Pset 2

1.(a) pf. $a_1<a_2$. Let's consider the place for a_3. If $a_3>a_2$, $a_1<a_2<a_3$ so there is a 3-chain. Therefore $a_3<a_2$. Let's consider $a_1<a_3<a_2$. When $a_4>a_3$, $a_1<a_3<a_4$ forms a 3-chain. When $a_3>a_4$, $a_2>a_3>a_4$ forms a 3-chain. So, there always are 3-chains possible when $a_1<a_3<a_2$. Therefore $a_3<a_1$ for there to be no 3-chain in our sequence.

(b) $a_1<a_2$ & no 3-chain in sequence $\Rightarrow$ We are guaranteed that $a_3<a_1<a_2$ by (a). When $a_4>a_2$, $a_1<a_2<a_4$, and when $a_3>a_4$, $a_2>a_3>a_4$ are available. Therefore to not have a 3-chain in our sequence, $a_3<a_4<a_2$.

(c) Assume $a_1<a_2$, $a_3<a_4<a_2$. If $a_5>a_2$, $a_3<a_4<a_5$, $a_1<a_2<a_5$ are available. When $a_4<a_5<a_2$, $a_3<a_4<a_5$ available. When $a_5<a_4$, $a_5<a_4<a_2$ is available. Therefore any value of a_5 result in a 3-chain given $a_1<a_2$ and $a_3<a_4<a_2$.

(d) pf. (by contradiction) In purpose let's assume that any sequence of five distinct integers may not contain a 3-chain. When $a_1<a_2$ this is not the case by (a)-(c). When $a_1>a_2$, we can consider the same steps with (a)-(c) by just substituting $>$ with $<$, $<$ with $>$. Therefore we reach a contradiction and conclude that any sequence of five distinct integers will contain a 3-chain.

2. pf. (by induction) Let $P(n)$ be proposition that for all nonnegative integers n. $\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

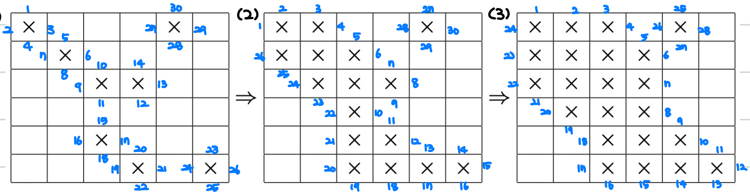
Basecase: $n=0$, $\sum_{i=0}^{0} i^3 = \frac{0 \cdot 1}{2} = 0$ V

Inductive Step: Assume $P(n)$ to prove $P(n+1)$ for $n \geq 0$

$$\sum_{i=0}^{n+1} i^3 = \sum_{i=0}^{n} i^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \frac{n^4+2n^3+n^2}{4} + \frac{4n^3+12n^2+12n+4}{4} = \frac{n^4+6n^3+13n^2+12n+4}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2 \Rightarrow P(n+1). \square \text{ V}$$

Since $P(0)$ is true and $P(n) \Rightarrow P(n+1)$ for $n \geq 0$, $P(n)$ is true for all nonnegative integers.

3. (1) (2) (3)



Let K be the number of edges with infected student only on one side, K=30 for (1), K=30 for (2), K=28 for (3). Let K_1 be value of K at the initial stage.

pf. (by induction) Let P(n) be the proposition that K is at most K_1 after n steps.

<u>Basecase</u>: P(0) is true because $K=K_1$ after 0 step. V

<u>Inductive step</u>: Let's assume P(n) to prove P(n+1) for $n \geq 0$

At (n+1)th step, for each non-infected students adjacent to at least 2 infected students, K reduces by at least 2, and increases at most 2. Therefore K cannot increase but decrease or stay the same K is at most I after n+1 steps. $\Rightarrow$ P(n+1) V.

Since P(0) is true and $P(n) \Rightarrow P(n+1)$ for $n \geq 0$, P(n) is true for all nonnegative n. □

When the whole class in an nxn grid is infected, K=4n. Therefore, K_1 must be at least 4n. Since there are 4 edges available per students, at least n students must be infected for $K_1 \geq 4n$.

∴ Thm: If fewer than n students in class are initially infected, the whole class will never be completely infected is true. □

4. Let's consider P(1), $a^1 = 1$ only when a equals 1 not for any nonzero real number. $P(0) \Rightarrow P(1)$ is not true. So pf by induction fails because the basecase fails, it is not true that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

*Another solution.

Inductive Step: By induction hypothesis, $a^k = 1$ for all $k \in \mathbb{N}$ such that $k \leq n$. But then

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1,$$

which implies that $P(n+1)$ holds. It follows by induction that $P(n)$ holds for all $n \in \mathbb{N}$, and in particular, $a^n = 1$ holds for all $n \in \mathbb{N}$. □

We cannot assume that $a^{n-1} = 1$ because we don't know whether n-1 is nonnegative integer or not. The inductive step must work for all n, but in this case it does not.

5. pf. (by strong induction) Let $P(n)$ be the proposition that $G_n = 3^n - 2^n$

<u>Basecase</u>: $P(0) = G_0 = 3^0 - 2^0 = 0$, $P(1) = G_1 = 3^1 - 2^1 = 1$ ✓

<u>Inductive step</u>: Assume $P(2), P(3), \cdots P(n)$ to prove $P(n+1)$

$P(n+1) = G_{n+1} = 5G_n - 6G_{n-1} = 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) = 15 \cdot 3^{n-1} - 10 \cdot 2^{n-1} - 6\, 3^{n-1} + 6 \cdot 2^{n-1} = 3^{n-1}(15-6) - 2^{n-1}(10-6) = 3^{n-1} \cdot 9 - 2^{n-1} \cdot 4 = 3^{n+1} - 2^{n+1} \Rightarrow P(n+1)$

Since $P(0), P(1)$ is true and $P(n) \Rightarrow P(n+1)$ for $n \geq 0$, $P(n)$ is true for all $n \in \overset{\{0,1,2,\cdots\}}{\mathbb{N}}$

6. (a) In a row move, we move an item from cell i to adjacent cell $i-1$ or $i+1$. Nothing else moves. Therefore the order of the tiles does not change. Row move cannot change the order of the tiles.

(b) In a column move, we move an item from cell i to adjacent cell $i-4$ or $i+4$. When an item moves 4 position, it changes order with 3 items, $(i-3, i-2, i-1)$ when moving upward, $(i+1, i+2, i+3)$ when moving downward. The column move changes relative order of 3 pairs of tiles.

(c) A row move have no effect on the parity of the number of inversions because row move does not change the order of tiles, as proven in (a)

(d) Column move change the order of 3 pairs of tiles

A. All three pairs were in order: # of inversions 3↑

B. Two of three pairs were in order: # of inversions 1↑

C. Two of three pairs were not in order: # of inversions 1↓

D. All three pairs were not in order: # of inversions 3↓

Therefore column move always change the parity of # of inversions.

(e) pf. (by induction) Let $P(n)$ be the proposition that after n moves, the parity of the number of inversions is different from the parity of the row containing the blank square.

<u>Basecase</u>: $P(0)$, after 0 move only O and N are inverted. # of inversions = 1 (odd) Row containing the blank square: 4 (even) So the parity differs. ✓

<u>Inductive Step</u>: For $n \geq 0$, assume $P(n)$ to prove $P(n+1)$

When $n+1^{th}$ move is row move, the parity of the # of inversions is different from the parity of the row containing the blank square by $P(n)$ because row move has no effect on the parity of the # of inversions and the row containing the blank square.

When $n+1^{th}$ move is column move the parity of # of inversions are opposite of the parity of # of inversions after n moves. The parity of row containing the blank square also is the opposite of the parity of row containing blank square after n moves. therefore, the parity of # inversions is different from the parity of the row containing the blank square after $n+1$ moves.

$P(n) \Rightarrow P(n+1)$ ✓. Since $P(0)$ is true and $P(n) \Rightarrow P(n+1)$ for $n \geq 0$, $P(n)$ is true. □

(f)

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	

even number of inversions (zero), and the parity of row containing the blank square is even (4)

The parity of the # of inversions and the parity of row containing the blank square is not different.

From (d) we have proven the Lemma: In every configurations reachable from

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

the parity of the # of inversions is different from the parity of the row containing the blank square.

Therefore transformation from

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

to

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	

is not possible. Therefore the theorem we originally set out to prove is true. □

7. pf. (by induction) Let $P(n)$ be the proposition that after n generations, the number of Z-lings will always be at most the number of B-lings.

<u>Basecase</u>: $P(0)$ is true because there are 200 Z-lings and 800 B-lings in the first generation. ✓

<u>Inductive step</u>: Assume $P(n)$ to prove $P(n+1)$ for $n \geq 0$.

Let Z_n be the number of Z-lings after n generations and B_n the number of B-lings after n generations. $Z_n \leq B_n$ by $P(n)$.

There will be Z_n pairs of Z-B parents and $\frac{B_n - Z_n}{2}$ pairs of B-B parents. Z_n pairs of Z-B parents reproduce Z_n Z-lings and Z_n B-lings. $\frac{B_n - Z_n}{2}$ pairs of B-lings reproduce $(B_n - Z_n)$ B-lings and $\frac{B_n - Z_n}{2}$ Z-lings. Therefore $Z_{n+1} = Z_n + \frac{B_n - Z_n}{2} = \frac{B_n + Z_n}{2}$, $B_{n+1} = B_n$. Since $Z_n \leq B_n$, $\frac{B_n + Z_n}{2} \leq B_n$, $Z_{n+1} \leq B_{n+1}$. $\Rightarrow P(n+1)$ ✓

Since $P(0)$ is true and $P(n) \Rightarrow P(n+1)$ for $n \geq 0$ $P(n)$ is true. □

Because $P(n)$ which had stronger hypothesis is true, the original hypothesis: The # of Z-lings will always be at most twice the # of B-lings is also true.