The Continuous Time Fourier Series (CT-FS)

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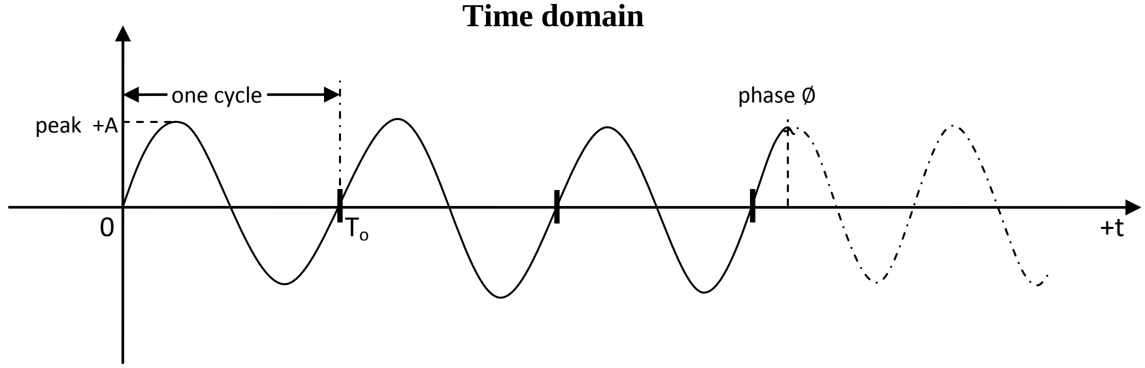
Contents

1. Finite Waves

Any ***finite wave*** propagating in the positive direction in the ***time domain*** can be completely identified by three pieces of information (parameters): ***amplitude, frequency*** and ***phase.*** We shall denote these by the triple (A, +ω, ∅).

Time domain



T_0 = fundamental period

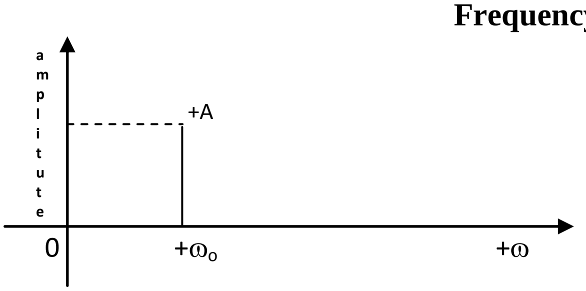
f = frequency = $1/T_0$ Hz

ω_0 = angular frequency = $2\pi/T_0$ [rads/sec]

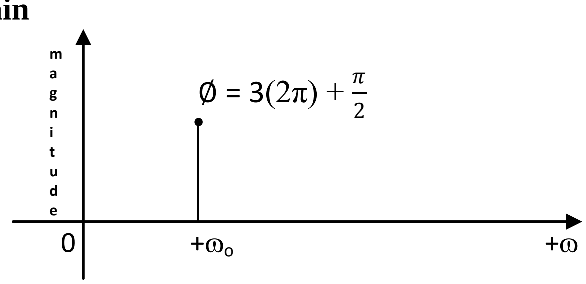
∅ = phase [rads] = fraction of a cycle

With the triple (A, +ω, ∅) we can describe the wave. More precisely, to completely describe this finite wave we will need to know the number of full cycles as well. The first cycle is generally assumed to start at time t = 0 as shown in the diagram. We may embed n, the number of cycles, in the phase: ∅ = {n(2π) + fraction of a cycle}. This is known as ***continuous phase.***

Frequency domain



Amplitude spectrum



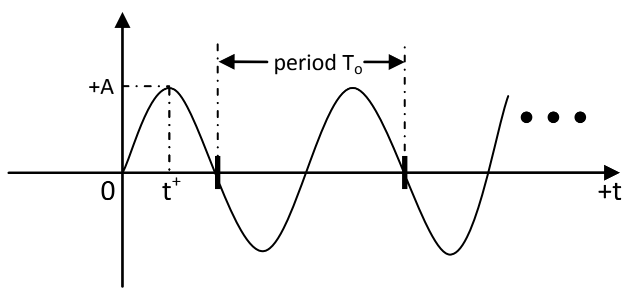
Continuous phase spectrum

With the triple (A, +ω, ∅) we can describe the wave and depict the information using the frequency axis in the ***frequency domain*** as shown above.

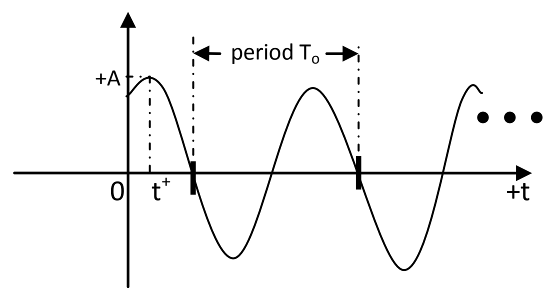
2. Finite Waves to Infinite Waves

Now we extend this ***finite wave*** to $+\infty$. The wave now propagates in the positive direction from 0 to $+\infty$. So the number of cycles becomes irrelevant. **But what happens to the phase?** When the wave extends to infinity there is no fraction of a cycle. Also the number of cycles is infinite.

Do we need the phase information? We can have two waves from 0 to $+\infty$ with the same peak amplitude +A and the same frequency $+\omega_o$ as shown below.



phase $\emptyset = \frac{\pi}{2} = t^{+}\frac{2\pi}{T_o}$ [rads]



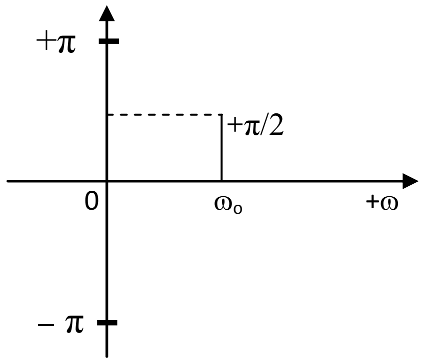
phase $\emptyset = \frac{\pi}{4} = t^{+}\frac{2\pi}{T_o}$ [rads]

How can we differentiate between these two infinite waves?

Both the waves have the same peak amplitude A and frequency $+\omega_o$. To distinguish between the two waves we need an additional piece of information. This information is t^{+}, the ***first positive peak*** in $[0, T_o]$. Hence we define phase as phase $\emptyset = t^{+}\frac{2\pi}{T_o}$ [rads]. This is known as ***modulo 2π phase.***

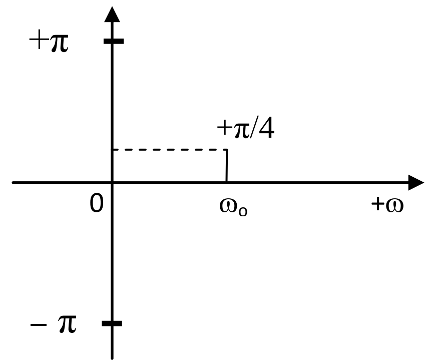
So the two infinite waves may now be described as (A, $+\omega_o$, $\pi/2$) and (A, $+\omega_o$, $\pi/4$). In the ***frequency domain*** we have separate spectrums for the phase information.

Frequency domain



Phase spectrum

Frequency domain



Phase spectrum

3. Negative Waves and Eternal Sinusoidals

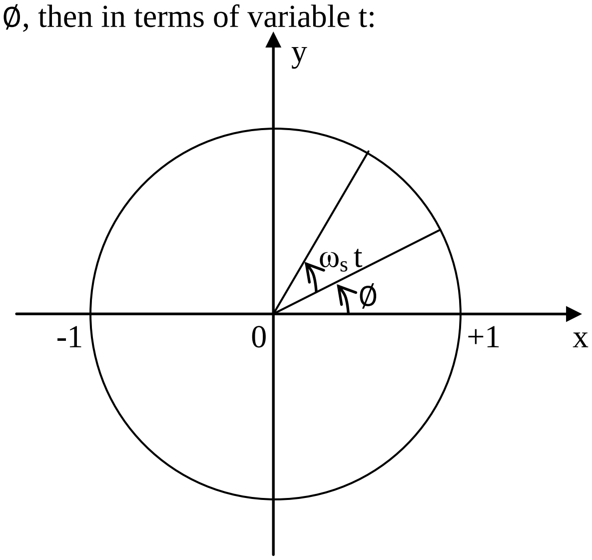
Let us now look at infinite waves propagating in the negative direction: from 0 to $-\infty$**.**

Angular displacement: If an object is moving in a circular path (unit radius) with constant angular speed ω_s and initial angular displacement $\emptyset$, then in terms of variable t:

angular displacement $\Theta = \omega_s t + \emptyset$

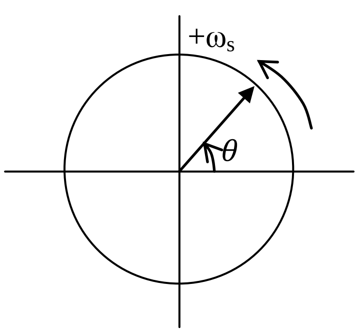
horizontal displacement = $\cos(\Theta)$

vertical displacement = $\sin(\Theta)$

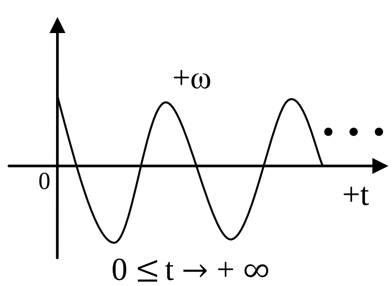


When a body is in ***uniform circular motion*** we know that its horizontal co-ordinate or x co-ordinate describes a cosine wave and its vertical co-ordinate or y co-ordinate describes a sine wave.

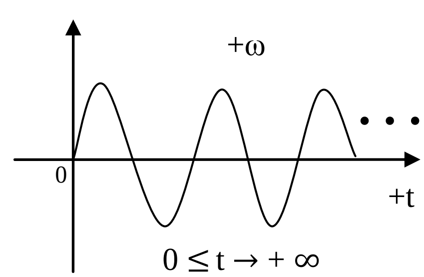
Counter-clockwise direction of rotation is defined as ***positive***. In ***uniform circular motion*** this corresponds to angular speed $+\omega_s$. In wave motion this corresponds to angular frequency $+\omega$ or to cos(t) and sin(t) with $t \geq 0$ progressing to the ***right***.



counter-clockwise

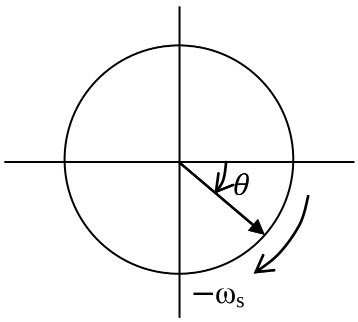


cosine wave

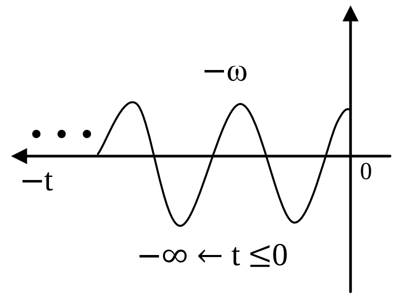


sine wave

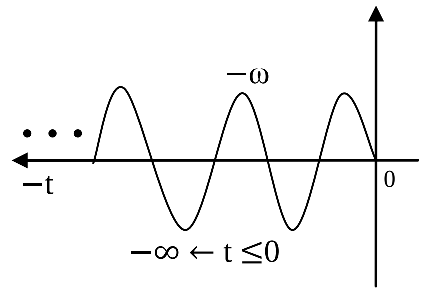
Clockwise direction of rotation is defined as ***negative***. In ***uniform circular motion*** this corresponds to angular speed $-\omega_s$. In wave motion this corresponds to angular frequency $-\omega$ or to cos(t) and sin(t) with $t \leq 0$ progressing to the ***left***.



clockwise

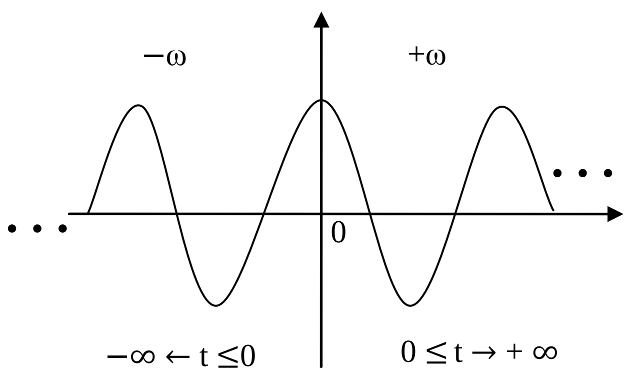


cosine wave

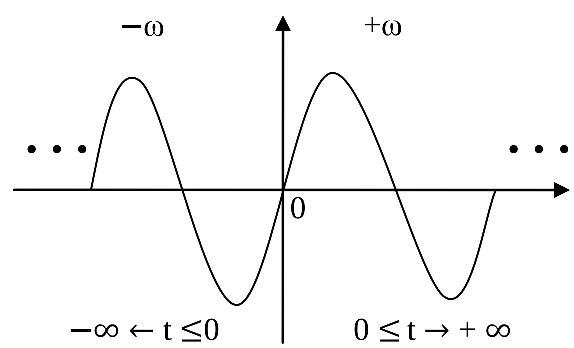


sine wave

Two frequencies: If we combine the intervals $t \geq 0$ and $t \leq 0$ we get the eternal (*san-antana*) cosine wave and eternal (*san-antana*) sine wave from $-\infty < t < +\infty$.



eternal cosine wave



eternal sine wave

Infinite waves from 0 to $+\infty$ are called sinusoidals.

Infinite waves from $-\infty$ to $+\infty$ are also called sinusoidals, more precisely eternal sinusoidals.

4. Negative Frequencies Analogy

Before we proceed further let us see an analogy. We are so used to working with numbers in the decimal ***digital*** form. We can express numbers in ***analog*** form or ***digital*** form. The number twenty nine in ***analog*** form may be expressed using twenty nine strokes. In ***digital*** form the expression depends on the ***digital system*** we use. The number twenty nine in the

		. . .	10^2	10^1	10^0	
DECIMAL DIGITAL SYSTEM is				2	9	

	. . . 2^4	2^3	2^2	2^1	2^0
BINARY DIGITAL SYSTEM is	1	1	1	0	1

	• • • 8^3	8^2	8^1	8^0
OCTAL DIGITAL SYSTEM is			3	5

We associate ***analog*** with the ***time domain*** and ***digital*** with the ***frequency domain***.

To transform numbers from ***time domain*** to ***frequency domain*** we need a ***base.*** The exponents of this base act as the ***digital frequencies***. This set of frequencies has to be $\mathbb{Z}$, the set of integers. All the frequencies should be present in order to express or represent any number (integer, rational or irrational) in digital form. This is the familiar ***place*** and ***place-value*** system we learned in class four. Here, we can think of the places: 10^0, 10^1, 10^2, • • • as the ***decimal digital frequencies*** and the ***values*** in each place as the ***amplitudes***.

	• • • 10^3	10^2	10^1	10^0
6 3 2 =		6	3	2

6 3 2 is decomposed into the ***sum of its frequency components***.

$$6\,3\,2 = 6 \times 10^2 + 3 \times 10^1 + 2 \times 10^0$$

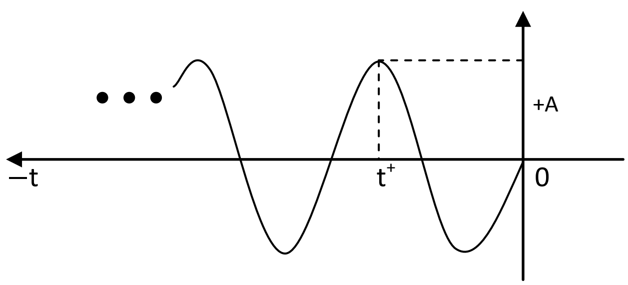
The values 6, 3 and 2 are the magnitudes or amplitude coefficients a_n at these frequencies.

We have no difficulty in writing down the rational 129/16 in decimal form

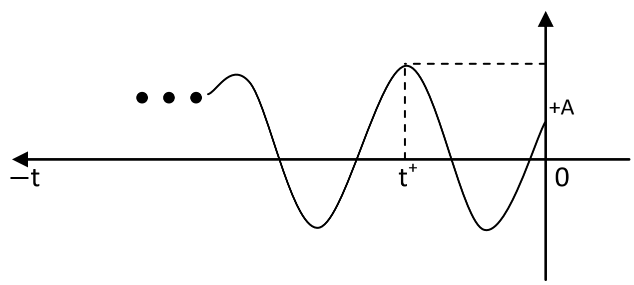
$$
\begin{array}{lccccccccc}
 & \cdots 10^{+2} & 10^{+1} & 10^{0} & & 10^{-1} & 10^{-2} & 10^{-3} & 10^{-4} \cdots & \\
129/16 \; = & & & 8 & \bullet & 0 & 6 & 2 & 5 & \\
 & \cdots\; a_{+2} & a_{+1} & a_{0} & & a_{-1} & a_{-2} & a_{-3} & a_{-4} \cdots & = \displaystyle\sum_{n=-\infty}^{+\infty} a_n \cdot 10^n
\end{array}
$$

Note the presence of negative decimal frequencies: 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, • • •. Again, 8.0625 is decomposed into the ***sum of its frequency components***.

So for the sinusoidals in the ***time domain*** with $t \geq 0$ the corresponding frequencies in the ***frequency domain*** are denoted by $+\omega$. And analogous to positive frequencies for ***discrete*** numbers: 10^0, 10^1, 10^2, 10^3, • • • we have ***positive frequencies***: $+\omega_0$, $+2\omega_0$, $+3\omega_0$, • • • for sinusoidals. Likewise, for the sinusoidals in the ***time domain*** with $t \leq 0$ the corresponding frequencies in the ***frequency domain*** are denoted by $-\omega$. So analogous to negative frequencies for ***discrete*** numbers: 10^{-1}, 10^{-2}, 10^{-3}, • • • we have ***negative frequencies***: $-\omega_0$, $-2\omega_0$, $-3\omega_0$, • • • for sinusoidals.



phase $\emptyset = \frac{3\pi}{2} = t^{+}\frac{2\pi}{T_o}$ [rads]



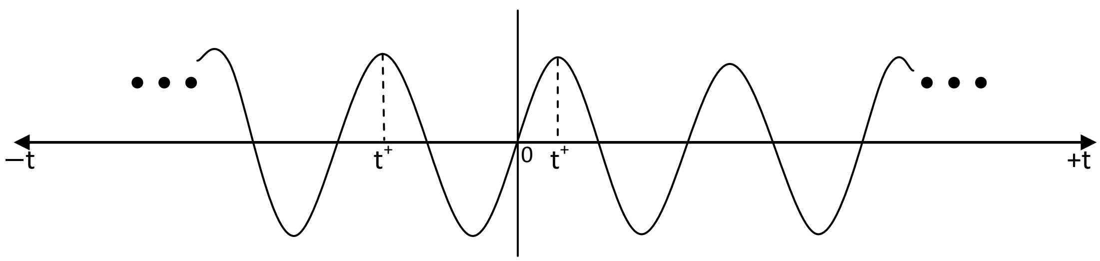
phase $\emptyset = \frac{7\pi}{4} = t^{+}\frac{2\pi}{T_o}$ [rads]

Because the waves propagate in the negative directions we say they have negative frequencies denoted by $-\omega$. Again we can completely describe these waves by the triples $(A, -\omega_0, 3\pi/2)$ and $(A, -\omega_0, 7\pi/4)$ where t^+ is the ***first positive peak*** in $[0, -T_o]$ and the phase is $\emptyset = t^{+}\frac{2\pi}{T_o}$ [rads].

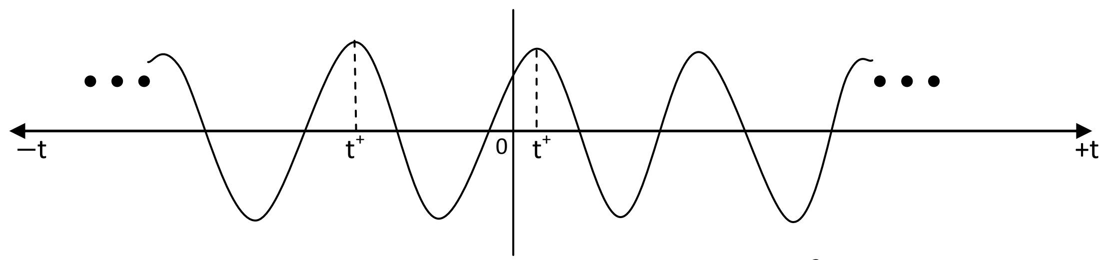
Later we shall see that, analogous to representing analog numbers in the decimal digital domain, we will need a discrete set, dense set or complete (continuous) set of frequencies to represent a signal in the frequency domain. The type of set depends on the type the signal.

5. Phase of Eternal Sinusoidals

Let us now combine the corresponding positive and negative waves as depicted below:



phase $\emptyset = \frac{3\pi}{2} = t^+ \frac{2\pi}{T_o}$ [rads]   phase $\emptyset = \frac{\pi}{2} = t^+ \frac{2\pi}{T_o}$ [rads]



phase $\emptyset = \frac{7\pi}{4} = t^+ \frac{2\pi}{T_o}$ [rads]   phase $\emptyset = \frac{\pi}{4} = t^+ \frac{2\pi}{T_o}$ [rads]

Now we have what are called eternal sinusoidals (or sometimes just referred to as sinusoidals), that is to say: waves extending from $-\infty$ to $+\infty$. We can see in the diagrams that each sinusoidal has two phases. **Which phase do we use?**

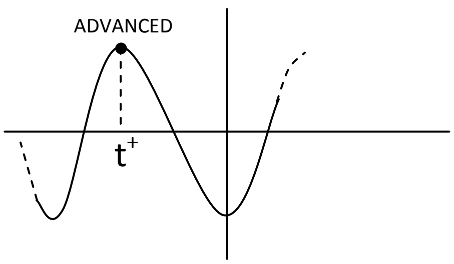
In this situation we define the phase $\emptyset = t^+ \frac{2\pi}{T_o}$ [rads] where t^+ is the ***first positive peak*** in $\left[\frac{-T_o}{2}, \frac{+T_o}{2}\right]$ and so the phase lies between $-\pi$ and $+\pi$. See diagrams on page 3.

Adding an integer multiple of 2π to this form of the phase does not change the sinusoidal. However, it can lead to discontinuities when depicting phase spectrum. For more information on this please refer to DSP by Steven Smith pages 164 to 168 and page 188.

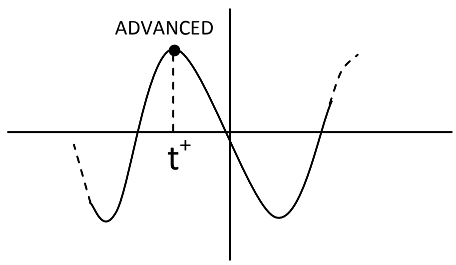
6. Phase Sign and Value

We still have more work to do regarding the phase. Phase has both ***sign*** and ***value***.

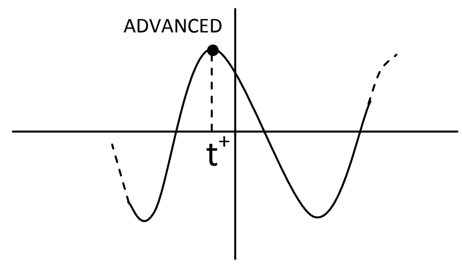
Let us see some examples.



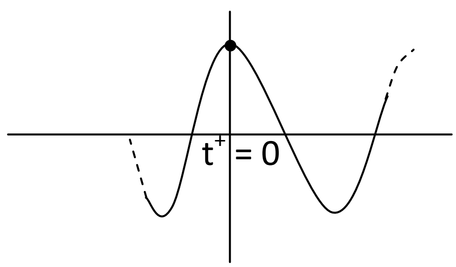
$\emptyset = +|t^+|\,\frac{2\pi}{T_o}$ [rads] $= +\pi$



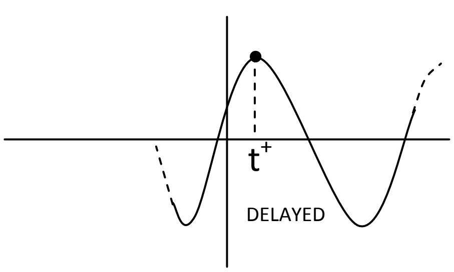
$\emptyset = +|t^+|\,\frac{2\pi}{T_o}$ [rads] $= +\frac{\pi}{2}$



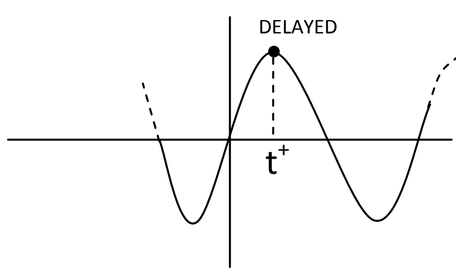
$\emptyset = +|t^+|\,\frac{2\pi}{T_o}$ [rads] $=+\frac{\pi}{4}$



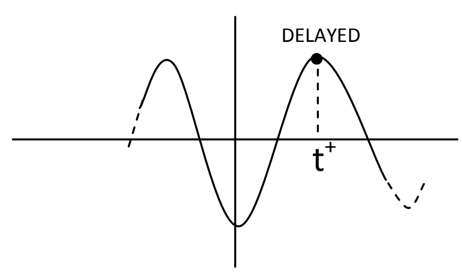
$\emptyset = |t^+|\,\frac{2\pi}{T_o}$ [rads] $= 0$



$\emptyset = -|t^+|\,\frac{2\pi}{T_o}$ [rads] $=-\frac{\pi}{4}$



$\emptyset = -|t^+|\,\frac{2\pi}{T_o}$ [rads] $=-\frac{\pi}{2}$



$\emptyset = -|t^+|\,\frac{2\pi}{T_o}$ [rads] $= -\pi$

Given a sinusoidal $\cos(\omega_0 t + \emptyset)$ the ***phase value*** $\emptyset = |t^+|\ 2\pi/T_0$ [rads] where t^+ = first positive peak closest to $t = 0$.

The ***phase sign*** in the ***single-sided phase spectrum*** (rectangular form) is defined by convention:

DELAY is ***negative*** when $t^+ > 0$

ADVANCE is ***positive*** when $t^+ < 0$

The ***phase sign*** in the ***double-sided phase spectrum*** (rectangular form) and in the ***phase spectrum*** (polar form) is defined by convention:

(sign of frequency ω_i) · {***phase sign*** in the ***single-sided phase spectrum***(rectangular form)}

$$\textbf{\textit{phase value}} = |t^+|\ \frac{2\pi}{T_o}\ \text{[rads]}$$

where t^+ is the ***first positive peak*** in $\left[\frac{-T_o}{2}, \frac{+T_o}{2}\right]$ and so the phase lies between $-\pi$ and $+\pi$.

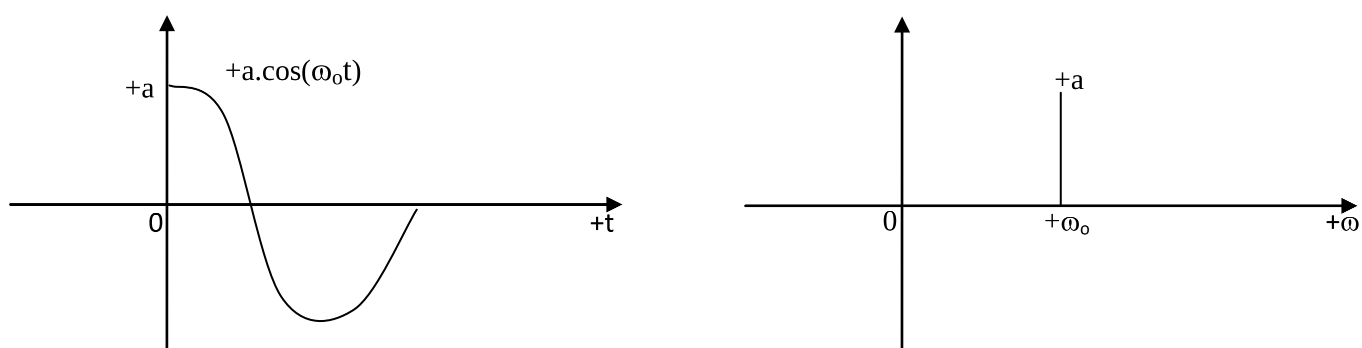
For the time being this is sufficient information on phase. We shall go into more detail on phase in another module.

7. Sinusoidals in Frequency Domain

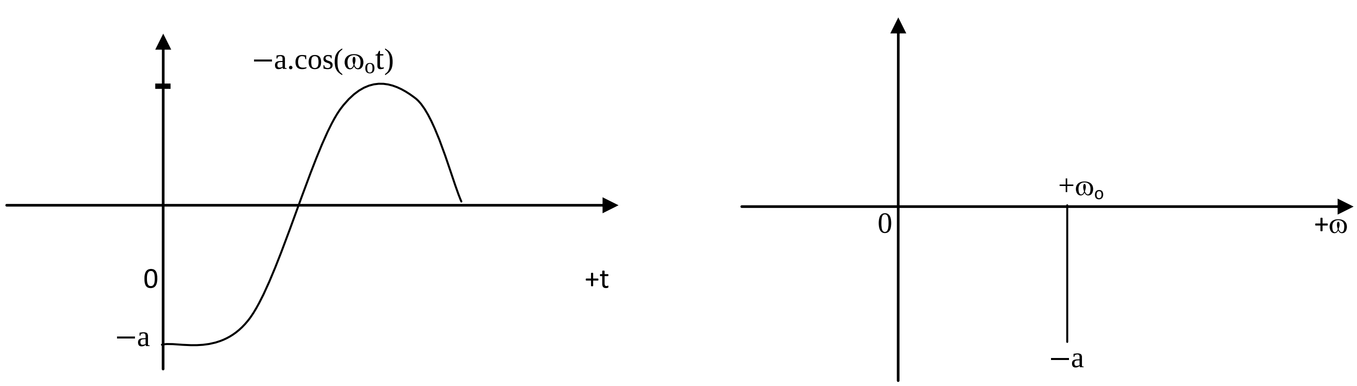
We have two forms of representing sinusoidals and signals in the frequency domain: ***rectangular form*** and ***polar form.*** In the ***rectangular form*** there are two ways of representation: ***single-sided amplitude spectrum*** which has only the positive frequency axis (+ϖ axis) and ***double-sided amplitude spectrum*** which has both the negative and positive frequency axii (−ϖ axis and +ϖ axis).

In the ***single-sided amplitude spectrum*** (as shown below) the symbol **a** is usually used to represent the ***peak amplitude*** of the cosine sinusoidal as in $\mathbf{a_i}.\cos(\varpi_i t)$. The amplitude co-efficients $\mathbf{a_i}$ may be positive or negative. Likewise the symbol **b** is usually used to represent the ***peak amplitude*** of the sine sinusoidal as in $\mathbf{b_i}.\sin(\varpi_i t)$. The amplitude co-efficients $\mathbf{b_i}$ may be positive or negative.

Time domain **Frequency domain**

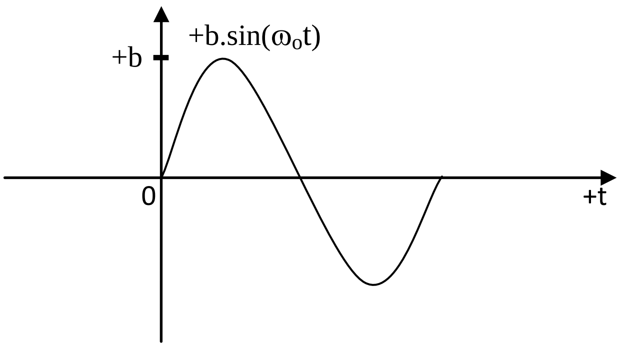


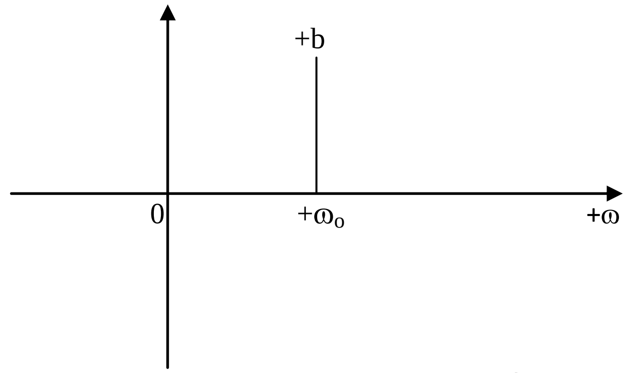
single-sided amplitude spectrum (rectangular form)



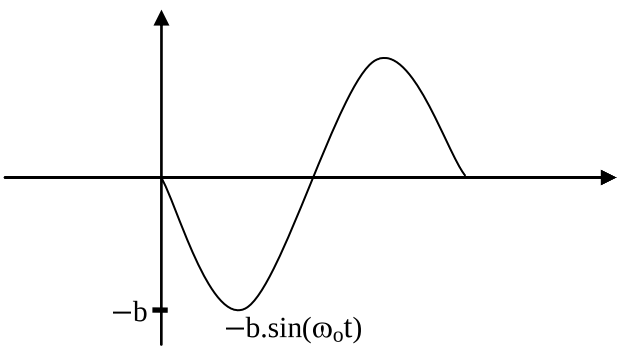
single-sided amplitude spectrum (rectangular form)

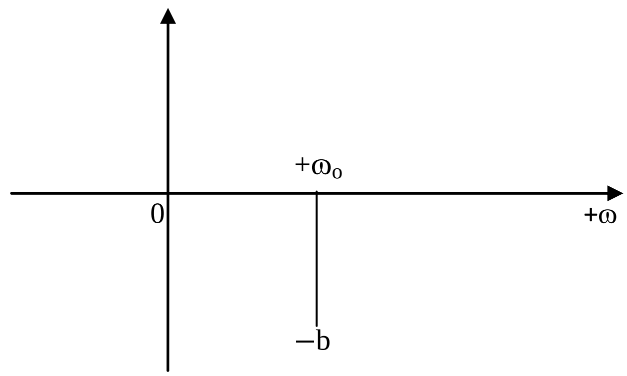
Time domain **Frequency domain**





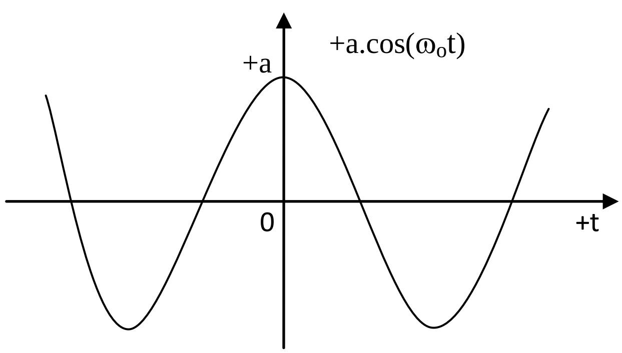
single-sided amplitude spectrum (rectangular form)

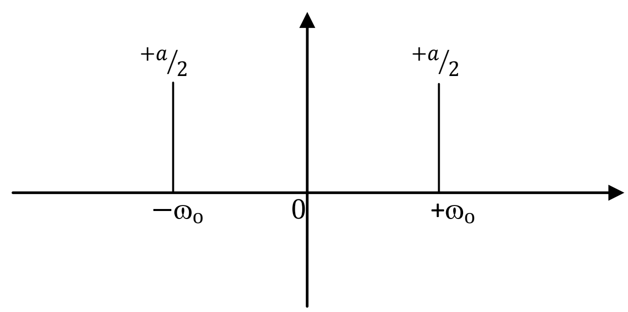




single-sided amplitude spectrum (rectangular form)

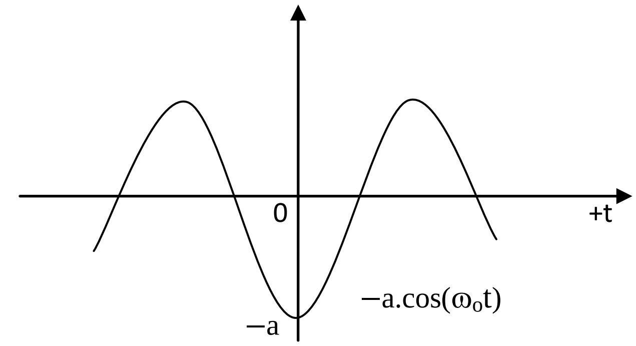
In the ***double-sided amplitude spectrum*** the amplitude is equally divided over both the positive and negative frequencies as shown below.

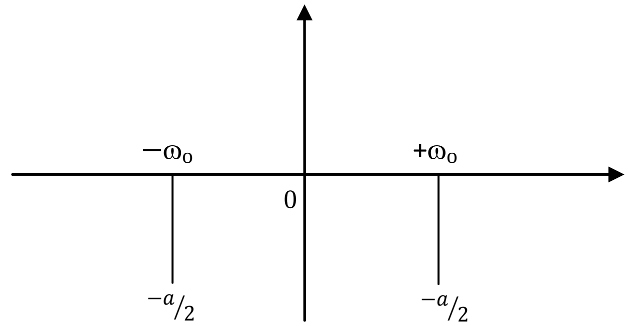




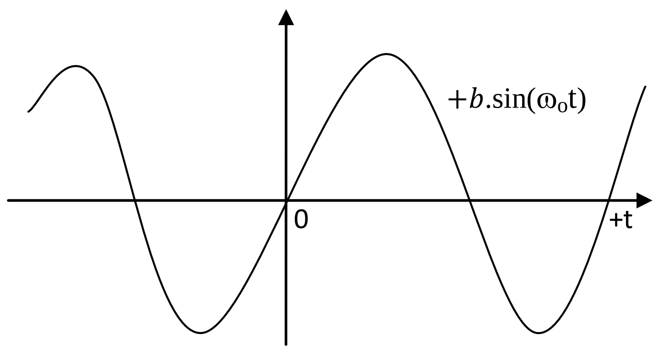
double-sided amplitude spectrum (rectangular form)

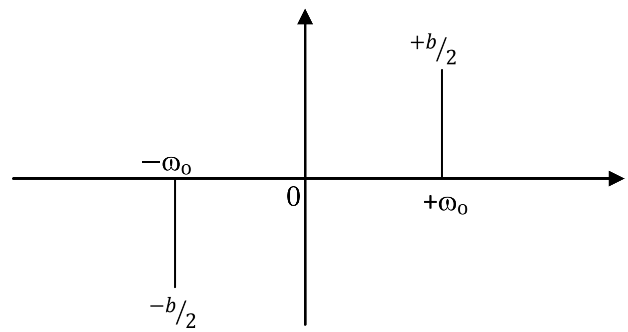
Time domain **Frequency domain**



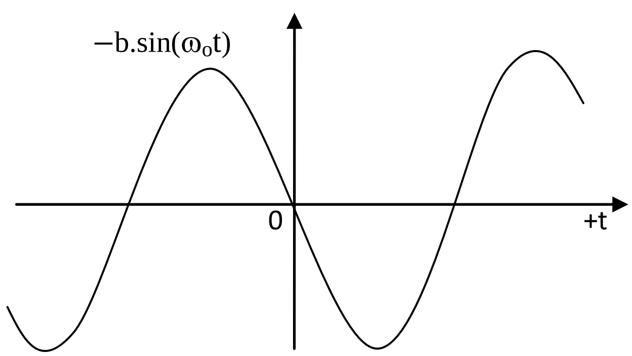


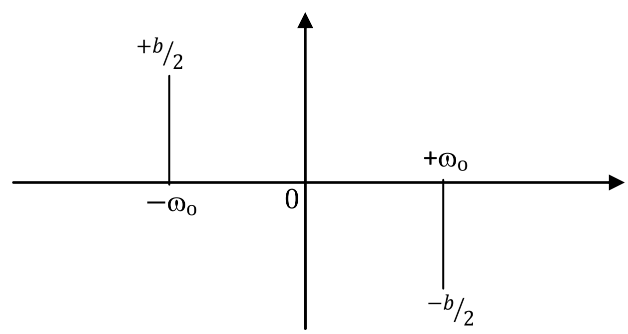
double-sided amplitude spectrum (rectangular form)





double-sided amplitude spectrum (rectangular form)





double-sided amplitude spectrum (rectangular form)

In the ***double-sided phase spectrum*** (rectangular form) the ***phase value*** is not divided because it represents the position of the first positive peak. Whereas the amplitude co-efficients in the ***double-sided amplitude spectrum*** (rectangular form) $\mathbf{a_i}$ and $\mathbf{b_i}$ represent energy which can be equally divided over both $+\omega_i$ and $-\omega_i$.

8. Polar form

In the ***polar form*** we have a magnitude spectrum and phase spectrum. The magnitude spectrum has both the negative and positive frequency axii ($-\varpi$ axis and $+\varpi$ axis). The magnitude **m** of a frequency component ϖ is **m** = sqrt($a^2 + b^2$). Hence the magnitude **m** is always positive. However, the magnitude $\mathbf{m_i}$ is the same for both the $+\varpi_i$ and $-\varpi_i$. In other words it is not equally divided like in the ***double-sided amplitude spectrum*** (rectangular form).

We have two types of eternal sinusoidals

sine sinusoidal from $-\infty$ to $+\infty$

cosine sinusoidals from $-\infty$ to $+\infty$

In ***rectangular form*** we need both types: sine as well as cosine. And the phase information is hidden in the amplitudes of the sinusoidals $a_i.\cos(\varpi_i t)$ and $b_i.\sin(\varpi_i t)$. This can be extracted: by

$$\tan \emptyset = \frac{b}{a}$$

$$arctan(\tan\emptyset) = \emptyset = arctan(\frac{b}{a})$$

In ***polar form*** we need only cosine type sinusoidals because when we add a cosine sinusoidal and a sine sinusoidal of the same frequency we get a cosine sinusoidal of the same frequency but with the different phase. The phase information is explicit.

$$a_i.\cos(\varpi_i t) + b_i.\sin(\varpi_i t) = m_i.\cos(\varpi_i t+\emptyset) \text{ where } m_i = \text{sqrt}(a_i^2 + b_i^2)$$

Here the addition is of the sine and cosine parts of a frequency component present in a signal. We may extract each frequency component as described in **Convolution & LTI Systems** page 82 **http://www.nitte.ac.in/nmamit/articles.php?linkId=131&parentId=20&mainId=20&facId=131#**

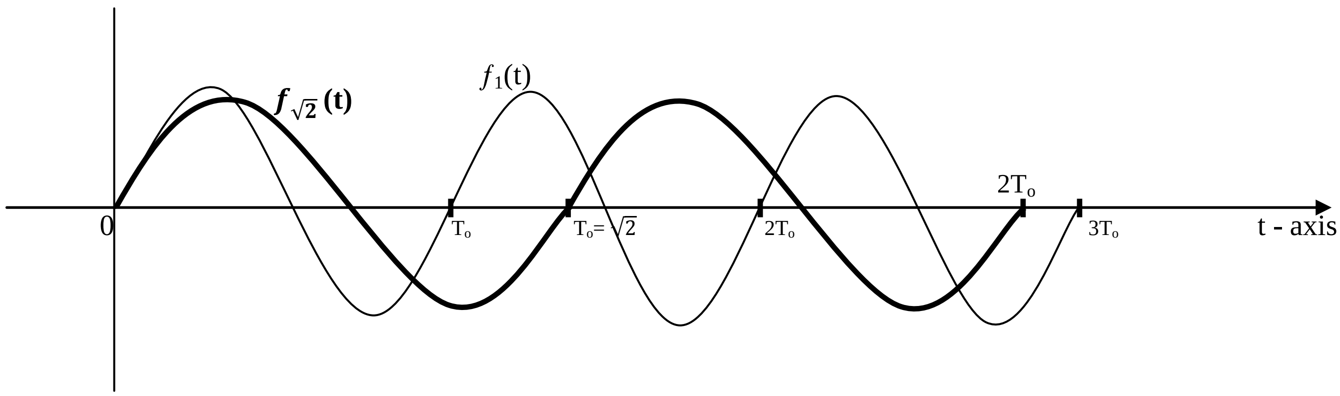
For more information on ***polar form*** please read DSP by Steven W Smith pages 161 to 164.

9. Periodic Signals

Let us see how we can form a periodic signal using sinusoidals. Are they any restrictions or conditions?

Let $f_1(t) = \sin(\omega_0 t)$ where $\omega_0 = 2\pi/1$, here the fundamental period $T_0 = 1$. Observe that $f_1(t)$ is periodic.

$f_{\sqrt{2}}(t) = \sin(\omega_0 t)$ where $\omega_0 = {}^{2\pi}/_{\sqrt{2}}$, here $T_0 = \sqrt{2}$. Observe that $f_{\sqrt{2}}(t)$ is also periodic.



Is $f(t) = f_1(t) + f_{\sqrt{2}}(t)$ periodic?

If $f(t)$ is periodic, then at some point in time, say T_f secs, the intercepts of $f_1(t)$ and $f_{\sqrt{2}}(t)$ on the t-axis will coincide for the first time. This will happen again at $2T_f$ secs, $3T_f$ secs and so on. So T_f will be the fundamental period of $f(t)$. But this cannot happen. Let us see why.

Proof by Way of Contradiction.

Suppose $f_1(t)$ and $f_{\sqrt{2}}(t)$ do coincide at some instant T_f on the time axis. This means that:

$$\text{m cycles of } f_1(t) = \text{n cycles of } f_{\sqrt{2}}(t)$$

$$m\frac{2\pi}{1} = n\frac{2\pi}{\sqrt{2}}$$

Therefore $\sqrt{2} = \frac{n}{m}$ implying $\sqrt{2}$ is rational. But $\sqrt{2}$ is an irrational number. So this cannot happen.

Let us look at the ***ratio of the periods*** of $f_1(t)$ and $f_{\sqrt{2}}(t)$**.**

$$\frac{\text{period of } f_1(t){:}T_0=1}{\text{period of } f_{\sqrt{2}}(t){:}T_0=\sqrt{2}} = \frac{1}{\sqrt{2}} \text{ is irrational}$$

10. Harmonics

If the ***ratio of the periods*** of the periodic functions is irrational, then their sum is NOT a periodic function.

Hence we can say: Given f(t) = g(t) + h(t) where both g(t) and h(t) are ***continuous*** and ***periodic***

f(t) is periodic $\Longleftrightarrow$ The ratio of the periods of g(t) and h(t) are rational

In other words: the fundamental frequencies of g(t) and h(t) must be rational multiples of each other.

In general, we can form a periodic signal f(t) by summing two or more periodic functions whose fundamental frequencies are rational multiples of each other.

Composing a periodic signal from sinusoidals whose fundamental frequencies are rational multiples of each other is known as ***synthesis***. These sinosoidals are called ***harmonics.***

Periodic f(t) $\Longleftrightarrow$ $\sum$ HARMONICS

Hence we may write continuous periodic f(t) as:

$$f(t) = \sum_{n=0}^{\infty}\{a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\}$$

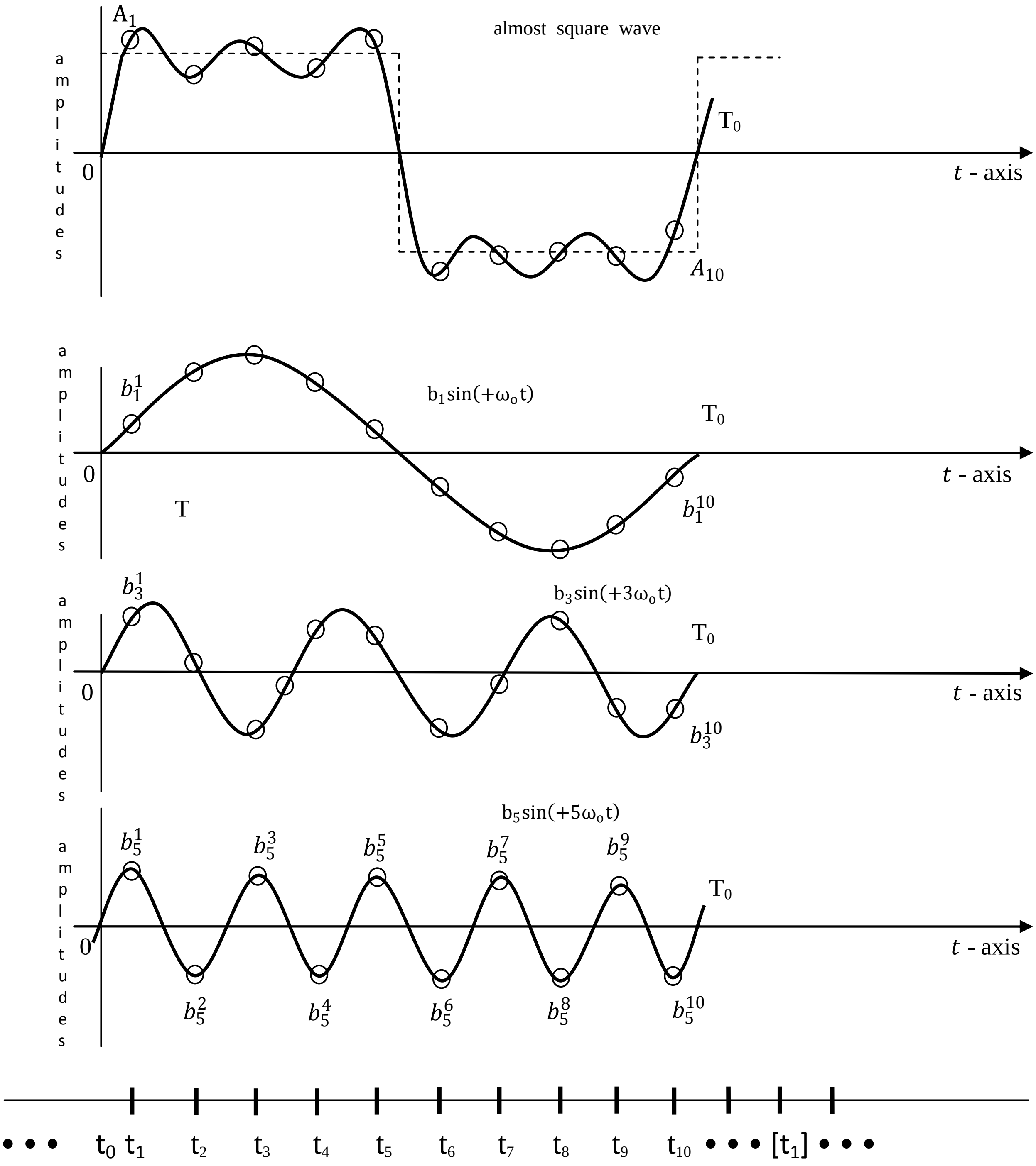
This is known as the Fourier Series or more precisely the CT-FS.

Let us now look at a periodic signal (the simplest type of signal) and see how we can extract the information of the frequency components present in it, that is to say the amplitude co-efficients a_i and b_i of the sinusoidals of the frequencies ω_i present in the signal.

11. The Almost Square Wave (ASqW)

Let us now compose the periodic signal ASqW from its three components.

ASqW = $b_1\sin(+1\sin\varpi_0 t) + b_3\sin(+3\varpi_0 t) + b_5\sin(+5\sin\varpi_0 t)$



12. Fourier Analysis

Given the periodic signal/function $f(t)$ = ASqW (in the TIME DOMAIN) how can we determine the components (sinusoidals) present and the (A, ω, ∅) of each component?

The amplitude coefficients a_i (of cosine sinusoidals) and b_i (of sine sinusoidals) are called the ***Fourier Coefficients***.

The operation of converting an operand from ***analog*** form in the TIME DOMAIN to ***digital*** form in the FREQUENCY DOMAIN is known as a ***transform***. Finding the amplitude coefficients in the FREQUENCY DOMAIN is knows as ***Fourier analysis.*** The conversion of an operand from ***digital*** form in the FREQUENCY DOMAIN to ***analog*** form in the TIME DOMAIN by the ***inverse transform*** operation is known as ***synthesis***. The ***transform*** and its ***inverse transform*** are called a ***transform pair***.

FOURIER SERIES: ∑ HARMONICS = $\sum_{n=0}^{\infty}\{a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\}$

$$[0 \le t \le T_o] \equiv \left[{}^{-T_o}/_2 \le t \le {}^{+T_o}/_2\right]$$

Periodic f(t) → ASqW

$\{\sin(\omega_0 t), \sin(3\omega_0 t), \sin(5\omega_0 t)\}$

CARRIERS

⊛

Linear Time Invariant System

→ $\{f(t) \circledast \sin(\omega_0 t),$

$f(t) \circledast \sin(3\omega_0 t),$

$f(t) \circledast \sin(5\omega_0 t)\}$

LINEAR: HOMOGENEITY AND EXPANSION AS SUM OF SINUSOIDALS

TIME/SHIFT INVARIANT: INPUT PERIOD = OUTPUT PERIOD

We shall find the ***Fourier coefficients*** in five steps. The first two steps are not usually shown in text books. It is important to see the central role of the ***circular convolution*** operation and the ***orthogonal property*** of sinusoidals in determining the coefficients.

1. **CIRCULAR CONVOLUTION**

$$f(\mathrm{t}) \circledast \sin(\omega_0 t) = \oint f(\mathrm{t}) \,.\, \sin(-\omega_0 t)\ dt$$

$$f(\mathrm{t}) \circledast \sin(3\omega_0 t) = \oint f(\mathrm{t}) \,.\, \sin(-3\omega_0 t)\ dt$$

$$f(\mathrm{t}) \circledast \sin(5\omega_0 t) = \oint f(\mathrm{t}) \,.\, \sin(-5\omega_0 t)\ dt$$

2. **EXPANSION**

$$\oint f(\mathrm{t}) \,.\, \sin(-\omega_0 t)\ dt = \oint \{\ \mathrm{b}_1 \sin(\omega_0 t) +\ \mathrm{b}_3 \sin(3\omega_0 t) +\ \mathrm{b}_5 \sin(5\omega_0 t)\}\ .\sin(-\omega_0 t)\, dt$$

The minus sign in $\sin(-\omega_0 t)$ on the L.H.S. will cancel out with the minus sign in $\sin(-\omega_0 t)$ on the R.H.S. The amplitude coefficients b_1, b_3, b_5 are unknown.

3. **ORTHOGONAL**

Due to the ORTHOGONAL PROPERTY, on R.H.S. we have:

L.H.S. R.H.S.

$$\oint f(\mathrm{t}) \,.\, \sin(1.\omega_0 t)\ dt\ =\ \oint b_1 \,.\, \sin^2(1\omega_0 t)\ dt$$

4. **TRIGOMETRIC MANIPULATION**

$$\sin^2(\omega_0 t) = \frac{1}{2}\{1 -\ \cos(2\omega_0 t)\}$$

5. **COMPLETE the $\int_{\text{over one period}}$ on R.H.S. The $\int$ came from the CIRCULAR CONVOLUTION.**

$$\oint \mathrm{b}_1 \sin^2(\omega_0 t)\, dt\ =\ \frac{\mathrm{b}_1}{2}\int_{\text{over } T_o}(1-\cos(2\omega_0 t))\ dt\ =\ \left\{\frac{\mathrm{b}_1}{2}\int_{\text{over } T_o} dt - \frac{\mathrm{b}_1}{2}\int_{\text{over } T_o}\cos(2\omega_0 t)\ dt\right\}$$

$$= \left\{\frac{\mathrm{b}_1 \mathrm{T}_0}{2} -\ 0\right\}\ =\ \frac{\mathrm{b}_1 \mathrm{T}_0}{2}$$

So: $\mathrm{b}_1 = \frac{2}{T_0}\oint \mathrm{b}_1 \sin^2(\omega_0 t)\ dt\ =\ \frac{2}{T_0}\int_{\text{over } T_o} f(\mathrm{t}).\sin(1.\omega_0 t)\, dt$

CT-FS coefficient $\mathbf{b_k}$ = $\frac{2}{T_0}\int_{\text{over } T_o} f(\mathrm{t}).\sin(k.\omega_0 t)\, dt$

Likewise we get

$$b_3 \sin^2(+3\omega_0 t) = \frac{b_3}{2}[\cos(2\omega_0 t) - \cos(4\omega_0 t)]$$

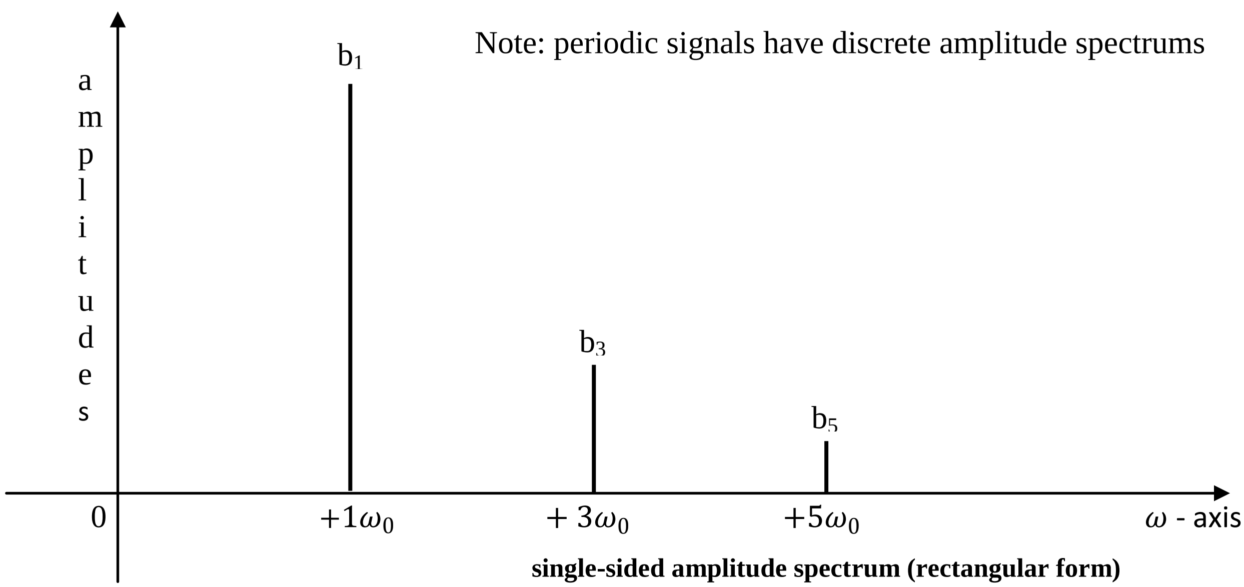
$$\text{So: } b_3 = \frac{2}{T_0}\int_{\text{over } T_o} f(t).\sin(+3\omega_0 t)dt$$

and

$$b_5 \sin^2(+5\omega_0 t) = \frac{b_5}{2}[\cos(4\omega_0 t) - \cos(6\omega_0 t)]$$

$$\text{So: } b_5 = \frac{2}{T_0}\int_{\text{over } T_o} f(t).\sin(+5\omega_0 t)dt$$

This is what the picture looks like in the FREQUENCY DOMAIN.



single-sided amplitude spectrum (rectangular form)

Usually for frequencies $\omega_1 < \omega_2 < \omega_3$, we have amplitude coefficients $b_1 > b_2 > b_3$. That is to say: the lower the frequency, the larger the amplitude. This is exactly what happens in nature. Because high frequency with large amplitude requires more power.

In the equation for period of the pendulum: $T = 2\pi\sqrt{L/g}$, we see that $L_2 < L_1$ implies $T_2 < T_1$. Now, $T_2 < T_1$ implies frequency $f_1 < f_2$. But $L_1 > L_2$ means amplitude $b_1 > b_2$. Hence frequency $f_1 < f_2$ implies amplitude $b_1 > b_2$.

The higher the frequency the smaller the amplitude.

13. The Orthogonal Property

$$\cos(A+B) + \cos(A-B) = 2\cos A.\cos B$$

Let A = $m\omega_0$ and B = $n\omega_0$. Then

$$\cos(m\omega_0 t).\cos(n\omega_0 t) = \frac{1}{2}\{\cos((m+n)\omega_0 t) + \cos((m-n)\omega_0 t)\}$$

$$\int \cos(m\omega_0 t).\cos(n\omega_0 t)\, dt = \frac{1}{2}\int \{\cos((m+n)\omega_0 t) + \cos((m-n)\omega_0 t)\}\, dt$$

$$\int \cos((m+n)\omega_0 t)\, dt = 0 \text{ when } m \neq n$$

$$\int \cos((m-n)\omega_0 t)\, dt = 0 \text{ when } m \neq n$$

$\int$ sinusoidal over one period = 0 because of equal area above and below the time axis.

When m = n : $\frac{1}{2}\int \{\cos((m+n)\omega_0 t) + \cos((m-n)\omega_0 t)\}dt$

$$= \frac{1}{2}\int \cos(2m.\omega_0 t)\, dt + \frac{1}{2}\int \cos(0.\omega_0 t)\, dt$$

$$= \quad 0 \quad + \frac{1}{2}\int 1.dt \quad = \frac{T_o}{2}$$

In short, the Orthogonal Property states that: ***the circular convolution of two sinusoidals of different frequencies is zero and of the same frequency is non-zero***. Recall that convolution by definition is the combined effect of each on each summed up over an interval. So from the Physics point of view we may say that when the frequencies of the sinusoidals are different they have no effect on each other. And when they are same there is some effect or response like resonance. This is what allows us to extract the individual frequency components present in a composite continuous periodic signal like the ASqW.