Particle methods:

Smooth Particle Hydrodynamics
and
Meshless Finite Element Method

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  - Introduction
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- Positive & negative sides
- Examples
- Discussion

Particle methods

- SPH

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- Reproducing Kernel Particle Method
  Liu, Jun, and Zhang (1995)

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- MFEM

SPH – Introduction

SPH was introduced in astrophysics by Lucy (1977) and Gingold and Monaghan (1977).

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Applications in fluid dynamics

- Dam break
- Breaking waves
- Interaction with structures
- Run-up
- Gravity currents
- Solitary waves
- . . .

SPH – Theory

Integral interpolant of $A(\boldsymbol{r})$:

$$A_I(\boldsymbol{r}) = \int_{\text{Space}} A(\boldsymbol{r}')W(\boldsymbol{r} - \boldsymbol{r}', h)\,\mathrm{d}\boldsymbol{r}'$$

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Kernel properties:

$$\int_{\text{Space}} W(\boldsymbol{r}-\boldsymbol{r}', h)\,\mathrm{d}\boldsymbol{r}' = 1$$

$$\lim_{h\to 0} W(\boldsymbol{r}-\boldsymbol{r}', h) = \delta(\boldsymbol{r}-\boldsymbol{r}')$$

SPH – Interpolant

Numerical interpolant of $A(\boldsymbol{r})$:

$$A_S(\boldsymbol{r}) = \sum_i m_i \frac{A_i}{\rho_i} W(\boldsymbol{r} - \boldsymbol{r}_i, h)$$

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Particle i has

- mass m_i
- position $\boldsymbol{r}_i$
- density ρ_i

SPH – Kernel

Kernel based on spline functions:

$$W(\boldsymbol{r}, h) = \frac{\sigma}{h^\nu} \begin{cases} (1 - \frac{3}{2}s^2 + \frac{3}{4}s^3), & 0 \le s \le 1 \\ \frac{1}{4}\left(2 - s\right)^3, & 1 \le s \le 2 \\ 0, & \text{otherwise} \end{cases}$$

where $s = r/h$, ν is the number of dimension and σ is a normalization constant.

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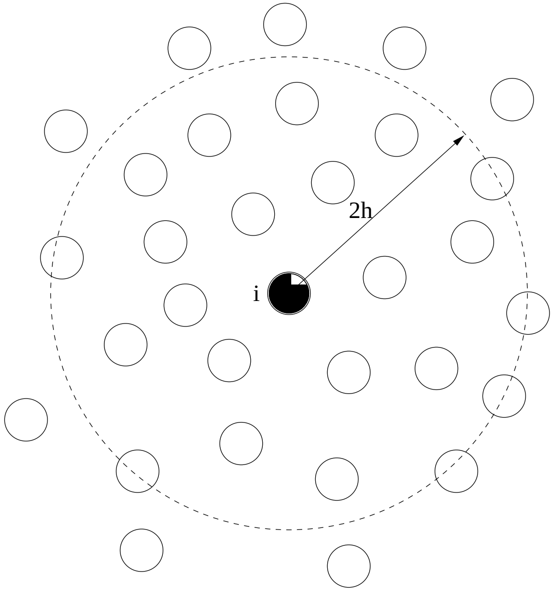

SPH – Gradient and integral

Gradient of $A(\boldsymbol{r})$:

$$\nabla A(\boldsymbol{r}) = \sum_i m_i \frac{A_i}{\rho_i} \nabla W(\boldsymbol{r} - \boldsymbol{r}_i, h)$$

SPH – Gradient and integral

Gradient of $A(\boldsymbol{r})$:

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Higher accuracy is obtained by using:

$$\rho \nabla A = \nabla(\rho A) - A \nabla \rho$$

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Integral:

$$\int_\Gamma A(\boldsymbol{r}) \, \mathrm{d}\Gamma = \sum_i m_i \frac{A_i}{\rho_i} \int_\Gamma W(\boldsymbol{r} - \boldsymbol{r}_i, h) \, \mathrm{d}\Gamma$$

SPH – The continuity equation

Alt. I:

$$
\begin{aligned}
\frac{d\rho}{dt} &= -\rho\nabla \cdot \boldsymbol{v} \\
&= -\sum_j m_j \frac{\rho}{\rho_j}\boldsymbol{v}_j \cdot \nabla W(\boldsymbol{r} - \boldsymbol{r}_j, h)
\end{aligned}
$$

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Alt. II:

$$\begin{aligned}\frac{d\rho}{dt} &= -\rho\nabla\cdot\boldsymbol{v}\\ &= \boldsymbol{v}\cdot\nabla\rho-\nabla\cdot(\rho\boldsymbol{v})\\ &= \sum_j m_j(\boldsymbol{v}-\boldsymbol{v}_j)\cdot\nabla W(\boldsymbol{r}-\boldsymbol{r}_j,h)\end{aligned}$$

SPH – The momentum equation

$$
\begin{aligned}
\frac{d\boldsymbol{v}}{dt} &= -\frac{\nabla p}{\rho} \\
&= -\left[\nabla\left(\frac{p}{\rho}\right) + \frac{p}{\rho^2}\nabla\rho\right] \\
&= -\sum_j m_j \left(\frac{p_j}{\rho_j^2} + \frac{p}{\rho^2}\right)\nabla W(\boldsymbol{r} - \boldsymbol{r}_j, h)
\end{aligned}
$$

SPH – Moving particles

$$\frac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i$$

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XSPH-variant

$$\frac{d\boldsymbol{r}_i}{dt} = \hat{\boldsymbol{v}}_i = \boldsymbol{v}_i + \epsilon \sum_j m_j \left(\frac{\boldsymbol{v}_j - \boldsymbol{v}_i}{\bar{\rho}_{ij}} \right) W(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

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where $\bar{\rho}_{ij} = (\rho_i + \rho_j)/2$ and ϵ is a constant $(0 \leq \epsilon \leq 1)$.

SPH – Boundary conditions

Rigid walls are modelled with

- Boundary particles and a repulsive force

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SPH – Boundary conditions

Rigid walls are modelled with

- Boundary particles and a repulsive force
- Perfect reflection
- A layer of fixed particles

SPH – Pos. and neg. sides

Negative

- Difficult to include boundary conditions
- Particle may penetrate the boundary
- Smoothing and accuracy

SPH – Pos. and neg. sides

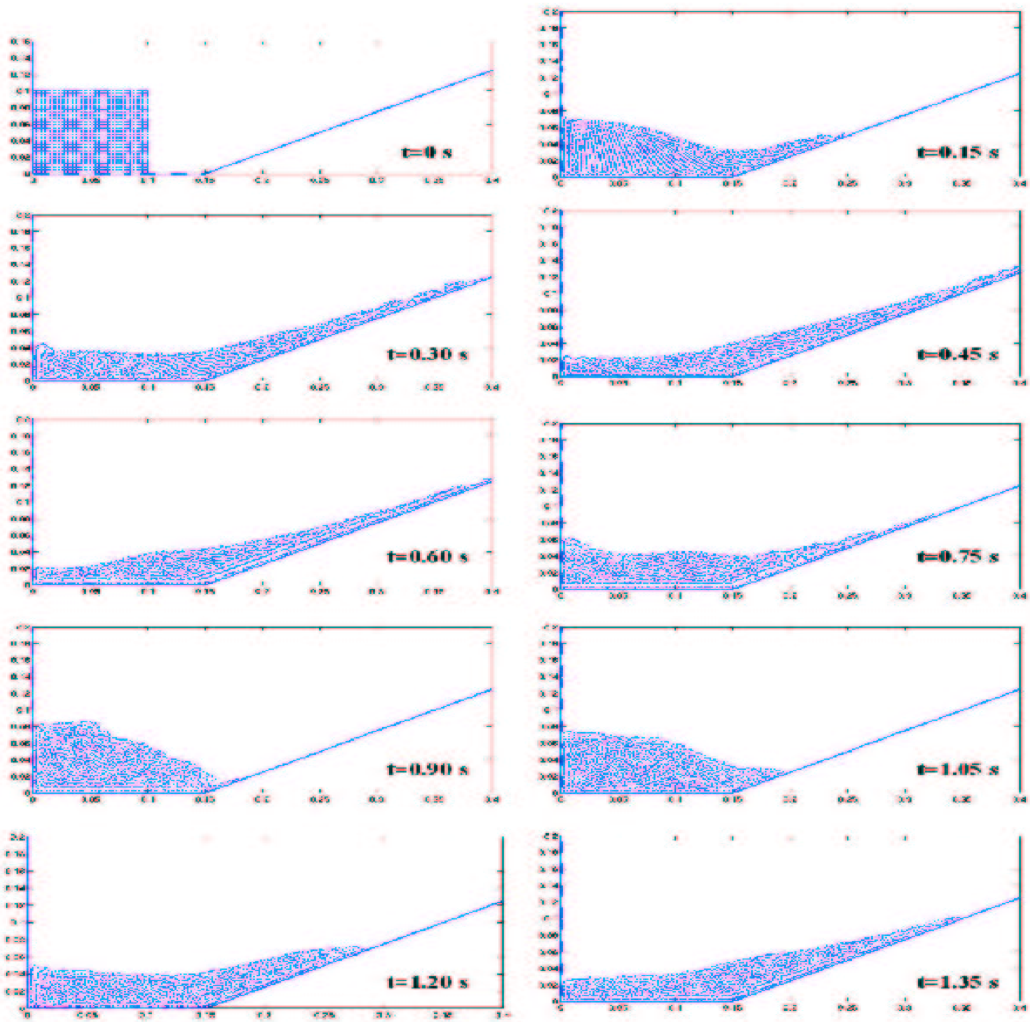
Negative

- Difficult to include boundary conditions
- Particle may penetrate the boundary
- Smoothing and accuracy

Positive

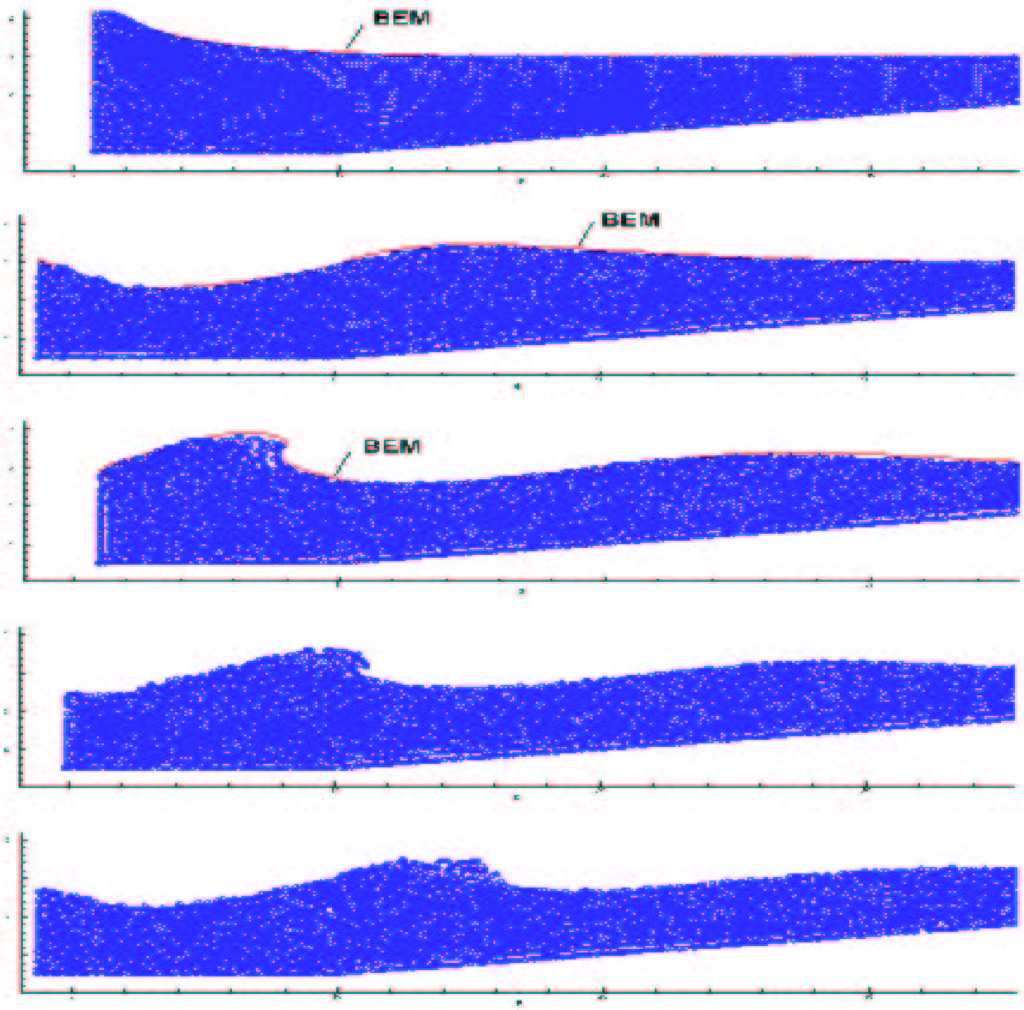
- Robust method
- Handles wave breaking
- Easy to parallize

SPH – Dam break



(Example by Mosqueira et al.(2002))

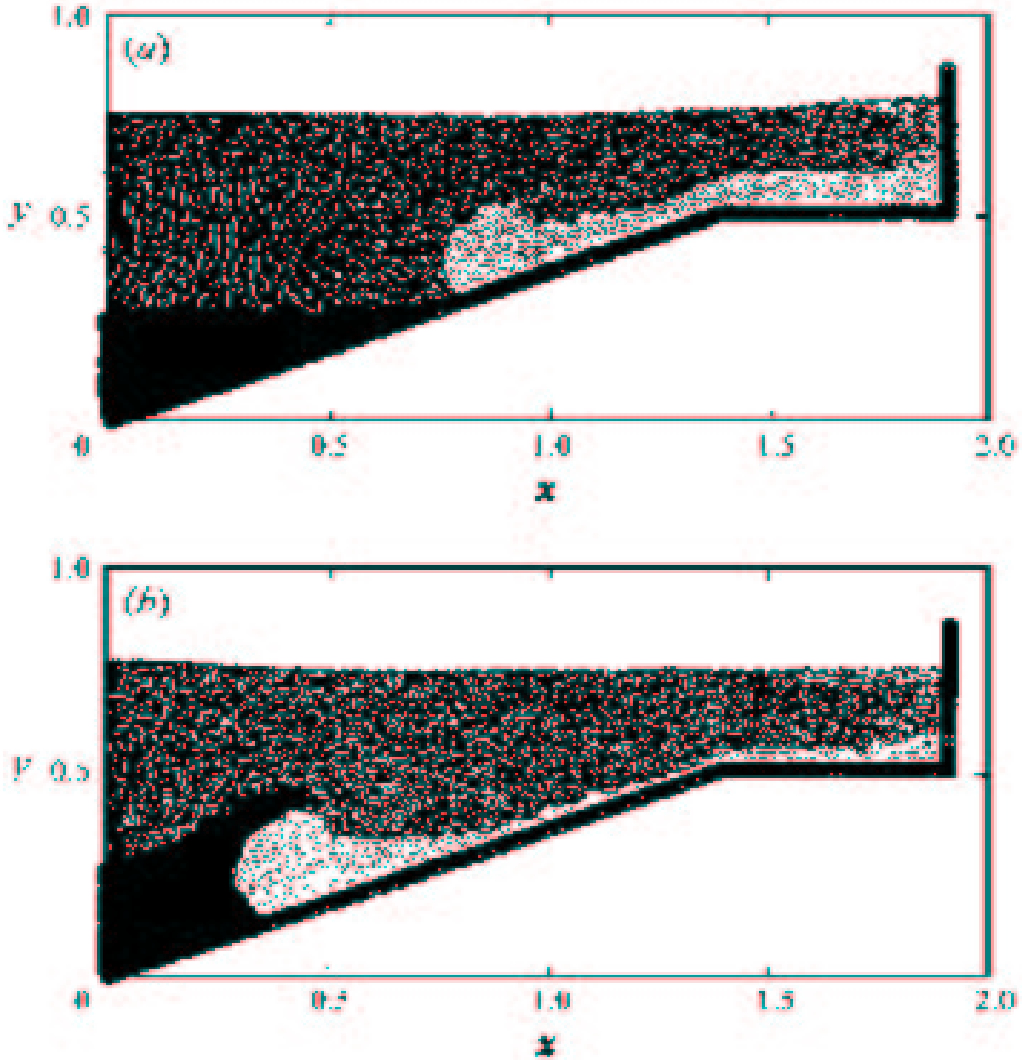
SPH – Breaking wave



Breaking wave in shallow water. Comparison between BEM and SPH.

(Example by Fontaine ())

SPH – Gravity current



Density:
Lock fluid: 1300 kg/m^3
Rest of tank: 1000 kg/m^3

(Example by
Monaghan et al.(1999))

MFEM – Introduction

The MFEM is described by
Idelsohn, Oñate, Calvo, and Del Pin (2002).

Definition of a meshless method:

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Definition of a meshless method:

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Definition of a meshless method:

1. The definition of the shape functions depends only on the node positions
2. The evaluation of the node connectivities is **bounded** in time, and this time depends exclusively on the total number of nodes in the domain

MFEM – Solution

The solution is approximated by

$$u(\boldsymbol{x}) \approx \hat{u}(\boldsymbol{x}) = \sum_i N_i(\boldsymbol{x}) u_i$$

where

$N_i(\boldsymbol{x})$: Shape functions

u_i: Nodal values

MFEM – Domain partition

Distinct nodes: $\boldsymbol{N} = \{\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_n\}$
The domain is partitioned by using

1. Voronoï diagram
2. Voronoï sphere
3. Extended Delaunay tessellation

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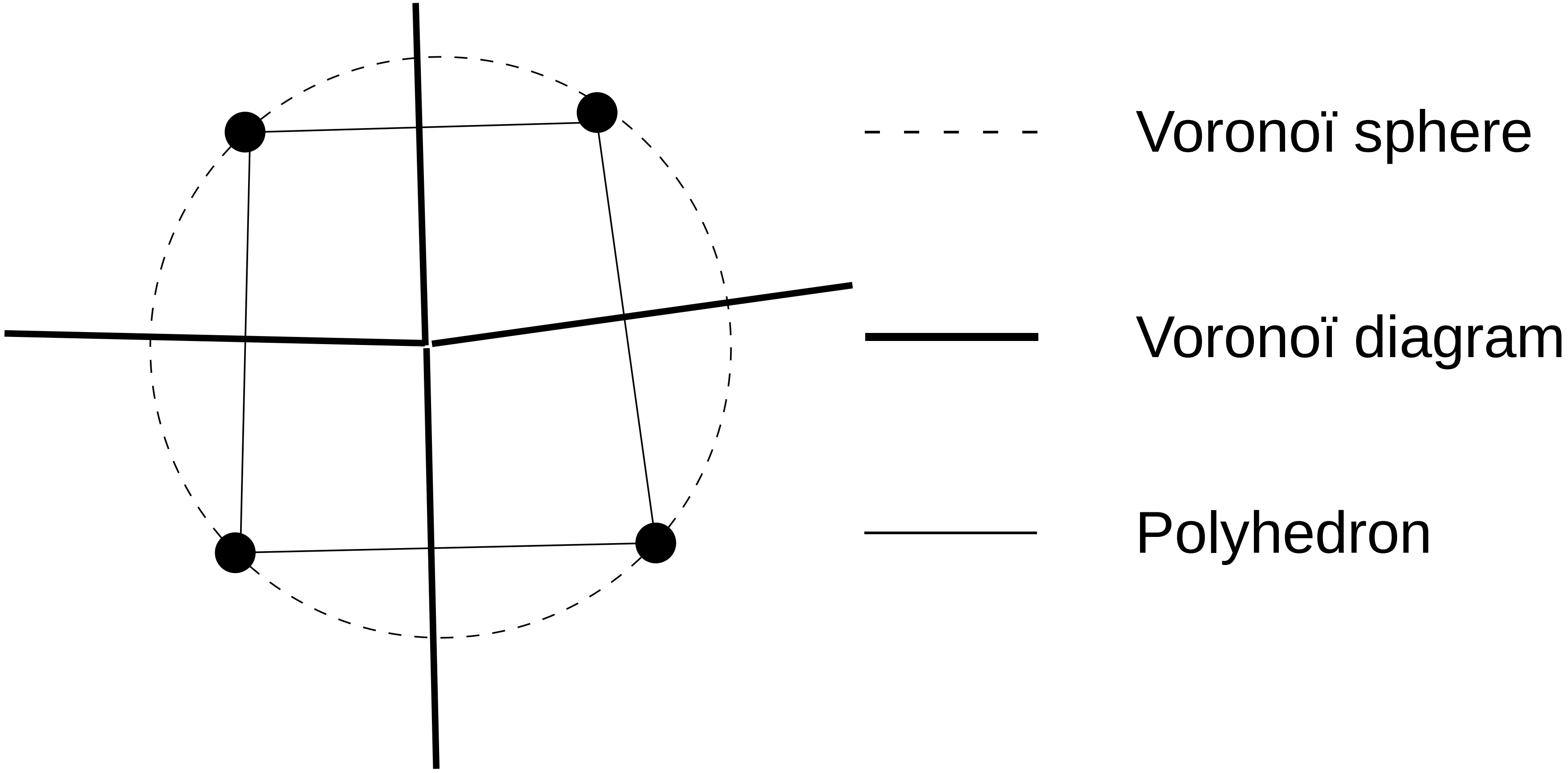
…at each timestep.

MFEM – Domain partition



MFEM – Shape functions

Non-Sibsonian interpolants, Belikov and Semenov (1998)

$$N_i(\boldsymbol{x}) = \frac{\dfrac{s_i(\boldsymbol{x})}{h_i(\boldsymbol{x})}}{\displaystyle\sum_{j=1}^{m} \frac{s_j(\boldsymbol{x})}{h_j(\boldsymbol{x})}}$$

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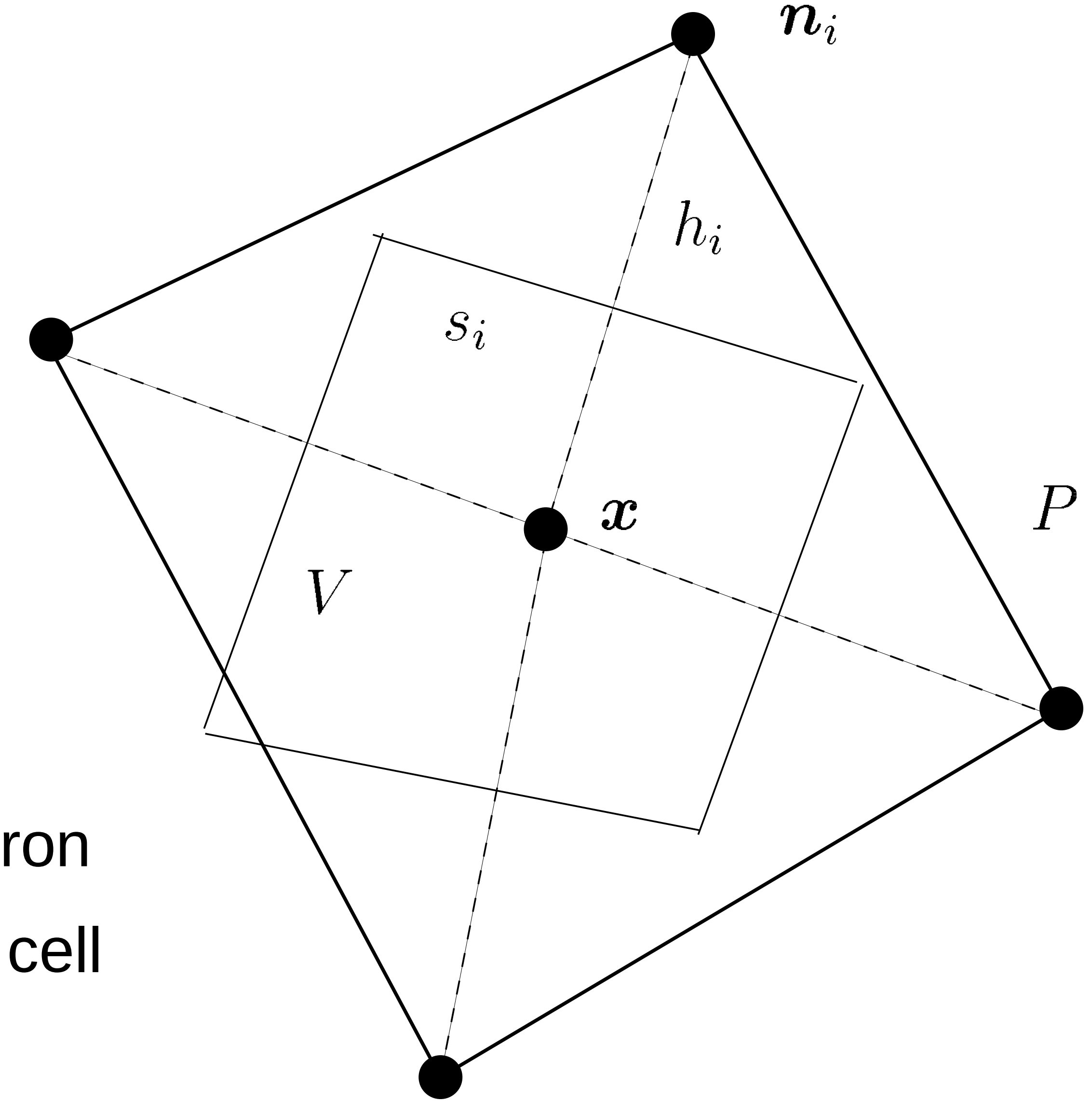
Properties:

1. $0 \leq N_i(\boldsymbol{x}) \leq 1$
2. $\sum_i N_i(\boldsymbol{x}) = 1$
3. $N_i(\boldsymbol{n}_j) = \delta_{ij}$
4. $\boldsymbol{x} = \sum_i N_i(\boldsymbol{x})\boldsymbol{n}_i$
5. Linear completeness: $f(\boldsymbol{x}) = \sum_i N_i(\boldsymbol{x}) f(\boldsymbol{n}_i)$

MFEM – Shape functions



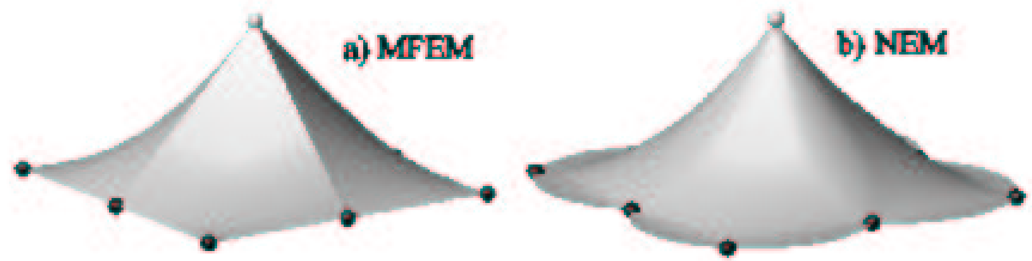
P: Polyhedron

V: Voronoï cell

MFEM – Shape functions

Shape functions in 2-D

MFEM NEM



C^0 continuity C^∞ continuity

MFEM – Gradient and integral

Function:

$$p(\boldsymbol{x}) = \sum_i N_i(\boldsymbol{x}) p_i$$

Integral:

$$\int_\Gamma p(\boldsymbol{x}) \, \mathrm{d}\Gamma = \sum_i p_i \int_\Gamma N_i(\boldsymbol{x}) \, \mathrm{d}\Gamma$$

MFEM – Boundary conditions

Procedure:

- Identify boundary nodes and surfaces
- Impose a value to the boundary nodes

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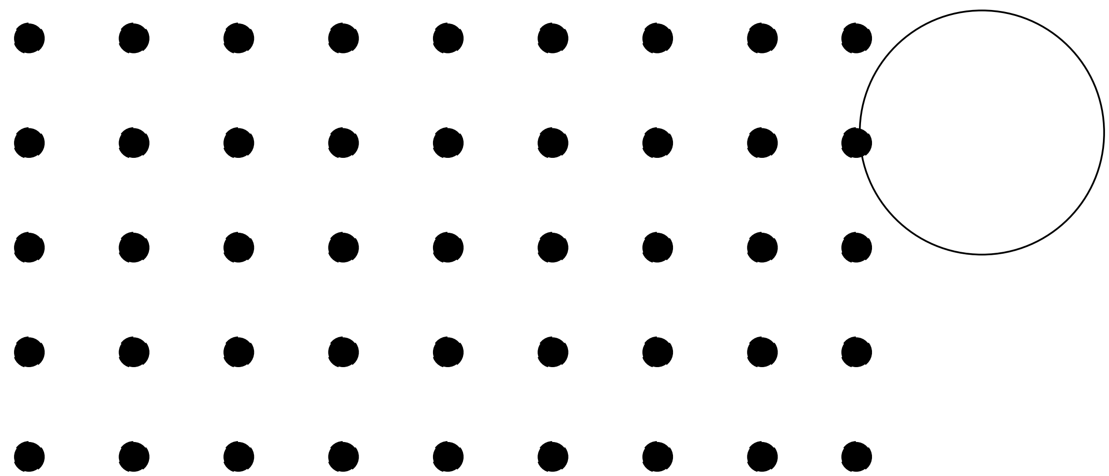

MFEM – Solution algorithm

Incompressible and non-viscous flow

Fractional time step method

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Assume solution is known at t^n

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2. Mom. eq., source part $\Rightarrow \boldsymbol{v}^*$ [Explicit]

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Incompressible and non-viscous flow

Fractional time step method

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3. Pressure: $\nabla^2 p^{n+1} = f(\boldsymbol{v}^*)$ [Galerkin method, weak form]

MFEM – Solution algorithm

Incompressible and non-viscous flow

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4. Mom. eq., pressure part $\Rightarrow \boldsymbol{v}^{n+1}$ [Explicit]

MFEM – Solution algorithm

Incompressible and non-viscous flow

Fractional time step method

Assume solution is known at t^n

1. Split step: $\boldsymbol{v}^{n+1}$ and $\boldsymbol{v}^*$
2. Mom. eq., source part $\Rightarrow \boldsymbol{v}^*$ [Explicit]
3. Pressure: $\nabla^2 p^{n+1} = f(\boldsymbol{v}^*)$ [Galerkin method, weak form]
4. Mom. eq., pressure part $\Rightarrow \boldsymbol{v}^{n+1}$ [Explicit]
5. Move particles: $\frac{d\boldsymbol{r}^{n+1}}{dt} = \boldsymbol{v}^{n+1}$ [Explicit]

MFEM – Pos. and neg. sides

Negative

- Documentation
- The Delaunay tesselation may give a bad partition (slivers)
- The Delaunay tesselation is time consuming?
- Error when deter-mining the boundary nodes/surfaces
- Difficult to handle incompressible flow (general problem)

MFEM – Pos. and neg. sides

Positive

- Robust method
- Handles wave breaking
- Easy to include boundary conditions
- Simple shape functions
- Allows treatment of material discontinu-ities
- Symmetric matrices
- Easy to implement(?):
  - Standard FEM program
  - Partition package
  - Need to implement shape functions

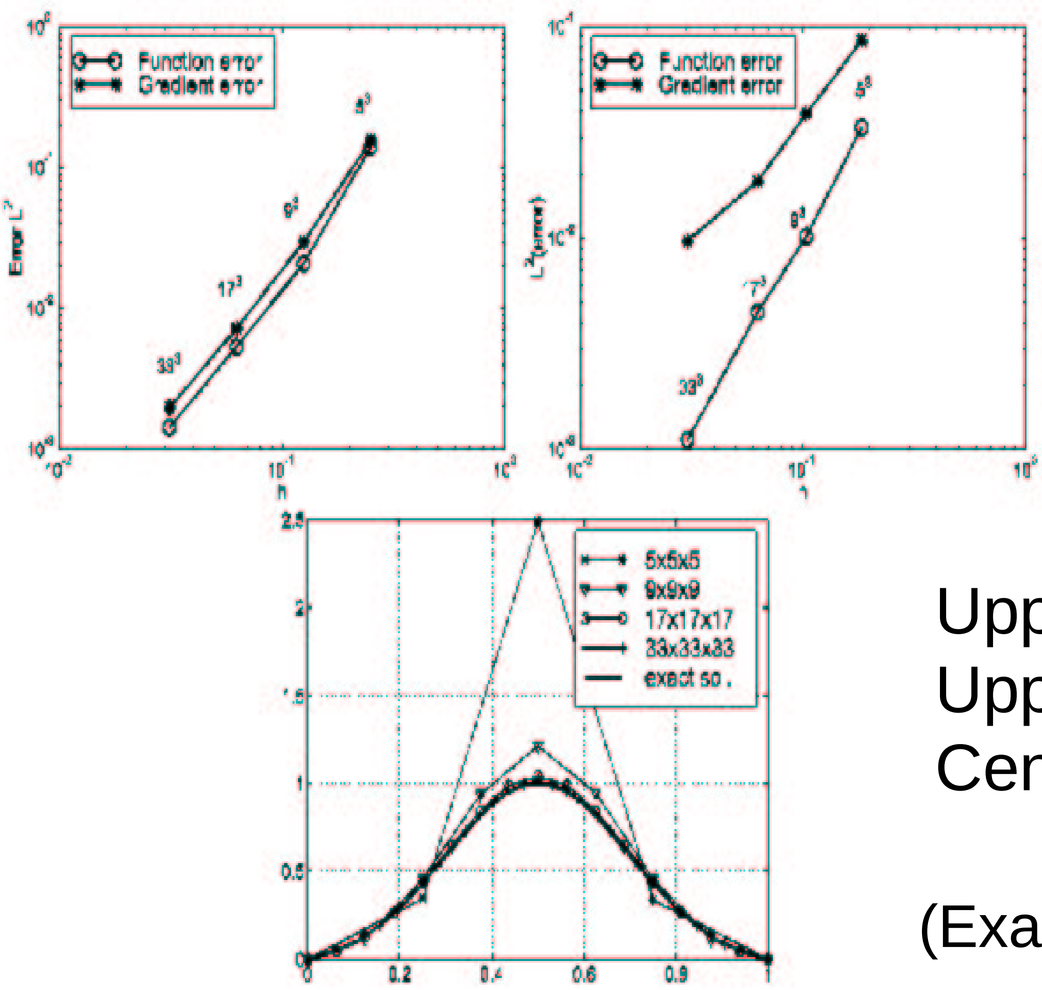
MFEM – Numerical test

Poisson equation:

$$\begin{aligned} \nabla^2 u = f(x,y,z) \quad & \text{in } \Omega \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned}$$

where Ω is a cube of unit side and f is an internal source.

MFEM – Numerical test



Upper left: Structured node distr.
Upper right: Non-structured node distr.
Center-line solution (str. node distr.)

(Example by Idelsohn et al. (2002))

Animations

SPH
Wavemaker <avi>
Planet <mpg>

MFEM
Dam break 2-D <gif>
Dam break 3-D <gif>
Channel <gif>
MFEM <gif>

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