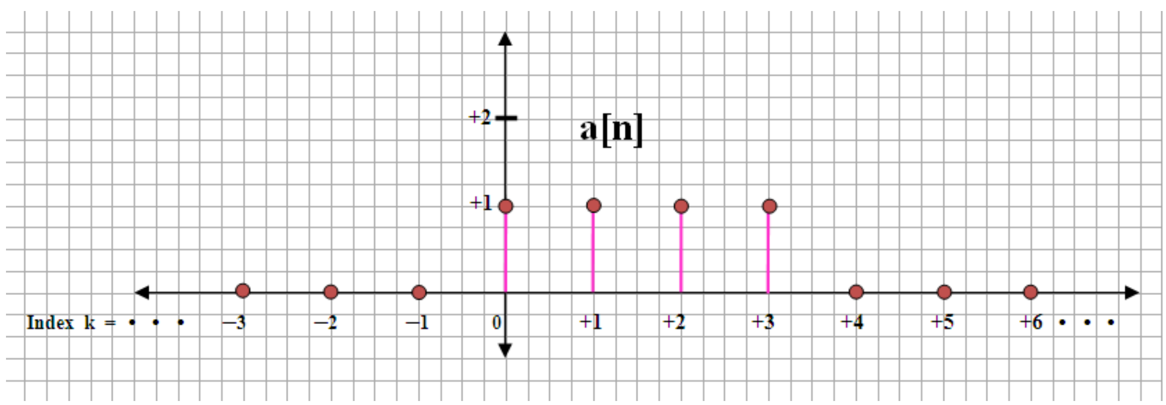


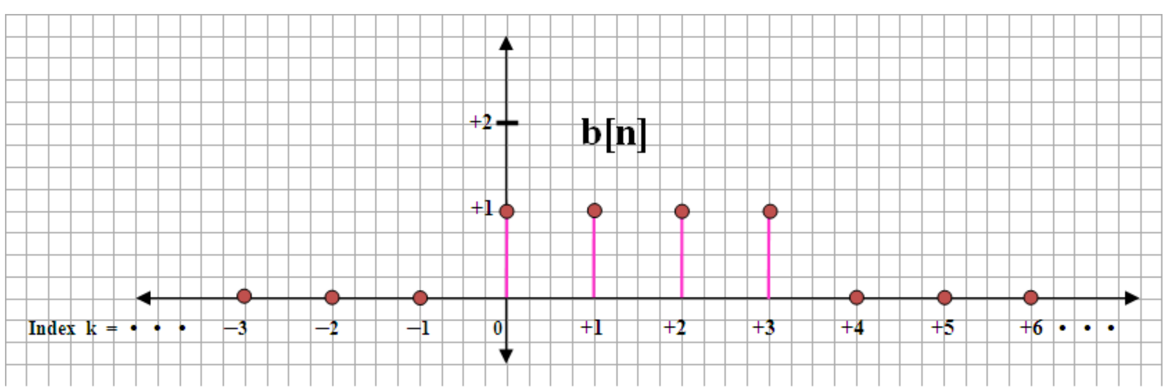


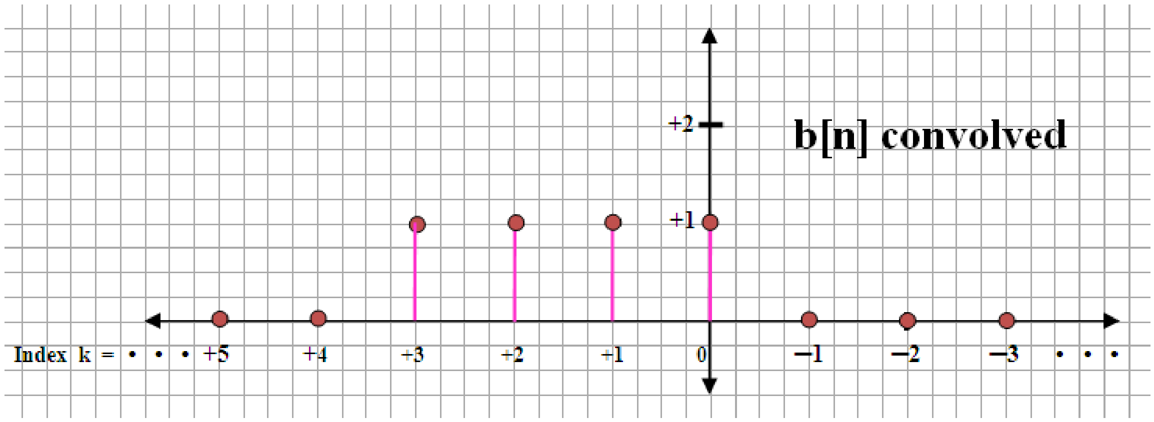
In general when making a function discrete, the index k is 0,1,2,3, • • • for the function over [0, +∞) and k = -1,-2,-3, • • • for the function over [0, -∞).

In the CONVOLUTION process, after we make $f_A(t)$ and $f_B(t)$ discrete choosing some particular sub-interval size T_s, we may treat the discrete values $f_A[t_k]$ as a pure sequence and $f_B[t_k]$ as a pure sequence. We then find:

$$f_C[t_n] = f_A[t_n] * f_B[t_n] \text{ more commonly written as: } c[n] = a[n] * b[n]$$







Notation: some books use the notation b[−n] to denote b[n] convolved.

There are several important aspects in the CONVOLUTION process. In the examples and exercises that follow we try to bring out these aspects

- SUPERPOSITON
- WEIGHTED
- SHIFTING
- OVERLAPPING INTERVALS
- EQUAL SPACING
- PADDING WITH ZEROS
- CONVOLUTION INTERVAL
- CONVOLUTION INDEX

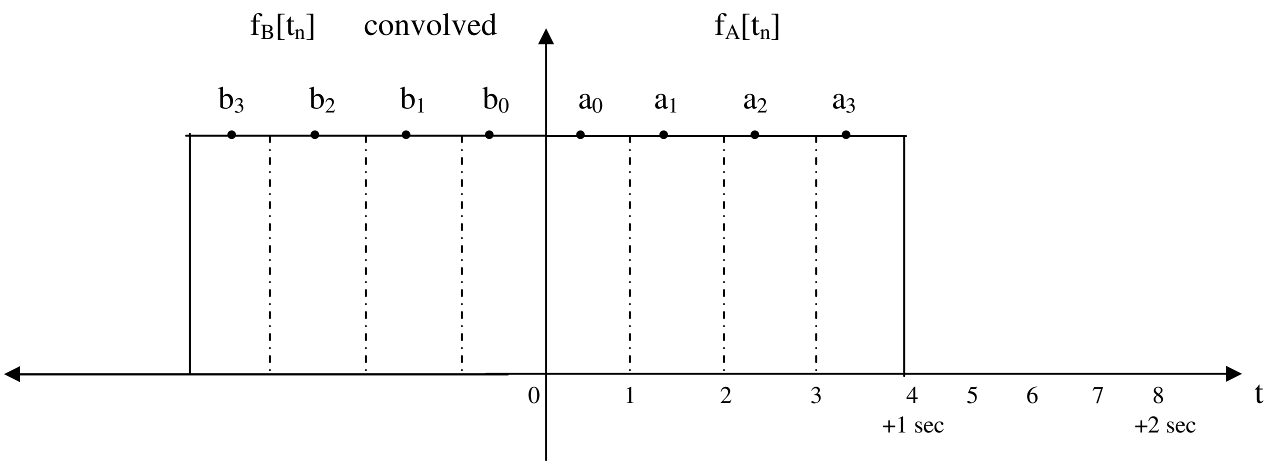
Once we have found a CONVOLUTION SUM c_i, we need to know:

What is the (COMBINED EFFECT SUM)i?

What is the ordinate on the time axis of the (COMBINED EFFECT SUM)i?

We know T_s. So (COMBINED EFFECT SUM)i = $c_i \bullet T_s$. And its ordinate will be at $i \bullet T_s$ on the time axis. So getting the CONVOLUTION INDEX i right is extremely important. And vice versa, if we know t on the time axis, we can determine the index i of c_i : $i = t/T_s$.

We are ready to start the convolution process by fixing $f_A[t_k]$ and SHIFTING CONVOLVED $f_B[t_k]$ from left to right. We could have equally well fixed the CONVOLVED $f_B[t_k]$ and SHIFTED the $f_A[t_k]$ from right to left. The sequence of CONVOLUTION SUMS c_0, c_1, c_2, • • • will appear in the same order. We know that: $f_A[t_k] * f_B[t_n] = f_B[t_n] * f_A[t_n]$.



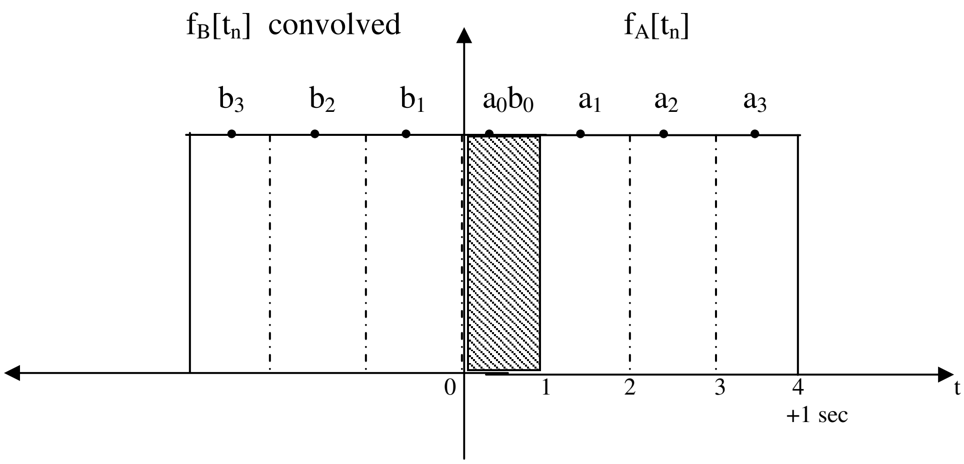
Note that the index k of b_k is still positive even though it is going in the negative direction (right to left). In the computation process we shift in the positive direction (left to right). So we make use of index of $f_A[t_n]$ to determine the index of the SHIFT and hence the CONVOLUTION INDEX.

$f_A[t_n]$: * $f_B[t_n]$ convolved: b_3 b_2 b_1 b_0	a_0	a_1	a_2	a_3				
CONVOLUTION SUM c_i with index i VALUES	c_0	c_1	c_2	c_3	c_4	c_5	c_6	

We show the computation of each CONVOLUTION SUM c_i step by step.

We have 7 = (4+4-1) = N+N-1 CONVOLUTION SUMS c_i to compute. The c_i are numbered from 0 to 2N-2, which is 0 to 6, using the CONVOLUTION INDEX i . We now SHIFT convolved $f_B[t_n]$ across $f_A[t_n]$ sub-interval by sub-interval.

CONVOLUTION INDEX $i = 0$



$$f_C[t_0] = f_A[t_0] \bullet f_B[t_0]$$

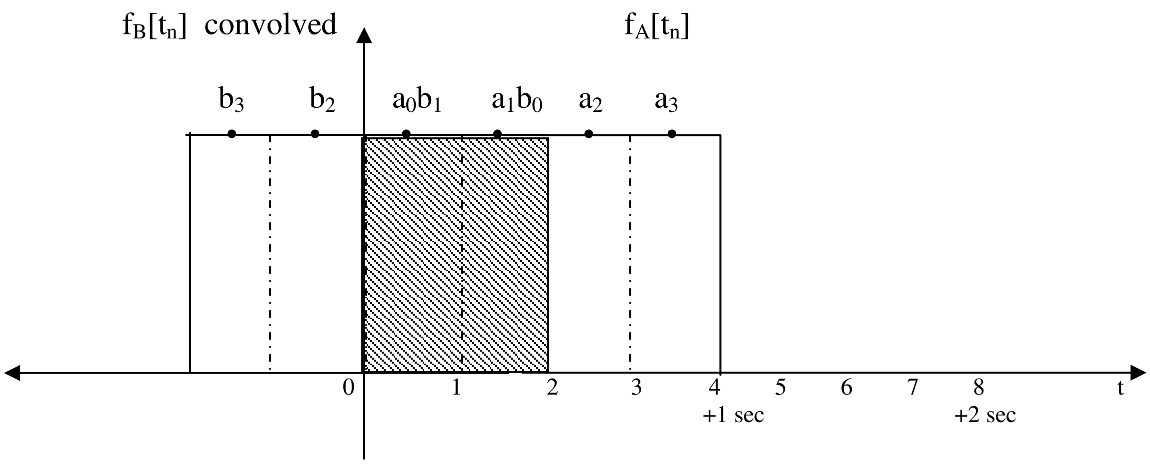
CONVOLUTION SUM $c_0 = a_0 \bullet b_0$ range of index k: 0

COMBINED EFFECT SUM $c_0 \bullet T_s = (a_0 \bullet b_0) \bullet T_s$ overlapping interval [0, 0.25] sec

$f_A[t_n]$:	a_0	a_1	a_2	a_3				
* $f_B[t_n]$convolved: b_3 b_2 b_1	b_0							
	$a_0 \bullet b_0$							
CONVOLUTION SUM c_i with index i	c_0	c_1	c_2	c_3	c_4	c_5	c_6	
VALUES	1							

$$c_i = \sum_{k=0}^{i} a_k \bullet b_{i-k}$$

CONVOLUTION INDEX $i = 1$



$$f_C[t_1] \;=\; f_A[t_0] \bullet f_B[t_1] + f_A[t_0] \bullet f_B[t_1]$$

CONVOLUTION SUM $c_1 \;=\; a_0 \bullet b_1 + a_1 \bullet b_0$ range of index k: 0, 1

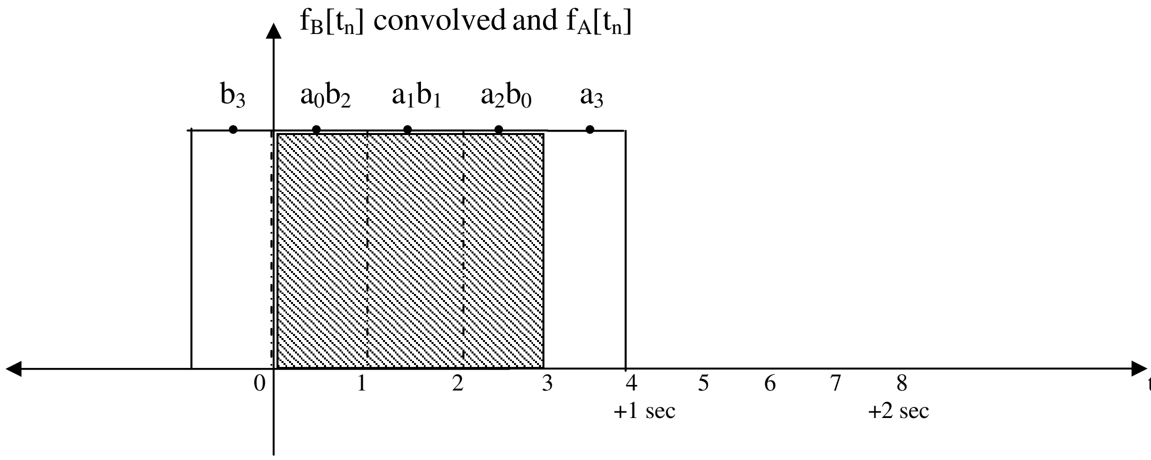
COMBINED EFFECT SUM $c_1 \bullet T_s = (a_0 \bullet b_1 + a_1 \bullet b_0) \bullet T_s$

overlapping interval [0, 0.50] sec

$f_A[t_n]$:	a_0	a_1	a_2	a_3			
*							
$f_B[t_n]$ convolved: b_3 b_2	b_1	b_0					
	$a_0 \bullet b_0$	$a_0 \bullet b_1$ $a_1 \bullet b_0$					
CONVOLUTION SUM c_i	c_0	c_1	c_2	c_3	c_4	c_5	c_6
VALUES	1	2					

$$c_i = \sum_{k=0}^{i} a_k \bullet b_{i-k}$$

CONVOLUTION INDEX $i = 2$



$$f_C[t_2] \;=\; f_A[t_0] \bullet f_B[t_2] \;+\; f_A[t_1] \bullet f_B[t_1] \;+\; f_A[t_2] \bullet f_B[t_0]$$

CONVOLUTION SUM $\quad c_2 \;=\; a_0 \bullet b_2 + a_1 \bullet b_1 + a_2 \bullet b_0$

range of index k: 0, 1, 2

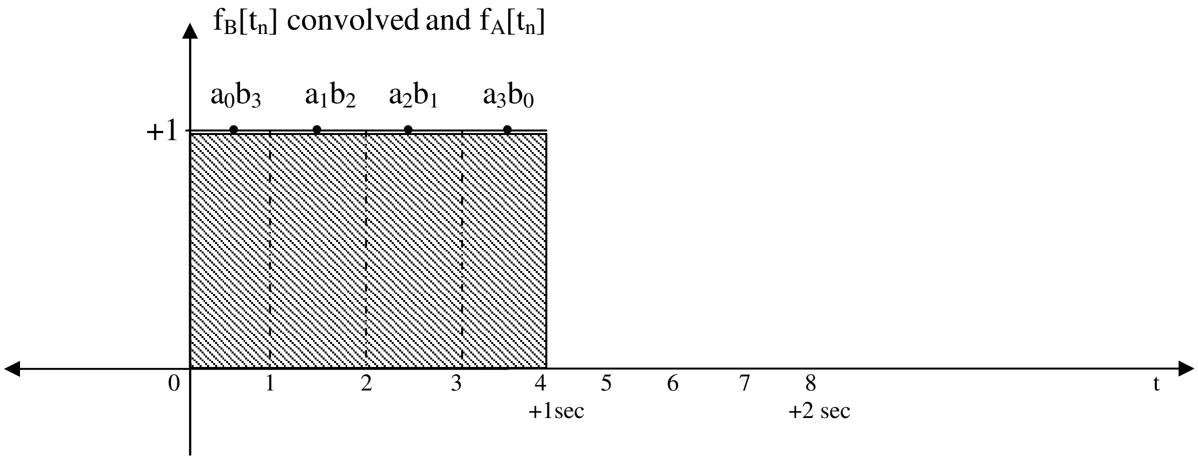
COMBINED EFFECT SUM $\quad c_2 \bullet T_s = (a_0 \bullet b_2 + a_1 \bullet b_1 + a_2 \bullet b_0) \bullet T_s$

overlapping interval [0, 0.75] sec

$f_A[t_n]$:		a_0	a_1	a_2	a_3			
	*							
$f_B[t_n]$ convolved:	b_3	b_2	b_1	b_0				
		$a_0 \bullet b_0$	$a_0 \bullet b_1$ $a_1 \bullet b_0$	$a_0 \bullet b_2$ $a_1 \bullet b_1$ $a_2 \bullet b_0$				
CONVOLUTION SUM VALUES	c_i	c_0 1	c_1 2	c_2 3	c_3	c_4	c_5	c_6

$$c_i = \sum_{k=0}^{i} a_k \bullet \, b_{i-k}$$

CONVOLUTION INDEX $i = 3$



$f_C[t_3] = f_A[t_0] \bullet f_B[t_3] + f_A[t_1] \bullet f_B[t_2] + f_A[t_2] \bullet f_B[t_1] + f_A[t_3] \bullet f_B[t_0]$

CONVOLUTION SUM $c_3 = a_0 \bullet b_3 + a_1 \bullet b_2 + a_2 \bullet b_1 + a_3 \bullet b_0$

range of index k: 0, 1, 2, 3

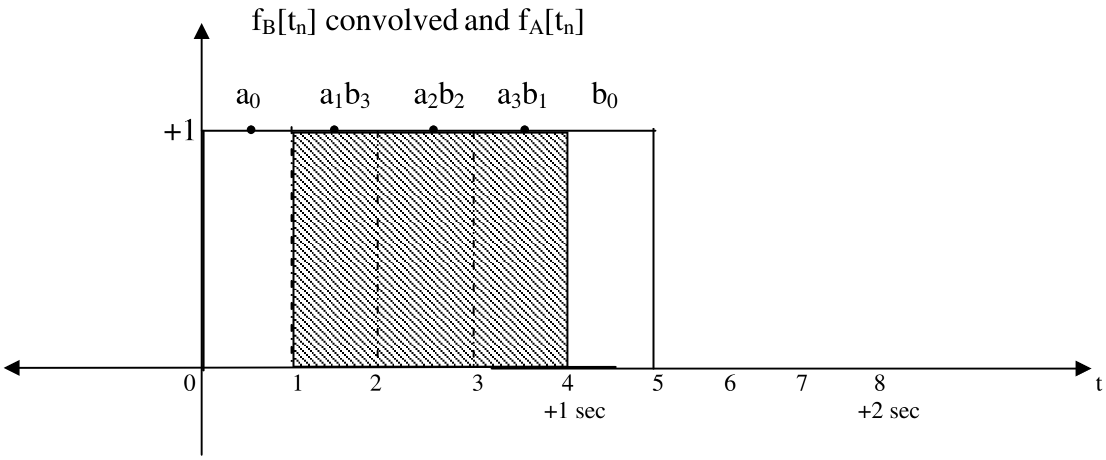
COMBINED EFFECT SUM $c_3 \bullet T_s = (a_0 \bullet b_3 + a_1 \bullet b_2 + a_2 \bullet b_1 + a_3 \bullet b_0) \bullet T_s$

overlapping interval [0, 1] sec

$f_A[t_n]$:	a_0	a_1	a_2	a_3			
*							
$f_B[t_n]$ convolved:	b_3	b_2	b_1	b_0			
	$a_0 \bullet b_0$	$a_0 \bullet b_1$ $a_1 \bullet b_0$	$a_0 \bullet b_2$ $a_1 \bullet b_1$ $a_2 \bullet b_0$	$a_0 \bullet b_3$ $a_1 \bullet b_2$ $a_2 \bullet b_1$ $a_3 \bullet b_0$			
CONVOLUTION SUM c_i	c_0	c_1	c_2	c_3	c_4	c_5	c_6
VALUES	1	2	3	4			

$$c_i = \sum_{k=0}^{i} a_k \bullet b_{i-k}$$

CONVOLUTION INDEX $i = 4$



$$f_C[t_4] \quad = \quad f_A[t_1] \bullet f_B[t_3] \; + \; f_A[t_2] \bullet \; f_B[t_2] \; + \; f_A[t_3] \bullet f_B[t_1]$$

CONVOLUTION SUM $\quad c_4 \quad = \quad a_1 \bullet b_3 + a_2 \bullet b_2 + a_3 \bullet b_1$

range of index k: 1, 2, 3

COMBINED EFFECT SUM $\;c_4 \bullet T_s = (a_1 \bullet b_3 + a_2 \bullet b_2 + a_3 \bullet b_1) \bullet T_s$

overlapping interval [0.25, 1.0] sec

$f_A[t_n]$:	a_0	a_1	a_2	a_3			
*							
$f_B[t_n]$ convolved:		b_3	b_2	b_1	b_0		
	$a_0 \bullet b_0$	$a_0 \bullet b_1$ $a_1 \bullet b_0$	$a_0 \bullet b_2$ $a_1 \bullet b_1$ $a_2 \bullet b_0$	$a_0 \bullet b_3$ $a_1 \bullet b_2$ $a_2 \bullet b_1$ $a_3 \bullet b_0$	$a_1 \bullet b_3$ $a_2 \bullet b_2$ $a_3 \bullet b_1$		
CONVOLUTION SUM c_i VALUES	c_0 1	c_1 2	c_2 3	c_3 4	c_4 3	c_5	c_6

Here we need to explain how PADDING WITH ZEROS allows us to use the same formula

$$c_i = \sum_{k=0}^{i} a_k \bullet \; b_{i-k}$$

instead of

$$\sum_{k=1}^{3} a_k \, b_{i-k}$$

where the range of k = 1, 2, 3 is the overlapping interval.

Notice the overlapping interval due to the SHIFT. While the CONVOLUTION INDEX i identifies the SHIFT position, the index k of the a_k and b_{i-k} lies within the overlapping interval and is used to find the products $a_k \bullet b_{i-k}$ listed in the column i.

Some very important points to note.

1. For each CONVOLUTION INDEX i pay close attention to the overlapping interval of $f_B[t_n]$ CONVOLVED and $f_A[t_n]$ on the time axis of $f_A[t_n]$. For this CONVOLUTION INDEX i = 4 the overlapping interval is [0.25, 1.0] seconds.
2. More importantly $i \bullet T_s = 1.0$ seconds. So on the CONVOLUTION CURVE the COMBINED EFFECT SUM with index i = 4 will be plotted for $t = i \bullet T_s = 1.0$ seconds on the time axis adjusted by $T_s/2$ to get the mid-point.
3. The index k of the super-positions $a_k \bullet b_{i-k}$ that appear in column i = 4 lie within this overlapping interval [0.25, 1.0] seconds on the time axis of $f_A[t_n]$. For this CONVOLUTION INDEX i = 4 the range of k = 1, 2, 3.

PADDING WITH ZEROS

Notice that outside the overlapping interval [0.25, 1.0] seconds on the time axis of $f_A[t_n]$ for this CONVOLUTION INDEX i = 4, the super-positions $a_k \bullet b_{i-k}$ are zero. Why?

Because outside the overlapping interval either a_k does not exist and hence is zero, or b_{i-k} does not exist and hence is zero. Hence these $a_k \bullet b_{i-k}$ products or super-positions are not shown in the respective columns of c_i. Also we do not plot them on the super-position curve. So a_4, a_5, a_6, b_4, b_5, b_6 are zero. This we call PADDING WITH ZEROS. PADDING WITH ZEROS allows us to use the same concise formula for the CONVOLUTION SUM and COMBINED EFFECT SUM.

The index k of the super-position has range 1, 2, 3 in the overlapping interval. But since $b_4 = 0$, this implies $a_0 \bullet b_4 = 0$. So we can extend the range of k left to 0. Likewise, since $a_4 = 0$, this implies $a_4 \bullet b_0 = 0$. Again, we can extend the range of k right to 4 = i. So now the range of k is 0, 1, 2, 3, 4 = i. We may use the same formula:

$$c_i = \sum_{k=0}^{i} a_k \bullet b_{i-k}$$

It is because of PADDING WITH ZEROS in both directions that we can have the same formula for c_i as the CONVOLUTION INDEX i varies over its entire range from 0 to 6. Also the formula matches the picture of the right edge of the convolved $f_B[t_n]$ being at CONVOLUTION INDEX i as it SHIFTS (continuous case: slides) left to right over $f_A[t_n]$.

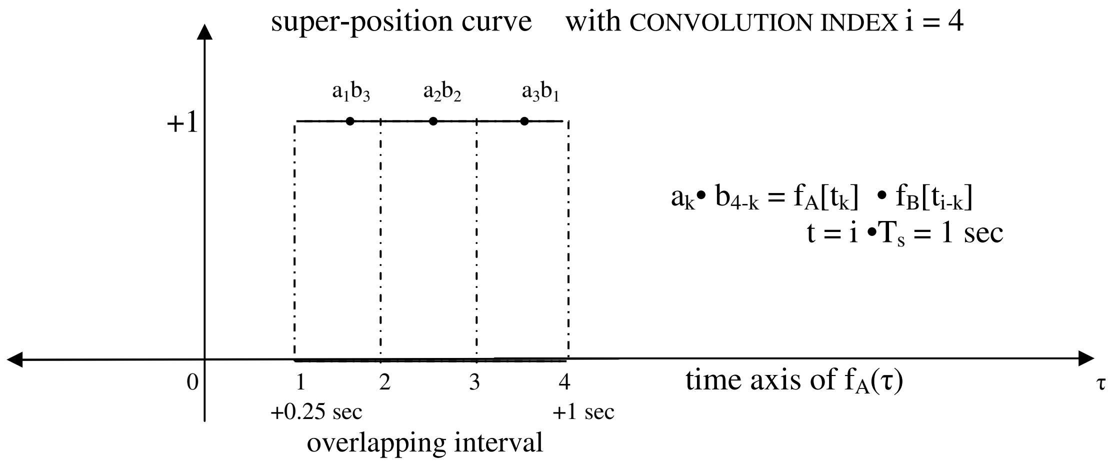
We may PAD WITH ZEROS $f_A[t_n]$ and $f_B[t_n]$ in both directions all the way to $-\infty$ and to $+\infty$. And now we may write the general form of the

$$\text{COMBINED EFFECT SUM } c_i \,.\, T_s = \sum_{k=-\infty}^{+\infty} a_k \,.\, b_{i-k} \,.\, T_s$$

Also notice that the overlapping interval is finite. Later, with $T_s \rightarrow 0$, we get an infinite number of products $a_k \bullet b_{i-k}$ in this finite overlapping interval. Then the discrete $\sum$ of these infinite number of products $a_k \bullet b_{i-k}$ in column i, when multiplied by $d\tau$ (representing limit $(T_s \rightarrow 0)$) becomes a continuous summation or an $\int$ over the overlapping interval. This is the CONVOLUTION INTEGRAL, the continuous version of what we call the COMBINED EFFECT SUM.

SUPER-POSITION CURVE

We now present an intermediate step which is usually not shown in text-books. We shall plot the ***super-position curve*** for this CONVOLUTION INDEX i = 4.



Geometrically, we can find the area under the ***super-position curve*** over the overlapping interval with $T_s = ¼$ sec.

(COMBINED EFFECT SUM)i = $c_i \bullet T_s$ = area under the ***super-position curve*** with index i.

Note: this example is too simple to notice the difference. But what do you think the super-position curve over the same ***overlapping interval*** of $f_B[t_n] * f_A[t_n]$ will look like? Will it look reflected?

Recall what we said in the example 632 * 521: the order of the products in each column of 521 * 632 will be inverted. But the sum of the products in each column in both cases will be the same.

Exercise: Repeat this example with $T_s = 1/8$ sec. Plot the ***super-position curve*** over the same ***overlapping interval*** and find the area.

What do you think will happen when T_s becomes smaller and smaller?

What will the ***super-position curve*** look like over the same ***overlapping interval***?

We have the CONVOLUTION INDEX i and the index k on the ***overlapping interval*** on the time axis of $f_A[t_n]$. We use this to plot the discrete ***super-position curve.***

discrete		continuous
$f_A[t_k] \bullet f_B[t_{i-k}]$	$\longleftarrow$	$f_A(t) \bullet f_B(t)$
$f_A[kT_s] \bullet f_B[(i-k) \bullet T_s]$	$\longrightarrow$	$f_A(\tau) \bullet f_B(t-\tau)$

As $T_s \rightarrow 0$ the discrete super-position curve becomes a continuous ***super-position curve.***

This gives the expression $f_A(\tau) \bullet f_B(t-\tau)$ for the ***super-position curve*** in the continuous case.

Where did the τ come from? And why do we need it? We shall explain when we get to the CONVOLUTION INTEGRAL

After we find the (COMBINED EFFECT SUM)i for each CONVOLUTION INDEX i, we may plot them for each ordinate $i \bullet T_s$ to get the CONVOLUTION CURVE.

Note that each point on the CONVOLUTION CURVE is $c_i \bullet T_s$ = COMBINED EFFECT SUM with index i for ordinate $i \bullet T_s$ on the time axis.

What do you think will happen when T_s becomes smaller and smaller?

What will the COMBINED EFFECT SUM $\sum a_k \bullet b_{i-k} \bullet T_s$ look like?

Will it become an integral?

$$\sum a_k \bullet b_{i-k} \bullet T_s \longrightarrow \int f_A(\tau) \bullet f_B(t-\tau) \bullet d\tau$$

In the continuous case, each point on the convolution curve is the CONVOLUTION INTEGRAL at instant t where:

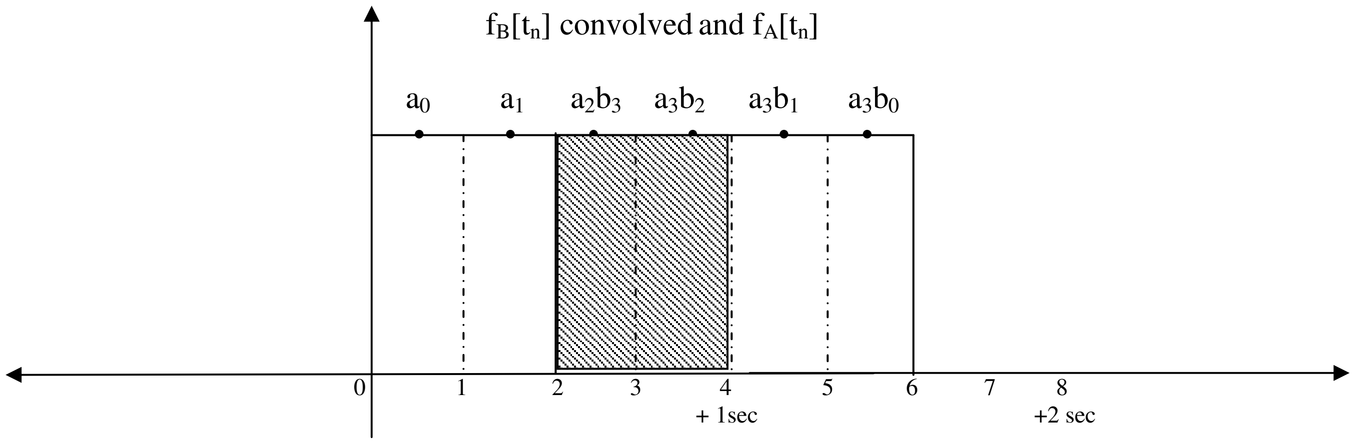
$$i \bullet T_s \rightarrow t \text{ as } T_s \rightarrow 0$$

$$k \bullet T_s \rightarrow \tau \text{ as } T_s \rightarrow 0$$

$$T_s \rightarrow d\tau \text{ as } T_s \rightarrow 0$$

Let us continue with our simple example

CONVOLUTION INDEX $i = 5$



$$f_C[t_5] = f_A[t_2] \bullet f_B[t_3] + f_A[t_3] \bullet f_B[t_2]$$

CONVOLUTION SUM $c_5 = a_2 \bullet b_3 + a_3 \bullet b_2$ range of index k: 2, 3

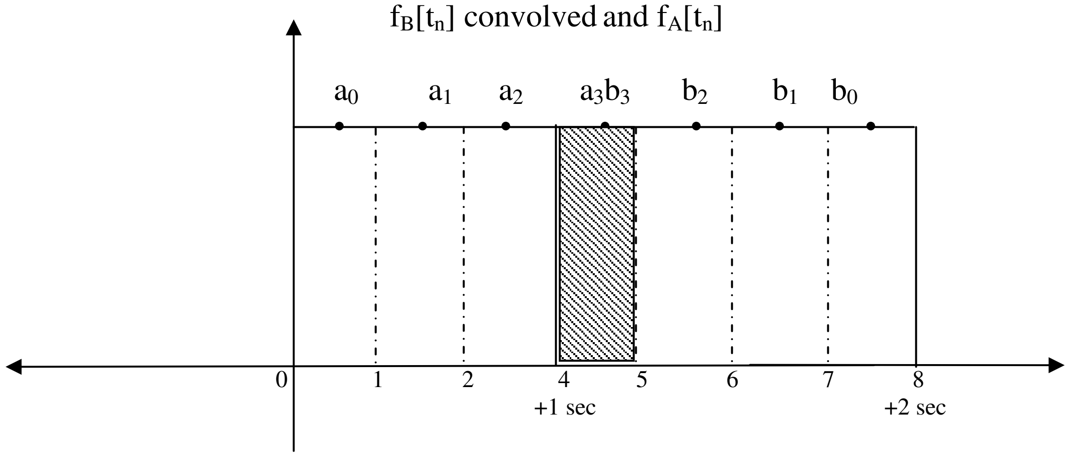
COMBINED EFFECT SUM $c_5 \bullet T_s = (a_2 \bullet b_3 + a_3 \bullet b_2) \bullet T_s$

overlapping interval [0.5, 1.0] sec

$f_A[t_n]$:	a_0	a_1	a_2	a_3				
*								
$f_B[t_n]$ convolved:			b_3	b_2	b_1	b_0		
	$a_0 \bullet b_0$	$a_0 \bullet b_1$ $a_1 \bullet b_0$	$a_0 \bullet b_2$ $a_1 \bullet b_1$ $a_2 \bullet b_0$	$a_0 \bullet b_3$ $a_1 \bullet b_2$ $a_2 \bullet b_1$ $a_3 \bullet b_0$	$a_1 \bullet b_3$ $a_2 \bullet b_2$ $a_3 \bullet b_1$	$a_2 \bullet b_3$ $a_3 \bullet b_2$		
CONV. SUM c_i	c_0	c_1	c_2	c_3	c_4	c_5	c_6	
VALUES	1	2	3	4	3	2		

$$c_i = \sum_{k=0}^{i} a_k \bullet b_{i-k}$$

CONVOLUTION INDEX $i = 6$



$f_C[t_6] \;=\; f_A[t_3] \bullet f_B[t_3]$

CONVOLUTION SUM $\quad c_6 = a_3 \bullet b_3 \qquad$ range of index k: 3

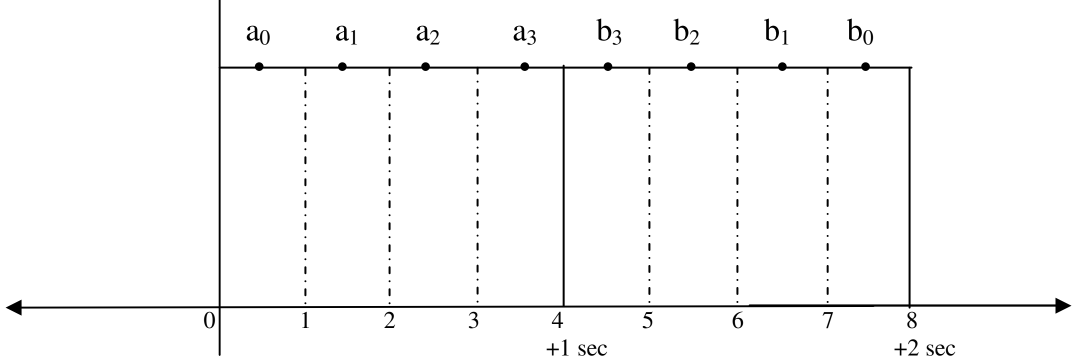
COMBINED EFFECT SUM $\quad c_6 \bullet T_s = (a_3 \bullet b_3) \bullet T_s \qquad$ overlapping interval [0.75, 1.0] sec

$f_A[t_n]$: * $f_B[t_n]$ convolved:	a_0	a_1	a_2	a_3 b_3	b_2	b_1	b_0
	$a_0 \bullet b_0$	$a_0 \bullet b_1$ $a_1 \bullet b_0$	$a_0 \bullet b_2$ $a_1 \bullet b_1$ $a_2 \bullet b_0$	$a_0 \bullet b_3$ $a_1 \bullet b_2$ $a_2 \bullet b_1$ $a_3 \bullet b_0$	$a_1 \bullet b_3$ $a_2 \bullet b_2$ $a_3 \bullet b_1$	$a_2 \bullet b_3$ $a_3 \bullet b_2$	$a_3 \bullet b_3$
CONV. SUM c_i	c_0	c_1	c_2	c_3	c_4	c_5	c_6
VALUES	1	2	3	4	3	2	1

$$c_i = \sum_{k=0}^{i} a_k \bullet b_{i-k}$$

CONVOLUTION INDEX $i = 7$

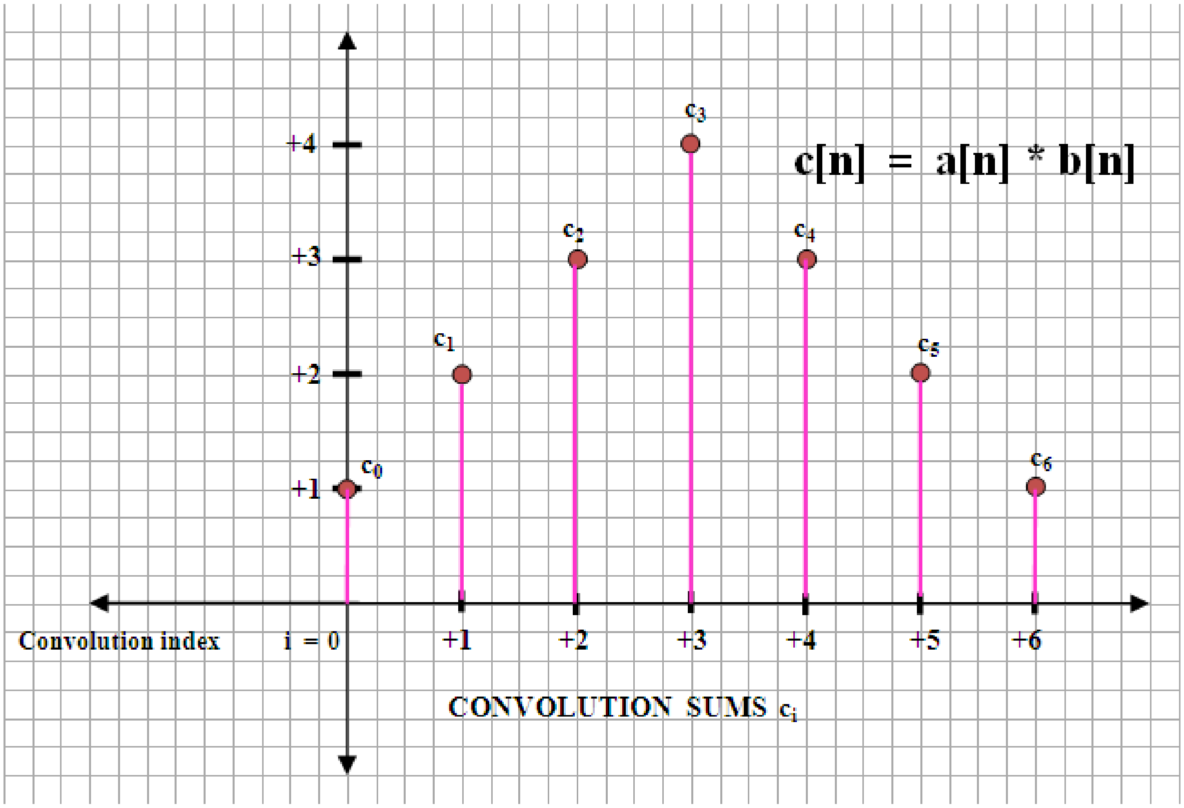
$f_B[t_n]$ convolved and $f_A[t_n]$

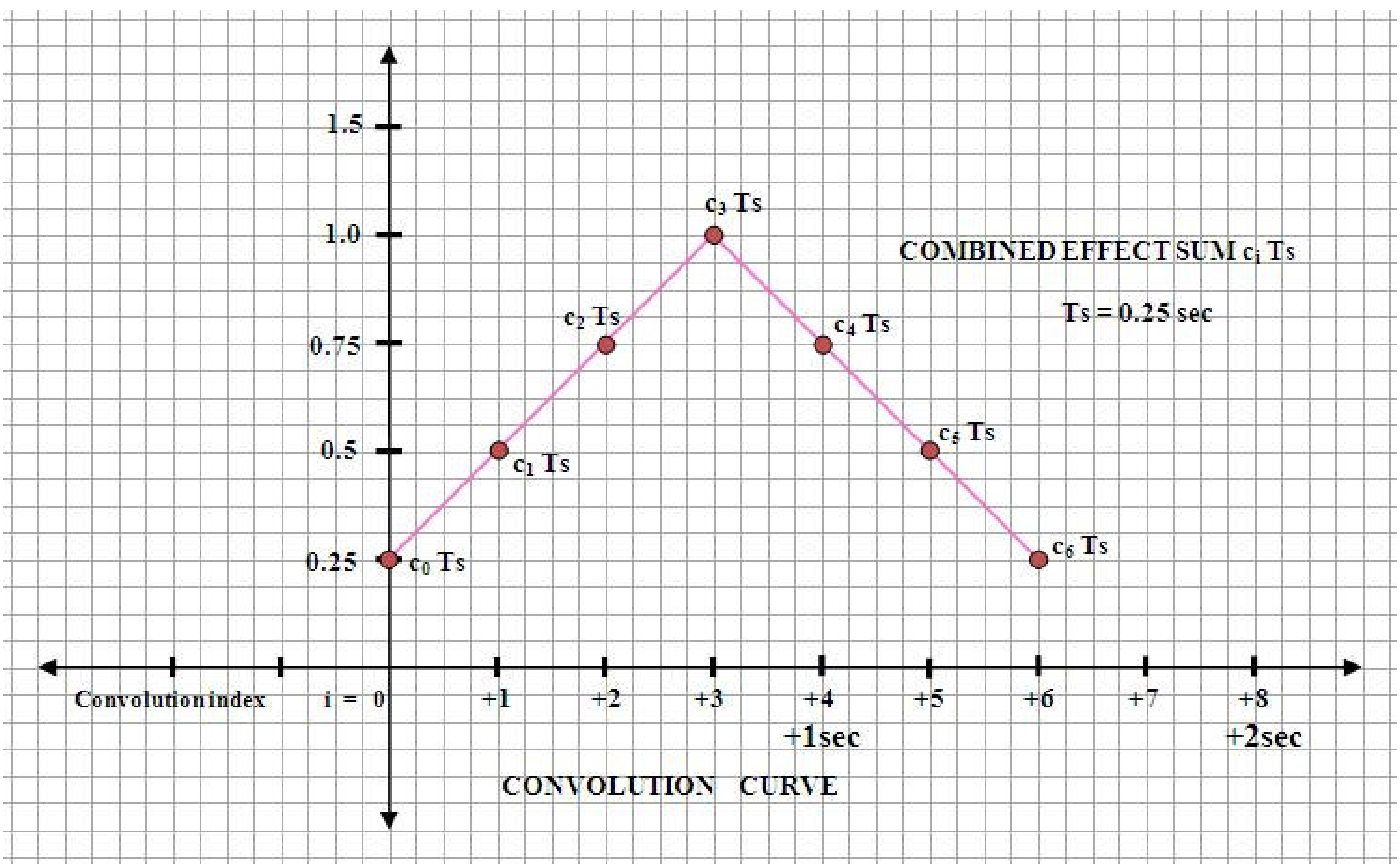


The convolution process is complete

$f_A[t_n]$:	a_0	a_1	a_2	a_3				
*								
$f_B[t_n]$ convolved:					b_3	b_2	b_1	b_0
	$a_0 \bullet b_0$	$a_0 \bullet b_1$ $a_1 \bullet b_0$	$a_0 \bullet b_2$ $a_1 \bullet b_1$ $a_2 \bullet b_0$	$a_0 \bullet b_3$ $a_1 \bullet b_2$ $a_2 \bullet b_1$ $a_3 \bullet b_0$	$a_1 \bullet b_3$ $a_2 \bullet b_2$ $a_3 \bullet b_1$	$a_2 \bullet b_3$ $a_3 \bullet b_2$	$a_3 \bullet b_3$	
CONV. SUM VALUES c_i	c_0 1	c_1 2	c_2 3	c_3 4	c_4 3	c_5 2	c_6 1	

CONVOLUTION SUMS AND CONVOLUTION CURVE





We prefer to use lollipop diagrams (seen in text books) for ***pure sequences,*** and graph paper (grid background diagrams) for discrete functions with the CONVOLUTION INDEX on the horizontal axis and the time indicated below.

Notice the difference in the diagrams. For ***pure sequences*** we do not have a time axis. We have a horizontal axis with the CONVOLUTION INDEX i:

$$i = -\infty, \ldots -3, -2, -1, 0, 1, 2, 3, \ldots, +\infty$$

For ***pure sequences*** like decimal digital numbers we write: c[n] = a[n] * b[n].

Each c_i or c[i] is called a CONVOLUTION SUM.

For sequences that are ***discrete functions***, like $f_A[t_n]$ and $f_B[t_n]$ got by discretising ***continuous functions*** $f_A(t)$ and $f_B(t)$ over some specified interval or intervals, we should be clear and use the notation:

$f_C[t_n] = f_A[t_n] * f_B[t_n]$

$c[t_n] \;\; = a[t_n] * b[t_n]$

$y[t_n] \;\; = \; x[t_n] * h[t_n]$ and so on.

Here what we compute on the right hand side is the COMBINED EFFECT SUM. We may use this expression to get to the CONVOLUTION INTEGRAL. We plot $c[t_i]$ over $i \bullet T_s$ on the time axis.

CONVOLUTION INTERVAL

See where the COMBINED EFFECT SUMS occur on the time axis with T=1 sec & T_s = ¼ sec.

To gain more insight, again with T = 1 sec choose T_s = 1/8 and plot the CONVOLUTION CURVE of: $f_C[t_n] = f_A[t_n] * f_B[t_n]$

How many COMBINED EFFECT SUMS will there be?

Where will you plot them?

What will be the CONVOLUTION INTERVAL?

The CONVOLUTION INTERVAL is [0, 1.5] secs.

PADDING WITH ZEROS

It is not necessary that the intervals over which $f_A(t)$ and $f_B(t)$ are defined be the same. We could have defined $f_A(t)$ over $[0, T_1]$ and $f_B(t)$ over $[0, T_2]$. Let us say $T_1 < T_2$. Then we PAD WITH ZEROS the values of $f_A(t)$ over $[T_1, T_2]$. $f_C(t)$ will be over $[0, T_1 + T_2]$.

Now when we say $f_C(t) = f_A(t) * f_B(t)$ we mean:

$f_C(t)$	=	$f_A(t)$	*	$f_B(t)$
over		over $[0, T_1]$		over $[0, T_2]$
$[0, T_1+T_2]$		padded with		padded with
		zeros over $[T_1, T_1+T_2]$		zeros over $[T_2, T_1+T_2]$

In the discrete versions we choose a certain T_s. Let $N = T_1/T_s$ and $M = T_2/T_s$.

Since $T_1 < T_2$ we will have N < M. So we pad the remaining (M-N) values of $f_A[t_k]$ with zeros and perform the CONVOLUTION. Now there will be (N+M-1) CONVOLUTION SUMS c_i or $f_C[t_i]$ with the CONVOLUTION INDEX i running from 0 to (N+M-2). We shall see more on this aspect when we talk about the CONVOLUTION INDEX.

Let us now formularize the CONVOLUTION SUM and COMBINED EFFECT SUM computation process:

1. CONVOLUTION SUM for CONVOLUTION INDEX i = 0, 1, 2, 3, . . . , 2N–2.

$$c_i = \sum_{k=0}^{i} a_k \,.\, b_{i-k} = \sum_{k=0}^{i} a_{i-k}\, b_k$$

$$f_C[i\,.\,T_s] = \sum_{k=0}^{i} f_A[k\,.\,T_s]\,.\,f_B[(i-k)\,.\,T_s] = \sum_{k=0}^{i} f_A[(i-k)\,T_s]\,.\,f_B[k\,.\,T_s]$$

2. COMBINED EFFECT SUM for CONVOLUTION INDEX i = 0, 1, 2, 3, . . . , 2N–2.

$$c_i\,.\,T_s = \sum_{k=0}^{i} (a_k\,.\,b_{i-k})\,.\,T_s = \sum_{k=0}^{i} (a_{i-k}\,.\,b_k)\,.\,T_s$$

$$f_C[i\,.\,T_s]\,.\,T_s = \sum_{k=0}^{i} (f_A[k\,.\,T_s]\,.\,f_B[(i-k)\,.\,T_s])\,.\,T_s = \sum_{k=0}^{i} (f_A[(i-k)\,.\,T_s]\,.\,f_B[k\,.\,T_s])\,.\,T_s$$

EQUAL SPACING

If we look closely at the summations on the right hand side we note that T_s appears in both $f_A[k \bullet T_s]$ and $f_B[(i- k) \bullet T_s]$. Because the sub-intervals are of EQUAL size we can perform the CONVOLUTION. The CONVOLUTION formula works. Hence we say: ***without equal spacing there is no CONVOLUTION***. This is the reason when sampling, we need to take equally spaced samples. When some of the samples are missing, the points or samples will not be equally spaced. Then what do we do?

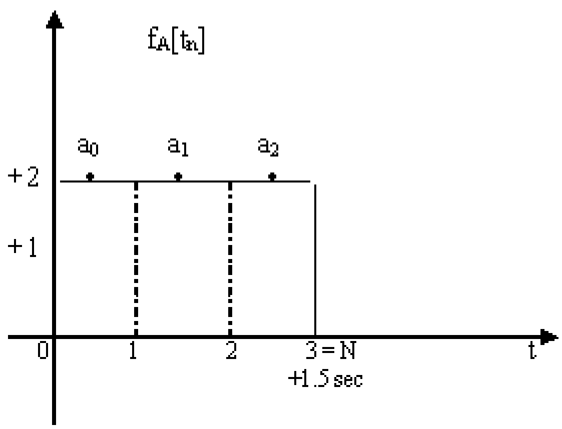
NORMALIZED CONVOLUTION

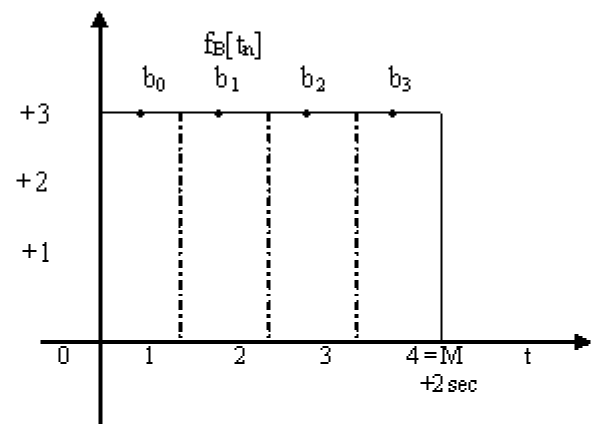
Further reading on the Internet: Normalized Convolution: A Tutorial by Roberta Piroddi and Maria Petrou (September 8, 2003).

Example of Calculation of Normalized Convolution by Bob Fisher (09 March 2003).

A WORD OF CAUTION

We were very careful to use the expression overlapping interval even though from the diagrams it looks like overlapping areas. In this simple example the difference is NOT noticeable. But it is crucial. Some authors, when explaining CONVOLUTION graphically use this simple example and say overlapping areas. But this is NOT correct. Moreover, it does NOT convey the sense of EFFECT computed by multiplication (superposition) over overlapping sub-intervals of size T_s. Lets us see a slightly different example.





Now when you CONVOLVE (reverse, reflect) one of them and SHIFT it over the other, the areas do not overlap. But contiguous sub-intervals do overlap. So we have an overlapping interval. The EFFECT $a_k \bullet b_{i-k}$ is computed as before, and is: $2 \bullet 3 = 6$.

Exercise: Work out this example in detail. Here $T_s = 0.5$ sec.

Pad $f_A[t_n]$ with zeros.

Plot the ***super-position curve*** with CONVOLUTION INDEX $i = 3$.

Plot the CONVOLUTION CURVE.

Exercise: Compute and plot $f_C(t_n) = f_A(t_n) * f_B(t_n)$ with T = 1 sec and T_s = ¼ sec

a) $f_A(t) = 1$, $f_B(t) = t$

b) $f_A((t) = t$, $f_B(t) = t$

Exercise: Repeat the above exercise with T_s = 1/8 sec.

It is very important to note that

- $f_A(t)$ is defined over [0, T].
- $f_B(t)$ is defined over [0, T].
- But $f_C(t) = f_A(t) * f_B(t)$ is defined over the CONVOLUTION INTERVAL [0, 2T] no matter how many points we may take, or regardless of the size of T_s.

To convey the notion that CONVOLUTION is being performed we may write:

$$f_C(t) = f_A(t) * f_B(t)$$

But we have to specify the interval over which the CONVOLUTION of $f_A(t)$ and $f_B(t)$ is taking place.

EXAMPLE FROM COMMUNICATION SYSTEMS

We shall work out one last example adapted from Communication Systems by B. P. Lathi (page 84, fig 1-43), John Wiley, NY 1965, so that all the aspects of DISCRETE CONVOLUTION can be highlighted.

- Making two continuous functions discrete.
- Using equal spacing $T_s = 0.5$ sec.
- Padding with zeros
- Then indexing the discrete functions correctly.
- Convolving one of the functions and shifting
- Finding the super-positions (products) $a_k \bullet b_{i-k}$.
- Finding the CONVOLUTION SUM c_i.
- Determining the picture for a chosen convolution index i or $t = i \bullet T_s$.

In the previous example we saw how PADDING WITH ZEROS allows us to use the same CONVOLUTION SUM formula:

$$c_i = \sum_{k=0}^{i} a_k \,.\, b_{i-k}$$

throughout the CONVOLUTION process. The starting CONVOLUTION INDEX i (left end) is the (starting index of $f_A[t_n]$) + (starting index of $f_B[t_n]$) which in the earlier example was $0 + 0 = 0$.

In this example $f_A(t) = t$ over [0, 2] sec and $f_B(t) = +2$ over [−1, +1] sec. Let $T_s = 0.5$ sec.

So $f_A[t_k] = t_k$ for k = 0, 1, 2, 3 and $f_B[t_k] = +2$ for k = −2, −1, 0, 1.

We shall PAD WITH ZEROS in two steps. Initially we PAD WITH ZEROS $f_A[t_k]$ for k = −2, −1.

And we PAD WITH ZEROS $f_B[t_k]$ for k = 2, 3.

So now the range of k for both $f_A[t_k]$ and $f_B[t_k]$ is k = −2, −1, 0, 1, 2, 3.

We do this for the sake of clarity and simplicity in the explanation that follows.

The CONVOLUTION INDEX i will range from −2 + (−2) = −4 on the left to 3 + 3 = 6 on the right.

Even though we do not show it in the diagrams, we further PAD WITH ZEROS both $f_A[t_k]$ and $f_B[t_k]$ from −5 to −∞ on the left and from +4 to +∞ on the right.

So now we can use the general form of the CONVOLUTION SUM throughout the example:

$$c_i = \sum_{k=-\infty}^{\infty} a_k b_{i-k}$$

Since we know that $c_i = 0$ for $i < -4$ and $i > 6$ we may use the form of the CONVOLUTION SUM:

$$c_i = \sum_{k=-4}^{6} a_k b_{i-k}$$

So now the CONVOLUTION INDEX i varies from −4 to +6. The CONVOLUTION INTERVAL is [−4, 6].

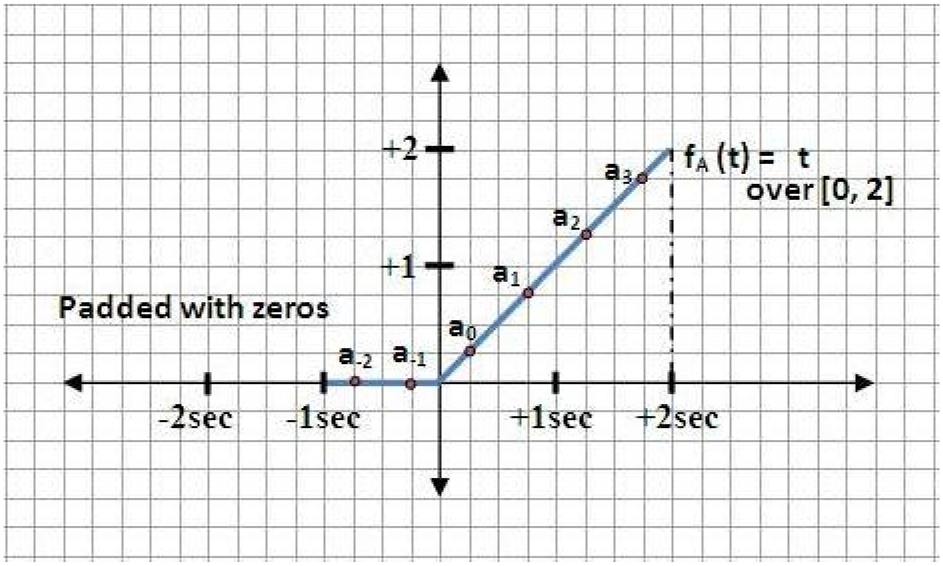
Since we want to focus on the ***overlapping interval*** and the ***super-position curve*** over the ***overlapping interval*** as convolved $f_B[t_k]$ SHIFTS over $f_A[t_k]$ we shall show only the range of k that lies in the ***overlapping interval***.

This way we can clearly depict the ***super-position curve*** over the ***overlapping interval*** as the shifting progresses.

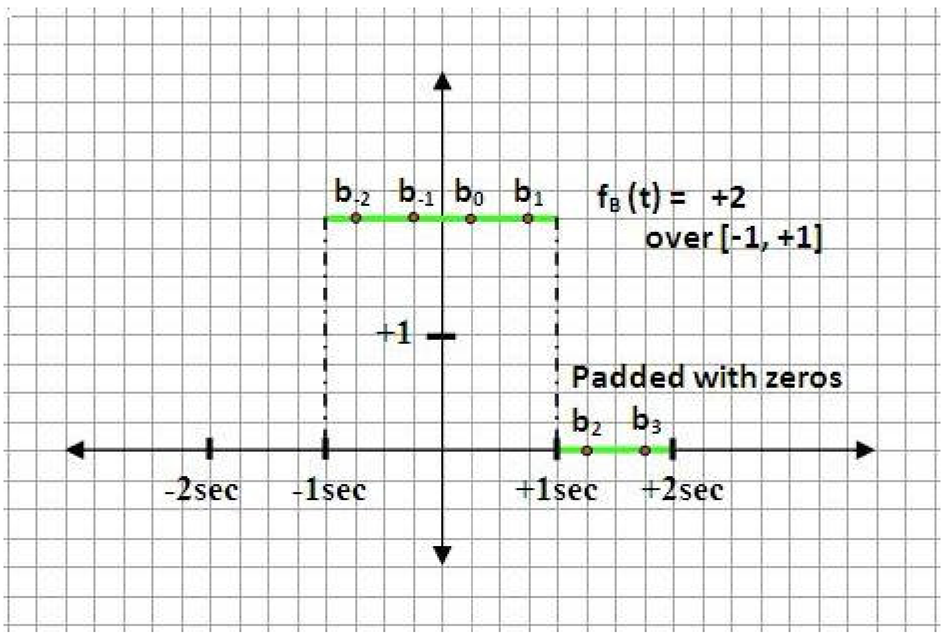
Also we can clearly see how we get the CONVOLUTION SUMS for the lollipop diagram and the COMBINED EFFECT SUMS for the CONVOLUTION CURVE.

$$\text{CONVOLUTION SUM } c_i = \sum a_k b_{i-k} \text{ for k in the } \boldsymbol{overlapping\ interval}$$

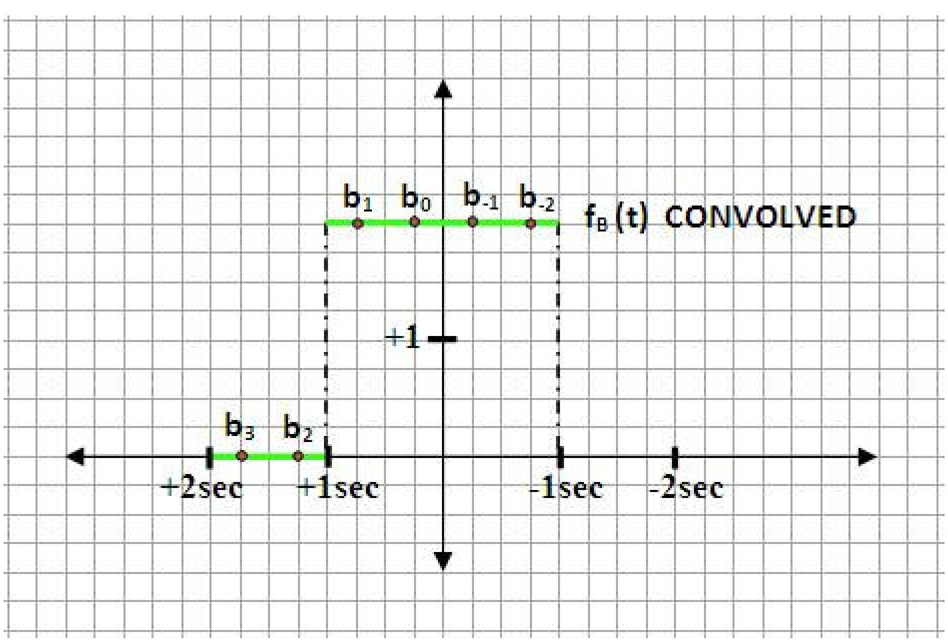
$$\text{COMBINED EFFECT SUM } c_i \bullet T_s = \sum a_k\, b_{i-k} \bullet T_s \text{ for k in the } \boldsymbol{overlapping\ interval}$$



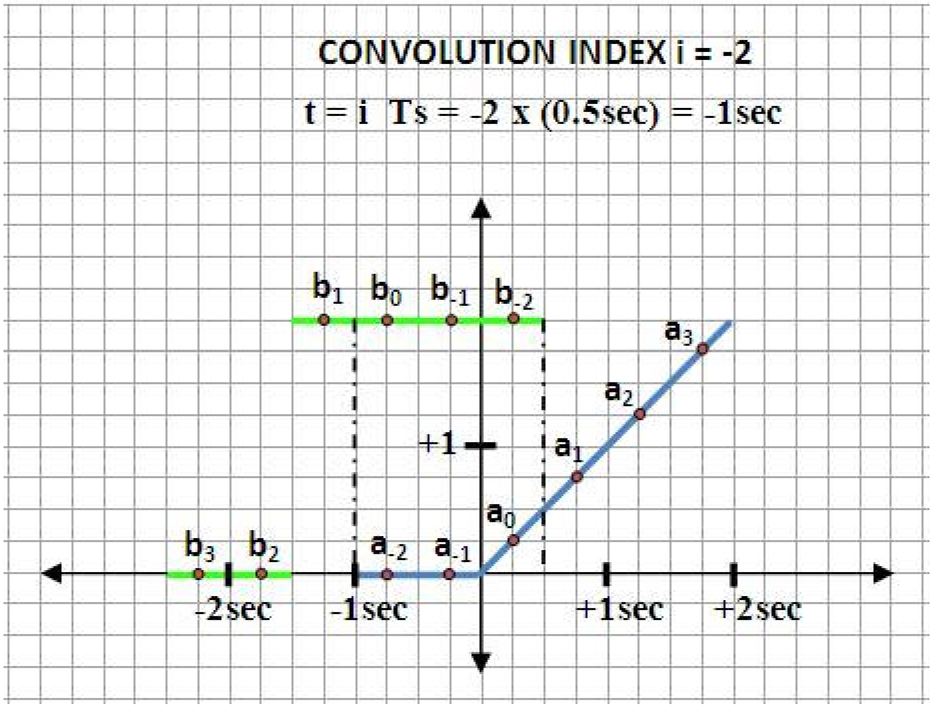
$f_A(t) = t$
over [0, 2]
Padded with zeros
a_{-2} a_{-1} a_0 a_1 a_2 a_3
+2
+1
-2sec
-1sec
+1sec
+2sec



b_{-2} b_{-1} b_0 b_1
$f_B(t) = +2$
over [-1, +1]
+1
Padded with zeros
b_2 b_3
-2sec
-1sec
+1sec
+2sec



b_1 b_0 b_{-1} b_{-2}
$f_B(t)$ CONVOLVED
+1
b_3 b_2
+2sec
+1sec
-1sec
-2sec

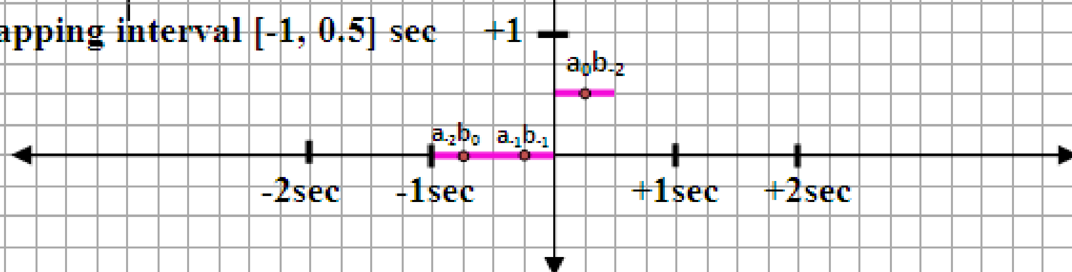


Discrete case $a_k \cdot b_{i-k}$

Range of index k = -2, -1, 0

Continuous case $f_A(\tau) \cdot f_B(t-\tau)$

τ in overlapping interval [-1, 0.5] sec



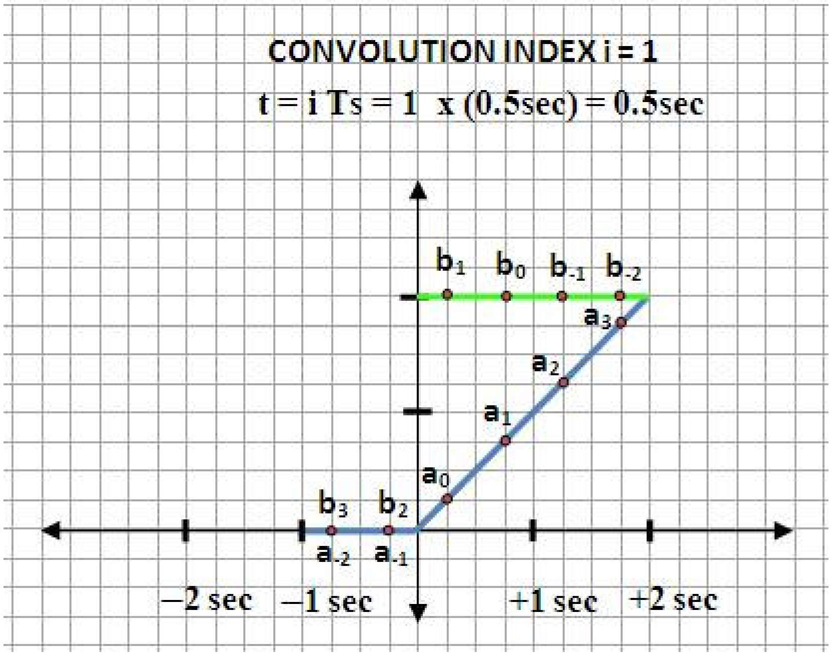
Super-position curve over overlapping interval [-1, 0.5] sec

Convolution Sum $\quad c_{-2} = \sum_{k=-2}^{0} (a_k \cdot b_{i-k}) = a_{-2}b_0 + a_{-1}b_{-1} + a_0b_{-2}$

$$= 0 + 0 + 0.5 = 0.5$$

Combined Effect Sum $\quad c_{-2}\ T_s = \sum_{k=-2}^{0} (a_k \cdot b_{i-k})\ T_s = 0.5 \times 0.5 = 0.25$

Convolution Integral $\quad \int_{-1}^{0.5} f_A(\tau) \cdot f_B(t-\tau)\, d\tau$ **with** $t = -1$**sec**

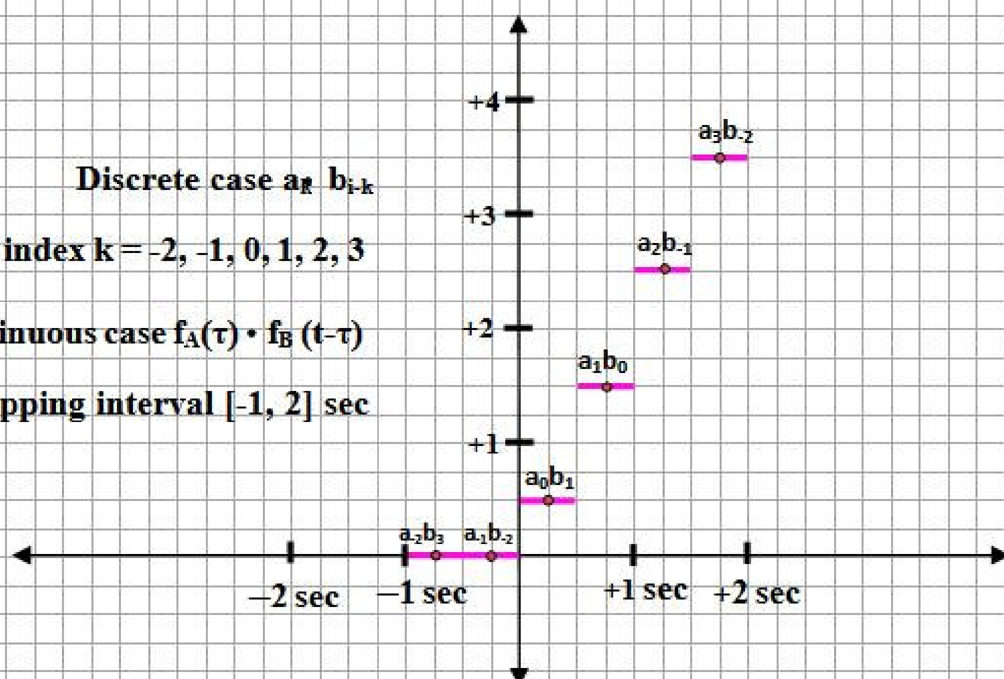


Discrete case $a_k \, b_{i-k}$

Range of index k = -2, -1, 0, 1, 2, 3

Continuous case $f_A(\tau) \cdot f_B\,(t-\tau)$

τ in overlapping interval [-1, 2] sec



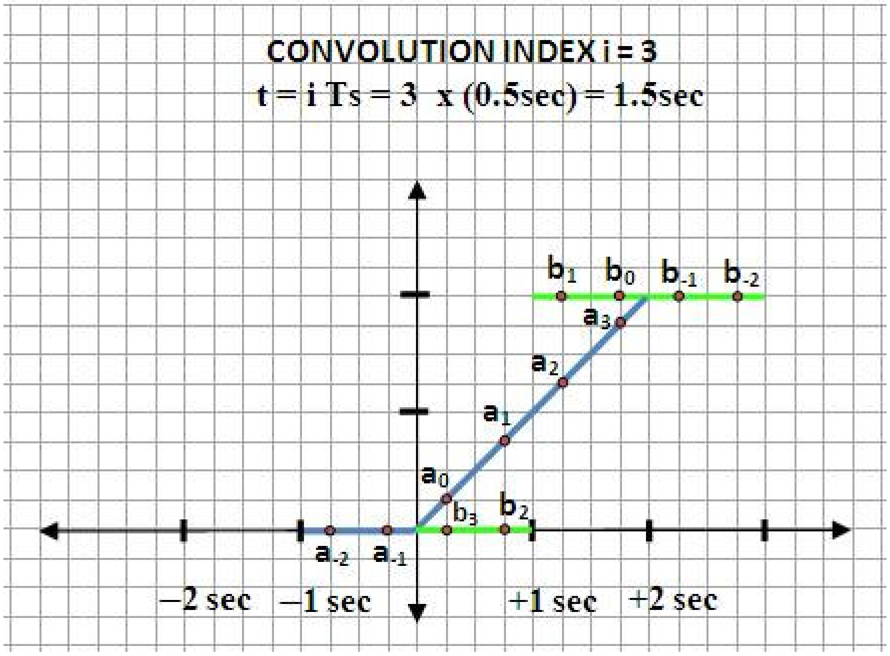
Super-position curve over overlapping interval [-1, 2] sec

Convolution Sum $c_1 = \sum_{k=-2}^{3} (a_k \cdot b_{i-k}) = a_{-2}b_3 + a_{-1}b_2 + a_0b_1 + a_1b_0 + a_2b_{-1} + a_3b_{-2}$

$= 0 + 0 + 0.5 + 1.5 + 2.5 + 3.5 = 8$

Combined Effect Sum $c_1 T_s = \sum_{k=-2}^{3} (a_k \cdot b_{i-k})\, T_s = 8 \times 0.5 = 4$

Convolutuion Integral $= \int_{-1}^{2} f_A(\tau) \cdot f_B\,(t-\tau)d\tau$ **with t = 0.5sec**

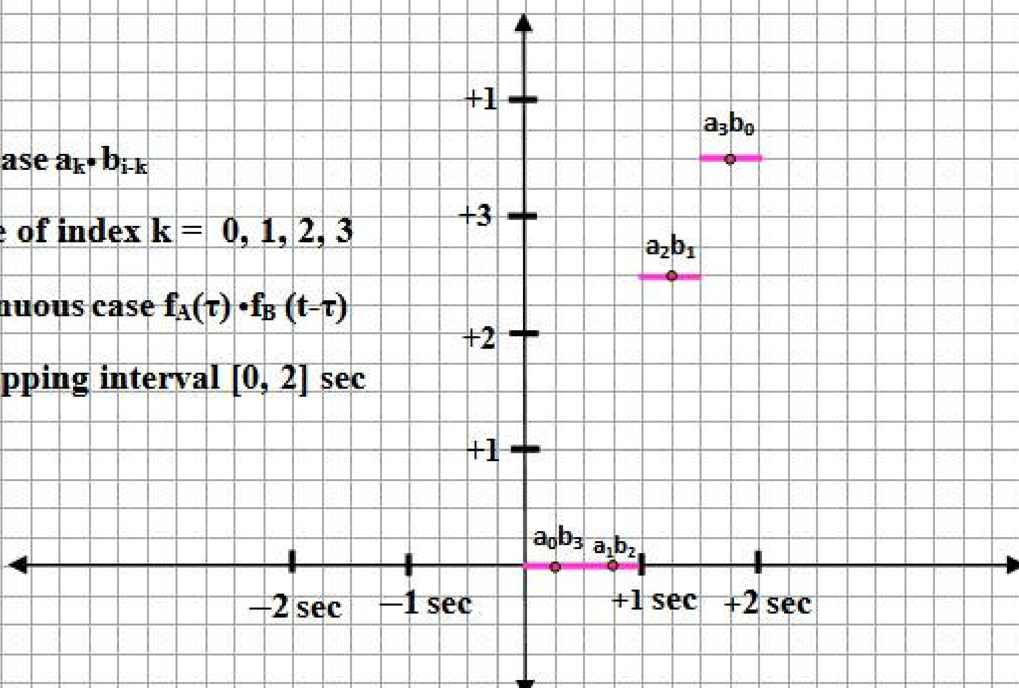


Discrete case $a_k \cdot b_{i-k}$

Range of index k = 0, 1, 2, 3

Continuous case $f_A(\tau) \cdot f_B(t-\tau)$

τ in overlapping interval [0, 2] sec



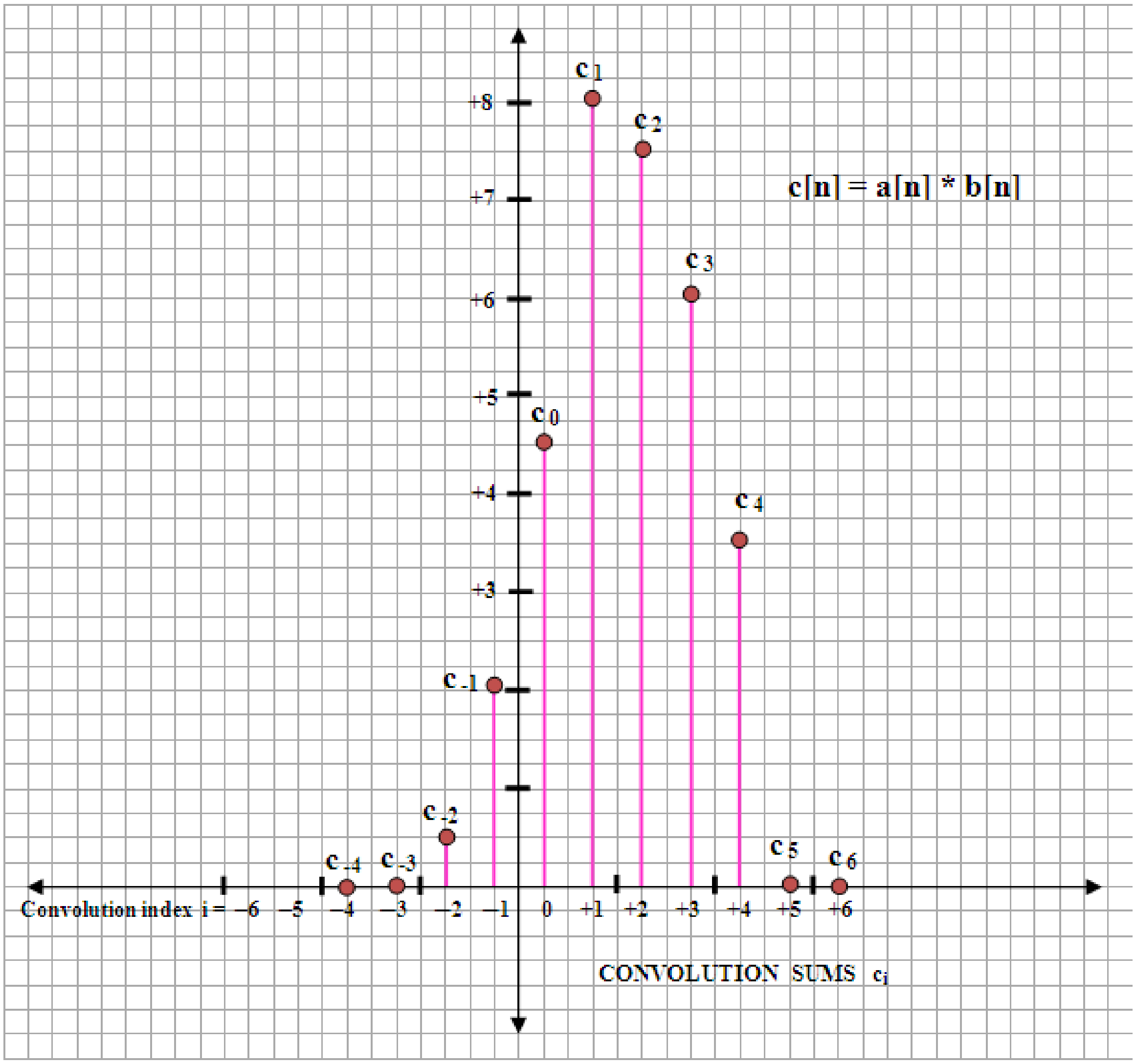
Super-position curve over overlapping interval [-1, 2] sec

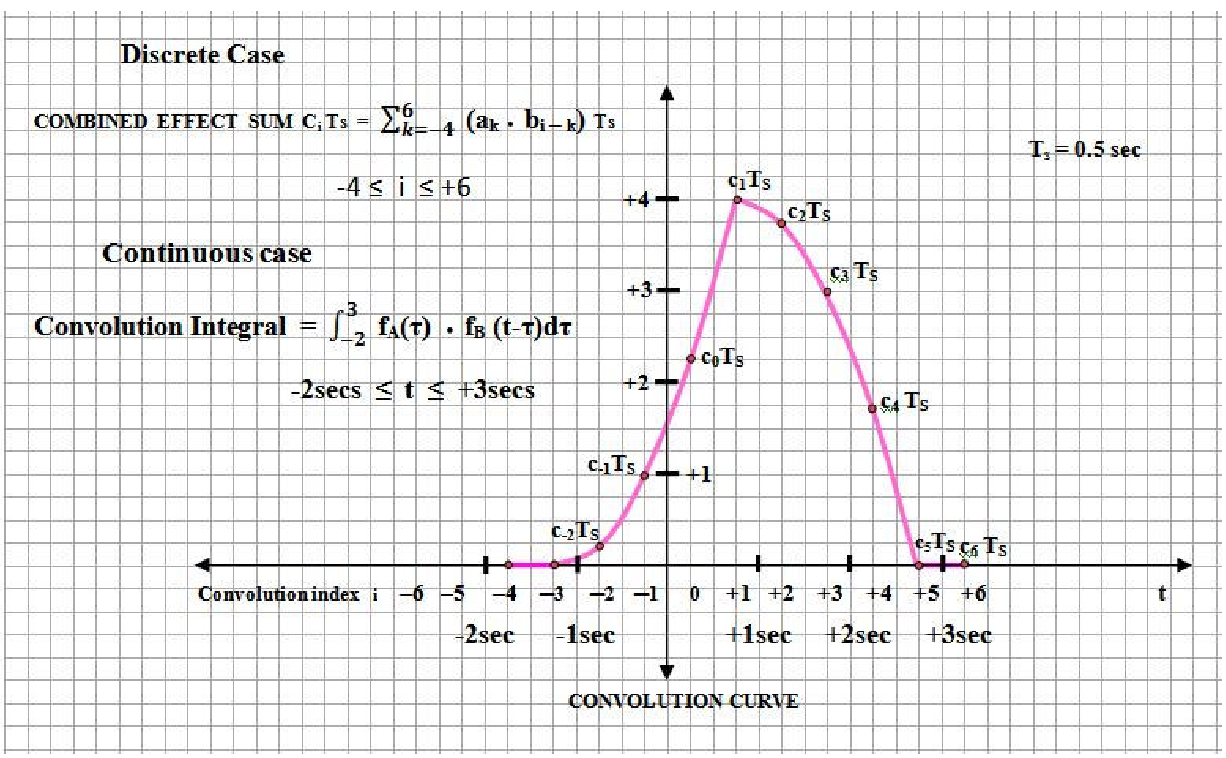
Convolution Sum $c_3 = \sum_{k=0}^{3} (a_k \cdot b_{i-k}) = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0$

$= 0 + 0 + 2.5 + 3.5 = 6$

Combined Effect Sum $c_3\, T_s = \sum_{k=0}^{3} (a_k \cdot b_{i-k})\, T_s = 6 \times 0.5 = 3$

Convolution Integral $= \int_0^2 f_A(\tau) \cdot f_B(t-\tau)d\tau$ **with t = 1.5sec**


c[n] = a[n] * b[n]
c 1
c 2
c 3
c 0
c 4
c -1
c -2
c -4
c -3
c 5
c 6
+8
+7
+6
+5
+4
+3
Convolution index i = −6 −5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5 +6
CONVOLUTION SUMS c_i


Discrete Case
COMBINED EFFECT SUM $C_i T_s = \sum_{k=-4}^{6} (a_k \cdot b_{i-k})\ T_s$
$-4 \leq i \leq +6$
Continuous case
Convolution Integral $= \int_{-2}^{3} f_A(\tau) \cdot f_B(t-\tau)d\tau$
$-2secs \leq t \leq +3secs$
$T_s = 0.5$ sec
c_1T_S
c_2T_S
c_3T_S
c_0T_S
c_4T_S
$c_{-1}T_S$
$c_{-2}T_S$
c_5T_S c_6T_S
+4
+3
+2
+1
Convolution index i −6 −5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5 +6 t
-2sec -1sec +1sec +2sec +3sec
CONVOLUTION CURVE

As an exercise find $f_C(t = 0.5 \text{ sec})$ with $f_A(t)$ convolved instead of $f_B(t)$.

Sketch the overlapping of the two functions at $t = 0.5$ sec.

What do you think the ***super-position curve*** $f_A(t-\tau) \bullet f_B(t)$ with $t = 0.5$ sec will look like?

Will this ***super-position curve*** look like the reflected version of the ***super-position curve*** with CONVOLUTION INDEX $i = 1$?

What is the area under the ***super-position curve?***

CONVOLUTION INTERVAL AND CONVOLUTION INDEX

We are familiar with the CONVOLUTION (long multiplication without carry) of two decimal digital numbers. For example:

6320 * 0.000521

The interval of the least significant and most significant NON-ZERO digits is [1, 3]

10^3 10^2 10^1

x[n] = . . . 0 0 6 3 2 0 . 0 0 0 . . .

x_k = 2, 3, 6 for k = 1, 2, 3 respectively

The INDEX interval of the least significant and most significant NON-ZERO digits is [−6, −4]

10^{-4} 10^{-5} 10^{-6}

y[n] = . . . 0 0 0 0 0 . 0 0 0 5 2 1 0 0 . . .

y_k = 5, 2, 1 for k = −4, −5, −6 respectively

After the CONVOLUTION we get z[n] = x[n] * y[n].

The least significant NON-ZERO digit will be under 10^{-5}

The most significant NON-ZERO digit will be under 10^{-1}

The CONVOLUTION INTERVAL is [1−6, 3−4] = [−5, −1].

The INDEX i of the CONVOLUTION SUMS we need to evaluate are: −5, −4, −3, −2, −1.

This is exactly the role of the CONVOLUTION INDEX even after PADDING WITH ZEROS.

And 632 * 521 yields the values z_i : 2, 7, 22, 27, 30 in the INDEXED order.

In fact we may compute the value of any chosen term z_i

$$z_i = \sum_{\text{all } k} x_k \,.\, y_{i-k}$$

Notice that k + (i-k) = i.

For example:

$$z_{-3} = (x_3.y_{-6}) + (x_2.y_{-5}) + (x_1.y_{-4})$$

$$= \quad 6 \quad + \quad 6 \quad + \quad 10$$

$$= \quad 22$$

PADDING WITH ZEROS to $-\infty$ and $+\infty$

Let us see a general example using two small finite sequences.

Example: Given two sequences

$x[n]= . . . .0, 0, 0, x_{-6}, x_{-5}, x_{-4}, x_{-3}, 0, 0, 0, . . .$

$y[n]= . . . .0, 0, 0, y_5, y_6, y_7, y_8, y_9, 0, 0, 0, . . .$

What is CONVOLUTION INTERVAL of: $z[n] = x[n]*y[n]$?

Let k^- be the INDEX of the left most NON-ZERO term.

k^+ be the INDEX of the right most NON-ZERO term.

$x[n]$ is in the interval $[k_x^-, k_x^+] = [-6, -3]$

$y[n]$ is in the interval $[k_y^-, k_y^+] = [5, 9]$

$z[n]$ is in the interval $[i_z^-, i_z^+] = [k_x^- + k_y^-, k_x^+ + k_y^+] = [-6+5, -3+9]$

The CONVOLUTION INTERVAL $= [i_z^-, i_z^+] = [-1, +6]$.

The first and last NON-ZERO CONVOLUTION SUMS are z_{-1} and z_6.

The CONVOLUTION INDEX and CONVOLUTION SUMS are:

$$. . . 0, 0, 0, z_{-1}, z_0, z_1, z_2, . . . , z_6, 0, 0, 0, . . .$$

From now on we PADDING WITH ZEROS $x[n]$ and $y[n]$ all the way to $-\infty$ and $+\infty$. We know exactly the INDEX of the first and last NON-ZERO CONVOLUTION SUMS of $z[n]$ and the CONVOLUTION INTERVAL.

How can we evaluate the CONVOLUTION SUMS?

Let $x[n] = . . . x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, . . .$

$y[n] = . . . y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, . . .$

To find $z[n]$ first CONVOLVE either one of the sequences and place it under the other.

$$x[n] : . . . x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, . . .$$

$y[-n] = y[n]$ CONVOLVED: $. . . y_3, y_2, y_1, y_0, y_{-1}, y_{-2}, y_{-3}, . . .$

Now $z[0] = . . . + (x_{-3}.y_3) + (x_{-2}.y_2) + . . . + (x_0.y_0) + . . . + (x_3.y_{-3}) + . . .$

$$= \sum_{k=-\infty}^{+\infty} x_k \cdot y_{0-k}$$

Here $i = 0$. Notice how the index k and index i-k of the products $(x_k \,.\, y_{i-k})$ add up to i.

To find z[1], SHIFT RIGHT y[-n] one step

$$x[n] : \ldots x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ x_1,\ x_2,\ x_3, \ldots$$

y[-n] shifted right one step: $\ldots y_3,\ y_2,\ y_1,\ y_0,\ y_{-1},\ y_{-2},\ y_{-3}, \ldots$

Now $z[1] = \ldots + (x_{-2} \bullet y_3) + (x_{-1} \bullet y_2) + (x_0 \bullet y_1) + (x_1 \bullet y_0) + (x_2 \bullet y_{-1}) + \ldots$

$$= \sum_{-\infty}^{+\infty} x_k \,.\, y_{1-k}$$

Here $i = 1$. Again notice how the index k and index i−k of the products $(x_k \,.\, y_{i-k})$ add up to i.
To find z[2], SHIFT RIGHT , y[−n] two steps. Find all the products and add up.

$$z[2] = \sum_{-\infty}^{+\infty} x_k \,.\, y_{2-k}$$

To find z[-3] SHIFT LEFT, y[-n] three steps. Find all the products and add up.

$$z[-3] = \sum_{-\infty}^{+\infty} x_k \,.\, y_{-3-k}$$

The advantage of this method is that once we know the sequences x[n] and y[n] we can find any chosen z_i.

When CONVOLVING discrete functions $f_A[t_k]$ and $f_B[t_k]$ using T_s , we may view these sequences a[n] and b[n] as pure sequences. We may then find c[n] = a[n] * b[n]. In fact we may find any chosen c_i.
What is $f_C[t_i]$?

$$f_C[t_i] = \sum_{k=-\infty}^{+\infty} f_A[t_k] \,.\, f_B[t_{i-k}] = \sum_{k=-\infty}^{+\infty} a_k \,.\, b_{i-k}$$

Where is t_i on the time axis? $t_i = i \,.\, T_s$

May be adjusted by $\frac{T_s}{2}$ to get the mid-point of the sub-interval.
This is useful in real time processing.

GENERAL FORM OF CONVOLUTION SUM

Notice that in the computation of each CONVOLUTION SUM z_i

$$z_i = \sum_{\text{all k}} x_k \, . \, y_{i-k} = \sum_{\text{all k}} x_{i-k} \, . \, y_k$$

there is an interval where the x_k and $y_{i\text{-}k}$ overlap. Outside this interval the products $x_k \, .y_{i-k}$ are zero. Why?

Because outside this ***overlapping interval***, either x_k does not exist and hence is zero or y_{i-k} does not exist and hence is zero.

So after we PAD WITH ZEROS the given finite sequences x[n] and y[n] in both directions all the way to $-\infty$ and $+\infty$ and make the sequences infinite, we may write the more general form of the CONVOLUTION SUM

$$z_i = \sum_{k=-\infty}^{+\infty} x_k \, . \, y_{i-k} = \sum_{k=-\infty}^{+\infty} x_{i-k} \, . \, y_k$$

or

$$c_i = \sum_{k=-\infty}^{+\infty} a_k \, b_{i-k} = \sum_{k=-\infty}^{+\infty} a_{i-k} \, b_k$$

CONVOLUTION INTEGRAL

We mentioned earlier that we are interested in each individual CONVOLUTION SUM. We are also interested in each individual COMBINED EFFECT SUM. The COMBINED EFFECT SUM at instant i • T_s is

$$f_C[i\,.\,T_s]\,.\,T_s = \sum_{k=-\infty}^{+\infty} (f_A[k\ .\ T_s]\,.\ f_B[(i-k)\,.\,T_s])\ \ .\ T_s$$

$$= \sum_{k=-\infty}^{+\infty} (f_A[(i-k)\,.\ T_s]\,.\ f_B[k\,.\,T_s])\ \ .\ T_s$$

where $f_A(t)$ and $f_B(t)$ are defined over [0, T].

From the exercises with T_s= 1/4 and T_s = 1/8 we get the sense that the smaller the sub-interval T_s, the discrete versions become more accurate or closer to the continuous versions.

Let t = i • T_s be the instant on the time axis between [0, 2T]. So now:

$$\text{COMBINED EFFECT SUM}\ f_C(t).\ T_s = \sum_{k=-\infty}^{+\infty} f_A(k\,.\,T_s)\,.\ f_B(t-k\,.\,T_s)\,.\ T_s$$

$$= \sum_{k=-\infty}^{+\infty} f_A(t-k\,.\,T_s)\,.\,f_B(k\,.\,T_s)\,.\ T_s$$

Now, instead of sub-dividing [0, T] into T_s= 1/4 or T_s = 1/8 sub-intervals, let $T_s \to 0$.

What happens? Limit ($T_s \to 0$) = dτ, k • T_s becomes τ on the time axis and the ∑ becomes an ∫ over the overlapping interval. The COMBINED EFFECT SUM at instant i • T_s on the left hand side becomes the CONVOLUTION INTEGRAL at instant t. We get:

$$\text{CONVOLUTION INTEGRAL at instant t} = \int_{-\infty}^{+\infty} f_A(\tau)\,.\ f_B(t-\tau)\ .\,d\tau$$

$$= \int_{-\infty}^{+\infty} f_A(t-\tau)\,.\ f_B(\tau)\ .\,d\tau$$

In going from ***discrete*** to ***continuous*** several things happen simultaneously:

- First we let $T_s \to 0$
- Then we take limit ($T_s \to 0$).
- In f_B[(i–k) • T_s] the i • $T_s \to$ t, the CONVOLUTION INSTANT.
- We have used the letter t for the CONVOLUTION INDEX i. So we need to introduce another variable, say τ, for the index k.
- And limit k • $T_s \to \tau$ the SUPER-POSITION INSTANT.
- And limit ($T_s \to 0$) = dτ.

We make use of a τ (Greek letter t) as the dummy variable of integration because, we already used $t = i \bullet T_s$.

On the right side we computed the continuous version of the discrete COMBINED EFFECT SUM. But the text books say (quite incorrectly) that it is the continuous version of the CONVOLUTION SUM.

This we call the CONVOLUTION INTEGRAL. Standard text-books say that the CONVOLUTION INTEGRAL is the continuous version of the discrete CONVOLUTION SUM. This is not quite correct. That is why we introduced some extra terminology: CONVOLUTION SUM for pure sequences and COMBINED EFFECT SUM for discrete functions. And then quite naturally and precisely: CONVOLUTION INTEGRAL for continuous functions.

It is important to see in the CONVOLUTION process:

1. When are we dealing with ***pure sequences*** where there is only a pure index on the horizontal axis? and, when are we dealing with ***discrete functions*** (sequences got from discretizing continuous functions over specified intervals on the time axis)? Here the CONVOLUTION INDEX i is to identify the sub-intervals with the specified intervals on the time axis.

2. In the case of the COMBINED EFFECT SUM we should know exactly where it occurs on the time axis within the specified CONVOLUTION INTERVAL. Likewise for the CONVOLUTION INTEGRAL.

In the case of the CONVOLUTION SUM (used when pure sequences are convolved) we should know the exact CONVOLUTION INDEX.

Exercises:

1. Complete the following table

$f_A(t)$	$f_B(t)$	$f_A(t) \bullet f_B(t)$	$\int f_A(t) \bullet f_B(t)$	$f_A(t) * f_B(t)$
1	1			
1	t			
t	t			

From the earlier related exercises can you see the COMBINED EFFECT SUM converging to the CONVOLUTION INTEGRAL?

2.

a) What is sin(t) • cos(t)?

b) What is sin(t) * cos(t) over one period [0, T].

3.

a)What is –sin(t) * 1? How is it related to d/dt sin(t)?

b) What is –cos(t) * 1? How is it related to d/dt cos(t)?

Can you achieve differentiation of sinusoidals using the CONVOLUTION operation?

LINEAR CONVOLUTION

So far what we have been doing is LINEAR CONVOLUTION. This works with finite length sequences. To keep the explanation simple, assume only non-negative index.

x[n] has length N with index k: 0 to N−1

y[n] has length M with index k: 0 to M−1

z[n] has length N+M-1 with index i: 0 to N+M−2

EXAMPLE:

index k: **-3, - 2, -1, 0, 1, 2, 3, 4**

x[n]: . . . . , 0, 0, 0, 2, 3, 0, 0, 0, . . . **length** N = 2

y[n]: . . . . . ,0, 0, 0, 4, 5, 0, 0, 0, ... **length** M = 2

index k: **-3, - 2, -1, 0, 1, 2, 3, 4**

x[n]: . . . , 0, 0, 0, 2, 3, 0, 0, 0, ...

y[-n]: . . ., 0, 0, 5, 4, 0, 0, 0, ...

Convolution index k: **-3, - 2, -1, 0, 1, 2, 3, 4**. . . . . **length** N+M−1 = 3

z[n] = x[n] * y[n] = . . ., 0, 0, 0, 8, 22, 15, 0, 0, ...

Linear convolution: perform the convolution as usual and write out all the z[i] for **convolution index** i from -∞ to +∞.

Now let us see what happens when we have to deal with infinite sequences. Infinite sequences may be of 2 types: **periodic** and **aperiodic.** Let us consider the **periodic** case. Let us make both x[n] and y[n] periodic.

So now: **index k**: **-3, - 2, -1, 0, 1, 2, 3, 4**

x[n]: . . . , 3, 2, 3, 2, 3, 2, 3, 2, . . . **period** N = 2

y[n]: . . . . ,5, 4, 5, 4, 5, 4, 5, 4, ... **period** M = 2

Now what is z[n]=x[n] * y[n] ?

Obviously each CONVOLUTION SUM z[i] is unbounded. What can we do in these situations? What kind of useful information can we extract?
Let us see another example where the periods of x[n] and y[n] are not the same.

index k: **-3, - 2, -1, 0, 1, 2, 3, 4**

x[n]: . . . . , 6, 7, 8, 2, 3, 6, 7, 8, . . . **period** N = 5

y[n]: . . . . . ,5, 4, 9, 5, 4, 9, 5, 4, ... **period** M = 3

index k: **-3, - 2, -1, 0, 1, 2, 3, 4**

We pad with zeroes y[n] to get: 5 4 9 0 0

Now both x[n] and y[n] have the same **period** N.

We take both x[n] and y[n] over one period, assume all the rest are PADDED WITH ZEROES all the way to $-\infty$ and $+\infty$.

index k: **-3, - 2, -1, 0, 1, 2, 3, 4 5 6 7**

x[n]: . . . . , 0, 0, 0, 2, 3, 6, 7, 8, 0, 0, 0, . . . **length** N = 5

y[n]: . . . . . ,0, 0, 0, 5, 4, 9, 0, 0, 0, ... **length** M = 3

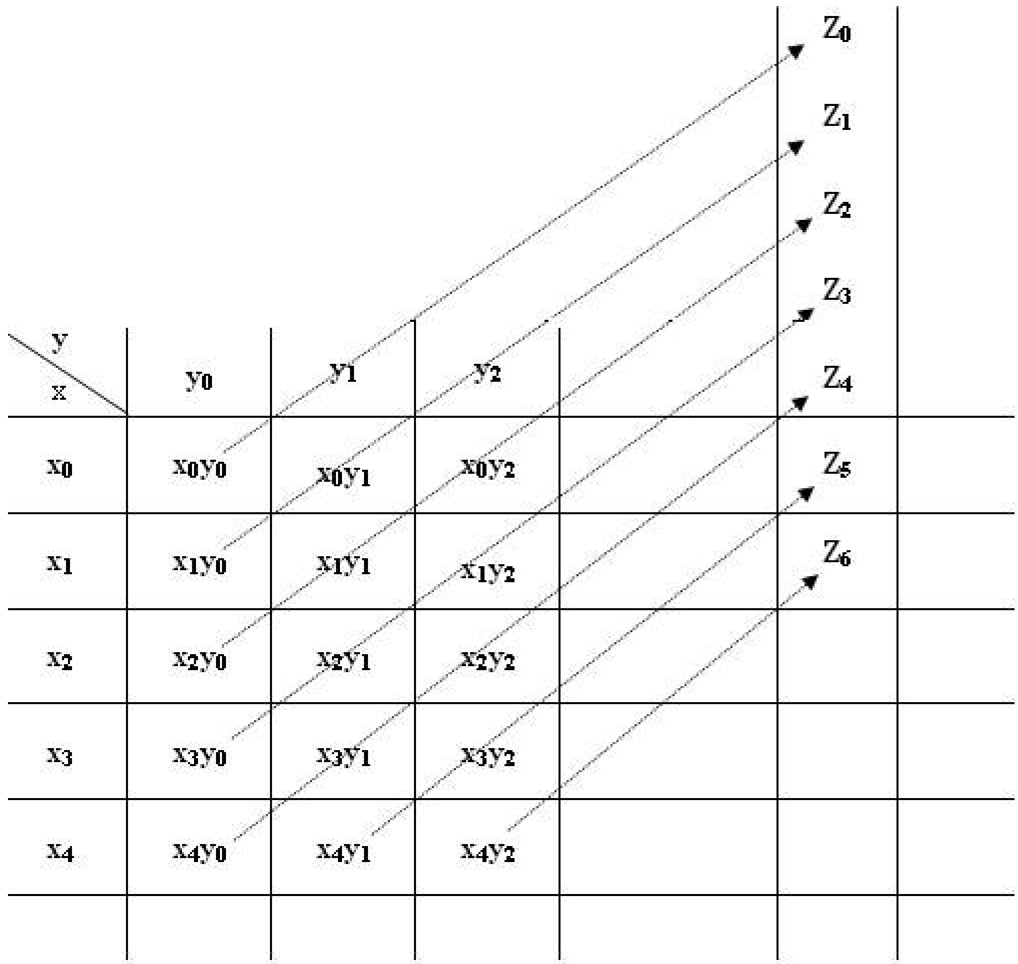
Then we find: $z[n] = \underset{[0,N-1]}{x[n]} * \underset{[0,N-1]}{y[n]}$

the usual **linear convolution** of two finite sequences.

z[n] = z[0], z[1], z[2], z[3], z[4], z[5], z[6]

z[n] has length N+M−1 = 7 with **linear** CONVOLUTION INDEX i: 0 to N+M−2 = 6.

MATRIX PICTURE FOR LINEAR CONVOLUTION



CIRCULAR CONVOLUTION

Suppose we want z[n] to be **periodic** with same period N, like x[n] and the PADDED WITH ZEROES y[n].

Instead of letting index i run from 0 to N+M−2, we take i_c = (i) congruence modulo N. We get

$z[i_c]$: (z[0c] = z[0]+z[5]), (z[1c] = z[1]+z[6]), (z[2c] = z[2]), (z[3c] = z[3]), (z[4c] = z[4])

where i_c denotes the **circular** CONVOLUTION INDEX. Note that the period of $z[i_c]$ is N = 5

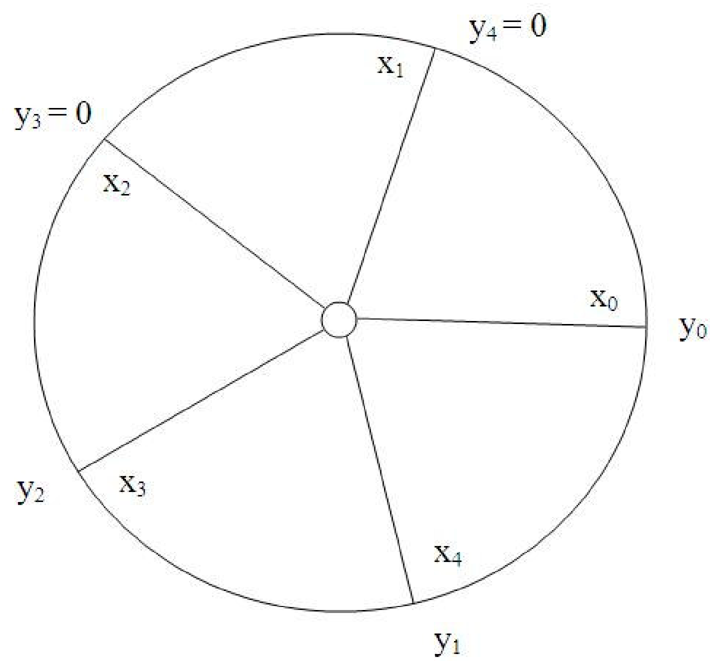
This we call **circular convolution** or **periodic convolution.** And it is written as:

$$z[n] = x[n] \circledast y[n].$$

This same method applies even when M = N

x[n] is written inside circle in the positive direction (counter-clockwise).

y[n] convolved is written outside the circle in the negative direction (clockwise).



i_c = 0: z[0c] = z[0] + z[5]

$= x_0 \,.\, y_0 + (x_4 \,.\, y_1 + x_3 \,.\, y_2)$

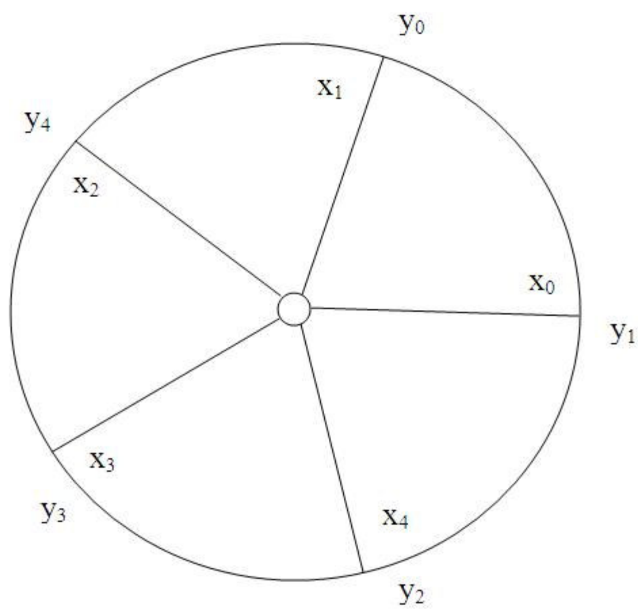
This is exactly what we get when we sum up the corresponding products in the circular picture with $i_c = 0$.

Now: ROTATE y[n] convolved counter-clockwise one step.

This is similar to a **right** SHIFT one step in **linear convolution**.

$i_c = 1$: z[1c] = z[1] + z[6]

$= (x_1 . y_0 + x_0 . y_1) + (x_4 . y_2)$



This is exactly what we get when we sum up the corresponding products in the circular picture with $i_c = +1$.

Notice the difference between **linear convolution** and **circular convolution**.

z[2], z[3] and z[4] of **linear convolution** and **circular convolution** are the same for **convolution index** from M to N-1.

But z[0c] $\neq$ z[0], and z[1c] $\neq$ z[1].

By now you must have realized how the $z[i_c]$ can be computed for positive i_c.

How do we compute $z[i_c]$ for negative i_c?

You guessed it! ROTATE y[n] convolved clockwise (negative direction).

This corresponds to a **left** SHIFT in **linear convolution.**

LINEAR CONVOLUTION of sin(ω_0t) and cos(ω_0t)

Let us work out the circular convolution of $a(t) = \sin(\omega_0 t)$ and $b(t) = \cos(\omega_0 t)$ over one period

$T_0 = 2\pi$ sec and $T_s = \pi/4$. So M = N = 8. Note that $\omega_0 = 2\pi/T_0$.

k	0	1	2	3	4	5	6	7
$a[t_k]$	0.0	+ 0.7	+ 1	+ 0.7	0	− 0.7	− 1	− 0.7
$b[t_k]$	+ 1	+ 0.7	0	− 0.7	− 1	− 0.7	0	+ 0.7

Let us first work out the LINEAR CONVOLUTION	**c[n] = a[n] * b[n]**
$c_0 = a_0b_0$	0
$c_1 = a_0b_1 + a_1b_0$	0.7
$c_2 = a_0b_2 + a_1b_1 + a_2b_0$	1.49
$c_3 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0$	1.4
$c_4 = a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0$	0
$c_5 = a_0b_5 + a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 + a_5b_0$	− 2.1
$c_6 = a_0b_6 + a_1b_5 + a_2b_4 + a_3b_3 + a_4b_2 + a_5b_1 + a_6b_0$	− 3.47
$c_7 = a_0b_7 + a_1b_6 + a_2b_5 + a_3b_4 + a_4b_3 + a_5b_2 + a_6b_1 + a_7b_0$	− 2.8
$c_8 = a_1b_7 + a_2b_6 + a_3b_5 + a_4b_4 + a_5b_3 + a_6b_2 + a_7b_1$	0
$c_9 = a_2b_7 + a_3b_6 + a_4b_5 + a_5b_4 + a_6b_3 + a_7b_2$	+ 2.1
$c_{10} = a_3b_7 + a_4b_6 + a_5b_5 + a_6b_4 + a_7b_3$	+ 2.47
$c_{11} = a_4b_7 + a_5b_6 + a_6b_5 + a_7b_4$	+ 1.4
$c_{12} = a_5b_7 + a_6b_6 + a_7b_5$	0
$c_{13} = a_6b_7 + a_7b_6$	− 0.7
$c_{14} = a_7b_7$	− 0.49

PERIODIC CONVOLUTION of $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$

Now let us computer the PERIODIC CONVOLUTION $d[n] = a[n] \circledast b[n]$ from the LINEAR CONVOLUTION $c[n]$

$d_0 = c_0 + c_8 = 0 + 0 \qquad = 0$

$d_1 = c_1 + c_9 = 0.7 + 2.1 \qquad = 2.8$

$d_2 = c_2 + c_{10} = 1.49 + 2.47 \qquad = 3.96$

$d_3 = c_3 + c_{11} = 1.4 + 1.4 \qquad = 2.8$

$d_4 = c_4 + c_{12} = 0 + 0 \qquad = 0$

$d_5 = c_5 + c_{13} = -\,2.1 - 0.7 \qquad = -\,2.8$

$d_6 = c_6 + c_{14} = -\,3.47 \;-0.49 \qquad = -\,3.96$

$d_7 = c_7 \qquad = -\,2.8$

This is the CIRCULAR CONVOLUTION of

$\sin[\omega_o(kT_s)] =$ ***discretised*** $\sin(\omega_o t)$

and

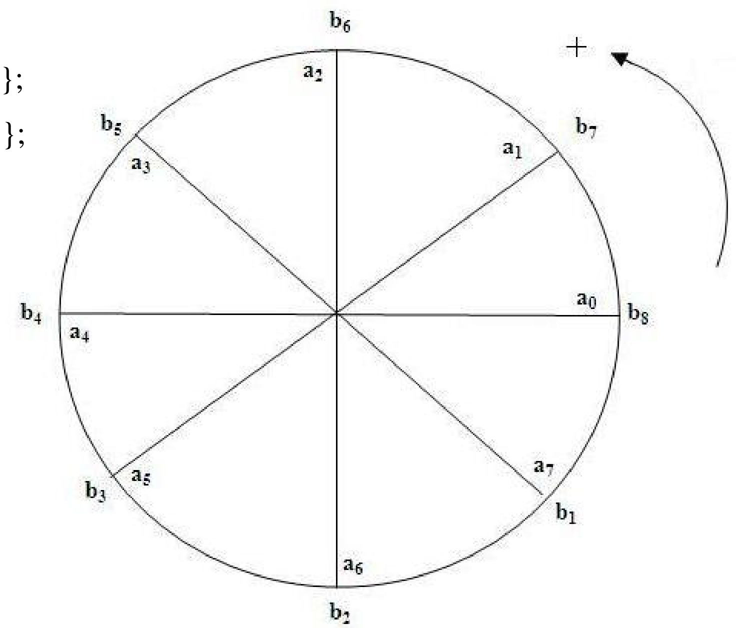
$\cos[\omega_o(kT_s)] =$ ***discretised*** $\cos(\omega_o t)$

with $T_s = \frac{\pi}{4}$.

Later, in LTI Systems we shall see the CIRCULAR CONVOLUTION of ***continuous*** $\sin(\omega_o t)$ and $\cos(\omega_o t)$.

We could have computed the circular convolution a[n] ⊛ b[n] directly as shown below:

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
float a[8]={0.0,0.7,1.0,0.7,0.0,-0.7,-1.0,-0.7};
float b[8]={1.0,0.7,0.0,-0.7,-1.0,-0.7,0.0,0.7};
int i,j,k;
float c[8],b0;
clrscr();
for(i=0;i<=7;i++)
{
  c[i]=a[0]*b[0];
  for(j=1;j<=7;j++)
  {
    k=8-j;
    c[i]=c[i]+a[j]*b[k];
  }
    printf("i=%d,c[i]=%6.4f\n",i,c[i]);
    //rotate circular convolved B counter clockwise = left circular shift of linear B not
convolved.
    b0=b[0];
    for(k=0;k<=6;k++)
    {
      b[k]=b[k+1];
    }
    b[7]=b0;
}
getch();
}
```



STARTING THE CIRCULAR CONVOLUTION

And the following is the output obtained. If we plot it we get a beautiful sine wave with same period T_0, only the amplitude and phase will be different.

i = 0, c[i] = 0.0000

i = 1, c[i] = 2.8000

i = 2, c[i] = 3.9600

i = 3, c[i] = 2.8000

i = 4, c[i] = 0.0000

i = 5, c[i] = −2.8000

i = 6, c[i] = −3.9600

i = 7, c[i] = −2.8000

Which do you think is easier to compute, the linear convolution or the circular convolution?

Linear convolution is also referred to as acyclic convolution.

Further reading:

Algebraic Methods for Signal Processing and Communications Coding
by Richard E. Blahut, Springer-Verlag, New York, 1991.

Chapter 1 and Chapter 5.

CIRCULAR CONVOLUTION OF TWO GUITARS EXAMPLE

Let us take our two guitars example that we started with PADDED WITH ZEROS from 5 to 8.

$g \; = s_0 \;\; s_1 \;\; s_2 \;\; s_3 \;\; s_4 \;\; s_5 \;\; s_6 \;\; s_7 \;\; s_8$

$G \; = S_0 \;\; S_1 \;\; S_2 \;\; S_3 \;\; S_4 \;\; S_5 \;\; S_6 \;\; S_7 \;\; S_8$

With index k and padded with zeroes.

Linear convolution g * G : s_0S_0 , 0 , s_1S_1 , 0, s_2S_2 , 0, s_3S_3, 0, s_4S_4

Convolution index i : 0 , 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8

linear convolution index i:		0	1	2	3	4
values	:	s_0S_0	0	s_1S_1	0	s_2S_2
values	:	0	s_3S_3	0	s_4S_4	
linear convolution index i :		5	6	7	8	
Cirular convolution						
$g \circledast G$ values	:	s_0S_0	s_3S_3	s_1S_1	s_4S_4	s_2S_2
circular convolution index i_c :		0_c	1_c	2_c	3_c	4_c

We can see that it is possible to extract the **linear convolution** result from the **circular convolution**. It is not always so easy or clear.

Further reading: Orthogonality and the Graham-Smith method in Linear Algebra by Gilbert Strang.

So now we can handle the convolution of infinite sequences that are **periodic**.

Later in DSP you will see:

Time **Frequency**

$$x[n] \xrightarrow{\text{DFT}} X[w]$$

$$y[n] \xrightarrow{\text{DFT}} Y[w]$$

Where x [n] and y[n] are periodic with period N.

$$z[n_c] = x[n] \circledast y[n] \underset{\text{INVERSE DFT}}{\overset{\text{DFT}}{\rightleftarrows}} X[w] \bullet Y[w]$$

which means:

"The multiplication of the two DFTS of 2 periodic sequences in the frequency domain is equivalent to their circular convolution in the time domain"

After achieving the **circular convolution** (for reasons of efficiency by transforming to the **frequency domain**, performing the multiplication of the DFTs in the **frequency domain** and then inverse transform to get $z[n_c]$ in the time domain) we need to extract $z[n]$ from $z[n_c]$.

How do we do this?

Exercises:

Given:

	-3	-2	-1	0	1	2	3	4	5
x[n] =		0	2	3	6	7	8		
y[n] =		5	4	9					

1. Find: $z[n] = x[n] * y[n]$
2. Compute circular $z[n_c]$ from linear $z[n]$ without the circular picture.
3. Find $z[n_c] = x[n] \circledast y[n]$, for $i_c \geq 0$ with the circular picture.

   Draw the circles, rotate +ve direction step by step and compute $z[i_c]$ for $i_c \geq 0$.

   Compare the values got in $z[n_c]$ in 2 above.
4. Find $z[n_c] = x[n] \circledast y[n]$ for $i_c < 0$.

   Draw the circles, rotate –ve direction step by step and compute $z[i_c]$ for $i_c < 0$.

   Compare the values got in $z[n_c]$ in 2 above.
5. Verify by example or counter example if $\circledast$ is **associative**.

   Is: $(a[n] \circledast b[n]) \circledast c[n] = a[n] \circledast (b[n] \circledast c[n])$?

QUIZ

1. Assume $\omega_0 = 2\pi/T_0$ where T_0 is the fundamental period. What is:

- $\sin(\omega_0 t) \circledast \cos(\omega_0 t)$ over one period T_0.
- $\sin(n\omega_0 t) \circledast \cos(m\,\omega_0 t)$ over one period T_0.
  When n = m? and when $n \neq m$?
- $\sin(n\omega_0 t) \circledast \sin(m\omega_0 t)$ over one period T_0.
  When n = m? and when $n \neq m$?
- $\cos(n\omega_0 t) \circledast \cos(m\omega_0 t)$ over one period T_0.
  When n = m? and when $n \neq m$?

2. When you mix two colors like blue and yellow to get green, is the operation:

blue + yellow, blue * yellow or blue · yellow?

3. What is white*black?

4. The **vector cross-product** formula is given as:

	i	j	k
$\vec{A}$:	a_1	a_2	a_3
$\vec{B}$:	b_1	b_2	b_3

$$\vec{A}\times\vec{B} = (a_2b_3-a_3b_2)\mathbf{i} +(a_3b_1-a_1b_3)\mathbf{j} +(a_1b_2-a_2b_1)\mathbf{k}$$

Work out $\vec{A} * \vec{B}$, the effect of EACH component on EACH component. Note that $i * j = k$, $j * k = i$, $k * i = j$ and $i * i = 0$, $j * j = 0$, $k * k = 0$

What does **vector cross-product** look like?

Does it have a physical interpretation?

Given $\vec{E}$ = electrical field and $\vec{B}$ = magnetic field what is $\vec{E} \times \vec{B}$ or rather $\vec{E} * \vec{B}$

Applying the right hand thumb rule?

Can you think of an algorithm to find the **cross-product** of two vectors with n>3 dimensions?

5. With CONVOLUTION in mind, what is **:**

CROSS CORRELATION**?**

AUTO CORRELATION**?**

Which one can be used in distance measuring**?**

What is echo-location**?**

6. DECONVOLUTION

Given c[n] = a[n]*b[n] (all finite pure sequences). If you know c[n] and a[n], how will you find b[n]**?**

7. Given $f_1(t) = t/1! - t^3/3! + t^5/5! - t^7/7! + \ldots\ldots$ t in $[0, \infty]$

$f_2(t) = 1 + (-at)/1! + (-a\,t)^2/2! + (-at)^3/3! + \ldots\ldots$ t in $[0, \infty]$

How will you find $f_1(t) * f_2(t)$**?**

What will the picture look like**?**

8. Which is more accurate: the CONVOLUTION SUM or the CONVOLUTION INTEGERAL**?**

Given that you have a computer to compute the CONVOLUTION SUM and also, you know the expression for the CONVOLUTION INTEGERAL, which one is easier to work with**?** Why**?**

9. In Calculus we learn: $\int_a^b + \int_b^c = \int_a^c$.

Can we say: $\underset{[a,\,b]}{\{f_A(t) * f_B(t)\}} + \underset{[b,\,c]}{\{f_A(t) * f_B(t)\}} = \underset{[a,\,c]}{f_A(t) * f_B(t)}$ **?**

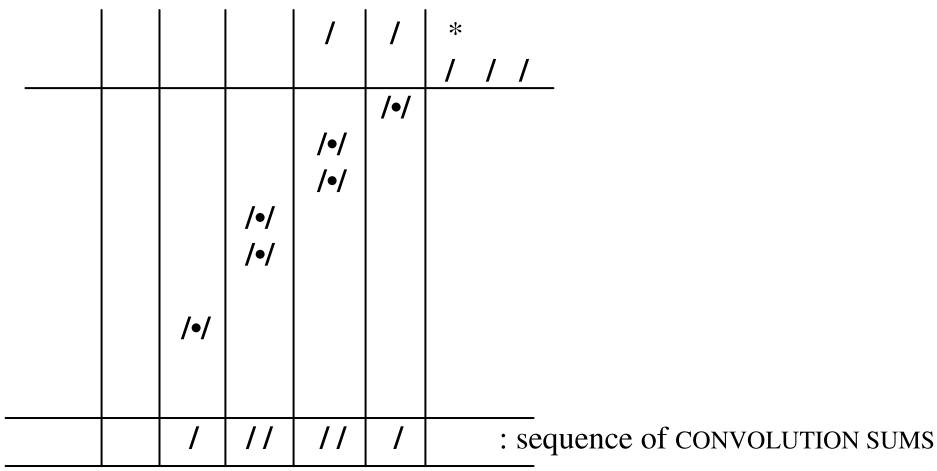
10. Given two periodic functions $f_A(t)$ and $f_B(t)$ with same period T_o ,when and how do you decide how many equally spaced points to take to perform $f_A[t_n] * f_B[t_n]$ **?**

LTI SYSTEMS

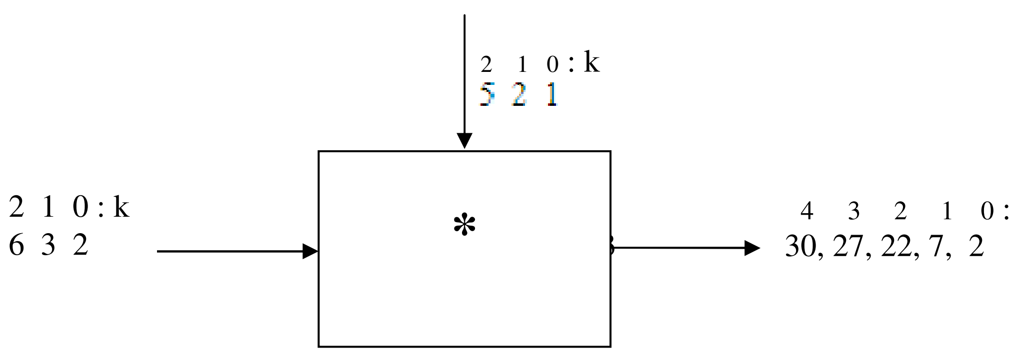
Now that we have learned a new operation, what can we do with it?

Obviously our first application is to replace multiplication denoted by • with convolution denoted by *. It is not necessary that we first have to convert ***analog*** operands (like representing the number of objects in a set in an abstract way by strokes) into decimal ***digital*** form before we perform the ***convolution***. Let us see a simple example

		analog		***digital***
a	=	/ /	≡	2
b	=	/ / /	≡	3

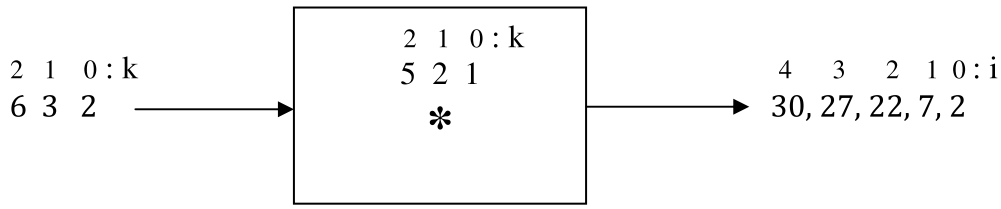


So now we can add this operation to a calculator. In fact we can program both versions, ***linear convolution*** denoted by * and ***circular convolution*** denoted by ⊛ into a calculator



CALCULATOR

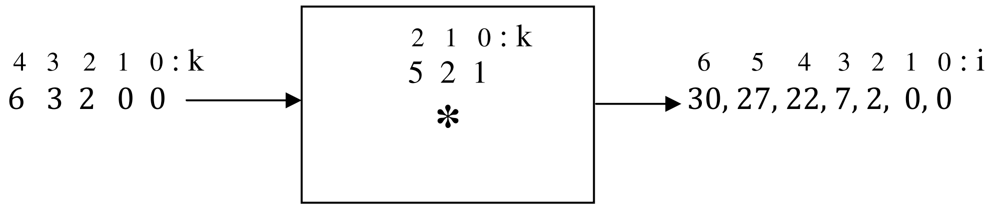
We can do something better. We can create a system by programming it with the ***convolution*** operation and an in-built operand. So now we have



CONVOLUTION SYSTEM

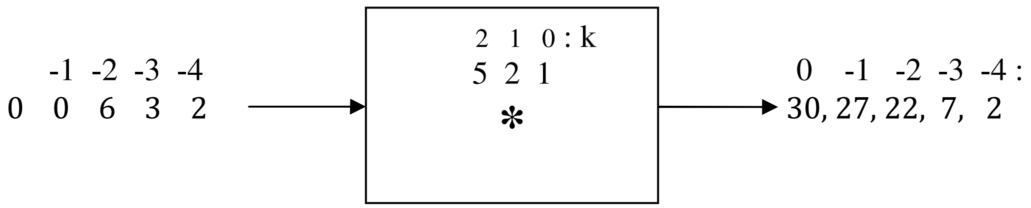
SHIFT INVARIANT

We already know this is SHIFT INVARIANT meaning



SHIFT INVARIANT SYSTEM

and

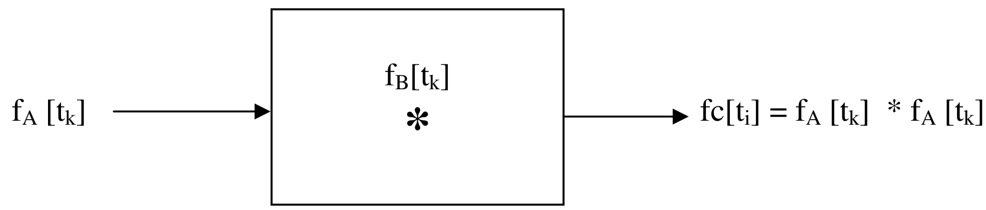


SHIFT INVARIANT SYSTEM

So we can have a SHIFT INVARIANT SYSTEM that can take in a SHIFTED ***pure sequence*** as input and operate on it to give as output an equally SHIFTED ***pure sequence.***

TIME INVARIANT

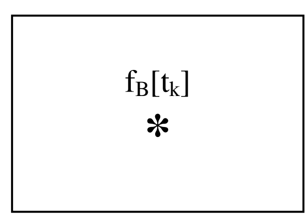
Now we know how to get ***discrete functions*** $f_A[t_k]$ and $f_B[t_k]$ from their ***continuous*** counterparts $f_A(t)$ and $f_B(t)$. What we do with ***pure sequences***, we can also do with ***discrete functions.***



SHIFT INVARIANT SYSTEM

What will happen if we SHIFT $f_A[t_k]$ along with the time axis: a little to the left (***advance***) or a little to the right (***delay***)?

The output $f_C[t_i]$ will also be SHIFTED by an equal amount. Because this shifting is taking place on the time axis, we call it TIME INVARIANT. Now we may call our system

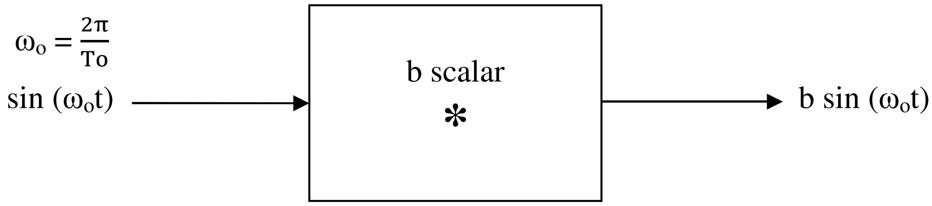


TIME INVARIANT SYSTEM

Usually the operand in the LTI system is denoted by h[n] or h(t). But this notation does not convey sufficient information. If h[n] is a ***pure sequence*** we need to know the index range over which it is defined. If h[n] is a ***discrete function*** $h[t_k]$ got from h(t), then we need to know the index range and T_s as well. And in the case of h(t) we need to know the interval over which it is defined.

FREQUENCY FIDELITY

This TIME INVARIANT property is more important than it seems. Let us see a few examples

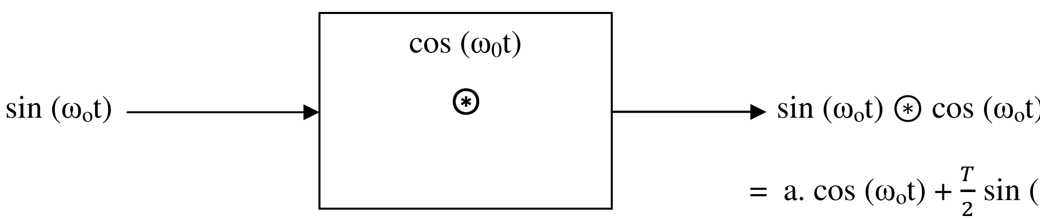


TIME INVARIANT SYSTEM

The input frequency = output frequency = ω_0 because the input period T_0 = output period T_0

due to the TIME INVARIANT property.

Let us take another look at circular or periodic convolution. We shall depict the ***frequency fidelity*** aspect of it.



$= \text{a.} \cos(\omega_o t) + \frac{T}{2}\sin(\omega_o t)$

$= \text{A.} \cos(\omega_o t + \text{ø})$

Again, the input frequency = output frequency = ω_0

We saw an example of this earlier in ***circular convolution.***

This is the single most important concept and calculation in DSP.

$\sin(\omega_0 t) \circledast \cos(\omega_0 t) = \int_0^T \cos[\omega_0 (t-\tau)].\sin(\omega_0 \tau)\, d\tau$ where T is the period.

Now: $\cos[\omega_0 (t-\tau)] = \cos(\omega_0 t).\cos(\omega_0 \tau) + \sin(\omega_0 t).\sin(\omega_0 \tau)$

$$\sin(\omega_0 t) \circledast \cos(\omega_0 t) = \int_0^T \sin(\omega_0 \tau).\{\cos(\omega_0 t).\cos(\omega_0 \tau) + \sin(\omega_0 t).\sin(\omega_0 \tau)\}\, .d\tau$$

$$= \cos(\omega_0 t) \int_T \sin(\omega_0 \tau).\cos(\omega_0 \tau).d\tau + \sin(\omega_0 t) \int_T \sin(\omega_0 \tau).\sin(\omega_0 \tau).d\tau$$

$$= \cos(\omega_0 t)\, .\left[\tfrac{1}{2\omega_0} \sin^2(\omega_0 \tau)\right]_0^T + \tfrac{T}{2} \sin(\omega_0 t)$$

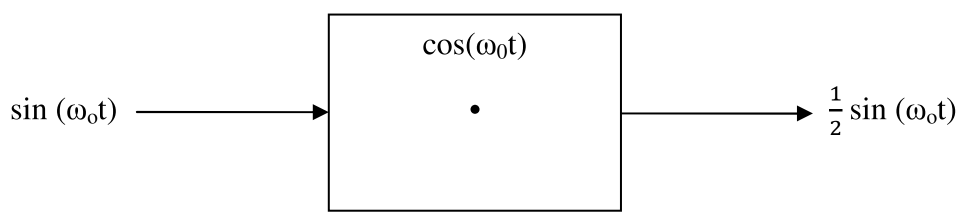
$$= a.\cos(\omega_0 t) + \tfrac{T}{2} \sin(\omega_0 t) \quad \text{where } a = \frac{1}{2} \sin^2(\omega_0 T)$$

$$= A.\cos(\omega_0 t + \emptyset)$$

So: $\sin(\omega_0 t) \circledast \cos(\omega_0 t) = A.\cos(\omega_0 t + \emptyset)$

Notice that the period of the ***periodic convolution*** is again T. This is how we get the frequency fidelity: the input frequency = output frequency = ω_0

So we have TIME INVARIANT SYSTEMS that allow us to work with sinusoidals. In contrast let us look at a NON-LINEAR SYSTEM.



AMPLITUDE MODULATION

Notice the change in frequency. The input frequency ≠ output frequency.

Further Reading: Now you may browse ***periodic convolution***

http://ocw.mit.edu

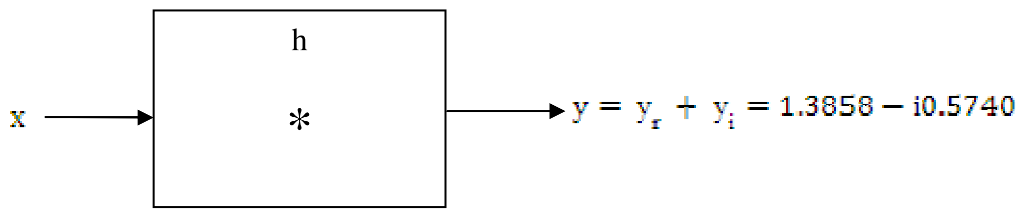
Signals and Systems, Fall 2003, Lect 6 (23 Sept 2003) 19 pages.

COMPLEX REPRESENTATION of an LTI System

Let us now see an example using complex numbers. This example is taken straight out of DSP by Steven SMITH page 562, but explained differently.

Let $x = x_r + x_i = 2.1213 + i2.1213$

$h = h_r + h_i = 0.1913 - i0.4619$



LTI SYSTEM

We may write x, h and y using sinusoidals as**:**

rectangular form **polar form**

$$x = 2.1213\cos(\omega t) + 2.1213\sin(\omega t) = 3\cos\left(\omega t + \frac{\Pi}{4}\right) \qquad = 3e^{-i\frac{\Pi}{4}}$$

$$h = 0.1913\cos(\omega t) - 0.4619\sin(\omega t) = 0.5\cos\left(\omega t - \frac{3\Pi}{8}\right) \qquad = 0.5e^{+i\frac{3\Pi}{8}}$$

$$y = 1.3858\cos(\omega t) - 20.5740\sin(\omega t) = 1.5\cos\left(\omega t - \frac{\Pi}{8}\right) \qquad = 1.5e^{+i\frac{\Pi}{8}}$$

Note how the complex number i = √-1 allows us to keep the cos and sin components separate even though it goes through an LTI System.

The output = (input * system function) in ***rectangular form.***

y = x * h in ***rectangular form.***

We could have achieved the same result with ***multiplication*** but using the ***polar form.***

$$y_{polar} = x_{polar} \cdot h_{polar}$$

$$1.5e^{i\frac{\Pi}{8}} = 3e^{-i\frac{\Pi}{4}} \,.\, 0.5e^{i\frac{3\Pi}{8}}$$

Note how the * operation was replaced by ▪ (***multiplication***). This is because the polar form (complex exponential form) is actually in frequency format as opposed to spatial format. Later we shall learn a very important theorem known as the CONVOLUTION THEOREM:

CONVOLUTION in ***time domain*** = MULTIPLICATION in ***frequency domain***.

And its counterpart:

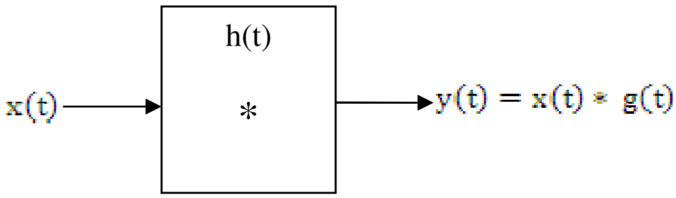
MULTIPLICATION in ***time domain*** = CONVOLUTION in ***frequency domain.***

These theorems allow us to go back and forth between Algebra (***discrete operands***) and Analysis (***continuous operands***). Obviously the Algebra is easy to work with.

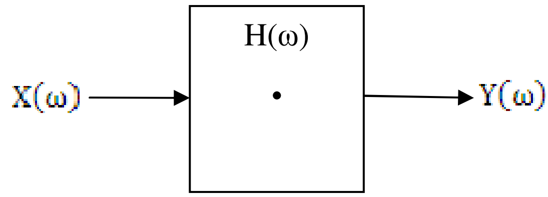
DECONVOLUTION

Recall what we said about DECONVOLUTION: This is what the picture looks like

$h(t)$ = system function | $H(\omega)$ = frequency response



LTI SYSTEM



LTI SYSTEM

Given $x(t)$ and $y(t) = x(t) * h(t)$, how can we DECONVOLVE $y(t)$ to find $h(t)$?

$$y(t) = x(t) * h(t) \xrightarrow{\text{TRANSFORM}} Y(\omega) = X(\omega)\,.\,H(\omega)$$

$$h(t) \xleftarrow[\text{TRANSFORM}]{\text{INVERSE}} H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

We ***transform*** $x(t) \longrightarrow X(\omega)$ and $y(t) \longrightarrow Y(\omega)$. We perform the ***division*** $Y(\omega)/X(\omega)$ to find $H(\omega)$. Then we take the ***inverse transform*** of $H(\omega)$ to find $h(t)$.

This is known as DECONVOLUTION.

LINEAR

Let us now look at two very important properties of CONVOLUTION

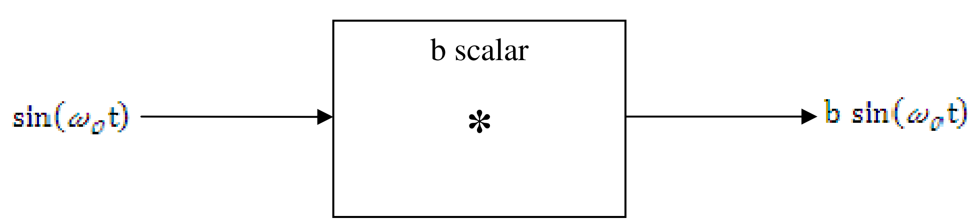
Homogeneity

The word ***homogeneity*** deals with form or shape. In optics, magnification is the scaling operation that does not change the form or shape of the object. In mathematics we may take any curve and multiply it by a constant and still get a curve of the same shape. A straight line function multiplied by a constant is still a straight line function. A quadratic function (degree 2) multiplied by a constant is still a quadratic function. Multiplying a sinusoidal by a constant does not change its frequency. It only changes the amplitude. The property of multiplying by a constant without changing the shape of the curve is known as ***homogeneity.***

The relation V = i.R tells us how the voltage across the resistor varies with the current. The voltage is the scaled version of the current, i.e. it is multiplied by the constant resistance R. A system that takes the current as input and gives the voltage as output is ***homogeneous.***

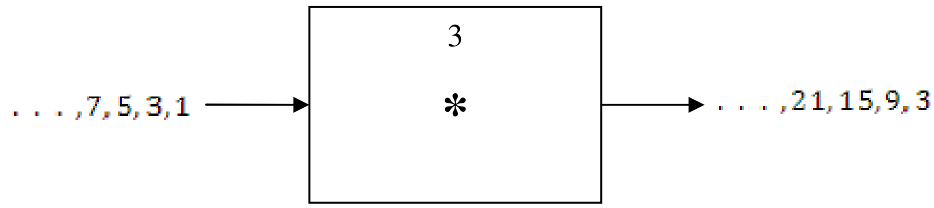
On the other the relation $P = i^2.R$ tells us how the power varies across the resistor. A system that takes the current as input and gives the power as output is ***non-homogeneous*** or ***non-linear.***

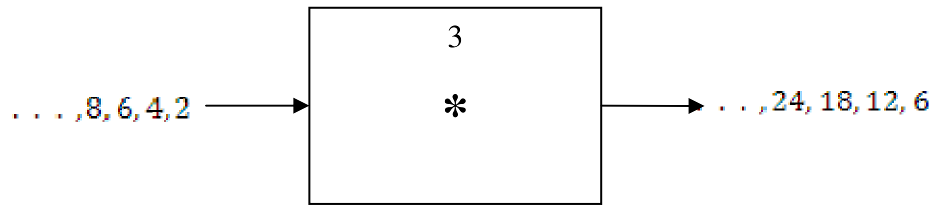
To engineers, a curve or a function is a signal. So if we input a signal to an LTI system the output will be a scaled version of the input.



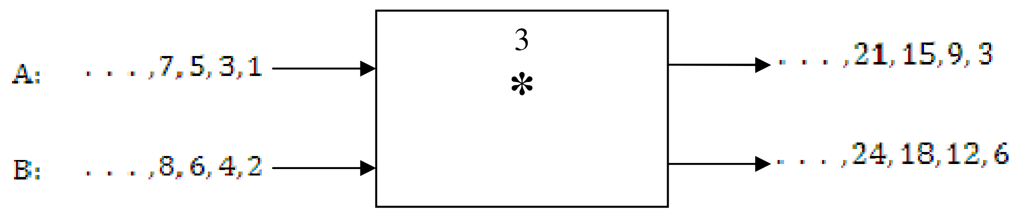
Additivity

An interesting aspect of ***linearity*** is illustrated below:



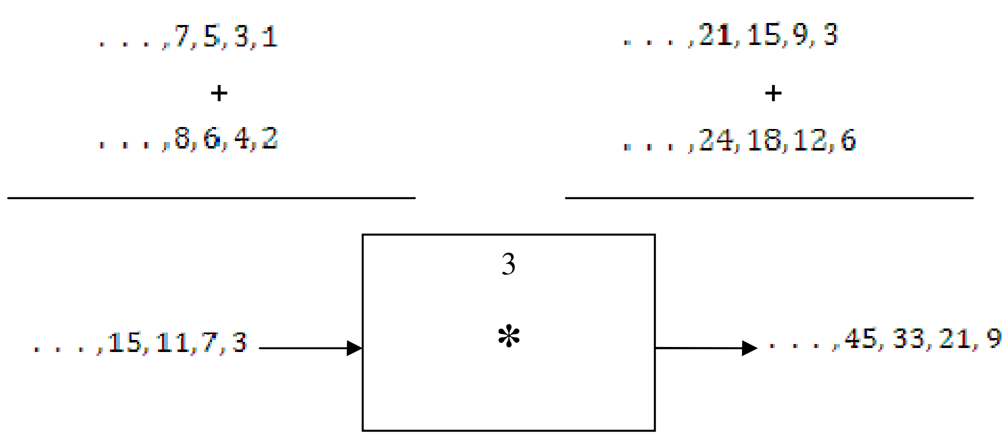


If we now input both sequences simultaneously, what would you expect?



This is exactly what happens. If two persons A and B were calling out the two sequences in sync into the microphone of a tape-recorder or a telephone, on playback or at the other end of the line we will hear the individual voices of both persons calling out the two sequences. The LTI system does not mix up the two voices to give an unintelligible output.

If we now add the two sequences and then feed it as input into the LTI system, we get



This is known as the ***additivity*** property. This is with ***pure sequences.***

We observe the same properties if our operands are sinusoidals. They may be:

- Single continuous sinusoidals.
- Single discrete sinusoidals

continuous periodic functions = addition of several ***continuous*** sinusoidals that are harmonics.

HARMONICS

Harmonics means the frequencies of the component sinusoidals (of the periodic function) are rational multiples of each other.

$f_A(t) = \{\sin(2\omega_0 t) + \sin(\sqrt{2}\omega_0 t)\}$ is **not** periodic because 2 is **not** a rational multiple of $\sqrt{2}$.

The proof is simple. If $f_A(t)$ were periodic, the for some positive integers m and n:

$$m \text{ cycles of } 2\omega_0 = n \text{ cycles of } \sqrt{2}\omega_0.$$

$$2m\omega_0 = \sqrt{2}n\omega_0.$$

$$\sqrt{2} = 2m/n, \text{ which is rational}$$

$f_A(t) = \{\sin(2\omega_0 t) + \sin(3\omega_0 t)\}$ is periodic because 2 is a rational multiple of 3.

$f_A(t) = \{\sin(\sqrt{2}\omega_0 t) + \sin(\sqrt{8}\omega_0 t)\}$ is periodic because $\sqrt{2}$ is a rational multiple of $\sqrt{8}$

We may say the same thing about ***discrete*** periodic functions.

discrete periodic functions = addition of several ***discrete*** sinusoidals provided we use the same ***equal spacing*** and same instants on the time axis for the component sinusoidals.

Let us elaborate a little more on this important point.
Let $f_A(t) = \sin(2\omega_0 t)$ and $f_B(t) = \sin(\sqrt{2}\,\omega_0 t)$

	continuous		***discrete***
periodic:	$f_A(t)$	$\longrightarrow$	$f_A[t_j]$ using equal spacing T_A
periodic:	$f_B(t)$	$\longrightarrow$	$f_B[t_k]$ using equal spacing T_B

Assume $T_A = T_B = T_s$ and $j.T_s = k.T_s$ (meaning the same instants on the time axis).

$$f_C[t_i] = f_A[t_j] + f_B[t_k] \text{ where } i = j = k.$$

We know that $f_C(t) = f_A(t) + f_B(t)$ is not periodic. However, $f_C[t_i] + f_A[t_j] + f_B[t_k]$ is periodic.

Now assume that $T_A = T_B$ but the instants on the time axis are not aligned. That is to say for $j = k$: $j \bullet T_s \neq k \bullet T_s$. $f_C[t_i]$ is still periodic. Even with $T_A \neq T_B$ $f_C[t_i]$ is still periodic.
Note that when $j \bullet T_A = k \bullet T_B$ $f_C[t_i]$ is periodic.
So when going from continuous to discrete we have to be careful. We shall see more of this later.

This subtle problem arises in the DFT and its fast version the FFT. This is the underlying problem referred to in page 178 of DSP by Steven W. Smith. The underlying mathematics is simple and is beautifully explained in chapter 2 section 1 Diophantine Equations, example 2.1.7 (page 78) in a gem of a book by the great Fredrick W. STEVENSON (University of Arizona) "EXPLORING THE REAL NUMBERS" © Prentice Hall 2000.

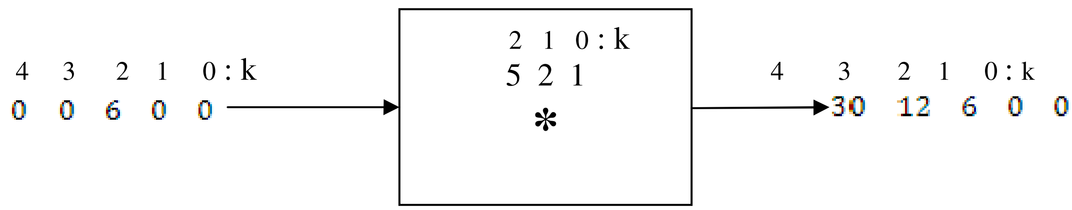
LINEAR DECOMPOSITION

Let us take a close look at the ***linear*** property in LTI a system linear implies ***homogeneity*** and ***additivity*. *Homogeneity*** means multiplication of an operand by a scalar – The form or shape or basic information in the operand is preserved, something akin to a magnifying glass. The object remains the same.

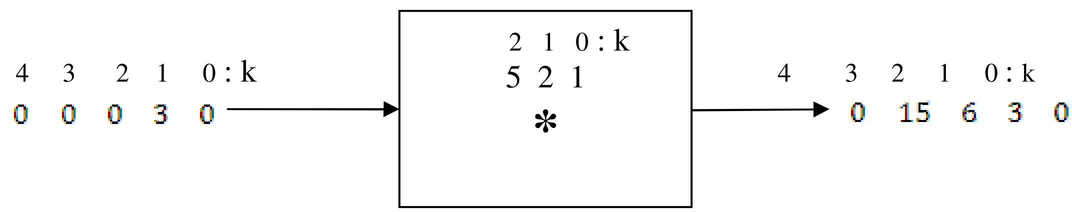
Additivity means an operand can be made up by the simple addition of several more simple or basic operands. For example $\underset{10^2}{6}\ \underset{10^1}{3}\ \underset{10^0}{2}$ is got by adding up the simple operands below:

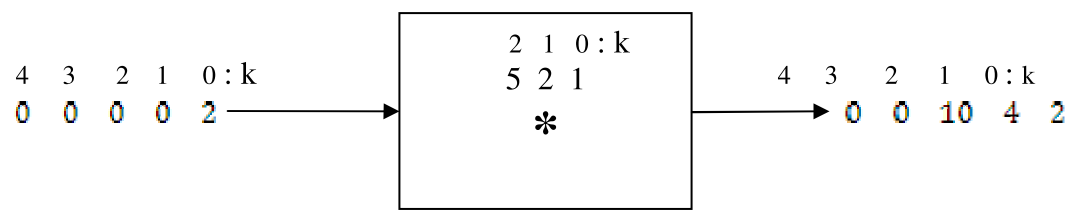
	$\ldots\ 10^4$	10^3	10^2	10^1	10^0	10^{-1}	$10^{-2}\ \ldots$
$\ldots$	0	0	6	0	0	0	0 $\cdots$
$+\ \ldots$	0	0	0	3	0	0	0 $\cdots$
$+\ \ldots$	0	0	0	0	2	0	0 $\cdots$

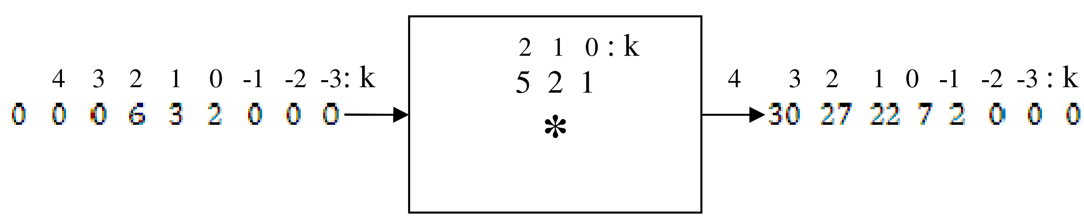
So this tells us that we can ***decompose*** an operand into its basic components, put it through an LTI system, one basic component at a time, and then add up the results. So:



LTI SYSTEM







UNIT IMPULSE FUNCTION

There is a very special ***pure sequence*** denoted by δ[n]. Its continuous counterpart is δ(t), the ***unit impulse function***. We shall see more of this later: For now:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{elsewhere} \end{cases}$$

It looks like:

. . . -3	-2	-1	0	1	2	3 . . .
0	0	0	1	0	0	0

$$\delta\,[n-k] = \begin{cases} 1 & \text{when } n = k \\ 0 & \text{elsewhere} \end{cases}$$

Example: k = 3(***delay***)**:** δ [n-3] =

. . . -3	-2	-1	0	1	2	3	4	5 . . .
0	0	0	0	0	0	1	0	0

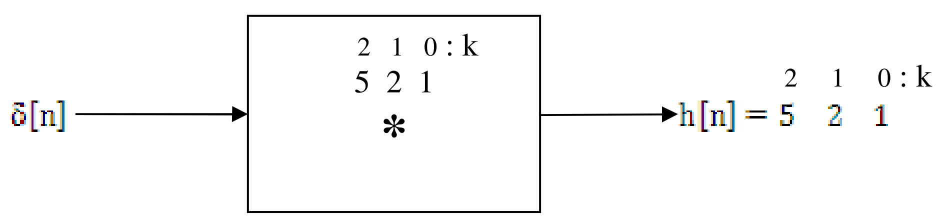
Example: k = -2 (***advance***)**:** δ [n-(-2)] =

. . . -3	-2	-1	0	1	2	3	4	5 . . .
0	1	0	0	0	.	.	.	

Note that a[n] * δ[n] = a[n]

$$\begin{array}{ccccccccc} 2 & 1 & 0:k & & & & 2 & 1 & 0:k \\ 6 & 3 & 2 & * & \delta[n] & = & 6 & 3 & 2 \end{array}$$

So we can use δ[n] to determine the operand in an LTI System.



LTI SYSTEM

Also we can use δ[n] to SHIFT a sequence

$$\begin{array}{ccc} 2 & 1 & 0:k \\ 6 & 3 & 2 \end{array} * \delta[n-1] = \begin{array}{cccc} 3 & 2 & 1 & 0:k \\ 6 & 3 & 2 & 0 \end{array} \quad \textit{\textbf{(delay)}}$$

$$\begin{array}{ccc} 2 & 1 & 0:k \\ 6 & 3 & 2 \end{array} * \delta[n-(-1)] = \begin{array}{ccc} 1 & 0 & -1:k \\ 6 & 3 & 2 \end{array} \quad \textit{\textbf{(advance)}}$$

In fact we can write 632 as a sum of SHIFTED and SCALED unit impulses**:**

$$\begin{array}{ccc} 2 & 1 & 0:k \\ 6 & 3 & 2 \end{array} = 6 \cdot \delta[n-2] + 3 \cdot \delta[n-1] + 2 \cdot \delta[n]$$

All three properties: ***time invariance, homogeneity*** and ***additivity*** gives us what we call an LTI System

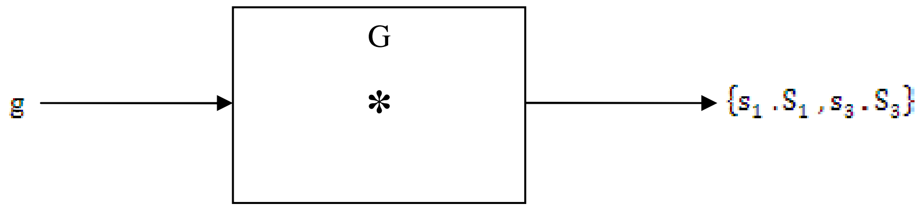
What can we do with LTI Systems?

FILTERING

Let us now take a look at the rudiments of filtering.

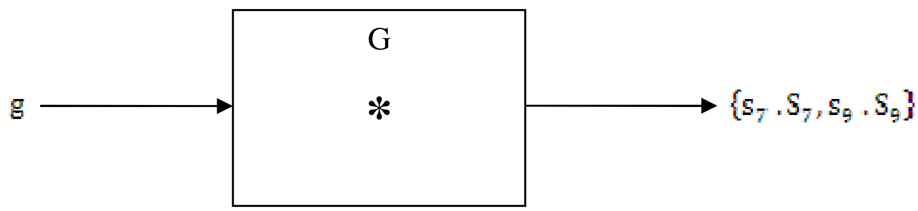
Let guitar, g = $\{s_1, s_3, s_5, s_7, s_9\}$

$$G = \{S_1, S_2, S_3\}$$



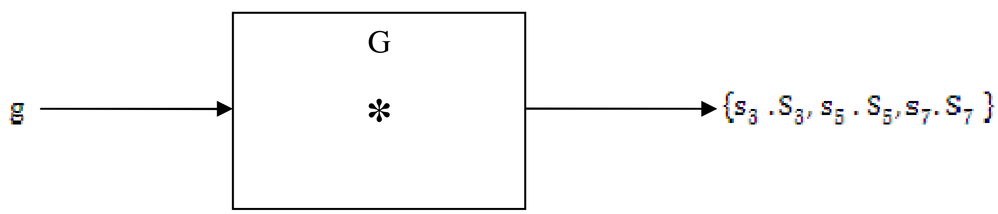
LOW PASS FILTER

$$G = \{S_7, S_8, S_9\}$$



HIGH PASS FILTER

$$G = \{S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$$



BAND PASS FILTER

In the module on periodic convolution we shall see a more mathematical description of this kind of filtering where the guitar strings are modeled by sinusoidals.

You may now browse “Mathematics of the DFT” with Audio applications

by Julius O Smith III

Center for Computer Research into Music and Acoustics

Stanford University

See the six examples on convolution.

We give you the link to example 1.

http://ccrma.stanford.edu/vjos/hdft/Convolution_Example1_smoothing.html

You should be able to immediately recognize it.

LINEAR CONSTANT COEFFICIENT DIFFERENCE EQUATIONS

Now where have we seen these properties of ***homogeneity*** and ***additivity*** before?

LINEAR ALGEBRA.

What is another class of problems that we can solve using LINEAR ALGEBRA?

Linear Constant Coefficient Difference Equations (the discrete counterpart of Linear Constant Coefficient Differential Equations)

Further reading: LINEAR ALGEBRA with Applications by the great GILBERT STRANG (Massachusetts Institute of Technology)

You may treat yourself to a series of majestic lectures by STRANG freely available in the MIT opencourseware program.

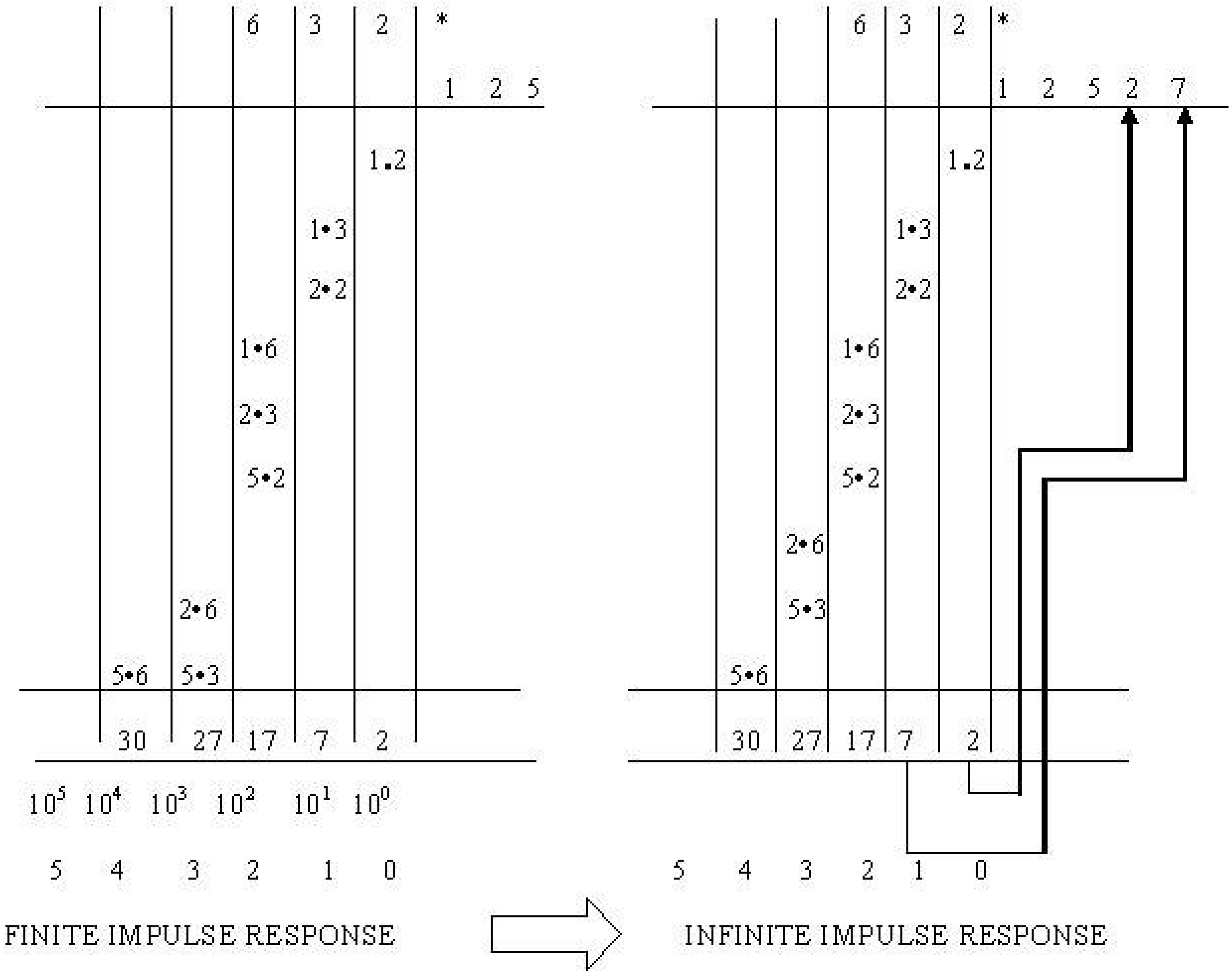
Chapter 5 deals with EIGEN VALUES.

After reading sections 1 and 2 of chapter 5 you may then proceed to read the chapter on the Laplace transform in DSP by Steven SMITH.

INFINITE IMPULSE RESPONSE

So far we have been dealing with finite length/duration operands. In any realizable system we are forced to deal with operands that are finite in length/duration and bounded in values. Let us see how we can make one of the operands infinite using feedback.

Let us look at 632 * 521



After the first clock pulse the feedback starts. The feedback sequence is 2, 7, 17 . . .

If the operand 521 in the LTI System is with feedback as shown above, it becomes infinite in length and we have an infinite sequence of impulse responses. This is known as INFINITE IMPULSE RESPONSE (IIR).

Further reading:

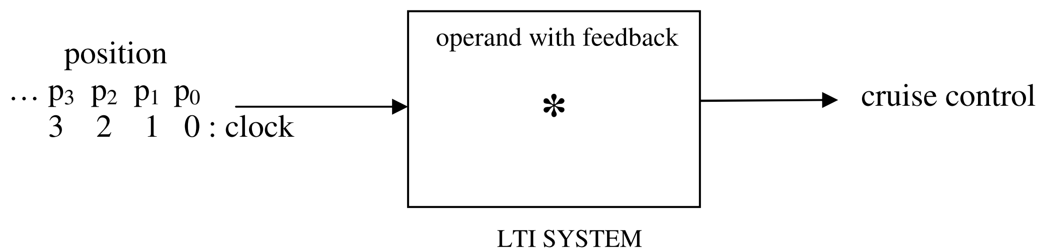
"FUNDAMENTALS of DSP " by LONNIE C. LUDEMAN©1986

Chapter1**:** Introduction (17 pages)

CONTROL SYSTEMS

One typical situation where this type of feedback is required is in Control Systems. Let us look at a simple example: speed control of a vehicle moving along a straight line in a fixed direction.

In order to control the speed of the vehicle we need 3 pieces of information: its current position, speed and acceleration. Suppose the only input we have is the position, say at every clock pulse.



With some simple feedback, we can find speeds: $v_0 = p_1 - p_0 / t_1 - t_0$ and $v_1 = p_2 - p_1 / t_2 - t_1$.

With a little more manipulation we can find acceleration $a_0 = v_1 - v_0 / t_1 - t_0$.

Then the LTI System (with feedback in the built operand) can output the desired cruise control command/data.

Further reading

"SIGNALS and SYSTEMS" by Alan V. OPPENHEIM and Alan S. WILLSKY with Ian T. YOUNG ©Prentice Hall 1987

Chapter11: Linear Feedback Systems (sections 1 & 2, some 16 pages)

What is the mathematics required to handle all this?

Higher Secondary Partial Fractions and Power Series Expansion with some Complex Algebra.

Well, and differentiation in the frequency domain.

Now you may read the chapter on the Z-transform in DSP by Steven W. SMITH

Reference:

"A Preview of Convolutions & Transforms" and Lecture Notes on DSP by S. Vadakkan.

Link to Calculus book:

http://www.nitte.ac.in/nmamit/articles.php?linkId=131&parentId=20&mainId=20&facId=131