Def: An axiom is a proposition that is "assumed" to be true

ex) If $a=b$ & $b=c$, then $a=c$

Euclidean Geometry: Given a line L & a point p not on L, there is exactly one line through p pararell to L.

Spherical Geometry: Given a line L & a point p not on L, there is no line through p pararell to L.

Hyperbolic Geometry: Given a line L & a point p not on L, there are infinitely many lines through p pararell to L.

Axioms should be 1. Consistent

2. Complete

Def: A set of axioms is consistent if no proposition can be proved T & F

A set of axioms is complete if it can be used to prove every proposition is T or F

Induction

Proof by contradiction

To prove P is true, we assume P is false (in other words $\sim P$ is true) & then use that hypotheses to derive a falsehood or contradiction

- If $\sim P = F$ is true, $\sim P$ is false, therefore P is true

can't be expressed as a ratio of integer

ex) Thm: $\sqrt{2}$ is irrational

pf. (by contradiction) Assume for purpose of contradiction that $\sqrt{2}$ is rational

$\Rightarrow \sqrt{2} = a/b$ (fraction in lowest terms)

divides

$\Rightarrow 2 = a^2/b^2 \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ is even $\Rightarrow a$ is even $(2|a)$

$\Rightarrow (4|a^2) \Rightarrow (4|2b^2) \Rightarrow (2|b^2)$

$\Rightarrow b$ is even $\Rightarrow a/b$ is not in lowest terms $\Rightarrow$ contradiction $\Rightarrow$ ※

$\Rightarrow \sqrt{2}$ is irrational. $\square$

Induction axiom

Let $P(n)$ be a predicate. If $P(0)$ is true and $\forall n \in \mathbb{N}\ (P(n) \Rightarrow P(n+1))$ is true then $\forall n \in \mathbb{N}\ P(n)$ true.

If $P(0), P(0) \Rightarrow P(1), P(1) \Rightarrow P(2) \cdots$ is true, then $P(0), P(1), P(2) \cdots$ are true ex) $\forall n \geq 0,\ 1+2+3+\cdots+n = \sum_{i=1}^{n} i = \sum_{1\leq i\leq n} i = \sum^{1\leq i\leq n} i = \frac{n(n+1)}{2}$

If $n=1,\ 1+2+\cdots+n=1$

If $n\leq 0,\ 1+2+\cdots+n=0$

pf. by induction, let $P(n)$ be proposition that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

<u>Basecase</u>: $P(0)$ is true: $\sum_{i=1}^{0} i = 0 = \frac{0(0+1)}{2} = 0$ V

<u>Inductive step</u>: For $n\geq 0$, Show $P(n)\Rightarrow P(n+1)$ is true.

Assume $P(n)$ is true for purpose of Induction.

(i.e assume $1+2+\cdots+n = \frac{n(n+1)}{2}$)

need to show $1+2+\cdots+(n+1) = \frac{(n+1)(n+2)}{2}$

$= \frac{n(n+1)}{2} + (n+1) = \frac{n^2+n+2n+2}{2} = \frac{(n+1)(n+2)}{2}$ V

ex) $\forall n\in\mathbb{N},\ 3|(n^3-n)$ ((n^3-n) is multiple of 3) ex) $n=5,\ 3|(125-5)=120$

pf let $P(n) = 3|(n^3-n)$

<u>Basecase</u>: $n=0,\ P(0) = 3|(0^3-0)$ V

<u>Inductive Step</u>: For $n\geq 0$, Show $P(n)\Rightarrow P(n+1)$ is true.

Assume $P(n)$ true, i.e $3|(n^3-n)$

Examine $(n+1)^3-(n+1) = n^3+3n^2+3n+1-(n+1) = n^3+3n^2+2n = n^3-n+3n^2+3n \Rightarrow 3|3n^2,\ 3|3n,\ 3|(n^3-n)$ by $P(n)$ $\therefore 3|(n+1)^3-(n+1)$ V

* <u>Basecase</u>: $P(b)$ is true

<u>Inductive step</u>: $\forall n\geq b\ P(n)\Rightarrow P(n+1)$

<u>Conclude</u>: $\forall n\geq b\ P(n)$

Thm (NOT!): All horses are the same color

pf: by induction

$P(n)$: in any set of $n (\geq 1)$ horses, the horses in the set are all the same color.

Basecase: $P(1)$ is true since just one horse.

Inductive step: Assume $P(n)$ to prove $P(n+1)$ *$P(1) \Rightarrow P(2)$?

H_1, H_2

Consider any set of $(n+1)$ horses $H_1, H_2, \cdots, H_n, H_{n+1}$.

H_1 H_2

Then $H_1, \cdots, H_n$ are the same color & $H_2, \cdots, H_{n+1}$ are the same color because they are set of n horses.

becomes empty set H_2

Since $\text{Color}(H_1) = \text{Color}(H_2, \cdots, H_n) = \text{Color}(H_{n+1}) \Rightarrow$ all $(n+1)$ are same color. $\Rightarrow P(n+1)$ $\square$.

$\Rightarrow$ therefore does not work

$P(1)$, $P(2) \Rightarrow P(3)$, $P(3) \Rightarrow P(4) \cdots$, $\forall n \geq 2$, $P(n) \Rightarrow P(n+1)$

$P(1) \Rightarrow P(2)$ is not true. Let the Basecase be $P(2)$ is true $\Rightarrow$ Basecase fails.

Thm $\forall n$, $\exists$ a way to tile a $2^n \times 2^n$ region with a center square missing (for a statue of wealthy donor) using L-shaped tile.

pf. by induction

2. any

$P(n)$: $\exists$ a way to tile a $2^n \times 2^n$ region with a center square missing (for a statue of wealthy donor Bill)

Basecase: $P(0)$, the $2^0 \times 2^0$ region is for Bill.

(stronger)

Inductive Step: For $n \geq 0$, assume $P(n)$ to verify $P(n+1) \Rightarrow$ hard to proof $\Rightarrow$ 1. What if we make $P(n)$ harder to proof?

3. $P(n) \Rightarrow P(n+1)$ because $P(n)$ gives a stronger assumption.

Consider a $2^{n+1} \times 2^{n+1}$ courtyard