A LITTLE BIT OF CALCULUS

by

S. VADAKKAN

For we are in God's hands . . .
our understanding and technical knowledge.

Wisdom VII : 16

I have held many things
in my hands, and lost them all.
But whatever I have placed
in God's hands I still possess.

Martin Luther King Jr.
(1929 - 1968)

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of Arabia

Preface

This is not a Text Book, Primer or Guide Book on Calculus. It is just an introduction to the fundamental concepts. Particular words and symbols are used to express these concepts. These concepts and words take time to sink in. It is hoped that in studying this booklet the student will become familiar with the concepts, terminology, notation and the kind of calculations that can be done: what to look for and what to expect.

Generally speaking, in most of the Sciences there are usually two main branches. Biology is divided into Botany (plant life) and Zoology (animal life). Chemistry is divided into Organic (carbon compounds) and Inorganic (non-carbon compounds). In Light we have the *discrete* particle photon and the *continuous* wave. Likewise, Mathematics is divided into two main branches: Algebra and Analysis.

In Algebra we deal with *discrete* operands, be they numbers like 1,2,3, . . . or symbols for numbers like x, y, z, ... The operands take on *discrete* values. The 7 operations are $+$ and its inverse $-$, $\times$ and its in its inverse $\div$, exponentiation $y = x^n$ and its two inverses $\sqrt[n]{y} = x$ and $\log_x y = n$. Usually the set of different values the operands may take on is finite, or at least the set of values can be counted or enumerated.

In Analysis we deal with *continuous* expressions or operands or functions. Take for example a bouncing ball: each position it bounces is *discrete* and can be counted. The ball may bounce indefinitely. On the other hand a ball rolling in a straight line has infinitely many positions in a *continuous* manner. We cannot even begin to count all the different positions. So rather than *discrete* identification of position, we have a *continuous* expression or function to describe its position or motion over time.

The fundamental concepts in Calculus are INFINITESIMAL and LIMIT which are used to develop the concepts of INSTANT, INSTANTANEOUS, CONTINUITY and DIFFERENTIABLITY. Once these concepts are in place we can talk of functions that are WELL BEHAVED: SINGLE VALUED, CONTINUOUS and DIFFERENTIABLE. We can then do two very beautiful calculations or operations:

1. Given a SINGLE VALUED, CONTINUOUS and DIFFERENTIABLE function that expresses CHANGE we can DIFFERENTIATE the function and get the new function that expresses the INSTANTANEOUS RATE OF CHANGE.

2. Given the function that expresses the INSTANTANEOUS RATE OF CHANGE, we can INTEGRATE the function and get the new function that expresses CHANGE.

Working with polynomials is easy since they are CONTINUOUS everywhere. The fundamental concepts and calculations (Differentiation and Integration) can be taught with ease and clarity. Getting more information about the function (increasing, decreasing, maximum, minimum, inflexion) from its Higher Order Derivatives can also be shown.

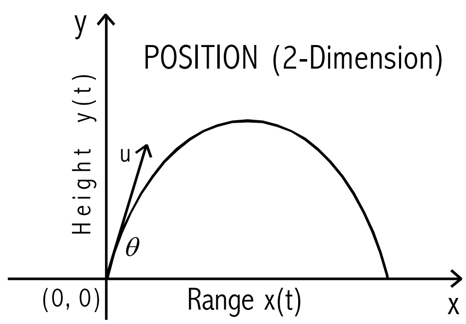
I have deliberately chosen very simple examples to illustrate the concepts. I have avoided all the rigorous details under which calculations are done. The theory of Real Analysis and Calculus deals with this. Rigour does not necessarily mean clarity and ease of understanding.

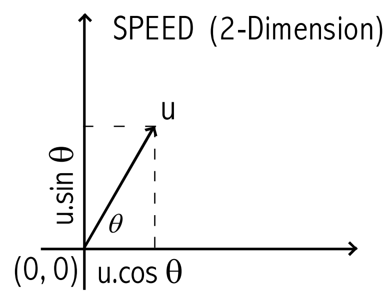
At the Class XI level this should be the approach. This approach will provide the confidence to deal with more difficult functions and the motivation to study the theory.

The notes are intended as thought evoking pointers for students who wish to study "A Littile More Calculus" by the author.

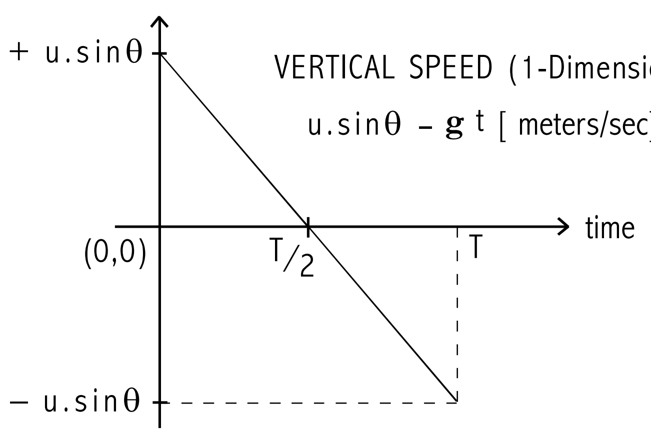
Throughout the book we make use of one main example, the projection of a ball, to illustrate all the concepts and demonstrate all the calculations. Non-science students may not be familiar with the projectile equation. So a brief explanation follows.

An object may be projected into space with an initial velocity u and an angle of projection θ . Let T be the time the object takes to go up and come down.

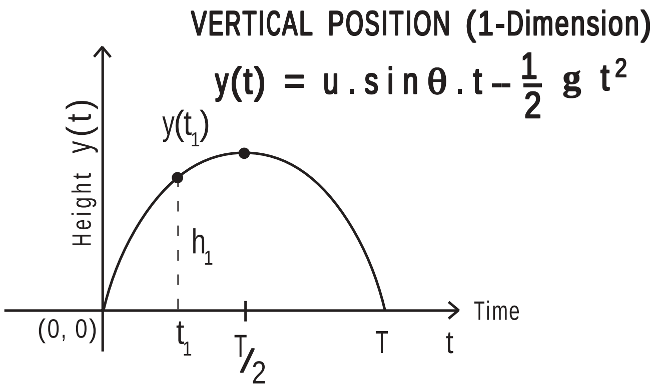




The initial speed u has two components: a *vertical* component $u.\sin\theta$ and a *horizontal* component $u.\cos\theta$. We know that gravity $\mathbf{g}$ acts in the downward or negative direction. So, with the appropriate *units of measure* approach in mind, we may see that the *vertical speed* must be $u.\sin\theta - \mathbf{g}\,t$ [meters/sec].



Since DISTANCE = SPEED X TIME, the ***vertical position*** or ***height*** must be of the form $A_1.(u.\sin\theta).t - A_2\, g\, t^2$ [meters]. The coefficients A_1 and A_2 may be determined **Analytically** (as we shall see in ***integration***) or by physical experiments. It turns out that $A_1 = 1$ and $A_2 = {}^1/_2$. So the function or expression that describes the ***vertical position*** or ***height*** is $u.\sin\theta.t - \frac{1}{2}\, g\, t^2$.



Even if the object is projected vertically up $(\theta = {}^{\pi}/_2)$, this is what the graph of the VERTICAL POSITION will look like over the time interval [0, T].

In $y(t) = u.\sin\theta.t - \frac{1}{2}\, g\, t^2$ **when** $\theta = {}^{\pi}/_2$

we get $y(t) = u.t - \frac{1}{2}\, g\, t^2$, again a parabolic function.

The ***differentiation*** operation allows us to find the VERTICAL SPEED from the VERTICAL POSITION. And the ***integration*** operation allows us to find the CHANGE in the VERTICAL POSITION from the VERTICAL SPEED.

The book is divided into 4 parts. On initial reading the student may skip part 3.

CONTENTS

Part 3 : ANALYTICAL GEOMETRY

Part 4 : INTEGRATION

1. INTRODUCTION

Microbiologists work with small things called cells that form the basic unit of matter they are dealing with. Physicists and chemists work with molecules representative of a compound or atoms representative of an element. Very often scientists work with entities smaller than cells or atoms. However, no matter how small it is, it is always something tangible.

While molecules, atoms and particles are DISCRETE objects, in Calculus we work with CONTINUOUS entities like the functions that describe the motion of an object - its position, speed, acceleration and so on. We call these CONTINUOUS operands.

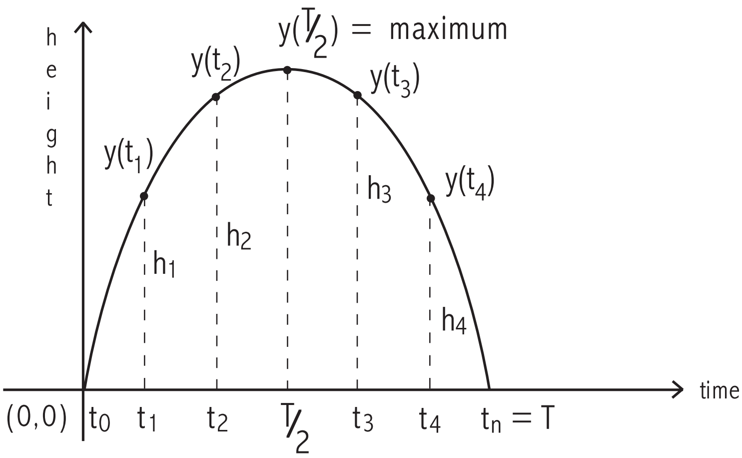
Mathematicians like to study the behavior of a function at a ***point*** or ***instant*** and over a very small interval around a ***point*** or ***instant***. The interval is so small that we have the special word ***infinitesimal*** to describe it. Understanding the behavior of a function over an ***infinitesimal*** interval will help us to:

1. Formulate an expression to describe its behavior at a ***point*** or ***instant***.

2. Use this expression to extract more information about the behavior of the function, i.e. is the function increasing or decreasing, at a maximum or minimum, or changing direction.

Let us see an example of a CONTINUOUS operand and the kind of operations we would like to perform.

MOTIVATION



Throw a ball up into the air. Suppose we know the height at any instant in time, i.e. we know the function $y(t)$ that describes the height. From this we can estimate the CHANGE in height with respect to time.

Example:

CHANGE in height from time t_1 to t_2 is simply : $h_2 - h_1$

which is : $y(t_2) - y(t_1)$

From the function $y(t)$ can we find out the RATE OF CHANGE in height at any chosen INSTANT ?

What is the vertical speed or RATE OF CHANGE in height at time instant t_1 ?

What is the ***expression*** of the INSTANTANEOUS RATE OF CHANGE in height ?

We know that vertical speed $= \dfrac{\textit{CHANGE in height}}{\textit{CHANGE in time}}$

Vertical speed between t_1 and $t_2 = \dfrac{y(t_2) - y(t_1)}{t_2 - t_1}$

This is the AVERAGE VERTICAL SPEED over the time interval t_1 to t_2.

No matter how close we take t_2 to t_1, i.e. no matter how small the time interval is, we still get only an AVERAGE VERTICAL SPEED. We do not get the vertical speed at time instant t_1.

We ask two fundamental questions.

Question 1:

> How can we go from **interval** $[t_1, t_2]$, no matter how small, to reach the **instant** t_1 ?

Question 2:

> How can we find the RATE OF CHANGE at **instant** t_1 from the AVERAGE RATE OF CHANGE over the **interval** $[t_1, t_2]$?

With Calculus, for functions that are WELL BEHAVED: SINGLE VALUED, CONTINUOUS and DIFFERENTIABLE, we are able to find the vertical speed at any chosen ***instant***, i.e. from the function that expresses the CHANGE we can find the INSTANTANEOUS RATE OF CHANGE.

From the ***Analysis*** point of view we would like to introduce some special notation to express what we are doing. We may say:

$$\Delta t = t_2 - t_1 \ . \quad \text{So} \quad t_2 = t_1 + \Delta t$$

$$\Delta y = y(t_2) - y(t_1) = y(t_1 + \Delta t) - y(t_1)$$

where the Greek letter Δ denotes a difference that is measurable.

1. AVERAGE step: So the AVERAGE VERTICAL SPEED may be denoted by :

$$\frac{\Delta y}{\Delta t} = \frac{y(t_2) - y(t_1)}{t_2 - t_1} = \frac{y(t_1 + \Delta t) - y(t_1)}{(t_1 + \Delta t) - t_1}$$

2. TENDS TO step: We then fix t_1 and let t_2 get closer and closer to t_1. The difference Δt between t_1 and t_2 becomes smaller and smaller. It becomes infinitely small. This kind of difference we denote using the Greek symbol δ. When t_2 gets closer and closer to t_1 we get a better or more accurate AVERAGE VERTICAL SPEED. This we denote by :

$$\frac{\delta y}{\delta t} = \frac{y(t_1 + \delta t) - y(t_1)}{(t_1 + \delta t) - t_1}$$

3. LIMIT step: Finally, when we let t_2 coincide with t_1, we get the INSTANTANEOUS VERTICAL SPEED at ***instant*** t_1. This we denote by :

$$\frac{dy}{dt} = \underset{t_2 \to t_1}{\text{LIMIT}} \frac{\delta y}{\delta t} = \underset{\delta t \to 0}{\text{LIMIT}} \frac{\delta y}{\delta t} = \underset{\delta t \to 0}{\text{LIMIT}} \frac{y(t_1 + \delta t) - y(t_1)}{(t_1 + \delta t) - t_1}$$

To do this we need to develop the concept of CONTINUOUS from the concepts that are described using the special vocabulary: TENDS TO, LIMIT, INFINITESIMAL and INSTANT. We shall then show that the TIME AXIS is a CONTINUOUS set of INSTANTS and that each INSTANT corresponds to a Real number. We shall then define CONTINUOUS functions.

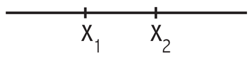
Part 1 : CONTINUITY

Overview

To show that the TIME AXIS is a ***continuous*** set of ***instants*** and that each ***instant*** corresponds to a Real number we shall proceed in three steps.

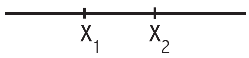
Step 1: We know from class 8 Geometry that a straight line is a ***continuous*** set of ***points***. We are also familiar with the different kinds of sets of numbers in Algebra: $\mathcal{N}$ (natural numbers), $\mathcal{Z}$ (integers), $\mathcal{Q}$ (rational numbers), Irrational numbers and $\mathcal{R}$ (real numbers). We present the ***Analysis*** concept of the COMPLETENESS property of the Real numbers.

We thus show the connection between the set $\mathcal{R}$ of real numbers in Algebra and a straight line (a ***continuous*** set of ***points***) in Geometry. Each point x_i on the number line corresponds to a Real number and vice versa. We may call this number line the x-axis. And when we say $x_1 < x_2$ for $x_1, x_2 \in \mathcal{R}$, the picture from the Geometry point of view is :

ALGEBRA		GEOMETRY
set $\mathcal{R}$	$\equiv$	x-axis
$x_1 < x_2$		

Step 2: In this step we learn the concepts: TENDS TO, LIMIT and INFINITESIMAL.

We shall use these concepts to take an ***Analysis*** view of the x-axis as a ***continuous*** set of ***points***. We shall then define an ***instant*** on the TIME AXIS. From this we shall show the equivalence between the x-axis as a ***continuous*** set of ***points*** and the TIME AXIS as a ***continuous*** set of ***instants***.

GEOMETRY		ANALYSIS
x-axis	$\equiv$	time axis
		

Each ***point*** x_i on the x-axis corresponds to an ***instant*** t_i on the TIME AXIS.

Step 3: Here we relate the TIME AXIS to the set of real numbers $\mathcal{R}$. Now when we say instant t_1 we mean some definite real number $t_1 \in \mathcal{R}$. And when we say $t_1 < t_2$ for t_1, t_2 in $\mathcal{R}$, the picture from the ***Analysis*** point of view is :

ALGEBRA		ANALYSIS
set $\mathcal{R}$	$\equiv$	time axis
$t_1 < t_2$		t_1 t_2

Since the TIME AXIS is a ***continuous*** set of ***instants***, we may say t_2 TENDS TO t_1 in a ***continuous*** manner. This is denoted by $t_2 \rightarrow t_1$. And each ***instant*** on the way to t_1 corresponds to a definite real number in $\mathcal{R}$.

We may even let t_2 coincide with t_1 . This is expressed in ***Analysis*** terminology as:

$$\underset{t_2 \rightarrow t_1}{\mathrm{LIMIT}}$$

After we develop the concept of the CONTINUOUS INFINITESIMAL δt, the $t_2 \rightarrow t_1$ may be used in the calculation as :

$$t_2 = t_1 + \delta t$$

And the concept of t_2 coinciding with t_1 may be expressed using the word LIMIT with δt in the calculation as:

$$\underset{t_2 \rightarrow t_1}{\mathrm{LIMIT}} \{ \text{expression involving } t_1 \text{ and } t_2 \} \equiv \underset{\delta t \rightarrow 0}{\mathrm{LIMIT}} \{ \text{expression involving } t_1 \text{ and } \delta t \}$$

We then extend these concepts from a ***particular*** instant t_1 to a ***general*** instant t simply by dropping the subscript.

2. NUMBERS AND THE NUMBER LINE

Natural numbers: $\mathcal{N} = \{0, 1, 2, 3, \ldots\}$*

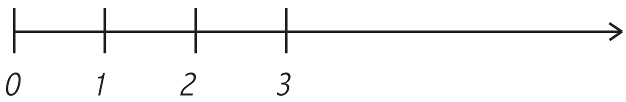
1. We note that between any two natural numbers there are FINITELY many natural numbers.

   e.g. between 1 and 5 there are exactly three natural numbers: 2, 3 and 4.

2. We cannot exhaust counting the natural numbers because there are infinitely many of them. But we can COUNT or ENUMERATE them in an orderly manner without missing any. This property we call COUNTABLE or ENUMERABLE.

A set of numbers with these two properties is called DISCRETE. The set of natural numbers $\mathcal{N}$ is DISCRETE.

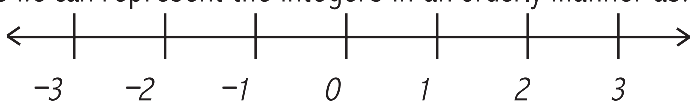
We can draw an infinite line and represent the natural numbers on this line in an orderly manner as individual points spaced evenly apart.



Integers: $\mathcal{Z} = \{\ldots\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}$

1. Between any two integers there are FINITELY many integers.
2. We may COUNT the intergers in an orderly manner without missing any as follows: $0, +1, -1, +2, -2, +3, -3, \ldots$. The set $\mathcal{Z}$ is COUNTABLE.

Hence the set $\mathcal{Z}$ is DISCRETE. To continue with our representation of numbers on an infinite line we can represent the integers in an orderly manner as:



with the line extending to infinity, ∞ , in both directions.

* Note: In some contexts zero is not considered a natural number. For our purposes it does not really matter. All we need to know is whether we can count the natural numbers or not. One number more or less does not make a difference.

Rationals: $\mathcal{Q} = \{ \ \frac{p}{q} \ni p,\ q \in \mathcal{Z},\ q \neq 0 \}$

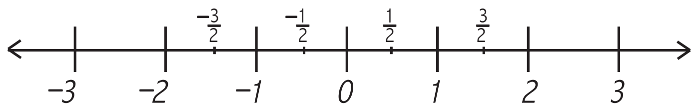
1. We know from class 4 Algebra between any two rational numbers there are INFINITELY many rational numbers.
2. Even though there are infinitely many rationals between any two rationals we can still COUNT or ENUMERATE the rationals in an orderly manner without missing any. The rationals are COUNTABLE. This is how we may count the rational numbers.

$$
\begin{array}{cccccccccc}
{}^{0}/_{1} & \rightarrow & {}^{\pm 1}/_{1} & & {}^{\pm 2}/_{1} & \rightarrow & {}^{\pm 3}/_{1} & & {}^{\pm 4}/_{1} & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & & & \\
{}^{\pm 1}/_{2} & & {}^{\pm 2}/_{2} & & {}^{\pm 3}/_{2} & & {}^{\pm 4}/_{2} & & {}^{\pm 5}/_{2} & \cdots \\
\downarrow & \nearrow & & \swarrow & & & & & & \\
{}^{\pm 1}/_{3} & & {}^{\pm 2}/_{3} & & {}^{\pm 3}/_{3} & & {}^{\pm 4}/_{3} & & {}^{\pm 5}/_{3} & \cdots \\
 & \swarrow & & & & & & & & \\
{}^{\pm 1}/_{4} & & {}^{\pm 2}/_{4} & & {}^{\pm 3}/_{4} & & {}^{\pm 4}/_{4} & & {}^{\pm 5}/_{4} & \cdots \\
\downarrow & & & & & & & & & \\
\ldots & \text{and so on.} & & & & & & & &
\end{array}
$$

The set of rational numbers $\mathcal{Q}$ is more than DISCRETE.

We say that the set of rational numbers $\mathcal{Q}$ is DENSE.

Let us look again at our representation of numbers on an infinite line.



If we try to plot the rationals as co-linear points in an orderly manner we would get an "***almost continuous***". The rational numbers by themselves do not exhaust, COMPLETELY cover, the number line.

From the Set Theory point of view we see that $\mathcal{N} \subset \mathcal{Z} \subset Q$.

Can we say that all the POINTS on the line between the rational number 1 and the rational number 2 are rational numbers ? There are numbers that are not rational numbers. Let us consider the square root of 2 denoted by $\sqrt{2}$.

If $\sqrt{2}$ is a rational number then $\sqrt{2} = \frac{p}{q} \ni p,\ q \in \mathcal{Z},\ q \neq 0$.

Further, let p, q be such that they have no factor in common.

$$2 = \frac{p^2}{q^2}$$

$$2q^2 = p^2$$

$2q^2$ is even $\Rightarrow$ p^2 is also even $\Rightarrow$ p is even.

Let $p = 2n$

Since $2q^2 = p^2 \Rightarrow 2q^2 = (2n)^2$

$\Rightarrow q^2 \Rightarrow 2n^2$.

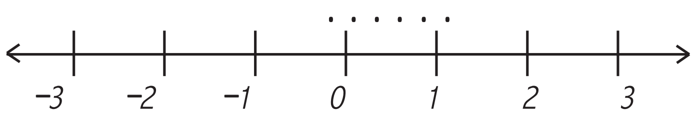
$2n^2$ is even $\Rightarrow$ q^2 is even $\Rightarrow$ q is even.

Thus p is even and q is even, implying they have as common factor 2. This contradicts our condition that p and q have no common factor. Therefore our assumption that $\sqrt{2}$ is rational is wrong. We say $\sqrt{2}$ is irrational. The term IRRATIONAL literally means " ***having no ratio*** ". For practical mensuration the whole range of the rationals is more than sufficient. But from the theoretical point of view this is not enough. For example, we need to define the length of the diagonal of a square of side of unit length.

In fact, between any two ***rational*** numbers there are infinitely many ***irrational*** numbers. So we cannot say that the set of rationals Q is COMPLETE. We cannot say that each and every point on the number line is some rational number. The set of rational numbers Q is DENSE but not COMPLETE.

Irrational numbers:

Look at the number line again and consider a portion of it, say from 0 to 1.



Try to visualize it as a collection of infinitely many points, so many that we cannot even begin to ENUMERATE them. Infinitely many points represent ***rational*** numbers and infinitely many points represent ***irrational*** numbers.

1. Between any two irrational numbers there are INFINITELY many irrational numbers. The numbers:

$$\sqrt{2},\ \sqrt{5},\ \sqrt[3]{\sqrt{3}+\sqrt{2}},\ \sqrt[3]{\sqrt{5}+\sqrt{7}}$$

   and many other expressions involving rational numbers under the radical sign $\sqrt{\ }$ are irrational. These irrational numbers are said to be expressed in terms of radicals.

The decimals help us classify the rational and irrational numbers. Rational numbers are represented by TERMINATING DECIMALS,

$$\text{e.g. } \frac{1}{4} = 0.25$$

or INFINITE REPEATING DECIMALS, e.g. $\frac{1}{3} = 0.333\ldots$

Irrational numbers are represented by NON-TERMINATING NON-REPEATING DECIMALS.

Examples:

$$\sqrt{2} = 1.41421356237\ldots$$

$$e = 2.71828182845904523536\ldots$$

$$\pi = 3.14159265358979323846264338327950\ldots$$

2. Are the irrationals COUNTABLE ?

Without giving a formal proof let us try to get an intuitive feel for how many ***irrational*** numbers there could be as campared to the COUNTABLE set of ***rationals*** Q .

How many ***rational*** numbers can you create with TERMINATING (finitely many digits) decimal expansion? Infinitely many.

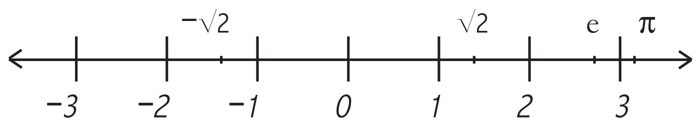
How many ***rational*** numbers can you create with NON-TERMINATING (infinitely many digits) and REPEATING decimal expansion? Again, infinitely many.

Now, imagine how many ***irrational*** numbers you can create with NON-TERMINATING NON-REPEATING decimal expansion?

Between the ***rationals*** $\frac{1}{4} = 0.25$ and $\frac{1}{3} = 0.333\ldots$ we can create innumerable ***irrational*** numbers with NON-TERMINATING NON-REPEATING patterns of decimal digits.

Again, between any two such ***irrational*** numbers created above we may create innumerably more ***irrational*** numbers in a similar manner. There is a virtual flood of ***irrational*** numbers. This should give us the feeling that the irrationals are NOT COUNTABLE. Since we cannot even begin to COUNT them, there is no sense in talking about a subscript to enumerate them.

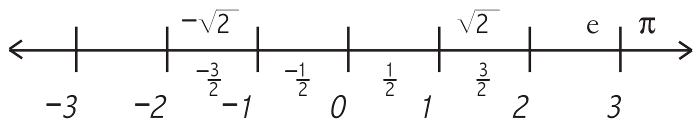
If the ***rationals*** are DENSE then the ***irrationals*** are more than DENSE. Each ***irrational*** number corresponds to a point on the number line. If we try to plot the ***irrationals*** as co-linear points in an orderly manner we would get an ***"almost continuous"*** line. However, the ***irrationals*** by themselves do not exhaust, COMPLETELY cover, the number line.



3. REALS, COMPLETE, CONTINUOUS

From the decimal expansion point of view, any number must have either a FINITE (terminating) number of digits or an INFINITE (non-terminating) number of digits. And, if it has an INFINITE number of digits, then the pattern of digits must be REPEATING (recurring) or NON-REPEATING. Can you think of any other possibility? Also, we can compare any two numbers (rational or irrational) and plot them on the number line in an orderly manner.

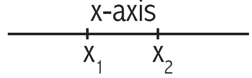
Real Numbers: It should now be at least intuitively clear that any point on the number line is either a ***rational*** or an ***irrational*** number. And together these two sets of numbers COMPLETELY cover or exhaust the whole number line extending in both directions, positive and negative. The union of the set of ***rational*** numbers and the set of ***irrational*** numbers is called the set of real numbers $\mathcal{R}$.



COMPLETENESS property of the Real numbers:

Corresponding to every ***point*** on the number line we have a unique ***real number*** and vice versa. Between any two real numbers on the number line each and every point corresponds to a real number. The set of real numbers $\mathcal{R}$ is COMPLETE. The real number line is smooth and CONTINUOUS. There are no gaps, breaks or bumps. This Real number line we call the **x-axis**.

We now see the connection between the set $\mathcal{R}$ of real numbers in Algebra and a straight line (a ***continuous*** set of ***points***) in Geometry. Each point x_i on the x-axis corresponds to a Real number and vice versa. And when we say $x_1 < x_2$ for $x_1, x_2 \in \mathcal{R}$ the picture from the Geometry point of view is :

ALGEBRA		GEOMETRY
set R	$\equiv$	x-axis
$x_1 < x_2$		

4. TENDS TO and LIMIT

In elementary geometry you learned the definition or meaning of a **point**. You also know how to name or label a **point**.

In Calculus a **point** on the number line or x-axis corresponds to a real number. Sometimes we know its exact value, e.g. 2. Sometimes we will know only the approximate value. However, we may denote it by a special name or label or symbol, e.g. $\sqrt{2}$, e, π. Regardless of knowing the exact value or not we know which **point** we are talking about.

In Calculus we prefer to use the word **instant** rather than **point**. We speak of the **instant** 2, or the **instant** $\sqrt{2}$, and so on. In Calculus we have another way to define an **instant**. Before we do this we need to know what an **infinitesimal** is.

In order to give a formal definition of an **infinitesimal** we need to use two concepts: TENDS TO and LIMIT.

TENDS TO:

Let $a = 0.5$ and $x = 0.4, 0.49, 0.499, 0.4999, \ldots$ progressively.

It is clear that x TENDS TO 0.5, we write this as: $x \to a$.

Will $x = a$? No. We will always have $0 < |x - a|$

LIMIT:

If we let $b = 0.51$ we are correct in saying: x TENDS TO 0.51, denoted by $x \to b$. Will $x = b$? No. $0 < |\mathbf{x - b}|$

What is the difference between $\mathbf{x \to a}$ and $\mathbf{x \to b}$?

In $x \to b$: we cannot choose **x** ***as close as we like to*** **b**. The difference $|x - b|$ cannot be made ***as small as we like***. We cannot have $|x - b| < \delta$ for any small quantity δ ***as small as we like***. For example:

if we choose $\delta = 0.001$ we will have $0 < \delta < 0.01 < |x - b|$.

In $x \to a$: we can choose **x** ***as close as we like to*** **a**. The difference $|x - a|$ can be made ***as small as we like***. If we choose $\delta > 0$, any small positive quantity ***as small as we like***, we can choose x such that $|x - a| < \delta$.

Since we can choose **x** ***as close as we like to*** **a**, and only to **a** but not to any other ***point***, we say: LIMIT as $x \to a$ is **a**, written as $\underset{x \to a}{\mathit{Limit}}\ x = a$

In x = 0.4, 0.49, 0.499, 0.4999, ... progressively with a = 0.5, since x < a, we can be more precise and say $x \to a$ from the **left.** We write this as $x \to a^-$.

We could have let x = 0.51, 0.501, 0.5001, 0.50001, ... progressively with a = 0.5 . Here too we have $x \to a$ but from the **right**. We write this as $x \to a^+$.

Sometimes we will be very specific and say:

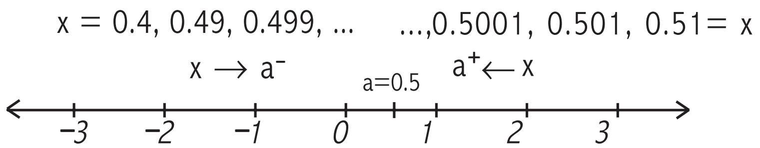
x TENDS TO a from the **left**, written as : $x \to a^-$

x TENDS TO a from the **right**, written as : $x \to a^+$

Which ever be the case, whether $x \to a^+$ (from the **right**) or $x \to a^-$ (from the **left**) we insist that $x \neq a$, i.e. $0 < |x - a|$. The case when x = a we will speak of separately.

Since we know that LIMIT ($x \to a^-$) is **a**, we write : $\underset{x \to a^-}{\mathit{Limit}}\ \{x\} = a$.

Since we know that LIMIT ($x \to a^+$) is **a**, we write : $\underset{x \to a^+}{\mathit{Limit}}\ \{x\} = a$.



5. INFINITESIMALS

Let us now focus on the difference $\Delta = |a - x|$ as $x \to a = 0.5$, be it from the ***left*** or from the ***right***.

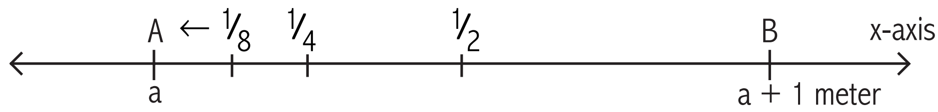
$$\Delta = |a - x| = 0.01,\ 0.001,\ 0.0001,\ 0.00001,\ \ldots$$

After a while it becomes ***infinitely small*** and we denote it by $\boldsymbol{\delta}$. We may say :

$$\boldsymbol{\delta} = 1/10^n \quad \text{as } n \to \infty$$

We can see that $\underset{n \to \infty}{\text{LIMIT}} \{ 1/10^n \} = 0$.

Let us look at another example. Imagine A and B located one meter apart on the x-axis . At each second let B jump $1/2$ distance towards A. We may say that $B \to A$.



The difference $\Delta = |A - B| = 1/2,\ 1/4,\ 1/8,\ \ldots$

After a while it becomes ***infinitely small*** and we denote it by $\boldsymbol{\delta}$. We may say :

$$\boldsymbol{\delta} = 1/2^n \quad \text{as } n \to \infty$$

We can see that $\underset{n \to \infty}{\text{LIMIT}} \{ 1/2^n \} = 0$.

By definition:

> An ***infinitely small*** quantity whose LIMIT is zero is called an INFINITESIMAL.

Note that the LIMIT is zero, i.e. the quantity $\boldsymbol{\delta}$ TENDS TO zero but does not become zero. $\boldsymbol{\delta}$ does not vanish.

We may now use the infinitesimal δ and say:

$x \rightarrow a^{+} \equiv x = a + \delta$

$x \rightarrow a^{-} \equiv x = a - \delta$

$B \rightarrow A \equiv B = a + \delta$

To express the concept x coincides with a, we may say :

from the ***right*** : $x = \underset{\delta \rightarrow 0}{Limit} \{ a + \delta \}$

from the ***left*** : $x = \underset{\delta \rightarrow 0}{Limit} \{ a - \delta \}$

Likewise, to express B coincides with A we may say : $B = \underset{\delta \rightarrow 0}{Limit} \{ a + \delta \}$

As B jumps from 1 to $1/2$ to $1/4$ to $1/8$ . . . to A it skips a lot ***points*** in between. Likewise, as $x \rightarrow a$ it skips a lot of ***points*** in between. This is because δ is a ***discrete infinitesimal***.

We can think of the number line or x-axis as smooth and CONTINUOUS.There are no gaps or breaks. In Calculus we want to study the behavior of a function at each and every **point** on the x-axis or **instant** on the TIME AXIS.

So we want x to TEND TO a **point** in a smooth and CONTINUOUS manner. For the infinitesimal that TENDS TO zero in a smooth and CONTINUOUS manner we have the special notation δx.

Geometrically, we can think of δx as a small CONTINUOUS line segment representing an infinitely small CHANGE in x. The length of this small line segment δx is always > 0 no matter where we are on the x-axis. δx gets smaller and smaller. The length of δx TENDS TO zero but does not become zero. δx does not vanish. However, the LIMIT of δx is zero.

We may let :

$$\delta x = {}^{1}\!/_{10^x} \quad \text{as } x \to \infty$$

$$\text{or} \quad \delta x = {}^{1}\!/_{2^x} \quad \text{as } x \to \infty$$

Now we may express $x \to a$ in a ***continuous*** manner by :

$$x \to a^+ \quad \equiv \quad x = a + \delta x$$

$$x \to a^- \quad \equiv \quad x = a - \delta x$$

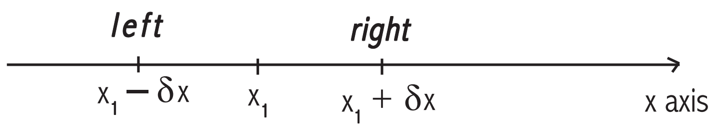
$$B \to A \quad \equiv \quad B = A + \delta x$$

To express the concept x coincides with **a** , we may say :

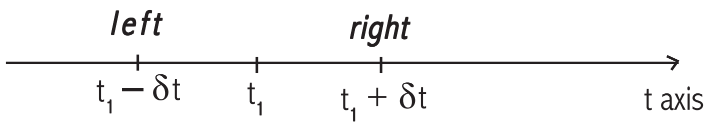
from the ***right*** : $x = \underset{\delta x \to 0}{Limit} \{ \mathbf{a} + \delta x \}$

from the ***left*** : $x = \underset{\delta x \to 0}{Limit} \{ \mathbf{a} - \delta x \}$

Likewise, to express B coincides with A we may say : $B = \underset{\delta x \to 0}{Limit} \{ A + \delta x \}$



Geometry: point x_1
Analysis: instant x_1
Algebra: real number x_1



Geometry: point t_1
Analysis: instant t_1
Algebra: real number t_1

6. δx and INSTANT

We may now take an Analysis view of the ***particular*** point x_1 on the x-axis and define it as :

> The ***particular point*** x_1 on the x-axis is where
> $$\underset{\delta x \to 0}{Limit}\,(x_1 - \delta x) = x_1 = \underset{\delta x \to 0}{Limit}\,(x_1 + \delta x)$$

And the ***general*** point x on the x-axis is defined by dropping the subscript.

> A ***general point*** x on the x-axis is where
> $$\underset{\delta x \to 0}{Limit}\,(x - \delta x) = x = \underset{\delta x \to 0}{Limit}\,(x + \delta x)$$

Likewise, we may also define the ***particular*** instant t_1 on the time axis as :

> The ***particular instant*** t_1 on the time axis is where
> $$\underset{\delta t \to 0}{Limit}\,(t_1 - \delta t) = t_1 = \underset{\delta t \to 0}{Limit}\,(t_1 + \delta t)$$

And the ***general*** instant t on the time axis is defined by dropping the subscript.

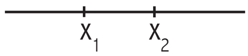
> A ***general instant*** t on the time axis is where
> $$\underset{\delta t \to 0}{Limit}\,(t - \delta t) = t = \underset{\delta t \to 0}{Limit}\,(t + \delta t)$$

Thus we see the TIME AXIS is identical to the x-axis except for a change in name.

GEOMETRY		ANALYSIS
x-axis	$\equiv$	time axis
x		t

Each ***point*** x on the x-axis corresponds to an ***instant*** t on the TIME AXIS.

When we relate this to the set of real numbers $\mathcal{R}$ in Algebra we have :

ALGEBRA		GEOMETRY and ANALYSIS	
$x_1 < x_2$ for x_1 , $x_2 \in \mathcal{R}$	$\equiv$		x-axis
$t_1 < t_2$ for t_1 , $t_2 \in \mathcal{R}$	$\equiv$		time axis

Now when we say t_2 TENDS TO t_1 on the TIME AXIS, denoted by $t_2 \to t_1$, we let ***instant*** t_2 go ***instant*** , ***instant*** , ***instant*** , . . . all the way to ***instant*** t_1 in a ***continuous*** manner. And each ***instant*** on the TIME AXIS corresponds to a definite real number in $\mathcal{R}$.

Likewise, when we say x_2 TENDS TO x_1 on the x-axis, denoted by $x_2 \to x_1$, we let ***point*** x_2 go ***point***, ***point***, ***point***, . . . all the way to ***point*** x_1 in a ***continuous*** manner. And each ***point*** on the x-axis corresponds to a definite real number in $\mathcal{R}$.

From the Algebra point of view $\mathcal{R}$ is a COMPLETE set of real numbers. From the Geometry point of view $\mathcal{R}$ is a CONTINUOUS line of POINTS. And from the Analysis point of view $\mathcal{R}$ is a CONTINUOUS set of INSTANTS.

We can now answer the fundamental Question 1 on page 3.

> In taking the Limit as $t_2 \to t_1$ we go from the **interval** $[\, t_1 \, , \, t_2 \,]$ and reach the **instant** t_1.

> In general, if t is any **instant**, by taking the Limit as $\delta t \to 0$ we can go from the **interval** $[t, t+\delta t]$ and reach the **instant** t.

What happens at an ***instant*** is ***instantaneous***.

In English, ***instantaneous*** = occuring or completed in an ***instant***.

7. SINGLE VALUED FUNCTIONS

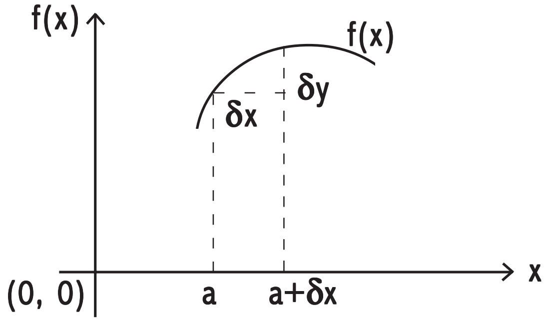
In the next few chapters we apply the concepts TENDS TO, LIMIT and CONTINUOUS INFINITESIMALS to look at the first two properties of WELL-BEHAVED functions, i.e. SINGLE VALUED and CONTINUOUS.

To know whether a function is CONTINUOUS or not, we need to know :

1. The LIMIT of a function.
2. The VALUE of a function.

With these two concepts in place we define CONTINUOUS functions.

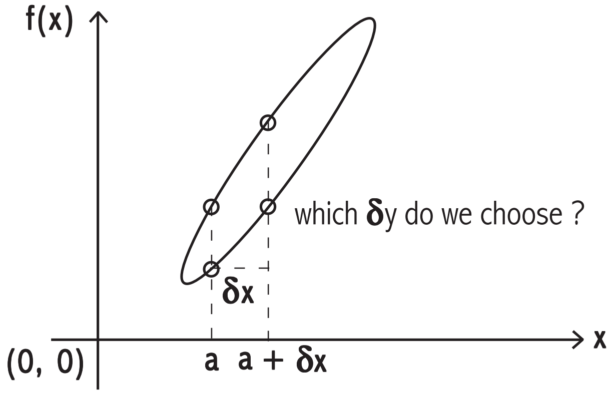
A function is said to be SINGLE VALUED* if at every **point** or **instant** on the real line where the function is defined it has one and only one value.



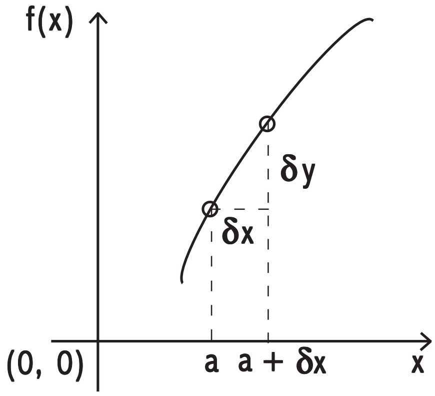
Moreover, for an infinitesimal CHANGE δx at the **instant a** (where the function is defined) there is exactly one corresponding CHANGE δy in the value of the function.

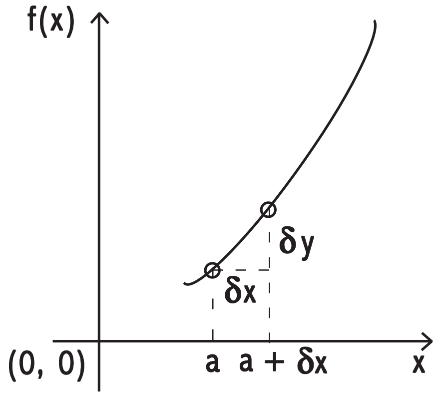
* Note: In Algebra the concept of a function as developed from a ***relation*** (set of ordered pairs) and ***mapping*** is single valued by definition. In Analysis the concept is not so strict. For calculation purposes we have to enforce this by restricting the range of the function.

In the figure below at $x = a$ there are two values of $f(x)$. So, for an infinitesimal CHANGE δx at $x = a$ we have two corresponding CHANGES δy.

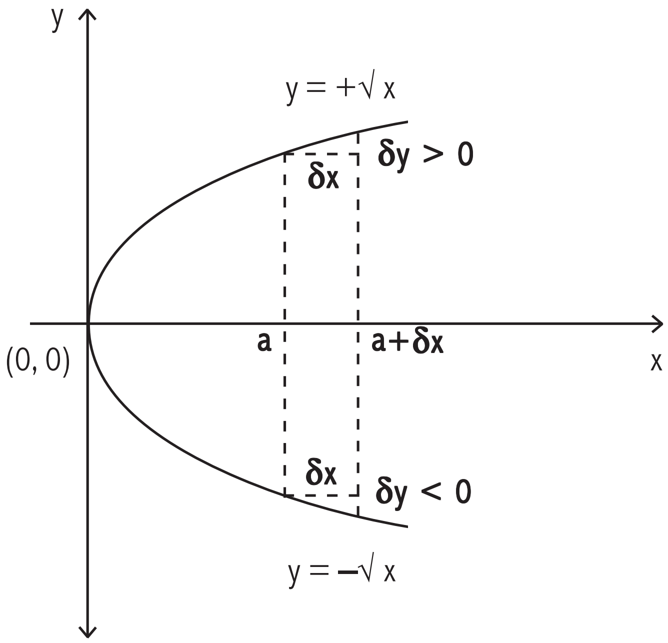


We may make $f(x)$ SINGLE VALUED by restricting the range as in either one of the diagrams below.





The function $y = \sqrt{x}$ for $x \geq 0$ is not SINGLE VALUED. It has two values: $+\sqrt{x}$ and $-\sqrt{x}$. We must specify which root we are using.



Since we are dealing with ***real*** valued functions over the ***real line*** we avoid cases where the function may take on complex values such as $\sqrt{x}$ for $x < 0$.

Also, the functions we deal with must take on ***well-defined*** real values. We avoid undefined entities such as $+\infty$, $-\infty$, division by zero and ∞/∞.

Functions of the form $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ are called ***polynomials***. These functions are ***single valued***.

8. LIMIT of a Function

To understand the behaviour of a function f(x) very close to some ***instant a***, that is to say near ***instant a*** and to the LEFT and RIGHT of ***instant a***, we must analyse :

$$\underset{x \to a^-}{Limit} \{ f(x) \} \quad \text{or} \quad \underset{\delta x \to 0}{Limit} \{ f(a - \delta x) \}$$

and

$$\underset{x \to a^+}{Limit} \{ f(x) \} \quad \text{or} \quad \underset{\delta x \to 0}{Limit} \{ f(a + \delta x) \}$$

If it turns out that there is some definitive real number b such that :

LIMIT from the LEFT $= b =$ LIMIT from the RIGHT

$$\underset{x \to a^-}{Limit} \{ f(x) \} = b = \underset{x \to a^+}{Limit} \{ f(x) \}$$

$$\underset{\delta x \to 0}{Limit} \{ f(a - \delta x) \} = b = \underset{\delta x \to 0}{Limit} \{ f(a + \delta x) \}$$

then we say : $\underset{x \to a}{Limit} \{ f(x) \} = b$.

Properly speaking, finding or calculating the $\underset{x \to a^+}{Limit} \{ f(x) \}$ is a 2 step process. This becomes clear when we use the infinitesimal δx.

1. TENDS TO step : as $x \to a^+$ we may say $x = a + \delta x$. We may substitute $(a + \delta x)$ for x in f(x). Then we do whatever expansion, regrouping of terms, cancellation and simplification possible. We may perform these operations because $\delta x \neq 0$. It only TENDS TO zero. ***There is no division by zero in this step***. Likewise, when $x \to a^-$ we may substitute $(a - \delta x)$ for x in f(x).

2. LIMIT step : here we let $\delta x = 0$ so that x coincides with a.

We combine both steps in one expression as: $\underset{x \to a^-}{Limit} \{f(x)\}$ or $\underset{x \to a^+}{Limit} \{f(x)\}$.

Then we combine both the above into one expression: $\underset{x \to a}{Limit} \{f(x)\}$ or $\underset{\delta x \to 0}{Limit} \{f(x)\}$.

Note: More precisely, ***near = as close as we like***.

f(x) at instant x = a is f(a). This we call the VALUE of f(x) at x = a, written as :

$$\underset{x = a}{\text{VALUE}} \{ f(x) \} = f(a)$$

Sometimes we may have : $\underset{x \to a}{\text{LIMIT}} \{ f(x) \} = b = \underset{x = a}{\text{VALUE}} \{ f(x) \}$

But this is not always the case as we shall see from the examples that follow.

Example 1: Consider the function f(x) = C, a constant.

What is the LIMIT of f(x) as x TENDS TO a ?

Limit from the LEFT

$$\underset{x \to a^-}{\text{LIMIT}} \{ f(x) \} = \underset{x \to a^-}{\text{LIMIT}} \{ C \} = C.$$

$$\underset{\delta x \to 0}{\text{LIMIT}} \{ f(a - \delta x) \} = \underset{\delta x \to 0}{\text{LIMIT}} \{ C \} = C.$$

Limit from the RIGHT

$$\underset{x \to a^+}{\text{LIMIT}} \{ f(x) \} = \underset{x \to a^+}{\text{LIMIT}} \{ C \} = C$$

$$\underset{\delta x \to 0}{\text{LIMIT}} \{ f(a + \delta x) \} = \underset{\delta x \to 0}{\text{LIMIT}} \{ C \} = C$$

Since, $\underset{x \to a^-}{\text{LIMIT}} \{ f(x) \} = C = \underset{x \to a^+}{\text{LIMIT}} \{ f(x) \}$, we say $\underset{x \to a}{\text{LIMIT}} \{ f(x) \} = C$.

$$\underset{x = a}{\text{VALUE}} \{ f(x) \} = f(a) = C .$$

Here we have : $\underset{x \to a}{\text{LIMIT}} \{ f(x) \} = C = \underset{x = a}{\text{VALUE}} \{ f(x) \}$.

Example 2: What is the LIMIT of $f(x) = x + a$ as x TENDS TO a ?

Limit from the LEFT

$$\underset{x \to a^-}{Limit} \{ f(x) \} = \underset{x \to a^-}{Limit} \{ x + a \} = \underset{x \to a^-}{Limit} \{ a + a \} = 2a$$

Let us find $\underset{x \to a^-}{Limit} \{f(x)\}$ in 2 steps using δx.

1. TENDS TO step : near **a** and to the ***left*** of **a** : $x = a - \delta x$

$f(x) = \{ (a - \delta x) + a \} = 2a - \delta x$

2. LIMIT step : $\underset{\delta x \to 0}{Limit} \{2a - \delta x \} = 2a$

Limit from the RIGHT

$$\underset{x \to a^+}{Limit} \{ f(x) \} = \underset{x \to a^+}{Limit} \{ x + a \} = \underset{x \to a^+}{Limit} \{ a + a \} = 2a$$

Let us find $\underset{x \to a^+}{Limit} \{f(x)\}$ in 2 steps using δx.

1. TENDS TO step : near **a** and to the ***right*** of **a** : $x = a + \delta x$

$f(x) = \{ (a + \delta x) + a \} = 2a + \delta x$

2. LIMIT step : $\underset{\delta x \to 0}{Limit} \{2a + \delta x\} = 2a$

Since, $\underset{x \to a^-}{Limit} \{ f(x) \} = 2a = \underset{x \to a^+}{Limit} \{ f(x) \}$, we say $\underset{x \to a}{Limit} \{ f(x) \} = 2a$.

$$\underset{x = a}{\text{VALUE}} \{ f(x) \} = f(a) = 2a .$$

Here we have :

$$\underset{x \to a}{\text{LIMIT}} \{ f(x) \} = 2a = \underset{x = a}{\text{VALUE}} \{ f(x) \} .$$

Now see the graph on page 180.

Example 3: What is the LIMIT of $f(x) = x^2$ as x TENDS TO a ?

Limit from the LEFT

$$\underset{x \to a^-}{Limit} \{ f(x) \} = \underset{x \to a^-}{Limit} \{ x^2 \} = \underset{x \to a^-}{Limit} \{ a^2 \} = a^2$$

Let us find $\underset{x \to a^-}{Limit} \{f(x)\}$ in 2 steps using δx.

1. TENDS TO step : near **a** and to the ***left*** of **a** : $x = a - \delta x$

$$f(x) = (a - \delta x)^2 = a^2 - 2a\delta x + \delta x^2$$

2. LIMIT step : $\underset{\delta x \to 0}{Limit} \{a^2 - 2a\delta x + \delta x^2\} = a^2$

Limit from the RIGHT

$$\underset{x \to a^+}{Limit} \{ f(x) \} = \underset{x \to a^+}{Limit} \{ x^2 \} = \underset{x \to a^+}{Limit} \{ a^2 \} = a^2$$

Let us find $\underset{x \to a^+}{Limit} \{f(x)\}$ in 2 steps using δx.

1. TENDS TO step : near **a** and to the ***right*** of **a** : $x = a + \delta x$

$$f(x) = (a + \delta x)^2 = a^2 + 2a\delta x + \delta x^2$$

2. LIMIT step : $\underset{\delta x \to 0}{Limit} \{a^2 + 2a\delta x + \delta x^2\} = a^2$

Since, $\underset{x \to a^-}{Limit} \{ f(x) \} = a^2 = \underset{x \to a^+}{Limit} \{ f(x) \}$, we say $\underset{x \to a}{Limit} \{ f(x) \} = a^2$.

$$\underset{x = a}{\text{VALUE}} \{ f(x) \} = f(a) = a^2 .$$

Here we have : $\underset{x \to a}{\text{LIMIT}} \{ f(x) \} = a^2 = \underset{x = a}{\text{VALUE}} \{ f(x) \}$.

We should begin to get the general feeling that for ***polynomials*** of the form :

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

we have: $\underset{x \to a}{\text{LIMIT}} \{ f(x) \} = \underset{x = a}{\text{VALUE}} \{ f(x) \}$.

Example 4: What is the LIMIT of $f(x) = \frac{x - a}{x - a}$ as x TENDS TO a ?

Limit from the LEFT

$$\underset{x \to a^-}{\text{LIMIT}} \{ f(x) \} = \underset{x \to a^-}{\text{LIMIT}} \{ \frac{x - a}{x - a} \} = \underset{x \to a^-}{\text{LIMIT}} \{ 1 \} = 1$$

Note how we ***simplify first and then take the limit***. We can simplify first because when $x \to a$ we have $(x - a) \neq 0$. Let us do this in 2 steps using δx.

1. TENDS TO step : near **a** and to the ***left*** of **a** : $x = a - \delta x$

$$f(x) = \{ \frac{(a - \delta x) - a}{(a - \delta x) - a} \} = \{ \frac{-\delta x}{-\delta x} \} = 1$$

Note how we ***simplify first and then take the limit***. We can simplify first because δx only TENDS TO zero. $\delta x \neq 0$. ***So there is no division by zero in this step.***

2. LIMIT step : $\underset{\delta x \to 0}{\text{LIMIT}} \{1\} = 1$

Limit from the RIGHT

$$\underset{x \to a^+}{\text{LIMIT}} \{ f(x) \} = \underset{x \to a^+}{\text{LIMIT}} \{ \frac{x - a}{x - a} \} = \underset{x \to a^+}{\text{LIMIT}} \{ 1 \} = 1$$

1. TENDS TO step : near **a** and to the ***right*** of **a** : $x = a + \delta x$

$$f(x) = \{ \frac{(a + \delta x) - a}{(a + \delta x) - a} \} = \{ \frac{+\delta x}{+\delta x} \} = 1$$

Note how we ***simplify first and then take the limit***. We can simplify first because δx only TENDS TO zero. $\delta x \neq 0$. ***So there is no division by zero in this step.***

2. LIMIT step : $\underset{\delta x \to 0}{\text{LIMIT}} \{1\} = 1$

Since, $\underset{x \to a^-}{\text{LIMIT}} \{ f(x) \} = 1 = \underset{x \to a^+}{\text{LIMIT}} \{ f(x) \}$, we say $\underset{x \to a}{\text{LIMIT}} \{ f(x) \} = 1$.

$$\underset{x = a}{\text{VALUE}} \{ f(x) \} = \underset{x = a}{\text{VALUE}} \{ \frac{x - a}{x - a} \} = 0/0$$, which is something undefined.

Example 5: What is the LIMIT of $f(x) = \dfrac{x^2 - a^2}{x - a}$ as x TENDS TO a ?

Limit from the LEFT

$$\underset{x \to a^-}{Limit}\{f(x)\} = \underset{x \to a^-}{Limit}\left\{\frac{x^2 - a^2}{x - a}\right\} = \underset{x \to a^-}{Limit}\left\{\frac{(x-a)(x+a)}{x-a}\right\} = \underset{x \to a^-}{Limit}\{x+a\} = 2a$$

Note how we ***simplify first and then take the limit***. We can simplify first because when $x \to a$ we have $(x - a) \neq 0$. Let us do this in 2 steps using δx.

1. TENDS TO step : near **a** and to the ***left*** of **a** : $x = a - \delta x$

$$f(x) = \left\{\frac{(a - \delta x)^2 - a^2}{(a - \delta x) - a}\right\} = \left\{\frac{-2a.\delta x + \delta x^2}{-\delta x}\right\} = 2a - \delta x$$

Note how we ***simplify first and then take the limit***. We can simplify first because δx only TENDS TO zero. $\delta x \neq 0$. ***There is no division by zero in this step.***

2. LIMIT step : $\underset{\delta x \to 0}{Limit}\{2a - \delta x\} = 2a$

Limit from the RIGHT

$$\underset{x \to a^+}{Limit}\{f(x)\} = \underset{x \to a^+}{Limit}\left\{\frac{x^2 - a^2}{x - a}\right\} = \underset{x \to a^+}{Limit}\left\{\frac{(x-a)(x+a)}{x-a}\right\} = \underset{x \to a^+}{Limit}\{x+a\} = 2a$$

1. TENDS TO step : near **a** and to the ***right*** of **a** : $x = a + \delta x$

$$f(x) = \left\{\frac{(a + \delta x)^2 - a^2}{(a + \delta x) - a}\right\} = \left\{\frac{2a.\delta x + \delta x^2}{\delta x}\right\} = 2a + \delta x$$

2. LIMIT step : $\underset{\delta x \to 0}{Limit}\{2a + \delta x\} = 2a$

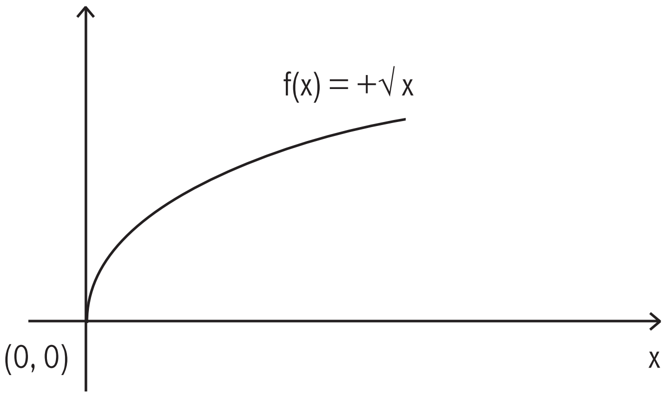
Since, $\underset{x \to a^-}{Limit}\{f(x)\} = 2a = \underset{x \to a^+}{Limit}\{f(x)\}$, we say $\underset{x \to a}{Limit}\{f(x)\} = 2a.$

$$\underset{x = a}{Value}\{f(x)\} = \underset{x = a}{Value}\left\{\frac{x^2 - a^2}{x - a}\right\} = 0/0$$, which is something undefined.

The reader must have guessed by now that $\underset{x \to a}{Limit}\left\{\frac{x^n - a^n}{x - a}\right\} = n\,a^{n-1}$.

We shall see this very special limit in its proper context when we study the ***derivative*** of $f(x) = x^n$.

Example 6 : What is the LIMIT of $f(x) = +\sqrt{x}$ as x TENDS TO 0 ?



Limit from the LEFT

1. TENDS TO step : near **0** and to the ***left*** of **0** : $x = 0 - \delta x$

$$f(x) = \sqrt{(0 - \delta x)} = \sqrt{-\delta x}$$

A negative real number under the radical sign is a complex number. We are dealing with ***real*** numbers and ***real*** valued functions. Here $\sqrt{-\delta x}$ is not ***real*** valued and so it is not defined. $f(x) = +\sqrt{x}$ is ***not defined*** for $x < 0$.

2. LIMIT step : The LIMIT from the ***left*** does not exist.

Limit from the RIGHT

1. TENDS TO step : near **0** and to the ***right*** of **0** : $x = 0 + \delta x$

$$f(x) = \sqrt{(0 + \delta x)} = \sqrt{\delta x}$$

2. LIMIT step : $\underset{\delta x \to 0}{Limit} \{\sqrt{\delta x}\} = 0$

Since $\underset{x \to 0^-}{Limit} f(x) \neq \underset{x \to 0^+}{Limit} f(x)$ we say : $\underset{x \to 0}{Limit} f(x)$ does NOT exist.

$$\underset{x = 0}{Value} f(x) = f(0) = \sqrt{0} = 0$$

Example 7: What is the LIMIT of $f(x) = {}^{1}\!/_{x}$ as x TENDS TO 0 ?

Limit from the LEFT

1. TENDS TO step : near **0** and to the ***left*** of **0** : $x = 0 - \delta x$

$f(x) = f(0 - \delta x) = \{ {}^{1}\!/_{(0 - \delta x)} \} = -{}^{1}\!/_{\delta x}$. Note that $\delta x \neq 0$.

2. LIMIT step : $\underset{\delta x \to 0}{\mathit{Limit}} \{ -{}^{1}\!/_{\delta x} \} = -{}^{1}\!/_{0} = -\infty$, something undefined.

Limit from the RIGHT

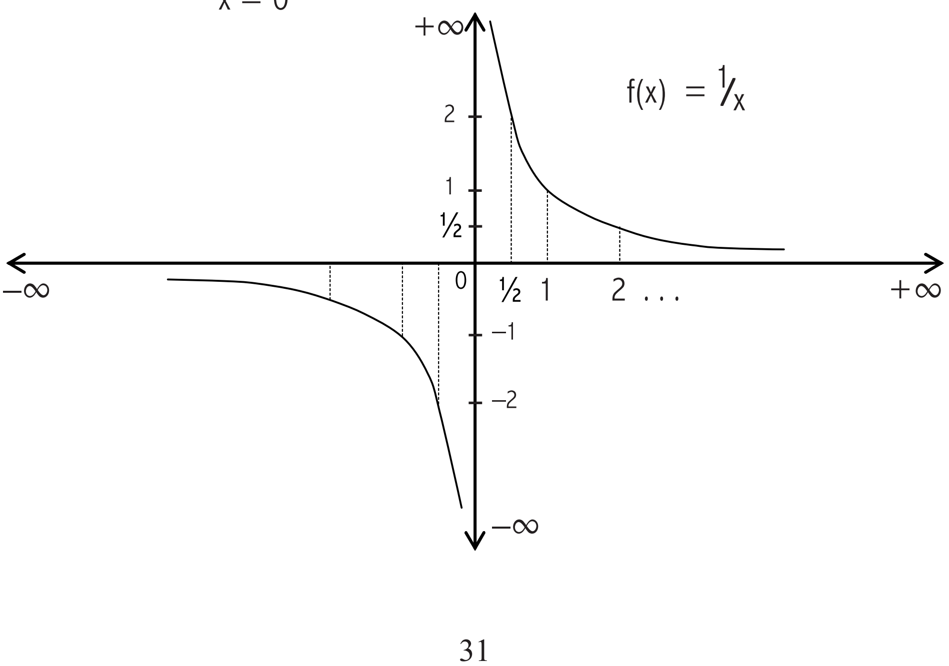
1. TENDS TO step : near **0** and to the ***right*** of **0** : $x = 0 + \delta x$

$f(x) = f(0 + \delta x) = \{ {}^{1}\!/_{(0 + \delta x)} \} = +{}^{1}\!/_{\delta x}$. Note that $\delta x \neq 0$.

2. LIMIT step : $\underset{\delta x \to 0}{\mathit{Limit}} \{ +{}^{1}\!/_{\delta x} \} = = +{}^{1}\!/_{0} = +\infty$, something undefined.

The LIMIT from the ***left*** does not exist. And, the LIMIT from the ***right*** does not exist.

$\underset{x = 0}{\text{VALUE}} \{ f(x) \} = f(0) = {}^{1}\!/_{0} = \infty$, something undefined.



Example 8: What is the LIMIT of $f(x) = \dfrac{\sin(x)}{x}$ as x TENDS TO 0 ?

We shall use the trigonometric identity $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.
Also, when x is very small, we may say* : $\sin(x) = x$.

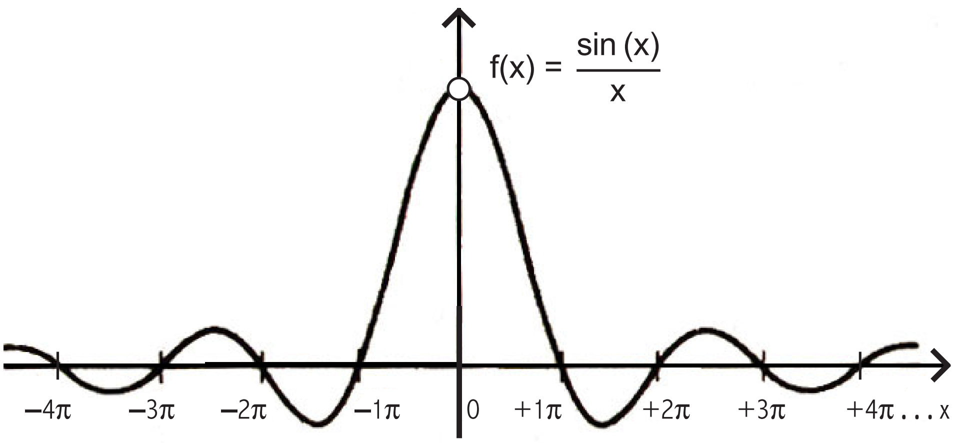
Limit from the LEFT

1. TENDS TO step : near **0** and to the ***left*** of **0** : $x = 0 - \delta x$

$$f(x) = \left\{\frac{\sin(0 - \delta x)}{(0 - \delta x)}\right\} = \left\{\frac{\sin(-\delta x)}{-\delta x}\right\} = \left\{\frac{-\delta x}{-\delta x}\right\} = 1$$

Note how we ***simplify first and then take the limit***. We can simplify first because δx only TENDS TO zero. $\delta x \neq 0$. ***There is no division by zero in this step***.

2. LIMIT step : $\underset{\delta x \to 0}{Limit} \{1\} = 1$



* Note: we may infer this from the basic definition of $\sin(\theta)$ = opposite side / hypoteneuse.
When θ is infinitely small, say $\delta\theta$, then $\sin(\delta\theta) = r\,\delta\theta/r = \delta\theta$. Another way is to look at the expansion of sin(x). When x becomes infinitely small, say δx , then only the first term in the expansion matters.

Limit from the RIGHT

1. TENDS TO step : near **0** and to the ***right*** of **0** : $x = 0 + \delta x$

$$f(x) = \left\{ \frac{\sin(0 + \delta x)}{(0 + \delta x)} \right\} = \left\{ \frac{\sin(+ \delta x)}{+ \delta x} \right\} = \left\{ \frac{\delta x}{\delta x} \right\} = 1$$

Note how we ***simplify first and then take the limit***. We can simplify first because δx only TENDS TO zero. $\delta x \neq 0$. ***There is no division by zero in this step.***

2. LIMIT step : $\underset{\delta x \to 0}{Limit} \{1\} = 1$

Since, $\underset{x \to 0^-}{Limit} \{ f(x) \} = 1 = \underset{x \to 0^+}{Limit} \{ f(x) \}$, we say $\underset{x \to 0}{Limit} \left\{\frac{\sin(x)}{x}\right\} = 1$.

$$\underset{x = 0}{Value} \{ f(x) \} = \underset{x = 0}{Value} \left\{\frac{\sin(x)}{x}\right\} = 0/0$$, which is something undefined.

Example 9 : We now present an example of a function f(x) where $\underset{x = a}{Value}$ f(x) exists everywhere (i.e. a can be any real number), but the $\underset{x \to a}{Limit}$ f(x) does NOT exist anywhere.

$$f(x) = \begin{cases} +1 \text{ if } x \text{ is } \textbf{\textit{rational}} \\ 0 \text{ if } x \text{ is } \textbf{\textit{irrational}} \end{cases}$$

Recall what we said about the rationals and irrationals. Between any two rationals there are infinitely many rationals. And between any two rationals there are also infinitely many irrationals. So we can see that as x TENDS TO a, the function f(x) will keep fluctuating. The function f(x) will be either 0 or 1. It will not take on a single specific value. Hence we can say that the $\underset{x \to a}{Limit}$ f(x) does NOT exist. Since a can be any point on the real number line, the *Limit* does NOT exist anywhere.

On first reading it is not necessary to study the Analysis of Limits. The reader may skip the next chapter without loss of continuity.

9. Analysis of Limits*

In general, we can talk about the LIMIT of a function f(x) as $x \to a$.

Does f(x) TEND TO any particular value b as x TENDS TO a ?

Can we choose f(x) ***as close as we like to*** b by choosing x ***sufficiently close*** to a ?

$$\text{Does } \underset{x \to a^-}{Limit} f(x) = b = \underset{x \to a^+}{Limit} f(x) \ ?$$

$$\text{Does } \underset{\delta x \to 0}{Limit} f(a-\delta x) = b = \underset{\delta x \to 0}{Limit} f(a+\delta x) \ ?$$

If it does, then we call this b the LIMIT of f(x) as x TENDS TO a and we write this as:

$$\underset{x \to a}{Limit} f(x) = b \quad \text{or} \quad \underset{\delta x \to 0}{Limit} f(a+\delta x) = b$$

We are not concerned with if $0 < |f(x) - b|$ or $0 = |f(x) - b|$

What we are concerned with is when $x \to a$, for any ***discrete infinitesimal*** ε as small as we like:

can we have $|f(x) - b| < \varepsilon$ or equivalently $|f(a+\delta x) - b| < \varepsilon$?

Again, we stress that $x \neq a$. f(x) at the ***instant*** $x = a$ as we said earlier, is

$$\underset{x = a}{Value} f(x) = f(a), \text{ if it exists.}$$

We may combine $x \to a^-$ and $x \to a^+$ into one expression : $|x - a| < \delta$.

Formally, we may define the LIMIT of a function f(x) as x TENDS TO a :

$\underset{x \to a}{Limit} f(x) = b$ if for any ***discrete infinitesimal*** ε as small as we like, we can find δ such that :

$$|f(x) - b| < \varepsilon \text{ when } |x - a| < \delta .$$

*Note : A more detailed and rigorous presentation is in A LITTLE MORE CALCULUS by the author.

Based on the formal definition of the ***"limit of a function"*** , we may analyse the behaviour of a function near some instant a, and see if it has a *Limit* or not. Let us look at the example 2 of the previous chapter.

Example 2: Consider the function $f(x) = x + a$

What is the *Limit* of f(x) as x TENDS TO a ? As $x \to a$ we have $f(x) \to 2a$
Since $0 < |x - a|$ we have $0 < |f(x) - 2a|$, i.e. $f(x) \neq 2a$.

We can choose f(x) ***as close as we like to*** 2a by choosing x ***sufficiently close*** to a. ***Sufficiently close*** means ***as close as necessary***.

If we want $0 < |f(x) - 2a| < \varepsilon$ some discrete infintesimal as small as we like, we can choose x ***sufficiently close*** to a. For example, we may choose :

$$0 < |x - a| < \delta \text{ , where } \delta = \frac{\varepsilon}{2}$$

Work it out:

$$0 < |x - a| < \delta \Rightarrow 0 < |x - a| < \frac{\varepsilon}{2} \Rightarrow a - \frac{\varepsilon}{2} < x < a + \frac{\varepsilon}{2}$$

$$f(x) = x + a = (a \pm \frac{\varepsilon}{2}) + a = 2a \pm \frac{\varepsilon}{2}$$

Now: $0 < |f(x) - 2a| = |(2a \pm \frac{\varepsilon}{2}) - 2a| = \frac{\varepsilon}{2}$ which is $< \varepsilon$

So when $|x - a| < \frac{\varepsilon}{2}$ we will have $|f(x) - 2a| < \varepsilon$

We can say: $\underset{x \to a}{Limit}\ f(x) = 2a$

Instead of saying: $\underset{x \to a}{Limit}\ f(x)$ which is $\underset{x \to a}{Limit}\ (x+a)$

we can use the infinitesimal δx and say:

$$\underset{x \to a^-}{Limit}(x+a) = \underset{\delta x \to 0}{Limit}((a-\delta x) + a) = 2a \qquad \text{(from the } \textbf{\textit{left}}\text{)}$$

and $$\underset{x \to a^+}{Limit}(x+a) = \underset{\delta x \to 0}{Limit}((a+\delta x) + a) = 2a \qquad \text{(from the } \textbf{\textit{right}}\text{)}$$

Let us now look at the example 9 of the previous chapter.

Example 9 :

$$f(x) = \begin{cases} +1 \text{ if } x \text{ is } \textbf{\textit{rational}} \\ 0 \text{ if } x \text{ is } \textbf{\textit{irrational}} \end{cases}$$

Suppose the $\underset{x \to a}{Limit}\ f(x) = b$. Let us try to choose an $\boldsymbol{\varepsilon}$ such that $|f(x) - b| > \boldsymbol{\varepsilon}$ for some x very close to a, that is to say for $0 < |x - a| < \delta$.

Recall what we said about the rationals and irrationals. Between any two rationals there are infinitely many ***rationals***. And between any two rationals there are also infinitely many ***irrationals***.

No matter *how close x is to a* there will always be infinitely many ***rational*** numbers and infinitely many ***irrational*** numbers. So we can see that as x TENDS TO a, the function f(x) will keep fluctuating : f(x) will be 0 or 1.

There are two possibilities : $b = 0$ or $b \neq 0$.

Case $b = 0$: choose $\boldsymbol{\varepsilon} = 1/2$

No matter *how close x is to a* , that is to say no matter how small the δ we choose, there will always be some ***rational*** numbers such that $0 < |x - a| < \delta$.

If x is rational then $|f(x) - b| = |1 - 0| = 1 > \boldsymbol{\varepsilon} = 1/2$.

So $b = 0$ cannot be the $\underset{x \to a}{Limit}\ f(x)$.

Case $b \neq 0$: choose $\boldsymbol{\varepsilon} = \frac{|1 - b|}{2}$

No matter *how close x is to a* , that is to say no matter how small the δ we choose, there will always be some ***rational*** numbers such that $0 < |x - a| < \delta$.

If x is rational then $|f(x) - b| = |1 - b| > \boldsymbol{\varepsilon} = \frac{|1 - b|}{2}$.

So $b \neq 0$ cannot be the $\underset{x \to a}{Limit}\ f(x)$.

Since *a* can be any point on the real number line, $\underset{x \to a}{Limit}\ f(x)$ does not exist anywhere.

10. VALUE versus LIMIT

So far when we took Limits of functions as $x \to a$ or $\delta x \to 0$ we ASSUMED that these Limits exist. But this is not so simple.

1. Does the Limit exist ?

2. Is the Limit from the right = Limit from the left ?

   i.e. is $\underset{x \to a^+}{Limit} f(x) = \underset{x \to a^-}{Limit} f(x)$

   $x > a$ right $x < a$ left

3. Is the $\underset{x = a}{Value} f(x) = \underset{x \to a}{Limit} f(x)$

In finding the *Limit* we first simplify and then put $x = a$ or $x = 0$ or whatever and evaluate. We can simplify because:

δx only TENDS TO 0, $\delta x \neq 0$,

x only TENDS TO a, $x \neq a$,

x only TENDS TO 0, $x \neq 0$.

Whereas in finding the *Value* we must directly put $x = a$ or $x = 0$ or whatever and evaluate.

Let us compare $\underset{x = a}{Value}$ f(x) and $\underset{x \to a}{Limit}$ f(x)

We have the following possibilities:

(i) **Value** and **Limit** both exist and are equal : $f(x) = x$

(ii) **Value** and **Limit** exist but are not equal.

$$\text{Let } f(x) = \begin{cases} 2 & \text{for } x = a \\ 1 & \text{for } x \neq a \end{cases}$$

$$\underset{x \to a^-}{Limit}\ f(x) = 1 = \underset{x \to a^+}{Limit}\ f(x),\quad \underset{x = a}{Value}\ f(x) = 2$$

Try drawing the graph of this function.

(iii) **Value** exists but **Limit** does not.

$$\text{Let } f(x) = \begin{cases} 1 - x & \text{for } x \leq 1 \\ 3 - x & \text{for } 1 < x \end{cases}$$

$$\underset{x \to 1^-}{Limit}\ f(x) = 0 \neq \underset{x \to 1^+}{Limit}\ f(x) = 2,\quad \underset{x = 1}{Value}\ f(x) = 0$$

Try drawing the graph of this function.

(iv) **Value** does not exist but **Limit** does.

$$\text{Let } f(x) = \frac{x^2 - a^2}{x - a},\quad \underset{x \to a}{Limit}\ f(x) = 2a,\quad \underset{x = a}{Value}\ f(x) = ?$$

$$\text{Let } f(x) = \frac{\sin x}{x},\quad \underset{x \to 0}{Limit}\ f(x) = 1,\quad \underset{x = 0}{Value}\ f(x) = ?$$

(v) **Value** and **Limit** do not exist.

$$\text{Let } f(x) = \frac{1}{x},\quad \underset{x \to 0}{Limit}\ f(x) = ?,\quad \underset{x = 0}{Value}\ f(x) = ?$$

11. CONTINUITY of a Function

Our common perception of ***continuous*** is so taken for granted that we seldom pause to give it a precise mathematical definition. In a string of beads, the string is ***continuous*** and the beads are ***discrete***. Yet, from ancient times philosopher-mathematicians were aware of the concept ***"continuous"*** and tried to define it.

In his ***Physik***, Aristotle (384-322 BC), Greek philosopher and student of Plato and tutor to Alexander the great, explained ***"continuous"*** as: ' *I say that something is continuous whenever the two extremities of their contiguous parts coincide, and as the name itself implies, they are kept together.* '

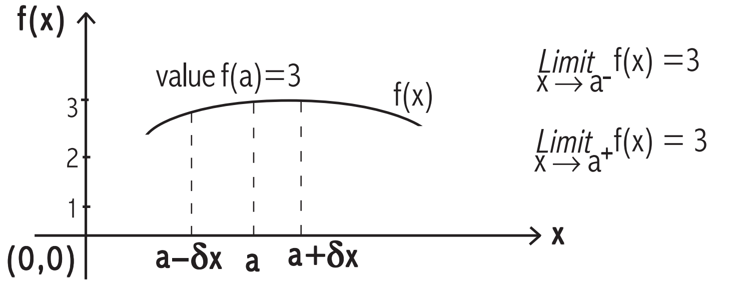
Gottfried Wilhelm von LEIBNIZ (1646-1716), co-inventor of Calculus and librarian and historian under Duke Johann Friedrich of Hanover, defined ***"continuous"*** as: '*The whole is said to be continuous, when any two component parts thereof (or more precisely any two parts which together make up the whole) have something in common, ... at the very least a common boundary.*'

It is interesting that Sir Isaac Newton did not have much to say on ***"continuous"***. More recently R. DEDEKIND (1872) defined ***"continuous"*** as: '*If the points of a line are divided into two classes, in such a way that each point of the first class lies to the left of every point of the second class, then there exists one and only one point of division which produces this particular sub-division into two classes, this cutting of the line into two parts.*'

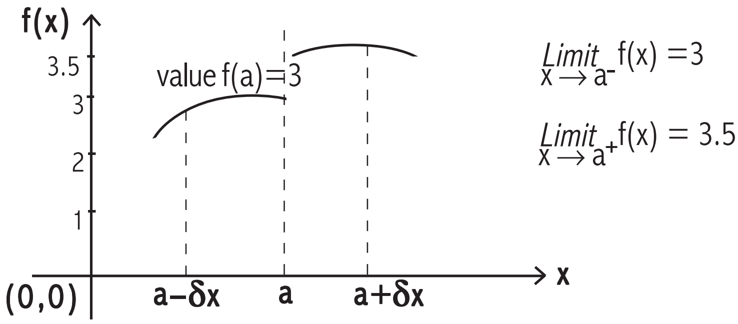
We saw the connection between the set of Real numbers being ***complete*** and the Real line being ***continuous***. The absence of a single point causes a cut or break or ***discontinuity*** in the Real line.

In Calculus we deal with ***continuous*** operands or functions. We now carry forward the concept of the Real line being ***continuous*** to functions over the Real line. We need to know the behaviour or ***value*** of a function at each and every ***point*** or ***instant***. Also, we need to know the behaviour of a function near any chosen instant $\boldsymbol{a}$: a little to the ***left*** of $\boldsymbol{a}$ or $(\boldsymbol{a} - \delta x)$ and a little to the ***right*** of $\boldsymbol{a}$ or $(\boldsymbol{a} + \delta x)$.

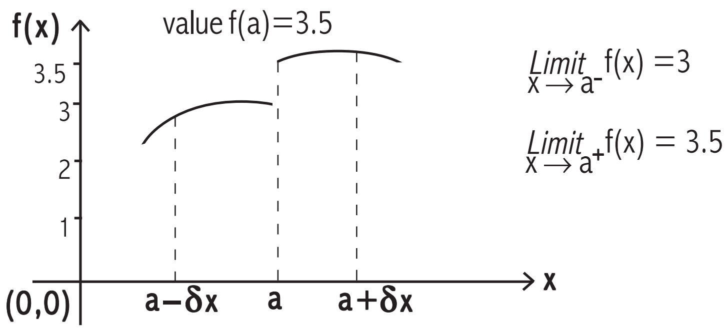
Geometrically when we say that a SINGLE VALUED function is CONTINUOUS around the point or **instant a**, we mean that the curve of the function is without breaks or gaps.



f(x) is CONTINUOUS at the point or instant a



f(x) is CONTINUOUS at instant a but only from the *left*



f(x) is CONTINUOUS at instant a but only from the *right*

Analytically: Assume $\underset{x = a}{\textit{Value}}\ f(x) = f(a)$ exists and is something definite. Then

1. if $\underset{x \to a^-}{\textit{Limit}}\ f(x)$ exists and is $= \underset{x = a}{\textit{Value}}\ f(x)$

   we say that f(x) is CONTINUOUS at the point x = a from the ***left***.

2. if $\underset{x \to a^+}{\textit{Limit}}\ f(x)$ exists and is $= \underset{x = a}{\textit{Value}}\ f(x)$

   we say that f(x) is CONTINUOUS at the point x = a from the ***right***.

> f(x) is said to be CONTINUOUS at the **point** or **instant** x = a
>
> if $\underset{x \to a^-}{\textit{Limit}}\ f(x) = f(a) = \underset{x \to a^+}{\textit{Limit}}\ f(x)$

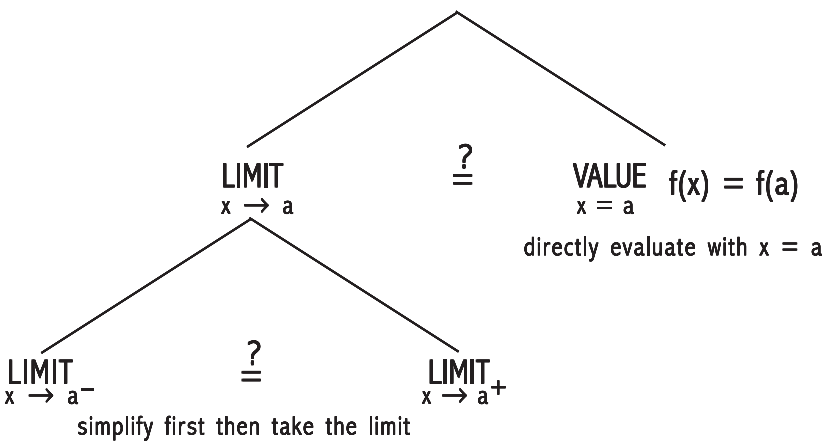
Using the infinitesimal δx we can write this as:

> f(x) is said to be CONTINUOUS at the **point** or **instant** x = a
>
> if $\underset{\delta x \to 0}{\textit{Limit}}\ f(a - \delta x) = f(a) = \underset{\delta x \to 0}{\textit{Limit}}\ f(a + \delta x)$

If f(x) is not CONTINUOUS at a we say f(x) is DISCONTINUOUS at a.

We may test the ***continuity of a function*** using the decision tree below.

f(x) CONTINUOUS at a ?



Are the ***polynomials*** CONTINUOUS everywhere ?

We can easily see that $f(x) = x^n$ for $n = 1, 2, 3, \ldots$

$$\underset{x \to a}{\mathit{Limit}}\ f(x) = \underset{x = a}{\mathit{Value}}\ f(x) = a^n$$

Since a can be any point on the real number line, these f(x) are ***continuous*** everywhere.

Likewise, $f(x) = a_n x^n$ for $n = 0, 1, 2, 3, \ldots$

where the coefficients a_n are real numbers, are also ***continuous*** everywhere.

Combining both we get the general polynomial f(x) :

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

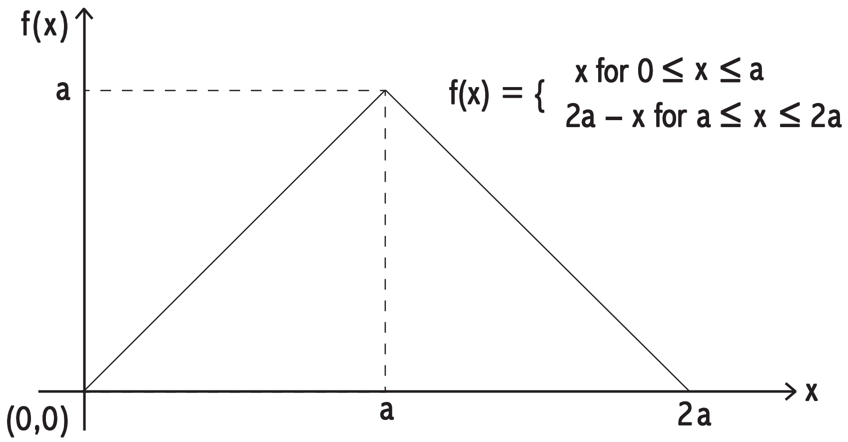
These are ***single valued*** and ***continuous*** everywhere.

$y(t) = u.\sin\theta.t - \frac{1}{2} g t^2$ is a polynomial of the form $a_0 + a_1 t + a_2 t^2$

with coefficients $a_0 = 0$, $a_1 = u.\sin\theta$ and $a_2 = -\frac{1}{2} g$. So the height function y(t) is ***single valued*** and ***continuous*** .

It is important to know that polynomials are to functions what rational numbers are to real numbers. We can approximate any real number to any required degree of accuracy by a suitable rational number. Likewise we can approximate almost any real value function by a suitable polynomial. For this we have Lagrange's method of interpolation.

Example 1: is f(x) ***continuous*** at $x = a$?



Approaching **a** from the ***left*** : $f(x) = x$

Near **a** and to the ***left*** of **a** : $x = a - \delta x$

$$\underset{x \to a^-}{Limit} f(x) = \underset{\delta x \to 0}{Limit} \{ a - \delta x\} = a$$

Approaching **a** from the ***right*** : $f(x) = 2a - x$

Near **a** and to the ***right*** of **a** : $x = a + \delta x$

$$\underset{x \to a^+}{Limit} f(x) = \underset{\delta x \to 0}{Limit} \{ 2a - (a + \delta x)\} = \underset{\delta x \to 0}{Limit} \{ 2a - a - \delta x \}$$

$$= \underset{\delta x \to 0}{Limit} \{ a - \delta x \} = a$$

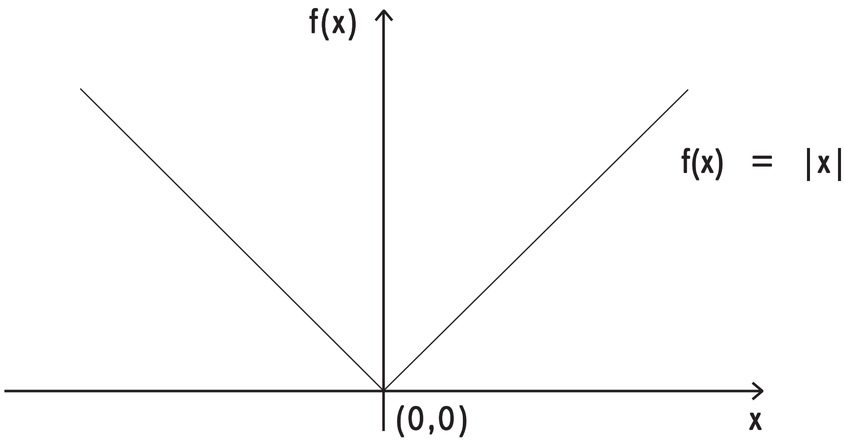
Hence, $\underset{x \to a^-}{Limit} f(x) = a = \underset{x \to a^+}{Limit} f(x)$

$$\underset{x = a}{Value} f(x) = f(a) = a$$

Since, $\underset{x \to a^-}{Limit} f(x) = \underset{x \to a^+}{Limit} f(x) = \underset{x = a}{Value} f(x) = a$

we say : f(x) is ***continuous*** at $x = a$.

Example 2 : is $f(x) = |x|$ ***continuous*** at $x = 0$?



Approaching **0** from the ***left*** : $f(x) = |x| = -x$

Near **0** and to the ***left*** of **0** : $x = 0 - \delta x$

$$\underset{x \to 0^-}{Limit}\ f(x) = \underset{\delta x \to 0}{Limit}\ \{-(0 - \delta x)\} = \underset{\delta x \to 0}{Limit}\ \{+\delta x\} = 0$$

Approaching **0** from the ***right*** : $f(x) = x$

Near **0** and to the ***right*** of **0** : $x = 0 + \delta x$

$$\underset{x \to 0^+}{Limit}\ f(x) = \underset{\delta x \to 0}{Limit}\ \{(0 + \delta x) = \underset{\delta x \to 0}{Limit}\ \{+\delta x\} = 0$$

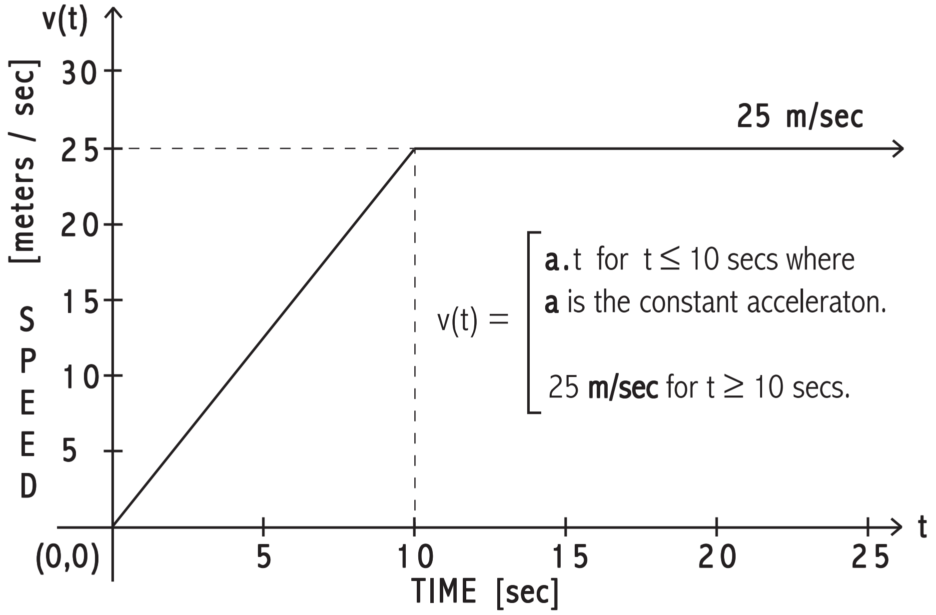
Hence, $\underset{x \to 0^-}{Limit}\ f(x) = 0 = \underset{x \to 0^+}{Limit}\ f(x)$

$$\underset{x = 0}{Value}\ f(x) = f(0) = 0$$

Since, $\underset{x \to 0^-}{Limit}\ f(x) = \underset{x \to 0^+}{Limit}\ f(x) = \underset{x = 0}{Value}\ f(x) = 0$

we say : $f(x)$ is ***continuous*** at $x = 0$.

Example 3: v(t) is the speed of a car weighing one ton that starts from rest (0 kmph) and accelerates steadily due East along the x-axis for 10 secs and then levels off at 90 kmph (25 m/sec). Is v(t) ***continuous*** at $t = 10$ secs ?



What acceleration are we going to use ? We still do not know how to ***differentiate*** the speed fuction v(t) to find the acceleration (***rate of change*** of speed) at ***any instant***. Since the acceleration is steady or constant we may use the ***average*** acceleration between any two instants , say $t = 0$ secs and $t = 10$ secs, where the speeds are known.

$$\mathbf{a} = \frac{v(t = 10\text{ sec}) - v(t = 0\text{ sec})}{10\text{ sec} - 0\text{ sec}} = \frac{25\text{ m/sec}}{10\text{ sec}} = 2.5\text{ m/sec}^2$$

Approaching **t = 10 secs** from the ***left*** : $v(t) = \mathbf{a}.t = 2.5\,t$ meter/sec^2

Near **t = 10 secs** and to the ***left*** of **t = 10 secs** : $t = 10 - \delta t$

$$\underset{t \to 10^-}{Limit}\ v(t) = \underset{\delta t \to 0}{Limit}\ \{\mathbf{a}\,(10 - \delta t)\} = \underset{\delta t \to 0}{Limit}\ \{10\mathbf{a} - 10\delta t\} = 10\mathbf{a} = 25\text{ m/sec}$$

Approaching **t = 10 secs** from the ***right***: v(t) = 25 meters/sec, constant.

Near **t = 10 secs** and to the ***right*** of **t = 10 secs** : t = 10 + δt

$$\underset{t \to 10^+}{Limit}\ v(t) = \underset{\delta t \to 0}{Limit}\ \{\ 25\ \text{meter/sec}\ \} = 25\ \text{m/sec}$$

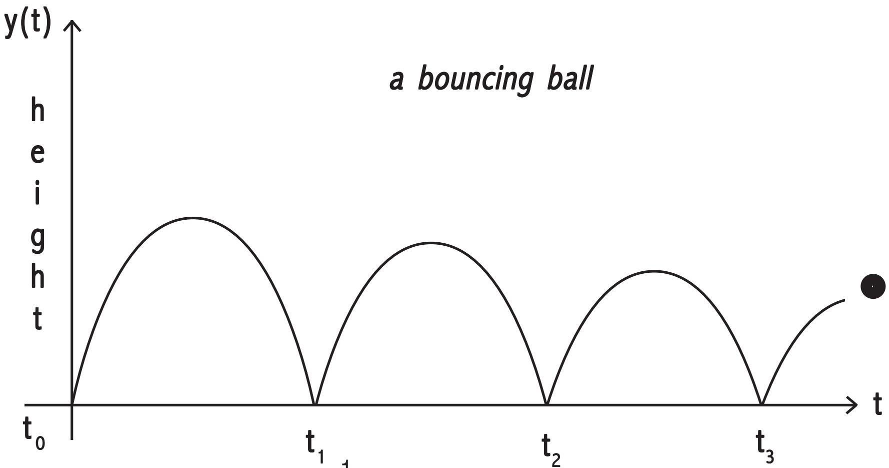
$$\text{Hence,}\quad \underset{t \to 10^-}{Limit}\ v(t) = 25\ \text{m/sec} = \underset{t \to 10^+}{Limit}\ v(t)$$

$$\underset{t\ =\ 10}{Value}\ v(t) = v(t = 10\ \text{secs}) = 25\ \text{meters / sec}$$

$$\text{Since,}\ \underset{t \to 10^-}{Limit}\ v(t) = \underset{t \to 10^+}{Limit}\ v(t) = \underset{t\ =\ 10}{Value}\ v(t) = 25\ \text{m/sec}$$

we say : v(t) is ***continuous*** at t = 10 secs.

Example 4: Is the function y(t), that describes the height of a bouncing ball, ***continuous*** at instants t_0, t_1, t_2, t_3, . . . ?



In general, $y(t) = u.\sin\theta.t - \frac{1}{2}g\,t^2$ where u is the initial velocity and θ is the angle of projection. In this example, over each interval $[t_0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$ and so on, we have different initial velocities u_0, u_1, u_2, u_3, . . . and angles of projection θ_0, θ_1, θ_2, θ_3, . . . respectively. So over each interval $[t_0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$ and so on, $y_i(t) = u_i.\sin\theta_i.t - \frac{1}{2}g\,t^2$ for i = 0, 1, 2, 3, . . . respectively.

Let us take the general equation $y(t) = u.\sin\theta.t - \frac{1}{2}gt^2$ that describes the height of a projectile and adapt it to each interval $[t_0,\ t_1]$, $[t_1,\ t_2]$, $[t_2,\ t_3]$.

Over $[t_0,\ t_1]$ the height function y(t) is : $y_0(t) = u_0.\sin\theta_0.(t - t_0) - \frac{1}{2}g(t - t_0)^2$.
Over $[t_1,\ t_2]$ the height function y(t) is : $y_1(t) = u_1.\sin\theta_1.(t - t_1) - \frac{1}{2}g(t - t_1)^2$.
Over $[t_2,\ t_3]$ the height function y(t) is : $y_2(t) = u_2.\sin\theta_2.(t - t_2) - \frac{1}{2}g(t - t_2)^2$
and so on. Let us first look at the behaviour around $t = t_0$.

Approaching t_0 from the ***left*** : y(t) is not defined. We do not know the position or height of the ball. Hence y(t) is ***not continuous*** from the ***left***. We say that y(t) is ***discontinuous*** at $t = t_0$.

We may remove this ***discontinuity*** by defining the height of the ball when at rest on the ground to be zero. In this case $y(t) = 0$ for $t \leq t_0$. So, approaching t_0 from the ***left*** $y(t) = 0$.

Now, approaching t_0 from the ***right*** : $y_0(t) = u_0.\sin\theta_0.(t - t_0) - \frac{1}{2}g(t - t_0)^2$.
Near t_0 and to the ***right*** of t_0 : $t = t_0 + \delta t$. So $(t - t_0) = \delta t$.

$$\underset{t \to t_0^+}{Limit}\ y_0(t) = \underset{\delta t \to 0}{Limit}\ y_0(t_0 + \delta t) = \underset{\delta t \to 0}{Limit}\ \{ u_0.\sin\theta_0.(\delta t) - \frac{1}{2}g(\delta t)^2 \} = 0$$

$$\text{Hence, } \underset{t \to t_0^-}{Limit}\ y_0(t) = 0 = \underset{t \to t_0^+}{Limit}\ y_0(t)$$

$$\underset{t = t_0}{Value}\ y_0(t) = y_0(t_0) = 0$$

$$\text{Since, } \underset{t \to t_0^-}{Limit}\ y_0(t) = \underset{t \to t_0^+}{Limit}\ y_0(t) = \underset{t = t_0}{Value}\ y_0(t) = 0$$

we say : $y_0(t)$ is ***continuous*** at $t = t_0$.

Let us now look at the behaviour around $t = t_1$.

Approaching t_1 from the ***left*** y(t) is : $y_0(t) = u_0.\sin\theta_0.(t - t_0) - \frac{1}{2}g(t - t_0)^2$.

Near t_1 and to the ***left*** of t_1 : $t = t_1 - \delta t$.

So, $y_0(t) = u_0.\sin\theta_0.(t_1 - \delta t - t_0) - \frac{1}{2}g(t_1 - \delta t - t_0)^2$.

$$\underset{t \to t_1^-}{Limit}\ y_0(t) = \underset{\delta t \to 0}{Limit}\ y_0(t) = u_0.\sin\theta_0.(t_1 - t_0) - \frac{1}{2}g(t_1 - t_0)^2 \text{ --- } \mathbf{(1)} \ .$$

We know from the projectile equation for $t_0 = 0$ we have $t_1 = 2u_0.\sin\theta_0 / g$.

Substituting $t_0 = 0$ and $t_1 = 2u_0.\sin\theta_0 / g$ in equation (1) above, we get :

$$\underset{t \to t_1^-}{Limit}\ y(t) = \underset{t \to t_1^-}{Limit}\ y_0(t) = \underset{\delta t \to 0}{Limit}\ y_0(t) = 0 \text{ --- } \mathbf{(2)} \ .$$

Approaching t_1 from the ***right*** y(t) is : $y_1(t) = u_1.\sin\theta_1.(t - t_1) - \frac{1}{2}g(t - t_1)^2$.

Near t_1 and to the ***right*** of t_1 : $t = t_1 + \delta t$.

So, $y_1(t) = u_1.\sin\theta_1.(t_1 + \delta t - t_1) - \frac{1}{2}g(t_1 + \delta t - t_1)^2 = u_1.\sin\theta_1.(\delta t) - \frac{1}{2}g(\delta t)^2$

$$\underset{t \to t_1^+}{Limit}\ y(t) = \underset{t \to t_1^+}{Limit}\ y_1(t) = \underset{\delta t \to 0}{Limit}\ \{u_1.\sin\theta_1.(\delta t) - \frac{1}{2}g(\delta t)^2\} = 0 \ .$$

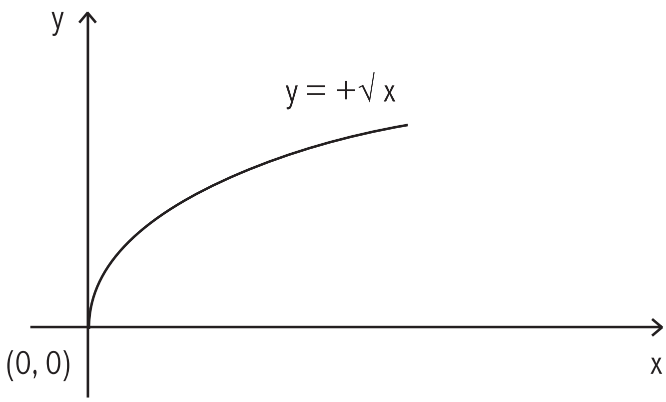
Hence, $\underset{t \to t_1^-}{Limit}\ y(t) = 0 = \underset{t \to t_1^+}{Limit}\ y(t)$

$$\underset{t = t_1}{Value}\ y(t) = 0$$

Since, $\underset{t \to t_1^-}{Limit}\ y(t) = \underset{t \to t_1^+}{Limit}\ y(t) = \underset{t = t_1}{Value}\ y(t) = 0$

we say : y(t) is ***continuous*** at $t = t_1$.

Example 5 : is $f(x) = +\sqrt{x}$ ***continuous*** at $x = 0$?



Near **0** and to the ***left*** of **0** : $x = 0 - \delta x$

$$\underset{x \to 0^-}{Limit} \; f(x) = \underset{\delta x \to 0}{Limit} \; \sqrt{(0 - \delta x)} = \underset{\delta x \to 0}{Limit} \; \sqrt{-\delta x}$$

A negative real number under the radical sign is a complex number. We are dealing with ***real*** numbers and ***real*** valued functions. Here $\sqrt{-\delta x}$ is not ***real*** valued and so it is not defined. $f(x) = +\sqrt{x}$ is ***not defined*** for $x < 0$.

$$\underset{x \to 0^+}{Limit} \; f(x) = \underset{\delta x \to 0}{Limit} \; \sqrt{(0 + \delta x)} = \underset{\delta x \to 0}{Limit} \; \sqrt{+\delta x} = 0$$

Since, $\underset{x \to 0^-}{Limit} f(x) \neq \underset{x \to 0^+}{Limit} f(x)$, we say $\underset{x \to 0}{Limit} f(x)$ does not exist.

$$\underset{x = 0}{Value} \; f(x) = f(0) = \sqrt{0} = 0$$

Since $\underset{x \to 0}{Limit} f(x)$ does not exist we say : $f(x)$ is ***discontinuous*** at $x = 0$.

We may remove this discontinuity by redefining f(x) to be : $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \sqrt{x} & \text{for } x \geq 0 \end{cases}$

Example 6: Is the function $f(x) = 1/x$ ***continuous*** at $x = 0$?

Limit from the LEFT

$$\underset{x \to 0^-}{Limit} \{ f(x) \} = \underset{x \to 0^-}{Limit} \{ 1/x \} = -1/0 = -\infty$$, something undefined.

$$\underset{\delta x \to 0}{Limit} \{ f(x) \} = \underset{\delta x \to 0}{Limit} \{f(0 - \delta x)\} = \underset{\delta x \to 0}{Limit} \{ 1/(0 - \delta x) \} = -1/0 = -\infty$$

Limit from the RIGHT

$$\underset{x \to 0^+}{Limit} \{ f(x) \} = \underset{x \to 0^+}{Limit} \{ 1/x \} = +1/0 = +\infty$$, something undefined.

$$\underset{\delta x \to 0}{Limit} \{ f(x) \} = \underset{\delta x \to 0}{Limit} \{f(0 + \delta x)\} = \underset{\delta x \to 0}{Limit} \{ 1/(0 + \delta x) \} = +1/0 = +\infty$$

The LIMIT from the ***left*** does not exist. And, the LIMIT from the ***right*** does not exist.

Also $\underset{x = 0}{Value} \{ f(x) \} = f(0) = 1/0 = \infty$, something undefined.

So on any one of the three counts, be it LIMIT from the ***left*** or LIMIT from the ***right*** or the VALUE, f(x) is ***discontinuous*** at x = 0. Moreover, we cannot remove this discontinuity.

Example 7: Is the function $f(x) = \frac{\sin(x)}{x}$ ***continuous*** at $x = 0$?

We shall use the trigonometric identity $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.
Also, when x is very small, we may say : $\sin(x) = x$.

Limit from the LEFT

$$\underset{\delta x \to 0}{\mathit{Limit}} \{f(x)\} = \underset{\delta x \to 0}{\mathit{Limit}} \left\{ \frac{\sin(0 - \delta x)}{(0 - \delta x)} \right\} = \underset{\delta x \to 0}{\mathit{Limit}} \left\{ \frac{\sin(-\delta x)}{-\delta x} \right\}$$

$$= \underset{\delta x \to 0}{\mathit{Limit}} \left\{ \frac{-\delta x}{-\delta x} \right\} = \underset{\delta x \to 0}{\mathit{Limit}} \{1\} = 1 \ .$$

Note how we ***simplify first and then take the limit***. We can simplify first because δx only TENDS TO zero. $\delta x \neq 0$.

Limit from the RIGHT

$$\underset{\delta x \to 0}{\mathit{Limit}} \{f(x)\} = \underset{\delta x \to 0}{\mathit{Limit}} \left\{ \frac{\sin(0 + \delta x)}{(0 + \delta x)} \right\} = \underset{\delta x \to 0}{\mathit{Limit}} \left\{ \frac{\sin(+\delta x)}{+\delta x} \right\}$$

$$= \underset{\delta x \to 0}{\mathit{Limit}} \left\{ \frac{\delta x}{\delta x} \right\} = \underset{\delta x \to 0}{\mathit{Limit}} \{1\} = 1 \ .$$

Note how we ***simplify first and then take the limit***. We can simplify first because δx only TENDS TO zero. $\delta x \neq 0$.

Since, $\underset{x \to 0^-}{\mathit{Limit}} \{ f(x) \} = 1 = \underset{x \to 0^+}{\mathit{Limit}} \{ f(x) \}$, we say $\underset{x \to 0}{\mathit{Limit}} \left\{ \frac{\sin(x)}{x} \right\} = 1$.

$$\underset{x = 0}{\mathit{Value}} \{ f(x) \} = \underset{x = 0}{\mathit{Value}} \left\{ \frac{\sin(x)}{x} \right\} = {}^{0}/_{0}$$, which is something undefined.

Since $\underset{x \to 0}{\mathit{Limit}} f(x) \neq \underset{x = 0}{\mathit{Value}} f(x)$ we say $f(x)$ is ***discontinuous*** at $x = 0$.

We may remove this discontinuity by redefining $f(x)$ to be : $f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$

Example 8 : Is the function f(x) defined below ***continuous*** anywhere ?

$$f(x) = \begin{cases} +1 \text{ if } x \text{ is } \textbf{\textit{rational}} \\ 0 \text{ if } x \text{ is } \textbf{\textit{irrational}} \end{cases}$$

We saw that the $\underset{x \to a}{\mathit{Limit}}$ f(x) does NOT exist anywhere.

However, $\underset{x = a}{\mathit{Value}}$ f(x) is well-defined everywhere.

Since $\underset{x \to a}{\mathit{Limit}}\ f(x) \neq \underset{x = a}{\mathit{Value}}\ f(x)$ we say that f(x) is ***not continuous*** at $x = a$.

But a can be any point on the Real number line. So f(x) is ***not continuous*** anywhere.

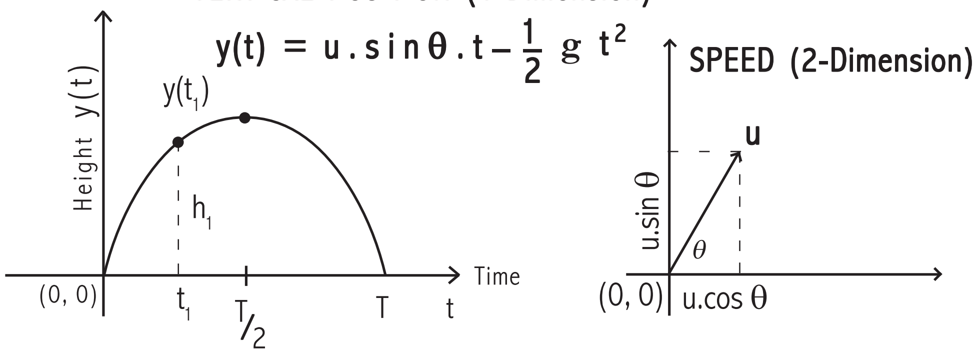
Part 2 : DIFFERENTIATION

Overview

Let us continue with our main example.

VERTICAL POSITION (1-Dimension)

$$y(t) = u . \sin\theta . t - \frac{1}{2} g\, t^2$$



When an object is projected into the air with a given initial velocity **u** and angle of projection or direction θ, the curve that describes its path is a parabola. There are two functions that describe the two components of its position. A VERTICAL function y(t) which describes its HEIGHT at any chosen INSTANT and a HORIZONTAL function x(t) which describes its RANGE at any chosen INSTANT (refer Preface iii).

We know from Dynamics that:

RANGE function $x(t) = u . \cos\theta . t$

HEIGHT function $y(t) = u . \sin\theta . t - \frac{1}{2} g\, t^2$

Let us concentrate on the VERTICAL function y(t). Let t_1 the time for the object to travel from $t = 0$ to $y(t_1)$. Let h_1 be the height or vertical displacement described in time t_1.

$$y(t_1) = u . \sin\theta . t_1 - \frac{1}{2} g\, t_1^{\,2}$$

This is a polynomial of the form: $f(x) = a_2 x^2 + a_1 x + a_0$ which is CONTINUOUS everywhere.

Now that we know the TIME AXIS is a CONTINUOUS set of INSTANTS and y(t) is a SINGLE VALUED and CONTINUOUS function over [A , B], we may proceed to describe the DIFFERENTIATION operation and perform the calculation.

Before we do any calulation (***differentiation***) we will develop the concept and definition of the FIRST DERIVATIVE or the INSTANTANEOUS RATE OF CHANGE.

Using our main example we will illustrate the concept to find the INSTANTANEOUS RATE OF CHANGE of the ***particular*** height function y(t) at some ***particular instant*** t_1.

$$\left(\frac{dy}{dt}\right)_{t_1} = \underset{t_2 \to t_1}{\text{LIMIT}} \frac{\delta y}{\delta t} = \underset{\delta t \to 0}{\text{LIMIT}} \frac{\delta y}{\delta t} = \underset{\delta t \to 0}{\text{LIMIT}} \frac{y(t_1 + \delta t) - y(t_1)}{(t_1 + \delta t) - t_1}$$

By dropping the subscript of t_1 , we extend this concept of the INSTANTANEOUS RATE OF CHANGE of the ***particular function*** y(t) at ***particular instant*** t_1 to any ***general instant*** t on the time axis.

$$\frac{dy}{dt} = \underset{\delta t \to 0}{\text{LIMIT}} \frac{\delta y}{\delta t} = \underset{\delta t \to 0}{\text{LIMIT}} \frac{y(t + \delta t) - y(t)}{(t + \delta t) - t}$$

With the concept now clear, we define the INSTANTANEOUS RATE OF CHANGE of some ***general function*** f(x) at some ***particular instant*** x_1 .

$$\left(\frac{df}{dx}\right)_{x_1} = \underset{x_2 \to x_1}{\text{LIMIT}} \frac{\delta f}{\delta x} = \underset{\delta x \to 0}{\text{LIMIT}} \frac{\delta f}{\delta x} = \underset{\delta x \to 0}{\text{LIMIT}} \frac{f(x_1 + \delta x) - f(x_1)}{(x_1 + \delta x) - x_1}$$

Again, dropping the subscript of x_1 , we extend this concept of the INSTANTANEOUS RATE OF CHANGE of the ***general function*** f(x) at ***particular instant*** x_1 to any ***general instant*** x on the x-axis.

$$\frac{df}{dx} = \underset{\delta x \to 0}{\text{LIMIT}} \frac{\delta f}{\delta x} = \underset{\delta x \to 0}{\text{LIMIT}} \frac{f(x + \delta x) - f(x)}{(x + \delta x) - x}$$

Here too we have to take the LIMIT both from the ***left*** and from the ***right*** . Even with SINGLE VALUED and CONTINUOUS functions, it is quite possible when taking the limits to end up with something undefined such as $+\infty$, $-\infty$, division by zero and ∞/∞ , or the ***left*** LIMIT $\neq$ ***right*** LIMIT. In this case we say the function f(x) is not DIFFERENTIABLE at that instant.

With the definition of the FIRST DERIVATIVE in mind we proceed to ***differentiate*** the general polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$.

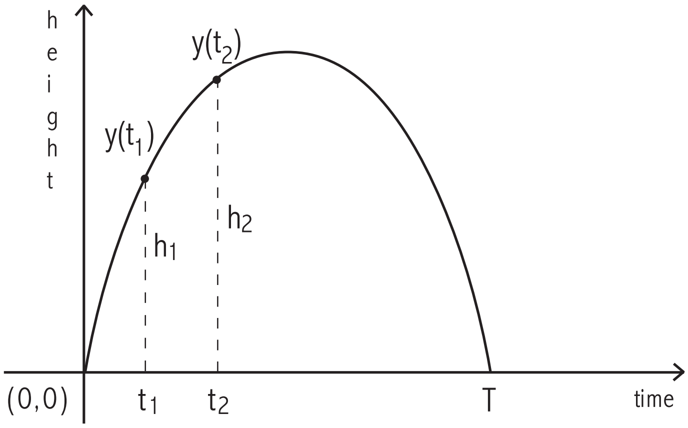
In fact we need to know only how to ***differentiate*** x^n , the general variable term of the general polynomial. We shall do this two ways : the Vedic way and the Western way. Once we know how to ***differentiate*** x^n , we may differentiate any polynomial. Also, we will be able to ***differentiate*** WELL-BEHAVED functions. Finding the DERIVATIVE by directly applying the definition is known as ***differentiation from first principles***.

It is not necessary to always ***differentiate*** from FIRST PRINCIPLES. We may differentiate each of the various functions just once and build a ***Table of Derivatives***. We then look at more complicated functions made up from elementary functions. To differentiate these combinations of other functions we have a ***Set of Rules*** .

We also look at the ***Units of Measure*** in ***differentiation*** and ***integration***. A calculation usually has two parts: an ***operation*** and a ***units of measure***. The operation with a ***function*** or ***expression*** as an input operand produces as output another ***function*** or ***expression***. When we evaluate this output expression we get a ***value***. The output ***expression*** or ***value*** has a ***units of measure*** associated with it. For example, we may perform an operation to find the expression πr^2 that tells us the area of a circle. When we evaluate this expression πr^2, say with $r = 2$, we get the value 4π. The expression πr^2 or the value 4π has ***units of measure*** **[meter^2]** associated with it.

We assume we are dealing with WELL-BEHAVED functions : SINGLE VALUED, CONTINUOUS and DIFFERENTIABLE. We have seen the first two properties (SINGLE VALUED and CONTINUOUS) and why they are necessary. We conclude this part by looking at the DIFFERENTIABLE property.

12. INSTANTANEOUS RATE OF CHANGE of y(t)



If we measure the height at any chosen instant, say t_1, then in terms of the height function $y(t) = u.\sin\theta.t - \frac{1}{2}gt^2$ the AVERAGE SPEED over time $t = 0$ to t_1 is :

$$\frac{\textit{distance}}{\textit{time}} = \frac{y(t)}{t} = u.\sin\theta - \frac{1}{2}g\,t$$

We may ask: what is the actual vertical speed at the time ***instant*** t_1 ?
We shall do this in three steps.

1. AVERAGE step: If we take a particular small time interval, say t_1 to t_2 , we could be more precise. AVERAGE VERTICAL SPEED from time t_1 to t_2 is :

$$\frac{\textit{CHANGE in height}}{\textit{CHANGE in time}} = \frac{\Delta y}{\Delta t} = \frac{y(t_2) - y(t_1)}{t_2 - t_1}$$

2. TENDS TO step: the smaller the time interval $t_2 - t_1$ the more accurate the calculation of the AVERAGE VERTICAL SPEED at time instant t_1. So we fix t_1 and let t_2 get closer and closer to t_1. The difference Δt between t_1 and t_2 becomes smaller and smaller. It becomes infinitely small. This kind of difference we denote using the Greek symbol δ. As $t_2 \to t_1$ we may say : $t_2 = t_1 + \delta t$.

So : $\qquad \delta t = t_2 - t_1 = (t_1 + \delta t) - t_1$

and $\qquad \delta y = y(t_2) - y(t_1) = y(t_1 + \delta t) - y(t_1)$

Hence, $\qquad \dfrac{\delta y}{\delta t} = \dfrac{y(t_1 + \delta t) - y(t_1)}{(t_1 + \delta t) - t_1}$

3. LIMIT step: Finally, when we let t_2 coincide with t_1, we get the INSTANTANEOUS VERTICAL SPEED at ***instant*** t_1. This we denote by :

$$\frac{dy}{dt} = \underset{t_2 \to t_1}{\text{LIMIT}} \frac{\delta y}{\delta t} = \underset{\delta t \to 0}{\text{LIMIT}} \frac{\delta y}{\delta t} = \underset{\delta t \to 0}{\text{LIMIT}} \frac{y(t_1 + \delta t) - y(t_1)}{(t_1 + \delta t) - t_1}$$

This vertical speed or ***"rate of change"*** in height got by taking the LIMIT is called the INSTANTANEOUS SPEED or INSTANTANEOUS RATE OF CHANGE in height at the time instant t_1 . We can now answer Question 2 on page 3.

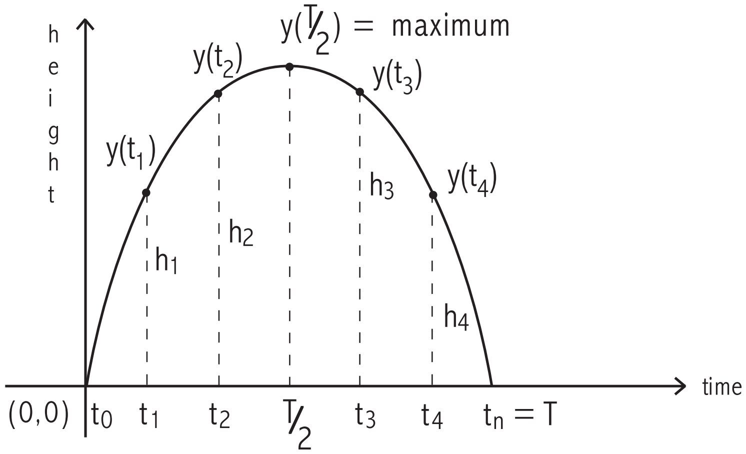
Limit of AVERAGE RATE OF CHANGE of $y(t)$ over the interval $[t_1, t_2]$ as t_2 TENDS TO t_1 = INSTANTANEOUS RATE OF CHANGE of $y(t)$ at the instant t_1

This is called the FIRST DERIVATIVE of $y(t)$ at ***instant*** t_1. There are several other notations for the FIRST DERIVATIVE :

$$\left(\frac{dy}{dt}\right)_{t_1}, \quad y'(t_1), \quad Dy(t_1), \quad \dot{y}(t_1)$$

We know from observation that the INSTANTANEOUS VERTICAL SPEED or INSTANTANEOUS RATE OF CHANGE in height is not the same at different instants t_1, t_2, t_3 and t_4 in time.

Using this same method of taking the $\underset{\delta t \to 0}{\mathit{Limit}}$ we can find the INSTANTANEOUS RATE OF CHANGE in height at any ***particular instant*** t_2 or t_3 or t_4, in the time interval t_0 to t_n.



Can we find the INSTANTANEOUS RATE OF CHANGE in height at ***any instant*** t in the time interval t_0 to t_n, rather than at ***particular instants*** t_1, t_2, t_3 or t_4 ? We may drop the subscript and let t be a ***general instant*** in $[t_0 , t_n]$.

$$\frac{dy}{dt}, \quad y'(t), \quad Dy(t), \quad \dot{y}(t)$$

So far we have only a NOTATION, some symbols that express the concept of what we are trying to do: find the INSTANTANEOUS RATE OF CHANGE of the particular function y(t) AT ANY INSTANT.

When we perform the calculation to find the INSTANTANEOUS RATE OF CHANGE of y(t) at any instant we get (refer Preface iii) :

$$\text{INSTANTANEOUS VERTICAL SPEED} = \frac{dy}{dt} = u\,.\sin\,\theta - g\,t$$

Compare this expression of INSTANTANEOUS VERTICAL SPEED to the earlier calculation of AVERAGE VERTICAL SPEED:

$$\text{AVERAGE VERTICAL SPEED} = \frac{y(t)}{t} = u\,.\sin\,\theta - \frac{1}{2}\,g\,t$$

When we compare the AVERAGE SPEED with the INSTANTANEOUS SPEED we make the profound distinction between an INTERVAL (no matter how small) and an INSTANT. Only with this distinction and the calculation using INSTANT are we able to get the INSTANTANEOUS SPEED.

δt is an INTERVAL. $\underset{\delta t \to 0}{\mathit{Limit}}\ (a + \delta t)$ is the INSTANT a.

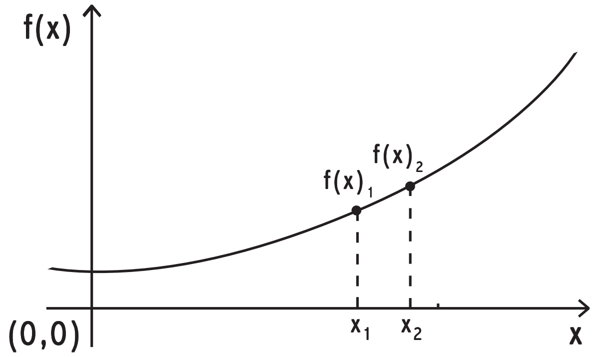
δx is an INTERVAL. $\underset{\delta x \to 0}{\mathit{Limit}}\ (x + \delta x)$ is any INSTANT x.

From the ***Analysis*** point of view : an ***interval*** δt or δx, no matter how small, is a ***continuous*** set of ***instants*** or ***points***.

From the ***Algebra*** point of view : an ***interval*** corresponds to a subset of the real numbers $\mathcal{R}$. This subset has UNCOUNTABLY many real numbers (both rational and irrational numbers) in a ***contiguous*** manner, ie; with none missing in between. Since we cannot even begin to COUNT them, there is no sense in talking about a subscript to enumerate them.

We shall use the words DIFFERENTIATE or FIND THE DERIVATIVE to denote the process of finding the expression for the INSTANTANEOUS RATE OF CHANGE of a function. Let us now use this concept to define the INSTANTANEOUS RATE OF CHANGE of a general function f(x).

13. INSTANTANEOUS RATE OF CHANGE of f(x)



What is the ***instantaneous rate of change*** of f(x) at particular instant x_1 ?
We shall define this in three steps.

1. AVERAGE step: If we take a particular small interval, say x_1 to x_2 , we could be more precise. AVERAGE RATE OF CHANGE from instant x_1 to x_2 is :

$$\frac{\textit{CHANGE in function}}{\textit{CHANGE in variable}} = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

2. TENDS TO step: the smaller the interval $x_2 - x_1$ the more accurate the calculation of the AVERAGE RATE OF CHANGE at ***particular instant*** x_1. So we fix x_1 and let x_2 get closer and closer to x_1. The difference Δx between x_1 and x_2 becomes smaller and smaller. It becomes infinitely small. This kind of difference we denote using the Greek symbol δ. As $x_2 \to x_1$ we may say : $x_2 = x_1 + \delta x$.

So : $\delta x = x_2 - x_1 = (x_1 + \delta x) - x_1$

and $\delta f = f(x_2) - f(x_1) = f(x_1 + \delta x) - f(x_1)$

Hence, $$\frac{\delta f}{\delta x} = \frac{f(x_1 + \delta x) - f(x_1)}{(x_1 + \delta x) - x_1}$$

3. LIMIT step: Finally, when we let x_2 coincide with x_1, we get the INSTANTANEOUS RATE OF CHANGE of f(x) at ***particular instant*** x_1. This we denote by :

$$\frac{df}{dx} = \underset{x_2 \to x_1}{\text{LIMIT}} \frac{\delta f}{\delta x} = \underset{\delta x \to 0}{\text{LIMIT}} \frac{\delta f}{\delta x}$$

This ***"rate of change"*** got by taking the LIMIT is called the INSTANTANEOUS SPEED or INSTANTANEOUS RATE OF CHANGE at the particular instant x_1 .

Limit of AVERAGE RATE OF CHANGE of f(x) over the interval $[x_1, x_2]$ as x_2 TENDS TO x_1 = INSTANTANEOUS RATE OF CHANGE of f(x) at the instant x_1

$$\underset{\delta x \to 0}{\mathit{Limit}} \frac{f(x_1 + \delta x) - f(x_1)}{(x_1 + \delta x) - x_1} = \underset{\delta x \to 0}{\mathit{Limit}} \frac{f(x_1 + \delta x) - f(x_1)}{\delta x}$$

This is denoted by $f'(x_1)$.

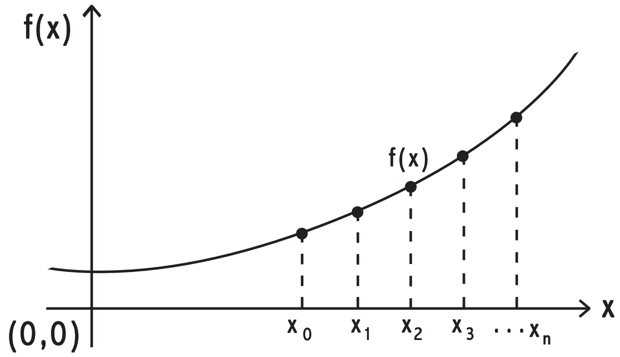
$$\boxed{f'(x_1) = \underset{\delta x \to 0}{\mathit{Limit}} \frac{f(x_1 + \delta x) - f(x_1)}{\delta x}}$$

This is called the FIRST DERIVATIVE of $f(x_1)$. There are several other notations for the FIRST DERIVATIVE :

$$\left(\frac{df}{dx}\right)_{x_1}, \quad f'(x_1), \quad Df(x_1), \quad \dot{f}(x_1)$$

Now, rather than choosing the ***particular instant*** x_1, we want the INSTANTANEOUS RATE OF CHANGE of the function f(x) at any ***general instant*** x .

Using this same method of taking the $\underset{\delta t \to 0}{Limit}$ we can find the INSTANTANEOUS RATE OF CHANGE at any ***particular instant*** x_2 or x_3 or x_4, in the time interval x_0 to x_n. We may drop the subscript and let x be a ***general instant*** in $[x_0 , x_n]$.



$$\underset{\delta x \to 0}{Limit} \frac{f(x + \delta x) - f(x)}{(x + \delta x) - x} = \underset{\delta x \to 0}{Limit} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Definition : instantaneous rate of change of f(x)

$$\boxed{f'(x) = \underset{\delta x \to 0}{Limit} \frac{f(x + \delta x) - f(x)}{\delta x}}$$

This is called the FIRST DERIVATIVE of $f(x_1)$. There are several other notations for the FIRST DERIVATIVE :

$$\frac{df}{dx}, \quad f'(x), \quad Df(x), \quad \dot{f}(x)$$

So far we have only a NOTATION, some symbols that express the idea of what we are trying to do: find an expression for the INSTANTANEOUS RATE OF CHANGE AT ANY INSTANT.

We shall use the words DIFFERENTIATE or FIND THE DERIVATIVE to denote the process of finding the expression for the INSTANTANEOUS RATE OF CHANGE of a function.

Properly speaking we should say :

$$\text{LEFT DERIVATIVE of } f(x) = \underset{\delta x \to 0}{Limit} \; \frac{f(x - \delta x) - f(x)}{-\delta x}$$

$$\text{RIGHT DERIVATIVE of } f(x) = \underset{\delta x \to 0}{Limit} \; \frac{f(x + \delta x) - f(x)}{\delta x}$$

If the LEFT DERIVATIVE = the RIGHT DERIVATIVE and is something ***well-defined***, then and only then we may combine both the LEFT DERIVATIVE and the RIGHT DERIVATIVE into one expression and say :

$$\text{DEFINITION :} \quad \boxed{f'(x) = \underset{\delta x \to 0}{Limit} \; \frac{f(x + \delta x) - f(x)}{\delta x}}$$

Even with SINGLE VALUED and CONTINUOUS functions, it is quite possible when taking the LIMITS to find the derivatives, we may end up with something undefined such as $+\infty$, $-\infty$, division by zero and ∞/∞ , or the LEFT DERIVATIVE $\neq$ RIGHT DERIVATIVE. In this case we say the function f(x) is not DIFFERENTIABLE at that instant.

Note : In A LITTLE MORE CALCULUS the student will learn the MEAN VALUE THEOREM which states that :

INSTANTANEOUS RATE OF CHANGE at some MEAN point (we do not know which) denoted by $\bar{x}$ in the interval [a,b]	=	AVERAGE RATE OF CHANGE over the interval [a,b]

$$f'(\bar{x}) = \frac{f(b) - f(a)}{b - a}$$

14. INSTANTANEOUS RATE OF CHANGE of x^n

Let us consider how to derive the general expression for INSTANTANEOUS RATE OF CHANGE of the general polynomial :

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

Observe that the terms of this function are of the form $a_n x^n$

for $n = 0, 1, 2, 3, \ldots$

To DIFFERENTIATE the general polynomial we should be able to find the INSTANTANEOUS RATE OF CHANGE of the general variable term x^n . We can think of x^n as the function x^n. In other words, we should be able to DIFFERENTIATE the function $f(x) = x^n$.

As usual we will first find the INSTANTANEOUS RATE OF CHANGE of the function $f(x) = x^n$ at ***particular point*** **a** . Then we will find the general expression of the INSTANTANEOUS RATE OF CHANGE $f(x) = x^n$ at ***any point*** x.

We note that :

$$\underset{x=a}{\mathit{Value}}\ f(x) = a^n$$

We can differentiate $f(x) = x^n$ at $x = a$ in two ways :

1. The Vedic way - without the use of δx .
2. The Western way - using the infinitesimal δx .

$$\textbf{Vedic way : } f'(a) = \underset{x \to a}{\textbf{Limit}} \; \frac{x^n - a^n}{x - a}$$

$$\text{AVERAGE RATE OF CHANGE close to a .} = \frac{\text{CHANGE in function}}{\text{CHANGE in variable}}$$

$$\text{So : } \frac{f(x) - f(a)}{x - a} = \frac{x^n - a^n}{x - a}$$

Properly speaking we have to compute both :

$$\underset{x \to a^-}{\mathit{Limit}} \; \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad \underset{x \to a^+}{\mathit{Limit}} \; \frac{f(x) - f(a)}{x - a}$$

But in the Vedic way the Algebraic notation is lacking. For $n = 1$ and $n = 2$ the reader may wish to review example 4 (page 28) and example 5 (page 29).

For n = 3 :

$$\underset{x \to a}{\mathit{Limit}} \; \frac{x^3 - a^3}{x - a} = \underset{x \to a}{\mathit{Limit}} \; \frac{(x-a)\;(x^2 + xa + a^2)}{(x-a)} = \underset{x \to a}{\mathit{Limit}} \; (x^2 + xa + a^2)$$

Substitute for $x = a$ to get this Limit $= 3a^2$

Let us do this in three steps for general n.

1. AVERAGE step : $\dfrac{f(x) - f(a)}{x - a} = \dfrac{x^n - a^n}{x - a}$

2. TENDS TO step :

$$\frac{x^n - a^n}{x - a} = \frac{(x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1})}{x - a}$$

$$= (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1})$$

We may simplify because $(x - a) \neq 0$.

3. LIMIT step : $\underset{x \to a}{\mathit{Limit}} \; (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1})$

Substitute for $x = a$ to get na^{n-1}

$$\textbf{Western way : } f'(a) = \lim_{\delta x \to 0} \frac{(a+\delta x)^n - a^n}{(a+\delta x) - a}$$

1. AVERAGE step : $\dfrac{f(x) - f(a)}{x - a} = \dfrac{(a+\delta x)^n - a^n}{(a+\delta x) - a}$ (from the right)

In the numerator **:** $(a+\delta x)^n = a^n(1+\dfrac{\delta x}{a})^n$

So**:** $(a+\delta x)^n - a^n = a^n[\,(1+\dfrac{\delta x}{a})^n - 1]$

In the denominator **:** $(a+\delta x) - a = \delta x$

So**:** $\dfrac{(a+\delta x)^n - a^n}{(a+\delta x) - a} = \dfrac{a^n}{\delta x}\,[\,(1+\dfrac{\delta x}{a})^n - 1]$

2. TENDS TO step : expanding by the Binominal Theorem **:**

$= \dfrac{a^n}{\delta x}\,[\,(1 + n\dfrac{\delta x}{a} + \dfrac{n(n-1)}{2!}[\dfrac{\delta x}{a}]^2 + \ldots$ terms with higher powers of δx) $- 1]$

The $+1$ and -1 will cancel out. We are left with :

$= \dfrac{a^n}{\delta x}\,[\,n\dfrac{\delta x}{a} + \dfrac{n(n-1)}{2!}[\dfrac{\delta x}{a}]^2 + \ldots$ terms with higher powers of δx]

Since $\delta x \to 0$ but $\delta x \neq 0$ we may simplify the terms with δx to get :

$= a^n\,[\,\dfrac{n}{a} + \dfrac{n(n-1)}{2!}[\dfrac{\delta x}{a^2}] + \ldots$ terms with higher powers of δx]

3. LIMIT step : now take the LIMIT as $\delta x \to 0$. $\underset{\delta x \to 0}{Limit}\ \delta x = 0$,

$\underset{\delta x \to 0}{Limit}\ \dfrac{\delta x}{a^2} = 0$ and $\underset{\delta x \to 0}{Limit}$ (terms with higher powers of δx) = 0.

We are left with $a^n\dfrac{n}{a} = na^{n-1}$.

$$\boxed{\underset{\delta x \to 0}{Limit}\ \frac{(a+\delta x)^n - a^n}{(a+\delta x) - a} = na^{n-1} \quad \text{(from the right)}}$$

In calculating $f'(a)$ when we wrote $\underset{x \to a}{Limit}$ we meant $\underset{x \to a^+}{Limit}$, i.e. from the ***right***.

We could have taken $\underset{x \to a^-}{Limit}$, i.e. from the ***left***.

What is $\underset{\delta x \to 0}{Limit} \dfrac{(a-\delta x)^n - a^n}{(a-\delta x) - a}$? (from the ***left***)

i.e. what is $\underset{\delta x \to 0}{Limit} \dfrac{(a-\delta x)^n - a^n}{-\delta x}$? (from the ***left***)

IMPORTANT: notice the $-\delta x$ in the denominator when approaching the instant a from the ***left*** :

$$x \to a^- = a - \delta x \text{ as } \delta x \to 0 \text{ and } (a - \delta x) - a = -\delta x.$$

Now, rather than choosing the particular ***instant* a**, we may want to find the INSTANTANEOUS RATE OF CHANGE of the function $f(x) = \mathbf{x^n}$ at ***any instant*** x . In general, if we let x be any ***instant*** over which the function is WELL- BEHAVED (SINGLE VALUED, CONTINUOUS and DIFFERENTIABLE):

$$\boxed{f'(x) = \underset{\delta x \to 0}{Limit} \frac{f(x + \delta x) - f(x)}{\delta x}}$$

Instead of using the symbols δx we will use the letter h. So, by our understanding, we can define the DERIVATIVE of a function to be:

$$f'(x) = \underset{h \to 0}{Limit} \frac{f(x + h) - f(x)}{h}$$

Now, let us use our definition of the FIRST DERIVATIVE to prove that :

$$\boxed{\frac{d\mathbf{x^n}}{dx} = n\,\mathbf{x}^{n-1}}$$

Proof :

$$f(x+h) = (x+h)^n = x^n (1 + \frac{h}{x})$$

$$f(x) = x^n$$

1. **AVERAGE step :** $\dfrac{f(x+h) - f(x)}{h} = \dfrac{1}{h}\left[x^n (1+\frac{h}{x})^n - x^n\right]$

$$= \frac{x^n}{h}\left[(1+\frac{h}{x})^n - 1\right]$$

2. TENDS TO step : expanding by the Binomial Theorem :

$$= \frac{x^n}{h}\left[(1 + n\frac{h}{x} + \frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2 + \ldots \text{ terms with higher powers of } h) - 1\right]$$

The $+1$ and -1 will cancel out.

$$= \frac{x^n}{h}\left[(n\frac{h}{x} + \frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2 + \ldots \text{ terms with higher powers of } h\right]$$

Since $h \to 0$ but $h \neq 0$ we may cancel h in the numerators and denominator.

$$= x^n\left[(\frac{n}{x} + \frac{n(n-1)}{2!}\left(\frac{h}{x^2}\right) + \ldots \text{ terms with higher powers of } h\right]$$

3. LIMIT step : now we can take the LIMIT as $h \to 0$.

$\underset{h \to 0}{\mathit{Limit}}\ h = 0$, $\underset{h \to 0}{\mathit{Limit}}\ \dfrac{h}{x^2} = 0$ and $\underset{h \to 0}{\mathit{Limit}}$ (terms with higher powers of h) $= 0$.

We are left with : $x^n \dfrac{n}{x} = n x^{n-1}$

> We denote this INSTANTANEOUS RATE OF CHANGE as:
>
> $$\frac{d}{dx}\mathbf{x^n} = nx^{n-1}$$
>
> This is also called the FIRST DERIVATIVE of $\mathbf{x^n}$.

Most of the confusion in Calculus arises out of the fear of dividing by zero. Limit δx TENDS TO zero is zero. Therefore, it appears in $\underset{\delta x \to 0}{Limit} \dfrac{\delta y}{\delta x}$ to be dividing by zero.

We must always keep in mind that δx ia attached to some ***particular instant*** **a** or ***general instant*** **x** . As $\delta x \to 0$ we approach that INSTANT computing as we go along the AVERAGE RATE OF CHANGE $\dfrac{\delta y}{\delta x}$.

When we take $\underset{\delta x \to 0}{Limit}$ of this AVERAGE RATE OF CHANGE $\dfrac{\delta y}{\delta x}$ we get the INSTANTANEOUS RATE OF CHANGE $\dfrac{dy}{dx}$ at the ***particular instant*** **a** or ***general instant*** **x**.

Exercise : Prove the INSTANTANEOUS RATE OF CHANGE of $a_n x^n = n a_n x^{n-1}$?

$$\text{Let } f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0$$

be any polynomial of degree n, then :

$$\frac{df(x)}{dx} = f'(x) = n.a_n x^{n-1} + (n-1).a_{n-1} x^{n-2} + \ldots + 2a_2 x^1 + a_1$$

You have now learned to DIFFERENTIATE x^n when the exponent n is an integral or whole number. What if n is a RATIONAL number ?

How would you DIFFERENTIATE $x^{1/2}$ or $x^{-3/2}$?

Mechanically, we may say:

$$\frac{dx^{1/2}}{dx} = \frac{1}{2} x^{-1/2} \quad \text{and} \quad \frac{dx^{-3/2}}{dx} = -\frac{3}{2} x^{-5/2}$$

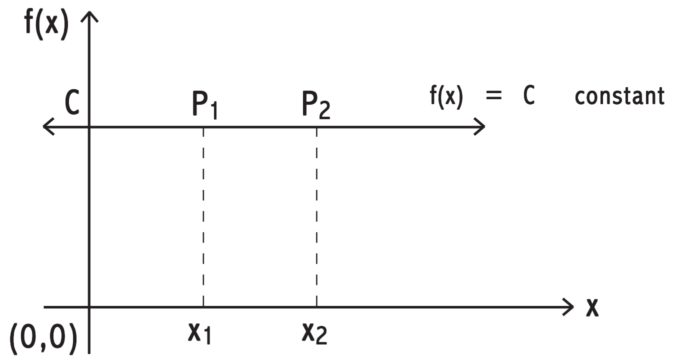
But we must be careful and add : for $x > 0$

The same rule applies. The proof is not required at this level.

15. Differentiation from FIRST PRINCIPLES

Let us now find the FIRST DERIVATIVE of some simple functions by directly applying the definition. This is known as ***differentiation from first principles***. Before we DIFFERENTIATE a function we must ensure that it is WELL-BEHAVED : SINGLE VALUED, CONTINUOUS and DIFFERENTIABLE.

Example 1: What is the ***instantaneous rate of change*** of $f(x) = \mathbf{C}$?



1. AVERAGE step : $\dfrac{P_2 - P_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{\mathbf{C} - \mathbf{C}}{x_2 - x_1} = \dfrac{0}{x_2 - x_1}$

2. TENDS TO step : since x_2 only TENDS TO x_1 : $x_2 \neq x_1$, we may simplify .

$$\frac{0}{x_2 - x_1} = 0$$

3. LIMIT step : now we may take the LIMIT by letting x_2 coincide with x_1 .

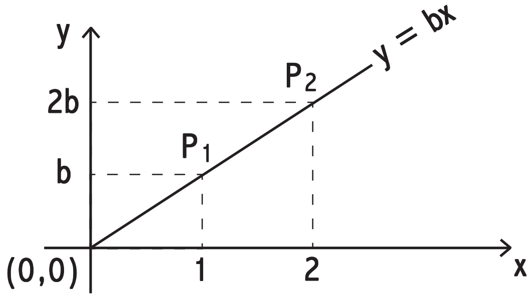
$$\underset{x_2 \to x_1}{Limit}\ 0 = 0$$

$$f'(x) = \frac{df(x)}{dx} = \frac{d\mathbf{C}}{dx} = 0$$

Since $f(x)$ is CONSTANT its ***rate of change*** is zero.

Example 2: What is the ***instantaneous rate of change*** of $y = bx$?

It is not necessary to find the derivative in 3 steps. We may combine all 3 steps in one expression.



$$\frac{dy}{dx} = \underset{x_2 \to x_1}{Limit} \frac{P_2 - P_1}{x_2 - x_1} = \underset{x_2 \to x_1}{Limit} \frac{y(x_2) - y(x_1)}{x_2 - x_1} = \underset{x_2 \to x_1}{Limit} \frac{bx_2 - bx_1}{x_2 - x_1}$$

Since x_2 only TENDS TO x_1 : $x_2 \neq x_1$, we can cancel out the $(x_2 - x_1) \neq 0$. We simplify first and then take the LIMIT. Observe how we may factor out the coefficient b.

$$= \underset{x_2 \to x_1}{Limit} \frac{b(x_2 - x_1)}{x_2 - x_1} = \underset{x_2 \to x_1}{Limit} b = b .$$

The ***instantaneous rate of change*** of y is CONSTANT.

$$\frac{dy}{dx} = \frac{d(bx)}{dx} = b$$

The general linear function is : $\mathbf{y = mx + C}$ where **m** is the ***slope*** .

So the DERIVATIVE of a linear function is its ***slope*** .

Exercise: What is the slope of $f(x) = x + a$? Draw the graph.
(Ref. page 26 Example 2 and the graph on page 180)

Example 3: Let us find the general expression for the INSTANTANEOUS RATE OF CHANGE in height of the ball from the function of its height.
We need to ***differentiate*** $y(t) = u.\sin\theta.t - \frac{1}{2}g\,t^2$ at ***general instant*** t.

1. AVERAGE step: $\dfrac{\Delta y}{\Delta t} = \dfrac{y(t+\Delta t) - y(t)}{(t+\Delta t) - t}$

$$= \frac{\{u.\sin\theta.(t+\Delta t) - \frac{1}{2}g\,(t+\Delta t)^2\} - \{u.\sin\theta.t - \frac{1}{2}g\,t^2\}}{\Delta t}$$

$$= \frac{\{u.\sin\theta.(t+\Delta t) - \frac{1}{2}g\,(t^2 + 2t\Delta t + \Delta t^2)\} - \{u.\sin\theta.t - \frac{1}{2}g\,t^2\}}{\Delta t}$$

$$= \frac{u.\sin\theta.\Delta t - g\,t\,\Delta t - \frac{1}{2}g\,\Delta t^2}{\Delta t}$$

2. TENDS TO step: $\dfrac{\delta y}{\delta t} = \dfrac{u.\sin\theta.\delta t - g\,t\,\delta t - \frac{1}{2}g\,\delta t^2}{\delta t}$

Here we may simplify since $\delta t \neq 0$.

$$\frac{\delta y}{\delta t} = u.\sin\theta - g\,t - \frac{1}{2}g\,\delta t$$

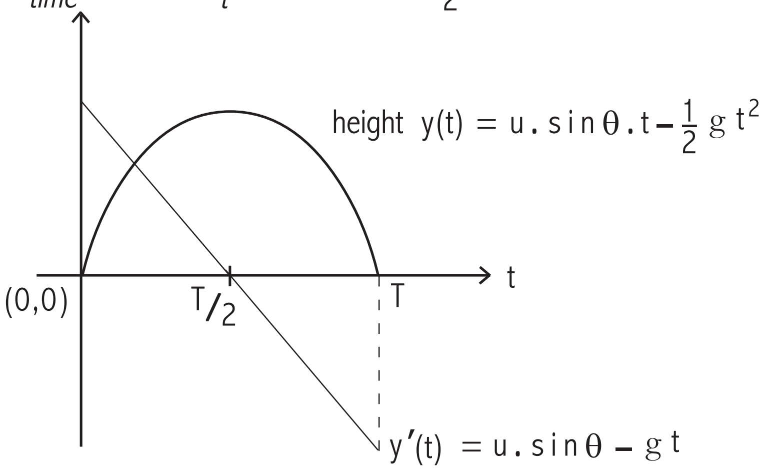
3. LIMIT step: Finally, we take the LIMIT as $\delta t \to 0$ to get the INSTANTANEOUS VERTICAL SPEED at ***general instant*** t.

$$\frac{dy}{dt} = \underset{\delta t \to 0}{\text{LIMIT}}\ \frac{\delta y}{\delta t} = \underset{\delta t \to 0}{\text{LIMIT}}\ \{u.\sin\theta - g\,t - \frac{1}{2}g\,\delta t\}$$

$y'(t) = u.\sin\theta - g\,t$ (refer Preface iii & iv)

We may compare this with the AVERAGE SPEED. If we measure the height at any chosen instant t, then in terms of the height function $y(t) = u.\sin\theta.t - \frac{1}{2}gt^2$ the AVERAGE SPEED over time t = 0 to t is :

$$\frac{distance}{time} = \frac{y(t)}{t} = u.\sin\theta - \frac{1}{2}gt$$



$y(t) = u.\sin\theta.t - \frac{1}{2}gt^2$ is a polynomial of the form $a_0 + a_1t + a_2t^2$ with coefficients $a_0 = 0$, $a_1 = u.\sin\theta$ and $a_2 = -\frac{1}{2}g$. We may directly apply our formula for differentiating polynomials to get :

Vertical speed $y'(t) = \frac{dy}{dt} = \frac{d}{dt}(u.\sin\theta.t - \frac{1}{2}gt^2) = u.\sin\theta - gt$

When we evaluate this expression for the INSTANTANEOUS RATE OF CHANGE at a particular instant in time we get a numerical value. In Calculus we also learn how to interpret this value. We will be able to say things such as the height is increasing, decreasing, is at a maximum or a minimum and so on.

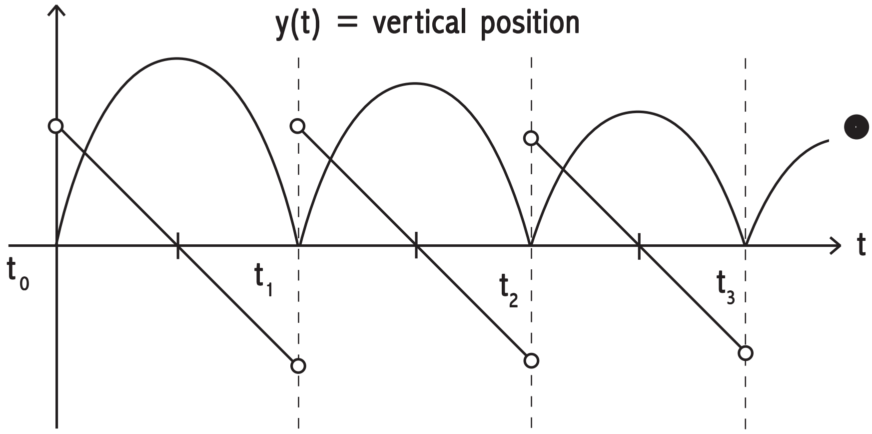
If the duration of flight is T, then from the physics of the situation we know the ***maximum*** height is reached when $t = T/2$. We also know that at the maximum height the vertical speed should be zero. Substituting $t = T/2$ in the vertical speed equation $u.\sin\theta - gt = 0$, we get $T = 2u.\sin\theta/g$.

Example 4: What is the ***instantaneous vertical speed*** of the bouncing ball

(i) over (t_0, t_1) ? (ii) over (t_1, t_2) ? (iii) at t_1 ?

Over $[t_0,\ t_1]$ the height y(t) is : $y_0(t) = u_0.\sin\theta_0.(t - t_0) - \frac{1}{2}g(t - t_0)^2$.

Over $[t_1,\ t_2]$ the height y(t) is : $y_1(t) = u_1.\sin\theta_1.(t - t_1) - \frac{1}{2}g(t - t_1)^2$.



(i) Over (t_0, t_1) the ***instantaneous vertical speed*** $y_0'(t) = u_0.\sin\theta_0 - g(t - t_0)$.

(ii) Over (t_1, t_2) the ***instantaneous vertical speed*** $y_1'(t) = u_1.\sin\theta_1 - g(t - t_1)$.

(iii) At t_1 we have two ***instantaneous vertical speeds*** : the speed on impact at t_1 while ***descending*** and the speed after impact at t_1 while ***rising***. We know from the previous example the duration of flight over $[t_0, t_1]$ is $T_0 = 2u_0.\sin\theta/g$.
To find the speed on impact at t_1 while ***descending*** we substitute $t_0 = 0$ and $t = T_0$ in $y_0'(t) = u_0.\sin\theta_0 - g(t - t_0)$ to get $y_0'(t_1^-) = -u_0.\sin\theta_0$.

After impact at t_1 we know the speed on ***rising*** is $y_1'(t_1^+) = +u_1.\sin\theta_1$. Alternatively, we may substitute $t = t_1$ in $y_1'(t) = u_1.\sin\theta_1 - g(t - t_1)$ to get the same result.

Example 5: Let us ***differentiate*** f(x) = sin (x)

1. AVERAGE step: $\dfrac{\Delta f(x)}{\Delta x} = \dfrac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

$$= \frac{\{\sin(x) . \cos(\Delta x) + \cos(x) . \sin(\Delta x)\} - \sin(x)}{\Delta x}$$

2. TENDS TO step: $\dfrac{\delta f(x)}{\delta x} = \dfrac{\{\sin(x) . \cos(\delta x) + \cos(x) . \sin(\delta x)\} - \sin(x)}{\delta x}$

When x is infinitely small, say δx , then as we said earlier, we may let :

$$\sin(\delta x) = \delta x \quad \text{and} \quad \cos(\delta x) = 1$$

$$\frac{\delta f(x)}{\delta x} = \frac{\{\sin(x) . 1 + \cos(x) . \delta x\} - \sin(x)}{\delta x}$$

$$= \frac{\cos(x) . \delta x}{\delta x}$$

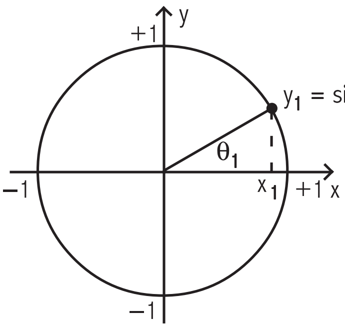
Here we may simplify since $\delta x \neq 0$: $\dfrac{\delta f(x)}{\delta x} = \cos(x)$

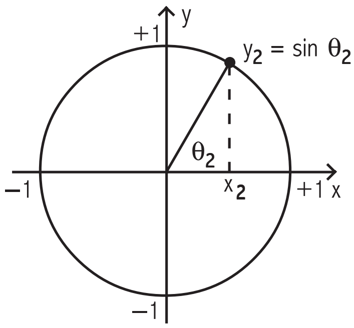
3. LIMIT step: Finally we take the LIMIT as $\delta x \to 0$.

$$\frac{df(x)}{dx} = \underset{\delta x \to 0}{\text{LIMIT}} \frac{\delta f}{\delta x} = \underset{\delta x \to 0}{\text{LIMIT}} \{\cos(x)\}$$

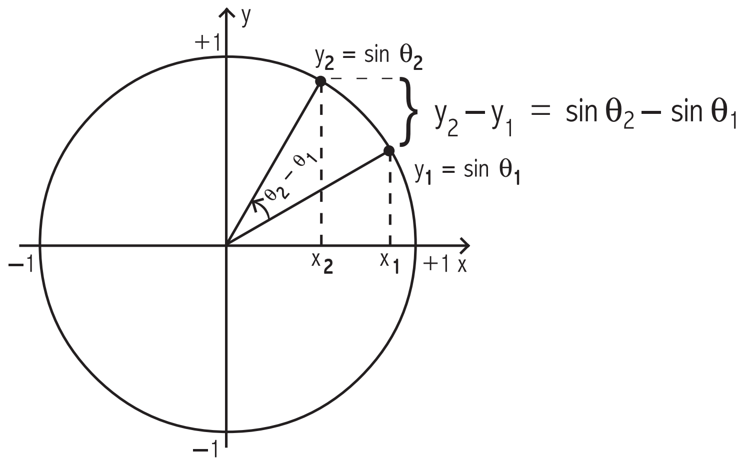
$$f'(x) = \cos(x)$$

Example 6: Let us now ***differentiate*** the function $y = \sin(\theta)$ the Vedic way.





Let us now magnify the two diagrams and superimpose them.



With the concept of arc length $= r\,\theta$, we can see from the diagram that :

$$x_2\,(\theta_2 - \theta_1) \le y_2 - y_1 \le x_1\,(\theta_2 - \theta_1)$$

Since $(\theta_2 - \theta_1) \ne 0$ we may divide throughout by $(\theta_2 - \theta_1)$.

AVERAGE RATE OF CHANGE of y with respect to θ : $\quad x_2 \le \dfrac{y_2 - y_1}{\theta_2 - \theta_1} \le x_1$

Now we may take the LIMIT as $\theta_2 \to \theta_1$ to get the INSTANTANEOUS RATE OF CHANGE of y with respect to θ :

$$\underset{\theta_2 \to \theta_1}{Limit}\ x_2 \le \underset{\theta_2 \to \theta_1}{Limit}\ \frac{y_2 - y_1}{\theta_2 - \theta_1} \le \underset{\theta_2 \to \theta_1}{Limit}\ x_1$$

As $\theta_2 \to \theta_1$ we can see that $x_2 \to x_1$:

$$x_1 \le \frac{dy}{d\theta} \text{ at } \theta_1 \le x_1$$

So : $\frac{dy}{d\theta}$ at $\theta_1 = \frac{d}{d\theta}(\sin\theta)$ at $\theta_1 = x_1 = \cos\theta$ at θ_1.

We may drop the subscript of particular angle θ_1 to generalise and get :

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

There is no need to DIFFERENTIATE a function from FIRST PRINCIPLES each and every time. We may differentiate the functions that we encounter most frequently and build a table. Then all we have to do is look up the TABLE OF DERIVATIVES.

16. Tables and Rules

A function f(x) expresses CHANGE. The CHANGE could be in position, velocity, acceleration, temperature, length, area, volume, pressure, charge on a capacitor, current, voltage, resistance, energy, cost, population, time, probability, or any other quantity that CHANGES. The calculation to find the ***derivative*** $f'(x)$, the expression of the INSTANTANEOUS RATE OF CHANGE, from the function f(x), the expression of CHANGE, is almost mechanical. Below is a partial ***Table of Derivatives*** of the more frequently encountered functions in standard form.

function **f(x)**	***derivative*** **f′(x)**
x^n (n is a rational number)	$n\,x^{n-1}$
e^x	e^x
log (x)	$1/x$
sin (x)	cos (x)
cos (x)	−sin (x)
tan (x)	$\sec^2(x)$
cosec (x)	−cosec (x) . cot (x)
sec (x)	sec (x) . tan (x)
cot (x)	$-\text{cosec}^2(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{\sqrt{1+x^2}}$

We may ***differentiate*** various combinations of functions using the following rules. Let u(x) and v(x) be functions of x, the independent variable.

function	***derivative***
C (constant)	0
C.u	$C\dfrac{du}{dx}$
u + v	$\dfrac{du}{dx}+\dfrac{dv}{dx}$
u − v	$\dfrac{du}{dx}-\dfrac{dv}{dx}$
u.v	$v\dfrac{du}{dx}+u\dfrac{dv}{dx}$
$^{u}/_{v}$	$\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

Chain rule

Let u(v) be a function of v and v(x) be a function of x, the independent variable.

$$\frac{du(v)}{dx} = \frac{du(v)}{dv}\cdot\frac{dv(x)}{dx}$$

Higher order derivatives

If y(x) is a function of x , then $y'(x) = \dfrac{dy}{dx}$ is also a function of x. If y′(x) is a ***well-behaved*** function, then the ***derivative*** of y′(x) is :

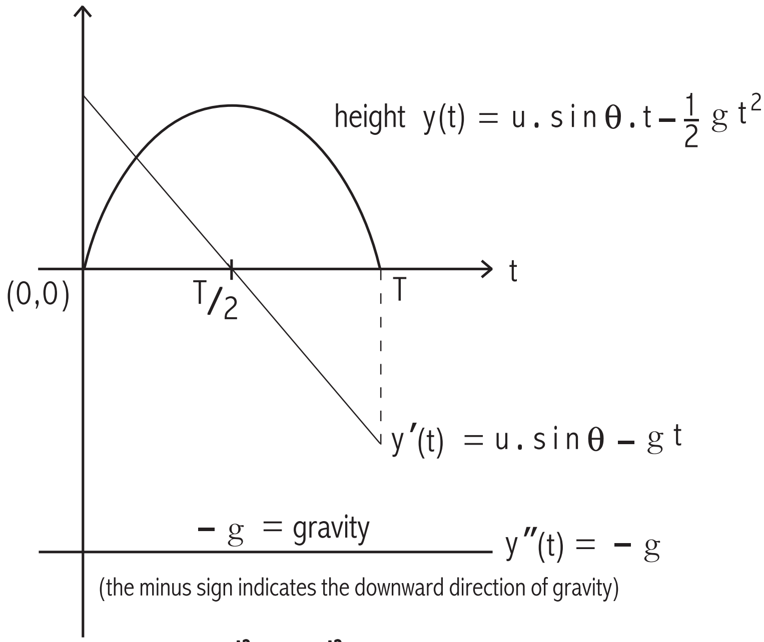
$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

y'' is the ***second derivative*** of y(x) with respect to x. This is known as ***successive differentiation***. In general : $\dfrac{d^ny}{dx^n} = \dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)$

Example : y(t) is the function that expresses CHANGE in height.

y′(t) is the INSTANTANEOUS RATE OF CHANGE in height or vertical speed.

We can think of y′(t) as the function that expresses CHANGE in vertical speed and y″(t) the INSTANTANEOUS RATE OF CHANGE in vertical speed, i.e. vertical acceleration, which in our case is g for gravity.



Vertical acceleration : $y''(t) = \frac{d^2y}{dt} = \frac{d^2}{dt^2}\ (u.\sin\theta.t - \frac{1}{2}gt^2) = -g$

As with any calculation there are two parts: the ***operation*** part and the ***units of measure*** part. Here the height is measured in [meters] and time **t** is measured in [secs]:

y = vertical position or height **[meters]**

y′ = speed **[meters / sec]**

y″ = acceleration **[meters / sec²]**

Derivative of the inverse function

Let $y = x^2$. Then the ***inverse function*** is : $x = y^{1/2}$. We may now differentiate the ***inverse function*** x with respect to y to get $^{dx}/_{dy} = {}^1/_{2y^{1/2}}$. Alternatively, we know that $^{dy}/_{dx} = 2x$. Hence:

$$\frac{1}{^{dy}/_{dx}} = \frac{dx}{dy} = \frac{1}{2x}$$

Now we may substitute $x = y^{1/2}$ to get the derivative of the ***inverse function*** x with respect to y : $^{dx}/_{dy} = {}^1/_{2y^{1/2}}$. So the derivative of the ***inverse function*** x is:

$$\frac{dx}{dy} = \frac{1}{^{dy}/_{dx}}$$

Differentiation in parametric form

We have now learned to find the INSTANTANEOUS RATE OF CHANGE of a function. The reader should ask the question: what about the INSTANTANEOUS RATE OF CHANGE of one function with respect to another function of the same parameter ?

A plane flying horizondally at a speed of 600 kms / hour descends at the rate of 1200 feet / minute. What is the ***glide ratio*** ?

horizontal ***speed*** $u'(t) = 600$ kms / hour $\cong$ 500 feet / sec

downward ***speed*** $v'(t) = 1200$ feet / minute = 20 feet / sec

$$\textbf{\textit{glide ratio}} = \frac{\text{horizontal } \textbf{\textit{speed}}\ u'(t)}{\text{downward } \textbf{\textit{speed}}\ v'(t)} = \frac{500 \text{ feet / sec}}{20 \text{ feet / sec}} = 25$$

That is to say, for every 25 feet forward the plane looses height (descends) by 1 foot. Notice the absence of the ***time*** units of measure. If we know the present altitude we can tell how far the plane will glide before it touches the surface.

Here the ***speed*** functions $u'(t)$ and $v'(t)$ are given. How will we find the ***glide ratio*** given only the horizontal ***position*** function $u(t)$ and vertical ***position*** function $v(t)$?

The ***glide ratio*** need not be a constant. Using our example of flight path of the ball one could ask: what is the INSTANTANEOUS RATE OF CHANGE of RANGE with respect to HEIGHT, both functions of the same parameter t.

$$\text{HEIGHT} = y(t) = u.\sin\theta.\,t - \frac{1}{2}gt^2$$

$$\text{RANGE} = x(t) = u.\cos\theta.\,t$$

We have Le HOSPITAL'S RULE : $\dfrac{du(x)}{dv(x)} = \dfrac{u'(x)}{v'(x)}$

Exerceise 1: Differentiate term by term the following:

a) $\sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots$

b) $\cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots$

c) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots$

Exercise 2: Differentiate term by term sin (ax), cos (ax), and e^{ax} .

Exercise 3: Differentiate sin (ax), cos (ax), and e^{ax} using the ***chain rule***.

It is fundamental for the student to understand how to find the INSTANTANEOUS RATE OF CHANGE. He should feel at ease finding the INSTANTANEOUS RATE OF CHANGE of elementary functions where the ***well-behaved*** properties SINGLE VALUED, CONTINUOUS and DIFFERENTIABLE are easily satisfied.

A function f(x) is an expression of CHANGE.
Given f(x) the expression of CHANGE,
we can DIFFERENTIATE f(x) and find f'(x) the expression of INSTANTANEOUS RATE OF CHANGE.

17. Units of Measure

A function may be dependent on more than one variable. For example f(x, y, z) may be a function of three independent variables x, y and z along the three orthogonal x-axis, y-axis and z-axis respectively. The function f(x, y, z) may describe the position of an object in 3-dimensional space.

With any calculation there are two parts: the ***operation*** part and the ***units of measure*** part. The ***differentiation*** operation gives the EXPRESSION of the INSTANTANEOUS RATE OF CHANGE. The ***integration*** operation gives us the EXPRESSION of CHANGE.

From the ***calculation*** point of view the symbol **dx** in the denominator of df/dx tells us with respect to which independent variable we differentiated the function*. If this independent variable happens to have a ***units of measure*** associated with it, then **dx** has the same ***units of measure***. For example, **dt** has the standard units of measure **[second]** along the time axis. Likewise, $\mathbf{dt^2}$ has $[\mathbf{sec^2}]$ as ***units of measure***.

From the ***Analysis*** point of view the symbol **dx** denotes the calculation at an ***instant*** rather than over an ***interval***. Later in ***integration*** we shall see the symbol **dx** in a similar role.

Let us look at some examples to understand the concept. Let **s** = position or distance, **v** = speed, **a** = acceleration, with distance measured in [meters] and time **t** measured in [secs].

$$\text{SPEED} = \frac{\mathbf{d}}{\mathbf{dt}}\text{POSITION} \quad [\mathbf{meter / sec}]$$

$$\text{ACCELERATION} = \frac{\mathbf{d}}{\mathbf{dt}}\text{SPEED} \quad [\mathbf{meter / sec^2}]$$

---

* Note: In A LITTLE MORE CALCULUS we shall see how to differentiate a function of more than one independent variable. This is known as ***partial differentiation***.

operation	***units of measure***
$v = \frac{\mathbf{d}s}{\mathbf{d}t}$	$[\text{meters / sec}] = \frac{\mathbf{d}[\text{meters}]}{\mathbf{d}[\text{sec}]}$
$a = \frac{\mathbf{d}v}{\mathbf{d}t} = \frac{\mathbf{d}^2 s}{\mathbf{d}t^2}$	$[\text{meters / sec}^2] = \frac{\mathbf{d}[\text{meters / sec}]}{\mathbf{d}[\text{sec}]} = \frac{\mathbf{d}^2[\text{meters}]}{\mathbf{d}[\text{sec}]^2}$
$v = \int a \,.\, \mathbf{d}t$	$[\text{meters / sec}] = \int [\text{meters / sec}^2] \,.\, \mathbf{d}[\text{sec}]$
$s = \int v \,.\, \mathbf{d}t$	$[\text{meters}] = \int [\text{meters / sec}] \,.\, \mathbf{d}[\text{sec}]$

$\text{Area} = \int \text{length} \,.\, \mathbf{d}\,\text{breadth} \qquad [\text{meters}]^2 = \int [\text{meters}] \,.\, \mathbf{d}[\text{meters}]$

$$\int \text{FORCE} \,.\, \mathbf{d}s = \text{ENERGY}$$

$$\frac{\mathbf{d}\,\text{ENERGY}}{\mathbf{d}t} = \text{POWER}$$

$$\int \text{POWER} \,.\, \mathbf{d}t = \text{ENERGY}$$

This format of ***units of measure*** calculation should serve as a template in any application.

18. DIFFERENTIABILITY

If you take a closer look at how we took the LIMIT in the FIRST DERIVATIVE of x^n, you will notice that when we took

$$\underset{h \to 0}{\mathit{Limit}} \ \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \underset{\delta x \to 0}{\mathit{Limit}} \ \frac{f(x+\delta x) - f(x)}{\delta x}$$

We were actually taking the limit from the right.

In other words, we were taking $\underset{x \to a^+}{\mathit{Limit}}$ i.e. approaching the instant a from the right.

We should ask: what would happen if we took $\underset{x \to a^-}{\mathit{Limit}}$ i.e. approaching the instant a from the left ?

$$\text{this is} \quad \underset{h \to 0}{\mathit{Limit}} \ \frac{f(x-h) - f(x)}{-h} \quad \text{or} \quad \underset{\delta x \to 0}{\mathit{Limit}} \ \frac{f(x-\delta x) - f(x)}{-\delta x}$$

because $x \to a^-$ in terms of δx is :

$$x \to a^- = a - \delta x \text{ as } \delta x \to 0 \text{ and } (a - \delta x) - a = -\delta x .$$

Will we get the same INSTANTANEOUS RATE OF CHANGE ?

In general, for any function f(x) we expect:

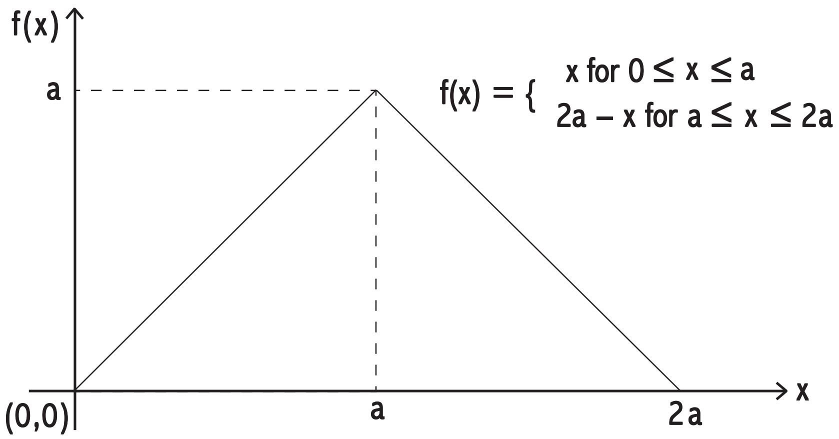
$$\begin{array}{c}\text{INSTANTANEOUS RATE OF CHANGE}\\ \text{approaching a from the left}\end{array} = \begin{array}{c}\text{INSTANTANEOUS RATE OF CHANGE}\\ \text{approaching a from the right}\end{array}$$

If they are equal then we say that f(x) is DIFFERENTIABLE at $x = a$.

$$\underset{x \to a^-}{\mathit{Limit}} \ \frac{f(x) - f(a)}{x - a} = \underset{x \to a^+}{\mathit{Limit}} \ \frac{f(x) - f(a)}{x - a}$$

$$\underset{\delta x \to 0}{\mathit{Limit}} \ \frac{f(a-\delta x) - f(a)}{-\delta x} = \underset{\delta x \to 0}{\mathit{Limit}} \ \frac{f(a+\delta x) - f(a)}{\delta x}$$

Example 1: is f(x) ***differentiable*** at $x = a$?



Approaching **a** from the ***left*** : $f(x) = x$

Near **a** and to the ***left*** of **a** : $x = a - \delta x$

$$\underset{x \to a^-}{Limit} \frac{f(x) - f(a)}{x - a} = \underset{\delta x \to 0}{Limit} \frac{f(a - \delta x) - f(a)}{(a - \delta x) - a}$$

$$= \underset{\delta x \to 0}{Limit} \frac{(a - \delta x) - a}{-\delta x} = \underset{\delta x \to 0}{Limit} \frac{-\delta x}{-\delta x} = +1$$

Approaching **a** from the ***right*** : $f(x) = 2a - x$

Near **a** and to the ***right*** of **a** : $x = a + \delta x$

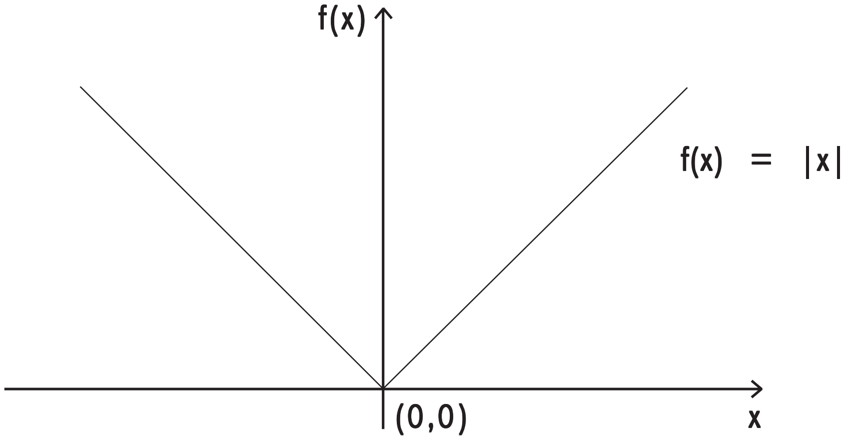
$$\underset{x \to a^+}{Limit} \frac{f(x) - f(a)}{x - a} = \underset{\delta x \to 0}{Limit} \frac{f(a + \delta x) - f(a)}{(a + \delta x) - a}$$

$$= \underset{\delta x \to 0}{Limit} \frac{\{2a - (a + \delta x)\} - \{2a - a\}}{+\delta x} = \underset{\delta x \to 0}{Limit} \frac{-\delta x}{+\delta x} = -1$$

left derivative $\underset{x \to a^-}{Limit} \frac{f(x) - f(a)}{x - a} \neq$ ***right derivative*** $\underset{x \to a^+}{Limit} \frac{f(x) - f(a)}{x - a}$

we say : f(x) is not ***differentiable*** at $x = a$.

Example 2 : is $f(x) = |x|$ ***differentiable*** at $x = 0$?



Approaching **0** from the ***left*** : $f(x) = |x| = -x$

Near **0** and to the ***left*** of **0** : $x = 0 - \delta x$

$$\underset{x \to 0^-}{Limit} \frac{f(x) - f(0)}{x - 0} = \underset{\delta x \to 0}{Limit} \frac{f(0 - \delta x) - f(0)}{(0 - \delta x) - 0}$$

$$= \underset{\delta x \to 0}{Limit} \frac{-(0 - \delta x) - 0}{-\delta x} = \underset{\delta x \to 0}{Limit} \frac{+\delta x}{-\delta x} = \underset{\delta x \to 0}{Limit} (-1) = -1$$

Approaching **0** from the ***right*** : $f(x) = x$

Near **0** and to the ***right*** of **0** : $x = 0 + \delta x$

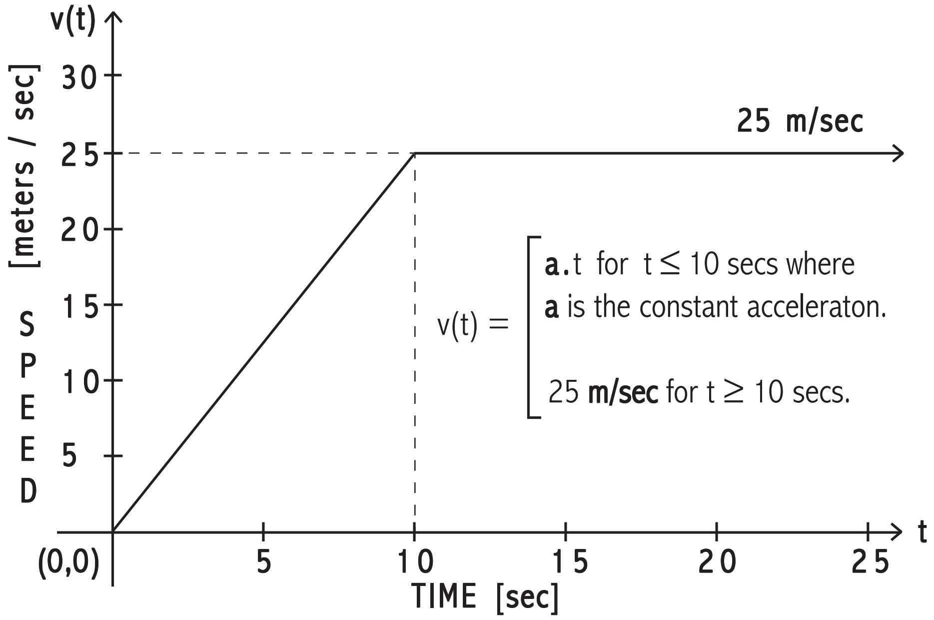
$$\underset{x \to 0^+}{Limit} \frac{f(x) - f(0)}{x - 0} = \underset{\delta x \to 0}{Limit} \frac{f(0 + \delta x) - f(0)}{(0 + \delta x) - 0}$$

$$= \underset{\delta x \to 0}{Limit} \frac{(0 + \delta x) - 0}{+\delta x} = \underset{\delta x \to 0}{Limit} \frac{+\delta x}{+\delta x} = \underset{\delta x \to 0}{Limit} (+1) = +1$$

left derivative $\underset{x \to 0^-}{Limit} \frac{f(x) - f(0)}{x - 0}$ $\neq$ ***right derivative*** $\underset{x \to 0^+}{Limit} \frac{f(x) - f(0)}{x - 0}$

we say : $f(x)$ is not ***differentiable*** at $x = 0$.

Example 3: v(t) is the speed of a car weighing one ton that starts from rest (0 kmph) and accelerates steadily due East (along the x-axis) for 10 secs and then levels off at 90 kmph (25 m/sec). Is v(t) ***differentiable*** at $t = 10$ secs?



Approaching **t = 10 secs** from the ***left*** : $v(t) = \mathbf{a}.t = 2.5\,t$ meter/sec^2

Near **t = 10 secs** and to the ***left*** of **t = 10 secs** : $t = 10 - \delta t$

$$\underset{t \to 10^-}{Limit} \frac{v(t) - v(10)}{t - 10} = \underset{\delta t \to 0}{Limit} \frac{\{2.5\,(10 - \delta t)\} - 25 \text{ m/sec}}{(10 - \delta t) - 10}$$

$$= \underset{\delta t \to 0}{Limit} \frac{(25 - 2.5\delta t) - 25}{-\delta t} = \underset{\delta t \to 0}{Limit} \frac{-2.5\delta t}{-\delta t} = \underset{\delta t \to 0}{Limit}\,(2.5)$$

$$= 2.5 \text{ m/sec}^2$$

So the ***acceleration*** from the left or ***left derivative*** = 2.5 m/sec^2 .

Approaching **t = 10 secs** from the ***right***: $v(t) = 25$ meters/sec, constant.

Near **t = 10 secs** and to the ***right*** of **t = 10 secs** : $t = 10 + \delta t$

$$\underset{t \to 10^+}{Limit} \frac{v(t) - v(10)}{t - 10} = \underset{\delta t \to 0}{Limit} \frac{25 - 25 \text{ m/sec}}{(10 + \delta t) - 10} = \underset{\delta t \to 0}{Limit} \frac{0}{\delta t} = 0 \text{ m/sec}^2$$

So the ***acceleration*** from the right or ***right derivative*** = 0 m/sec^2, which is what we expect since the speed or $v(t) = 25$ m/sec for $t \geq 10$ seconds is constant.

left derivative $\underset{t \to 10^-}{Limit} \frac{v(t) - v(10)}{t - 10}$ $\neq$ ***right derivative*** $\underset{t \to 10^+}{Limit} \frac{v(t) - v(10)}{t - 10}$

we say : $v(t)$ is not ***differentiable*** at $t = 10$ secs.

Even though $v(t)$ is not ***differentiable*** at $t = 10$ secs, we can still compute the ***left derivative*** and ***right derivative*** and interpret them according to the situation.

Exercise1: Given $f'(x)$ = AVERAGE RATE OF CHANGE of $f(x)$, what is $f(x)$?

a) e^x

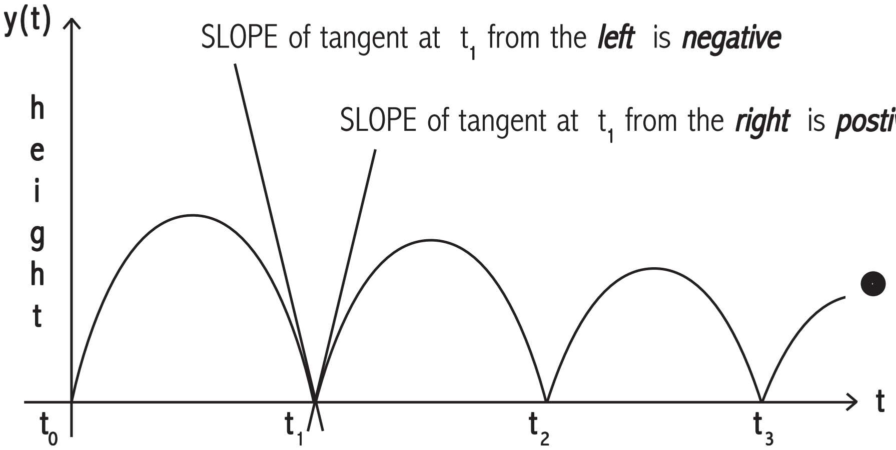
b) $|x|$

c) constant

d) x

e) $ax + b$

Example 4: Is the function y(t), that describes the height of a ***bouncing ball***, ***differentiable*** at instants t_0, t_1, t_2, t_3, . . . ?



At t_1 we have two ***instantaneous vertical speeds*** : the speed on impact at t_1 while ***descending*** and the speed after impact at t_1 while ***rising***. We know the duration of flight $T = 2u_0.\sin\theta/g$ (see page 75). To find the speed on impact at t_1 while ***descending*** we substitute T = 0 and $t = T_0$ in $y_0'(t) = u_0.\sin\theta_0 - g(t - t_0)$ to get $y_0'(t_1^-) = -u_0.\sin\theta_0$.

After impact at t_1 we know this speed on ***rising*** is $y_1'(t_1^+) = +u_1.\sin\theta_1$. Alternatively, we may substitute $t = t_1$ in $y_1'(t) = u_1.\sin\theta_1 - g(t - t_1)$ to get the same result.

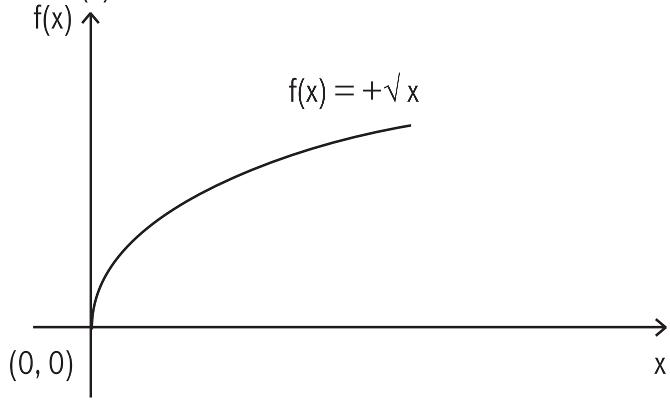
Geometrically, we can see that we have two tangents at t_1.

left derivative $y_0'(t_1^-) = -u_0.\sin\theta_0 \neq y_1'(t_1^+) = +u_1.\sin\theta_1$ ***right derivative***

we say : y(t) is not ***differentiable*** at $t = t_1$ secs.

Even though y(t) is not ***differentiable*** at $t = t_0$, t_1, t_2, t_3, . . . , we can still compute the ***left derivative*** and ***right derivative*** and interpret them according to the situation.

Example 5 : is $f(x) = +\sqrt{x}$ ***differentiable*** at $x = 0$?



We know that $f(x) = +\sqrt{x}$ is not defined for $x < 0$ and hence f(x) is not ***continuous*** for $x < 0$. So we may straight away say that f(x) is not ***differentiable*** at $x = 0$. Another way to think about this is : since f(x) is not defined for $x < 0$ then certainly $f'(x)$ is not defined for $x \leq 0$.

Let us make f(x) ***continuous*** at $x = 0$ by defining $f(x) = 0$ for $x < 0$. Now :

$$f(x) = \begin{cases} +\sqrt{x} & \text{for } x \geq 0 \\ 0, \text{ a constant} & \text{for } x < 0 \end{cases}$$

We may now apply the method of finding the derivative from ***first principles*** as we have been doing so far. Alternatively, we may mechanically apply the formula to find the derivative of x^n when n is rational. Either way we get :

$$f'(x) = \begin{cases} 1/2\sqrt{x} & \text{for } x \geq 0 \text{ is the \textbf{\textit{derivative}} from the \textbf{\textit{right}}} \\ 0 & \text{for } x < 0 \text{ is the \textbf{\textit{derivative}} from the \textbf{\textit{left}}} \end{cases}$$

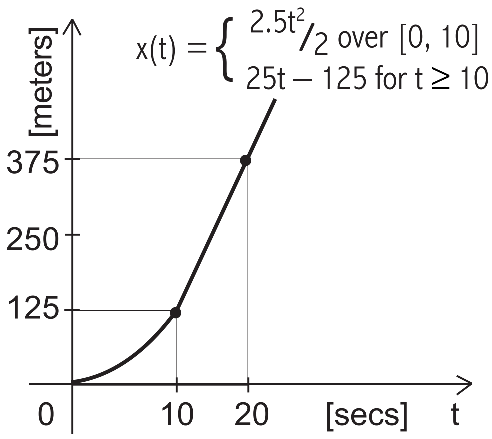
At $x = 0$: the ***right derivative*** $f'(x) = 1/2\sqrt{x}$ is not defined. So at $x = 0$:

left derivative $\neq$ ***right derivative***

We say : f(x) is not ***differentiable*** at $x = 0$.

When finding the derivatives of functions with ***rational exponents***, we must be cautious and check to see if f(x) and $f'(x)$ are defined at $x = 0$ and $x < 0$.

Exercise : A car weighing one ton and starting from rest (0 kmph) accelerates steadily due EAST (along the x-axis) for 10 secs and levels off at 90 kmph (25 m/sec). Its position function x(t) and the graph is shown below.



What is the speed $x'(t)$ over speed over [0, 10] ?

What is the speed $x'(t)$ over speed over $t \geq 10$?

Is $x'(t)$ DIFFERENTIABLE at $t = 10$ secs ?

Draw the speed graph $x'(t)$ over [0, 25].

What is the acceleration $x''(t)$ over [0, 10] ?

What is the acceleration $x''(t)$ over $t \geq 10$?

Is $x''(t)$ DIFFERENTIABLE at $t = 10$ secs ?

Draw the acceleration graph $x''(t)$ over [0, 25].

(Hint : see example 3 page 89 and example 2 page 181) .

Before finding the INSTANTANEOUS RATE OF CHANGE of a function the ***well-behaved*** properties SINGLE VALUED, CONTINUOUS and DIFFERENTIABLE must be satisfied. The first part of the study of Calculus deals with this.

VEDIC mathematicians understood the need for finding the INSTANTANEOUS RATE OF CHANGE known in Sanskrit as ***Chal-na Kal-na***. This is illustrated in the proof to find the INSTANTANEOUS RATE OF CHANGE OF x^n at $x = a$.

The genius of this approach is in its simplicity. First DIVIDE to find the ***rate of change*** then SUBSTITUTE for the ***instant***. This method works almost always, especially if we note that almost all functions can be approximated by finite polynomials or infinite polynomial series.

Polynomials are to functions what rationals are to real numbers.

Without elaborating by analysis the concept, they developed formulas or ***sutras***: ***Gunaka-samuccaya-sutras*** for differentiating various functions.

The methods were encoded in mnemonics (mental one-liners) known as ***slokas*** or ***stotras*** (aphorisms). The more difficult functions could be decomposed using the ***Paravartya Sutra***.

It was up to the genius of the western mind, the great Newton and Leibniz, to develop the concepts of INFINITESIMAL and LIMIT. These concepts are illustrated in the proof to find the INSTANTANEOUS RATE OF CHANGE of x^n.

It is only with these concepts of INFINITESIMAL and LIMIT that we can develop the concepts of CONTINUOUS and DIFFERENTIABLE. While the concept of CONTINUOUS can be intuitively understood the concept of DIFFERENTIABLE is not so obvious. The reader should take heart that even the great Cauchy, master of rigorous analysis, failed to grasp the difference. Put another way :

f(x) is DIFFERENTIABLE if and only if f′(x) is CONTINUOUS.

Part 3 : ANALYTICAL GEOMETRY

Overview

We are familiar with the concept of the ***tangent*** to a curve: a straight line that touches the curve at a point. As a rule, for WELL-BEHAVED functions, to each point on the graph or curve there is exactly one ***tangent***. However, there are exceptions to this rule. We are also familiar with the sign of the trigonometric functions in the 4 quadrants with the angle θ measured in the counter-clockwise (positive) direction.

sin	**all**
tan	**cos**

Let $y(x)$ be a function and let $z_1(x)$ be the ***tangent*** to the curve of $y(x)$ at the point x_1. Let θ_1 be the angle that the ***tangent*** $z_1(x)$ makes with the x-axis. From the sign of $\mathbf{tan\ \theta_1}$ we can gain more information about $y(x)$ at x_1. We can tell if $y(x)$ is increasing, decreasing, at an ***extremum*** (maximum or minimum), or changing direction (flexing) at the point x_1.

From the ***Analysis*** point of view, since the ***tangent*** $z_1(x)$ is a straight line, its equation must be of the form $z_1(x) = m_1 x + c_1$, where $m_1 = \tan\theta_1$ is the SLOPE of the ***tangent***.

So, if we know θ_1, then we may say that: $z_1(x) = \mathbf{tan\ \theta_1} \,.\, x + c_1$.

With a little ***Analysis*** we can show that FIRST DERIVATIVE $y'(x_1) = \mathbf{tan\ \theta_1}$.

So instead of ***geometrically*** :

1. drawing the graph of $y(x)$.
2. drawing the ***tangent*** $z_1(x)$ to $y(x)$ at some particular point $y(x_1)$.
3. measuring the angle θ_1 of the ***tangent*** $z_1(x)$ at $y(x_1)$.
4. computing $\mathbf{tan\ \theta_1}$

to tell the behaviour of $y(x)$ at x_1 , we may straight away use the FIRST DERIVATIVE $y'(x_1)$ and analyse the behaviour of $y(x)$ at x_1.

We may apply this method of ***Analysis*** to two intersecting curves and analytically compute the angle between two intersecting curves.

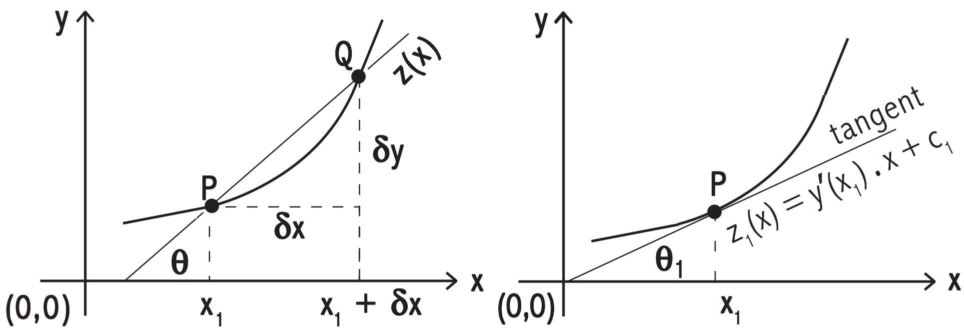
The most useful result of this part is to be able to use the FIRST DERIVATIVE to find the points at which y(x) is a maximum or minimum.

Another important insight is:

if we know the ***tangents*** $z_1(x), z_2(x), z_3(x), \ldots, z_n(x)$, (first derivatives of y(x)) at sufficiently many points $x_1, x_2, x_3, \ldots, x_n$, we may reconstruct the curve y(x). From the ***Analysis*** point of view, this insight tells us that we should be able to calculate y(x) from y′(x). This ***inverse operation*** to find y(x) from its ***derivative*** y′(x) is known as ***integration***.

The reader may skip this part on first reading and go directly to Part 4 without loss of continuity.

19. FIRST DERIVATIVE = Slope of Tangent



We observe that the line z(x) thru P and Q makes an angle θ with the x-axis.

We also observe that : $\tan\theta = \dfrac{\delta y}{\delta x}$

What is the ***Limit*** of the line z(x) as $Q \to P$?

The ***tangent*** to the curve of the function y(x) at the point P or at $x = x_1$.

Let us call it : $z_1(x) = m_1 x + c_1$, where m_1 is the SLOPE .

Note that at P : $\boxed{z_1(x_1) = y(x_1)}$

This ***tangent*** to the curve of the function y(x) at the point P intersects the x-axis at an angle. Let us call this angle θ_1 .

So, when we take the ***Limit*** as $Q \to P$ the line z(x) becomes the ***tangent*** to the curve of y(x) at the point P. So : $z_1(x) = \tan\theta_1 \,.\, x + c_1$.

This is equivalent to saying: $\underset{\delta x \to 0}{Limit} \dfrac{\delta y}{\delta x}$ at the point P $= \tan\theta_1$

$= \text{SLOPE of TANGENT } z_1(x)$ to the curve of y(x) at $x = x_1$.

From the ***Analysis*** point of view :

$\dfrac{\delta y}{\delta x}$ is the AVERAGE RATE OF CHANGE of the function y(x) over the small interval δx.

$y'(x) = \underset{\delta x \to 0}{Limit} \dfrac{\delta y}{\delta x}$ is the INSTANTANEOUS RATE OF CHANGE.

$y'(x_1)$ = FIRST DERIVATIVE of the function y(x) at $x = x_1$

= GRADIENT or SLOPE of the TANGENT $z_1(x)$ to the curve of y(x) at $x = x_1$.

Hence :

$$\boxed{y'(x_1) = \tan\theta_1}$$

On closer observation of the diagrams, as $\delta x \to 0$ it would appear that δy also TENDS TO zero. Hence:

$$\underset{P \to Q}{Limit} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Limit} \frac{\delta y}{\delta x} \text{ should } = {}^0\!/_0 .$$

What we can depict in a diagram has its limitations. So let us look at what is happening from an ***Analysis*** point of view.

Let $y(x) = x^3$

What is δy ? $(x + \delta x)^3 - x^3$

$$= \{ x^3 + 3x^2\,\delta x + 3x\,\delta x^2 + \delta x^3 \} - x^3$$

$$= 3x^2\,\delta x + 3x\,\delta x^2 + \delta x^3$$

So :

$$\frac{\delta y}{\delta x} = \frac{3x^2\,\delta x + 3x\,\delta x^2 + \delta x^3}{\delta x}$$

$$= 3x^2 + 3x\,\delta x + \delta x^2$$

We may simplify because $\delta x \neq 0$. It only TENDS TO zero as $P \to Q$. Now we take the LIMIT as $P \to Q$.

$$\underset{P \to Q}{Limit} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Limit} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Limit} \{ 3x^2 + 3x\,\delta x + \delta x^2 \} = 3x^2$$

It is important to note that in no way we are saying the FIRST DERIVATIVE is always a straight line function of the form mx + c.

Example: Let $y(x) = x^3$. Then $y'(x) = 3x^2$ is not a straight line function.

Draw the curve of the function $y(x) = x^3$. Let P be a point on the curve of the function x^3. Let $\mathbf{x}_1$ be the x co-ordinate of the point P. The value of the FIRST DERIVATIVE at $x = \mathbf{x}_1$ of the function x^3 is:

$$y'(\mathbf{x}_1) = 3\mathbf{x}_1^2.$$

This numerical value $3\mathbf{x}_1^2 = \tan \theta_1$, where θ_1 is the angle that the ***tangent*** at the point P on the curve of the function x^3 makes with the x-axis.

Draw the ***tangent*** $z_1(x)$ to the curve at the point $y(x_1 = 1) = 1^3 = 1$.

Measure the angle θ_1 that the ***tangent*** $z_1(x)$ makes with the x-axis.

It should be just about $72°$. And $\tan \theta_1 = \tan 72° = 3.078$.

At the point on the curve $x^3 = 1$ the x co-ordinate is $x_1 = 1$.

Evaluate the FIRST DERIVATIVE of x^3 at $x_1 = 1$.

$\tan 72° = y'(x_1) = 3 =$ SLOPE of TANGENT to the curve at $x_1 = 1$.

Try this again at the point on the curve $y(x_2 = 2) = 2^3 = 8$.

Draw the ***tangent*** $z_2(x)$ to the curve at this point.

Measure the angle θ_2 that the ***tangent*** $z_2(x)$ makes with the x-axis.

It should be a little more than $85°$. And $\tan \theta_2 = \tan 85° = 12$.

At the point on the curve $x^3 = 8$ the x co-ordinate is $x_2 = 2$.

Evaluate the FIRST DERIVATIVE of x^3 at $x_2 = 2$.

$\tan 85° = y'(x_2) = 12 =$ SLOPE of TANGENT to the curve at $x_2 = 2$.

$$z_i(x_i) = y(x_i)$$

At $x = x_i$ ***slope of tangent*** $z_i(x) = y'(x_i)$

$z_i(x) = y'(x_i) \cdot x + c_i$ is the ***tangent*** to $y(x)$ at $x = x_i$

The constant $c_i = z_i(x_i) - y'(x_i) \cdot x_i$

In ***point-slope*** form the linear equation of the ***tangent*** is :

$$z_i(x) - z_i(x_i) = y'(x_i) \cdot (x - x_i)$$

At $x = 1$ the equation of ***tangent*** to $y = x^3$ is : $z_1(x) = 3x - 2$

At $x = 2$ the equation of ***tangent*** to $y = x^3$ is : $z_2(x) = 12x - 16$

At $x = 3$ the equation of ***tangent*** to $y = x^3$ is : $z_3(x) = 27x - 54$

At $x = 4$ the equation of ***tangent*** to $y = x^3$ is : $z_4(x) = 48x - 128$

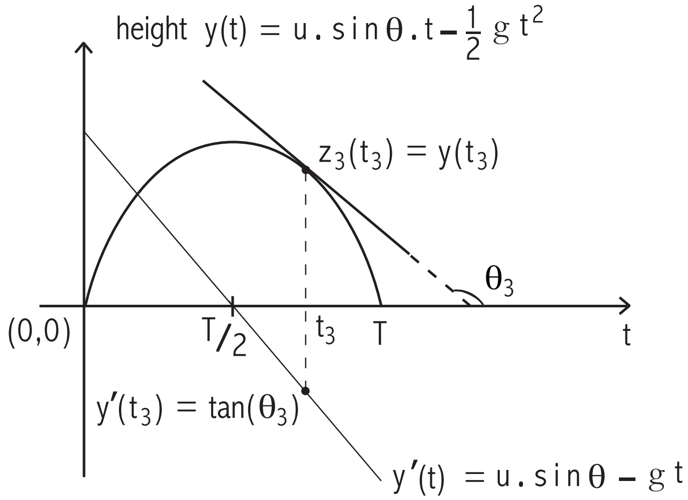
Excercise 1 : Draw the graph of $\mathbf{y = x^3}$ and complete the following table.

x_i	approx. θ_i	$\tan\theta_i$	$y'(x_i)$	$y(x_i)$	$z_i(x) = y'(x_i) \cdot x + c_i$
1	72°	3.078	3	3	$3x - 2$
2	85°		12	8	$12x - 16$
3	87°		27	27	$27x - 54$
4	88°		48	64	$48x - 128$

Exercise 2: Repeat exercise 1 with $y(x) = {}^1\!/_2\, x^2$.

What if $\tan\theta = 0$?, that is to say the tangent to the curve is parallel to the x-axis. We shall deal with this in the next few chapters.

Exercise 3: In our main example of the projectile depicted below with $u = 2\sqrt{2}$ and $\theta = {}^{\pi}\!/_4$ draw the graph and complete the table as in exercise 1 for instants $t_1 = {}^T\!/_4$, $t_2 = {}^T\!/_2$, and $t_3 = {}^{3T}\!/_4$.



It does not make sense to talk about a ***tangent*** to a WELL-BEHAVED linear function. However, its SLOPE = FIRST DERIVATE = $\tan\theta$, when θ is the ***angle of intersection*** the linear function makes with the x-axis.

With the concept of the ***tangent*** in mind, the reader may wish to review the ***bouncing ball*** example on page 91 and gain an ***Analytical geometry*** view of ***differentiability.***

When a SINGLE-VALUED and CONTINUOUS function is NOT ***differentiable*** at a particular point or instant, it may have more than one tangent at that instant. In example 1 on page 87, f(x) is SINGLE-VALUED and CONTINUOUS. But f(x) is NOT ***differentiable*** at $x = a$. At the co-ordinates (a, f(a)) we may draw infinitely many tangents.

We may relate the ***Analysis*** view of ***differentiable*** and the ***Analytical geometry*** property of the ***tangent*** and say :

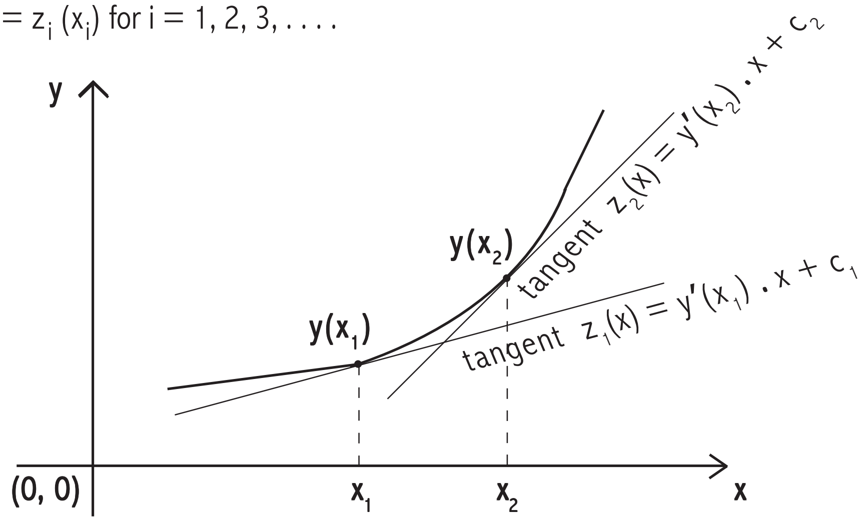
f(x) is DIFFERENTIABLE at a $\Longleftrightarrow$ f(x) has a UNIQUE TANGENT at f(a).

Suppose we know the tangents $z_1(x), z_2(x), z_3(x), \ldots$ at $x_1, x_2, x_3, \ldots$ respectively, can we find the curve y(x)?

We may consider all the tangents $z_i(x)$ to a particular curve y(x) as belonging to a single family. We say that the curve y(x) ENVELOPES this family $z_i(x)$ of straight lines. The curve y(x) is touched by all the straight lines of this family.

From the ***Analytical geometry*** point of view we ask: if we know the ***tangents*** at each point to a curve y(x) can we determine the curve ?

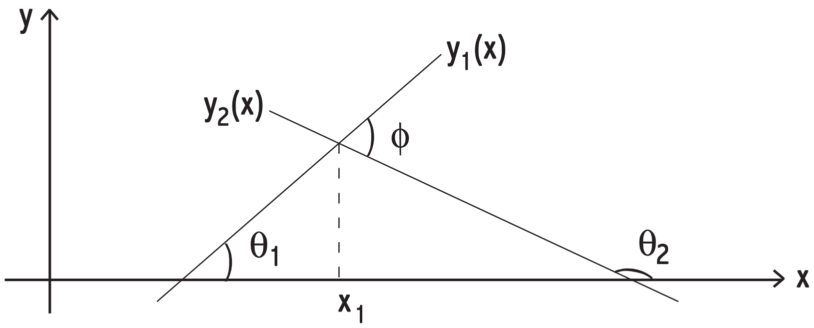
If we know the tangents $z_i(x) = y'(x_i) \cdot x + c_i$ for $i = 1, 2, 3, \ldots$, then we can find $y(x_i) = z_i(x_i)$ for $i = 1, 2, 3, \ldots$.



If we know the ***tangents*** $z_1(x), z_2(x), z_3(x), \ldots, z_n(x)$, (first derivatives of y(x)) at sufficiently many points $x_1, x_2, x_3, \ldots, x_n$, we may reconstruct the curve y(x). From the ***Analysis*** point of view, this insight tells us that we should be able to calculate y(x) from y′(x). This ***inverse operation*** to find y(x) from its ***derivative*** y′(x) is known as ***integration***.

20. Angle of Intersection

Let $y_1(x) = m_1x + c_1$ and $y_2(x) = m_2x + c_2$ be two linear functions.



The ***angle of intersection*** between $y_1(x)$ and $y_2(x)$ is ϕ . Usually the smaller angle is taken. Hence :

$$\phi = \textbf{minimum}\ (|\theta_1 - \theta_2|,\ 180^\circ - |\theta_1 - \theta_2|).$$

With our understanding that **SLOPE = tanθ** , we may say that :

$$m_1 = \tan\theta_1 \text{ and } m_2 = \tan\theta_2 .$$

Now if $y_1(x)$ and $y_2(x)$ are **orthogonal** (**perpendicular** to each other) then:

$$\boxed{m_1 \cdot m_2 = -1}$$

Proof : $\tan \theta_1 = {}^{\sin \theta_1}/_{\cos \theta_1}$

If y_1 and y_2 are **orthogonal** then : $\theta_2 = 90^\circ + \theta_1$

$$\text{So : } \tan \theta_2 = \frac{\sin (90^\circ + \theta_1)}{\cos (90^\circ + \theta_1)} = \frac{\sin 90^\circ . \cos\theta_1 + \cos 90^\circ . \sin\theta_1}{\cos 90^\circ . \cos\theta_1 - \sin 90^\circ . \sin\theta_1}$$

$$= \frac{\cos\theta_1}{-\sin\theta_1}$$

$$\text{Therefore } m_1 . m_2 = \tan \theta_1 . \tan \theta_2 = \frac{\sin\theta_1}{\cos\theta_1} . \frac{\cos\theta_1}{-\sin\theta_1} = -1$$

Hence : $\boxed{m_2 = {}^{-1}/_{m_1}}$

From the ***Analysis*** point of view :

$$y_1'(x_1) = \text{SLOPE } m_1 = \tan \theta_1$$

$$y_2'(x_1) = \text{SLOPE } m_2 = \tan \theta_2$$

Hence, without drawing any graphs, we may directly ***analyse*** and say :

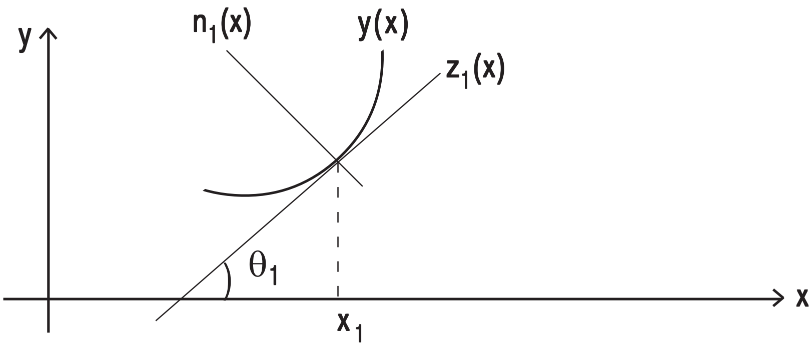
$$\text{if } y_1'(x_1) . y_2'(x_1) = -1$$

then $y_1(x)$ and $y_2(x)$ must be **orthogonal** (**perpendicular** to each other) at x_1.

Also $\boxed{y_2'(x_1) = {}^{-1}/_{y_1'(x_1)}}$

Equations of the tangent and normal

Let $y(x)$ be a general function with ***tangent*** $z_1(x)$ and ***normal*** $n_1(x)$ at $(x_1, y(x_1))$ as depicted below.



From the ***Analysis*** point of view the equations of the ***tangent*** and ***normal*** are :

$$\textit{tangent} \quad z_1(x) - y(x_1) = y'(x_1)\,(x - x_1)$$

$$\textit{normal} \quad n_1(x) - y(x_1) = {}^{-1}/_{y'(x_1)}\,(x - x_1)$$

There are two special cases for the ***tangent*** .

1. If $y'(x_1) = 0$ then the ***tangent*** at $(x_1, y(x_1))$ is PARALLEL to the x-axis. So $z_1(x) = y(x_1)$. Try to imagine what the picture of $y(x)$ will look like around $x = x_1$. Can we say that $y(x_1)$ is either a ***maximum*** or a ***mimimum*** ?

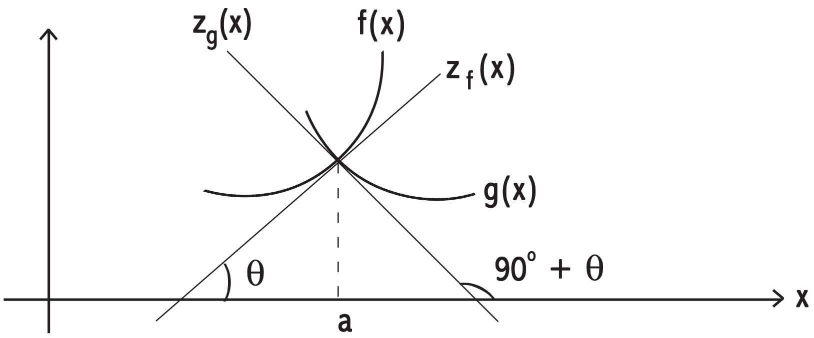
2. If $y'(x_1) = \infty$ then the ***tangent*** at $(x_1, y(x_1))$ is PARALLEL to the y-axis. So $z_1(x) = x_1$.

Likewise, there are two special cases for the ***normal*** .

1. If $y'(x_1) = 0$ then the ***normal*** at $(x_1, y(x_1))$ is PARALLEL to the y-axis. So $n_1(x) = x_1$.

2. If $y'(x_1) = \infty$ then the ***normal*** at $(x_1, y(x_1))$ is PARALLEL to the x-axis. So $n_1(x) = y(x_1)$.

We may extend this thinking to find the angle of intersection between any two functions f(x) and g(x).



We may draw the ***tangents*** to f(x) and g(x) at the point of intersection **a** . Let :

$$z_f(x) = m_f \cdot x + c_f \quad \text{and} \quad z_g(x) = m_g \cdot x + c_g$$

be the equations of these ***tangents*** . Then :

$$m_f = \tan \theta_f \quad \text{and} \quad m_g = \tan \theta_g$$

So : $\phi_{f,\, g} = \text{minimum} \; (|\theta_f - \theta_g|, \; 180° - |\theta_f - \theta_g|).$

And, if f(x) and g(x) intersect at right angles at **a**, then : $m_f \cdot m_g = -1$

This is the ***Analytical geometry*** point of view. From the ***Analysis*** point of view, we note that at the point of intersection a :

$$f'(a) = m_f = \tan \theta_f$$

$$g'(a) = m_g = \tan \theta_g$$

So, to know if the two curves f(x) and g(x) intersect at right angles at a , all we have to do is verify if :

$$\boxed{f'(a) \cdot g'(a) = -1}$$

21. Increasing, MAXIMUM, Decreasing

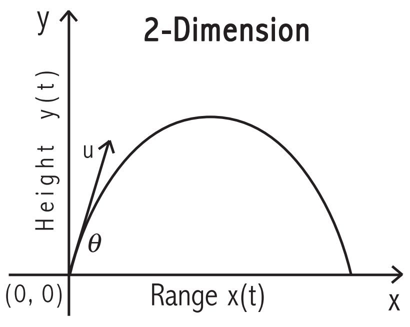
MOTIVATION

You are standing on the ground with a height measuring instrument. Your friend is projected into the air in a module equipped with only a vertical speedometer. Using your height measuring instrument you can measure the height of the module at any instant: so many meters above the ground. Your friend, by looking at the vertical speedometer, can know his vertical speed at any instant: climbing at so many meters per second or descending at so many meters per second.

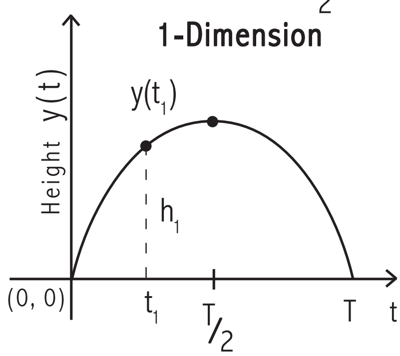
Without communicating with your friend and using only the information you have - the height at any instant, how can you know what your friend knows - the vertical speed of the module at any instant ? Can you tell when the vertical speed is zero ?

Vice versa, how can your friend, with only the information he has - his vertical speed at any instant, find out what you know - his height at any instant ? Can your friend tell when he has reached a maximum height and exactly what is the maximum height ?

$$y(t) = u \,.\, \sin\theta \,.\, t - \frac{1}{2} g t^2$$



POSITION



VERTICAL POSITION

When an object is projected into the air with a given initial velocity u and angle of projection θ, there are two functions that describe its position. A VERTICAL function y(t) which describes its HEIGHT at any INSTANT and a HORIZONTAL function x(t) which describes its RANGE at any INSTANT.

We know from Dynamics that :

Range function $x(t) = u . \cos \theta . t$

Height function $y(t) = u . \sin \theta . t - \frac{1}{2} g t^2$

Geometrically, by looking at the picture that describes the flight path, we can see that the height increases, reaches a maximum and then decreases.

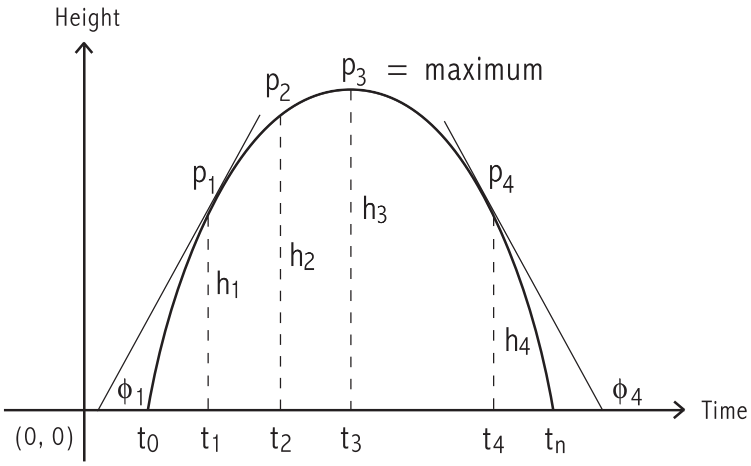
Analytically, by looking at the expression of the HEIGHT function y(t), how can we tell when the height is increasing, decreasing and at a maximum ?

With a little thinking you can infer that when the height is maximum the vertical speed must be zero.

Conversely, your friend, by closely watching the vertical speedometer, can infer that when the vertical speed is zero (while he is still in the air) he has reached the maximum height.

Analytically, can you and your friend tell at which instant in time the maximum height is reached ?

Analytically, can your friend find out the maximum height ?



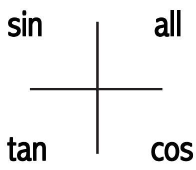
The expression for the INSTANTANEOUS RATE OF CHANGE in height is

$$\frac{d\,y(t)}{dt} = y'(t) = u \,.\, \sin\theta - g\,t$$

We know from observation that the INSTANTANEOUS RATE OF CHANGE in height or ***vertical speed*** is not the same at different INSTANTS t_1, t_2, t_3 and t_4 in time and this is reflected in the expression.

When we evaluate this expression for the INSTANTANEOUS RATE OF CHANGE in height at any particular instant t_i we get a numerical value = **tan** ϕ_i . We shall now learn how to interpret the SIGN of this value.

We shall make use of the SIGN of the **tan** ϕ_i in the 4 quadrants with the angle ϕ_i measured in the counter-clockwise (positive) direction.



and the fact that $\boxed{y'(t_i) = \tan\phi_i}$

From the ***Analytical geometry*** point of view, if we draw the **tangent** to the curve at the point P_1 it will intersect the x-axis at an acute angle. Let us call this angle ϕ_1.

The sign of tan ϕ_1 is > 0, **positive.**

The **first derivative** of y(t) evaluated at time instant t_1

$y'(t_1) = \tan \phi_1 = u. \sin \theta - g\, t_1$ must be **posititve**.

Over the interval $[t_0, t_3)$, from t_0 upto but not including t_3, we note that the height y(t) is **increasing**. We also note that the SIGN of the **tangent** or equivalently the value of the **first derivative** $y'(t)$ evaluated at any instant over $[t_0, t_3)$ is **positive.**

We can say : if y(t) is **increasing** then $y'(t)$ is **positive** (>0).
Conversely : if $y'(t)$ is **positive** then y(t) must be **increasing.**

At P_3, where the object reaches its **maximum** height, the **tangent** to the curve is PARALLEL to the x-axis . Hence $\phi_3 = 0$.

The **first derivative** of y(t) evaluated at time instant t_3 is

$y'(t_3) = \tan \phi_3 = u. \sin \theta - g\, t_3 = 0$

From the ***Analytical geometry*** point of view, if we draw the **tangent** to the curve at the point P_4 it will intersect the x-axis at an obtuse angle ϕ_4 .

The sign of tan ϕ_4 is < 0, **negative.**

The **first derivative** of y(t) evaluated at time instant t_4

$y'(t_4) = \tan \phi_4 = u. \sin \theta - g\, t_4$ must be **negative**.

Over the interval $(t_3, t_n]$, from but not including t_3 to t_n, we note that the height y(t) is **decreasing**. We also note that the SIGN of the **tangent** or equivalently the value of the **first derivative** $y'(t)$ evaluated at any instant over $(t_3, t_n]$ is **negative**.

We can say : if $y(t)$ is **decreasing** then $y'(t)$ is **negative** (< 0).
Conversely : if $y'(t)$ is **negative** then $y(t)$ must be **decreasing.**

In general, let $f(x)$ be a function and **a** the point where $f(x)$ is **maximum**.

At a **maximum** point **a** the value of the function is **greater** than the values around it. Let δx be a small change in x.

Then $f(\mathbf{a} - \delta x) < f(\mathbf{a})$ and $f(\mathbf{a} + \delta x) < f(\mathbf{a})$

At a **maximum** point $f'(x)$ is
positive before, zero, negative after.

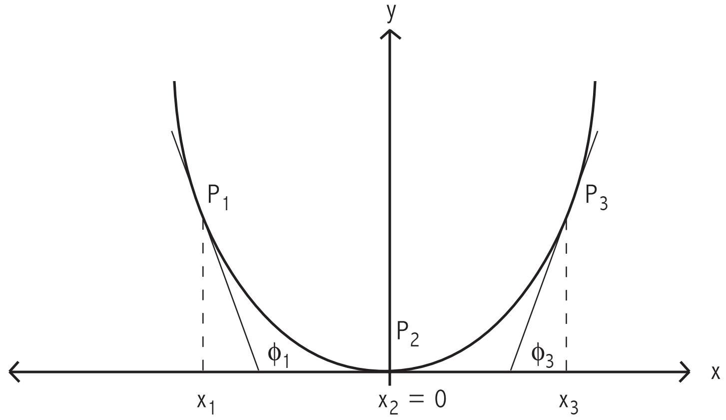
What can you say about $f''(\mathbf{a})$?

Exercise 1: Given that you know the initial velocity **u** and the angle of projection θ, how long will it take to reach a **maximum** height ?

Exercise 2: What is the duration of flight ?

Exercise 3: What is the range ?

22. Decreasing, MINIMUM, Increasing



Consider the function $y = x^2$

$$\frac{dy}{dx} = y'(x) = 2x$$

The **tangent** to the curve of $y = x^2$ at P_1 will intersect the x-axis at obtuse angle ϕ_1.

> The SIGN of $\tan \phi_1$ is < 0, **negative.**
>
> The **first derivative** of $y = x^2$ evaluated at instant x_1
>
> $y'(x_1) = \tan \phi_1 = 2x_1$ must be **negative.**

When $x < 0$ we note that the function $y = x^2$ is **decreasing.** We also note that the SIGN of the **tangent** or equivalently the value of the **first derivative** $y'(x)$ evaluated at any instant $x < 0$ is **negative.**

We can say: if $y(x)$ is **decreasing** then $y'(x)$ is **negative** (<0).
Conversely : if $y'(x)$ is **negative** then $y(x)$ must be **decreasing**.

At P_2, where $y = x^2$ is **minimum**, the **tangent** to the curve is PARALLEL to the x-axis. Hence $\phi_2 = 0$. So the **first derivative** of $y = x^2$ evaluated at instant x_2 is :

$$y'(x_2) = \tan \phi_2 = 2x_2 = 0$$

The **tangent** to the curve of $y = x^2$ at P_3 will intersect the x-axis at acute angle ϕ_3.

The SIGN of $\tan \phi_3$ is > 0, **positive**.

The **first derivative** of $y = x^2$ evaluated at instant x_3

$y'(x_3) = \tan \phi_3 = 2x_3$ must be **positive**.

When $x > 0$ we note that the function $y = x^2$ is **increasing**. We also note that the SIGN of the tangent or equivalently the value of the **first derivative** $y'(x)$ evaluated at any instant $x > 0$ is **positive**.

We can say : if $y(x)$ is **increasing** then $y'(x)$ is **positive** (>0).
Conversely : if $y'(x)$ is **positive** then $y(x)$ is **increasing**.

Let $f(x)$ be a function and **b** the point where $f(x)$ is **minimum**.

At a **minimum** point **b** the value of the function is **lesser** than the values around it. Let δx be a small change in x.

Then $f(\mathbf{b} - \delta x) > f(\mathbf{b})$ and $f(\mathbf{b} + \delta x) > f(\mathbf{b})$

At a **minimum** point $f'(x)$ is
negative before, zero, positive after.

What can you say about $f''(\mathbf{b})$?

Exercise 1: Find an instant where $f(x) = x^2 + x$ has an extreme value (maximum or minimum).

Exercise 2: Find an instant where $f(x) = -x^2 + x$ has an extreme value (maximum or minimum).

Exercise 3: Draw the curves of the functions below and answer the following :

a) $f(x) = x^2$ for $x < 0$. The shape of the curve is concave upwards and falling.

b) $f(x) = -x^2$ for $x > 0$. The shape of the curve is concave downwards and falling.

c) $f(x) = x^2$ for $x > 0$. The shape of the curve is concave upwards and rising.

d) $f(x) = -x^2$ for $x < 0$. The shape of the curve concave downwards and rising.

i) What kind of angle does the tangent to the curve make ? (acute, obtuse) . What is the sign of the tangent ?

ii) What is the sign of the first derivative of the function ? (positive, negative)

iii) Is the function increasing or decreasing ?

Exercise 4: Repeat exercise 3 for the functions below and describe the shape of the curve.

a) $f(x) = x^3$ for $x < 0$.

b) $f(x) = x^3$ for $x > 0$.

c) $f(x) = -x^3$ for $x < 0$.

d) $f(x) = -x^3$ for $x > 0$.

23. MAXIMA and MINIMA

Given a function f(x) how can we find the instants where f(x) has **maximum** or **minimum** values ?

If the point **a** is an **extreme** then the first derivative must equal zero.

Solve the equation $f'(\mathbf{a}) = 0$ to find **a** .

This information is necessary but not sufficient to know whether the point **a** is a **maximum** point or a **minimum** point. There are three methods.

Method 1 : We need to further check around **a** if :

$f(\mathbf{a} - \delta x) < f(\mathbf{a})$ and $f(\mathbf{a}+\delta x) < f(\mathbf{a})$
implying that **a** is a **maximum** point

or

$f(\mathbf{a} - \delta x) > f(\mathbf{a})$ and $f(\mathbf{a}+\delta x) > f(\mathbf{a})$
implying that **a** is a **minimum** point.

These calculations may be tedious.

If we know the graph of the function then we can determine whether **a** is **maximum** or **minimum**. This ***Geometric*** method may not be practical. So let us try to find an ***Analytical*** method.

We shall use the results :

if f(x) is **increasing** then f′(x) is **positive**.

if f′(x) is **positive** then f(x) is **increasing**.

if f(x) is **decreasing** then f′(x) is **negative**.

if f′(x) is **negative** then f(x) is **decreasing.**

Method 2: At the **maximum** point **a**: f(x) is **increasing before** and **decreasing after.**

Correspondingly, $f'(x)$ is **positive before** and **negative after.**

We need to check around **a** if :

$f'(\mathbf{a} - \delta x) > 0$ and $f'(\mathbf{a} + \delta x) < 0$.

At the **minimum** point **a**: f(x) is **decreasing before** and **increasing after.**

Correspondingly, $f'(x)$ is **negative before** and **positive after.**

We need to check around **a** if :

$f'(\mathbf{a} - \delta x) < 0$ and $f'(\mathbf{a} + \delta x) > 0$.

This again may be tedious.

Method 3: We know that at the **maximum** point **a**: $f'(x)$ is **positive before** and **negative after.**

Going from **positive** to **negative** means $f'(x)$ must be **decreasing.**

We also know that if a function is **decreasing** its first derivative must be **negative.** We take the first derivative of $f'(x)$ which is $f''(x)$, the second derivative of f(x), and compute $f''(\mathbf{a})$.

$f''(\mathbf{a}) < 0$ implies **a** is **maximum.**

Similarly, we know that at the **minimum** point **a**: $f'(x)$ is **negative before** and **positive after.**

Going from **negative** to **positive** means $f'(x)$ must be **increasing.**

We also know that if a function is **increasing** its first derivative must be **positive.** We take the first derivative of $f'(x)$ which is $f''(x)$, the second derivative of f(x), and compute $f''(\mathbf{a})$.

$f''(\mathbf{a}) > 0$ implies **a** is **minimum.**

This is usually the easiest and most useful method. But there are exceptions.

Exercise 1: For the function $f(x) = x^3$ $f'(x) = 3x^2$.

At $x = 0 : f'(0) = 0$ implies zero is an extreme point.

Is $x = 0$ a maximum point or minimum point ?

Use all three methods to determine this.

Exercise 2: For the following functions find the extreme points and determine if they are maximum or minimum.

a) $x + 2$

b) x^2

c) $-x^2$

d) x^3

e) $-x^3$

f) x^4

g) $-x^4$

From the number of times a function intersects the x-axis (the number of real roots) can you estimate the number of extreme points ?

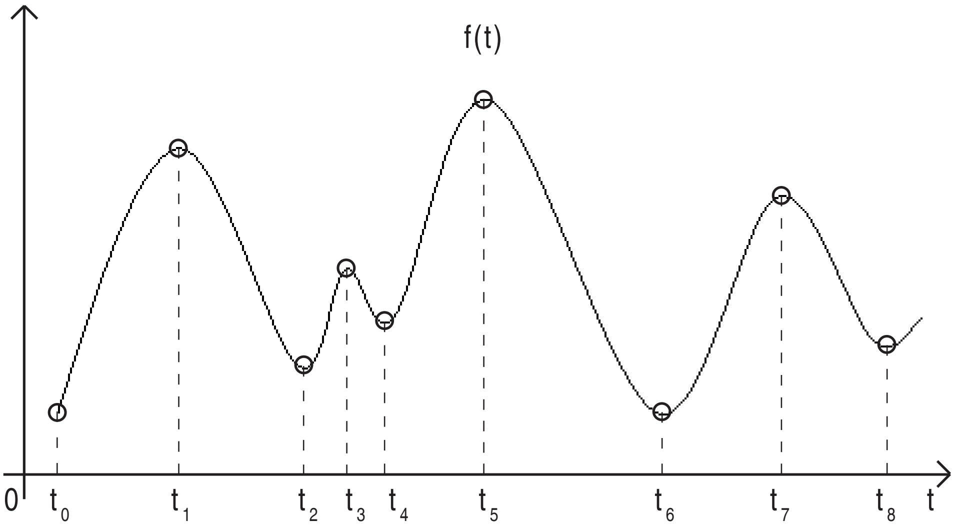
Exercise 3: Divide 10 into two parts such that the product is a maximum.

Exercise 4: Divide 5 into two parts such that the product is a maximum. Draw the graph and verify.

Exercise 5: Divide 2 into two parts such that the product is a maximum. Draw the graph and verify.

Absolute Maximum and Absolute Minimum

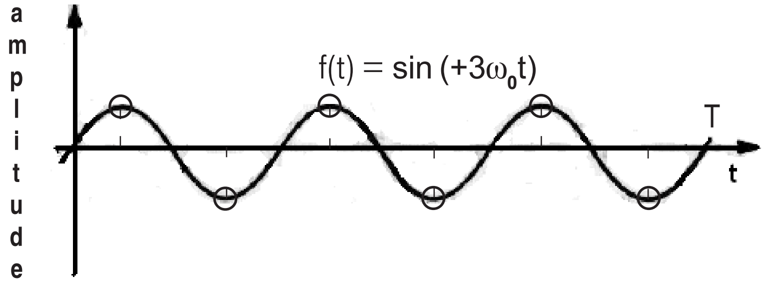
It is possible to have more than one ***maximum*** and one ***minimum*** as in f(t) below.



In this case we will find the overall or ***absolute maximum*** and the overall or ***absolute minimum*** . By solving f(t) = 0 we find all the ***maximum*** and ***minimum*** points. Each ***maximum*** point is called a local ***maximum*** . And each ***minimum*** point is called a local ***minimum*** .

absolute maximum = ***maximum*** (all ***local maximum***)
absolute minimum = ***minimum*** (all ***local minimum***)

Exercise : Find the maximum and minimum points of f(t) = sin $(+3\omega_0 t)$.



At a **maximum**, **minimum** or **inflexion** point, the ***tangent*** to the curve is PARALLEL to the x-axis and hence $\tan\theta = 0$. Conversely, if the ***tangent*** to the curve at a point is zero ($\tan\theta = 0$) then that point must be a **maximum**, **minimum** or **inflexion** point.

We mentioned earlier that ***polynomials are to functions what rationals are to real numbers***. We can do all our calculations of reals to any degree of accuracy using the rationals. Likewise we may approximate any single-valued function by a suitable polynomial. In fact, thinking more deeply on what we just covered in the last few chapters, we may be more specific. We may piece-wise approximate any single-valued function by a suitable choice of polynomials of maximum degree 3 or cubic.

We just saw how a function is either increasing, decreasing or changing direction. Polynomials of degree 0,1,2, and 3 are sufficient to cover all these variations. We only need to identify the **maximum**, **minimum**, and **inflexion** points of the function thru which the approximating polynomials must interpolate.

Note : Later the student will learn ROLLE'S THEOREM, which is a special case of the MEAN VALUE THEOREM. In the special case when $f(a) = f(b)$ then there exists some MEAN point $\overline{x}$ in the interval $[a, b]$ such that $f'(\overline{x}) = 0$. Is it possible to find the point $\overline{x}$ now ? (see note page 64)

24. Points of INFLEXION

A point at which a curve changes its shape in known as a point of **inflexion.**

inflexion = a bending in the curve, a change in curvature or shape, a change in direction, modulation of sound. (flexible, inflexible)

We have seen in the previous sections that at a **maximum** or **minimum** point a curve changes its shape.

maximum : concave down and going left to right from rising to falling.

minimum : concave up and going left to right from falling to rising.

There is another type of change in shape possible: concave up to concave down or vice versa concave down to concave up.

Consider the function $f(x) = x^3 + 1$. Please see the graph.

$$f'(x) = 3x^2 \text{ and } f''(x) = 6x$$

At $x = 0$: $f(0) = +1$ and $f'(0) = 0$, implying $x=0$ may be a **maximum** or a **minimum** point. From the graph we can see it is neither. f(x) is continuously increasing. However, the curve is changing shape from **concave down** to **concave up**.

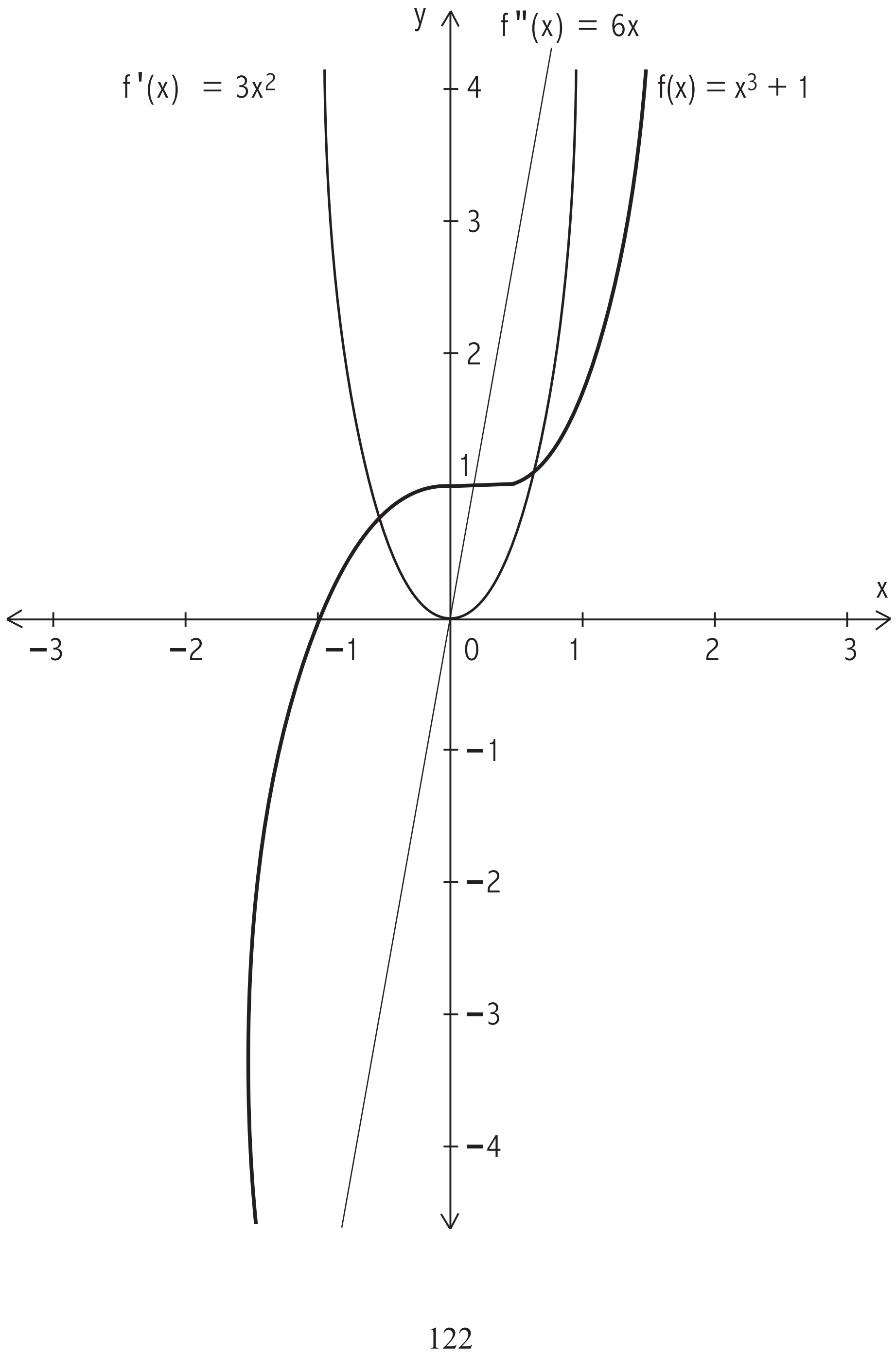
At $x=0$: $f''(0) = 0$ and $f''(x)$ is **negative before** and **positive after.**

Consider the function $f(x) = -x^3 + 1$. Please draw the graph.

$$f'(x) = -3x^2 \text{ and } f''(x) = -6x$$

At $x = 0$: $f(0) = +1$ and $f'(0) = 0$, implying $x = 0$ may be a **maximum** or a **minimum** point. From the graph we can see it is neither. f(x) is continuously decreasing. However, the curve is changing shape from **concave up** to **concave down**.

At $x = 0$: $f''(0) = 0$ and $f''(x)$ is **positive before** and **negative after**.


y
f"(x) = 6x
f'(x) = 3x2
f(x) = x3 + 1
x
4
3
2
1
0
−1
−2
−3
−4
−3
−2
−1
1
2
3

In both cases, $f(x) = x^3 + 1$ and $f(x) = -x^3 + 1$, we have $f'(0) = 0$.

But $x = 0$ is neither a **maximum** nor a **minimum**. How do we determine what kind of point x is ?

$f'(x) = 0$ and $f''(x)$ negative $\Rightarrow$ **maximum**.

$f'(x) = 0$ and $f''(x)$ positive $\Rightarrow$ **minimum**.

$f'(x) = 0$ and $f''(x) = 0$ and changing sign $\Rightarrow$ **inflexion**.

Let us now summarize in tabular form the results of these last few chapters. For any **well-behaved** function $y = f(x)$, with $dy/dx = f'(x)$ and $d^2y/dx^2 = f''(x)$, we have:

Type of point	$\frac{dy}{dx}$ = tan θ = ***slope of tangent***			$\frac{d^2y}{dx^2}$
		before	after	
Maximum	0	+	−	− or 0
Minimum	0	−	+	+ or 0
Inflexion (turning)	0	±	±	0 and changing sign

Reviewing the example of the bouncing ball at the instants t_1, t_2, t_3, **...** we can analyse and say:

1. The function (height) is **decreasing** before and **increasing** after .
2. The first derivatives are **negative** before and **positive** after i.e. changing sign.
3. $f'(x) = 0$. What can you say about $f''(x)$? Is the curve changing shape ?

It makes sense to say that some object is going up and coming down at shorter and shorter time intervals.

If we now note that the maximum height in succeeding time intervals (t_0, t_1), (t_1, t_2), (t_2, t_3), ... is decreasing, we may reasonably interpret this to mean that the object is bouncing in some kind of force (gravitational) field which is causing it to slow down.

If on the other hand the time intervals were of the same duration $(t_0, t_1) = (t_1, t_2) = (t_2, t_3) = ...$ and the maximum height in succeeding intervals kept decreasing, how would you interpret this ? A pendulum oscillating ?

Is it not amazing that Galileo knew so much about falling objects, pendulums, projectiles and planetary motion, yet missed discovering gravity **!**

Part 4 : INTEGRATION

VEDIC mathematicians treated Integration in the *Ekadikha Sutra* as the natural inverse of Differentiation: the ANTIDERIVATIVE. The application was esoteric in nature and more of aesthetic value (***Bijaganita = Algebra***)rather than geometric (***Patiganita***): finding the area under the curve.

While various approximations were employed in different ancient civilizations to find the areas enclosed by certain special curves, the credit is usually given to Archimedes (c.287-212B.C.), of unsinkable fame, for the embryonic idea of the general method of INTEGRATION for finding the area under a curve.

This concept remained dormant till the Renaissance mathematicians Cavalieri (1591-1647) and Torricelli (1608-1647), both students of Galileo (1564-1642), made it germinate.

Issac Barrow (1630-1677), a teacher of the genius Sir Isaac Newton (1642-1727), was the first to make the connection between the slope of the TANGENT (***derivative***) and "***area under the curve***" in 1659. This idea was abstracted and developed both by Newton and Leibniz to establish firmly the relationship between Differential and Integral Calculus.

Gottfried Wilhelm Leibniz (1646-1716) in his manuscript dated 29 October 1673, was the first to use the integral sign $\int$, the elongated or stretched S as we know and use it today, to denote ***continuous*** summation. Leibniz called the expression $\int f(x)\,dx$ the INTEGRAL from the Latin ***integralis*** or whole.

However, both the concepts of defining the relationship between Differential and Integral Calculus: DERIVATIVE & ANTIDERIVATIVE and TANGENT and "***area under the curve***", are useful but limited. The correct concept that corresponds to the relationship is INSTANTANEOUS RATE OF CHANGE and CHANGE.

Overview

We may define a WELL-BEHAVED function F(x) and on ***differentiation*** let :

$$f(x) = \frac{d\,F(x)}{dx}$$

Hence f(x) is the DERIVATIVE of F(x). If we write $f(x)dx = dF(x)$ then we call f(x) the DIFFERENTIAL COEFFICIENT.

We may now define the ***inverse operation*** of ***differentiation*** as finding the ANTIDERIVATIVE . So :

$$F(x) = \text{ANTIDERIVATIVE } \{f(x)\}$$

Just like with ***differentiation*** we may construct a table of ANTIDERIVATIVES. The calculation becomes almost mechanical. This is purely from the ***calculation*** point of view.

From the ***Analysis*** point of view we may infer that : if f(x) is the expression of the INSTANTANEOUS RATE OF CHANGE of F(x), then F(x) must be the expression of CHANGE.

From the ***Geometric*** point of view we have an even more interesting relationship. It seems almost magical.

$$\begin{matrix}\text{area under } f(x) \\ \text{over interval } [a, b]\end{matrix} = \begin{matrix}\text{CHANGE in } F(x) \\ \text{over interval } [a, b]\end{matrix} = F(b) - F(a)$$

Integral Calculus developed out of the need for a general method to find areas, volumes and centres of gravity. Computing the area under f(x) involves a ***continuous summation*** . It is from this process of ***continuous summation*** that we have the operation expressed as ***integration*** derived from the Latin ***integralis*** and the Mathematical notation or symbol $\int$ to define it.

In ***Algebra*** we are familiar with the concept of ***discrete summation*** of a finite number of terms. For example :

$$\sum_{i=0}^{n} \frac{1}{2^i} \qquad \text{where } n \text{ is finite.}$$

We are also familiar with the ***discrete summation*** of an infinite number of terms. For example, the geometric series :

$$\sum_{i=0}^{\infty} \frac{1}{2^i}$$

We could have defined a more general case like :

$$\sum_{i=0}^{\infty} f(x_i)$$

Notice that even though the summation involves infinitely many terms, they are COUNTABLE. Hence we may identify each term using a ***discrete*** subscript. Recall the difference between the set of natural numbers $\mathcal{N}$ that is COUNTABLE, and the ***contiguous*** set of real numbers $\mathcal{R}$ that is UNCOUNTABLE. In the ***continuous summation*** over an interval [a, b] we must include each and every real number or point or instant in [a, b]. Since there are UNCOUNTABLY many of them it does NOT make sense to use a subscript. We know only the end points **a** and **b** of the interval [a, b]. So we evolve the notation for the ***continuous summation*** in 3 steps :

$$\sum_{i=0}^{n} \quad \Rightarrow \quad \sum_{i=0}^{n \to \infty} \quad \Rightarrow \quad \int_a^b$$

discrete summation	***discrete summation***	***continuous summation***
of FINITELY many terms	of COUNTABLY many terms	of UNCOUNTABLY many terms

For the ***continuous summation*** to be of practical value it must converge to some definite finite quantity. This fundamental property, in an intuitive way, is known as ***integrable***. We give an informal definition of this.

25. F(x) = ANTIDERIVATIVE {f(x)}

Suppose we know the function that expresses the INSTANTANEOUS RATE OF CHANGE, are we able to determine the function that expresses the CHANGE ? Suppose we know that the function $y'(t)$ that describes the vertical speed i.e. the INSTANTANEOUS RATE OF CHANGE in height is :

$$y'(t) = \frac{dy(t)}{dt} = u.\sin\theta - gt$$

Are we able to determine the function $y(t) = u.\sin\theta . t - \frac{1}{2} g t^2$

that describes the CHANGE in height ?

We could mechanically say: $\frac{dy(t)}{dt}$ is a ***polynomial*** of the form $ax + b$

So y(t) must be a ***polynomial*** of the form $\frac{1}{2} ax^2 + bx + C$

For : $\frac{dy(t)}{dt} = -gt + u.\sin\theta$

$y(t) = \frac{-1}{2} gt^2 + u.\sin\theta . t$ + may be some constant C.

if at time $t = 0$ the ball was thrown from the ground level (= 0) : $C = 0$

if at time $t = 0$ the ball was thrown from a height of 5 meters: $C = +5m$

This mechanical method of calculation works when the functions are INTEGRABLE. It is known as finding the ANTIDERIVATIVE. There are tables which you can look up to find the matching ANTIDERIVATIVE of the given DERIVATIVE.

Analytically, if f(x) is the DERIVATIVE of the function F(x), then the function F(x) is called the ANTIDERIVATIVE of the function f(x). We may write:

$$f(x) = \frac{d\,F(x)}{dx}$$

If we write $f(x)dx = dF(x)$ then we call f(x) the DIFFERENTIAL COEFFICIENT.

Below is a partial ***Table of Integrals*** of the more frequently encountered functions in standard form. The ***derivative = integrand*** and ***antiderivative = integral***.

DERIVATIVE f(x)	ANTIDERIVATIVE F(x)
x^n	$\frac{x^{n+1}}{n+1} \quad n \neq -1$
$1/x$	$\log \lvert x \rvert$
e^x	e^x
a^x	$\frac{a^x}{\log a} \quad a > 0, a \neq 1$
$\sin(x)$	$-\cos x$
$\cos(x)$	$\sin(x)$
$\sec^2(x)$	$\tan(x)$
$\mathrm{cosec}^2(x)$	$-\cot(x)$
$\tan(x) . \sec(x)$	$\sec(x)$
$\cot(x) . \mathrm{cosec}(x)$	$-\mathrm{cosec}(x)$
$\tan(x)$	$\log \lvert \sec(x) \rvert$
$\cot(x)$	$\log \lvert \sin(x) \rvert$
$\sec(x)$	$\log \lvert \sec(x) + \tan(x) \rvert = \log \lvert \tan(\pi/4 + x/2) \rvert$
$\mathrm{cosec}(x)$	$\log \lvert \mathrm{cosec}(x) - \cot(x) \rvert$ or $\log \lvert \tan x/2 \rvert$
$\frac{1}{a^2+x^2}$	$1/a \tan^{-1}(x/a) \quad a \neq 0$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \log \left\lvert \frac{a+x}{a-x} \right\rvert$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a} \log \left\lvert \frac{x-a}{x+a} \right\rvert$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{\lvert a \rvert}\right) \quad x^2 < a^2$

26. $F(x) = \text{area under } f(x) = \int f(x)dx$

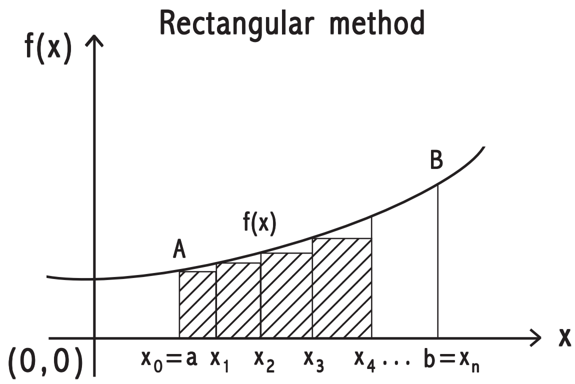
Let us look at the original concept of the ***integral*** - finding the ***area under a curve*** . For the time being we denote the area function as F(x).

F(x) = area under the curve of f(x)

In the next chapter we shall prove that :

$$F(x) = \text{ANTIDERIVATIVE } \{f(x)\} = \int f(x)dx$$

We now present 2 methods to find the ***"area under the curve"*** f(x) over the interval [a,b].



Let f(x) be the curve between x = a and x = b with f(a) = A and f(b) = B.

Divide the interval [a,b] into n sub-intervals Δx_i and construct rectangles in step-like fashion as shown in the figure. We shall compute F(x) in three steps :

$$\sum_{i=0}^{n-1} f(x_i).\Delta x_i \Rightarrow \sum_{i=0}^{n\to\infty} f(x_i).\delta x_i \Rightarrow \textbf{Limit} \sum_{i=0}^{n\to\infty} f(x_i).\delta x_i = \int_a^b f(x).dx$$

discrete summation of FINITELY many terms — ***discrete summation*** of COUNTABLY many terms — ***continuous summation*** of UNCOUNTABLY many terms

1. APPROXIMATE AREA step : *discrete summation* of FINITELY many terms.

An approximation of the actual area function F(x) is

$$F(x) = f(x_0)\,(x_1 - a) + f\,(x_1)\,(x_2 - x_1) + \ldots + f(x_{n-1})\,(b - x_{n-1}) = \sum_{i=0}^{n-1} f(x_i).\Delta x_i$$

2. TENDS TO step : *discrete summation* as $n \to \infty$ of COUNTABLY many terms.

We let n be very, very large by letting $n \to \infty$. The sub-intervals $\Delta x_i = x_1 - a$, $x_2 - x_1, \ldots, b - x_{n-1}$ will be very, very small. We can denote Δx_i by δx_i for $i = 0, 1, 2, \ldots, n \to \infty$.

$\delta x_0 = x_1 - a, \quad \delta x_1 = x_2 - x_1, \quad \ldots, \quad \delta x_{n-1} = b - x_{n-1}$

So: $F(x) = f(x_0)\,\delta x_0 + f(x_1)\,\delta x_1 + \ldots + f(x_{n-1})\,\delta x_{n-1} = \sum_{i=0}^{n \to \infty} f(x_i).\delta x_i$

As n gets larger and larger we can see that the sum is a better and better approximation of the area function F(x).

3. LIMIT step : *continuous summation* of UNCOUNTABLY many terms.

Geometrically we can think of δx_i as a small ***continuous*** line segment. As $n \to \infty$ the δx_i gets smaller and smaller. The LIMIT of δx_i as $n \to \infty$ is a point or instant. We can denote δx_i by dx .

Now we take the LIMIT of the sum as $n \to \infty$.

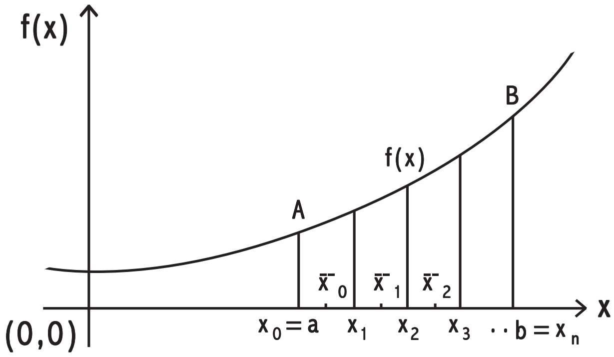
$$F(x) = \textbf{\textit{Limit}} \sum f(x_i)\,.dx \text{ for } i = 0, 1, 2, \ldots, n \to \infty \,.$$

$$F(x) = \textbf{\textit{Limit}} \sum_{i=0}^{n \to \infty} f(x_i)\,.dx = \int_a^b f(x).\,dx$$

We use the special symbol $\int$ which is the initial letter of the word ***summa*** or sum, to distinguish the ***continuous summation*** of UNCOUNTABLY many terms from the ***discrete summation*** of a finite number of terms using + or COUNTABLY many terms using Σ .

The ***dx*** in $^{dy}/_{dx}$ of ***differentiation*** refers to the calculation at a ***point*** or ***instant***. The $\int f(x).\,dx$ of ***integration*** refers to the ***continuous summation*** over an ***interval*** of ***contiguous*** instants ***dx*** .

Trapezoidal method



Instead of approximating the ***area under the curve*** f(x) by rectangular strips we could use trapezoidal strips.

Area of a trapezoid is: $\left(\frac{1}{2}\text{ sum of the parallel sides}\right) * (\text{perpendicular distance})$

1. APPROXIMATE AREA step : ***discrete summation*** of FINITELY many terms.

$$F(x) = \frac{f(x_1) + f(a)}{2} \cdot (x_1 - a) + \frac{f(x_2) + f(x_1)}{2} \cdot (x_2 - x_1) + \ldots$$
$$+ \frac{f(b) + f(x_{n-1})}{2} \cdot (b - x_{n-1})$$

It is not difficult to see that in sub-interval $\Delta x_0 = [a, x_1]$ there is some MEAN point (see note on next page) $\bar{x}_0$ such that:

$$f(\bar{x}_0) = \frac{f(x_1) + f(a)}{2}$$

Similarly, in sub-interval $\Delta x_1 = [x_1, x_2]$ there some MEAN point $\bar{x}_1$ such that:

$$f(\bar{x}_1) = \frac{f(x_2) + f(x_1)}{2}$$

and so on. So we can write:

$$F(x) = f(\bar{x}_0)\,(x_1 - a) + f(\bar{x}_1)\,(x_2 - x_1) + \ldots + f(\bar{x}_{n-1})\,(b - x_{n-1}) = \sum_{i=0}^{n-1} f(\bar{x}_i).\Delta x_i$$

2. TENDS TO step : ***discrete summation*** as $n \to \infty$ of COUNTABLY many terms.

All we have to do now is let n get larger and larger by letting $n \to \infty$. The sub-intervals Δx_i will get smaller and smaller. We can denote the sub-intervals Δx_i by δx_i for $i = 0, 1, 2, \ldots, n \to \infty$:

$$\delta x_0 = (x_1 - a), \quad \delta x_1 = (x_2 - x_1), \ldots, \quad \delta x_{n-1} = (b - x_{n-1})$$

$$F(x) = f(\overline{x}_0)\,\delta x_0 + f(\overline{x}_1)\,\delta x_1 + \ldots + f(\overline{x}_{n-1})(\delta x_{n-1}) = \sum_{i=0}^{n \to \infty} f(\overline{x}_i) \cdot \delta x_i$$

As n gets larger and larger we can see that the sum is a better and better approximation of the area function F(x).

3. LIMIT step : ***continuous summation*** of UNCOUNTABLY many terms.

We can denote δx_i by dx when we take the LIMIT of the sum as $n \to \infty$.

$$F(x) = \textbf{\textit{Limit}} \sum f(\overline{x}_i) \cdot dx \quad \text{for } i = 0, 1, 2, \ldots, n \to \infty .$$

$$F(x) = \textbf{\textit{Limit}} \sum_{i=0}^{n \to \infty} f(\overline{x}_i) \cdot dx = \int_a^b f(x) \cdot dx$$

F(x) = ***continuous summation*** of $f(x_i)$ dx over interval [a, b].

Note:

We should be careful when we say " that in the interval [a, x_1] there is some MEAN point $\overline{x}_0 \ldots$ ". This may NOT always be the case. Let the set S = (3, 4, 5, 7).

For the elements 3 and 5 there is the element 4 in S, such that 4 is the MEAN of 3 and 5.

i.e. $4 = \frac{3+5}{2}$. However, there is no element 6 in S as the MEAN of 5 and 7.

$6 = \frac{5+7}{2}$ is not an element of S.

Since we are dealing only with CONTINUOUS functions we can say " that in the interval [a, x_1] there is some MEAN point $\overline{x}_0 \ldots$ ".

For example, for $f(x) = x^2$: f(2) = 4, and f(4) = 16 .

The mean of f(2) and f(4) is $= \frac{f(2) + f(4)}{2} = \frac{4 + 16}{2} = 10$.

There is some MEAN point $\overline{x} = \sqrt{10}$ in the interval [2, 4] such that $f(\overline{x}) = 10$.

It should now be clear that whichever method you choose (rectangular, trapezoidal, . . .) the $\underset{n \to \infty}{Limit}\ F_n(x)$ is the same and is the area function F(x) of the ***"area under the curve"*** f(x) over the interval [a,b].

The area function $F(x) = \int f(x)dx$.

Since we know the interval over which we are finding the area we could be more DEFINITE and use the (Fourier refinement of Leibniz) notation:

$$[F(x)]_a^b = \int_a^b f(x)dx$$

With the understanding that F(x) = the area under f(x) = $\int f(x)dx$ the reader may wish to review the chapter on ***Units of Measure*** .

So far we have only a notation for the area function : $F(x) = \int f(x)dx$

We do NOT know what F(x) is, much less even begin to evaluate it. What is the area function F(x) ?

Is there a relationship between f(x) and F(x) ? We claim that $f(x) = F'(x)$.

F(x) = ANTIDERIVATIVE of {f(x)}.

And once we prove this, we call f(x) the ***integrand*** and F(x) the ***integral*** .

integrand = the operand of the integration operation.

integral = $\int$***integrand*** .

Thus our Table of DERIVATIVES and ANTIDERIVATIVES becomes a **Table of Integrals.**

27. F(x) = ANTIDERIVATIVE{f(x)} = $\int$ f(x)dx

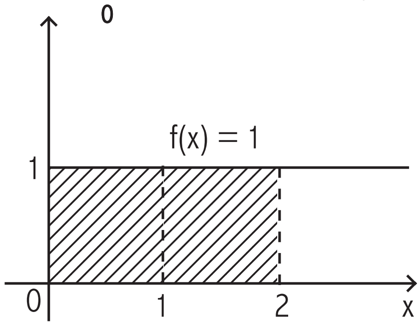
Let us look at two examples to see the relation between :

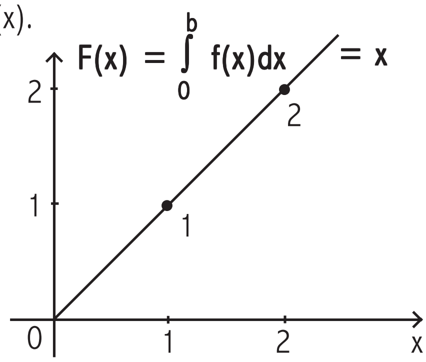
F(x) = area under the curve of f(x) = $\int$ f(x)dx

and **F(x) = ANTIDERIVATIVE {f(x)} .**

Example 1: Let f(x) = 1. For **b** = 1, 2, 3, . . . we may compute the area function

F(x) = $\int_0^b$ **f(x)dx** and plot the graph of F(x).

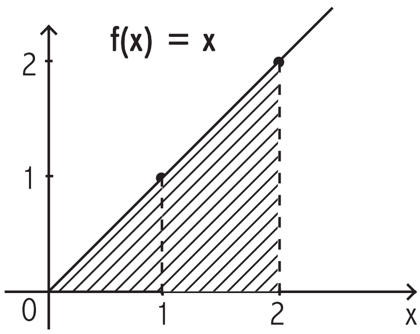


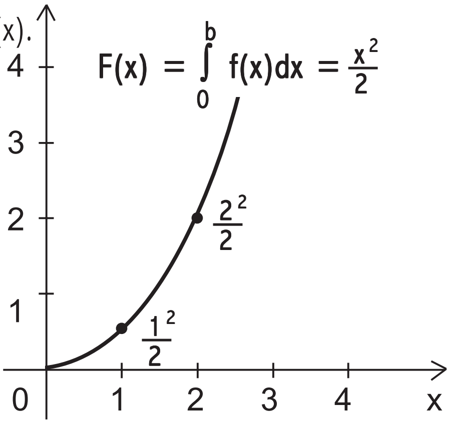


From the diagrams above it is clear that F(x) = x and f(x) = $\dfrac{d\,F(x)}{dx}$ = 1.

Example 2: Let f(x) = x. For x = 1, 2, 3, . . . we may compute the area function

F(x) = $\int_0^b$ **f(x)dx** and plot the graph of F(x).





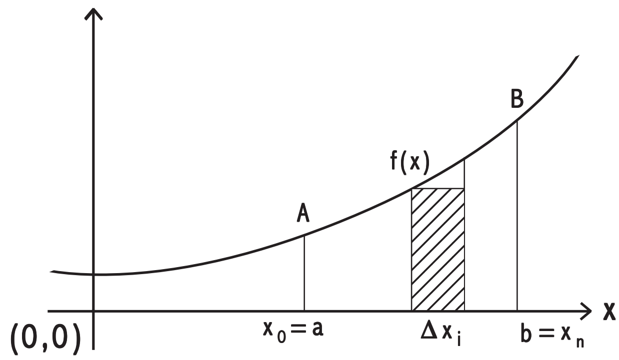
From the diagrams above it is clear that F(x) = $\frac{1}{2}x^2$ and f(x) = $\dfrac{d\,F(x)}{dx}$ = x .

From these two particular examples it is reasonable to infer that :

area function F(x) = ANTIDERIVATIVE {f(x)}.

Proof : Let f(x) be the curve between x = a and x = b with f(a) = A and f(b) = B. Divide the interval [a,b] into n sub-intervals Δx_i and construct rectangles in step-like fashion as shown in the figure.

Let F(x) = the ***area under the curve*** of f(x).



1. APPROXIMATE AREA step : ***discrete summation*** of FINITELY many terms.

An approximation of the actual area function $F(x) = \sum_{i=0}^{n-1} \Delta F(x_i)$

$$= f(x_0)(x_1 - a) + f(x_1)(x_2 - x_1) + \ldots + f(x_{n-1})(b - x_{n-1}) = \sum_{i=0}^{n-1} f(x_i).\Delta x_i$$

where $\Delta F(x_i) = f(x_i).\Delta x_i$ an ***element*** of area .

2. TENDS TO step : ***discrete summation*** as $n \to \infty$ of COUNTABLY many terms.

$$\sum_{i=0}^{n\to\infty} \delta F(x_i) = f(x_0)\,\delta x_0 + f(x_1)\,\delta x_1 + \ldots + f(x_{n-1})\,\delta x_{n-1} = \sum_{i=0}^{n\to\infty} f(x_i).\delta x_i$$

where $\delta F(x_i) = f(x_i).\delta x_i$ an ***infinitesimal element*** of area .

3. LIMIT step : ***continuous summation*** of UNCOUNTABLY many terms.

$$F(x) = \textbf{\textit{Limit}} \sum_{i=0}^{n\to\infty} \delta F(x_i) = \int_a^b dF(x) = \textbf{\textit{Limit}} \sum_{i=0}^{n\to\infty} f(x_i)\,.dx = \int_a^b f(x).\,dx$$

where $dF(x) = f(x).dx$ an ***instantaneous element*** of area .

So $f(x) = \dfrac{d\,F(x)}{dx}$. **The area function F(x) = ANTIDERIVATIVE {f(x)}.**

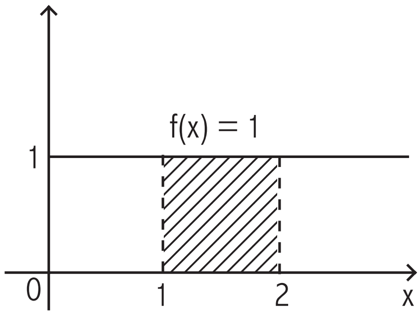
28. $\int_a^b f(x)dx$ = CHANGE in F(x)

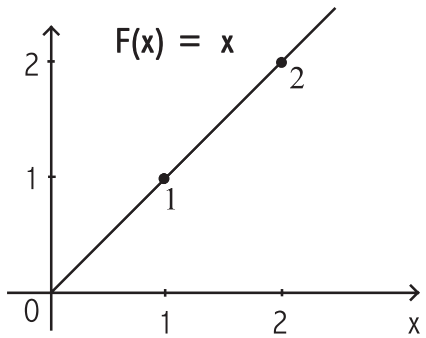
From the CALCULATION point of view : **$F(x)$ = ANTIDERIVATIVE $\{f(x)\}$.**

From the GEOMETRIC point of view : **$F(x) = \int_a^b f(x)dx$ = area under the curve of $f(x)$** over $[a, b]$**.**

Now let us see ***integration*** from an ANALYSIS point of view. Let us look at two examples to see : $\int_a^b f(x)dx$ **= CHANGE in $F(x)$ = $F(b) - F(a)$.**

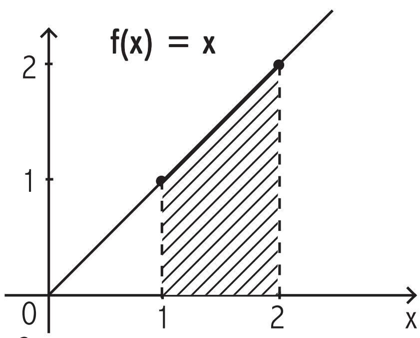
Example 1:

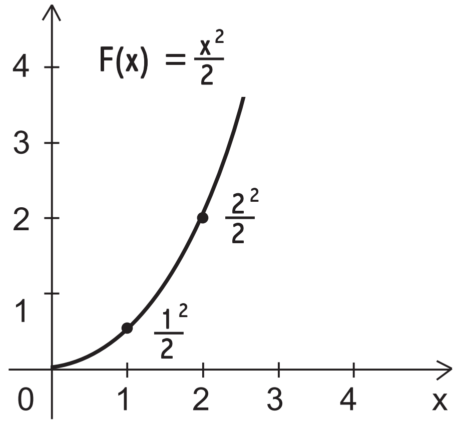




$$\int_1^2 f(x)dx = [x]_1^2 = F(2) - F(1) = 2 - 1 = 1 .$$

Example 2:





$$\int_1^2 f(x)dx = [\tfrac{1}{2} x^2]_1^2 = F(2) - F(1) = \tfrac{1}{2} (2^2) - \tfrac{1}{2} (1^2) = \tfrac{3}{2} .$$

In general:

$$\textbf{CHANGE in } F(x) = \int_a^b f(x)dx = F(b) - F(a)$$

What is $\int_b^a f(x)dx$? It must be $F(a) - F(b)$.

Apply this to the two examples we just saw.

$$\int_2^1 f(x)dx = [x]_2^1 = F(1) - F(2) = 1 - 2 = -1 \ .$$

$$\int_2^1 f(x)dx = [\tfrac{1}{2}x^2]_2^1 = F(1) - F(2) = \tfrac{1}{2}(1^2) - \tfrac{1}{2}(2^2) = -\tfrac{3}{2} \ .$$

Implying that the ***area under the curve*** of f(x) is NEGATIVE. We know from middle school Geometry that there is no such thing as a **negative area**. So how do we explain this result ?

In ***integration*** the DIRECTION OF THE INTEGRATION has a role to play as we shall see in the next chapter. Moreover, this aspect gives the correct physical interpretation.

Calculation: Derivative Anti-Derivative

$$f(x) \xleftarrow[\text{DIFFERENTIATION}]{\frac{d\,F(x)}{dx}} F(x)$$

Geometric: Integrand Integral

$$f(x) \xrightarrow[\text{INTEGRATION}]{\int f(x)dx} F(x)$$

Analysis: Instantaneous Rate of Change $\longleftrightarrow$ Change

29. Direction of Integration and CHANGE in F(x)

Let $F(x) = \int f(x)dx$

When we **integrate** the function f(x) over the interval [a,b] we get a value or NUMBER. From the Calculus point of view this NUMBER represents a CHANGE in the value of the integral F(x) from a to b.

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

The NUMBER may be positive (> 0), meaning an INCREASE in the value of F(x) from a to b. The NUMBER may be negative (< 0), meaning a DECREASE in the value of F(x) from a to b. The NUMBER may be zero, meaning there is NO CHANGE in the value of F(x) from a to b .

Geometrically speaking there is no such thing as a negative area or negative length. We may use the ***"area under the curve"*** f(x) over the interval [a, b] calculation to find the CHANGE in F(x). We follow the simple rules in the table below to determine the SIGN of the ***"area under the curve"*** f(x) in our calculation.

SIGN of ***"area under the curve"***	integrating f(x) with $a < b$	
	left to ***right*** $\int_a^b$ + direction	***right*** to ***left*** $\int_b^a$ − direction
curve f(x) is + above the x axis	+	−
curve f(x) is − below the x axis	−	+

This is the usual law of signs under multiplication. So the SIGN of the ***area under the curve*** , which is also the SIGN of the CHANGE, depends on both SIGN of the function and the ***direction of integration*** .

Integration has five parts :

1. The ***integration*** operation to find the ***integral*** F(x)

   F(x) = the expression of CHANGE = ANTIDERIVATIVE {f(x)} .

2. The ***interval***, say [a, b] with a < b, over which the operand or ***integrand*** f(x) is to be integrated.

3. The ***direction of integration*** over the interval [a, b] with a < b, where the integration is being done :

$$\int_a^b = \textit{positive direction} = a \text{ to } b \text{ with } a < b$$

$$\int_b^a = \textit{negative direction} = b \text{ to } a \text{ with } a < b$$

4. Evaluating the ***integral*** F(x) over the interval [a, b] in the given direction to get something ***definite*** = CHANGE .

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

$$\int_b^a f(x)dx = [F(x)]_b^a = F(a) - F(b)$$

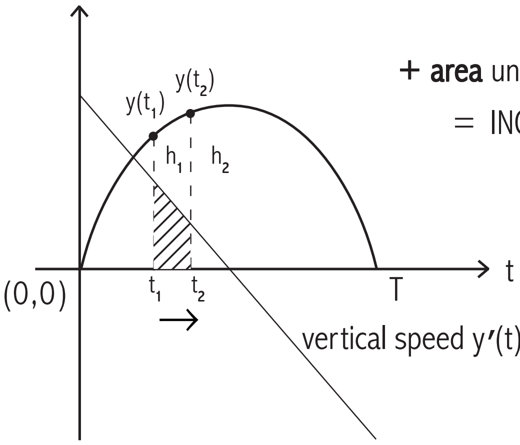
5. Finding the CONSTANT OF INTEGRATION.

To get a clear picture and firm grip of the relationship between :

1. ***direction of integration*** and SIGN of the ***area under the curve*** ,
2. SIGN of the ***area under the curve*** and the SIGN of the CHANGE ,
3. the correct physical interpretation ,

let us look at all 4 cases in the table using our main example.

1. (positive f(x)) . (positive direction) = positive CHANGE



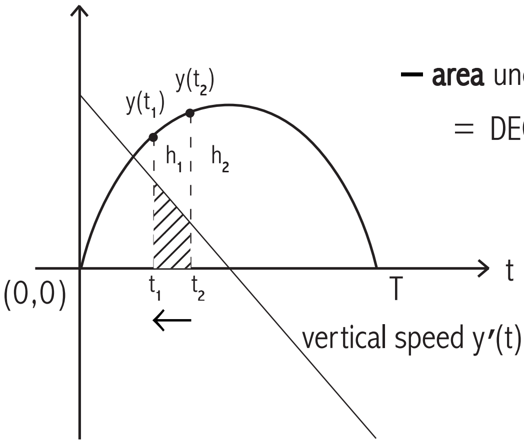
+ area under $y'(t)$ over $[t_1, t_2]$

= INCREASE in height from t_1 to t_2

vertical speed $y'(t) = u \cdot \sin\theta - g\,t$

$$\int_{t_1}^{t_2} y'(t)dt = \Big[y(t)\Big]_{t_1}^{t_2} = y(t_2) - y(t_1) = \text{INCREASE in height}$$

2. (positive f(x)) . (negative direction) = negative CHANGE



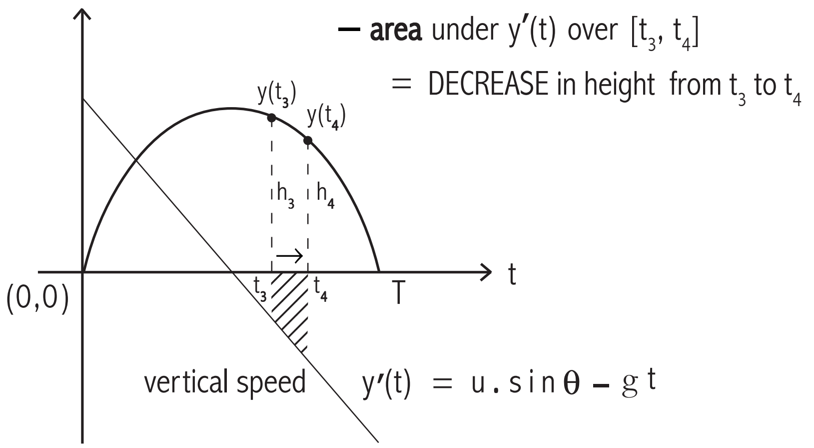
− area under $y'(t)$ over $[t_1, t_2]$

= DECREASE in height from t_2 to t_1

vertical speed $y'(t) = u \cdot \sin\theta - g\,t$

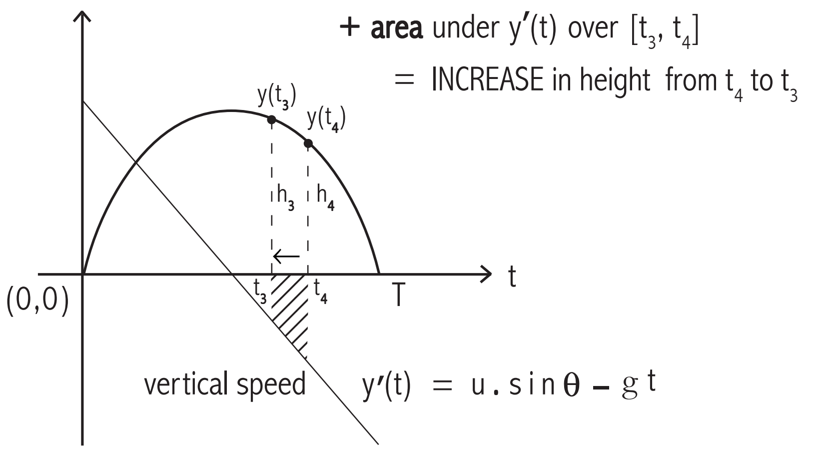
$$\int_{t_2}^{t_1} y'(t)dt = \Big[y(t)\Big]_{t_2}^{t_1} = y(t_1) - y(t_2) = \text{DECREASE in height}$$

3. (negative f(x)) . (positive direction) = negative CHANGE



$$\int_{t_3}^{t_4} y'(t)dt = \left[y(t)\right]_{t_3}^{t_4} = y(t_4) - y(t_3) = \text{DECREASE in height}$$

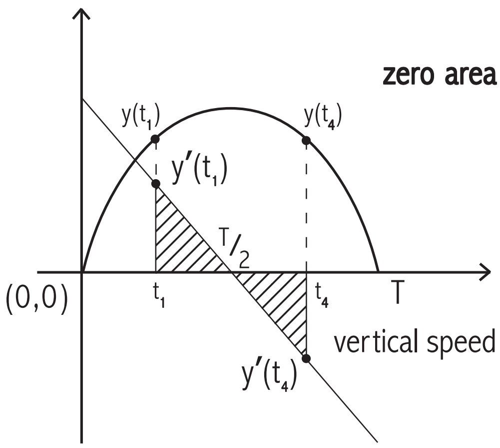
4. (negative f(x)) . (negative direction) = positive CHANGE



$$\int_{t_4}^{t_3} y'(t)dt = \left[y(t)\right]_{t_4}^{t_3} = y(t_3) - y(t_4) = \text{INCREASE in height}$$

What does zero area under the curve $y'(t)$ imply ? NO CHANGE in height.

In the diagram below let the intervals $[\,t_1, {}^{T}\!/_{2}\,]$ and $[\,{}^{T}\!/_{2}\,,\ t_4\,]$ be equal in length. Also let $(\,t_1, y'(t_1))$ and $(\,t_4, y'(t_4))$ be equal in length. So from the ***mensuration*** point of view the shaded triangles are equal in area. Let us call this area A.



zero area under $y'(t)$ over $[t_1, t_4]$

= NO CHANGE in height

vertical speed $y'(t) = u \,.\, \sin\theta - g\,t$

From the ***Calculation*** point of view :

Notice that $y'(t)$ changes SIGN at ${}^{T}\!/_{2}$. So we may break up the integral into 2 parts.

$$\int_{t_1}^{t_4} y'(t)dt = \int_{t_1}^{T/2} y'(t)dt + \int_{T/2}^{t_4} y'(t)dt$$

$$= [y(t)]_{t_1}^{T/2} + [\,y(t)\,]_{T/2}^{t_4}$$

$$= \{\,y(T/2) - y(t_1)\,\} + \{\,y(t_4) - y(T/2)\}$$

$$= y(t_4) - y(t_1)$$

From the ***Geometric*** point of view :

$$\int_{t_1}^{t_4} y'(t)dt = \int_{t_1}^{T/2} y'(t)dt + \int_{T/2}^{t_4} y'(t)dt$$

$\int_{t_1}^{T/2} y'(t)dt$ = (positive function) **.** (positive direction) = positive area : **+** A

$\int_{T/2}^{t_4} y'(t)dt$ = (negative function) **.** (positive direction) = negative area : **–** A

So : $$\int_{t_1}^{t_4} y'(t)dt = +A - A = 0$$

From the ***Analytical*** point of view :

$$\int_{t_1}^{t_4} y'(t)dt = y(t_4) - y(t_1) = 0$$

There is NO CHANGE in the height y(t) over interval [t_1, t_4] .

Now let us REVERSE the direction of integration over interval [t_1, t_4].

From the ***Calculation*** point of view :

Notice that $y'(t)$ changes SIGN at $T/2$. So we may break up the integral into 2 parts.

$$\int_{t_4}^{t_1} y'(t)dt = \int_{t_4}^{T/2} y'(t)dt + \int_{T/2}^{t_1} y'(t)dt$$

$$= [y(t)]_{t_4}^{T/2} + [y(t)]_{T/2}^{t_1}$$

$$= \{ y(T/2) - y(t_4) \} + \{ y(t_1) - y(T/2) \}$$

$$= y(t_1) - y(t_4)$$

From the ***Geometric*** point of view :

$$\int_{t_4}^{t_1} y'(t)dt = \int_{t_4}^{T/2} y'(t)dt + \int_{T/2}^{t_1} y'(t)dt$$

$\int_{t_4}^{T/2} y'(t)dt$ = (negative function) **.** (negative direction) = positive area : $+$ A

$\int_{T/2}^{t_1} y'(t)dt$ = (positive function) **.** (negative direction) = negative area : $-$ A

So : $$\int_{t_4}^{t_1} y'(t)dt = + A - A = 0$$

From the ***Analytical*** point of view :

$$\int_{t_4}^{t_1} y'(t)dt = y(t_1) - y(t_4) = 0$$

There is NO CHANGE in the height y(t) over interval [t_1, t_4] .

In general, let f(x) be a ***continuous*** function over the interval [a, b] and F(x) = ANTIDERIVATIVE {f(x)} . Then :

area under f(x) = 0 **over the interval [a, b]**	$\Longleftrightarrow$	**CHANGE in F(x) = 0** **over the interval [a, b]**

We note that, even if f(x) changes sign several times over the interval [a, b] , we may still mechanically look up our ***Table of Integrals*** to find F(x) and directly compute the CHANGE. It is NOT necessary to break up the integration of f(x) into positive and negative areas as we did above. Let us see a few more examples.

Consider $F(x) = x^2$. Then $f(x) = F'(x) = 2x$. See the graph.

Example1: Integrate ***continuous*** $f(x)$ over the interval $[1, 2]$ going LEFT to RIGHT in the ***positive direction*** .

Using the ***trapezoidal method*** a narrow strip or ***infinitesimal element*** of area is :

$$f(\overline{x}_i) \cdot \delta x_i = \frac{[f(x_i) + f(x_{i-1})] \cdot \delta x_i}{2} \quad \text{where } \delta x_i = x_i - x_{i-1}$$

and $\overline{x}_i$ is the MEAN POINT in $[x_{i-1}, x_i]$ such that $f(\overline{x}_i) = \dfrac{f(x_i) + f(x_{i-1})}{2}$.

This is the area of a narrow trapezium where the parallel sides are $f(x_i)$ and $f(x_{i-1})$, and the perpendicular distance is δx_i . We do not need to know $\overline{x}_i$.

$$F(x) = \int f(x)dx = \underset{i=1}{\overset{n \to \infty}{\text{LIMIT}}} \; \Sigma \, f(\overline{x}_i) \cdot \delta x_i$$

In the LIMIT as $n \to \infty$ each trapezoidal ***infinitesimal element*** of area $f(\overline{x}_i) \cdot \delta x_i$ becomes an ***instantaneous element*** of area $f(x)dx$.

$$F(x) = \int f(x)dx = \int 2x \, dx = x^2$$

The AREA under $f(x)$ over the interval $[1, 2]$ going LEFT to RIGHT in the ***positive direction*** is :

$$\int_1^2 f(x)dx = \int_1^2 2x \, dx = [x^2]_1^2$$

$$[x^2]_1^2 = (2)^2 - (1)^2 = +3$$

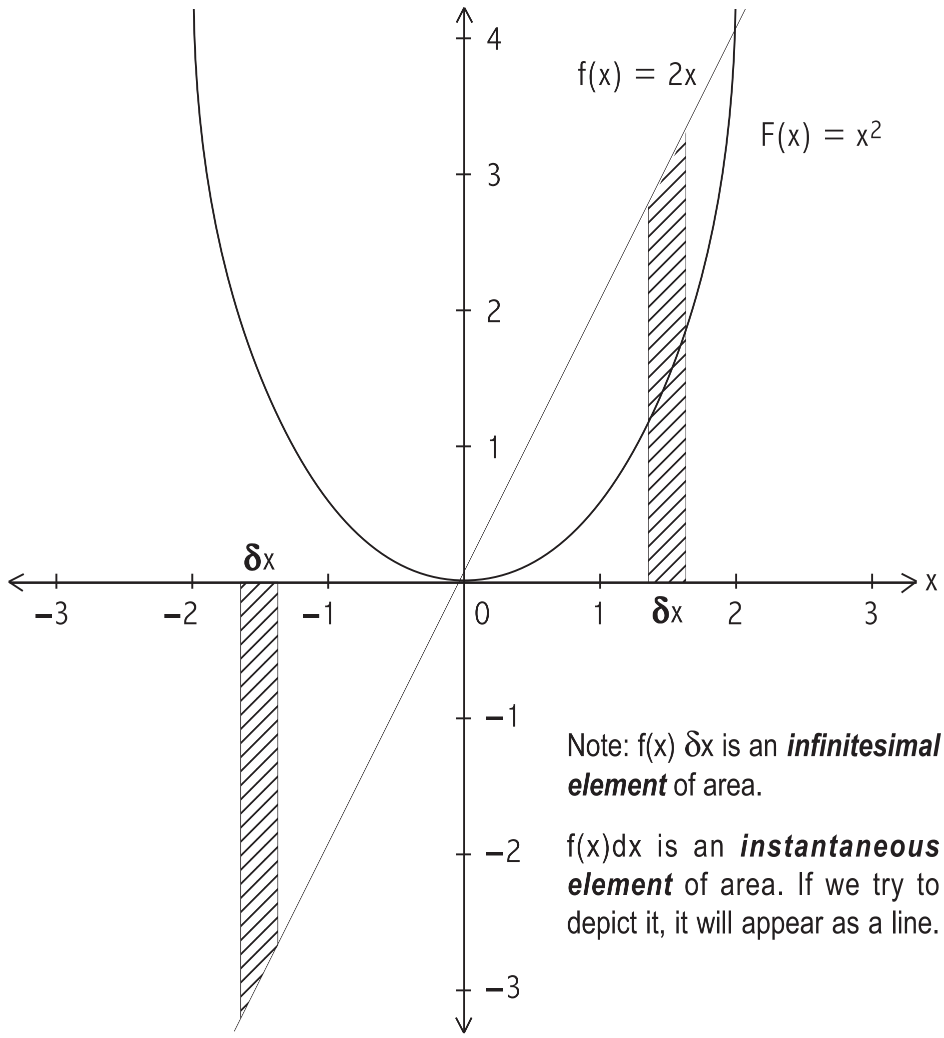
The AREA is 3 in magnitude.

The **+ SIGN** is due to (positive $f(x)$) **.** (***positive direction***) .

$$F(2) - F(1) = (2)^2 - (1)^2 = +3$$

The CHANGE in $F(x)$ is 3 in magnitude.

The **+ SIGN** reflects an INCREASE in $F(x)$ over $[1, 2]$ in the ***positive direction***.



Note: f(x) δx is an ***infinitesimal element*** of area.

f(x)dx is an ***instantaneous element*** of area. If we try to depict it, it will appear as a line.

We know from ***mensuration*** that the area of a trapezoid is $= \frac{1}{2}$ (sum of the parallel sides) **.** (perpendicular distance). This is the ***trapezoidal method*** of NUMERICAL INTEGRATION. We do not need to know $\bar{x}_i$. However, we must be careful to ensure that the parallel sides $f(x_{i-1})$ and $f(x_i)$ of the trapezium are of the same SIGN. So we must watch out for the points where f(x) CHANGES SIGN. In this example f(x) CHANGES SIGN at x = 0.

Example 2: Integrate ***continuous*** f(x) over the interval [1, 2] going RIGHT to LEFT in the ***negative direction*** .

The AREA under f(x) over the interval [1, 2] going RIGHT to LEFT in the ***negative direction*** is :

$$\int_2^1 f(x)dx = \int_2^1 2x\ dx = [x^2]_2^1$$

$$[x^2]_2^1 = (1)^2 - (2)^2 = -3$$

The AREA is 3 in magnitude.

The **– SIGN** is due to (positive f(x)) **.** (***negative direction***) .

$$F(1) - F(2) = (1)^2 - (2)^2 = -3$$

The CHANGE in F(x) is 3 in magnitude.

The **– SIGN** reflects a DECREASE in F(x) over [1, 2] in the ***negative direction***.

Example 3: Integrate ***continuous*** f(x) over the interval [–2, –1] going LEFT to RIGHT in the ***positive direction*** .

The AREA under f(x) over the interval [–2, –1] going LEFT to RIGHT in the ***positive direction*** is :

$$\int_{-2}^{-1} f(x)dx = \int_{-2}^{-1} 2x\ dx = [x^2]_{-2}^{-1}$$

$$[x^2]_{-2}^{-1} = (-1)^2 - (-2)^2 = -3$$

The AREA is 3 in magnitude.

The **– SIGN** is due to (negative f(x)) **.** (***positive direction***) .

$$F(-1) - F(-2) = (-1)^2 - (-2)^2 = -3$$

The CHANGE in F(x) is 3 in magnitude.

The **– SIGN** reflects a DECREASE in F(x) over [–2, –1] in the ***positive direction***.

Example 4: Integrate ***continuous*** f(x) over the interval [–2, –1] going RIGHT to LEFT in the ***negative direction*** .

The AREA under f(x) over the interval [–2, –1] going RIGHT to LEFT in the in the ***negative direction*** is :

$$\int_{-1}^{-2} f(x)dx = \int_{-1}^{-2} 2x\ dx = [x^2]_{-1}^{-2}$$

$$[x^2]_{-1}^{-2} = (-2)^2 - (-1)^2 = +3$$

The AREA is 3 in magnitude.

The **+ SIGN** is due to (negative f(x)) **.** (***negative direction***) .

$$F(-2) - F(-1) = (-2)^2 - (-1)^2 = +3$$

The CHANGE in F(x) is 3 in magnitude.

The **+ SIGN** reflects an INCREASE in F(x) over [–2, –1] in the ***negative direction***.

Let us now consider $F(x) = -x^2$. Then $f(x) = F'(x) = -2x$ and verify our calculations with the graph.

Example 5: Integrate ***continuous*** f(x) over the interval [1, 2] going LEFT to RIGHT in the ***positive direction*** .

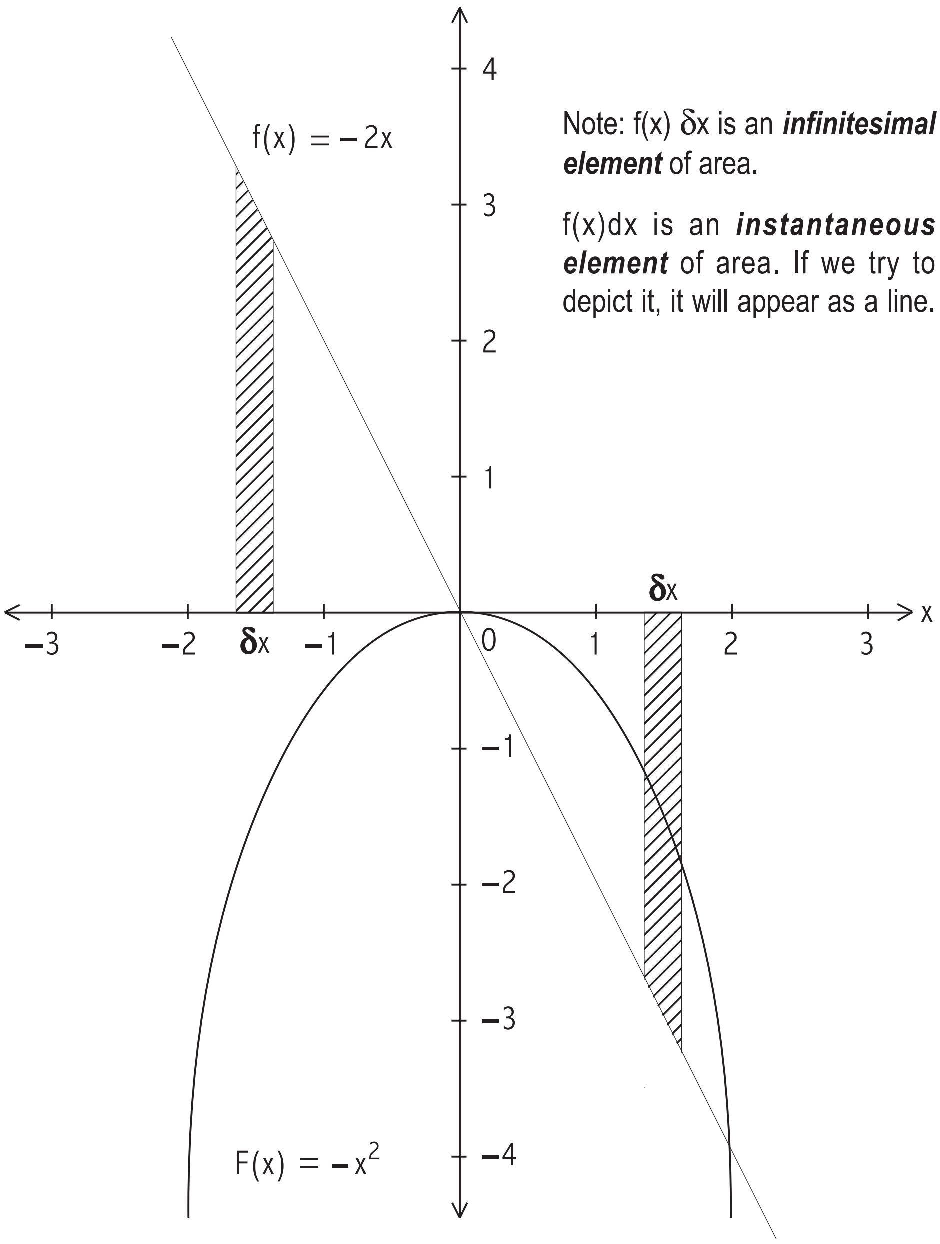
The AREA under f(x) over the interval [1, 2] going LEFT to RIGHT in the ***positive direction*** is :

$$\int_{1}^{2} f(x)dx = \int_{1}^{2} -2x\ dx = [-x^2]_{1}^{2}$$

$$[-x^2]_{1}^{2} = -(2)^2 - (-(1)^2) = -4 - (-1) = -3$$

The AREA is 3 in magnitude.

The **– SIGN** is due to (negative f(x)) **.** (***positive direction***) .



Note: f(x) δx is an ***infinitesimal element*** of area.

f(x)dx is an ***instantaneous element*** of area. If we try to depict it, it will appear as a line.

$$F(2) - F(1) = -(2)^2 - (-(1)^2) = -3$$

The CHANGE in F(x) is 3 in magnitude.

The **– SIGN** reflects a DECREASE in F(x) over [1, 2] in the ***positive direction*** .

Example 6: Integrate ***continuous*** f(x) over the interval [1, 2] going RIGHT to LEFT in the ***negative direction*** .

The AREA under f(x) over the interval [1, 2] going RIGHT to LEFT in the ***negative direction*** is :

$$\int_2^1 f(x)dx = \int_2^1 -2x\,dx = [-x^2]_2^1$$

$$[-x^2]_2^1 = -(1)^2 - (-(2)^2) = -1 + 4 = +3$$

The AREA is 3 in magnitude.

The **+ SIGN** is due to (negative f(x)) . (***negative direction***) .

$$F(1) - F(2) = -(1)^2 - (-(2)^2) = +3$$

The CHANGE in F(x) is 3 in magnitude.

The **+ SIGN** reflects an INCREASE in F(x) over [1, 2] in the ***negative direction***.

$$\boxed{\int_b^a f(x)dx = -\int_a^b f(x)dx}$$

RIGHT to LEFT over [a,b]	= –	LEFT to RIGHT over [a,b]

In all the previous examples f(x) did not CHANGE SIGN over the interval of integration. Now let us see two examples where f(x) CHANGES SIGN over the interval of integration.

Example 7: Integrate ***continuous*** f(x) = 2x over the interval [−1, 2] going LEFT to RIGHT in the ***positive direction*** . See diagram page 148.

The AREA under f(x) = 2x over the interval [−1, 2] going LEFT to RIGHT in the ***positive direction*** is :

$$\int_{-1}^{2} 2x\,dx = \int_{-1}^{0} 2x\,dx + \int_{0}^{2} 2x\,dx$$

We must break this up into 2 parts because f(x) CHANGES SIGN at x = 0 .

$$\int_{-1}^{0} 2x\,dx = [x^2]_{-1}^{0} = (0)^2 - (-1)^2 = 0 - 1 = -1$$

The AREA is 1 in magnitude.

The **− SIGN** is due to (negative f(x)) **.** (***positive direction***) .

$$\int_{0}^{2} 2x\,dx = [x^2]_{0}^{2} = (2)^2 - (0)^2 = 4 - 0 = +4$$

The AREA is 4 in magnitude.

The **+ SIGN** is due to (positive f(x)) **.** (***positive direction***) .

So: $\int_{-1}^{2} 2x\,dx = -1 + 4 = +3$

$$F(2) - F(-1) = \int_{-1}^{2} 2x\,dx = [x^2]_{-1}^{2}$$

$$= (2)^2 - (-1)^2 = 4 - 1 = +3$$

The CHANGE in F(x) is 3 in magnitude.

The **+ SIGN** reflects an INCREASE in F(x) over [− 1, 2] in the ***positive direction***.

Example 8: Integrate ***continuous*** $f(x) = 2x$ over the interval $[-1, 2]$ going RIGHT to LEFT in the ***negative direction*** . See diagram page 148.

The AREA under $f(x) = 2x$ over the interval $[-1, 2]$ going RIGHT to LEFT in the ***negative direction*** is :

$$\int_{2}^{-1} 2x \, dx = \int_{2}^{0} 2x \, dx + \int_{0}^{-1} 2x \, dx$$

We must break this up into 2 parts because f(x) CHANGES SIGN at $x = 0$.

$$\int_{2}^{0} 2x \, dx = [x^2]_{-1}^{0} = (0)^2 - (2)^2 = 0 - 4 = -4$$

The AREA is 4 in magnitude.

The **– SIGN** is due to (positive f(x)) . (***negative direction***) .

$$\int_{0}^{-1} 2x \, dx = [x^2]_{0}^{2} = (-1)^2 - (0)^2 = 1 - 0 = +1$$

The AREA is 1 in magnitude.

The **+ SIGN** is due to (negative f(x)) . (***negative direction***) .

So: $$\int_{2}^{-1} 2x \, dx = -4 + 1 = -3$$

$$F(-1) - F(2) = \int_{2}^{-1} 2x \, dx = [x^2]_{-1}^{2}$$

$$= (-1)^2 - (2)^2 = 1 - 4 = -3$$

The CHANGE in F(x) is 3 in magnitude.

The **– SIGN** reflects a DECREASE in F(x) over $[-1, 2]$ in the ***negative direction***.

A note on AREA and INTEGRATION

Several attempts have been made to equate AREA under f(x) over [a, b] from a ***mensuration*** point of view with the CHANGE in the INTEGRAL {F(b) − F(a)} for all possisble f(x). These attempts have not been successful because they failed to take into account the influence of the ***direction of integration*** over [a, b].

We may use the INTEGRAL F(x) to find the AREA under continuous f(x) over [a, b] when :

1. f(x) is ***positive*** (above the x-axis) and the ***direction of integration*** is ***positive***. In this case :

$$\begin{matrix}\text{AREA under } f(x) \\ \text{over } [a, b]\end{matrix} = \begin{matrix}\text{CHANGE in the INTEGRAL } F(x) \\ \text{over } [a, b]\end{matrix} = F(b) - F(a)$$

In both SIGN and MAGNITUDE as we saw in the examples 1 and 2.

2. If f(x) is ***negative*** (below the x-axis) then :

$$\begin{matrix}\text{AREA under } f(x) \\ \text{over } [a, b]\end{matrix} = [\,F(b) - F(a)\,] \text{ when the } \textbf{\textit{direction of integration}} \text{ over } [a, b] \text{ is } \textbf{\textit{positive}}$$

In both SIGN and MAGNITUDE as we saw in the examples 3 and 5.

$$\begin{matrix}\text{AREA under } f(x) \\ \text{over } [a, b]\end{matrix} = -[\,F(b) - F(a)\,] \text{ when the } \textbf{\textit{direction of integration}} \text{ over } [a, b] \text{ is } \textbf{\textit{negative}}$$

In both SIGN and MAGNITUDE as we saw in the examples 4 and 6.

We may use the AREA under continuous f(x) over [a, b] to find the CHANGE in the INTEGRAL F(x) over [a, b] in the positive direction as follows.

1. We must break up the AREA under f(x) over [a, b] into segments of ***positive areas*** and ***negative areas*** as we did in example 7.

positive area = area above the x-axis
negative area = area below the x-axis

Here we implicitly assumed that the ***direction of integration*** is ***positive*** .

2. We SUM UP all the ***negative*** and ***positive*** areas to get a net result, say A . Then:

CHANGE in F(x) over [a, b] in the ***positive direction*** = F(b) − F(a) = A

EXERCISES

The exercises are to show the relationship between the ***area under the curve*** $f(x) = F'(x)$ and the CHANGE IN VALUE of the Integral F(x).

Exercise 1: Repeat examples 7 and 8 with $f(x) = -2x$

Exercise 2: Integrate $f(x) = -2x$ over the interval $[-2, -1]$ from LEFT to RIGHT. Verify it against the CHANGE in $F(x) = -x^2$ from -2 to -1. What is the area under the curve ?

Exercise 3: Integrate $f(x) = -2x$ over the interval $[-2, -1]$ from RIGHT to LEFT. Verify it against the CHANGE in $F(x) = -x^2$ from -1 to -2. What is the area under the curve ?

Exercise 4: What is the CHANGE in the value of the function sin (x) over the intervals $[0, \pi/2]$, $[\pi/2, 3\pi/2]$, $[\pi, 2\pi]$, $[0, 2\pi]$. Use the fact that the INTEGRAL sin (x) is the ANTIDERIVATIVE of cos (x). Draw the curve of cos (x) over $[0, 2\pi]$. For each interval verify that the CHANGE IN AREA is equal to the CHANGE IN VALUE of sin (x) over the interval going LEFT to RIGHT. Repeat the exercise going RIGHT to LEFT.

30. Area under f(x) and Plotting $F_C(x)$

In the previous chapter we saw the relation between the ***area under the curve*** of ***continuous*** f(x) over the interval [a, b] and the CHANGE in F(x) over [a, b].

Let: $F_{-1}(x) = x^2 - 1 \Rightarrow f_{-1}(x) = F'_{-1}(x) = 2x$

$F_0(x) = x^2 \Rightarrow f_0(x) = F'_0(x) = 2x$

and $F_{+1}(x) = x^2 + 1 \Rightarrow f_{+1}(x) = F'_{+1}(x) = 2x$

In all 3 cases : $f(x) = F'(x) = 2x.$

Also, in all 3 cases: F(x) = ANTIDERIVATIVE {f(x)} = $\int f(x)\, dx = x^2$.

Again, in all 3 cases the CHANGE over the interval [a, b] is the same:

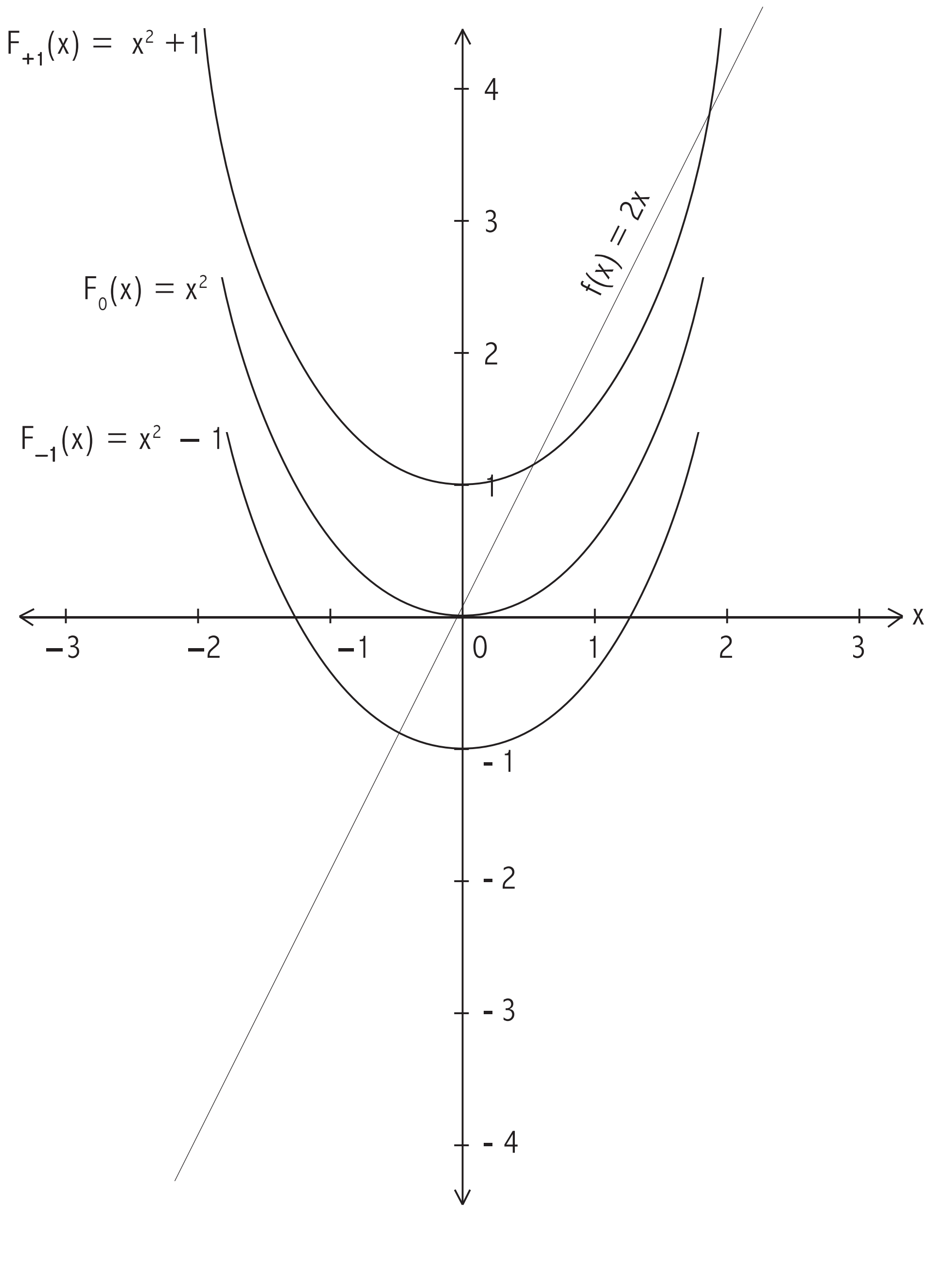
$$F_{-1}(b) - F_{-1}(a) = F_0(b) - F_0(a) = F_{+1}(b) - F_{+1}(a) = b^2 - a^2 .$$

$$F_{-1}(a) - F_{-1}(b) = F_0(a) - F_0(b) = F_{+1}(a) - F_{+1}(b) = a^2 - b^2 .$$

So given only f(x) how can we compute $F_C(x)$?

Given ***continuous*** f(x) we can find $F_0(x)$ and plot the graph of $F_0(x)$ directly. We saw the plot of $F_0(x)$ in the 2 examples in chapter 27. But we cannot plot the graphs of $F_{-1}(x)$ and $F_{+1}(x)$ directly. This is because we do not know the constant term. The constant -1 in $F_{-1}(x)$ and $+1$ in $F_{+1}(x)$ is known as the CONSTANT OF INTEGRATION and is denoted by **C** . In the case of $F_0(x)$ the constant **C** $= 0$.

For each value of **C** we have a curve similar to the curve of $F_0(x)$. In fact, $F_C(x) = F(x) + \mathbf{C}$ is a family of curves. This is the ***Geometric*** view of the role of **C** , the CONSTANT OF INTEGRATION.



$F_{+1}(x) = x^2 + 1$
$F_0(x) = x^2$
$F_{-1}(x) = x^2 - 1$
$f(x) = 2x$
x
−3
−2
−1
0
1
2
3
4
3
2
1
- 1
- 2
- 3
- 4

To know $F_c(x)$ we need to know **C**. To calculate **C** we need to know $F_c(x)$ at some point, say x_0. Then from $F_c(x) = F(x) + \mathbf{C}$ we have : $\mathbf{C} = F_c(x_0) - F(x_0)$. In $F_0(x) = x^2$ the constant $\mathbf{C} = 0$ since $F_0(x) = F(x)$ everywhere. If we know $F(x)$ and **C** then it is easy to plot $F_c(x)$.

Suppose we do NOT know F(x) and C . We only know f(x) , x_0 and $F_c(x_0)$.

How can we plot $F_c(x)$ without finding the expression $F_c(x)$?

If **F(a)** is known, then from **a** going LEFT to RIGHT in the ***positive direction*** to **b** :

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$F(b) = \int_a^b f(x)dx + F(a)$$

If **F(b)** is known, then from **b** going RIGHT to LEFT in the ***negative direction*** to **a**:

$$\int_b^a f(x)dx = F(a) - F(b)$$

$$F(a) = \int_b^a f(x)dx + F(b)$$

Using the relations above we may compute $F_c(x)$. We must choose a small interval Δx . Then $\{f(x).\ \Delta x\}$ is an element of ***area under the curve*** of f(x) over the interval Δx . The smaller the Δx the more accurate the approximation of the ***area under the curve*** of f(x) over the interval Δx . Another reason why we have to choose Δx very small is that we do not skip or miss the EXTREMUM points (maximum and minimum) in plotting $F_c(x)$. These are the points where f(x) CHANGES SIGN.

next $F_c(x)$ = CHANGE in $F_c(x)$ over next Δx + current value of $F_c(x)$

CHANGE in $F_c(x)$ over next Δx = ***area under the curve*** of f(x) over next Δx

Going RIGHT (in the *postive direction*) from x_0 with $i = 1$:

$$F_C(x_i) = \int_{x_{i-1}}^{x_{i-1}+\Delta x} f(x)\,dx + F_C(x_{i-1})$$

$$F_C(x_i) \cong f(x_i).(+\Delta x) + F_C(x_{i-1}) \text{ for } i = 1, 2, 3, \ldots$$

The + SIGN in ($+\Delta x$) is due to the ***postive direction*** in the integration.

Going LEFT (in the *negative direction*) from x_0 with $i = -1$:

$$F_C(x_i) = \int_{x_{i+1}}^{x_{i+1}-\Delta x} f(x)\,dx + F_C(x_{i+1})$$

$$F_C(x_i) \cong f(x_i).(-\Delta x) + F_C(x_{i+1}) \text{ for } i = -1, -2, -3, \ldots$$

The − SIGN in ($-\Delta x$) is due to the ***negative direction*** in the integration.

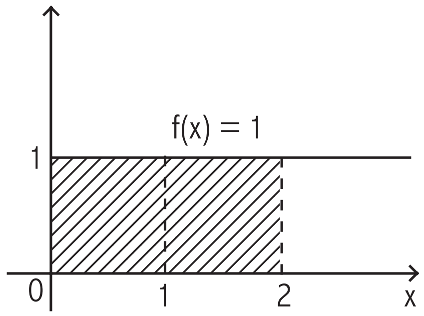
This is the ***rectangular method*** of NUMERICAL INTEGRATION. In the next chapter we shall see how to compute and interpret the meaning of the CONSTANT OF INTEGRATION from the nature of the problem.

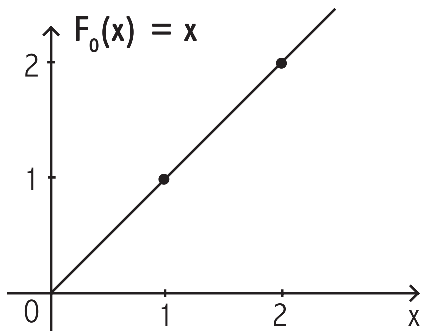
EXERCISES

The exercises are to show how to plot $F_c(x)$ when only f(x), x_0 and $F_c(x_0)$ are known and the constant of integration C is not known. Also, we do not need to know the expression of F(x). We make use of the relationship between the "***area under the curve***" f(x) = F′(x) and the CHANGE IN VALUE of the Integral F(x). On a computer this can be demonstrated very elegantly.

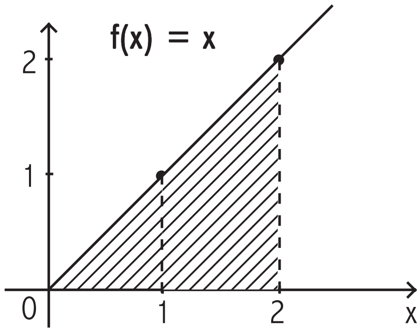
Draw the curve of the First Derivative f(x). Shade the area you are integrating and simultaneously plot its value. You will get the curve of the Integral $F_c(x)$. Do this LEFT to RIGHT(in the ***postive direction***) from x_0 and then RIGHT to LEFT (in the ***negative direction***) from x_0 . Choose a suitable Δx .

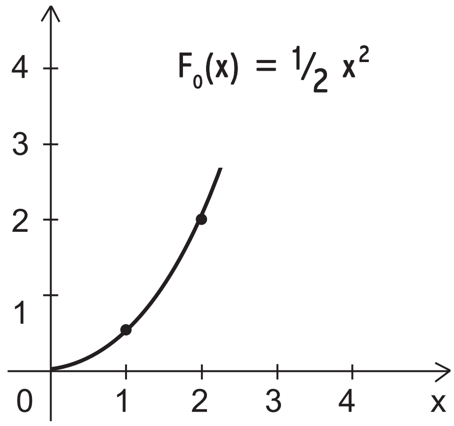
Exercise 1: Given f(x) = 1 use the ***rectangular method*** to plot $F_0(x)$ over [−2, 0].





Exercise 2: Given f(x) = x use the ***trapezoidal method*** to plot $F_0(x)$ over [−2, 0].





Exercise 3: On a graph paper draw the curve of the function f(x) = 2x over the interval [−4, 4]. Let $x_0 = 1$ and $F_c(x_0) = 2$. Plot $F_c(x)$ over interval [−4, 4].

Exercise 4: Repeat exercise 3 over the interval [−2, 2] using a smaller Δx.

Exercise 5: Repeat exercises 3 and 4 this time using $f(x) = x^2$. Compare the graph you get with the graph of $F(x) = x^3/3 + 1$.

In any kind of NUMERICAL INTEGRATION to find the CHANGE in F(x) or to find $F_c(x)$ we must take into consideration :

1. The instants or points where f(x) CHANGES SIGN.
2. The ***direction of integration***.

31. Constant of Integration

Any continuous function f(x) has an infinity of ANTIDERIVATIVES. If F(x) is one of them, then any other one may be given by the expression F(x) + C, where C is constant. From a ***Geometric*** point of view we have a family of curves.

$$\int \text{RATE OF CHANGE } f(x) = \text{CHANGE } F(x) + C \quad (\text{ a constant })$$

The constant C is called the CONSTANT OF INTEGRATION. C could be an arbitrary constant. But in general it depends on the situation.

In general: $\int f^{n}(x)dx = f^{n-1}(x) + C_{n-1}$

Let us see how we determine the constant of integration C_{n-1} by looking at three examples.

Example1: We know y″(t), the function that expresses the INSTANTANEOUS RATE OF CHANGE in vertical speed (i.e. acceleration). We can integrate y″(t) and get y′(t) the function that expresses the CHANGE in vertical speed.

We may think of y′(t), the function that expresses the CHANGE in vertical speed, as the function that expresses the INSTANTANEOUS RATE OF CHANGE in height. We can integrate y′(t) and get y(t) the function that expresses the CHANGE in height.

We know : $y''(t) = -g$ [meters / sec ^{2}]

$$y'(t) = \int y''(t)\,dt = \int (-g)\,dt = -gt + C_1 \quad [\text{ meters/ sec }].$$

The constant $C_1 = u.\sin\theta$ is the initial vertical velocity.

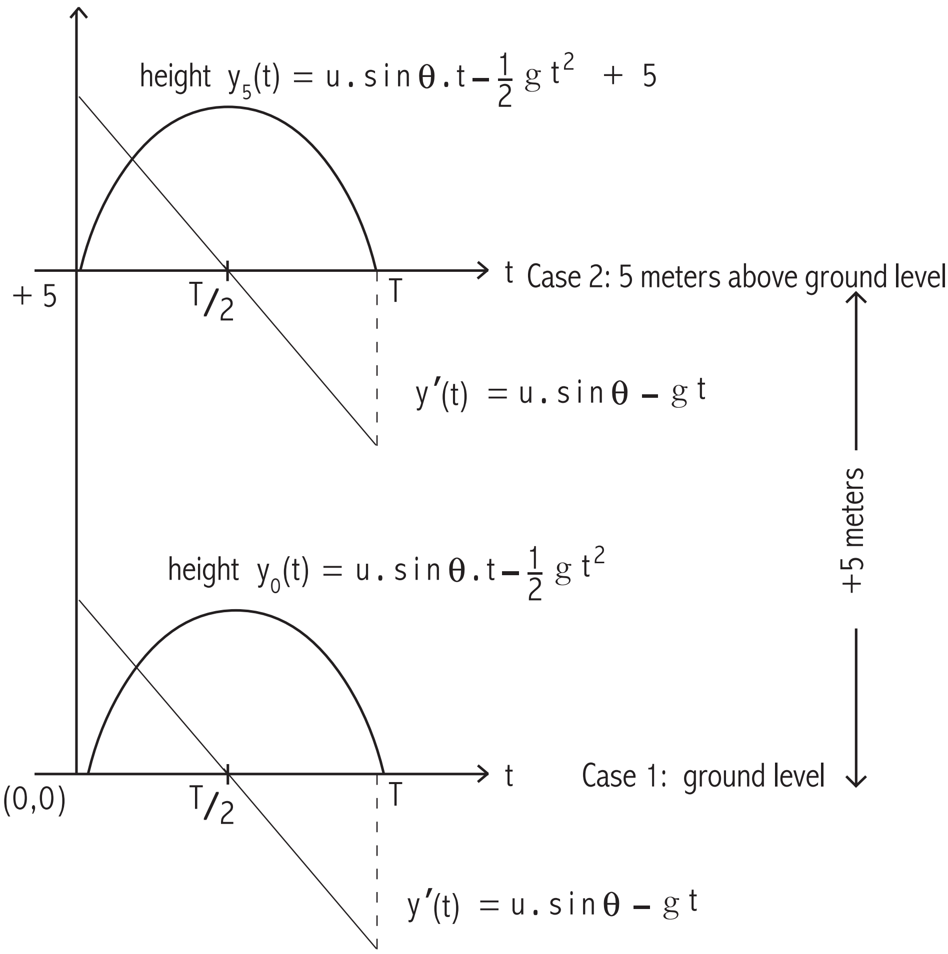
So $y'(t) = u.\sin\theta - gt$ [meters/ sec].

Now: $y(t) = \int y'(t)\,dt = \int (u.\sin\theta - gt)\,dt$

$$= u.\sin\theta . t - \frac{1}{2} g t^2 + C_0 \quad [\text{ meters }].$$

The constant C_0 is the initial height.

How do you determine the constant of integration C_0?



In both cases $y'(t)$ is the same. So : $y(t) = \int y'(t)\,dt = \int (u.\sin\theta - gt)\,dt$

So: $y(t) = u.\sin\theta.t - \frac{1}{2}gt^2$ is also the same.

We need to know something more about the initial conditions. For example :

Case 1: $C_0 = 0$ at ground level

Case 2: $C_0 = +5$ meters above ground level.

$y(t_i) = \int_0^{t_i} y'(t) \mathbf{.} dt$ gives us only the CHANGE in height. It does NOT tell us the height at instant t_i . We must know the initial height at time $t = 0$, which in this case is the constant of integration C_0 . Then :

$$\text{HEIGHT at instant } t_i = \text{CHANGE in height} + \text{INITIAL height}$$

$$= y(t_i) + C_0$$

Example 2: We know the relationship between temperature **t** measured in °F and °C .

$$°F\,(t) = {}^9\!/_5\; t\,°C + 32$$

$$°F\,'(t) = \frac{d°F(t)}{dt} = {}^9\!/_5$$

$$\int °F\,'(t)\; dt = {}^9\!/_5\; t\,°C + C_0$$

How do we determine the constant of integration C_0? We must know °F (t) at $t = 0$. We may conduct an experiment. We immerse the °F thermometer and the °C thermometer in a beaker of ice which we know is 0 °C . Then we read the °F thermometer to get $C_0 = 32$. So:

$$°F\,(t) = {}^9\!/_5\; t\,°C + 32$$

Example 3: A pump delivers with a rate of flow $f'(\mathbf{t}) = 60$ litres/minute.

What is the volume of water in the tank after 5 minutes ?

$$f(t) = \int f'(\mathbf{t}) \;.\; dt = 60\,\mathbf{t} + C_0$$

$$\int_{t=0}^{t=5} f'(\mathbf{t}) \;\mathbf{.}\; dt = \int_{t=0}^{t=5} 60 \;\mathbf{.}\; dt = [60\,\mathbf{t}\,]_0^5 \text{ litres} = 300 \text{ litres}$$

However, this is NOT the volume of water in the tank. This is only the CHANGE in volume. We must know the INITIAL volume, say $C_0 = 500$ litres. Then the volume of water in the tank at $\mathbf{t} =$ 5 minutes is :

$$\begin{aligned}\text{VOLUME (at } \mathbf{t} = 5 \text{ minutes)} &= \text{CHANGE in volume} + \text{INITIAL volume} \\ &= 300 \text{ litres} + 500 \text{ litres} \\ &= 800 \text{ litres.}\end{aligned}$$

This example may be easily adapted to compute the charge on a capacitor and also the time taken to charge or discharge the capacitor given $f'(\mathbf{t})$ = the rate of flow of charge = current $\boldsymbol{i} = {}^{dq}/_{dt}$. Let us look at this very interesting in example in a more general way .

$$\int_{\text{START TIME}}^{\text{STOP TIME}} (\text{ RATE of FLOW}).\, dt = \text{CHANGE in VOLUME}$$

There are 4 entities involved: $f'(\mathbf{t})$ = RATE of FLOW, CHANGE in VOLUME, START TIME and STOP TIME. Usually we know $f'(\mathbf{t})$. Now we have 3 possibilities.

1. If we know the START TIME and STOP TIME we may compute the CHANGE in VOLUME for the given RATE of FLOW $f'(\mathbf{t})$.

2. If we know the START TIME and CHANGE in VOLUME we may determine the STOP TIME. Then (STOP TIME – START TIME) will tell us how much time it took to pump a certain volume of water, or charge a capacitor from the given START TIME.

3. If we know the STOP TIME and CHANGE in VOLUME, then we may determine the START TIME for the given RATE of FLOW $f'(\mathbf{t})$.

32. INDEFINITE and DEFINITE INTEGRAL

Let $f(x) = x^2$. When we ***differentiate*** f(x) to get the ***derivative*** $f'(x) = 2x$, we do not use the term ***"indefinite"*** to say : $f'(x)$ is the ***"indefinite derivative"*** of f(x). Also, when we evaluate $f'(x) = 2x$ at some instant ***a*** to get ***2a***, we do not use the term ***"definite"*** to say : ***2a*** is the ***"definite derivative"*** of f(x) at x = a.

When it comes to ***integration*** we have the terms ***indefinite*** and ***definite***.

$F(x) = \text{ANTIDERIVATIVE}\{f(x)\} = \int f(x)dx$ is called the INDEFINITE INTEGRAL of the ***integrand*** f(x). The INDEFINITE INTEGRAL F(x) of a given function f(x) is the most general form of its ANTIDERIVATIVE.

We may evaluate the INDEFINITE INTEGRAL F(x) over and interval [a, b].

$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ which is the CHANGE in F(x) from a to b.

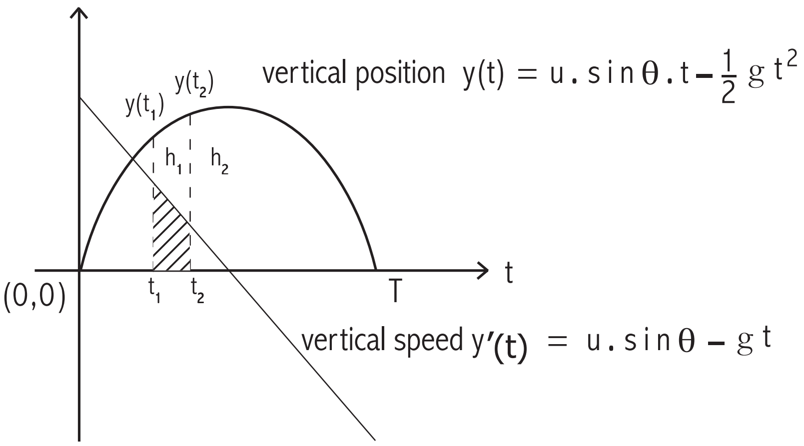
$\int_a^b f(x)dx$ is also called the DEFINITE INTEGRAL of the ***integrand*** f(x).

We may evaluate $F_C(x) = F(x) + \mathbf{C}$ over and interval [a, b]. In the process the CONSTANT OF INTEGRATION **C** disappears.

$\int_a^b f(x)dx = [F_C(x)]_a^b = \{F(b) + \mathbf{C}\} - \{F(a) + \mathbf{C}\} = F(b) - F(a)$ which is again only the CHANGE in $F_C(x)$ from a to b.

If F(x) is the expression of CHANGE in ***position***, then evaluation of the INDEFINITE INTEGRAL F(x) over the ***interval*** [a, b] is only the CHANGE in ***position*** from a to b = F(b) − F(a).

To find the ***position*** at any chosen ***instant*** x_i starting from instant a, we need to know F(a), the the position at instant a, and the CHANGE or DEFINITE INTEGRAL $\int_a^{x_i} f(x)dx$.



INDEFINITE INTEGRAL $y(t) = \int y'(t)dt = \int \{u.\sin\theta - g t\}dt$

$= u.\sin\theta.t - \frac{1}{2} g t^2$

$=$ ANTI-DERIVATIVE of $y'(t)$

$=$ area under $y'(t)$

$=$ expression of CHANGE in $y(t)$.

$\int_{t_1}^{t_2} y'(t)dt = y(t_2) - y(t_1) =$ shaded area under vertical speed curve $y'(t)$

$=$ CHANGE in $y(t)$ from t_1 to t_2

$= h_2 - h_1 =$ CHANGE in height from t_1 to t_2.

DEFINITE INTEGRAL $y(t)$ over $[t_1, t_2] = \int_{t_1}^{t_2} y'(t)dt$

The concept or view of integration as ***"area under the curve"*** is useful in getting started. The most general and correct concept is : the ***indefinite integral*** F(x) is always the expression of CHANGE. When we evaluate the ***indefinite integral*** F(x) over a given interval and in a given direction, the value or number we get is called the ***change*** .

We have three points of view :

Calculation : ANTIDERIVATIVE = $\int$ DERIVATIVE

$y(t)$ = ANTI-DERIVATIVE of $y'(t) = \int y'(t).dt$

$F(x) = \int f(x)dx$ where $f(x) = F'(x)$

$F_c(x) = F(x) + \mathbf{C}$

Geometric : ***area under the curve*** of the INSTANTANEOUS RATE OF CHANGE

$y(t)$ = ***area under the curve*** $y'(t) = \int y'(t).dt$

$F(x)$ = ***area under the curve*** $f(x) = \int f(x)dx$

Analytical : EXPRESSION OF CHANGE = $\int$ INSTANTANEOUS RATE OF CHANGE

CHANGE in $y(t)$ over $[t_1, t_2] = y(t_2) - y(t_1) = \int_{t_1}^{t_2} y'(t).dt$

$F(x) = \int f(x)dx$ the ***indefinite integral***

$\int_a^b f(x)dx = F(b) - F(a)$ = CHANGE in $F(x)$ from a to b

is the ***definite integral*** .

A function F(x) is an expression of CHANGE. In working with INFINITESIMALS we have a relationship between CHANGE and INSTANTANEOUS RATE OF CHANGE. If we know f(x) the expression of the INSTANTANEOUS RATE OF CHANGE we can INTEGRATE f(x) and find F(x) the expression of CHANGE.

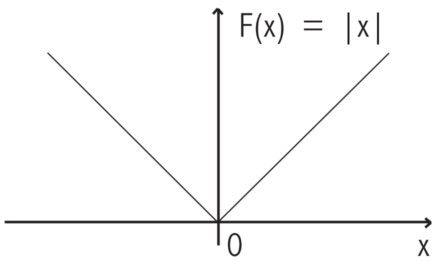
33. INTEGRABLE

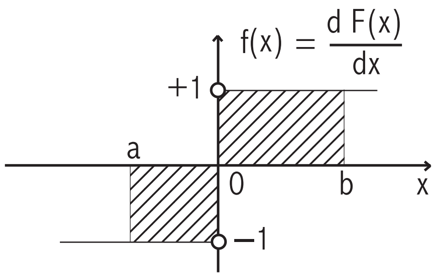
We have used the word INTEGRABLE without giving a formal definition. Here we give an informal definition with a **geometrical** explanation. We also present an intepretation from a **Physics** point of view with the focus on ***position*** and ***energy*** .

Let us start with a strict definition. Let $f(x) = \dfrac{d\,F(x)}{dx}$.

Then f(x) is INTEGRABLE ***if and only if*** F(x) is WELL-BEHAVED - ***single valued, continuous*** and ***differentiable***. The emphasis here is on the ***differentiable*** property.

We may be less strict by letting F(x) be just ***single valued*** and c***ontinuous.*** In this case it is possible that f(x) will be ***discontinuous*** at some instants as in the example below. At such instants where f(x) is ***discontinuous*** we require f(x) to have two definite values.





We see that f(x) is defined over the sub-intervals $[a, 0^-)$ and $(0^+, b]$. Notice that f(x) is ***discontinuous*** at 0 but it has two definite values.

$$f(0^-) = -1 \quad \text{and} \quad f(0^+) = +1$$

With this type of discontinuity we may still integrate f(x) over [a, b] .

$$\int_a^b f(x)\,dx = \int_a^{0^-} f(x)\,dx + \int_{0^+}^b f(x)\,dx$$

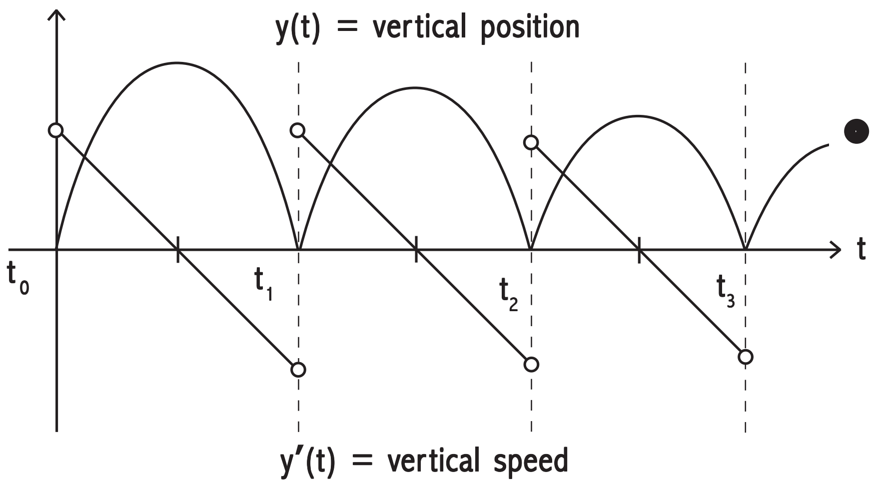
With the understanding that $\int$ is a ***continuous summation***, we want the sum to be FINITE or CONVERGE to some definite value. In this case we can see that both $\int_a^{0^-} f(x)\, dx = S_1 = -a$ is something finite, and $\int_{0^+}^{b} f(x)\, dx = S_2 = +b$ is something finite. Also their sum does add up to a definite value.

$$\left\{ \int_a^{0^-} f(x)\, dx \right\} + \left\{ \int_{0^+}^{b} f(x) dx \right\} = S_1 + S_2 = -a + b$$

We state 4 basic conditions for f(x) to be ***integrable***.

1. f(x) is defined over [a, b] (that is to say SINGLE VALUED) except at the points of discontinuity. And f(x) has only FINITELY many n discontinuities over [a, b].
2. An each such point of ***discontinuity*** f(x) has 2 definite values.
3. Over each of the (n + 1) sub-intervals the ***continuous summation*** must be something finite, say : $S_0, S_1, S_2, \ldots, S_n$.
4. The combined ***discrete summation*** $S = \{S_0 + S_1 + S_2 + \ldots + S_n\}$ must also be something finite.

We may relax the first condition be letting f(x) be discontinuous at COUNTABLY many points. But at each such point of ***discontinuity*** f(x) must have 2 definite values. With this in mind let us review the example of the ***bouncing ball***.



At each of the instants $t_0, t_1, t_2, \ . \ . \ . \ ,$ $y'(t_i)$ is ***discontinuous*** and has 2 definite values $y'(t_i^-)$ and $y'(t_i^+)$. The ball bounces indefinitely. So there are infinitely many such instants of discontinuity. However, these instants are COUNTABLE. So we may integrate $y'(t)$ over each of the sub-intervals (t_0, t_1), (t_1, t_2), (t_2, t_3), ... to get something finite. (In this example the integration of $y'(t)$over each sub-intervals is zero, because as we can see the CHANGE in height $y(t)$ over each sub-intervals is zero. But this may not generally be the case). The ***continuous summation*** will CONVERGE. This means that $y(t)$ will be something definite. From a practical or ***Physics*** point of view we know the ***vertical position*** of the ball at any instant.

THEORETICAL CONVERGENCE from the **Physics** point of view means that the ball will come to rest. And at each bounce the ball loses a little ***energy*** . There are two ways in which the ball may come to rest.

1. The ball may bounce FINITELY many times and then come to rest in a FINITE time interval $[0, T]$. Correspondingly, $y'(t)$ will have only FINITELY many points of discontinuity.
2. The ball may bounce indefinitely and "**eventually**" or "**in the LIMIT**" the ball will come to rest. Here too there are two possibilities.
   (i) The ball may bounce indefinitely over a FINITE time interval. Correspondingly, $y'(t)$ will have COUNTABLY many points of discontinuity over a FINITE time interval $[0, T]$.
   (ii) The ball may bounce indefinitely over an INFINITE time interval. Correspondingly, $y'(t)$ will have COUNTABLY many points of discontinuity over the INFINITE time interval $[0, +\infty)$. So eventually the ball will run out of ***energy*** and come to rest. From a practical point of view we may say that the ball disappears into TIME.

We know that ***energy*** E is a function of displacement. In simple terms $\mathbf{E = m.a.s}$, or more precisely $\mathbf{E = m.a.ds}$, where $\mathbf{m}$ = mass, $\mathbf{a}$ = acceleration, and $\mathbf{ds}$ = an element of displacement. In the case of the bouncing ball $\mathbf{ds} = \mathbf{d}y(t) = y'(t)\mathbf{d}t$.

Depending on how the ball comes to rest, in the integral $\int_a^b$ to compute the energy E, b = some finite instant T or b = $+\infty$. Now comes the question:

Does the integral $\int_a^b$ CONVERGE ?

If it does CONVERGE then the energy function E is INTEGRABLE.

When there is such a loss of ***energy*** (damping) on each bounce we can see that:

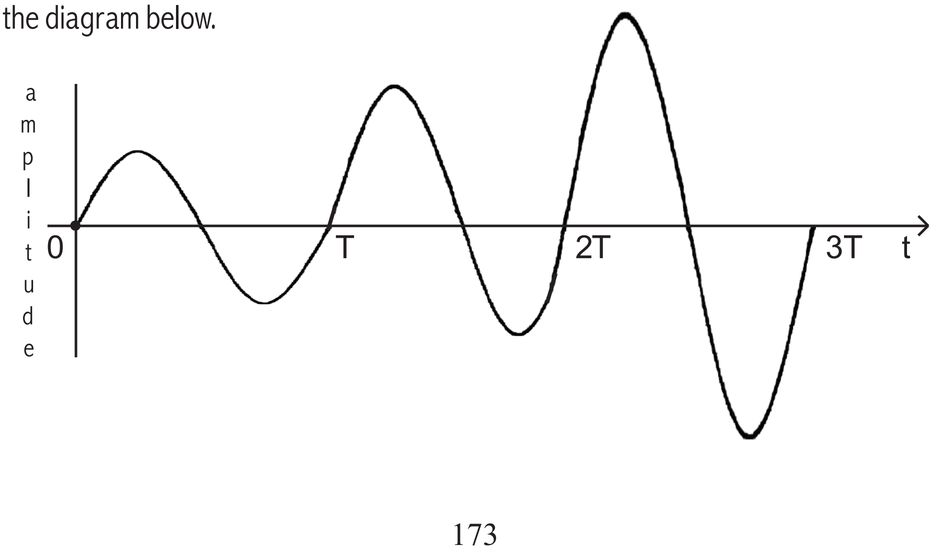
$$E = \int m.a.ds \text{ must CONVERGE.}$$

From the ***Physics*** point of view this CONVERGENCE means that the ***energy*** E required for the ball to bounce indefinitely is finite.

What if the ball bounces without loss or gain of ***energy*** ? The energy will change form between kinetic and potential. But the total energy will be CONSTANT. From the engineering point of view we will have a perpetual motion machine.

We have a similar situation in the hydrogen atom. The electron revolves around the proton in an elliptical orbit known as the path or curve of CONSTANT energy.

Now let us look at the situation where the ball gains energy on each bounce. Eventually the ball will disappear from sight. From a practical point of view we may say that the ball disappears into SPACE. This is similar to saying a swing or pendulum increases in amplitude with each oscillation. The displacement of such a pendulum is described in the diagram below.



In this case it is easy to see that the ***energy*** E, as a function of displacement, will be infinite. And hence:

$$\mathbf{E = \int m.a.ds} \text{ will not CONVERGE.}$$

Here we implicitly assumed that the length of the pendulum was infinite. What if the pendulum was of finite length ? Does it mean that the pendulum will swing all the way round in a circular orbit ?

What if the ball bounces indefinitely over infinite time interval $[0, +\infty)$ with gain in energy on each bounce ? Will the ball disappear both in TIME and SPACE ?

It is of fundamental importance to see the relationship between the abstract calculation on the one hand and the physical meaning (the reality - what is happening in nature) on the other hand.

With all this mind we now give informal definition of INTEGRABLE.

1. f(x) is defined over [a, b] (that is to say SINGLE VALUED) except at the points of discontinuity. And f(x) has at most COUNTABLY many discontinuities over [a, b]. So between each pair of consecutive such points of discontinuity we have a sub-interval.

2. An each such point of ***discontinuity*** f(x) has 2 definite values.

3. Over each of the sub-intervals in [a, b] the ***continuous summation*** $\int f(x)dx$ must be something FINITE, say : $S_0, S_1, S_2, \ldots, S_n, \ldots$.

4. The combined ***discrete summation*** $S = \{S_0 + S_1 + S_2 + \ldots + S_n + \ldots\}$ over all the COUNTABLY many sub-intervals must also be something FINITE so that $\int_a^b f(x)dx$ CONVERGES.

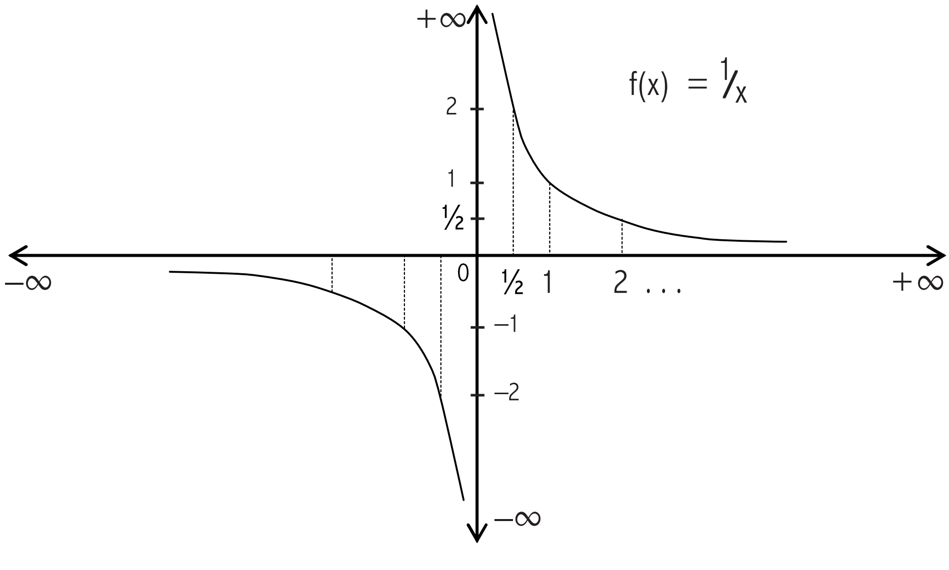
Then we say f(x) is INTEGRABLE.

A word on CONVERGES

The fundamental requirement in ***integration*** is that the ***continuous summation*** CONVERGES. Let us compare two examples of ***discrete summation*** and ***continuous summation***.

The ***discrete summation*** $\sum_{n=0}^{\infty} 1/n$ does NOT CONVERGE.

The integral $\int_0^{+1} \{1/x\}\, dx$ is a ***continuous summation*** over the finite interval [0, +1]. But the summation does not CONVERGE. So it is not INTEGRABLE.



We can see from the diagram that the area under the curve $f(x) = 1/x$ over the finite interval [0, +1] is NOT FINITE. From a **Geometric** point of view we may define INTEGRABLE as :

f(x) is INTEGRABLE $\Leftrightarrow$ the area under f(x) is FINITE

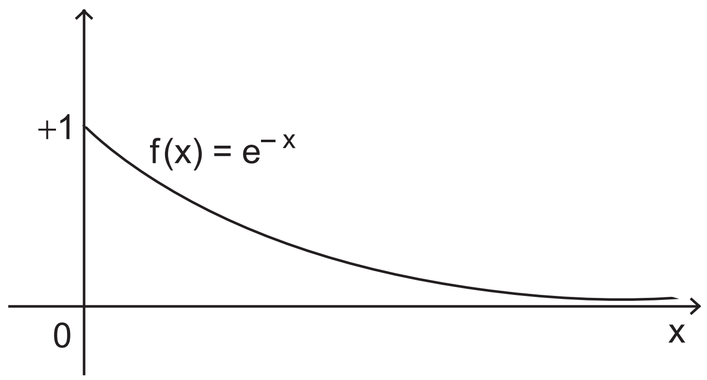
The ***discrete summation*** $\sum_{i=0}^{\infty} 1/2^i$ CONVERGES. The terms are finite in value.

However, the subscript **i** varies over the infinite interval $[0, +\infty)$.

Likewise the ***discrete summation*** $\sum_{i=0}^{\infty} e^{-i}$ also CONVERGES.

It is possible to have an integral or a ***continuous summaiton*** over an infinite interval.

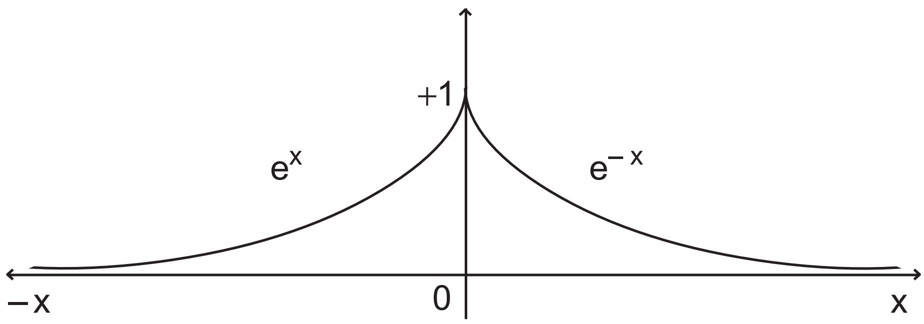
It is now easy to see that the ***continuous summation*** $\int_0^{+\infty} \{e^{-x}\}\, dx$ must CONVERGE even though the interval $[0, +\infty)$ is infinite. Hence $f(x) = e^{-x}$ is INTEGRABLE.



$$\int_0^{+\infty} \{e^{-x}\}\, dx = \left[-e^{-x}\right]_0^{+\infty} = (-e^{-\infty}) - (-e^{-0}) = 0 - (-1) = +1$$

We may also integrate over $(-\infty, +\infty)$ as in the example below. Let

$$f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ e^{x} & \text{for } x \leq 0 \end{cases}$$



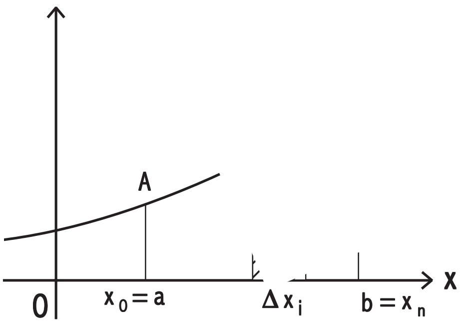
$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{+\infty} f(x)\,dx = +1 + 1 = +2$$

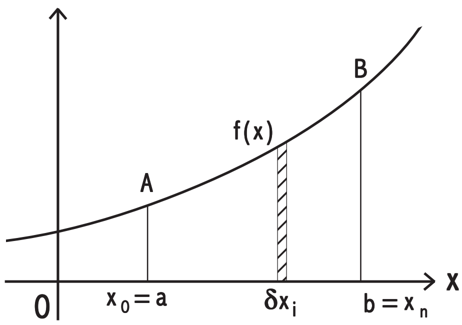
We are familiar with the ***associativity*** property in addition : $(a+b)+c = a+(b+c)$. This property holds good when the number of terms are finite. It also holds good for a ***discrete summation*** of COUNTABLY many terms if all the terms are of same sign. But it breaks down when there are COUNTABLY many terms of both positive and negative sign. The ***alternating series*** $+1 -1 +1 -1 + \ldots$ does NOT CONVERGE. Depending on how we do the summation we have 3 possible answers : -1, 0, and $+1$. So one can imagine the difficulties involved in a ***continuous summation*** with UNCOUNTABLY many terms of both signs. Proving CONVERGENCE becomes very, very difficult.

A more formal and rigorous treatment of INTEGRABLE dealing with the problems of CONVERGENCE - ***discontinuities***, ***finiteness*** and ***associativity*** - is covered in A LITTILE MORE CALCULUS.

34. Applications of Integration

There are numerous applications of integration. In this chapter we cover a few examples to illustrate some of the techniques in integration. Let us review the 3 in integration.





1. APPROXIMATE AREA step : ***discrete summation*** of FINITELY many terms.

$$F(x) = \sum_{i=0}^{n-1} \Delta F(x_i) = \sum_{i=0}^{n-1} f(x_i).\Delta x_i$$

So : $\Delta F(x_i) = f(x_i).\Delta x_i$ is an ***element*** of area.

2. TENDS TO step : ***discrete summation*** as $n \rightarrow \infty$ of COUNTABLY many terms.

$$F(x) = \sum_{i=0}^{n\rightarrow\infty} \delta F(x_i) = \sum_{i=0}^{n\rightarrow\infty} f(x_i).\delta x_i$$

So : $\delta F(x_i) = f(x_i).\delta x_i$ is an ***infinitesimal element*** of area .

3. LIMIT step : ***continuous summation*** of UNCOUNTABLY many terms.

$$F(x) = \textit{Limit} \sum_{i=0}^{n\rightarrow\infty} \delta F(x_i) = \int_a^b dF(x) = \textit{Limit} \sum_{i=0}^{n\rightarrow\infty} f(x_i).dx = \int_a^b f(x).\,dx$$

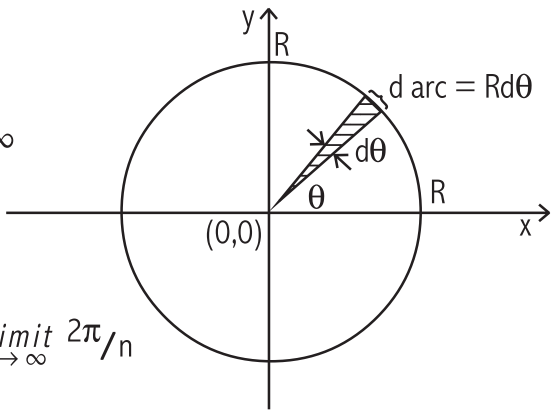
Hence : $dF(x) = f(x).dx$ is an ***instantaneous element*** of area .

In the diagrams in the examples that follow we cannot depict an ***instantaneous element*** f(x)dx . However, in keeping with the standard notation in text books we have used the label **dx** of the ***instantaneous element*** rather than **δx** of the ***infinitesimal element*** . This should NOT give the wrong impression or wrong notion that **dx** is an ***infinitesimal interval*** . The calculation with respect to **dx** be it ***differentiation*** or ***integration*** is always ***instantaneous*** .

From the diagrams the student should be able to construct an ***infinitesimal element*** , say **f(x) δx** , and then see that indeed **f(x) dx** $= \underset{n \to \infty}{Limit}$ **f(x) δx** .

$$\Delta\theta = {}^{2\pi}/_{n}$$

$$\delta\theta = {}^{2\pi}/_{n} \text{ as } n \to \infty$$



$$d\theta = \underset{n \to \infty}{Limit}\ \delta\theta = \underset{n \to \infty}{Limit}\ {}^{2\pi}/_{n}$$

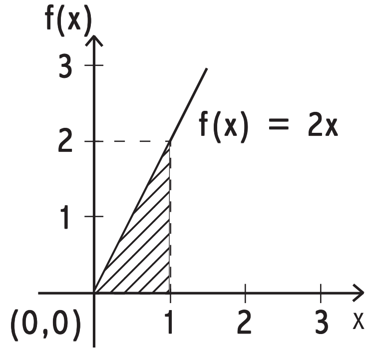
In the diagram above an ***infinitesimal element*** of area **δ**A is a triangle (wedge shape) of magnitude

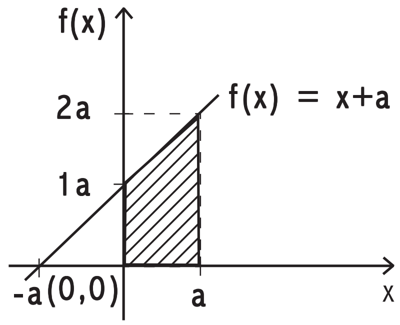
$$\boldsymbol{\delta}A = {}^{1}/_{2}(R.\delta\theta).R = {}^{1}/_{2}R^2\delta\theta$$

$$\mathbf{d}A = \underset{n \to \infty}{Limit} \{ {}^{1}/_{2}(R.\delta\theta).R \} = \underset{n \to \infty}{Limit} \{ {}^{1}/_{2}R^2\delta\theta \} = {}^{1}/_{2}R^2 d\theta$$

is an ***instantaneuos element*** of area.

Example 1: Find the ***area under the curve*** $f(x) = 2x$ over the interval [0, 1].





Using Integral Calculus: $F'(x) = f(x) = 2x$

The INTEGRAL $F(x)$ = ANTIDERIVATIVE of $f(x)$

What is the ANTIDERIVATIVE of $f(x) = 2x$? $x^2 + C$

$$F(x) = \int f(x)\, dx = \int 2x dx = x^2 + C$$

In this case the constant $C = 0$. Can you think why ?

The ***area under the curve*** $f(x) = 2x$ from $x = 0$ to $x = 1$ is:

$$\int_0^1 f(x)dx = \int_0^1 2xdx = [x^2]_0^1 = 1^2 - 0^2 = 1.$$

Evaluation of the INTEGRAL $F(x) = x^2$ in the ***positive direction*** over the interval [0, 1] is the CHANGE

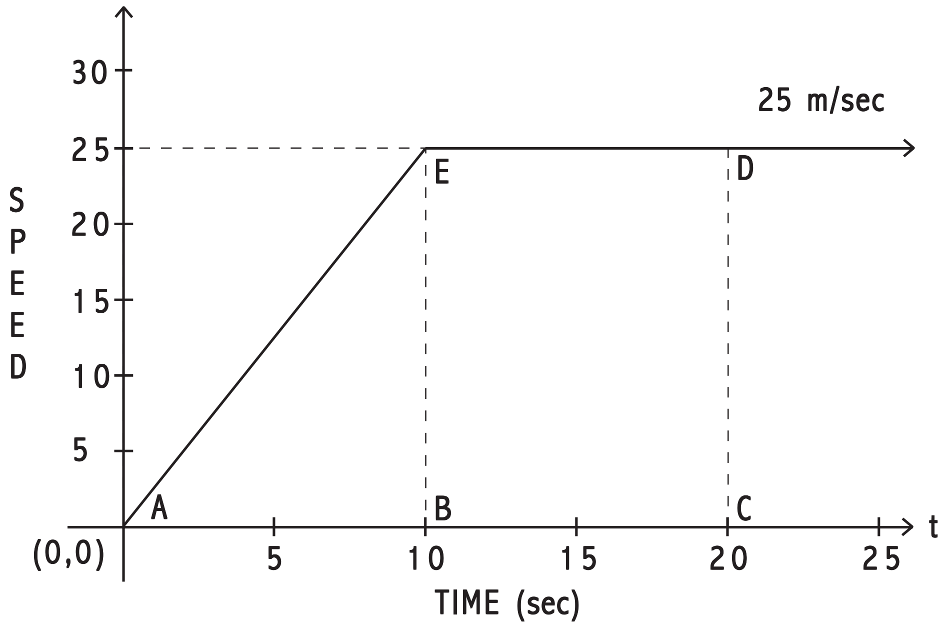
$$[F(x)]_0^1 = [x^2]_0^1 = 1^2 - 0^2 = 1.$$

Note: $[F_c(x)]_0^1 = [x^2+c]_0^1 = [1^2+c] - [0^2+c] =$ again 1.

Exercise: What is area under $f(x) = x + a$ over the interval [0,a]. Ref. page72.

In Physics the most important entity we are interested in is ***energy***. Any ***work done*** requires ***energy*** and causes CHANGE and any CHANGE involves ***energy***. The sense of the word CHANGE in Calculus is not synonymous with the word CHANGE (= ***work done*** = ***energy***) in Physics.

Example 2: A car weighing one ton and starting from rest (0 kmph) accelerates steadily due EAST (along the x-axis) for 10 secs and levels off at 90 kmph (25 m/sec). What is the ***distance travelled*** from t = 10 secs to t = 20 secs and also from t = 0 secs to t = 10 secs ?



$$\text{DISTANCE} = \text{SPEED} * \text{TIME} \quad [\text{meters} = \frac{\text{meters}}{\text{seconds}} * \text{seconds}]$$

over the interval [t = 10 secs, t = 20 secs] the SPEED = 25 m/sec (constant).

Distance travelled from t = 10 secs to t = 20 secs is 25m/sec * 10 sec = 250m which is the ***area under the speed curve*** = area of rectangle BCDE.

In the case of ***distance travelled*** between t = 0 secs and t = 10 secs, the SPEED is NOT constant. What SPEED are you going to use?

By looking at the graph we know the SPEED at each and every INSTANT between $t = 0$ secs and $t = 10$ secs. So we have to take the product of SPEED and TIME at each and every INSTANT and sum it up. This in effect will give the ***area under the speed curve*** from $t = 0$ secs to $t = 10$ secs which is the area of triangle ABE.

This is $\frac{1}{2}$ base $*$ height $= \frac{1}{2} * 10$ secs $* 25$ m / sec $= 125$ meters .

Conceptually we note that from the ***speed curve*** (INSTANTANEOUS RATE OF CHANGE of position graph) we got the **CHANGE in position**. CHANGE in position has both sign and magnitude. In this case we have denoted going East (right) as ***positive*** and going West (left) as ***negative*** on the x-axis**.** We may denote the SPEED of the car by $x'(t)$ and hence its POSITION on the x-axis will be $x(t)$.

Because the acceleration is steady or constant over [0, 10) we may use the average acceleration over [0, 10).

acceleration over [0, 10) : $x''(t) = \dfrac{x'(10) - x'(0)}{10 \text{ secs}} = 2.5 \text{m}/\text{sec}^2$

acceleration over $t \geq 10$: $x''(t) = 0$

speed over [0, 10] :

$$x'(t) = \int x''(t)dt = 2.5t \text{ m}/\text{sec}$$

speed over $t \geq 10$:

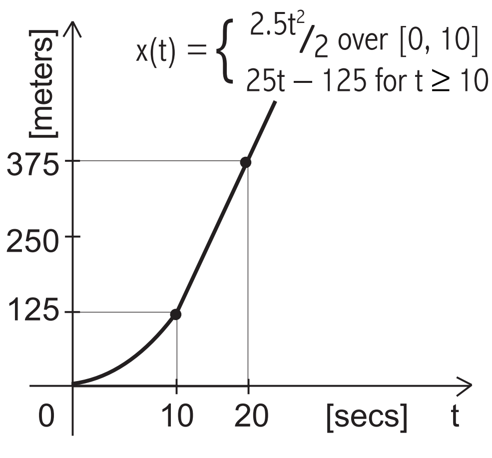
$$x'(t) = 25 \text{ m}/\text{sec}$$

position over [0, 10] :

$$x(t) = \int 2.5t \, dt = 2.5t^2/2$$

position over $t \geq 10$:

$$x(t) = \int 25 \, dt = 25t - 125$$



From the nature of the problem at $t = 10$ seconds we may determine that the CONSTANT OF INTEGRATION is -125 meters .

We need to be more clear about the terms CHANGE in position, ***displacement***, and ***distance travelled*** in **A)** 1-Dimension in ***fixed direction***, **B)** ***fixed axis*** or 1-Dimension with CHANGE in direction and **C)** 2-Dimensions. These terms need to be explained from both the **Calculus** point of view and the **Physics** (what is happening in nature) point of view and how they are related.

A) 1-Dimension in ***fixed direction*** : in the example the car travelled East in a fixed direction, that is to say moving along a straight line without reversing or changing direction. So in this very speical case from the **Physics** point of view:

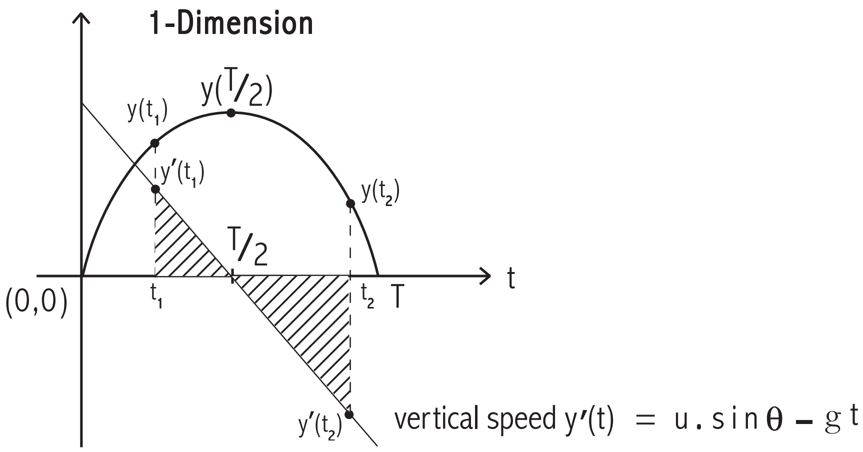
CHANGE in position = ***displacement*** = ***distance travelled*** .

Also, from the **Calculus** point of view, over a given interval, say $[t_1, t_2]$:

$$\text{CHANGE in position} = \int_{t_1}^{t_2} x'(t)dt = \textit{area under the speed curve}$$

This is because the SPEED function did NOT CHANGE SIGN. In **1-Dimensional** motion the SPEED function will CHANGE SIGN only if there is a CHANGE in direction.

B) ***fixed axis*** or 1-Dimension with CHANGE in direction : Let us review an example of the projected ball.



CHANGE in position over $[t_1, t_2] = \int_{t_1}^{t_2} y'(t)dt = y(t_2) - y(t_1)$

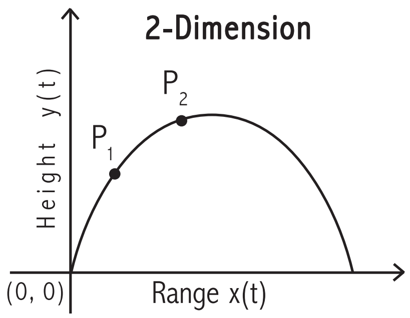
dispalcement $= y(t_2) - y(t_1)$

distance travelled $= \left|\int_{t_1}^{T/2} y'(t)dt\right| + \left|\int_{T/2}^{t_2} y'(t)dt\right|$

$= |y(T/2)| - y(t_1) \,| + |\, y(t_2) - y(T/2)|$

The ***distance travelled*** (odometer reading or length of path) in the usual (non-mathematical) sense is used to convey the notion **"CHANGE in position"**. But in Calculus these two are different. **"CHANGE in position"** is always with reference to a co-ordinate system. If an object moves in a circle of radius r thru one complete circle, then the **CHANGE in position** = 0. But the ***distance travelled*** $= 2\pi r$.

C) 2-Dimensions: What if the object is moving in **2-Dimension** as in the case of a projected ball ?



This type of problem may be split into two **1-Dimension** problems.

$x'(t) = u \,.\, \cos\theta$ **1-Dimension** in ***fixed direction***.

$y'(t) = u \,.\, \sin\theta - gt$ **1-Dimension** with CHANGE in direction.

CHANGE in position is from $P_1 = (x(t_1), y(t_1))$ to $P_2 = (x(t_2), y(t_2)) = P_2 - P_1$

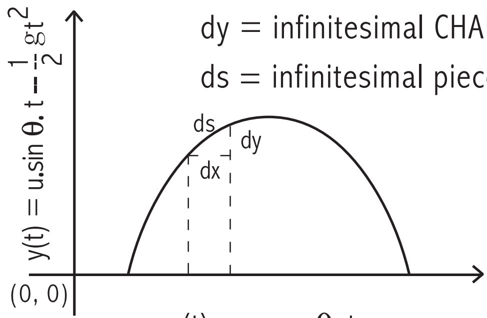
displacement $= \{[x(t_2) - x(t_1)]^2 + [y(t_2) - y(t_1)]^2\}^{1/2}$

Example 3: What is the ***distance travelled*** (length of path) in **2-Dimension** ?

dx = infinitesimal CHANGE in range.

dy = infinitesimal CHANGE in height.

ds = infinitesimal piece of the path.



$$\frac{dx}{dt} = x'(t) = u.\cos\theta\,. \quad \text{So } dx = x'(t)dt = u.\cos\theta.\,dt$$

$$\frac{dy}{dt} = y'(t) = u.\sin\theta - gt\,. \text{ So } dy = y'(t)dt = (u.\sin\theta - gt).\,dt$$

By Pythagoras theorem : $ds = \left[(dx)^2 + (dy)^2\right]^{½} = \left[(x'(t))^2 + (y'(t))^2\right]^{½}.dt$

$$s = \int ds = \int \left[\,(x'(t))^2 + (y'(t))^2\right]^{½}.dt$$

We should now look up the ***Table of Integrals*** to find the appropriate integral where the integrand from the given x(t) and y(t) is in the above form.

In the case of the vertical dimension of the projected ball the range x(t) is absent. So the above formula reduces to : $s = \int ds = \int y'(t).dt$

Adapting this technique to find the ***distance travelled*** (length of flight path) of a plane gliding forward at the rate of 500 $^{\text{feet}}/$second and downward at the rate of 20 $^{\text{feet}}/$second (see page 82) we note that :

$$\frac{dx}{dt} = x'(t) = 500\ ^{\text{feet}}/\text{second.} \quad \text{So } dx = 500\,.\,dt$$

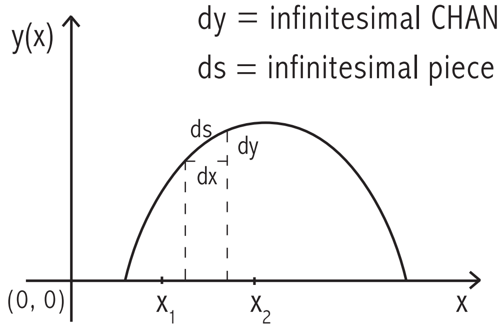
$$\frac{dy}{dt} = y'(t) = -20\ ^{\text{feet}}/\text{second.} \quad \text{So } dy = -20\,.\,dt$$

$$s = \int ds = \int \left[(x'(t))^2 + (y'(t))^2\right]^{½}.\,dt = \int \left[\,(500)^2 + (-20)^2\right]^{½}.\,dt$$

Let us now apply this method to find the **length of a curve** where y is a function of x.

Example 4: What is the length of the curve y(x) over the interval $[x_1, x_2]$?

dx = infinitesimal CHANGE along the x-axis.
dy = infinitesimal CHANGE along the y-axis.
ds = infinitesimal piece of the curve.



$$\frac{dy}{dx} = y'(x)\ .\ \text{So } dy = y'(x).dx$$

By Pythagoras theorem: $ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = \left[(dx)^2 + (y'(x).dx)^2\right]^{1/2}$

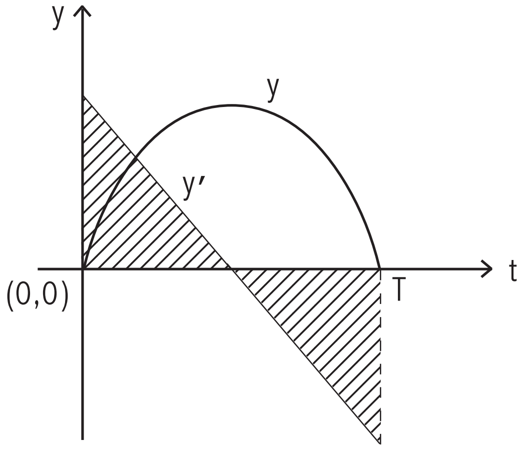
$$= \left[\ 1 + (y'(x))^2\ \right]^{1/2}.dx = \left[\ 1 + \left(\frac{dy}{dx}\right)^2\ \right]^{1/2}.dx$$

$$\boxed{s = \int ds = \int_{x_1}^{x_2} \left[\ 1 + \left(\frac{dy}{dx}\right)^2\ \right]^{1/2}.dx}$$

We should now look up the ***Table of Integrals*** to find the appropriate integral where the integrand from the given WELL-BEHAVED y(x) is in the above form.

Apply this method to find the length of the curve where f(x) = 2x over the interval [0, 1] as in example 1 (see page 180). Here f′(x) = 2. The result should tally with the result $\sqrt{5}$ that we get using the Pythagoras Theorem.

Example 5: What is the CHANGE in height $y(t)$ of the ball over $[\ 0, T\ ]$?



Height $y(t) = u \,.\, \sin\theta \,.\, t - \frac{1}{2} g t^2$.

Vertical speed $y'(t) = u \,.\, \sin\theta - g t$.

What is the shaded area under $y'(t)$ over $[\ 0, T\ ]$?

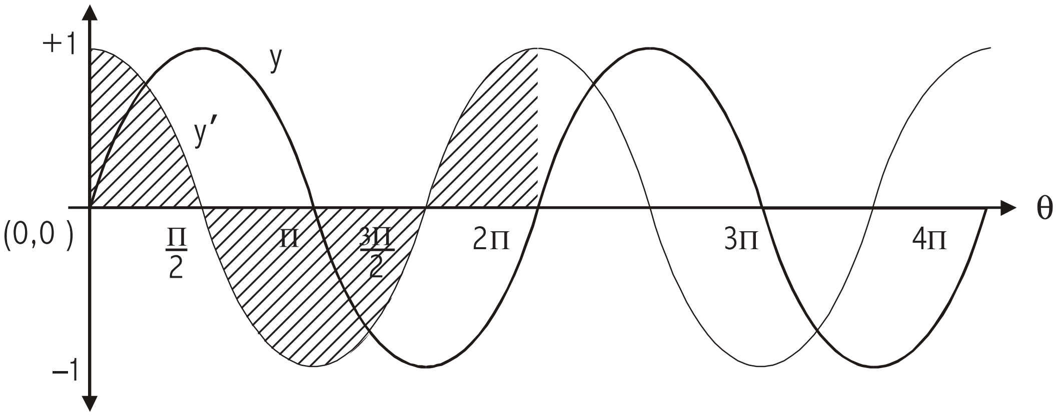
The area of the two shaded triangles is zero. $y(T) - y(0) = 0$.

The CHANGE in height is zero. This does not necessarily mean that the present height above the ground level (= zero) at $t = T$ seconds is zero. If initially at $t = 0$ seconds the ball was at height 5 meters above the ground level (= zero), then the difference or CHANGE in height from $t = 0$ secs to $t = T$ secs is zero. So the ball is back at its initial height of 5 meters above the ground level (= zero).

$$\int y'(t) \,.\, dt = y(t) + C$$

In this case the constant of integration C = + 5 meters.

Example 6: ***periodic functions***



$y = \sin\theta$ is shown by the solid line. $y' = \cos\theta$ is shown by the lighter line.

What is the area under y' over $[\ 0, 2\pi\ ]$? Zero.

What is the CHANGE in y over $[\ 0, 2\pi\ ]$? Zero.

We observe how $y = \sin\theta$ repeats itself over every interval of length 2π. y is said to be PERIODIC with period 2π. Similarly, y' is PERIODIC with period 2π.

Let y and y' be PERIODIC with period T.

Let m and n be whole numbers.

$$\int_{mT}^{nT} y' \cdot dt = [\ y\ ]_{mT}^{nT} = \text{CHANGE in y from mT to nT} = 0$$

Geometrically speaking, the area under y' from mT to nT is zero.

Example 7: A pump delivers at the rate of 60 **t** litres/minute.

a) How many litres will it pump from **t** = 5 minutes to **t** = 10 minutes ?

b) Starting at **t** = 0 minutes, how long will it take to pump 3000 litres ?

a)
$$f'(t) = 60\,\mathbf{t} \text{ litres/minute}$$

$$\int_{t=5}^{t=10} f'(\mathbf{t})\,.\,dt = \int_{t=5}^{t=10} 60\,\mathbf{t}\,.\,dt = \left[\frac{60t^2}{2}\right]_5^{10} \text{ litres}$$

$$= \left[\frac{6000 - 1500}{2}\right] \text{ litres} = 2250 \text{ litres}$$

b)
$$\int_{t=0}^{t=T} f'(\mathbf{t})\,.\,dt = \int_{t=0}^{t=T} 60\,\mathbf{t}\,.\,dt = \left[\frac{60t^2}{2}\right]_0^{T}$$

$$= \frac{60T^2}{2} = 3000$$

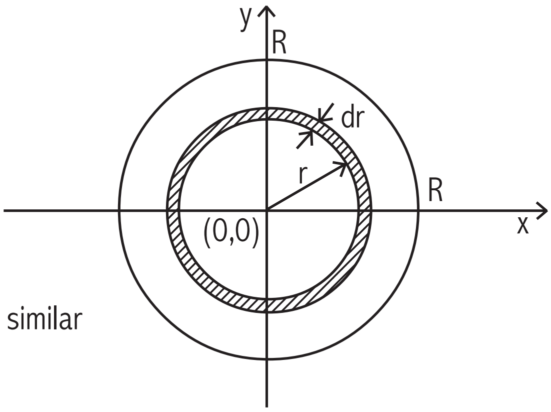
$$T^2 = 100 \text{ minutes}$$

$$T = 10 \text{ minutes}$$

This example may be easily adapted to compute the charge on a capacitor and also the time taken to charge or discharge the capacitor given the rate of flow of charge = current $\boldsymbol{i} = {}^{dq}/_{dt}$.

Example 8: *area of a circle*

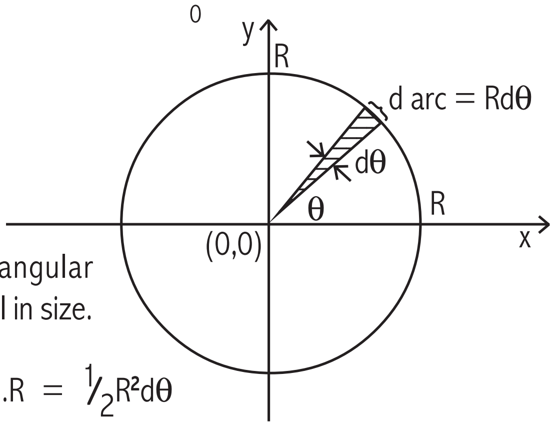
It is not always necessary to work with ***rectangular co-ordinates***. Let us find the area of a circle using ***polar co-ordinates*** (around a point). Here we take advantage of the symmetry of the ***elements of area*** around the origin. The contiguous ***elements of area*** when summed up continuously or integrated form the whole area.



The ***elements of area*** are rings similar in shape but varying in size.

element of area $\mathbf{d}A = 2\pi r dr$

area of circle $A = \int \mathbf{d}A = \int_0^R 2\pi r dr = [\pi r^2]_0^R = \pi R^2$

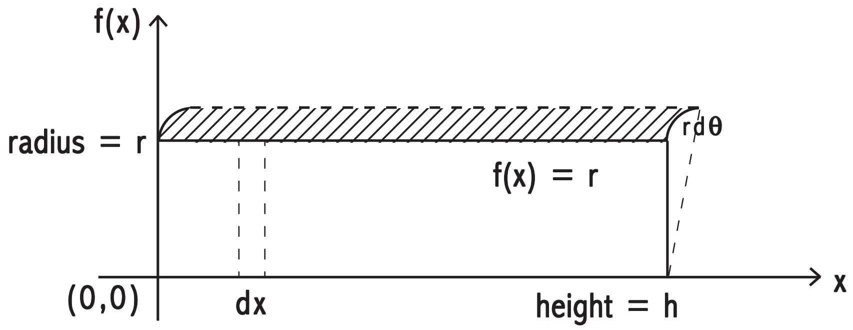


The ***elements of area*** are triangular wedges similar in shape and equal in size.

element of area $\mathbf{d}A = \frac{1}{2}(Rd\theta).R = \frac{1}{2}R^2 d\theta$

area of ciricle $A = \int \mathbf{d}A = \int_0^{2\pi} \frac{1}{2}R^2 d\theta = [\frac{R^2\theta}{2}]_0^{2\pi} = \pi R^2$

Example 9: *volume of a cylinder*



The ***"area under the curve"*** $f(x) = r$ over the interval [0, h] is :

$$\int_0^h f(x) \, . \, dx = \int_0^h r \, dx = [r x]_0^h = r \, . \, h$$

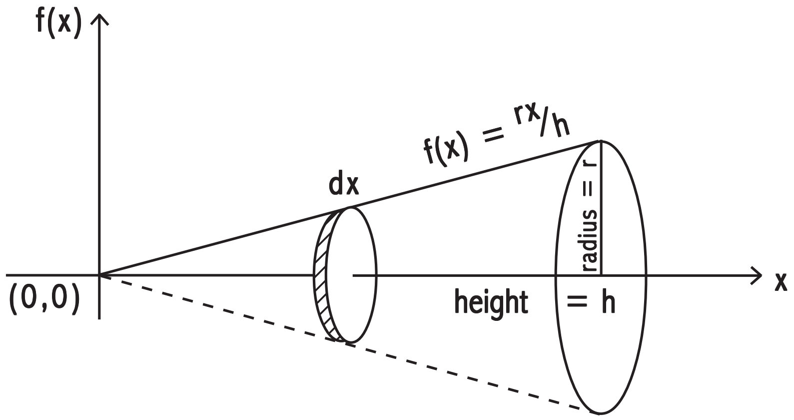
This is the area of a cross-section. If we rotate this rectangular cross-section area $r \, . \, h$ thru an infinitesimal angle $d\theta$ around the x-axis, we will get a wedge shaped block of volume dV.

$$\text{volume of wedge } dV = \tfrac{1}{2} \, (\text{cross-section area} \, . \, r \, d\theta) = \tfrac{1}{2} \, r^2 \, . \, h \, . \, d\theta$$

We may now rotate this wedge dV from $\theta = 0$ to $\theta = 2\pi$ radians to get the solid cylinder of height h and radius r.

$$\text{volume of cylinder } V = \int dV = \int_0^{2\pi} \tfrac{1}{2} r^2 h \, . \, d\theta = \left[\frac{r^2 h \, \theta}{2} \right]_0^{2\pi} = \pi r^2 h$$

Example 10: *volume of a cone*



The cylinder is known as a ***solid of rotation***. The technique used in example 8 to find the volume of a cylinder cannot be extended to other ***solids of rotation***. Because we cannot get an accurate measure of the ***element of volume*** dV. We need an accurate measure of dV.

In the case of a cylinder we may take an ***element of volume*** dV to be a solid disc of radius f(x) and thickness dx. Then :

$$dV = \pi \{f(x)\}^2.dx = \pi r^2.dx$$

$$\text{volume of a cylinder} = \int dV = \int_0^h \pi \{f(x)\}^2.dx = \int_0^h \pi r^2.dx = \left[\pi r^2 x\right]_0^h = \pi r^2 h$$

This technique may now be applied to other ***solids of rotation***.

$$\text{volume of cone} = \int dV = \int_0^h \pi \{f(x)\}^2.dx = \int_0^h \pi({}^{rx}/_{h})^2.dx$$

$$= \left[\pi({}^{r^2x^3}/_{3h^2})\right]_0^h = {}^1/_3\, \pi r^2 h$$

The physical entities like ***position***, ***distance***, ***speed***, ***acceleration***, ***area*** and ***volume*** are easy to visualize. Let us now look at an application of integration where the physical entity being measured is not so obvious.

Moment of Inertia

In ***linear motion*** the distribution of the mass (density) of the rigid body does not matter. We may take the ***center of mass*** of the rigid body and treat it as concentrated mass or ***point mass***. So we have only one position vector for the rigid body. Hence the ***orientation*** of the rigid body does not affect the equation of energy of linear motion. In ***rotational motion*** the ***orientation*** of the body does affect the equation of energy of rotational motion.

orientation = distance of the individual ***elements of mass*** of the rigid body from a particular point or axis.

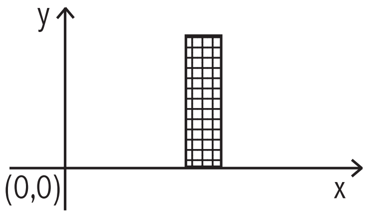
elements of mass = similar segments or cross-sections of the mass, more than just a particle.

Contiguous ***elements of mass*** when summed up continuously or integrated form the entire mass of the rigid body . To keep our calculations simple at the high school level, we assume the mass of the rigid body to be of ***uniform density***. Hence they are of equal mass. Each such ***element of mass*** may be viewed as a ***point mass*** with its individual position vector. Also, at the high school, we work with easy examples where a cross-section of the solid is uniform about an axis or at least symmetric about an axis.

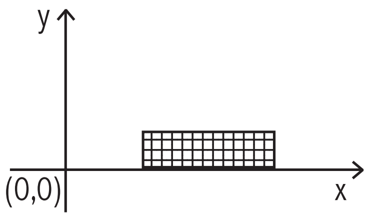
The ***moment of inertia*** $\mathbf{I}$ is defined as the measure of the ***tendency to rotate*** of a rigid body.

About an axis A : $\mathbf{I}_A$ = the ***axial*** moment of inertia.

About a point P : $\mathbf{I}_P$ = the ***polar*** moment of inertia.



greater ***tendency to rotate*** about x-axis



lesser ***tendency to rotate*** about x-axis

In the diagrams above the same brick of uniform density, depending on its ***orientation***, has a different ***tendency to rotate*** or ***moment of inertia*** about the x-axis. It is important to note that the brick is not in motion. The brick is at rest or in a state of ***inertia***. Therefore the ***moment of inertia*** of a rigid body depends only on the ***orientation*** or distance of the individual ***elements of mass*** from the axis or point of rotation.

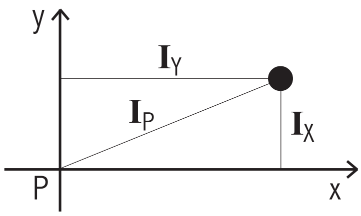
Measures may be formulated in different ways. Whatever be the measure, it should give a sense and proportion of the entity being measured. Thus the formula for the ***moment of inertia*** about an axis is defined as:

$\mathbf{I}_A =$ (mass) **.** (square of perpendicular distance to the axis of rotation)

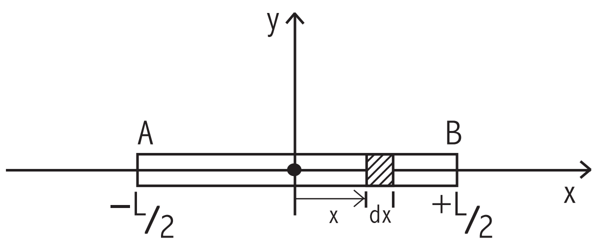
This has the advantage of taking into account the ***orientation*** without being affected by which side of the axis, negative or positive, the rigid body is positioned or located. The quantity $\mathbf{I}$ can be used directly in the equation of the kinetic energy of rotation. The ***moment of inertia*** is always a positive quantity. The measure of the ***tendency to rotate*** does not specify or include the direction of rotation as in positive for counter-clockwise and negative for clockwise.

By applying the Pythagoras Theorem with P as the origin:

$$\mathbf{I}_P = \mathbf{I}_X + \mathbf{I}_Y .$$



Example 11: Consider a thin rod AB of mass M of uniform density and length L along the x-axis. The rod is free to rotate about the y-axis passing through the center of mass and perpendicular to its length.



Since the mass M is of uniform density we have: mass per unit length $= M/L$.

So an ***element*** of the rod has mass $\mathbf{d}M = M/L \cdot \mathbf{d}x$

The ***moment of inertia*** of this ***element*** of the rod about the y-axis is dependent on the distance x from the y-axis.

Therefore: $\mathbf{dI}_Y = (M/L\,\mathbf{d}x) \cdot x^2 = M/L\,x^2\mathbf{d}x$.

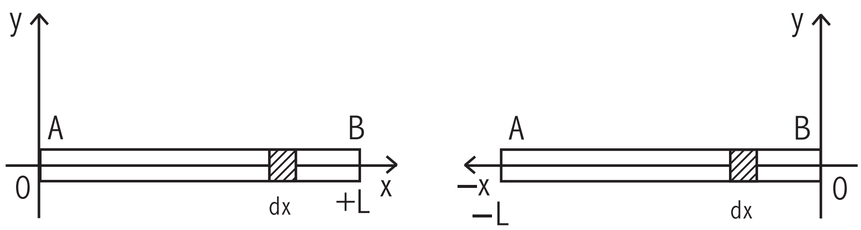
The ***moment of inertia*** of the entire rod about the y-axis is got by continuously summing up or integrating over the range of x from $-L/2$ to $+L/2$.

$$\mathbf{I}_Y = \int \mathbf{dI}_Y = \int_{-L/2}^{+L/2} M/L\,x^2\mathbf{d}x = M/L\left[x^3/3\right]_{-L/2}^{+L/2}$$

$$= M/3L\left[(+L/2)^3 - (-L/2)^3\right]$$

$$\mathbf{I}_Y = ML^2/12$$

If we now shift the rod so that the axis of rotation (y-axis) passes thru one end of the rod, what is the ***moment of inertia*** $\mathbf{I}_Y$?



In both cases $\mathbf{I}_Y = ML^2/3$

A position vector has both magnitude (distance) and direction. The direction may be positive or negative depending on which side of the axis the rigid body is located. It is important to note that the ***measure of the tendency to rotate*** is defined independent of the side of the axis the mass is located. The ***orientation*** is the distance of the mass from the axis or point of rotation.

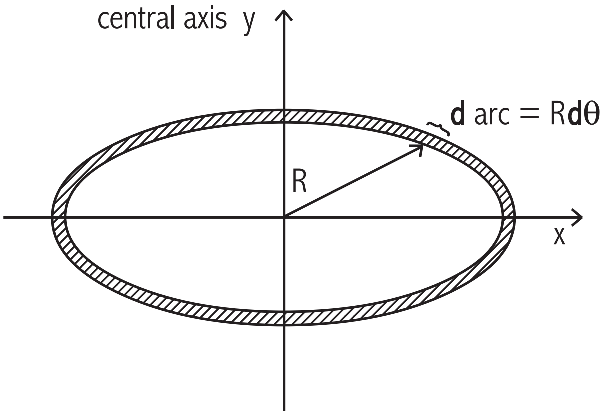
We observe that the greater the distance of the mass from the axis or point of rotation, the greater the ***moment of inertia***. Hence the easier it is to rotate or turn the rigid body and so lesser energy is required.

If the rigid body happens to be in rotational motion about an axis or point then the ***elements*** of mass of the rigid body, regardless of the distance from the axis or point of rotation, have the same angular velocity. If the angular velocity is ω then :

$$\text{kinetic energy of rotation} = \tfrac{1}{2}\mathbf{I}\omega^2 .$$

This is analogous to the kinetic energy of linear motion $= \tfrac{1}{2}mv^2$.

Example 12: Consider a ring of mass M of uniform density and radius R. Let us find the moment of inertia about the y-axis passing thru its center and perpendicular to its plane.



The circumference of the ring is $2\pi R$.

Since the mass is of uniform density we have: mass per unit length $= M/_{2\pi R}$.

The mass of an ***element*** (= **d** arc) of the ring is: $\mathbf{d}M = M/_{2\pi R} \cdot (R\,\mathbf{d}\theta)$.

The ***moment of inertia*** of this ***element*** about the central y-axis is:

$$\mathbf{dI}_Y = M/_{2\pi R} \cdot (R\,\mathbf{d}\theta) \cdot R^2 = MR^2/_{2\pi} \cdot \mathbf{d}\theta$$

The ***moment of inertia*** of the entire ring about the central y-axis is :

$$\mathbf{I}_Y = \int \mathbf{dI}_Y = \int_0^{2\pi} M/_{2\pi} R^2 \mathbf{d}\theta = MR^2/_{2\pi} \cdot \left[\theta\right]_0^{2\pi}$$

$$\mathbf{I}_Y = MR^2$$

Example 13: Let us now find the ***moment of inertia*** of a disc of mass M of uniform density and radius R.

The area of the disc is πR^2.

Since the mass is of uniform density we have: mass per unit area $= M/\pi R^2$.

We may think the disc to be made up of ***elements*** of concentric rings of radius r with $0 \leq r \leq R$ and thickness $\mathbf{d}r$. The mass of such an ***element*** of the disc is:

$$\mathbf{d}M = (M/\pi R^2) \,.\, 2\pi r\mathbf{d}r = 2M/R^2 \; r\mathbf{d}r$$

The ***moment of inertia*** of such an ***element*** of the disc, we know from example 2, is:

$$\mathbf{dI}_Y = \mathbf{d}M \,.\, r^2 = (2M/R^2 \; r\mathbf{d}r)r^2 = 2M/R^2 \; r^3\mathbf{d}r$$

Therefore the ***moment of inertia*** of the entire disc is:

$$\mathbf{I}_Y = \int \mathbf{dI}_Y = 2M/R^2 \int_0^R r^3\mathbf{d}r = 2M/R^2 \,.\, \left[r^4/4\right]_0^R$$

$$\mathbf{I}_Y = MR^2/2 \,.$$

To a casual observer a disc rotating at high speed about its central axis may seem stationary. Yet there is ***energy*** involved and work being done. The ***kinetic energy of rotation*** in this case may be viewed as a form of stored energy or ***potential energy***. The rotating disc when used to drive something is known as an impeller.

In comparing ring $\mathbf{I}_Y = MR^2$ with the disc $\mathbf{I}_Y = MR^2/2$ about the central y-axis, we see the reason for using rims (rings) with spokes instead of solid discs as wheels for cycles and bikes.

Two satellites A and B, each weighing one tonne, orbit the earth with the same angular velocity ω. The radii of orbit are $R_A = 100$ kms and $R_B = 1000$ kms. Which of the satellites uses more energy ?

Satellites are usually not of uniform density. Since the dimensions of a satellite are small (just a couple of meters across) in comparison to the radius of orbit we may view the mass of the satellite to be a sphere of uniform density and hence treat it like a point mass.

The satellite in lower orbit will require more energy. To this we must add some more energy to keep the satellite in stable orbit to counter the effect of gravity.

In the hydrogen atom the lighter electron revolves around the much heavier proton. What would be the difference if the heavier proton were to orbit the lighter electron ?

In the satellite question the masses of the satellites where the same, but the radii of orbit were different. Here the masses are different:

$$\text{mass of electron} = 9.1096 \text{ X } 10^{-31} \text{ kg}$$
$$\text{mass of proton} = 1.6726 \text{ X } 10^{-27} \text{ kg}$$

If the proton is made to orbit the electron, then it must orbit at a much higher angular velocity to prevent it from collapsing inwards. Hence the total energy of this system will be greater than the electron orbiting the proton system.

We see that ***nature always tends to find its lowest state of energy***. If so, why is the universe expanding ? We learn from the **Second Law of Thermodynamics** that ***the entropy*** (randomness) ***of the universe increases with every spontaneous change***.

We conclude this chapter with a few exercises that may require extra thought.

Exercise1: Given $f(x) = {}^{1}\!/_{a}\, f'(x)$ and $f'(x) = a\, f(x)$ where $a \neq 0$ is constant, what is f(x) ?

Exercise2: What is $\int dx$?

a) an interval on the x - axis

b) the independent variable x

c) 1

d) the expression of CHANGE $F(x) = x$

Exercise 3: A pump delivers at the rate of 60T litres/minute. What is the delivery rate in litres/second ? (T is time).

a) T b) T/120 c) T/60 d) $T / 60^2$

Exercise 4: A car moves at a speed of 60T kms/hour. What is its speed in meters/second ? (T is time)

a) 50T /3 b) $T/(2 \times 6^3)$ c) $T/(60 \times 6^3)$ d) $T/6^3$

Exercise 5: $\frac{4}{3}\pi r^3$ is the volume of a solid sphere of radius r. Its derivative is $4\pi r^2$ which is the surface area of the sphere of radius r. What is $8\pi r$, the derivative of the surface area of the sphere of radius r ? (Hint : circles.)

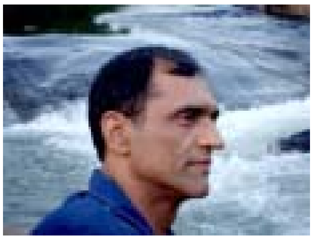
Exercise 6: A pump delivers at the rate of $T^{3/2}$ litres/minute. What is the delivery rate in litres/second ? (T is time).

Exercise 7: With the bouncing ball example in mind, is it correct to say :

area under curve of $f(x) = 0 \iff$ CHANGE in $F(x) = 0$?

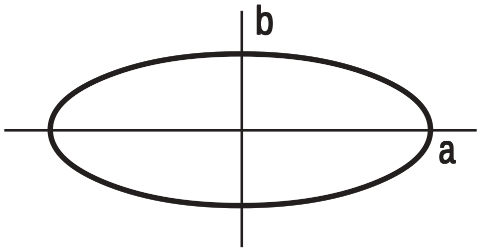
" GENIUS is asking the right question. INTELLIGENCE is finding the right answer."

Stephen Vadakkan



Prof. Stephen VADAKKAN

After completing his Bachelor of Science at Kuwait University in 1977 with distinction of rank holder, he went on to do two masters degree in Computer Science and Mathematics at the University of Arizona in Tucson, U.S.A. Later he studied Cryptography and Communications Security under the world renowned Prof. F.C. Piper of the University of London. He returned to Kuwait and worked as a Research Associate at the Kuwait Institute for Scientific Research (KISR) and for Kuwait Air Force/Air Defense, and also as a consultant for various commercial ventures. For a brief period he was an Honorary Visiting Faculty member, Reader (Associate Professor) of Computer Engineering and Visiting Adjunct Professor-Innovation Center at the Manipal Institute of Technology-Manipal; and Associate Professor of Computer Engineering at NMAMIT-Nitte.



Vadakkan formula for

$$\tfrac{1}{4}\text{ perimeter of ellipse} = \sqrt{(a-b)^2 + \left(\tfrac{\pi}{2}\right)^2 ab}$$