Dimensional Model Reduction

In non-linear finite element dynamics of solids and structures

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(1) We introduce the model reduction of general lagrangian systems → By this method the structure is preserved for mechanical systems

(2)We use the methods intruduced in (1) for reduction of finite-element models.

(1) Structure-preserving model reduction for mechanical systems:

- Let the space of configurations of a mechanical system be a differentiable manifold: Q
- The Lagrangian is a function L:TQ $\longrightarrow$ R. And is typically given by the difference between the kinetic and potential energy of the system:

$$L(q,\dot{q})=T(q,\dot{q})-V(q).$$

- The equation of motion: *Euler lagrange equation* $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}-\frac{\partial L}{\partial q^i}=0$

➔A traditional approach for model reduction:

Rewrite the Euler-Lagrange equation in first order:

$$\dot{x}(t)=f(x(t)),$$

construct a lower dimensional dynamical system:

$$\dot{y}(t)=g(y(t)),$$

where y(t) is a k-dimentional k<<n and a reconstruction function h:y $\mapsto$ x such that given y(t) the reconstruction $\tilde{x}(t)=h(y(t))$ is a good approximation to x(t).

We can find a set of basis functions to describe the state evolution and perform a linear decomposition

$$x(t)=\sum_{i=1}^{k} a_i(t)\phi_i$$

To construct the reduced dynamics on a subspace $S\subset\mathbb{R}^n$ we can use many different methods like Galerkin Projection. Which simply projects the vector field locally onto the subspace S, leading to a reduced-dimension set of ordinary differential equation.

- Mechanical problem formulation:

As our starting point for model reduction:

1) For a given configuration space Q, we find a submanifold $Q_r \subset Q$
2) We construct a new mechanical system whose configuration space is Q_r
3) Instead of Galerkin projection, which does not in general preserve the mechanical structure, we use *constraints.*

The original Lagrangian L is restricted to the constraint submanifold Q_r to give a new Lagrangian $L|_{Q_r}$ on Q_r $\Rightarrow$ Construct Euler-Lagrange equations for the reduced dynamics on Q_r. These reduced dynamics will then be truly mechanical and satisfy all of the structural properties of such systems.

- All that remains is to take a good choice of the submanifold Q_r

Finite-element models of elasticity

- <u>Original system</u>

$$L(q,\dot{q})=\frac{1}{2}\dot{q}^{*}M\dot{q}-V(q)$$

$$M\ddot{q}=-DV(q)$$

- <u>Reduced system</u>

- Characterize Q_r as the span of orthonormal basis vectors $\{\phi_1\cdots,\phi_k\}$ and define the k*r matrix P whose rows are the ϕ_i.

- The the constraint is that $(1-P^{*}P)q=0$

$$L_r=\frac{1}{2}\dot{y}^{*}PMP^{*}\dot{y}-V(P^{*}y)$$

- Where y are the corresponding generalized coordinates on the constraint surface: $q=P^{*}y \quad y=Pq$

$$PMP^{*}\ddot{y}=-P\,DV(P^{*}y)$$

- <u>Original system</u>

- Implicit method: Newmark method, which is the mapping from $(q_n, \dot{q}_n)$ to $(q_{n+1}, \dot{q}_{n+1})$ defined by:

$$q_{n+1} = q_n + h\dot{q}_n + \frac{1}{2}(h^2)((1-2\beta)\ddot{q}_n + 2\beta\ddot{q}_{n+1}),$$
$$\dot{q}_{n+1} = \dot{q}_n + h((1-\gamma)\ddot{q}_n + \gamma\ddot{q}_{n+1}),$$

- The discrete accelerations are defined by the Euler-Lagrange equ: $M\ddot{q}_n = -DV(q_n)$

- For a given $(q_n, \dot{q}_n)$ we have to solve the implicit equation at each time-step:

$$R(q_{n+1}) = M\ddot{q}_{n+1} + DV(q_{n+1}) = 0$$

$$where: \quad R(q_{n+1}) := \frac{1}{h^2\beta} M q_{n+1} + DV(q_{n+1}) - \frac{1}{h^2\beta} M(q_n + h\dot{q}_n) + \frac{1-2\beta}{2\beta} DV(q_n).$$

- <u>Reduced system</u>

- Implicit method: Newmark method

- Application of the Newmark integration algorithm to this equation gives the residual:

$$\tilde{R}(q_{n+1}) := \frac{1}{h^2\beta} PMP^* y_{n+1} + P\,DV(P^* y_{n+1}) - \frac{1}{h^2\beta} PMP^*(y_n + h\dot{y}_n) + \frac{1-2\beta}{2\beta} P\,DV(P^* y_n).$$

- ## <u>Original system</u>

- The solution of the equation is approximated via Newton-Raphson iteration

$$q_{n+1}^{i+1} = q_{n+1}^{i} + \Delta q_{n+1}^{i}$$
$$where \quad \Delta q_{n+1}^{i} = -\frac{R(q_{n+1}^{i})}{dR(q_{n+1}^{i})/dq_{n+1}^{i}}$$

- From the residual equation we can have effective stiffness as:

$$dR(q)/dq = \frac{1}{h^2\beta} M + D^2 V(q) \quad for\ q \in Q$$

- ## <u>Reduced system</u>

- Using Newton Ralphson iteration:

$$y_{n+1}^{i+1} = y_{n+1}^{i} + \Delta y_{n+1}^{i}$$
$$where \quad \Delta y_{n+1}^{i} = -\frac{\tilde{R}(q_{n+1}^{i})}{d\tilde{R}(y_{n+1}^{i})/dy_{n+1}^{i}}$$

- The effective stiffness:

$$dR(y)/dy = \frac{1}{h^2\beta} PMP^* + P D^2 V(P^* y) P^* \quad for\ y \in Q_r$$

(2) Using this method for Non-linear FEM

- <u>Original system</u>

- Formulation of FEM:

$$\textit{deformation of body}: \ \phi(\boldsymbol{X}, t) = \boldsymbol{x}$$
$$\textit{the displacement}: \ u(\boldsymbol{X}, t) = \phi(\boldsymbol{X}, t) - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{X}$$

- Governing equation: conservation of momentum:

$$\textit{strong form}: \begin{cases} (A_0 P)_{,X} + \rho_0 A_0 b = \rho_0 A_0 \ddot{u} \rightarrow \textit{Lagrangian form} \\ (A\sigma)_{,x} + \rho A b = \rho A\, dv/dt \rightarrow \textit{updated Lagrangian} \end{cases}$$

$$\textit{weak form}: \begin{cases} \int_{X_a}^{X_b} w[(A_0 P)_{,X} + \rho_0 A_0 b - \rho_0 A_0 \ddot{u}]\, dX = 0 \rightarrow \textit{Lagrangian form} \\ \int_{x_a}^{x_b} w[(A\sigma)_{,x} + \rho A b - \rho A\, dv/dt]\, dx = 0 \rightarrow \textit{updated Lagrangian} \end{cases}$$

$$\textit{weak form}: \begin{cases} \underbrace{\int_{X_a}^{X_b} \delta F\, A_0 P\, dX}_{\delta W_{\text{int}}} + \underbrace{\int_{X_a}^{X_b} \delta u\, \rho_0 A_0 \ddot{u}\, dX}_{\delta W_{kin}} = \underbrace{\int_{X_a}^{X_b} \delta u\, \rho_0 A_0 b\, dX + [\delta u\, A_0 t_X^0]_{\Gamma_t}}_{\delta W_{ext}} \rightarrow \textit{Lagrangian form} \\ \underbrace{\int_{\Omega} \delta D\, A \sigma\, d\Omega}_{\delta p_{\text{int}}} + \underbrace{\int_{\Omega} \delta v \rho A \dot{v}\, d\Omega}_{\delta p_{kin}} = \underbrace{\int_{\Omega} \delta v \rho A b\, d\Omega + [\delta v\, A t_x]_{\Gamma_t}}_{\delta p_{ext}} \rightarrow \textit{updated Lagrangian} \end{cases}$$

- <u>Reduced system</u>

- <u>Original system</u>

- Galerkin appriximation:

$$\left\{\begin{array}{l} u_h(\boldsymbol{X},t)=\sum_{j=1}^{n} N_j(\boldsymbol{X})u_j(t) \rightarrow \textit{ Lagrangian form} \\ v_h(\boldsymbol{x},t)=\sum_{j=1}^{n} N_j(\boldsymbol{x})v_j(t) \rightarrow \textit{ updated Lagrangian} \end{array}\right\}$$

- Discrete Equations:

$$\left\{\begin{array}{l} f_{\text{int}}=\int_{\Omega} B^T \sigma \, d\Omega \\ f_{ext}=\int_{\Omega} \rho N^T b \, d\Omega + [N^T A t_x]_{\Gamma_t} \\ M=\int_{\Omega} \rho_0 N^T N \, d\Omega_0 \\ \boldsymbol{M}\ddot{\boldsymbol{u}}=\boldsymbol{f}_{ext}-\boldsymbol{f}_{\text{int}} \end{array}\right\}$$

a system of non-linear second-order ordinary differential equations

- <u>Reduced system</u>

- <u>Original system</u>

- This can be integrated in time by an implicit integrator/ Newmark time-stepping algorithm

- <u>Reduced system</u>

- It seems quite natural to look for more brief representation of the configuration :

$$\hat{\zeta}_h \subset \zeta_h$$

$$with: \ \hat{u}_h(\boldsymbol{X},t)=\sum_j^M \underbrace{\phi^j(\boldsymbol{X})}_{=\sum_I^N N_I(\boldsymbol{X})\phi_I^j} \ p_j(t)$$

$$=\sum_{j\,I}^{MN} \phi_I^j N_I(\boldsymbol{X}) p_j(t), \qquad M \ll N$$

$$\underbrace{u_h(\boldsymbol{X},t)=\sum_I^N \ N^I(\boldsymbol{X}) u_I(t)}_{\rightarrow\rightarrow\rightarrow\rightarrow} \quad \begin{aligned} &\sum_j^M \phi_I^j p_j(t)=u_I(t) \\ &\sum_j^M \phi_I^j \Delta p_j(t)=\Delta u_I(t) \\ &\sum_j^M \phi_I^j \delta p_j(t)=\delta u_I(t) \end{aligned} \quad \forall I \ on\, boundary$$

- <u>Original system</u>
- Implicit Newmark integration:

$$\left\{\begin{array}{l} Ma_{t+\Delta t}=f^{ext}_{t+\Delta t}-f^{\text{int}}_{t+\Delta t} \\ u_{t+\Delta t}=u_t+\Delta t\, v_t+\frac{\Delta t^2}{2}[(1-2\beta)a_t+2\beta a_{t+\Delta t}] \\ v_{t+\Delta t}=v_t+\Delta t[(1-\gamma)a_t+\gamma a_{t+\Delta t}] \end{array}\right\}$$

- Newton-Raphson iteration:

$$\left\{\begin{array}{l} R^{(i)}(u_{t+\Delta t})=Ma^{(i)}_{t+\Delta t}-(f^{ext(i)}_{t+\Delta t}-f^{\text{int}(i)}_{t+\Delta t}) \\ u^{(i)}_{t+\Delta t}=u^{(i-1)}_{t+\Delta t}+\Delta u^{(i)} \\ \Delta u^{(i)}=\frac{-R^{(i)}(u_{t+\Delta t})}{dR/du^{(i)}_{t+\Delta t}} \end{array}\right\}$$

- By linearizing f^{ext} and f^{int} we can find stiffness matrices

$$effective\ stiffness: \boldsymbol{K}_{eff}=\frac{dR}{du_{t+\Delta t}}=\boldsymbol{M}\frac{da_{t+\Delta t}}{du_{t+\Delta t}}-\frac{d}{du_{t+\Delta t}}f^{ext}_{t+\Delta t}+\frac{d}{du_{t+\Delta t}}f^{\text{int}}_{t+\Delta t}$$

$$=\frac{M}{\beta\Delta t^2}-\frac{d}{du_{t+\Delta t}}f^{ext}_{t+\Delta t}+\frac{d}{du_{t+\Delta t}}f^{\text{int}}_{t+\Delta t}=\frac{M}{\beta\Delta t^2}-(K^m-K^\sigma)+\underbrace{\acute{K}}_{=0}$$

- <u>Reduced system</u>
- Implicit Newmark integration and
- Newton-Raphson iteration

$$\Delta\boldsymbol{u}^{(i)}=\boldsymbol{\Phi}(\boldsymbol{X})\Delta\boldsymbol{p}^{(i)}=\boldsymbol{K}^{(i)\ -1}_{eff}\boldsymbol{R}^{(i)}$$

$$\tilde{\boldsymbol{K}}_{eff}=\boldsymbol{\Phi}^T\boldsymbol{K}_{eff}\boldsymbol{\Phi}=\boldsymbol{\Phi}^T\left(\frac{M}{\beta\Delta t^2}-(K^m-K^\sigma)\right)\boldsymbol{\Phi}$$

$$\boldsymbol{K}^{(i)\ -1}_{eff}=\boldsymbol{\Phi}(\boldsymbol{\Phi}^T\boldsymbol{K}_{eff}\boldsymbol{\Phi})\boldsymbol{\Phi}^T$$

- <u>Original system</u>

$$\left.\begin{array}{l} i \leftarrow 0 \\ u_{t+\Delta t}^{(i)} = u_t \\ a_{t+\Delta t}^{(i)} = \frac{-1}{\beta \Delta t} v_t + \left(1 - \frac{1}{2\beta}\right) a_t \\ v_{t+\Delta t}^{(i)} = v_t + \Delta t \left[(1-\gamma) a_t + \gamma a_{t+\Delta t}^{(i)} \right] \end{array}\right\} \text{Predictor}$$

$$i \leftarrow i+1$$

$$\boldsymbol{K}_{\boldsymbol{eff}}^{(i)} = \frac{1}{\beta \Delta t^2} \boldsymbol{M} + \boldsymbol{K}^{m(i)} + \boldsymbol{K}^{\sigma(i)}$$

$$\boldsymbol{R}^{(i)} = \boldsymbol{M} a_{t+\Delta t}^{(i)} - f_{t+\Delta t}^{ext\ (i)} + f_{t+\Delta t}^{\text{int}\ (i)}$$

$$\Delta \boldsymbol{u}^{(i)} = \boldsymbol{K}_{\boldsymbol{eff}}^{(i)\ -1} \boldsymbol{R}^{(i)}$$

$$\left.\begin{array}{l} u_{t+\Delta t}^{(i)} = u_{t+\Delta t}^{(i-1)} + \Delta u^{(i)} \\ v_{t+\Delta t}^{(i)} = v_{t+\Delta t}^{(i-1)} + \frac{\gamma}{\beta \Delta t} \Delta u^{(i)} \\ a_{t+\Delta t}^{(i)} = a_{t+\Delta t}^{(i-1)} + \frac{\gamma}{\beta \Delta t^2} \Delta u^{(i)} \end{array}\right\} \text{Corrector}$$

$$if \, \|\boldsymbol{R}^{(i)}\| > \epsilon \|\boldsymbol{R}^{(0)}\| \quad repeat\ for\ next\ iteration\,;$$

$$else\ t \leftarrow t + \Delta t \text{ and } go \text{ to } the\ top$$

- <u>Reduced system</u>

$$\left.\begin{array}{l} i \leftarrow 0 \\ u_{t+\Delta t}^{(i)} = u_t \\ a_{t+\Delta t}^{(i)} = \frac{-1}{\beta \Delta t} v_t + \left(1 - \frac{1}{2\beta}\right) a_t \\ v_{t+\Delta t}^{(i)} = v_t + \Delta t \left[(1-\gamma) a_t + \gamma a_{t+\Delta t}^{(i)} \right] \end{array}\right\} \text{Predictor}$$

$$i \leftarrow i+1$$

$$\boxed{\boldsymbol{\Phi}^T \boldsymbol{K}_{\boldsymbol{eff}}^{(i)} \boldsymbol{\Phi} = \boldsymbol{\Phi}^T \left[\frac{1}{\beta \Delta t^2} \boldsymbol{M} + \boldsymbol{K}^{m(i)} + \boldsymbol{K}^{\sigma(i)} \right] \boldsymbol{\Phi}}$$

$$\boldsymbol{R}^{(i)} = \boldsymbol{M} a_{t+\Delta t}^{(i)} - f_{t+\Delta t}^{ext\ (i)} + f_{t+\Delta t}^{\text{int}\ (i)}$$

$$\boxed{\Delta \boldsymbol{u}^{(i)} = \boldsymbol{\Phi} \left(\boldsymbol{\Phi}^T \boldsymbol{K}_{\boldsymbol{eff}}^{(i)} \boldsymbol{\Phi} \right)^{-1} \boldsymbol{\Phi}^T \boldsymbol{R}^{(i)}}$$

$$\left.\begin{array}{l} u_{t+\Delta t}^{(i)} = u_{t+\Delta t}^{(i-1)} + \Delta u^{(i)} \\ v_{t+\Delta t}^{(i)} = v_{t+\Delta t}^{(i-1)} + \frac{\gamma}{\beta \Delta t} \Delta u^{(i)} \\ a_{t+\Delta t}^{(i)} = a_{t+\Delta t}^{(i-1)} + \frac{\gamma}{\beta \Delta t^2} \Delta u^{(i)} \end{array}\right\} \text{Corrector}$$

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$$else\ t \leftarrow t + \Delta t \text{ and } go \text{ to } the\ top$$

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