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Arena Sand/Soil Model Theory Manual

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A typical input file

The inputs for the Arena soil model are typically specified as follows.

```
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```

The governing equations

The governing equations are based on the aprroacj used by Uzuoka and Borja (Uzuoka and Borja, 2012).

Momentum balance

The momentum balance equation for the mixture is

$$\rho \mathbf{a}^s = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b}^s \quad \text{where} \quad \rho = \sum_\alpha \rho^\alpha \tag{1}$$

where ρ is the mass density of the mixture, $\mathbf{a}^s$ is the acceleration of the solid skeleton, $\boldsymbol{\sigma}$ is the total Cauchy stress in the mixture, $\mathbf{b}^s$ is the body force experienced by the mixture, and $\alpha = \{s, w, a\}$ are the three phases (solid, water, air). The effective stress is defined as

$$\boldsymbol{\sigma}_{\text{eff}} = \boldsymbol{\sigma} + B\overline{p^w}\boldsymbol{I} = \boldsymbol{\sigma} - \boldsymbol{\alpha} \tag{2}$$

where $\overline{p^w} = -p^w$ is the pore pressure, $\boldsymbol{\alpha} := -B\overline{p^w}\boldsymbol{I}$ can be treated as a backstress, and B is the Biot coefficient which is defined as

$$B := 1 - \frac{K_d(\overline{p^s})}{K_s(\overline{p^s})}\,. \tag{3}$$

Here p^s is the intrinsic pressure in the solid skeleton, K_d is the bulk modulus of the drained solid skeleton, K_s is the bulk modulus of the solid grains, and we assume that $\overline{p^s} = \overline{p^w}$. All unbarred quantities are positive in tension. For example, $\boldsymbol{\sigma}$ and p^w are positive in tension while $\overline{p^w}$ is positive in compression.

Mass balance

As shown in the previous report, with the modifications shown in Appendix C, the mass balance equations can be expressed as a rate equation for the pore pressure:

$$\frac{d\overline{p^w}}{d\overline{\varepsilon_v}} = \frac{1}{\mathcal{B}} \tag{4}$$

where $\overline{\varepsilon_v}$ is the total volumetric strain, and

$$\begin{aligned}\mathcal{B} := \frac{1}{(1-S_0)\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a}) + S_0\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})}\Bigg[-\frac{(B-\phi)\phi}{B\phi_0}\left(\frac{S_w}{K_w} + \frac{1-S_w}{K_a}\right) + \\ \frac{1-S_0}{K_a}\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a}) + \frac{S_0}{K_w}\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})\Bigg]\,.\end{aligned} \tag{5}$$

In the above expression, B is the Biot coefficient, ϕ is the current total porosity, ϕ_0 is the initial total porosity, S_w is the total saturation, S_0 is the initial total saturation, K_w is the pressure-dependent bulk modulus of water, K_a is the pressure-dependent bulk modulus of air, $\overline{\varepsilon_v^a}$ is the total strain in the air phase, and $\overline{\varepsilon_v^w}$ is the total strain in the water phase.

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Energy balance

The energy balance equation of the mixture is

$$\rho \frac{D^s e}{Dt} - \boldsymbol{\sigma}_{\text{eff}} : \boldsymbol{d}^s + \overline{p^w} \left[\frac{S_w \phi}{K_w(\overline{p^w})} - \frac{(1 - S_w)\phi}{K_a(\overline{p^w})} \right] \frac{D^s \overline{p^w}}{Dt} + \nabla \cdot \mathbf{q} - \rho h = 0 \tag{6}$$

where e is the internal energy density, $\boldsymbol{d}^s$ is the symmetric part of the velocity gradient in the solid skeleton, $\mathbf{q}$ is the heat flux, and h is a heat source. This equation may also be written in the alternative form (Hassanizadeh and Gray, 1990)

$$\rho \frac{D^s e}{Dt} - \boldsymbol{\sigma} : \boldsymbol{d}^s + \nabla \cdot \mathbf{q} - \rho h = 0 \,. \tag{7}$$

Constitutive model

We follow the approach used by Homel, Guilkey, and Brannon, (2015) where the total stress is used in the constitutive model instead of the effective stress. In rate form, and after rigid body rotations have been eliminated, the constitutive model has the form

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}_{\text{eff}} + \dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\sigma}}_{\text{eff}} - \dot{B}\overline{p^w}\boldsymbol{I} - B\dot{\overline{p^w}}\boldsymbol{I} = g_1(\boldsymbol{\sigma}, \boldsymbol{\eta}, \boldsymbol{d}^s) - g_2(\boldsymbol{\sigma}, \boldsymbol{\eta}, \boldsymbol{d}^s)\overline{p^w} - g_3(\boldsymbol{\sigma}, \boldsymbol{\eta}, \boldsymbol{d}^s)B \tag{8}$$

where g_1, g_2, and g_3 are assumed constitutive relations that can be expressed as

$$\begin{aligned}
g_1(\boldsymbol{\sigma}, \boldsymbol{\eta}, \boldsymbol{d}^s) &= \dot{\boldsymbol{\sigma}}_{\text{eff}} = \mathbf{C}^{\text{pe}}_{\text{eff}} : \boldsymbol{d}^s \\
g_2(\boldsymbol{\sigma}, \boldsymbol{\eta}, \boldsymbol{d}^s) &= \dot{B} = -\frac{1}{K_s}\dot{\overline{K}}_s + \frac{K_d}{K_s^2}\dot{K}_s = -\frac{1}{K_s \mathcal{B}}\left[\frac{\partial K_d}{\partial \overline{p^w}} - \frac{K_d}{K_s}\frac{\partial K_s}{\partial \overline{p^w}}\right] \operatorname{tr}(\boldsymbol{d}^s) \\
g_3(\boldsymbol{\sigma}, \boldsymbol{\eta}, \boldsymbol{d}^s) &= \dot{\overline{p^w}} = \frac{1}{\mathcal{B}} \operatorname{tr}(\boldsymbol{d}^s)
\end{aligned} \tag{9}$$

where $\mathbf{C}^{\text{pe}}_{\text{eff}}(\boldsymbol{\sigma}, \boldsymbol{\eta})$ is an effective elastic-plastic tangent modulus, $K_d(\boldsymbol{\sigma}, \boldsymbol{\eta})$ and $K_s(\boldsymbol{\sigma}, \boldsymbol{\eta})$ are the bulk moduli that contribute to the Biot parameter, and $\mathcal{B}(\boldsymbol{\sigma}, \boldsymbol{\eta})$ is defined in equation (5). Equation (8) suggests that the bulk modulus may also have to be treated as an internal variable if the backstress is treated as an internal variable.

As discussed in the previous report, we assume that

$$\boldsymbol{d}^s = \boldsymbol{d}^{\text{e}} + \boldsymbol{d}^{\text{p}} = \dot{\boldsymbol{\varepsilon}}^{\text{e}} + \dot{\boldsymbol{\varepsilon}}^{\text{p}} \tag{10}$$

where $\boldsymbol{\varepsilon}^{\text{e}}$ is an elastic strain and $\boldsymbol{\varepsilon}^{\text{p}}$ is an plastic strain. In addition to the plastic strain, we consider the following internal variables ($\boldsymbol{\eta}$) for the partially saturated soil model:

$$\boldsymbol{\eta} = \{\overline{p^w_{\text{e}}}, \overline{p^w_{\text{p}}}, B^{\text{e}}, B^{\text{p}}, \phi^{\text{e}}, \phi^{\text{p}}, S^{\text{e}}_w, S^{\text{p}}_w, \overline{X}^{\text{e}}, \overline{X}^{\text{p}}\} \tag{11}$$

where $\overline{p^w_{\text{e}}}$ is the part of the pore pressure that goes to zero after a load that leads to plastic deformation is removed, $\overline{p^w_{\text{p}}} = \overline{p^w} - \overline{p^w_{\text{e}}}$ is the unrecoverable pore pressure which remains nonzero after removal of the load, B^{e}, B^{p} are the the Biot parameter values associated with elastic and inelastic deformation, ϕ^{e} is the recoverable porosity when the load is removed, ϕ^{p} is the unrecoverable or "unloaded" porosity, S^{e}_w is the recoverable saturation, S^{p}_w is the unrecoverable saturation, and $\overline{X}^{\text{e}}$, $\overline{X}^{\text{p}}$ are the hydrostatic compressive strengths associated with elastic and inelastic deformation, respectively.

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Rate-independent plasticity

Coupled elasticity We assume that during elastic deformation the elastic behavior of the soil can be coupled to the plastic behavior. The rate equation for elastic behavior can then be written as (see Brannon, (2007))

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^{\mathrm{e}} : \boldsymbol{d}^{\mathrm{e}} - \dot{\lambda}\boldsymbol{Z} \tag{12}$$

where the purely elastic stiffness tensor, $\mathbf{C}^{\mathrm{e}}(\boldsymbol{\sigma}, \overline{p_{\mathrm{e}}^{w}}, B^{\mathrm{e}}, \phi^{\mathrm{e}}, S_{w}^{\mathrm{e}}, \overline{X}^{\mathrm{e}})$, is defined using

$$\begin{aligned}\mathbf{C}^{\mathrm{e}} : \boldsymbol{d}^{\mathrm{e}} &= \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} : \boldsymbol{d}^{\mathrm{e}} + \frac{\partial \boldsymbol{\sigma}}{\partial \overline{p_{\mathrm{e}}^{w}}}\,\dot{\overline{p_{\mathrm{e}}^{w}}} + \frac{\partial \boldsymbol{\sigma}}{\partial B^{\mathrm{e}}}\,\dot{B}^{\mathrm{e}} + \frac{\partial \boldsymbol{\sigma}}{\partial \phi^{\mathrm{e}}}\,\dot{\phi}^{\mathrm{e}} + \frac{\partial \boldsymbol{\sigma}}{\partial S_{w}^{\mathrm{e}}}\,\dot{S}_{w}^{\mathrm{e}} + \frac{\partial \boldsymbol{\sigma}}{\partial \overline{X}^{\mathrm{e}}}\,\dot{\overline{X}}^{\mathrm{e}} \\ &= \left[\frac{\partial \boldsymbol{\sigma}_{\mathrm{eff}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial \overline{p_{\mathrm{e}}^{w}}} \otimes \frac{\partial \overline{p_{\mathrm{e}}^{w}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial B^{\mathrm{e}}} \otimes \frac{\partial B^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial \phi^{\mathrm{e}}} \otimes \frac{\partial \phi^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial S_{w}^{\mathrm{e}}} \otimes \frac{\partial S_{w}^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial \overline{X}^{\mathrm{e}}} \otimes \frac{\partial \overline{X}^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}}\right] : \boldsymbol{d}^{\mathrm{e}}\end{aligned} \tag{13}$$

or

$$\mathbf{C}^{\mathrm{e}} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial \overline{p_{\mathrm{e}}^{w}}} \otimes \frac{\partial \overline{p_{\mathrm{e}}^{w}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial B^{\mathrm{e}}} \otimes \frac{\partial B^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial \phi^{\mathrm{e}}} \otimes \frac{\partial \phi^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial S_{w}^{\mathrm{e}}} \otimes \frac{\partial S_{w}^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} + \frac{\partial \boldsymbol{\sigma}}{\partial \overline{X}^{\mathrm{e}}} \otimes \frac{\partial \overline{X}^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}}\,. \tag{14}$$

where $\boldsymbol{\varepsilon}^{\mathrm{e}}$ is the elastic part of the strain that is energy conjugate to the *unrotated* Cauchy stress (see Norris, (2008) for possible strain measures). In the plastic coupling term in (12), $\dot{\lambda}$ is the plastic flow rate parameter, and $\boldsymbol{Z}(\boldsymbol{\sigma}, \overline{p_{\mathrm{p}}^{w}}, B^{\mathrm{p}}, \phi^{\mathrm{p}}, S_{w}^{\mathrm{p}}, \overline{X}^{\mathrm{p}}; \boldsymbol{\varepsilon}^{\mathrm{p}})$ is a rank-2 elastic-plastic coupling tensor given by

$$\boldsymbol{Z} = h_{p}\frac{\partial \boldsymbol{\sigma}}{\partial \overline{p_{\mathrm{p}}^{w}}} + h_{B}\frac{\partial \boldsymbol{\sigma}}{\partial B^{\mathrm{p}}} + h_{\phi}\frac{\partial \boldsymbol{\sigma}}{\partial \phi^{\mathrm{p}}} + h_{S_{w}}\frac{\partial \boldsymbol{\sigma}}{\partial S_{w}^{\mathrm{p}}} + h_{X}\frac{\partial \boldsymbol{\sigma}}{\partial \overline{X}^{\mathrm{p}}} \tag{15}$$

where

$$\dot{\overline{p_{\mathrm{p}}^{w}}} = \dot{\lambda}h_{p}\,,\quad \dot{B}^{\mathrm{p}} = \dot{\lambda}h_{B}\,,\quad \dot{\phi}^{\mathrm{p}} = \dot{\lambda}h_{\phi}\,,\quad \dot{S}_{w}^{\mathrm{p}} = \dot{\lambda}h_{S_{w}}\,,\quad \dot{\overline{X}}^{\mathrm{p}} = \dot{\lambda}h_{X}\,. \tag{16}$$

The "hardening" functions h_p, h_b, h_ϕ, h_{S_w}, and h_X require additional constitutive assumptions. During purely elastic loading, the coupling term is zero because $\dot{\lambda}$ is zero.

Flow rule and consistency The plastic flow rule is

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \boldsymbol{d}^{\mathrm{p}} = \dot{\lambda}\boldsymbol{M} \tag{17}$$

where $\boldsymbol{M}$ is the unit tensor in the direction of the plastic rate of deformation. The consistency condition can be written as

$$\widehat{\boldsymbol{N}} : \dot{\boldsymbol{\sigma}} = \dot{\lambda}H \tag{18}$$

where $\widehat{\boldsymbol{N}}$ is the unit normal to the yield surface, and H is the ensemble hardening modulus.

The plastic flow rate parameter The find the consistency parameter, we write the equation for coupled elasticity as

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^{\mathrm{e}} : (\boldsymbol{d}^{s} - \boldsymbol{d}^{\mathrm{p}}) - \dot{\lambda}\boldsymbol{Z}\,. \tag{19}$$

After substituting the flow rule into the above relation and defining

$$\dot{\boldsymbol{\sigma}}_{\mathrm{trial}} := \mathbf{C}^{\mathrm{e}} : \boldsymbol{d}^{s} \quad \text{and} \quad \boldsymbol{P} := \mathbf{C}^{\mathrm{e}} : \boldsymbol{M} + \boldsymbol{Z}\,. \tag{20}$$

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we get

$$\dot{\lambda} = \frac{\widehat{\boldsymbol{N}} : \dot{\boldsymbol{\sigma}}_{\text{trial}}}{\widehat{\boldsymbol{N}} : \boldsymbol{P} + H} \,. \tag{21}$$

Therefore,

$$\dot{\boldsymbol{\sigma}} = \left[\mathbf{I}^{(s)} - \frac{\boldsymbol{P} \otimes \widehat{\boldsymbol{N}}}{\widehat{\boldsymbol{N}} : \boldsymbol{P} + H} \right] : \dot{\boldsymbol{\sigma}}_{\text{trial}} =: \left(\mathbf{C}^{\text{p}} : \mathbf{C}^{\text{e}} \right) : \boldsymbol{d}^{s} = \mathbf{C}^{\text{pe}} : \boldsymbol{d}^{s} \,. \tag{22}$$

Yield condition For continued plastic loading on the yield surface, the yield condition is

$$f(\boldsymbol{\sigma}, \overline{p_{\text{e}}^{w}}, \overline{p_{\text{p}}^{w}}, B^{\text{e}}, B^{\text{p}}, \phi^{\text{e}}, \phi^{\text{p}}, S_{w}^{\text{e}}, S_{w}^{\text{p}}, \overline{X}^{\text{e}}, \overline{X}^{\text{p}}; \boldsymbol{\varepsilon}^{\text{p}}) = 0 \tag{23}$$

where $f(\dots)$ is the yield function.

Consistency condition The consistency condition implies that for continued plastic loading

$$\dot{f}(\dots) = 0 \,. \tag{24}$$

Assuming appropriate smoothness of $f(\dots)$, we have

$$\begin{aligned} \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \overline{p_{\text{e}}^{w}}} \dot{\overline{p_{\text{e}}^{w}}} + \frac{\partial f}{\partial B^{\text{e}}} \dot{B}^{\text{e}} + \frac{\partial f}{\partial \phi^{\text{e}}} \dot{\phi}^{\text{e}} + \frac{\partial f}{\partial S_{w}^{\text{e}}} \dot{S}_{w}^{\text{e}} + \frac{\partial f}{\partial \overline{X}^{\text{e}}} \dot{\overline{X}}^{\text{e}} + \\ \dot{\lambda} \left[h_{p} \frac{\partial f}{\partial \overline{p_{\text{p}}^{w}}} + h_{B} \frac{\partial f}{\partial B^{\text{p}}} + h_{\phi} \frac{\partial f}{\partial \phi^{\text{p}}} + h_{S_{w}} \frac{\partial f}{\partial S_{w}^{\text{p}}} + h_{X} \frac{\partial f}{\partial \overline{X}^{\text{p}}} \right] = 0 \,. \end{aligned} \tag{25}$$

The normal to the yield surface is

$$\boldsymbol{N} = \frac{\partial f}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial \overline{p_{\text{e}}^{w}}} \frac{\partial \overline{p_{\text{e}}^{w}}}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial B^{\text{e}}} \frac{\partial B^{\text{e}}}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial \phi^{\text{e}}} \frac{\partial \phi^{\text{e}}}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial S_{w}^{\text{e}}} \frac{\partial S_{w}^{\text{e}}}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial \overline{X}^{\text{e}}} \frac{\partial \overline{X}^{\text{e}}}{\partial \boldsymbol{\sigma}} \tag{26}$$

and the ensemble hardening modulus is

$$H = -\left[h_{p} \frac{\partial f}{\partial \overline{p_{\text{p}}^{w}}} + h_{B} \frac{\partial f}{\partial B^{\text{p}}} + h_{\phi} \frac{\partial f}{\partial \phi^{\text{p}}} + h_{S_{w}} \frac{\partial f}{\partial S_{w}^{\text{p}}} + h_{X} \frac{\partial f}{\partial \overline{X}^{\text{p}}} \right] / \|\boldsymbol{N}\| \,. \tag{27}$$

Therefore equation (25) can be written in the consistency condition form

$$\widehat{\boldsymbol{N}} : \dot{\boldsymbol{\sigma}} = \dot{\lambda} H \quad \text{where} \quad \widehat{\boldsymbol{N}} = \boldsymbol{N} / \|\boldsymbol{N}\| \,. \tag{28}$$

Hardening functions If we assume that

$$\overline{p_{\text{p}}^{w}} \equiv \overline{p_{\text{p}}^{w}}(\boldsymbol{\varepsilon}^{\text{p}}) \,, \quad B^{\text{p}} \equiv B^{\text{p}}(\boldsymbol{\varepsilon}^{\text{p}}) \,, \quad \phi^{\text{p}} \equiv \phi^{\text{p}}(\boldsymbol{\varepsilon}^{\text{p}}) \,, \quad S_{w}^{\text{p}} \equiv S_{w}^{\text{p}}(\boldsymbol{\varepsilon}^{\text{p}}) \,, \quad \overline{X}^{\text{p}} \equiv \overline{X}^{\text{p}}(\boldsymbol{\varepsilon}^{\text{p}}) \tag{29}$$

we can write

$$\dot{\overline{p_{\text{p}}^{w}}} = \frac{d\overline{p_{\text{p}}^{w}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{d}^{p} \,, \quad \dot{B}^{\text{p}} = \frac{dB^{\text{p}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{d}^{p} \,, \quad \dot{\phi}^{\text{p}} = \frac{d\phi^{\text{p}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{d}^{p} \,, \quad \dot{S}_{w}^{\text{p}} = \frac{dS_{w}^{\text{p}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{d}^{p} \,, \quad \dot{\overline{X}}^{\text{p}} = \frac{d\overline{X}^{\text{p}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{d}^{p} \,. \tag{30}$$

Using the flow rule (17), we have

$$\dot{\overline{p_{\text{p}}^{w}}} = \dot{\lambda} \frac{d\overline{p_{\text{p}}^{w}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{M} \,, \quad \dot{B}^{\text{p}} = \dot{\lambda} \frac{dB^{\text{p}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{M} \,, \quad \dot{\phi}^{\text{p}} = \dot{\lambda} \frac{d\phi^{\text{p}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{M} \,, \quad \dot{S}_{w}^{\text{p}} = \dot{\lambda} \frac{dS_{w}^{\text{p}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{M} \,, \quad \dot{\overline{X}}^{\text{p}} = \dot{\lambda} \frac{d\overline{X}^{\text{p}}}{d\boldsymbol{\varepsilon}^{\text{p}}} : \boldsymbol{M} \,. \tag{31}$$

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Therefore, comparing the above with (16), we have

$$h_p = \frac{d\overline{p^w_{\rm p}}}{d\boldsymbol{\varepsilon}^{\rm p}} : \boldsymbol{M}\,, \quad h_B = \frac{dB^{\rm p}}{d\boldsymbol{\varepsilon}^{\rm p}} : \boldsymbol{M}\,, \quad h_\phi = \frac{d\phi^{\rm p}}{d\boldsymbol{\varepsilon}^{\rm p}} : \boldsymbol{M}\,, \quad h_{S_w} = \frac{dS^{\rm p}_w}{d\boldsymbol{\varepsilon}^{\rm p}} : \boldsymbol{M}\,, \quad h_X = \frac{d\overline{X}^{\rm p}}{d\boldsymbol{\varepsilon}^{\rm p}} : \boldsymbol{M}\,. \tag{32}$$

Noting that the derivative of the trace of a rank-2 tensor with respect to the tensor is equal to the rank-2 identity tensor, we have

$$h_p = \frac{d\overline{p^w_{\rm p}}}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,, \quad h_B = \frac{dB^{\rm p}}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,, \quad h_\phi = \frac{d\phi^{\rm p}}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,, \quad h_{S_w} = \frac{dS^{\rm p}_w}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,, \quad h_X = \frac{d\overline{X}^{\rm p}}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,. \tag{33}$$

Expressions that have this form are derived later in this report.

Yield function If the volumetric and deviatoric components of the total stress are

$$\bar{p} := -\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) \quad \text{and} \quad \boldsymbol{s} := \boldsymbol{\sigma} + \bar{p}\boldsymbol{I} \tag{34}$$

we can define

$$\bar{p}_{\rm eff} := -\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}_{\rm eff}) = \bar{p} - B\overline{p^w}\,, \quad \boldsymbol{s}_{\rm eff} := \boldsymbol{\sigma}_{\rm eff} + \bar{p}_{\rm eff}\boldsymbol{I} = \boldsymbol{\sigma} + \bar{p}\boldsymbol{I} = \mathbf{s} \quad \text{and} \quad J_2^{\rm eff} := \tfrac{1}{2}\boldsymbol{s}_{\rm eff} : \boldsymbol{s}_{\rm eff}\,. \tag{35}$$

Then the Arena yield function is

$$f(\boldsymbol{\sigma}, B, \overline{p^w}, \overline{X}) = \beta\sqrt{J_2^{\rm eff}} - F_f(\bar{p}_{\rm eff})\, F_c(\bar{p}_{\rm eff}, \overline{X}) \tag{36}$$

where

$$F_f(\bar{p}_{\rm eff}) = a_1 - a_3\exp[-3a_2\bar{p}_{\rm eff}] + 3a_4\bar{p}_{\rm eff} \tag{37}$$

and

$$F_c(\bar{p}_{\rm eff}, \overline{X}) = \begin{cases} 1 & \text{for} \quad 3\bar{p}_{\rm eff} \le \bar{\kappa} \\ \sqrt{1 - \left(\dfrac{3\bar{p}_{\rm eff} - \bar{\kappa}}{\overline{X}_{\rm eff} - \bar{\kappa}}\right)^2} & \text{for} \quad 3\bar{p}_{\rm eff} > \bar{\kappa}\,. \end{cases} \tag{38}$$

Here a_i are material parameters, $\overline{X}_{\rm eff}(\boldsymbol{\varepsilon}^{\rm p}, B, \overline{p^w}) = \overline{X} - 3B\overline{p^w}$ is the shifted form of the apparent hydrostatic compressive strength ($\overline{X}/3$) of the partially saturated material, and $\bar{\kappa}$ is the branch point at which the cap function F_c starts decreasing until it reaches the hydrostatic strength point ($\overline{X}$):

$$\bar{\kappa} = 3\bar{p}^{\rm peak}_{\rm eff} - (3\bar{p}^{\rm peak}_{\rm eff} - \overline{X}_{\rm eff})R_c \tag{39}$$

where $\bar{p}^{\rm peak}_{\rm eff}$ is the maximum hydrostatic tensile stress that the material can support and R_c is a cap ratio. Non-associativity is modeled using the parameter β that modifies $\sqrt{J_2^{\rm eff}}$.

Hydrostatic compressive strength The drained hydrostatic compressive strength ($\overline{X}_d/3$) is found from the empirical drained material crush curve using

$$\overline{X}_d(\overline{\varepsilon^p_v}, \phi_0) - p_0 = p_1\left[\frac{1-\exp(-p_3)}{1-\exp(-p_3+\overline{\varepsilon^p_v})} - 1\right]^{1/p_2}\,, \quad p_3 := -\ln(1-\phi_0) \tag{40}$$

where $\overline{\varepsilon^{\rm p}_v} = -\mathrm{tr}(\boldsymbol{\varepsilon}^{\rm p})$ is the volumetric plastic strain, $\overline{X}_d/3$ is the drained hydrostatic compressive strength, p_0, p_1, p_2, p_3 are model parameters, and ϕ_0 is the initial porosity. The derivative of $\overline{X}_d$ is

$$\frac{d\overline{X}_d}{d\overline{\varepsilon^p_v}} = \frac{p_1[1-\exp(-p_3)]\exp(-p_3+\overline{\varepsilon^p_v})}{p_2[1-\exp(-p_3+\overline{\varepsilon^p_v})]^2}\left[\frac{1-\exp(-p_3)}{1-\exp(-p_3+\overline{\varepsilon^p_v})} - 1\right]^{1/p_2-1}\,. \tag{41}$$

The effective hydrostatic compressive strength for a partially saturated material is expected to be different from the drained value (we do not have any direct experimental data which supports that conjecture). We follow the approach used by Grujicic et al., (2009) and assume a model of the form

$$\overline{X}_{\text{eff}}(\overline{\varepsilon_v^p}, \phi_0, S_w) - p_0 = \left[(1 - S_w) + p_1^{\text{sat}} S_w\right](\overline{X}_d - p_0)\,; \quad \overline{X} = \overline{X}_{\text{eff}} + 3B\overline{p^w} \tag{42}$$

where $p_1 \times p_1^{\text{sat}}$ is the value of p_1 in a fully saturated material. Note that the Jan 2016 monthly report has results that suggest that $p_1^{\text{sat}} < 1$.

Porosity, saturation, and volumetric strain Recall that the saturation is defined as

$$S_w = \frac{v^w}{v^a + v^w} = 1 - S_a \quad \Longrightarrow \quad \frac{v^a}{v^w} = \frac{1 - S_w}{S_w} \tag{43}$$

where v^α is the volume occupied by phase α in the pore volume. Also recall that the porosity is defined as

$$\phi = \frac{v^a + v^w}{v^s + v^a + v^w} \quad \Longrightarrow \quad 1 - \phi = \frac{v^s}{v^s + v^a + v^w}\,. \tag{44}$$

The volumetric strain in each phase is defined as

$$\varepsilon_v^\alpha = \ln\left(\frac{v^\alpha}{v_0^\alpha}\right) \quad \text{where} \quad \alpha = \{s, w, a\} \tag{45}$$

where v_0^α is the initial volume of phase α. The total volumetric strain of the mixture is

$$\exp(\varepsilon_v) = (1 - S_0)\phi_0 \exp(\varepsilon_v^a) + S_0\phi_0 \exp(\varepsilon_v^w) + (1 - \phi_0)\exp(\varepsilon_v^s) \tag{46}$$

where S_0 is the initial saturation and ϕ_0 is the initial porosity. With the assumption that that the total volumetric strain can be additively decomposed into elastic and plastic parts, we have

$$\varepsilon_v = \varepsilon_v^{\text{e}} + \varepsilon_v^{\text{p}}\,. \tag{47}$$

Therefore,

$$\exp(\varepsilon_v^{\text{p}}) = (1 - S_0)\phi_0 \frac{\exp(\varepsilon_v^a)}{\exp(\varepsilon_v^{\text{e}})} + S_0\phi_0 \frac{\exp(\varepsilon_v^w)}{\exp(\varepsilon_v^{\text{e}})} + (1 - \phi_0)\frac{\exp(\varepsilon_v^s)}{\exp(\varepsilon_v^{\text{e}})}\,. \tag{48}$$

Saturation equation The saturation equation when $\overline{p^w} = \overline{p^a}$ is

$$S_w(\overline{p^w}) = \frac{\mathcal{C}}{1 - S_0 + \mathcal{C}}\,, \quad \mathcal{C} := S_0 \exp\left[\overline{\varepsilon_v^a}(\overline{p^w}) - \overline{\varepsilon_v^w}(\overline{p^w})\right]. \tag{49}$$

For a soil that has undergone compressive loading leading to plastic volumetric deformation, the continuity of the phases implies that, after unloading, the saturation is different from the initial value of saturation even when the hydraulic conductivity of the material is zero. Equation (49) then implies that the residual saturation is a function non-zero strains in the water and air phases. Because the air and the water are assumed to be perfectly elastic, the saturation is a function of a residual pressure which can be thought of as a "plastic" pore-pressure which is a function of the plastic volumetric strain.

One way of defining the "plastic" pore pressure can be that the unrecoverable part of the saturation is

$$S_w^{\text{p}}(\overline{p_{\text{p}}^w}) = \frac{\mathcal{C}^{\text{p}}}{1 - S_0 + \mathcal{C}^{\text{p}}}\,, \quad \mathcal{C}^{\text{p}} := S_0 \exp\left[\overline{\varepsilon_v^a}(\overline{p_{\text{p}}^w}) - \overline{\varepsilon_v^w}(\overline{p_{\text{p}}^w})\right]. \tag{50}$$

Porosity equation Recall that the porosity is given by

$$1-\phi = \frac{v^s}{v} \quad \Longrightarrow \quad \frac{1-\phi}{1-\phi_0} = \frac{v^s}{v_0^s}\frac{v_0}{v} = \exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^s}) = \frac{\exp(\varepsilon_v^s)}{\exp(\varepsilon_v)}\,. \tag{51}$$

If we know the total volumetric strain and the volumetric strains in the fluids, we can use equation (46) to write the above as

$$1-\phi = \frac{\exp(\varepsilon_v) - (1-S_0)\phi_0\exp(\varepsilon_v^a) - S_0\phi_0\exp(\varepsilon_v^w)}{\exp(\varepsilon_v)} \tag{52}$$

When the pore water and air pressure are equal to each other, and also equal to the intrinsic pressure in the solid grains), the porosity equation (52) can be expressed as

$$\phi(\overline{p^w}) = (1-S_0)\phi_0\exp\left[\overline{\varepsilon_v}(\overline{p^w}) - \overline{\varepsilon_v^a}(\overline{p^w})\right] + S_0\phi_0\exp\left[\overline{\varepsilon_v}(\overline{p^w}) - \overline{\varepsilon_v^w}(\overline{p^w})\right] \tag{53}$$

The primary cause of the residual pressure in a soil that has undergone volumetric plastic deformation is the irreversible change in porosity. Let the unrecoverable part of the porosity (called the "unloaded" porosity by Brannon, 2007) be ϕ^{p}. We could define the "unloaded" or "plastic" porosity as

$$\phi^{\mathrm{p}}(\overline{p_{\mathrm{p}}^w}) = (1-S_0)\phi_0\exp\left[\overline{\varepsilon_v}(\overline{p_{\mathrm{p}}^w}) - \overline{\varepsilon_v^a}(\overline{p_{\mathrm{p}}^w})\right] + S_0\phi_0\exp\left[\overline{\varepsilon_v}(\overline{p_{\mathrm{p}}^w}) - \overline{\varepsilon_v^w}(\overline{p_{\mathrm{p}}^w})\right] \tag{54}$$

Pore pressure evolution As discussed in the March-April 2016 report, an expression for the evolution of the pore pressure can be found from the mass balance equations:

$$\dot{\overline{p^w}} = \frac{1}{\mathcal{B}}\mathrm{tr}(\boldsymbol{d}^s) \tag{55}$$

where $\mathcal{B} \equiv \mathcal{B}(\overline{p^w}, B, \phi, S_w)$ is given in equation (5). If we express the above equation in terms of elastic and plastic components, we have

$$\dot{\overline{p_{\mathrm{e}}^w}} + \dot{\overline{p_{\mathrm{p}}^w}} = \frac{1}{\mathcal{B}}\left[\mathrm{tr}(\boldsymbol{d}^{\mathrm{e}}) + \mathrm{tr}(\boldsymbol{d}^{\mathrm{p}})\right]\,. \tag{56}$$

We define the "elastic" and "plastic" pore pressure evolution as

$$\dot{\overline{p_{\mathrm{e}}^w}} := \frac{1}{\mathcal{B}}\mathrm{tr}(\boldsymbol{d}^{\mathrm{e}}) \quad \text{and} \quad \dot{\overline{p_{\mathrm{p}}^w}} := \frac{1}{\mathcal{B}}\mathrm{tr}(\boldsymbol{d}^{\mathrm{p}}) = \frac{\dot\lambda}{\mathcal{B}}\mathrm{tr}(\boldsymbol{M})\,. \tag{57}$$

Biot ratio evolution If we assume that $\overline{p^s} = \overline{p^w} = \overline{p^a}$, the material time derivative of the Biot ratio (3) is

$$\dot B := -\frac{1}{K_s}\left(\dot{\tilde K}_s - \frac{K_d}{K_s}\dot K_s\right) = -\frac{1}{K_s}\left(\frac{dK_d}{d\overline{p^w}} - \frac{K_d}{K_s}\frac{dK_s}{d\overline{p^w}}\right)\dot{\overline{p^w}}\,. \tag{58}$$

We can define "elastic" and "plastic" parts of the rate of the Biot ratio as $\dot B = \dot B^{\mathrm{e}} + \dot B^{\mathrm{p}}$ where

$$\dot B^{\mathrm{e}} := \frac{1}{\mathcal{B}K_s}\left(\frac{K_d}{K_s}\frac{dK_d}{d\overline{p^w}} - \frac{dK_s}{d\overline{p^w}}\right)\mathrm{tr}(\boldsymbol{d}^{\mathrm{e}}) \quad \text{and} \quad \dot B^{\mathrm{p}} := \dot\lambda\left[\frac{1}{\mathcal{B}K_s}\left(\frac{K_d}{K_s}\frac{dK_d}{d\overline{p^w}} - \frac{dK_s}{d\overline{p^w}}\right)\right]\mathrm{tr}(\boldsymbol{M})\,. \tag{59}$$

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Saturation evolution From equation (49) we have

$$\dot{S}_w = \frac{(1-S_0)\dot{\mathcal{C}}}{(1-S_0+\mathcal{C})^2} = \frac{1-S_0}{(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}}\dot{\overline{p^w}} = \frac{1-S_0}{(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}}\left(\dot{\overline{p^w_e}} + \frac{\dot{\lambda}}{\mathcal{B}}\operatorname{tr}(\boldsymbol{M})\right). \tag{60}$$

We can express the above as $\dot{S}_w = \dot{S}^e_w + \dot{S}^p_w$, where the evolution equations for the "elastic" and "plastic" parts of the saturation are

$$\dot{S}^e_w := \left[\frac{1-S_0}{\mathcal{B}(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}}\right]\operatorname{tr}(\boldsymbol{d}^e) \quad \text{and} \quad \dot{S}^p_w := \dot{\lambda}\left[\frac{1-S_0}{\mathcal{B}(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}}\right]\operatorname{tr}(\boldsymbol{M}). \tag{61}$$

Porosity evolution If we take the material time derivative of equation (53), we get

$$\dot{\phi} = (1-S_0)\phi_0\exp\left(\overline{\varepsilon_v}-\overline{\varepsilon^a_v}\right)\left(\dot{\overline{\varepsilon_v}} - \frac{d\overline{\varepsilon^a_v}}{d\overline{p^w}}\dot{\overline{p^w}}\right) + S_0\phi_0\exp\left(\overline{\varepsilon_v}-\overline{\varepsilon^w_v}\right)\left(\dot{\overline{\varepsilon_v}} - \frac{d\overline{\varepsilon^w_v}}{d\overline{p^w}}\dot{\overline{p^w}}\right) \tag{62}$$

or

$$\dot{\phi} = -\left[(1-S_0)\phi_0\exp\left(\overline{\varepsilon_v}-\overline{\varepsilon^a_v}\right)\left(1+\frac{1}{K_a\mathcal{B}}\right) + S_0\phi_0\exp\left(\overline{\varepsilon_v}-\overline{\varepsilon^w_v}\right)\left(1+\frac{1}{K_w\mathcal{B}}\right)\right]\operatorname{tr}(\boldsymbol{d}^s). \tag{63}$$

We can express the above as $\dot{\phi} = \dot{\phi}^e + \dot{\phi}^p$, where

$$\begin{aligned}
\dot{\phi}^e &= -\left[(1-S_0)\phi_0\exp\left(\overline{\varepsilon_v}-\overline{\varepsilon^a_v}\right)\left(1+\frac{1}{K_a\mathcal{B}}\right) + S_0\phi_0\exp\left(\overline{\varepsilon_v}-\overline{\varepsilon^w_v}\right)\left(1+\frac{1}{K_w\mathcal{B}}\right)\right]\operatorname{tr}(\boldsymbol{d}^e)\\
\dot{\phi}^p &= -\dot{\lambda}\left[(1-S_0)\phi_0\exp\left(\overline{\varepsilon_v}-\overline{\varepsilon^a_v}\right)\left(1+\frac{1}{K_a\mathcal{B}}\right) + S_0\phi_0\exp\left(\overline{\varepsilon_v}-\overline{\varepsilon^w_v}\right)\left(1+\frac{1}{K_w\mathcal{B}}\right)\right]\operatorname{tr}(\boldsymbol{M}).
\end{aligned} \tag{64}$$

Hydrostatic compressive strength evolution The rate form of the hydrostatic compressive strength equation (42) is

$$\begin{aligned}
\dot{\overline{X}} &= \dot{\overline{X}}_{\text{eff}} + 3(\dot{B}\overline{p^w} + B\dot{\overline{p^w}})\\
&= \left[(1-S_w) + p_1^{\text{sat}}S_w\right]\dot{\overline{X}}_d + (p_1^{\text{sat}}-1)\dot{S}_w\overline{X}_d + 3(\dot{B}\overline{p^w} + B\dot{\overline{p^w}})\\
&= \left[(1-S_w) + p_1^{\text{sat}}S_w\right]\frac{d\overline{X}_d}{d\overline{\varepsilon^p_v}}\operatorname{tr}(\boldsymbol{d}^p) + (p_1^{\text{sat}}-1)\left(\dot{S}^e_w + \dot{S}^p_w\right)\overline{X}_d + 3[\dot{B}\overline{p^w} + B(\dot{\overline{p^w_p}} + \dot{\overline{p^w_e}})]\\
&= \left[(1-S_w) + p_1^{\text{sat}}S_w\right]\frac{d\overline{X}_d}{d\overline{\varepsilon^p_v}}[\dot{\lambda}\operatorname{tr}(\boldsymbol{M})] + \left[\frac{\overline{X}_d(p_1^{\text{sat}}-1)(1-S_0)}{\mathcal{B}(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}}\right]\left[\operatorname{tr}(\boldsymbol{d}^e) + \dot{\lambda}\operatorname{tr}(\boldsymbol{M})\right] +\\
&\quad 3\left[\frac{\overline{p^w}}{\mathcal{B}K_s}\left(\frac{K_d}{K_s}\frac{dK_d}{d\overline{p^w}} - \frac{dK_s}{d\overline{p^w}}\right)\left[\operatorname{tr}(\boldsymbol{d}^e) + \dot{\lambda}\operatorname{tr}(\boldsymbol{M})\right] + \frac{B}{\mathcal{B}}\operatorname{tr}(\boldsymbol{d}^e) + \frac{\dot{\lambda}B}{\mathcal{B}}\operatorname{tr}(\boldsymbol{M})\right]\\
&= \left[\frac{\overline{X}_d(p_1^{\text{sat}}-1)(1-S_0)}{\mathcal{B}(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}} + \frac{3\overline{p^w}}{\mathcal{B}K_s}\left(\frac{K_d}{K_s}\frac{dK_d}{d\overline{p^w}} - \frac{dK_s}{d\overline{p^w}}\right) + \frac{3B}{\mathcal{B}}\right]\operatorname{tr}(\boldsymbol{d}^e) +\\
&\quad \dot{\lambda}\left[\left[(1-S_w) + p_1^{\text{sat}}S_w\right]\frac{d\overline{X}_d}{d\overline{\varepsilon^p_v}} + \frac{\overline{X}_d(p_1^{\text{sat}}-1)(1-S_0)}{\mathcal{B}(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}} + \frac{3\overline{p^w}}{\mathcal{B}K_s}\left(\frac{K_d}{K_s}\frac{dK_d}{d\overline{p^w}} - \frac{dK_s}{d\overline{p^w}}\right) + \frac{3B}{\mathcal{B}}\right]\operatorname{tr}(\boldsymbol{M}).
\end{aligned} \tag{65}$$

As before, we could define "elastic" and "plastic" parts such that $\dot{\overline{X}} = \dot{\overline{X}}^e + \dot{\overline{X}}^p$, where

$$\begin{aligned}
\dot{\overline{X^{\mathrm{e}}}} &:= \left[\frac{\overline{X}_d(p_1^{\mathrm{sat}}-1)(1-S_0)}{\mathcal{B}(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}} + \frac{3\overline{p^w}}{\mathcal{B}K_s}\left(\frac{K_d}{K_s}\frac{dK_d}{d\overline{p^w}} - \frac{dK_s}{d\overline{p^w}}\right) + \frac{3B}{\mathcal{B}}\right]\mathrm{tr}(\boldsymbol{d}^{\mathrm{e}}) \\
\dot{\overline{X^{\mathrm{p}}}} &:= \dot{\lambda}\left[\left[(1-S_w) + p_1^{\mathrm{sat}}S_w\right]\frac{d\overline{X}_d}{d\overline{\varepsilon_v^{\mathrm{p}}}} + \right. \\
&\qquad \left.\frac{\overline{X}_d(p_1^{\mathrm{sat}}-1)(1-S_0)}{\mathcal{B}(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}} + \frac{3\overline{p^w}}{\mathcal{B}K_s}\left(\frac{K_d}{K_s}\frac{dK_d}{d\overline{p^w}} - \frac{dK_s}{d\overline{p^w}}\right) + \frac{3B}{\mathcal{B}}\right]\mathrm{tr}(\boldsymbol{M})\,.
\end{aligned} \tag{66}$$

Rate-dependent plasticity

Rate-dependent plastic effects are determined after the rate-independent solution has been found. An additive decomposition of the (unrotated) rate of deformation into elastic and viscoplastic parts is assumed, i.e.,

$$\boldsymbol{d}^s = \boldsymbol{d}^{\mathrm{e}} + \boldsymbol{d}^{\mathrm{vp}}\,. \tag{67}$$

The Perzyna formulation for the viscoplastic rate of deformation is used (Brannon, 2007):

$$\boldsymbol{d}^{\mathrm{vp}} = \left\langle\frac{f(\boldsymbol{\sigma},\boldsymbol{\eta})}{\tau}\right\rangle = \begin{cases}\dfrac{f(\boldsymbol{\sigma},\boldsymbol{\eta})}{\tau}\boldsymbol{M} & \text{if } f(\boldsymbol{\sigma},\boldsymbol{\eta})>0\\ 0 & \text{otherwise}\end{cases} \tag{68}$$

where $f(\dots)$ is the yield function and τ is a relaxation time. Note that the above relation can be expressed in Duvaut-Lion form as

$$\boldsymbol{d}^{\mathrm{vp}} = \begin{cases}(\mathbf{C}^{\mathrm{e}})^{-1} : \left(\dfrac{\boldsymbol{\sigma}-\mathcal{P}\boldsymbol{\sigma}}{\tau}\right) & \text{if } f(\boldsymbol{\sigma},\boldsymbol{\eta})>0\\ 0 & \text{otherwise}\end{cases} \tag{69}$$

where $\mathbf{C}^{\mathrm{e}}$ is the elastic stiffness tensor and $\mathcal{P}\boldsymbol{\sigma}$ is the closest point projection of the stress state on to the boundary of the region that bounds all possible elastic stress states. Clearly, $\mathcal{P}\boldsymbol{\sigma} = \boldsymbol{\sigma}_{qs}$, and for stress states outside the yield surface we have

$$\boldsymbol{d}^{\mathrm{vp}} = \frac{1}{\tau}(\mathbf{C}^{\mathrm{e}})^{-1} : \left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{qs}\right)\,. \tag{70}$$

Recall from equation (12) that

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^{\mathrm{e}} : \boldsymbol{d}^{\mathrm{e}} - \dot{\lambda}\boldsymbol{Z}\,. \tag{71}$$

If we ignore the $\boldsymbol{Z}$ term and use equations (20) and (70), we have

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^{\mathrm{e}} : (\boldsymbol{d}^s - \boldsymbol{d}^{\mathrm{vp}}) = \mathbf{C}^{\mathrm{e}} : \boldsymbol{d}^s - \frac{1}{\tau}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{qs}) = \dot{\boldsymbol{\sigma}}^{\mathrm{trial}} - \frac{1}{\tau}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{qs})\,. \tag{72}$$

We can subtract the rate of the quasistatic stress from both sides to get an equation for the viscoplastic overstress contribution ($\boldsymbol{\sigma}_{\mathrm{over}} := \boldsymbol{\sigma}-\boldsymbol{\sigma}_{qs}$):

$$\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_{qs} = \dot{\boldsymbol{\sigma}}^{\mathrm{trial}} - \dot{\boldsymbol{\sigma}}_{qs} - \frac{1}{\tau}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{qs}) \tag{73}$$

or

$$\dot{\boldsymbol{\sigma}}_{\mathrm{over}} = \left(\dot{\boldsymbol{\sigma}}^{\mathrm{trial}} - \dot{\boldsymbol{\sigma}}_{qs}\right) - \frac{1}{\tau}\boldsymbol{\sigma}_{\mathrm{over}}\,. \tag{74}$$

From equation (22) we have

$$\dot{\boldsymbol{\sigma}}_{qs} = \dot{\boldsymbol{\sigma}}_{\mathrm{trial}} - \frac{\boldsymbol{P}\otimes\widehat{\boldsymbol{N}}}{\widehat{\boldsymbol{N}}:\boldsymbol{P}+H} : \dot{\boldsymbol{\sigma}}_{\mathrm{trial}}\,. \tag{75}$$

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Therefore, (74) can be written as

$$\dot{\boldsymbol{\sigma}}_{\text{over}} = \frac{\boldsymbol{P} \otimes \widehat{\boldsymbol{N}}}{\widehat{\boldsymbol{N}} : \boldsymbol{P} + H} : \dot{\boldsymbol{\sigma}}_{\text{trial}} - \frac{1}{\tau}\,\boldsymbol{\sigma}_{\text{over}} =: \dot{\boldsymbol{\sigma}}_{\text{proj}} - \frac{1}{\tau}\,\boldsymbol{\sigma}_{\text{over}} \tag{76}$$

where

$$\dot{\boldsymbol{\sigma}}_{\text{proj}} := \dot{\boldsymbol{\sigma}}^{\text{trial}} - \dot{\boldsymbol{\sigma}}_{qs} = \frac{\boldsymbol{P} \otimes \widehat{\boldsymbol{N}}}{\widehat{\boldsymbol{N}} : \boldsymbol{P} + H} : \dot{\boldsymbol{\sigma}}_{\text{trial}}\,. \tag{77}$$

The above ordinary differential equation has the solution

$$\boldsymbol{\sigma}_{\text{over}}(t) = \frac{\int \exp\left[\int^t \frac{1}{\tau(t')}\,dt'\right] \dot{\boldsymbol{\sigma}}_{\text{proj}}(t)\,dt + C}{\exp\left[\int^t \frac{1}{\tau(t')}\,dt'\right]} \tag{78}$$

where C is a constant. If we assume that τ is constant over a timestep, we have

$$\boldsymbol{\sigma}_{\text{over}}(t) = \frac{\int \exp\left[\frac{t}{\tau}\right] \dot{\boldsymbol{\sigma}}_{\text{proj}}(t)\,dt + C}{\exp\left[\frac{t}{\tau}\right]} \tag{79}$$

In addition, if we assume that $\dot{\boldsymbol{\sigma}}_{\text{proj}}$ is also constant over a timestep, we have

$$\boldsymbol{\sigma}_{\text{over}}(t) = \frac{\tau \exp\left[\frac{t}{\tau}\right] \dot{\boldsymbol{\sigma}}_{\text{proj}} + C}{\exp\left[\frac{t}{\tau}\right]}. \tag{80}$$

If the initial overstress at $t = t_0$ is $\boldsymbol{\sigma}_{\text{over}}(t_0)$, we have

$$C = \exp\left[\frac{t_0}{\tau}\right] \boldsymbol{\sigma}_{\text{over}}(t_0) - \tau \exp\left[\frac{t_0}{\tau}\right] \dot{\boldsymbol{\sigma}}_{\text{proj}}\,. \tag{81}$$

Therefore

$$\boldsymbol{\sigma}_{\text{over}}(t) = \tau\left[1 - \exp\left(\frac{t_0 - t}{\tau}\right)\right] \dot{\boldsymbol{\sigma}}_{\text{proj}} + \exp\left(\frac{t_0 - t}{\tau}\right) \boldsymbol{\sigma}_{\text{over}}(t_0)\,. \tag{82}$$

The model used for the material relaxation time in Arensica is

$$\tau = T_1\, \dot{\varepsilon}^{-T_2} \quad \text{where} \quad \dot{\varepsilon} = \|\boldsymbol{d}^s\| \tag{83}$$

and T_1, T_2 are material parameters.

Arena Partially Saturated Soil Model: Computing the stress

The stress tensor computation procedure calls the computeStressTensor routine that is specific to each constitutive model.

Algorithm 1 Computing the stress tensor

Require: Δt, $\mathbf{x}_p^n$, m_p, V_p^{n+1}, $\mathbf{h}_p^n$, $\boldsymbol{l}_p^{n+1}$, $\boldsymbol{F}_p^{n+1}$, $\boldsymbol{\sigma}_p^n$, $\boldsymbol{\eta}_p^n$, ρ_0, `materialList`, `mpmFlags`, `constitutiveModel`

procedure computeStressTensor
  for `matl` **in** `materialList` **do**
    $\boldsymbol{\sigma}_p^{n+1}, \boldsymbol{\eta}_p^{n+1} \leftarrow$ `constitutiveModel[matl]`.computeStressTensor(Δt, $\mathbf{x}_p^n$, m_p, V_p^{n+1}, $\mathbf{h}_p^n$, $\boldsymbol{l}_p^{n+1}$, $\boldsymbol{F}_p^{n+1}$, $\boldsymbol{\sigma}_p^n$, $\boldsymbol{\eta}_p^n$, ρ_0, `mpmFlags`) ▷*Update the stress and any internal variables needed by the constitutive model*
  end for
  return $\boldsymbol{\sigma}_p^{n+1}$, $\boldsymbol{\eta}_p^{n+1}$
end procedure

Initialization of the model

The model is initialized in two steps. In the first step, the constitutive model object is created followed by initialization of the stress (and the deformation gradient if needed). Here

- ϕ_0 : The initial porosity.
- S_0 : The initial porosity.
- $n_{\max}$: The maximum number of subcycles in the plastic return algorithm.
- $\varepsilon_{v,p}^{e,n}$: The elastic volumetric strain at a particle at $t = t_n$.
- $\boldsymbol{\sigma}_p^n$: The dynamic Cauchy stress at a particle at $t = t_n$.
- $\boldsymbol{\sigma}_{qs,p}^n$: The quasistatic Cauchy stress at a particle at $t = t_n$.
- $\boldsymbol{\sigma}^0$: The initial Cauchy stress at a particle.

Algorithm 2 Creating the Arena constitutive model object

Require: `mpmFlags`, `xmlProblemSpec`

procedure createConstitutiveModel
  `elasticityModel` ← `ElasticModuliModelFactory`.create(`xmlProblemSpec`)
  `yieldCondition` ← `YieldConditionFactory`.create(`xmlProblemSpec`)
  $p_0, p_1, p_1^{\text{sat}}, p_2 \leftarrow$ readCrushCurveParameters(`xmlProblemSpec`)
  $\phi_0, S_0, \overline{p_0^w} \leftarrow$ readInitialPorosityAndSaturation(`xmlProblemSpec`)
  $n_{\max} \leftarrow$ readSubcyclingCharacteristicNumber(`xmlProblemSpec`)
  return `elasticityModel`, `yieldCondition`, ϕ_0, S_0, $\overline{p_0^w}$, $n_{\max}$, $p_0, p_1, p_1^{\text{sat}}, p_2$
end procedure

Algorithm 3 Initializing the Arena particle variables

Require: ϕ_0, S_0, $\overline{p_0^w}$, `particleList`, V_p^0, m_p^0, $\mathbf{v}_p^0$, `fluidParams`, `elasticityModel`, `yieldCondition`

procedure initialize
  `yieldCondition`.initialize(`particleList`, V_p^0)
  `yieldParams` ← `yieldCondition`.getParameters()
  for `part` **in** `particleList` **do**
    ϕ_p^0[`part`] ← ϕ_0
    $S_{w,p}^0$[`part`] ← S_0
    χ_p^0[`part`] ← 0
    $\varepsilon_{v,p}^{e,0}$[`part`] ← 0

$\boldsymbol{\sigma}_p^0$[part] ← (fluidParams.$\overline{p_0^w}$) $\boldsymbol{I}$
$\boldsymbol{\sigma}_{qs,p}^0$[part] ← $\boldsymbol{\sigma}^0$
end for
Δt ← computeStableTimestep(V_p^0, m_p, $\mathbf{v}_p^0$, elasticityModel)
return ϕ_p^0, $S_{w,p}^0$, χ_p^0, $\varepsilon_{v,p}^{e,0}$, $\boldsymbol{\sigma}_{qs,p}^0$, $\boldsymbol{\sigma}_p^0$, Δt
end procedure

Computing the stress and internal variables

The computeStressTensor routine in the partially saturated Arena model assumes that the Biot coefficient is $B = 1$, and has the following form. Here we introduce the new variables

- ϕ_p^n, ϕ_p^{n+1} : The porosity at $t = t_n$ and $t = t_{n+1}$.
- $S_{w,p}^n, S_{w,p}^{n+1}$: The saturation at $t = t_n$ and $t = t_{n+1}$.
- $a_{1,p}, a_{2,p}, a_{3,p}, a_{4,p}$: Yield condition parameters at each particle.
- $p_{3,p}^n$: The particle crush curve parameter p_3 at $t = t_n$.
- X_p^n : The particle hydrostatic compressive strength at $t = t_n$.
- κ_p^n : The yield function branch point at $t = t_n$.
- $\boldsymbol{\varepsilon}_p^{p,n}$: The particle plastic strain tensor at $t = t_n$.
- $\boldsymbol{\alpha}_p^n$: The particle backstress tensor at $t = t_n$.
- $\mathbf{d}^n$: The particle rate of deformation tensor at $t = t_n$.
- $\boldsymbol{R}^n, \boldsymbol{U}^n$: The particle rotation and stretch tensors at $t = t_n$.
- K^n, G^n : The particle bulk and shear modulus at $t = t_n$.

Algorithm 4 Computing the Arena stress tensor

Require: Δt, $\mathbf{x}_p^n$, m_p, V_p^{n+1}, $\mathbf{h}_p^n$, $\boldsymbol{l}_p^{n+1}$, $\boldsymbol{F}_p^n$, $\boldsymbol{F}_p^{n+1}$, X_p^n, κ_p^n, $\varepsilon_{v,p}^n$, $p_{3,p}^n$, $\boldsymbol{\varepsilon}_p^{p,n}$, $\boldsymbol{\alpha}_p^n$ ϕ_p^n, $S_{w,p}^n$, χ_p^n, $\varepsilon_{v,p}^{e,n}$, $\boldsymbol{\sigma}_{qs,p}^n$, $\boldsymbol{\sigma}_p^n$, ρ_0, particleList, mpmFlags, elasticityModel, yieldCondition

procedure computeStressTensor
  yieldParams ← yieldCondition.getParameters()
  $a_{1,p}, a_{2,p}, a_{3,p}, a_{4,p}, I_{1,p}^{peak}, R_{c,p}$ ← yieldCondition.getLocalVariables()
    ↪ ▷*Yield condition parameters vary at each particle.*
    ↪ *Get the per-particle values of these parameters.*
  for part **in** particleList **do**
    χ_p^{n+1}[part] ← χ_p^n[part] ▷*Copy over failure indicator variable*
    $\boldsymbol{d}^{n+1}$ ← $[\boldsymbol{l}_p^{n+1}$[part] $+(\boldsymbol{l}_p^{n+1}$[part]$)^T]/2$ ▷*Compute rate of deformation*
    $\boldsymbol{R}^n, \boldsymbol{U}^n$ ← polarDecomposition($\boldsymbol{F}_p^n$[part]) ▷*Compute rotation and stretch tensors*
    $\boldsymbol{d}_{\text{unrot}}^{n+1}$ ← $(\boldsymbol{R}^n)^T \cdot \boldsymbol{d}^{n+1} \cdot \boldsymbol{R}^n$ ▷*Unrotate the rate of deformation tensor*
    $\boldsymbol{\sigma}_{qs,\text{unrot}}^n$ ← $(\boldsymbol{R}^n)^T \cdot \boldsymbol{\sigma}_{qs,p}^n$[part]$\cdot \boldsymbol{R}^n$ ▷*Unrotate the quasistatic stress*
    $\boldsymbol{\sigma}_{\text{unrot}}^n$ ← $(\boldsymbol{R}^n)^T \cdot \boldsymbol{\sigma}_p^n$[part]$\cdot \boldsymbol{R}^n$ ▷*Unrotate the total stress*
    $K^n, G^n, \boldsymbol{s}^n, (\overline{p^w})^n, I_1^{\text{eff},n}, \sqrt{J_2^n}, r^n, z_{\text{eff}}^n, \varepsilon_v^{p,n}$ ←
      ↪ computeElasticProperties($\boldsymbol{\sigma}_{qs,\text{unrot}}^n$[part],
      ↪ ϕ_p^n[part], $S_{w,p}^n$[part], $\boldsymbol{\varepsilon}_p^{p,n}$[part], $\boldsymbol{\alpha}_p^n$[part], $p_{3,p}^n$[part])
      ↪ ▷*Compute elastic properties and stress invariants*
    isSuccess, $\boldsymbol{\sigma}_{qs,\text{unrot}}^{n+1}$, ϕ_{qs}^{n+1}, $S_{w,qs}^{n+1}$, X_p^{n+1}[part], $\boldsymbol{\alpha}_p^{n+1}$[part], $\boldsymbol{\varepsilon}_p^{p,n+1}$[part] ←
      ↪ rateIndependentPlasticUpdate(Δt, $\boldsymbol{d}_{\text{unrot}}^{n+1}$, K^n, G^n, $\boldsymbol{s}^n$, $(\overline{p^w})^n$, $I_1^{\text{eff},n}$, $\sqrt{J_2^n}$, r^n, z_{eff}^n,
      ↪ $\varepsilon_v^{p,n}$, $\boldsymbol{\sigma}_{qs,\text{unrot}}^n$, ϕ_p^n[part], $S_{w,p}^n$[part], X_p^n[part], $\boldsymbol{\alpha}_p^n$[part], $\boldsymbol{\varepsilon}_p^{p,n}$[part], $p_{3,p}^n$[part],
      ↪ $a_{1,p}$[part], $a_{2,p}$[part], $a_{3,p}$[part], $a_{4,p}$[part])
      ↪ ▷*Compute updated quasistatic state using the consistency bisection algorithm*
    if isSuccess = FALSE **then**

flagParticleForDeletion(part)
end if
$\boldsymbol{\sigma}^{n+1}_{\text{unrot}} \leftarrow$ rateDependentPlasticUpdate(Δt, $\boldsymbol{d}^{n+1}_{\text{unrot}}$, $\boldsymbol{\sigma}^{n}_{qs,\text{unrot}}$, $\boldsymbol{\sigma}^{n+1}_{qs,\text{unrot}}$, ϕ^n_p[part], ϕ^{n+1}_{qs}, $S^n_{w,p}$[part], $S^{n+1}_{w,qs}$, X^n_p[part], X^{n+1}_p[part], $\boldsymbol{\alpha}^n_p$[part], $\boldsymbol{\alpha}^{n+1}_p$[part], $\boldsymbol{\varepsilon}^{\text{p},n}_p$[part], $\boldsymbol{\varepsilon}^{\text{p},n+1}_p$[part], $p^n_{3,p}$[part], $a_{1,p}$[part], $a_{2,p}$[part], $a_{3,p}$[part], $a_{4,p}$[part])
$\boldsymbol{R}^{n+1}, \boldsymbol{U}^{n+1} \leftarrow$ polarDecomposition($\boldsymbol{F}^{n+1}_p$[part]) ▷*Compute rotation and stretch tensors*
$\boldsymbol{\sigma}^{n+1}_{p,qs}$[part] $\leftarrow \boldsymbol{R}^{n+1} \cdot \boldsymbol{\sigma}^{n+1}_{qs,\text{unrot}} \cdot (\boldsymbol{R}^{n+1})^T$ ▷*Rotate the quasistatic stress*
$\boldsymbol{\sigma}^{n+1}_{p}$[part] $\leftarrow \boldsymbol{R}^{n+1} \cdot \boldsymbol{\sigma}^{n+1}_{\text{unrot}} \cdot (\boldsymbol{R}^{n+1})^T$ ▷*Rotate the dynamic stress*
end for
$\Delta t^{n+1} \leftarrow$ computeStableTimestep(V^{n+1}_p, m_p, $\boldsymbol{\sigma}^{n+1}_p$, $\boldsymbol{\sigma}^{n}_p$, $\boldsymbol{l}^{n+1}_p$, elasticityModel)
return $\boldsymbol{\sigma}^{n+1}_p$, $\boldsymbol{\sigma}^{n+1}_{p,qs}$, ϕ^{n+1}_p, $S^{n+1}_{w,p}$, $\boldsymbol{\alpha}^{n+1}_p$, X^{n+1}_p, $\boldsymbol{\varepsilon}^{\text{p},n+1}_p$, Δt^{n+1}.
end procedure

Compute elastic properties

The pseudocode for the generic computeElasticProperties is listed below. The function has side effects beyond computing the elastic properties and should be used carefully. Note that the subscript p has been dropped for simplicity because all quantities are particle-based.

Algorithm 5 Computing the elastic properties

Require: $\boldsymbol{\sigma}$, ϕ, S_w, $\boldsymbol{\varepsilon}^{\text{p}}$, $\boldsymbol{\alpha}$, p_3, elasticityModel
procedure computeElasticProperties
$\boldsymbol{s}$, $\overline{p^w}$, I_1^{eff}, $\sqrt{J_2}$, r, $z_{\text{eff}} \leftarrow$ updateStressInvariants($\boldsymbol{\sigma}$, $\boldsymbol{\alpha}$) ▷*Compute the deviatoric stress and the invariants of the input stress tensor.*
$\varepsilon^{\text{p}}_v \leftarrow$ updateVolumetricPlasticStrain($\boldsymbol{\varepsilon}^{\text{p}}$) ▷*Compute the volumetric plastic strain from the input plastic strain tensor.*
K, $G \leftarrow$ elasticityModel.getCurrentElasticModuli(I_1^{eff}, $\overline{p^w}$, ε^{p}_v, ϕ, S_w) ▷*Compute the elastic moduli corresponding to the input state.*
if useDisaggregationAlgorithm = TRUE **then**
scale = MAX(exp[$-(p_3 + \varepsilon^{\text{p}}_v)$], 10^{-5})
$K \leftarrow K^*$scale , $G \leftarrow G^*$scale
end if
return K, G, $\boldsymbol{s}$, $\overline{p^w}$, I_1^{eff}, $\sqrt{J_2}$, r, z_{eff}, ε^{p}_v
end procedure

Algorithm 6 Updating the stress invariants

procedure updateStressInvariants($\boldsymbol{\sigma}$, $\boldsymbol{\alpha}$)
$I_1 \leftarrow \text{tr}(\boldsymbol{\sigma})$
$\boldsymbol{s} \leftarrow \boldsymbol{\sigma} - (I_1/3)\boldsymbol{I}$ ▷*Compute deviatoric stress*
$\overline{p^w} \leftarrow -\text{tr}(\boldsymbol{\alpha})/3$ ▷*Compute pore pressure*
$I_1^{\text{eff}} \leftarrow I_1 + 3\overline{p^w}$, $\sqrt{J_2} \leftarrow \sqrt{(1/2)\boldsymbol{s} : \boldsymbol{s}}$ ▷*Compute invariants of the effective stress*
$r \leftarrow \sqrt{2J_2}$, $z_{\text{eff}} \leftarrow I_1^{\text{eff}}/\sqrt{3}$ ▷*Compute Lode coordinates of the effective stress*
return $\boldsymbol{s}$, $\overline{p^w}$, I_1^{eff}, $\sqrt{J_2}$, r, z_{eff}
end procedure

Bulk modulus - Dry soil: Fits to experimental data on sands suggest that the pressure model for dry soils has the form

$$\frac{\bar{p}_{\text{eff}}}{K_s(\bar{p}_{\text{eff}})} = b_0\,\overline{\varepsilon^e_v} + \frac{b_1(\overline{\varepsilon^e_v})^{b_4}}{b_2(\overline{\varepsilon^e_v})^{b_4} + b_3} \tag{84}$$

where the material parameters are $b_0 > 0, b_1 > 0, b_2 > 0, b_3 > 0, b_4 > 1$, $\overline{\varepsilon_v^e}$ is the volumetric elastic strain in the matrix, K_s is the bulk modulus of the solid grain material, and

$$\bar{p}_{\text{eff}} := -\tfrac{1}{3}\text{tr}(\boldsymbol{\sigma}_{\text{eff}}) = -\tfrac{1}{3}\text{tr}(\boldsymbol{\sigma} - \boldsymbol{\alpha})\,. \tag{85}$$

The bulk modulus of the solid grains is assumed to be given by

$$K_s(\overline{p^s}) = K_{s0} + n_s\,(\overline{p^s} - \overline{p_0^s}) \tag{86}$$

where K_{s0} and n_s are material properties, and $\overline{p_0^s}$ is a reference pressure.

The tangent bulk modulus of the dry soil is defined as

$$K_d(\bar{p}_{\text{eff}}) := \frac{d\bar{p}_{\text{eff}}}{d\overline{\varepsilon_v^e}}\,. \tag{87}$$

Then, using (84),

$$K_d(\bar{p}_{\text{eff}}) = \frac{[K_s(\bar{p}_{\text{eff}})]^2}{[K_s(\bar{p}_{\text{eff}}) - n_s\bar{p}_{\text{eff}}]}\left[b_0 + \frac{b_1 b_3 b_4 (\overline{\varepsilon_v^e})^{b_4-1}}{\left[b_2(\overline{\varepsilon_v^e})^{b_4} + b_3\right]^2}\right]. \tag{88}$$

To express (88) in closed-form in terms of $\bar{p}$ we have to eliminate $\bar{\varepsilon}_v^e$. But a closed form expression for the volumetric elastic strain cannot be derived from the pressure model. So we find an approximate form of (84) by assuming $b_0 \to 0$. This approximation is valid at moderate to large strains. Then, from (84) with $b_0 = 0$, we have

$$\overline{\varepsilon_v^e} \approx \left[\frac{b_3\bar{p}_{\text{eff}}}{b_1 K_s(\bar{p}_{\text{eff}}) - b_2\bar{p}_{\text{eff}}}\right]^{1/b_4}. \tag{89}$$

Shear modulus - Dry soil: The shear modulus is typically assumed to be constant. However, a variable shear modulus may be needed to fit experimental data and to prevent negative values of Poisson's ratio in the simulations. In those situations a variable Poisson's ratio (ν) is defined as

$$\nu(\bar{p}_{\text{eff}}) = \nu_1 + \nu_2\,\exp\left[-\frac{K_d(\bar{p}_{\text{eff}})}{K_s(\bar{p}_{\text{eff}})}\right] \tag{90}$$

where ν_1 and ν_2 are material parameters. The shear modulus is computed using the Poisson's ratio and the dry bulk modulus:

$$G_d(\bar{p}_{\text{eff}}) = \frac{3K_d(\bar{p}_{\text{eff}})(1-2\nu)}{2(1+\nu)}\,. \tag{91}$$

Bulk modulus - Partially saturated soil: The tangent bulk modulus of the partially saturated soil is found using a variation on the Gassman model for fully saturated rocks (Dvorkin et al., 1999):

$$\widehat{K}(\bar{p}_{\text{eff}}, \overline{p^w}, \overline{\varepsilon_v^p}, \phi, S_w) = K_d(\bar{p}_{\text{eff}}) + \frac{\left(1 - \dfrac{K_d(\bar{p}_{\text{eff}})}{K_s(\bar{p}_{\text{eff}})}\right)^2}{\dfrac{1}{K_s(\bar{p}_{\text{eff}})}\left(1 - \dfrac{K_d(\bar{p}_{\text{eff}})}{K_s(\bar{p}_{\text{eff}})}\right) + \phi\left(\dfrac{1}{K_f(\overline{p^w})} - \dfrac{1}{K_s(\bar{p}_{\text{eff}})}\right)} \tag{92}$$

where $\widehat{K}$ is the effective bulk modulus of the partially saturated soil, K_d is the bulk modulus of the drained soil, K_f is the bulk modulus of the pore fluid, and K_s is the bulk modulus of the solid grains. At partial saturation, we compute the pore fluid bulk modulus using a harmonic mean (lower bound) on the air and water bulk moduli (K_a, K_w):

$$\frac{1}{K_f(\overline{p^w})} = \frac{S_w}{K_w(\overline{p^w})} + \frac{1 - S_w}{K_a(\overline{p^w})} \,. \tag{93}$$

Shear modulus - Partially saturated soil: The shear modulus of the saturated soil is assumed to be the same as that of the dry soil.

$$\widehat{G}(\bar{p}_{\text{eff}}, \overline{p^w}, \overline{\varepsilon_v^p}, \phi, S_w) = G_d(\bar{p}_{\text{eff}}) \,. \tag{94}$$

The elastic modulus computation procedures: The functions used to compute the moduli are listed in the pseudocode below.

Algorithm 7 Computing the current elastic moduli

Require: $I_1^{\text{eff}}, \overline{p^w}, \varepsilon_v^{\text{p}}, \phi, S_w$

procedure GETCURRENTELASTICMODULI
  $\overline{I_1}^{\text{eff}} \leftarrow -I_1^{\text{eff}}, \overline{\varepsilon_v^{\text{p}}} \leftarrow -\varepsilon_v^{\text{p}},$
  $K \leftarrow 0, G \leftarrow 0$
  if $S_w > 0$ **then**
    $K, G \leftarrow$ COMPUTEPARTIALSATURATEDMODULI$(\overline{I_1}^{\text{eff}}, \overline{p^w}, \overline{\varepsilon_v^{\text{p}}}, \phi, S_w)$
  else
    $K, G \leftarrow$ COMPUTEDRAINEDMODULI$(\overline{I_1}^{\text{eff}}, \overline{\varepsilon_v^{\text{p}}})$
  end if
  return K, G
end procedure

Algorithm 8 Computing the partially saturated elastic moduli

Require: $K_{s0}, n_s, \bar{p}_{s0}, K_{w0}, n_w, \bar{p}_{w0}, \gamma, \bar{p}_r$

procedure COMPUTEPARTIALSATURATEDMODULI$(\overline{I_1}^{\text{eff}}, \overline{p^w}, \overline{\varepsilon_v^{\text{p}}}, \phi, S_w)$
  if $\overline{I_1}^{\text{eff}} > 0$ **then**
    $\bar{p}^{\text{eff}} \leftarrow \overline{I_1}^{\text{eff}}/3$
    $K_s \leftarrow K_{s0} + n_s(\bar{p}^{\text{eff}} - \bar{p}_{s0})$
    $K_w \leftarrow K_{w0} + n_w(\overline{p^w} - \bar{p}_{w0})$
    $K_a \leftarrow \gamma(\overline{p^w} + \bar{p}_r)$
    $K_d, G \leftarrow$ COMPUTEDRAINEDMODULI$(\overline{I_1}^{\text{eff}}, \overline{\varepsilon_v^{\text{p}}})$
    $K_f \leftarrow 1/\left[S_w/K_w + (1 - S_w)/K_a\right]$ ▷*Bulk modulus of air + water mixture*
    numer $\leftarrow (1 - K_d/K_s)^2$
    denom $\leftarrow (1/K_s)\,(1 - K_d/K_s) + \phi\,(1/K_f - 1/K_s)$
    $K \leftarrow K_d +$ numer/denom ▷*Bulk modulus of partially saturated material (Biot-Gassman model)*
  else
    $K, G \leftarrow$ COMPUTEDRAINEDMODULI$(\overline{I_1}, \overline{\varepsilon_v^{\text{p}}})$
  end if
  return K, G
end procedure

Algorithm 9 Computing the drained elastic moduli

Require: K_{s0}, n_s, $\bar{p}_{s0}$, b_0, b_1, b_2, b_3, b_4, G_0, ν_1, ν_2

procedure COMPUTEDRAINEDMODULI($\bar{I}_1^{\text{eff}}$, $\bar{\varepsilon}_v^{\text{p}}$)
  if $\bar{I}_1^{\text{eff}} > 0$ **then**
    $\bar{p}^{\text{eff}} \leftarrow \bar{I}_1^{\text{eff}}/3$
    $K_s \leftarrow K_{s0} + n_s(\bar{p}^{\text{eff}} - \bar{p}_{s0})$
    $K_s^{\text{ratio}} \leftarrow K_s/(1 - n_s\bar{p}^{\text{eff}}/K_s)$
    $\varepsilon_v^{\text{e}} \leftarrow \text{POW}((b_3\bar{p}^{\text{eff}})/(b_1K_s - b_2\bar{p}^{\text{eff}}), (1/b_4))$;
    $y \leftarrow \text{POW}(\varepsilon_v^{\text{e}}, b_4)$
    $z \leftarrow b_2 y + b_3$
    $K \leftarrow K_s^{\text{ratio}}[b_0 + (1/\varepsilon_v^{\text{e}})b_1b_3b_4y/z^2]$; ▷ *Compute compressive bulk modulus*
    $\nu = \nu_1 + \nu_2 \exp(-K/K_s)$
    $G \leftarrow G_0$
    if $\nu > 0$ **then**
      $G \leftarrow 1.5K\,(1 - 2\nu)/(1 + \nu)$ ▷*Update the shear modulus (if ν_1, $\nu_2 > 0$)*
    end if
  else
    $K \leftarrow b_0K_{s0}$ ▷*Tensile bulk modulus = Bulk modulus at $p = 0$*
    $G \leftarrow G_0$ ▷*Tensile shear modulus*
  end if
  return K, G
end procedure

Rate-independent stress update

The partially saturated soil model uses Michael Homel's "consistency bisection" algorithm (Homel, Guilkey, and Brannon, 2015) to find the plastic strain direction and to update the internal state variables. A closest-point return algorithm in transformed stress space is used to project the trial stress state on to the yield surface. Because of the nonlinearities in the material models, it is easier to solve the problem by dividing the strain increment into substeps.

Recall from equation (20) and (14) that

$$\dot{\boldsymbol{\sigma}}_{\text{trial}} := \mathbf{C}^{\text{e}} : \boldsymbol{d}^{s} \tag{95}$$

where

$$\mathbf{C}^{\text{e}} = \frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}^{\text{e}}} + \frac{\partial\boldsymbol{\sigma}}{\partial\overline{p_{\text{e}}^{w}}} \otimes \frac{\partial\overline{p_{\text{e}}^{w}}}{\partial\boldsymbol{\varepsilon}^{\text{e}}} + \frac{\partial\boldsymbol{\sigma}}{\partial B^{\text{e}}} \otimes \frac{\partial B^{\text{e}}}{\partial\boldsymbol{\varepsilon}^{\text{e}}} + \frac{\partial\boldsymbol{\sigma}}{\partial\phi^{\text{e}}} \otimes \frac{\partial\phi^{\text{e}}}{\partial\boldsymbol{\varepsilon}^{\text{e}}} + \frac{\partial\boldsymbol{\sigma}}{\partial S_w^{\text{e}}} \otimes \frac{\partial S_w^{\text{e}}}{\partial\boldsymbol{\varepsilon}^{\text{e}}} + \frac{\partial\boldsymbol{\sigma}}{\partial\overline{X}^{\text{e}}} \otimes \frac{\partial\overline{X}^{\text{e}}}{\partial\boldsymbol{\varepsilon}^{\text{e}}}\,. \tag{96}$$

We assume that

$$\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}^{e}} = \mathbf{C}_{\sigma}^{\text{e}} = \left(\widehat{K} - \tfrac{2}{3}\widehat{G}\right)\boldsymbol{I}\otimes\boldsymbol{I} + 2\widehat{G}\,\mathbf{I}^{(s)} \tag{97}$$

where $\widehat{K}(\boldsymbol{\sigma}, \overline{p^{w}}, B, \phi, S_w)$ is the tangent bulk modulus of the mixture, $\widehat{G}(\boldsymbol{\sigma}, \overline{p^{w}}, B, \phi, S_w)$ is the tangent shear modulus of the mixture, $\boldsymbol{I}$ is rank-2 identity tensor, and $\mathbf{I}^{(s)}$ is the symmetric part of the rank-4 identity tensor. All other quantities in the expression for $\mathbf{C}^{\text{e}}$ are assumed to be negligible, i.e., $\mathbf{C}^{\text{e}} = \mathbf{C}_{\sigma}^{\text{e}}$.

The update algorithm uses the standard predictor-corrector approach of hypoelastic-plasticity where a trial predictor stress is computed first and then a corrector return algorithm is used to locate the

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position of the correct stress on the yield surface. This approach requires that the trial stress ($\boldsymbol{\sigma}_{\text{trial}}$) is computed using the relation

$$\boldsymbol{\sigma}_{\text{trial}} = \boldsymbol{\sigma}^n + \mathbf{C}^{\text{e}} : (\boldsymbol{d}^s \, \Delta t) \, . \tag{98}$$

After the trial stress is computed, the timestep is subdivided into substeps based on the characteristic dimension of the yield surface relative to the magnitude of the trial stress increment ($\boldsymbol{\sigma}_{\text{trial}} - \boldsymbol{\sigma}^n$). The substep size is then recomputed by comparing the elastic bulk modulus $\widehat{K}(\boldsymbol{\sigma}_{\text{trial}}, \dots)$ with that at $\boldsymbol{\sigma}^n$ to make sure that the nonlinear elastic solution is accurate.

Recall that the parameters of the yield function, not all of which are independent, are a_1, a_2, a_3, a_4, R_c, β, $I_1^{\text{peak}} = 3p_{\text{eff}}^{\text{peak}}$. The code uses the parameters $\alpha_1 :=$ `PEAKI1`, $\alpha_2 :=$ `FSLOPE`, $\alpha_3 :=$ `YSLOPE`, and $\alpha_4 :=$ `STREN` in some places. The relationship between these sets of parameters is:

$$\begin{aligned} I_1^{\text{peak}} &\leftarrow \alpha_1 = \texttt{PEAKI1} \\ a_1 &\leftarrow \alpha_4 = \texttt{STREN} \\ a_2 &\leftarrow \frac{\alpha_2 - \alpha_3}{\alpha_4 - \alpha_1\alpha_3} \\ a_3 &\leftarrow (\alpha_4 - \alpha_1\alpha_3)\exp(-a_2\alpha_1) \\ a_4 &\leftarrow \alpha_3 = \texttt{YSLOPE} \, . \end{aligned} \tag{99}$$

The pseudocode for the algorithm is given below. All quantities are particle-based and the subscript p has been dropped for convenience.

Algorithm 10 The rate-independent stress and internal variable update algorithm

Require: Δt, $\boldsymbol{d}^{n+1}$, K^n, G^n, $\boldsymbol{s}^n$, $(\overline{p^w})^n$, $I_1^{\text{eff},n}$, $\sqrt{J_2^n}$, r^n, z_{eff}^n, $\varepsilon_v^{\text{p},n}$, a_1, a_2, a_3, a_4, p_3^n, $\boldsymbol{\sigma}_{qs}^n$, ϕ^n, S_w^n, X^n, $\boldsymbol{\alpha}^n$, $\boldsymbol{\varepsilon}^{\text{p},n}$, $n_{\max}$, ϵ_{sub}, $\chi_{\max}$

procedure RATEINDEPENDENTPLASTICUPDATE
  $\boldsymbol{\sigma}_{\text{trial}} \leftarrow$ COMPUTETRIALSTRESS($\boldsymbol{\sigma}_{qs}^n$, K^n, G^n, $\boldsymbol{d}^{n+1}$, Δt) ▷*Compute trial stress*
  $\boldsymbol{\alpha}^{\text{trial}} \leftarrow \boldsymbol{\alpha}^n$, $p_3^{\text{trial}} \leftarrow p_3^n$, $\phi^{\text{trial}} \leftarrow \phi^n$, $S_w^{\text{trial}} \leftarrow S_w^n$, $X^{\text{trial}} \leftarrow X^n$, $\boldsymbol{\varepsilon}^{\text{p,trial}} \leftarrow \boldsymbol{\varepsilon}^{\text{p},n}$ ▷*Set all other trial quantities to the values at the beginning of the timestep*
  K^{trial}, G^{trial}, $\boldsymbol{s}^{\text{trial}}$, $(\overline{p^w})^{\text{trial}}$, $I_1^{\text{eff,trial}}$, $\sqrt{J_2^{\text{trial}}}$, r^{trial}, $z_{\text{eff}}^{\text{trial}}$, $\varepsilon_v^{\text{p,trial}} \leftarrow$ COMPUTEELASTICPROPERTIES($\boldsymbol{\sigma}_{\text{trial}}$, ϕ^{trial}, S_w^{trial}, $\boldsymbol{\varepsilon}^{\text{p,trial}}$, $\boldsymbol{\alpha}^{\text{trial}}$, p_3^{trial}) ▷*Update the trial values of the moduli and compute the invariants of the trial stress*
  $n_{\text{sub}} \leftarrow$ COMPUTESTEPDIVISIONS($n_{\max}$, ϵ_{sub}, K^n, K^{trial}, I_1^{peak}, a_1, X^n, $\boldsymbol{\sigma}_{qs}^n$, $\boldsymbol{\sigma}_{\text{trial}}$) ▷*Compute number of substeps used by the return algorithm*
  if $n_{\text{sub}} < 0$ **then**
    return isSuccess = FALSE
  end if
  $\delta t \leftarrow \dfrac{\Delta t}{n_{\text{sub}}}$ ▷*Substep timestep*
  $\chi \leftarrow 1$, $t_{\text{local}} \leftarrow 0$ ▷*Initialize substep multiplier and accumulated time increment*
  $\boldsymbol{\sigma}^k \leftarrow \boldsymbol{\sigma}_{qs}^n$, $\boldsymbol{\varepsilon}^{\text{p},k} \leftarrow \boldsymbol{\varepsilon}^{\text{p},n}$, $\phi^k \leftarrow \phi^n$, $S_w^k \leftarrow S_w^n$, $X^k \leftarrow X^n$, $\boldsymbol{\alpha}^k \leftarrow \boldsymbol{\alpha}^n$, $K^k \leftarrow K^n$, $G^k \leftarrow G^n$, $p_3^k \leftarrow p_3^n$, $\boldsymbol{s}^k \leftarrow \boldsymbol{s}^n$, $(\overline{p^w})^k \leftarrow (\overline{p^w})^n$, $I_1^{\text{eff},k} \leftarrow I_1^{\text{eff},n}$, $\sqrt{J_2^k} \leftarrow \sqrt{J_2^n}$, $r^k \leftarrow r^n$, $z_{\text{eff}}^k \leftarrow z_{\text{eff}}^n$, $\varepsilon_v^{\text{p},k} \leftarrow \varepsilon_v^{\text{p},n}$
  repeat
    isSuccess, $\boldsymbol{\sigma}^{k+1}$, $\boldsymbol{\varepsilon}^{\text{p},k+1}$, ϕ^{k+1}, S_w^{k+1}, X^{k+1}, $\boldsymbol{\alpha}^{k+1}$, K^{k+1}, G^{k+1}, $p_3^{k+1} \leftarrow$ COMPUTESUBSTEP($\boldsymbol{\sigma}^k$, $\boldsymbol{\varepsilon}^{\text{p},k}$, ϕ^k, S_w^k, X^k, $\boldsymbol{\alpha}^k$, K^k, G^k, $\boldsymbol{s}^k$, $(\overline{p^w})^k$, $I_1^{\text{eff},k}$, $\sqrt{J_2^k}$, r^k, z_{eff}^k, $\varepsilon_v^{\text{p},k}$, p_3^k, $\boldsymbol{d}^{n+1}$, δt) ▷*Compute updated stress and internal variables for the current substep*

if isSuccess = TRUE **then**
$t_{\text{local}} \leftarrow t_{\text{local}} + \delta t$
$\boldsymbol{\sigma}^k \leftarrow \boldsymbol{\sigma}^{k+1}$, $\boldsymbol{\varepsilon}^{\text{p},k} \leftarrow \boldsymbol{\varepsilon}^{\text{p},k+1}$, $\phi^k \leftarrow \phi^{k+1}$, $S_w^k \leftarrow S_w^{k+1}$, $X^k \leftarrow X^{k+1}$, $\boldsymbol{\alpha}^k \leftarrow \boldsymbol{\alpha}^{k+1}$
$K^k \leftarrow K^{k+1}$, $G^k \leftarrow G^{k+1}$, $p_3^k \leftarrow p_3^{k+1}$
else
$\delta t \leftarrow \delta t/2$ ▷*Halve the timestep*
$\chi \leftarrow 2\chi$ ▷*Keep a count of how many times the timestep has been halved.*
if $\chi > \chi_{\max}$ **then**
return isSuccess = FALSE, $\boldsymbol{\sigma}^k$, ϕ^k, S_w^k, X^k, $\boldsymbol{\alpha}^k$, $\boldsymbol{\varepsilon}^{\text{p},k}$, K^k, G^k, p_3^k
↪ ▷*Algorithm has failed to converge*
end if
end if
until $t_{\text{local}} \geq \Delta t$
$\boldsymbol{\sigma}_{qs}^{n+1} \leftarrow \boldsymbol{\sigma}^{k+1}$, $\boldsymbol{\alpha}^{n+1} \leftarrow \boldsymbol{\alpha}^{k+1}$, $\boldsymbol{\varepsilon}^{\text{p},n+1} \leftarrow \boldsymbol{\varepsilon}^{\text{p},k+1}$, $\phi^{n+1} \leftarrow \phi^{k+1}$, $S_w^{n+1} \leftarrow S_w^{k+1}$, $X^{n+1} \leftarrow X^{k+1}$
$K^{n+1} \leftarrow K^{k+1}$, $G^{n+1} \leftarrow G^{k+1}$, $p_3^{n+1} \leftarrow p_3^{k+1}$
return isSuccess = TRUE, $\boldsymbol{\sigma}_{qs}^{n+1}$, ϕ^{n+1}, S_w^{n+1}, X^{n+1}, $\boldsymbol{\alpha}^{n+1}$, $\boldsymbol{\varepsilon}^{\text{p},n+1}$, K^{n+1}, G^{n+1}, p_3^{n+1}
↪ ▷*Algorithm has converged*
end procedure

Computing the trial stress

The trial stress is computed on the basis of the total stress rather than the effective stress. Recall that the trial stress ($\boldsymbol{\sigma}_{\text{trial}}$) is computed using the relation

$$\boldsymbol{\sigma}_{\text{trial}} = \boldsymbol{\sigma}^n + \mathbf{C}^{\text{e}} : (\boldsymbol{d}^s\, \Delta t) =: \boldsymbol{\sigma}^n + \mathbf{C}^{\text{e}} : \Delta\boldsymbol{\varepsilon} \tag{100}$$

where

$$\mathbf{C}^{\text{e}} = \left(\widehat{K} - \tfrac{2}{3}\widehat{G}\right) \boldsymbol{I} \otimes \boldsymbol{I} + 2\widehat{G}\, \mathbf{I}^{(s)}\,. \tag{101}$$

Therefore

$$\boldsymbol{\sigma}_{\text{trial}} = \boldsymbol{\sigma}^n + K\text{tr}(\Delta\boldsymbol{\varepsilon})\boldsymbol{I} + 2G[\Delta\boldsymbol{\varepsilon} - \tfrac{1}{3}\text{tr}(\Delta\boldsymbol{\varepsilon})\boldsymbol{I}]\,. \tag{102}$$

The pseudocode of the trial stress algorithm is given below.

Algorithm 11 Computing the trial stress

procedure COMPUTETRIALSTRESS($\boldsymbol{\sigma}_{qs}^n$, K^n, G^n, $\boldsymbol{d}^{n+1}$, Δt)
$\Delta\boldsymbol{\varepsilon} \leftarrow \boldsymbol{d}^{n+1}\, \Delta t$ ▷*Total strain increment*
$\Delta\boldsymbol{\varepsilon}^{\text{iso}} \leftarrow \frac{1}{3}\text{tr}(\Delta\boldsymbol{\varepsilon})\boldsymbol{I}$
$\Delta\boldsymbol{\varepsilon}^{\text{dev}} \leftarrow \Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{\text{iso}}$
$\boldsymbol{\sigma}_{\text{trial}} \leftarrow \boldsymbol{\sigma}_{qs}^n + 3K^n\Delta\boldsymbol{\varepsilon}^{\text{iso}} + 2G^n\Delta\boldsymbol{\varepsilon}^{\text{dev}}$
return $\boldsymbol{\sigma}_{\text{trial}}$
end procedure

Computing the number of subcycles in the return algorithm

To allow for nonlinear parameter variations, the algorithm breaks a trial loading step into subcycles. The algorithm below, determines the number of substeps based on the magnitude of the trial stress increment relative to the characteristic dimensions of the yield surface. Another comparison uses the value of the pressure dependent elastic properties at $\boldsymbol{\sigma}_{qs}^n$ and $\boldsymbol{\sigma}_{\text{trial}}$ and adjusts the number of substeps if there is a large change in elastic moduli. This ensures an accurate solution for nonlinear elasticity even with fully elastic loading.

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The number of substeps based on the bulk modulus is given by

$$n_{\text{bulk}} = \left\lceil \frac{\| K^n - K^{\text{trial}} \|}{K^n} \right\rceil . \tag{103}$$

The number of substeps based on the size of the yield surface is computed as

$$n_{\text{yield}} = \left\lceil \frac{\epsilon \left\| \boldsymbol{\sigma}_{\text{trial}} - \boldsymbol{\sigma}_{qs}^n \right\|}{\ell_{\text{yield}}} \right\rceil \tag{104}$$

where ϵ is a constant, and the characteristic length of the yield surface is

$$\ell_{\text{yield}} = \text{MIN}\left(a_1, \frac{I_1^{\text{peak}} - X^n}{2} \right) . \tag{105}$$

The number of substeps used in the actual calculation is

$$n_{\text{sub}} = \text{MAX}\left(n_{\text{bulk}}, n_{\text{yield}} \right) . \tag{106}$$

Algorithm 12 Computing the number of subcycles

Require: $n_{\max}, \epsilon_{\text{sub}} \leftarrow 10^{-4}, K^n, K^{\text{trial}}, I_1^{\text{peak}}, a_1, X^n, \boldsymbol{\sigma}_{qs}^n, \boldsymbol{\sigma}_{\text{trial}}$

procedure COMPUTESTEPDIVISIONS
  $n_{\text{bulk}} \leftarrow \lceil \left| K^n - K^{\text{trial}} \right| / K^n \rceil$ ▷*Compute change in bulk modulus*
  $\Delta\boldsymbol{\sigma} \leftarrow \boldsymbol{\sigma}_{\text{trial}} - \boldsymbol{\sigma}_{qs}^n$
  $L \leftarrow \frac{1}{2}(I_1^{\text{peak}} - X^n)$
  if $a_1 > 0$ **then**
    $L \leftarrow \text{MIN}(L, a_1)$
  end if
  $n_{\text{yield}} \leftarrow \lceil \epsilon_{\text{sub}} \times \|\Delta\boldsymbol{\sigma}\| / L \rceil$ ▷*Compute trial stress increment relative to yield surface size*
  $n_{\text{sub}} \leftarrow \text{MAX}(n_{\text{bulk}}, n_{\text{yield}})$ ▷*n_{sub} is the maximum of the two values*
  if $n_{\text{sub}} > n_{\max}$ **then**
    $n_{\text{sub}} \leftarrow -1$
  end if
  return n_{sub}
end procedure

Updating the stress for a substep: consistency bisection

This procedure computes the updated stress state for a substep that may be either elastic, plastic, or partially elastic. It uses Homel's consistency bisection and non-hardening return concepts (Homel, Guilkey, and Brannon, 2015).

Algorithm 13 Computing the stress and internal variable update for a substep

Require: $\boldsymbol{d}^{n+1}, \delta t, \boldsymbol{\sigma}^k, \boldsymbol{\varepsilon}^{\text{p},k}, \phi^k, S_w^k, X^k, \boldsymbol{\alpha}^k, K^k, G^k, \boldsymbol{s}^k, (\overline{p^w})^k, I_1^{\text{eff},k}, \sqrt{J_2^k}, r^k, z_{\text{eff}}^k, \varepsilon_v^{\text{p},k}, p_3^k, a_1, a_2, a_3, a_4, I_1^{\text{peak}}, R_c, \beta,$ `yieldCondition`

procedure COMPUTESUBSTEP
  $\delta\boldsymbol{\varepsilon} \leftarrow \boldsymbol{d}^{n+1}\delta t$ ▷*Compute strain increment*
  $\boldsymbol{\sigma}_{\text{trial}} \leftarrow \text{COMPUTETRIALSTRESS}(\boldsymbol{\sigma}^k, K^k, G^k, \boldsymbol{d}^{n+1}, \Delta t)$ ▷*Compute substep trial stress*
  $\boldsymbol{\alpha}^{\text{trial}} \leftarrow \boldsymbol{\alpha}^k, K^{\text{trial}} \leftarrow K^k, G^{\text{trial}} \leftarrow G^k, p_3^{\text{trial}} \leftarrow p_3^k, \phi^{\text{trial}} \leftarrow \phi^k, S_w^{\text{trial}} \leftarrow S_w^k,$
  ↪ $X^{\text{trial}} \leftarrow X^k, \boldsymbol{\varepsilon}^{\text{p,trial}} \leftarrow \boldsymbol{\varepsilon}^{\text{p},k}$ ▷*Set all other trial quantities to the values*
  ↪ *at the beginning of the substep*

5: $K^{\text{trial}}, G^{\text{trial}}, \boldsymbol{s}^{\text{trial}}, (\overline{p^w})^{\text{trial}}, I_1^{\text{eff,trial}}, \sqrt{J_2^{\text{trial}}}, r^{\text{trial}}, z_{\text{eff}}^{\text{trial}}, \varepsilon_v^{\text{p,trial}} \leftarrow$
↪ computeElasticProperties($\boldsymbol{\sigma}_{\text{trial}}, \phi^{\text{trial}}, S_w^{\text{trial}}, \boldsymbol{\varepsilon}^{\text{p,trial}}, \boldsymbol{\alpha}^{\text{trial}}, p_3^{\text{trial}}$)
↪ ▷*Compute elastic moduli and stress invariants for the trial state*

6: isElastic ← `yieldCondition`.evalYieldCondition($I_1^{\text{eff,trial}}, \sqrt{J_2^{\text{trial}}}, X^{\text{trial}}, (\overline{p^w})^{\text{trial}}, \phi^{\text{trial}}, S_w^{\text{trial}},$
↪ $a_1, a_2, a_3, a_4, I_1^{\text{peak}}, R_c, \beta$)
↪ ▷*Determine whether the trial stress is elastic or not*

7: **if** isElastic = TRUE **then**

8: $\boldsymbol{\sigma}^{k+1} \leftarrow \boldsymbol{\sigma}_{\text{trial}}, \boldsymbol{\varepsilon}^{\text{p},k+1} \leftarrow \boldsymbol{\varepsilon}^{\text{p,trial}}, \phi^{k+1} \leftarrow \phi^{\text{trial}}, S_w^{k+1} \leftarrow S_w^{\text{trial}}, X^{k+1} \leftarrow X^{\text{trial}}, \boldsymbol{\alpha}^{k+1} \leftarrow \boldsymbol{\alpha}^{\text{trial}}$

9: $K^{k+1} \leftarrow K^{\text{trial}}, G^{k+1} \leftarrow G^{\text{trial}}, p_3^{k+1} \leftarrow p_3^{\text{trial}}$
↪ ▷*This is an elastic substep. Update the state to the trial value.*

10: isSuccess = TRUE

11: **return** isSuccess, $\boldsymbol{\sigma}^{k+1}, \boldsymbol{\varepsilon}^{\text{p},k+1}\ \phi^{k+1}, S_w^{k+1}, X^{k+1}, \boldsymbol{\alpha}^{k+1}, K^{k+1}, G^{k+1}, p_3^{k+1}$

12: **end if**

13: $\boldsymbol{\sigma}_{\text{fixed}}, \delta\boldsymbol{\varepsilon}_{\text{fixed}}^{\text{p}} \leftarrow$ nonHardeningReturn($\boldsymbol{\sigma}^k, \delta\boldsymbol{\varepsilon}, X^k, K^k, G^k, (\overline{p^w})^k,$
↪ $\boldsymbol{s}^{\text{trial}}, \sqrt{J_2^{\text{trial}}}, r^{\text{trial}}, z_{\text{eff}}^{\text{trial}}, a_1, a_2, a_3, a_4, I_1^{\text{peak}}, R_c, \beta$)
↪ ▷*Compute return to updated yield surface (no hardening)*

14: isSuccess, $\boldsymbol{\sigma}^{k+1}, \boldsymbol{\varepsilon}^{\text{p},k+1}, \boldsymbol{\alpha}^{k+1}, (\overline{p^w})^{k+1}, \phi^{k+1}, S_w^{k+1}, X^{k+1}, K^{k+1}, G^{k+1}, \boldsymbol{s}^{k+1}, (\overline{p^w})^{k+1}, I_1^{\text{eff},k+1}, \sqrt{J_2^{k+1}},$
↪ $r^{k+1}, z_{\text{eff}}^{k+1}, \varepsilon_v^{\text{p},k+1} \leftarrow$ consistencyBisection($\delta\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\text{p},k}, \boldsymbol{\sigma}^k, K^k, G^k, (\overline{p^w})^k, \phi^k, S_w^k, X^k,$
↪ $\boldsymbol{s}^{\text{trial}}, I_1^{\text{eff,trial}}, \sqrt{J_2^{\text{trial}}}, r^{\text{trial}}, z_{\text{eff}}^{\text{trial}}, \varepsilon_v^{\text{p,trial}}, p_3^{\text{trial}}, a_1, a_2, a_3, a_4, I_1^{\text{peak}}, R_c, \beta, i_{\max}, j_{\max},$
↪ $\boldsymbol{\sigma}_{\text{fixed}}, \delta\boldsymbol{\varepsilon}_{\text{fixed}}^{\text{p}}$) ▷*The bisection return algorithm to take care of yield surface hardening.*

15: **if** iSuccess = FALSE **then**

16: **return** isSuccess, $\boldsymbol{\sigma}^k, \boldsymbol{\varepsilon}^{\text{p},k}, \phi^k, S_w^k, X^k, \boldsymbol{\alpha}^k, K^k, G^k, p_3^k$

17: **end if**

18: **return** isSuccess, $\boldsymbol{\sigma}^{k+1}, \boldsymbol{\varepsilon}^{\text{p},k+1}, \phi^{k+1}, S_w^{k+1}, X^{k+1}, \boldsymbol{\alpha}^{k+1}, K^{k+1}, G^{k+1}, p_3^{k+1}$

19: **end procedure**

The nonhardening return algorithm

Let the plastic flow direction be $\boldsymbol{M}$. Then

$$\dot{\boldsymbol{\varepsilon}^p} = \dot{\lambda}\boldsymbol{M}\,. \tag{107}$$

The nonhardening return algorithm uses a transformed space (see Homel, Guilkey, and Brannon, (2015)) where the computation is carried out in special Lode coordinates (z_{eff}, r') where

$$z_{\text{eff}} := \frac{\text{tr}(\boldsymbol{\sigma} - \boldsymbol{\alpha})}{\sqrt{3}} \quad \text{and} \quad r' = \beta r\sqrt{\frac{3K}{2G}}\,, \quad r := \sqrt{2J_2}\,. \tag{108}$$

If the flow rule is non-associative, the yield surface parameter $\beta \neq 1$.

The closest point from the trial stress state to the yield surface in this transformed space, $(z_{\text{eff}}^{\text{close}}, r'_{\text{close}})$, is computed using a geometric algorithm discussed later. After the closest point has been found, the coordinates of the closest point are transformed back to the (z, r) coordinates and the stress state is reconstructed using

$$\boldsymbol{\sigma}^{\text{fixed}} = z^{\text{close}}\frac{\boldsymbol{I}}{\|\boldsymbol{I}\|} + r^{\text{close}}\frac{\boldsymbol{s}^{\text{trial}}}{\|\boldsymbol{s}^{\text{trial}}\|}\,. \tag{109}$$

Let the stress at the beginning of the load step be $\boldsymbol{\sigma}^k$ and let the trial stress be $\boldsymbol{\sigma}_{\text{trial}}$. Assume the yield surface is fixed and let the correct projection of the trial stress on to the fixed yield surface be $\boldsymbol{\sigma}_{\text{fixed}}^{k+1}$.

The increment of stress for the load step ($\Delta\boldsymbol{\sigma}_{\text{fixed}}$) is related to the elastic strain increment ($\Delta\boldsymbol{\varepsilon}^{\text{e}}_{\text{fixed}}$) by

$$\Delta\boldsymbol{\sigma}_{\text{fixed}} = \boldsymbol{\sigma}^{k+1}_{\text{fixed}} - \boldsymbol{\sigma}^{k} = \mathbf{C}^{\text{e}}_{\text{fixed}} : \Delta\boldsymbol{\varepsilon}^{\text{e}}_{\text{fixed}} \tag{110}$$

where $\mathbf{C}^{\text{e}}_{\text{fixed}}$ is a constant elastic modulus tensor. The elastic modulus tensor can be assumed to be an average value of the nonlinear tangent modulus for the load step. If we know $\mathbf{C}^{\text{e}}_{\text{fixed}}$, we can compute the elastic strain increment using

$$\Delta\boldsymbol{\varepsilon}^{\text{e}}_{\text{fixed}} = (\mathbf{C}^{\text{e}}_{\text{fixed}})^{-1} : \Delta\boldsymbol{\sigma}_{\text{fixed}} \,. \tag{111}$$

For the situation where $\mathbf{C}^{\text{e}}$ is given by equation (101), we have

$$(\mathbf{C}^{\text{e}})^{-1} = \left(\frac{1}{9K^k} - \frac{1}{6G^k}\right)\boldsymbol{I}\otimes\boldsymbol{I} + \frac{1}{2G^k}\mathbf{I}^{(s)} \,. \tag{112}$$

Therefore,

$$\begin{aligned}\Delta\boldsymbol{\varepsilon}^{\text{e}}_{\text{fixed}} &= \left(\frac{1}{9K^k} - \frac{1}{6G^k}\right)\text{tr}(\Delta\boldsymbol{\sigma}_{\text{fixed}})\boldsymbol{I} + \frac{1}{2G^k}\Delta\boldsymbol{\sigma}_{\text{fixed}} \\ &= \frac{1}{3K^k}\left[\frac{1}{3}\text{tr}(\Delta\boldsymbol{\sigma}_{\text{fixed}})\boldsymbol{I}\right] + \frac{1}{2G^k}\left[\Delta\boldsymbol{\sigma}_{\text{fixed}} - \frac{1}{3}\text{tr}(\Delta\boldsymbol{\sigma}_{\text{fixed}})\boldsymbol{I}\right]\end{aligned} \tag{113}$$

or

$$\Delta\boldsymbol{\varepsilon}^{\text{e}}_{\text{fixed}} = \frac{1}{3K^k}\Delta\boldsymbol{\sigma}^{\text{iso}}_{\text{fixed}} + \frac{1}{2G^k}\Delta\boldsymbol{\sigma}^{\text{dev}}_{\text{fixed}} \,. \tag{114}$$

For a strain driven update algorithm, the total strain increment $\Delta\boldsymbol{\varepsilon}$ is known. Assuming that the total strain increment can be additively decomposed into an elastic and a plastic part, we can find the plastic strain increment ($\Delta\boldsymbol{\varepsilon}^{\text{p}}_{\text{fixed}}$) using

$$\Delta\boldsymbol{\varepsilon}^{\text{p}}_{\text{fixed}} = \Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{\text{e}}_{\text{fixed}} \,. \tag{115}$$

The nonhardening return algorithm pseudocode is listed below:

Algorithm 14 Non-hardening return algorithm

Require: $\boldsymbol{\sigma}^k, \delta\boldsymbol{\varepsilon}, X^k, K^k, G^k, (\overline{p^w})^k, \boldsymbol{s}^{\text{trial}}, \sqrt{J_2^{\text{trial}}}, r^{\text{trial}}, z^{\text{trial}}_{\text{eff}}, a_1, a_2, a_3, a_4, I_1^{\text{peak}}, R_c, \beta,$ `yieldCondition`

procedure NONHARDENINGRETURN
- $r'_{\text{trial}} \leftarrow \beta\, r^{\text{trial}}\sqrt{\frac{3K^k}{2G^k}}$ ▷*Transform the trial r coordinate*
- $X^k_{\text{eff}} \leftarrow X^k + 3(\overline{p^w})^k$
- $z^{\text{close}}_{\text{eff}}, r'_{\text{close}} \leftarrow$ `yieldCondition`.GETCLOSESTPOINT(K^k, G^k, X^k_{eff}, a_1, a_2, a_3, a_4, I_1^{peak}, R_c, β, $\hookrightarrow$ $z^{\text{trial}}_{\text{eff}}, r'_{\text{trial}}$)
- $I_1^{\text{close}} \leftarrow \sqrt{3}z^{\text{close}}_{\text{eff}} - 3(\overline{p^w})^k$, $\sqrt{J_2^{\text{close}}} \leftarrow \frac{1}{\beta}\sqrt{\frac{G^k}{3K^k}}\, r'_{\text{close}}$
- **if** $\sqrt{J_2^{\text{trial}}} > 0$ **then**
  - $\boldsymbol{\sigma}^{\text{fixed}} = \frac{1}{3}I_1^{\text{close}}\boldsymbol{I} + \frac{\sqrt{J_2^{\text{close}}}}{\sqrt{J_2^{\text{trial}}}}\boldsymbol{s}^{\text{trial}}$ ▷*Compute updated total stress*
- **else**
  - $\boldsymbol{\sigma}^{\text{fixed}} = \frac{1}{3}I_1^{\text{close}}\boldsymbol{I} + \boldsymbol{s}^{\text{trial}}$ ▷*Compute updated total stress when the trial stress is hydrostatic*
- **end if**
- $\delta\boldsymbol{\sigma}_{\text{fixed}} \leftarrow \boldsymbol{\sigma}^{\text{fixed}} - \boldsymbol{\sigma}^k$ ▷*Compute stress increment*
- $\delta\boldsymbol{\sigma}^{\text{iso}}_{\text{fixed}} \leftarrow \frac{1}{3}\text{tr}(\delta\boldsymbol{\sigma}_{\text{fixed}})\boldsymbol{I}$, $\delta\boldsymbol{\sigma}^{\text{dev}}_{\text{fixed}} \leftarrow \delta\boldsymbol{\sigma}_{\text{fixed}} - \delta\boldsymbol{\sigma}^{\text{iso}}_{\text{fixed}}$
- $\delta\boldsymbol{\varepsilon}^{\text{p,fixed}} = \delta\boldsymbol{\varepsilon} - \frac{1}{3K^k}\delta\boldsymbol{\sigma}^{\text{iso}}_{\text{fixed}} - \frac{1}{2G^k}\delta\boldsymbol{\sigma}^{\text{dev}}_{\text{fixed}}$ ▷*Compute plastic strain increment*
- **return** $\boldsymbol{\sigma}^{\text{fixed}}, \delta\boldsymbol{\varepsilon}^{\text{p,fixed}}$

end procedure

Finding the closest point in transformed space

Algorithm 15 Compute the closest point from the trial state to transformed non-hardening yield surface

Require: K^k, G^k, X^k_{eff}, a_1, a_2, a_3, a_4, I_1^{peak}, R_c, β, $z_{\text{eff}}^{\text{trial}}$, r'_{trial}

procedure GETCLOSESTPOINT
  $n_{\text{poly}} \leftarrow 1000$
  $\mathbf{x}_{\text{poly}} \leftarrow$ GETYIELDSURFACEPOINTSALL_RPRIMEZ(n_{poly}, K^k, G^k, X^k_{eff}, a_1, a_2, a_3, a_4, I_1^{peak}, R_c, β) ▷*Get the polygon that represents the yield surface in z_{eff}-r' space.*
  $\mathbf{x}_{\text{seg}} \leftarrow$ GETCLOSESTSEGMENTS($z_{\text{eff}}^{\text{trial}}$, r'_{trial}, $\mathbf{x}_{\text{poly}}$) ▷*Find two yield surface segments that are closest to the trial stress state.*
  $\mathbf{x}_{\text{segpoly}} \leftarrow$ GETYIELDSURFACEPOINTSSEGMENT_RPRIMEZ(n_{poly}, K^k, G^k, X^k_{eff}, a_1, a_2, a_3, a_4, I_1^{peak}, R_c, β, $\mathbf{x}_{\text{seg}}[1]$, $\mathbf{x}_{\text{seg}}[3]$) ▷*Discretize the two closest segments.*
  $\mathbf{x}_{\text{close}} \leftarrow$ FINDCLOSESTPOINT($z_{\text{eff}}^{\text{trial}}$, r'_{trial}, $\mathbf{x}_{\text{segpoly}}$) ▷*Find the closest point in the discretized segments to the trial stress state.*
  return isSuccess = TRUE, $\mathbf{x}_{\text{close}}.z_{\text{eff}}$, $\mathbf{x}_{\text{close}}.r'$
end procedure

Finding the yield surface polygon in z_{eff}-r' space

Algorithm 16 Find points in a closed polygon that describes the yield surface in z_{eff}-r' space

Require: n_{poly}, K^k, G^k, X^k_{eff}, a_1, a_2, a_3, a_4, I_1^{peak}, R_c, β

procedure GETYIELDSURFACEPOINTSALL_RPRIMEZ
  $\kappa \leftarrow I_1^{\text{peak}} - R_c(I_1^{\text{peak}} - X^k_{\text{eff}})$ ▷*Compute κ.*
  $I_1^{\text{eff}} \leftarrow$ LINSPACE(`from` = X^k_{eff}, `to` = I_1^{peak}, `points` = n_{poly}) ▷*Create an equally spaced set of I_1^{eff} values.*
  for I_1 **in** I_1^{eff} **do**
    $F_f = a_1 - a_3 \exp(a_2 I_1) - a_4 I_1$; ▷*Compute F_f.*
    $F_c^2 \leftarrow 1$
    if $I_1 < \kappa$ **and** $X^k_{\text{eff}} < I_1$ **then**
      $F_c^2 = 1 - \left[\frac{\kappa - I_1}{\kappa - X^k_{\text{eff}}}\right]^2$; ▷*Compute F_c.*
    end if
    $J_2 = F_f^2 F_c^2$ ▷*Compute J_2 and push into a vector*
  end for
  $z_{\text{eff}} \leftarrow I_1^{\text{eff}}/\sqrt{3}$, $r' \leftarrow \beta\sqrt{\frac{3K^k}{2G^k}}\sqrt{2J_2}$
  $\mathbf{x}_{\text{poly}}.z_{\text{eff}} \leftarrow z_{\text{eff}} \cup$ REVERSE(z_{eff}) , $\mathbf{x}_{\text{poly}}.r' \leftarrow r' \cup$ REVERSE$(-r')$ ▷*Add the points on the negative r' side of the polygon*
  $\mathbf{x}_{\text{poly}}[2n_{\text{poly}} + 1] \leftarrow \mathbf{x}_{\text{poly}}[1]$ ▷*Add the first point to close the polygon*
  return $\mathbf{x}_{\text{poly}}$
end procedure

Locating the closest segments of the yield surface polygon in z_{eff}-r' space

Algorithm 17 Locate two closest segments of the yield surface polygon to the trial stress state

Require: $z_{\text{eff}}^{\text{trial}}$, r'_{trial}, $\mathbf{x}_{\text{poly}}$

procedure GETCLOSESTSEGMENTS
  $n_{\text{poly}} \leftarrow$ LENGTH($\mathbf{x}_{\text{poly}}$) ▷*Get the number of points in the closed polygon*

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$\mathbf{x}_{prev} \leftarrow \mathbf{x}_{poly}[n_{poly} - 1]$ ▷*Get the second to last point in the closed polygon*
$i_{next} \leftarrow 2$, $\mathbf{x}_{next} \leftarrow \mathbf{x}_{poly}[i_{next}]$ ▷*Get the second point in the closed polygon*
$d^2_{min} \leftarrow$ DOUBLE_MAX
$\mathbf{x}_{min} \leftarrow \mathbf{0}$, $\mathbf{x}^{min}_{prev} \leftarrow \mathbf{0}$, $\mathbf{x}^{min}_{next} \leftarrow \mathbf{0}$
for $\mathbf{x}_{cur}$ **in** $\mathbf{x}_{poly}$ **do**
  $d^2 \leftarrow$ DISTANCESQ(z^{trial}_{eff}, r'_{trial}, $\mathbf{x}_{cur}.z_{eff}$, $\mathbf{x}_{cur}.r'$)
  if $d^2 < d^2_{min}$ **then**
    $d^2_{min} \leftarrow d^2$
    $\mathbf{x}_{min} \leftarrow \mathbf{x}_{cur}$, $\mathbf{x}^{min}_{prev} \leftarrow \mathbf{x}_{prev}$, $\mathbf{x}^{min}_{next} \leftarrow \mathbf{x}_{next}$
  end if
  $i_{next} \leftarrow i_{next} + 1$
  if $i_{next} = n_{poly}$ **then**
    break ▷*Since the polygon is closed, ignore the last point*
  end if
  $\mathbf{x}_{prev} \leftarrow \mathbf{x}_{cur}$, $\mathbf{x}_{next} \leftarrow \mathbf{x}_{poly}[i_{next}]$
end for
$\mathbf{x}_{seg} \leftarrow \{\mathbf{x}^{min}_{prev}, \mathbf{x}_{min}, \mathbf{x}^{min}_{next}\}$
return $\mathbf{x}_{seg}$
end procedure

Finding the segments of the surface polygon in z_{eff}-r' space

Algorithm 18 Find points in an open segment that describes the yield surface in z_{eff}-r' space

Require: n_{poly}, K^k, G^k, X^k_{eff}, a_1, a_2, a_3, a_4, I^{peak}_1, R_c, β, $\mathbf{x}_{start}$, $\mathbf{x}_{end}$
procedure GETYIELDSURFACEPOINTSSEGMENT_RPRIMEZ
  $\kappa \leftarrow I^{peak}_1 - R_c(I^{peak}_1 - X^k_{eff})$ ▷*Compute κ.*
  $I^{start}_1 \leftarrow \sqrt{3}\mathbf{x}_{start}.z_{eff}$, $I^{end}_1 \leftarrow \sqrt{3}\mathbf{x}_{end}.z_{eff}$
  $I^{eff}_1 \leftarrow$ LINSPACE(from = I^{start}_1, to = I^{end}_1, points = n_{poly})
    ↪ ▷*Create an equally spaced set of I^{eff}_1 values.*
  for I_1 **in** I^{eff}_1 **do**
    $F_f = a_1 - a_3 \exp(a_2 I_1) - a_4 I_1$; ▷*Compute F_f.*
    $F^2_c \leftarrow 1$
    if $I_1 < \kappa$ **and** $X^k_{eff} < I_1$ **then**
      $F^2_c = 1 - \left[\frac{\kappa - I_1}{\kappa - X^k_{eff}}\right]^2$; ▷*Compute F_c.*
    end if
    $J_2 = F^2_f F^2_c$ ▷*Compute J_2 and push into a vector*
  end for
  $z_{eff} \leftarrow I^{eff}_1/\sqrt{3}$, $r' \leftarrow \beta\sqrt{\frac{3K^k}{2G^k}}\sqrt{2J_2}$
  $\mathbf{x}_{segpoly}.z_{eff} \leftarrow z_{eff} \cup$ REVERSE(z_{eff}) , $\mathbf{x}_{segpoly}.r' \leftarrow r' \cup$ REVERSE$(-r')$
    ↪ ▷*Add the points on the negative r' side of the polygon*
  return $\mathbf{x}_{segpoly}$
end procedure

Finding the closest point on yield surface segments in z_{eff}-r' space

Algorithm 19 Find the closest point from the trial stress state on the polyline describing the yield surface

Require: $\mathbf{x}_{trial}.z_{eff}$, $\mathbf{x}_{trial}.r'$, $\mathbf{x}_{segpoly}$
procedure FINDCLOSESTPOINT
  $i \leftarrow 0$

```
    for {x_segstart, x_segend} in x_segpoly do
        x_seg ← x_segend − x_segstart
        x_proj ← x_trial − x_segstart
        t ← (x_proj · x_seg) / ‖x_seg‖
        i ← i + 1
        if t < 0 then
            x[i] ← x_segstart
        else if t > 1 then
            x[i] ← x_segend
        else
            x[i] ← x_segstart + t x_seg
        end if
    end for
    d²_min ← DOUBLE_MAX
    x_close ← 0
    for x_i in x do
        d² ← DISTANCESQ(x_i, x_trial)
        if d² < d²_min then
            d²_min ← d²
            x_close ← x_i
        end if
    end for
    return x_close
end procedure
```

Consistency bisection algorithm

Now, if we allow the yield surface to harden, the distance between the trial stress point and its projection on to the yield surface decreases compared to that for a fixed yield surface. If $\Delta\boldsymbol{\varepsilon}^{\mathrm{p}}$ is the plastic strain increment for a hardening yield surface, we have

$$\Delta\boldsymbol{\varepsilon}^{\mathrm{p}} < \Delta\boldsymbol{\varepsilon}^{\mathrm{p}}_{\mathrm{fixed}} \tag{116}$$

where the inequality can be evaluated using an appropriate Euclidean norm. Note that this distance is proportional to the consistency parameter $\dot{\lambda}$.

Fully saturated model In Homel's *fully saturated* version of the Arenisca model (Homel, Guilkey, and Brannon, 2015), the internal variables are the hydrostatic compressive strength (X) and the scalar isotropic backstress ($\boldsymbol{\alpha}$). These depend only on the *volumetric* plastic strain increment

$$\Delta\varepsilon^{\mathrm{p}}_{v} = \mathrm{tr}(\Delta\boldsymbol{\varepsilon}^{\mathrm{p}})\,. \tag{117}$$

Because $\Delta\varepsilon^{\mathrm{p}}_{v} < \Delta\varepsilon^{\mathrm{p}}_{v,\mathrm{fixed}}$ we can define a parameter, $\eta \in (0,1)$, such that

$$\eta := \frac{\Delta\varepsilon^{\mathrm{p}}_{v}}{\Delta\varepsilon^{\mathrm{p}}_{v,\mathrm{fixed}}}. \tag{118}$$

Because the solution is bounded by the fixed yield surface, a **bisection algorithm can be used to find the parameter η.**

Partially saturated model To use the same approach for our *partially saturated model*, the internal variables have to be of the form

$$\overline{p^{w}_{\mathrm{p}}} \equiv \overline{p^{w}_{\mathrm{p}}}(\varepsilon^{\mathrm{p}}_{v})\,,\quad B^{\mathrm{p}} \equiv B^{\mathrm{p}}(\varepsilon^{\mathrm{p}}_{v})\,,\quad \phi^{\mathrm{p}} \equiv \phi^{\mathrm{p}}(\varepsilon^{\mathrm{p}}_{v})\,,\quad S^{\mathrm{p}}_{w} \equiv S^{\mathrm{p}}_{w}(\varepsilon^{\mathrm{p}}_{v})\,,\quad \overline{X^{\mathrm{p}}} \equiv \overline{X}^{\mathrm{p}}(\varepsilon^{\mathrm{p}}_{v}) \tag{119}$$

so that

$$\dot{\overline{p^w_{\rm p}}} = \dot\lambda \frac{d\overline{p^w_{\rm p}}}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,,\quad \dot{B^{\rm p}} = \dot\lambda \frac{dB^{\rm p}}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,,\quad \dot{\phi^{\rm p}} = \dot\lambda \frac{d\phi^{\rm p}}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,,\quad \dot{S^{\rm p}_w} = \dot\lambda \frac{dS^{\rm p}_w}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,,\quad \dot{\overline{X^{\rm p}}} = \dot\lambda \frac{d\overline{X}^{\rm p}}{d\varepsilon^{\rm p}_v}\mathrm{tr}(\boldsymbol{M})\,. \tag{120}$$

Recall that

$$\dot{\overline{p^w_{\rm p}}} = \frac{1}{\mathcal{B}}\mathrm{tr}(\boldsymbol{d}^{\rm p}) = \frac{\dot\lambda}{\mathcal{B}}\mathrm{tr}(\boldsymbol{M}) \tag{121}$$

$$\dot{B^{\rm p}} = \dot\lambda\left[\frac{1}{\mathcal{B}K_s}\left(\frac{K_d}{K_s}\frac{dK_d}{d\overline{p^w}} - \frac{dK_s}{d\overline{p^w}}\right)\right]\mathrm{tr}(\boldsymbol{M}) \tag{122}$$

$$\dot{S^{\rm p}_w} = \dot\lambda\left[\frac{1-S_0}{\mathcal{B}(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}}\right]\mathrm{tr}(\boldsymbol{M}) \tag{123}$$

$$\dot{\phi^{\rm p}} = -\dot\lambda\left[(1-S_0)\phi_0\exp\left(\overline{\varepsilon_v} - \overline{\varepsilon^a_v}\right)\left(1+\frac{1}{K_a\mathcal{B}}\right) + S_0\phi_0\exp\left(\overline{\varepsilon_v} - \overline{\varepsilon^w_v}\right)\left(1+\frac{1}{K_w\mathcal{B}}\right)\right]\mathrm{tr}(\boldsymbol{M}) \tag{124}$$

$$\dot{\overline{X^{\rm p}}} := \dot\lambda\left[\left[(1-S_w) + p_1^{\rm sat}S_w\right]\frac{d\overline{X}_d}{d\overline{\varepsilon^{\rm p}_v}} + \frac{\overline{X}_d(p_1^{\rm sat}-1)(1-S_0)}{\mathcal{B}(1-S_0+\mathcal{C})^2}\frac{d\mathcal{C}}{d\overline{p^w}} + \frac{3\overline{p^w}}{\mathcal{B}K_s}\left(\frac{K_d}{K_s}\frac{dK_d}{d\overline{p^w}} - \frac{dK_s}{d\overline{p^w}}\right) + \frac{3B}{\mathcal{B}}\right]\mathrm{tr}(\boldsymbol{M}) \tag{125}$$

where

$$\begin{aligned}\mathcal{B} &:= \frac{1}{(1-S_0)\exp[\overline{\varepsilon_v} - \overline{\varepsilon^a_v}(\overline{p^w})] + S_0\exp[\overline{\varepsilon_v} - \overline{\varepsilon^w_v}(\overline{p^w})]}\left[-\frac{(B-\phi)\phi}{B\phi_0}\left(\frac{S_w}{K_w} + \frac{1-S_w}{K_a}\right) + \right.\\ &\qquad \left.\frac{1-S_0}{K_a}\exp[\overline{\varepsilon_v} - \overline{\varepsilon^a_v}(\overline{p^w})] + \frac{S_0}{K_w}\exp[\overline{\varepsilon_v} - \overline{\varepsilon^w_v}(\overline{p^w})]\right]\\ \mathcal{C} &:= S_0\exp\left[\overline{\varepsilon^a_v}(\overline{p^w}) - \overline{\varepsilon^w_v}(\overline{p^w})\right]\,.\end{aligned} \tag{126}$$

These equation are consistent with Homel's assumptions (allowing for the use of a bisection algorithm) only if

$$\overline{p^w} = \overline{p^w_{\rm p}}\,,\quad B = B^{\rm p}\,,\quad \phi = \phi^{\rm p}\,,\quad S_w = S^{\rm p}_w\,,\quad \overline{X} = \overline{X}^{\rm p}\,. \tag{127}$$

This is a strong assumption and may violate the balance of mass. However, to keep the algorithm efficient, we make the assumptions in (127) and hence can use the bisection algorithm discussed later.

Now we can compare equations (120) and (121), and use the assumption that $B \approx 1$, to get the following set of coupled ordinary differential equations for the internal variables.

$$\begin{aligned}\frac{d\overline{p^w_{\rm p}}}{d\varepsilon^{\rm p}_v} &= \frac{1}{\mathcal{B}^{\rm p}}\\ \frac{d\phi^{\rm p}}{d\varepsilon^{\rm p}_v} &= (S_0-1)\phi_0\exp[\overline{\varepsilon^{\rm p}_v} - \overline{\varepsilon^a_v}(\overline{p^w_{\rm p}})]\left(1+\frac{1}{K_a(\overline{p^w_{\rm p}})\mathcal{B}^{\rm p}}\right) - S_0\phi_0\exp[\overline{\varepsilon^{\rm p}_v} - \overline{\varepsilon^w_v}(\overline{p^w_{\rm p}})]\left(1+\frac{1}{K_w(\overline{p^w_{\rm p}})\mathcal{B}^{\rm p}}\right)\\ \frac{dS^{\rm p}_w}{d\varepsilon^{\rm p}_v} &= \frac{1-S_0}{\mathcal{B}^{\rm p}(1-S_0+\mathcal{C}^{\rm p})^2}\frac{d\mathcal{C}^{\rm p}}{d\overline{p^w_{\rm p}}}\\ \frac{d\overline{X}^{\rm p}}{d\varepsilon^{\rm p}_v} &= \left[(1-S^{\rm p}_w) + p_1^{\rm sat}S^{\rm p}_w\right]\frac{d\overline{X}_d}{d\overline{\varepsilon^{\rm p}_v}} + \frac{\overline{X}_d(p_1^{\rm sat}-1)(1-S_0)}{\mathcal{B}^{\rm p}(1-S_0+\mathcal{C}^{\rm p})^2}\frac{d\mathcal{C}^{\rm p}}{d\overline{p^w_{\rm p}}} + \frac{3}{\mathcal{B}^{\rm p}}\end{aligned} \tag{128}$$

where

$$\mathcal{B}^{\mathrm{p}} := \frac{1}{(1-S_0)\exp[\overline{\varepsilon_v^{\mathrm{p}}} - \overline{\varepsilon_v^a}(\overline{p_{\mathrm{p}}^w})] + S_0\exp[\overline{\varepsilon_v^{\mathrm{p}}} - \overline{\varepsilon_v^w}(\overline{p_{\mathrm{p}}^w})]}\left[-\frac{(1-\phi^{\mathrm{p}})\phi^{\mathrm{p}}}{\phi_0}\left(\frac{S_w^{\mathrm{p}}}{K_w(\overline{p_{\mathrm{p}}^w})} + \frac{1-S_w^{\mathrm{p}}}{K_a(\overline{p_{\mathrm{p}}^w})}\right) + \frac{1-S_0}{K_a(\overline{p_{\mathrm{p}}^w})}\exp[\overline{\varepsilon_v^{\mathrm{p}}} - \overline{\varepsilon_v^a}(\overline{p_{\mathrm{p}}^w})] + \frac{S_0}{K_w(\overline{p_{\mathrm{p}}^w})}\exp[\overline{\varepsilon_v^{\mathrm{p}}} - \overline{\varepsilon_v^w}(\overline{p_{\mathrm{p}}^w})]\right]$$

$$\mathcal{C}^{\mathrm{p}} := S_0\exp\left[\overline{\varepsilon_v^a}(\overline{p_{\mathrm{p}}^w}) - \overline{\varepsilon_v^w}(\overline{p_{\mathrm{p}}^w})\right], \quad \frac{d\mathcal{C}^{\mathrm{p}}}{d\overline{p_{\mathrm{p}}^w}} = \mathcal{C}^{\mathrm{p}}\left[\frac{1}{K_a(\overline{p_{\mathrm{p}}^w})} - \frac{1}{K_w(\overline{p_{\mathrm{p}}^w})}\right]. \tag{129}$$

If the vector of internal variables is $\boldsymbol{\eta} := \{\overline{p_{\mathrm{p}}^w}, \phi^{\mathrm{p}}, S_w^{\mathrm{p}}, \overline{X}^{\mathrm{p}}\}$, we can express the above system as

$$\frac{d\boldsymbol{\eta}}{d\varepsilon_v^{\mathrm{p}}} = \mathbf{F}(\boldsymbol{\eta}, \varepsilon_v^{\mathrm{p}}). \tag{130}$$

We will use a forward Euler integration approach to update the internal variables:

$$\boldsymbol{\eta}^{n+1} = \boldsymbol{\eta}^n + \mathbf{F}[\boldsymbol{\eta}^n, (\varepsilon_v^{\mathrm{p}})^n]\Delta\varepsilon_v^{\mathrm{p}}. \tag{131}$$

Since the porosity and saturation can be computed in closed form, we use the following in our implementation

$$\begin{aligned}\phi^{\mathrm{p}}(\overline{p_{\mathrm{p}}^w}) &= (1-S_0)\phi_0\exp\left[\overline{\varepsilon_v^{\mathrm{p}}} - \overline{\varepsilon_v^a}(\overline{p_{\mathrm{p}}^w})\right] + S_0\phi_0\exp\left[\overline{\varepsilon_v^{\mathrm{p}}} - \overline{\varepsilon_v^w}(\overline{p_{\mathrm{p}}^w})\right]\\ S_w^{\mathrm{p}}(\overline{p_{\mathrm{p}}^w}) &= \frac{\mathcal{C}^{\mathrm{p}}}{1-S_0+\mathcal{C}^{\mathrm{p}}}, \quad \mathcal{C}^{\mathrm{p}} := S_0\exp\left[\overline{\varepsilon_v^a}(\overline{p_{\mathrm{p}}^w}) - \overline{\varepsilon_v^w}(\overline{p_{\mathrm{p}}^w})\right].\end{aligned} \tag{132}$$

The hardening consistency bisection algorithm

Algorithm 20 The consistency bisection algorithm for partially saturated materials

Require: $\delta\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^{\mathrm{p},k}$, $\boldsymbol{\sigma}^k$, K^k, G^k, $(\overline{p^w})^k$, ϕ^k, S_w^k, X^k, $\boldsymbol{s}^{\mathrm{trial}}$, $I_1^{\mathrm{eff,trial}}$, $\sqrt{J_2^{\mathrm{trial}}}$, r^{trial}, $z_{\mathrm{eff}}^{\mathrm{trial}}$, $\varepsilon_v^{\mathrm{p,trial}}$, p_3^{trial}, a_1, a_2, a_3, a_4, I_1^{peak}, R_c, β, $i_{\max}$, $j_{\max}$, $\boldsymbol{\sigma}_{\mathrm{fixed}}$, $\delta\boldsymbol{\varepsilon}_{\mathrm{fixed}}^{\mathrm{p}}$, `yieldCondition`

procedure CONSISTENCYBISECTION
  $\delta\varepsilon_v^{\mathrm{p,fixed}} \leftarrow \mathrm{tr}(\delta\boldsymbol{\varepsilon}_{\mathrm{fixed}}^{\mathrm{p}})$
  $i \leftarrow 1$
  $\eta^{\mathrm{in}} \leftarrow 0$, $\eta^{\mathrm{out}} \leftarrow 1$
  while ABS($\eta^{\mathrm{out}} - \eta^{\mathrm{in}}$) > TOLERANCE **do**
    $j \leftarrow 1$
    isElastic ← TRUE
    while isElastic = TRUE **do**
      $\eta^{\mathrm{mid}} \leftarrow \frac{1}{2}(\eta^{\mathrm{in}} + \eta^{\mathrm{out}})$
      $\delta\varepsilon_v^{\mathrm{p,mid}} \leftarrow \eta^{\mathrm{mid}}\delta\varepsilon_v^{\mathrm{p,fixed}}$
      $(\overline{p^w})^{\mathrm{mid}}, \phi^{\mathrm{mid}}, S_w^{\mathrm{mid}}, X^{\mathrm{mid}} \leftarrow$ COMPUTEINTERNALVARIABLES(K^k, G^k, $(\overline{p^w})^k$, ϕ^k, S_w^k, X^k, $\delta\varepsilon_v^{\mathrm{p,mid}}$) ▷*Update the internal variables using the bisected increment of the volumetric plastic strain*
      isElastic ← `yieldCondition`.EVALYIELDCONDITION($I_1^{\mathrm{eff,trial}}$, $\sqrt{J_2^{\mathrm{trial}}}$, X^{mid}, $(\overline{p^w})^{\mathrm{mid}}$, ϕ^{mid}, S_w^{mid}, a_1, a_2, a_3, a_4, I_1^{peak}, R_c, β) ▷*Determine whether the trial stress is elastic or not*

if isElastic = TRUE **then**
$\eta^{\text{out}} \leftarrow \eta^{\text{mid}}$ ▷*If the local trial state is inside the updated yield surface, the yield condition evaluates to "elastic". We need to reduce the size of the yield surface by decreasing the plastic strain increment.*
$j \leftarrow j + 1$
if $j \geq j_{\max}$ **then**
return isSuccess ← FALSE ▷*The bisection algorithm failed because of too many iterations.*
end if
end if
end while
$\boldsymbol{\sigma}^{\text{new}}_{\text{fixed}}, \delta\boldsymbol{\varepsilon}^{\text{p,new}}_{\text{fixed}} \leftarrow$ NONHARDENINGRETURN$(\boldsymbol{\sigma}^k, \delta\boldsymbol{\varepsilon}, X^{\text{mid}}, K^k, G^k, (\overline{p^w})^{\text{mid}}, \boldsymbol{s}^{\text{trial}}, \sqrt{J_2^{\text{trial}}}, r^{\text{trial}}, z^{\text{trial}}_{\text{eff}}, a_1, a_2, a_3, a_4, I_1^{\text{peak}}, R_c, \beta)$ ▷*Compute return to updated yield surface (no hardening)*
if SIGN$(\text{tr}(\boldsymbol{\sigma}_{\text{trial}} - \boldsymbol{\sigma}^{\text{new}}_{\text{fixed}}))$ ≠ SIGN$(\text{tr}(\boldsymbol{\sigma}_{\text{trial}} - \boldsymbol{\sigma}_{\text{fixed}}))$ **then**
$\eta^{\text{out}} \leftarrow \eta^{\text{mid}}$ ▷*Too much plastic strain*
continue
end if
if $\left\|\delta\boldsymbol{\varepsilon}^{\text{p,new}}_{\text{fixed}}\right\| > \eta^{\text{mid}} \left\|\delta\boldsymbol{\varepsilon}^{\text{p}}_{\text{fixed}}\right\|$ **then**
$\eta^{\text{in}} \leftarrow \eta^{\text{mid}}$ ▷*Too little plastic strain*
else
$\eta^{\text{out}} \leftarrow \eta^{\text{mid}}$ ▷*Too much plastic strain*
end if
$i \leftarrow i + 1$
if $i \geq i_{\max}$ **then**
return isSuccess ← FALSE ▷*Too many iterations*
end if
end while
$\delta\varepsilon^{\text{p},k+1}_{v,\text{fixed}} \leftarrow \text{tr}(\delta\boldsymbol{\varepsilon}^{\text{p},k+1}_{\text{fixed}})$
$(\overline{p^w})^{k+1}, \phi^{k+1}, S_w^{k+1}, X^{k+1} \leftarrow$ COMPUTEINTERNALVARIABLES$(K^k, G^k, (\overline{p^w})^k, \phi^k, S_w^k, X^k, \delta\varepsilon^{\text{p},k+1}_{v,\text{fixed}})$ ▷*Update the internal variables using the bisected increment of the volumetric plastic strain*
$\boldsymbol{\sigma}^{k+1} \leftarrow \boldsymbol{\sigma}^{\text{new}}_{\text{fixed}}$, $\boldsymbol{\alpha}^{k+1} \leftarrow -(\overline{p^w})^{k+1}\boldsymbol{I}$, $p_3^{k+1} \leftarrow p_3^{\text{trial}}$
$\boldsymbol{\varepsilon}^{\text{p},k+1} = \boldsymbol{\varepsilon}^{\text{p},k} + \delta\boldsymbol{\varepsilon}^{\text{p},k+1}$ ▷*Update the plastic strain*
$K^{k+1}, G^{k+1}, \boldsymbol{s}^{k+1}, (\overline{p^w})^{k+1}, I_1^{\text{eff},k+1}, \sqrt{J_2^{k+1}}, r^{k+1}, z^{k+1}_{\text{eff}}, \varepsilon_v^{\text{p},k+1} \leftarrow$ COMPUTEELASTICPROPERTIES$(\boldsymbol{\sigma}^{k+1}, \phi^{k+1}, S_w^{k+1}, \boldsymbol{\varepsilon}^{\text{p},k+1}, \boldsymbol{\alpha}^{k+1}, p_3^{k+1})$ ▷*Compute elastic moduli and stress invariants for the new state*
return isSuccess ← TRUE, $\boldsymbol{\sigma}^{k+1}, \boldsymbol{\varepsilon}^{\text{p},k+1}, \boldsymbol{\alpha}^{k+1}, (\overline{p^w})^{k+1}, \phi^{k+1}, S_w^{k+1}, X^{k+1}, K^{k+1}, G^{k+1}, \boldsymbol{s}^{k+1}, (\overline{p^w})^{k+1}, I_1^{\text{eff},k+1}, \sqrt{J_2^{k+1}}, r^{k+1}, z^{k+1}_{\text{eff}}, \varepsilon_v^{\text{p},k+1}, p_3^{k+1}$
end procedure

Updating the internal variables

Algorithm 21 Updating the internal variables for partially saturated materials

Require: $\boldsymbol{\sigma}^k, \varepsilon_v^{\text{p},k}, K^k, G^k, (\overline{p^w})^k, \phi^k, S_w^k, X^k, \delta\varepsilon_v^{\text{p}}$, `fluidParams`, `crushParams`, `airModel`, `waterModel`
procedure COMPUTEINTERNALVARIABLES
$\overline{\varepsilon_v^{\text{p},k}} \leftarrow -\varepsilon_v^{\text{p},k}$, $\delta\overline{\varepsilon_v^{\text{p}}} \leftarrow -\delta\varepsilon_v^{\text{p}}$
$\overline{p_0^w} \leftarrow$ `fluidParams`.$\overline{p_0^w}$, $S_0 \leftarrow$ `fluidParams`.S_0, $\phi_0 \leftarrow$ `fluidParams`.ϕ_0, $p_1^{\text{sat}} \leftarrow$ `crushParams`.p_1^{sat}
$K_a \leftarrow$ `airModel`.COMPUTEBULKMODULUS$((\overline{p^w})^k)$
$K_w \leftarrow$ `waterModel`.COMPUTEBULKMODULUS$((\overline{p^w})^k)$

$\varepsilon_v^{a,0} \leftarrow$ `airModel`.computeElasticVolumetricStrain($\overline{p_0^w}$)
$\varepsilon_v^{a} \leftarrow$ `airModel`.computeElasticVolumetricStrain($(\overline{p^w})^k$)
$\varepsilon_v^{w} \leftarrow$ `waterModel`.computeElasticVolumetricStrain($(\overline{p^w})^k$, $\overline{p_0^w}$)
$\overline{\varepsilon_v^a} \leftarrow -(\varepsilon_v^a - \varepsilon_v^{a,0})$, $\overline{\varepsilon_v^w} \leftarrow -\varepsilon_v^w$
$\mathcal{C}_p \leftarrow S_0 \exp(\overline{\varepsilon_v^a} - \overline{\varepsilon_v^w})$
$\mathcal{D}_p \leftarrow \frac{1-S_0}{(1-S_0+\mathcal{C}_p)^2}$
$\frac{d\mathcal{C}_p}{d\overline{p^w}} \leftarrow \mathcal{C}_p\left[\frac{1}{K_a} - \frac{1}{K_w}\right]$
$\mathcal{G}_a \leftarrow \exp(\overline{\varepsilon_v^{p,k}} - \overline{\varepsilon_v^a})$, $\mathcal{G}_w \leftarrow \exp(\overline{\varepsilon_v^{p,k}} - \overline{\varepsilon_v^w})$
$\mathcal{B}_p \leftarrow \frac{1}{(1-S_0)\mathcal{G}_a + S_0\mathcal{G}_w}\left[-\frac{(1-\phi^k)\phi^k}{\phi_0}\left(\frac{S_w^k}{K_w} + \frac{1-S_w^k}{K_a}\right) + \frac{1-S_0}{K_a}\mathcal{G}_a + \frac{S_0}{K_w}\mathcal{G}_w\right]$
$(\overline{p^w})^{k+1} \leftarrow \max\left[(\overline{p^w})^k + \frac{1}{\mathcal{B}_p}\delta\varepsilon_v^p, 0\right]$ ▷*Update the pore pressure making sure that pressure does not become negative during dilatative plastic deformations.*
$\overline{X}_d, \frac{d\overline{X}_d}{d\varepsilon_v^p} \leftarrow$ computeDrainedHydrostaticStrengthAndDeriv($\overline{\varepsilon_v^{p,k}}$) ▷*Compute the drained hydrostatic compressive strength and its derivative*
$\overline{X}^{k+1} = -X^k + \left[(1 - S_w^k + p_1^{sat} S_w^k)\frac{d\overline{X}_d}{d\varepsilon_v^p} + \overline{X}_d(p_1^{sat} - 1)\frac{\mathcal{D}_p}{\mathcal{B}_p}\frac{d\mathcal{C}_p}{d\overline{p^w}} + \frac{3}{\mathcal{B}_p}\right]\delta\varepsilon_v^p$ ▷*Update the hydrostatic compressive strength*
$\overline{X^{k+1}} \leftarrow -\overline{X}^{k+1}$
$\overline{\varepsilon_v^{p,k+1}} \leftarrow \overline{\varepsilon_v^{p,k}} + \delta\overline{\varepsilon_v^p}$ ▷*Compute the updated volumetric plastic strain.*
$\varepsilon_v^{a,k+1} \leftarrow$ `airModel`.computeElasticVolumetricStrain($(\overline{p^w})^{k+1}$)
$\varepsilon_v^{w,k+1} \leftarrow$ `waterModel`.computeElasticVolumetricStrain($(\overline{p^w})^{k+1}$, $\overline{p_0^w}$)
$\overline{\varepsilon_v^{a,k+1}} \leftarrow -(\varepsilon_v^{a,k+1} - \varepsilon_v^{a,0})$, $\overline{\varepsilon_v^{w,k+1}} \leftarrow -\varepsilon_v^{w,k+1}$ ▷*The updated strains in the fluid phases.*
$\mathcal{C}_p^{k+1} \leftarrow S_0 \exp(\overline{\varepsilon_v^{a,k+1}} - \overline{\varepsilon_v^{w,k+1}})$
$S_w^{k+1} \leftarrow \frac{\mathcal{C}_p^{k+1}}{1-S_0+\mathcal{C}_p^{k+1}}$ ▷*Update the saturation*
$\mathcal{G}_a^{k+1} \leftarrow \exp(\overline{\varepsilon_v^{p,k+1}} - \overline{\varepsilon_v^{a,k+1}})$, $\mathcal{G}_w^{k+1} \leftarrow \exp(\overline{\varepsilon_v^{p,k+1}} - \overline{\varepsilon_v^{w,k+1}})$
$\phi^{k+1} \leftarrow (1 - S_0)\phi_0\mathcal{G}_a^{k+1} + S_0\phi_0\mathcal{G}_w^{k+1}$ ▷*Update the porosity*
return $(\overline{p^w})^{k+1}$, ϕ^{k+1}, S_w^{k+1}, X^{k+1}
end procedure

Algorithm 22 Computing the drained hydrostatic strength and its derivative

Require: $\overline{\varepsilon_v^{p,k}}$, `fluidParams`, `crushParams`
procedure computeDrainedHydrostaticStrengthAndDeriv
$\phi_0 \leftarrow$ `fluidParams`.ϕ_0
$p_0 \leftarrow$ `crushParams`.p_0, $p_1 \leftarrow$ `crushParams`.p_1, $p_1^{sat} \leftarrow$ `crushParams`.p_1^{sat}, $p_2 \leftarrow$ `crushParams`.p_2
$p_3 \leftarrow -\log(1 - \phi_0)$
$\overline{X}_d \leftarrow \max(p_0, 1000)$; ▷$\overline{X}_d$ *has a minimum value of 1000 pressure units*
$\frac{d\overline{X}_d}{d\varepsilon_v^p} \leftarrow 0$
if $\overline{\varepsilon_v^{p,k}} > 0$ **then**
$\phi_{temp} \leftarrow \exp(-p_3 + \overline{\varepsilon_v^{p,k}})$
$\phi \leftarrow 1 - \phi_{temp}$
$\bar{\xi} \leftarrow p_1 \text{ pow}\left(\frac{\phi_0}{\phi} - 1, \frac{1}{p_2}\right)$
$\overline{X}_d \leftarrow \overline{X}_d + \bar{\xi}$

$\frac{d\overline{X}_d}{d\varepsilon_v^{\mathrm{p}}} \leftarrow \frac{1}{p_2}\frac{\phi_0}{\phi}\phi_{\mathrm{temp}}\frac{\bar{\xi}}{\phi\left(\frac{\phi_0}{\phi}-1\right)}$

end if

return $\overline{X}_d, \frac{d\overline{X}_d}{d\varepsilon_v^{\mathrm{p}}}$

end procedure

Rate-dependent plastic update

Recall from (74) and (77) that

$$\dot{\boldsymbol{\sigma}}_{\mathrm{over}} = \dot{\boldsymbol{\sigma}}_{\mathrm{proj}} - \frac{1}{\tau}\boldsymbol{\sigma}_{\mathrm{over}} \quad \text{where} \quad \dot{\boldsymbol{\sigma}}_{\mathrm{over}} := \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_{qs} \quad \text{and} \quad \dot{\boldsymbol{\sigma}}_{\mathrm{proj}} := \dot{\boldsymbol{\sigma}}^{\mathrm{trial}} - \dot{\boldsymbol{\sigma}}_{qs}\,. \tag{133}$$

and the exact solution (82) when τ and $\dot{\boldsymbol{\sigma}}_{\mathrm{proj}}$ are constant:

$$\boldsymbol{\sigma}_{\mathrm{over}}(t) = \tau\left[1-\exp\left(\frac{t_0-t}{\tau}\right)\right]\dot{\boldsymbol{\sigma}}_{\mathrm{proj}} + \exp\left(\frac{t_0-t}{\tau}\right)\boldsymbol{\sigma}_{\mathrm{over}}(t_0)\,. \tag{134}$$

For a timestep $\Delta t = t_{n+1} - t_n$, we can write the above as

$$\boldsymbol{\sigma}_{\mathrm{over}}(t_{n+1}) = \tau(t_n)\left[1-\exp\left(\frac{t_n-t_{n+1}}{\tau(t_n)}\right)\right]\dot{\boldsymbol{\sigma}}_{\mathrm{proj}}(t_n) + \exp\left(\frac{t_n-t_{n+1}}{\tau(t_n)}\right)\boldsymbol{\sigma}_{\mathrm{over}}(t_n) \tag{135}$$

or

$$\boldsymbol{\sigma}_{\mathrm{over}}^{n+1} = \left[\frac{1-\exp\left(-\Delta t/\tau^n\right)}{\Delta t/\tau^n}\right]\left(\Delta t\,\dot{\boldsymbol{\sigma}}_{\mathrm{proj}}^{n}\right) + \exp\left(-\Delta t/\tau^n\right)\boldsymbol{\sigma}_{\mathrm{over}}^{n}\,. \tag{136}$$

If we approximate the rate of $\boldsymbol{\sigma}_{\mathrm{proj}}$ via approximations of the rates of the trial and quasistatic stresses using

$$\dot{\boldsymbol{\sigma}}_{\mathrm{proj}} \approx \frac{\boldsymbol{\sigma}_{n+1}^{\mathrm{trial}} - \boldsymbol{\sigma}_{n}^{\mathrm{trial}}}{\Delta t} - \frac{\boldsymbol{\sigma}_{qs}^{n+1} - \boldsymbol{\sigma}_{qs}^{n}}{\Delta t} \tag{137}$$

noting that

$$\boldsymbol{\sigma}_{n}^{\mathrm{trial}} = \boldsymbol{\sigma}^{n}, \quad \boldsymbol{\sigma}_{n+1}^{\mathrm{trial}} = \boldsymbol{\sigma}^{n} + (\mathbf{C}^{\mathrm{e}} : \boldsymbol{d}^{s})\Delta t\,, \tag{138}$$

and using the definition of $\boldsymbol{\sigma}_{\mathrm{over}}$, we have

$$\boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}_{qs}^{n+1} = \left[\frac{1-\exp\left(-\Delta t/\tau^n\right)}{\Delta t/\tau^n}\right]\left[\left(\boldsymbol{\sigma}_{n+1}^{\mathrm{trial}} - \boldsymbol{\sigma}^{n}\right) - \left(\boldsymbol{\sigma}_{qs}^{n+1} - \boldsymbol{\sigma}_{qs}^{n}\right)\right] + \exp\left(-\Delta t/\tau^n\right)\left(\boldsymbol{\sigma}^{n} - \boldsymbol{\sigma}_{qs}^{n}\right). \tag{139}$$

If we define

$$R_H := \frac{1-\exp\left(-\Delta t/\tau^n\right)}{\Delta t/\tau^n} \quad \text{and} \quad r_h := \exp\left(-\Delta t/\tau^n\right) \tag{140}$$

we have the dynamic stress state

$$\boldsymbol{\sigma}^{n+1} = \boldsymbol{\sigma}_{qs}^{n+1} + \left[\left(\boldsymbol{\sigma}_{n+1}^{\mathrm{trial}} - \boldsymbol{\sigma}^{n}\right) - \left(\boldsymbol{\sigma}_{qs}^{n+1} - \boldsymbol{\sigma}_{qs}^{n}\right)\right]R_H + \left(\boldsymbol{\sigma}^{n} - \boldsymbol{\sigma}_{qs}^{n}\right)r_h\,. \tag{141}$$

Our implementation does not consider rate-dependent updates of the internal variables. Equation 141 requires that we compute the trial stress for the step, but this is not known since the bulk modulus is evolving through the substeps. It would be necessary to to loop through the substeps to compute the trial stress assuming nonlinear elasticity, but instead we approximate the trial stress using the average of the elastic moduli at the start and end of the step.

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Algorithm 23 Computing the correction to the stress due to rate-dependent plasticity

Require: Δt, $\boldsymbol{d}^{n+1}$, $\boldsymbol{\sigma}^n$, K^n, G^n, ϕ^n, S_w^n, X^n, $\boldsymbol{\alpha}^n$, $\boldsymbol{\varepsilon}^{\mathrm{p},n}$, p_3^n, $\boldsymbol{\sigma}_{qs}^n$, $\boldsymbol{\sigma}_{qs}^{n+1}$, K^{n+1}, G^{n+1}, ϕ^{n+1}, S_w^{n+1}, X^{n+1}, $\boldsymbol{\alpha}^{n+1}$, $\boldsymbol{\varepsilon}^{\mathrm{p},n+1}$, p_3^{n+1}, a_1, a_2, a_3, a_4, I_1^{peak}, R_c, `yieldParams`

procedure RATEDEPENDENTPLASTICUPDATE
  $T_1 \leftarrow$ `yieldParams`.T_1, $T_2 \leftarrow$ `yieldParams`.T_2
  if $T_1 = 0$ **or** $T_2 = 0$ **then** ▷*Check if rate-dependent plasticity has been turned on*
    return isRateDependent ← FALSE, $\boldsymbol{\sigma}_{qs}^{n+1}$
  end if
  $K_{\mathrm{dyn}} \leftarrow \frac{1}{2}(K^n + K^{n+1})$, $G_{\mathrm{dyn}} \leftarrow \frac{1}{2}(G^n + G^{n+1})$ ▷*Compute mid-step bulk and shear modulus*
  $\Delta\boldsymbol{\varepsilon} \leftarrow \Delta t\, \boldsymbol{d}^{n+1}$
  $\boldsymbol{\sigma}_{\mathrm{trial,dyn}} \leftarrow$ COMPUTETRIALSTRESS($\boldsymbol{\sigma}^n$, K_{dyn}, G_{dyn}, $\boldsymbol{d}^{n+1}$, Δt) ▷*Compute substep trial stress*
  $\dot{\varepsilon} \leftarrow$ MAX($\|\boldsymbol{d}^{n+1}\|$, ABS_DOUBLE_MIN)
  $\tau \leftarrow T_1$ POW($\dot{\varepsilon}$, T_2) ▷*The characteristic time is defined from the rate-dependence input parameters and the magnitude of the strain rate*
  $r_h \leftarrow \exp\left(-\frac{\Delta t}{\tau}\right)$
  $R_H \leftarrow \frac{1-r_h}{\frac{\Delta t}{\tau}}$
  $\boldsymbol{\sigma}^{n+1} \leftarrow \boldsymbol{\sigma}_{qs}^{n+1} + \left[(\boldsymbol{\sigma}_{\mathrm{trial,dyn}} - \boldsymbol{\sigma}^n) - (\boldsymbol{\sigma}_{qs}^{n+1} - \boldsymbol{\sigma}_{qs}^n)\right] R_H + (\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_{qs}^n)\, r_h$ ▷*Stress update*
  return $\boldsymbol{\sigma}^{n+1}$, isRateDependent ← TRUE
end procedure

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Appendices

Governing equations from mixture theory

An alternative way of determining the governing equations is using mixture theory, of which there are several alternatives. We explore the Borja et al. (Borja, 2004; Li, Borja, and Regueiro, 2004; Borja, 2006; Borja and White, 2010; Uzuoka and Borja, 2012; Song and Borja, 2014b; Song and Borja, 2014a) approach in this appendix.

Volume fraction, porosity, saturation, density

Let us once again consider a three-phase porous medium with constituents $\alpha = \{s, w, a\}$ where s is the solid skeleton, w is water, and a is air. Let $\mathbf{X}$ and $\mathbf{x}$ be the reference and current positions, respectively, of a point inside a body Ω. Each point may be considered to be the centroid of a representative volume element (RVE) region (ω) of volume dv and mass dm. The region ($\omega^\alpha \subset \omega$) occupied by phase α at each point in the body has a volume dv^α and mass dm^α such that

$$dv = dv^s + dv^w + dv^a \quad \text{and} \quad dm = dm^s + dm^w + dm^a \,. \tag{142}$$

. The volume fraction (f^α) of phase α at each point is defined as

$$f^\alpha = \frac{dv^\alpha}{dv} \quad \text{where} \quad \sum_\alpha f^\alpha = 1 \,. \tag{143}$$

The volume fraction of the solid skeleton is

$$f^s = 1 - \phi \,, \quad \phi = \frac{dv^w + dv^a}{dv} \tag{144}$$

where the ϕ is the porosity. The volume fractions of water and air are

$$f^w = \phi S_w \,, \quad f^a = \phi S_a = \phi(1 - S_w) \,, \quad S_w = \frac{dv^w}{dv^w + dv^a} \tag{145}$$

where S_w is the saturation. Let the mixture mass density (ρ) and the mass density of each phase (ρ^α) be defined as

$$\rho = \frac{dm}{dv} \quad \text{and} \quad \rho^\alpha = \frac{dm^\alpha}{dv^\alpha} \,. \tag{146}$$

We define the *partial mass density* of each phase as

$$\langle \rho^\alpha \rangle = f^\alpha \rho^\alpha \quad \text{such that} \quad \langle \rho^s \rangle + \langle \rho^w \rangle + \langle \rho^a \rangle = \rho \,. \tag{147}$$

Material derivatives

As before, the material time derivative of $\mathbf{f}^\alpha$ with respect to phase α is defined as

$$\frac{D^\alpha \mathbf{f}^\alpha}{Dt} = \frac{\partial \mathbf{f}^\alpha}{\partial t} + \nabla \mathbf{f}^\alpha \cdot \mathbf{v}^\alpha \tag{148}$$

where $\mathbf{v}^\alpha$ is the velocity of phase α. The relative material derivative of phase β with respect to phase α is

$$\frac{D^\alpha \mathbf{f}^\beta}{Dt} = \frac{D^\beta \mathbf{f}^\beta}{Dt} + \nabla \mathbf{f}^\beta \cdot (\mathbf{v}^\alpha - \mathbf{v}^\beta) = \frac{D^\beta \mathbf{f}^\beta}{Dt} + \nabla \mathbf{f}^\beta \cdot \mathbf{v}^{\alpha\beta} \,. \tag{149}$$

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Mass balance

As discussed in the appendix on the averaging approach, the phase mass balance equation for the RVE in the absence of mass exchange between the phases is

$$\frac{\partial}{\partial t}\left(\langle\rho^{\alpha}\rangle\right) + \nabla\cdot\left(\langle\rho^{\alpha}\rangle\mathbf{v}^{\alpha}\right) = 0\,. \tag{150}$$

Using the definition of the material derivative, we can write the mass balance equations as

$$\frac{D^{\alpha}}{Dt}\left(\langle\rho^{\alpha}\rangle\right) + \langle\rho^{\alpha}\rangle\nabla\cdot\mathbf{v}^{\alpha} = 0\,. \tag{151}$$

Alternatively, relative to phase β,

$$\frac{D^{\beta}}{Dt}\left(\langle\rho^{\alpha}\rangle\right) - \nabla\langle\rho^{\alpha}\rangle\cdot\mathbf{v}^{\beta\alpha} + \langle\rho^{\alpha}\rangle\nabla\cdot\mathbf{v}^{\alpha} = 0\,. \tag{152}$$

Solid phase

For the solid phase,

$$\frac{D^{s}}{Dt}\left(\langle\rho^{s}\rangle\right) + \langle\rho^{s}\rangle\nabla\cdot\mathbf{v}^{s} = 0\,. \tag{153}$$

Expanding out in terms of volume fractions,

$$\frac{D^{s}f^{s}}{Dt} + \frac{f^{s}}{\rho^{s}}\frac{D^{s}\rho^{s}}{Dt} + f^{s}\nabla\cdot\mathbf{v}^{s} = 0\,. \tag{154}$$

Assume a smooth function $\rho^{s} = \rho^{s}(p^{s})$, where p^{s} is the intrinsic pressure in the solid phase (actual force per unit area acting on the solid phase - without considering voids). Then

$$\frac{D^{s}f^{s}}{Dt} + \frac{f^{s}}{\rho^{s}}\frac{d\rho^{s}}{dp^{s}}\frac{D^{s}p^{s}}{Dt} + f^{s}\nabla\cdot\mathbf{v}^{s} = 0\,. \tag{155}$$

The bulk modulus of the solid phase is defined as

$$K_{s}(p^{s}) = -\frac{dp^{s}}{d\varepsilon_{v}^{e}} = -\frac{dp^{s}}{d\rho^{s}}\frac{d\rho^{s}}{d\varepsilon_{v}^{e}} = \rho^{s}\frac{dp^{s}}{d\rho^{s}} \quad \text{where} \quad \varepsilon_{v}^{e} = \ln\left(\frac{v^{s}}{v_{0}^{s}}\right) = -\ln\left(\frac{\rho^{s}}{\rho_{0}^{s}}\right) = \ln J^{s}\,. \tag{156}$$

We can use the above definition to get

$$\frac{D^{s}f^{s}}{Dt} + \frac{f^{s}}{K_{s}(p^{s})}\frac{D^{s}p^{s}}{Dt} + f^{s}\nabla\cdot\mathbf{v}^{s} = 0\,. \tag{157}$$

Note that there is some ambiguity here about the conservation of intrinsic mass and the definition of intrinsic volume (v^{s}).

For our purposes, it is more convenient to work with a solid phase pressure that is defined over a RVE containing voids. Borja and coworkers define this pressure as

$$p^{s} = \widetilde{p}^{s}(\rho^{s}, f^{s}) \tag{158}$$

such that

$$\frac{D^{s}p^{s}}{Dt} = \frac{\partial\widetilde{p}^{s}}{\partial\rho^{s}}\frac{D^{s}\rho^{s}}{Dt} + \frac{\partial\widetilde{p}^{s}}{\partial f^{s}}\frac{D^{s}f^{s}}{Dt}\,. \tag{159}$$

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Now,

$$\frac{D^s \varepsilon_v^e}{Dt} = -\frac{1}{\rho_s}\frac{D^s \rho^s}{Dt} = \frac{D^s}{Dt}(\ln J^s) = \frac{1}{J_s}\frac{D^s J^s}{Dt}. \tag{160}$$

Recall that

$$\frac{D^s J^s}{Dt} = J^s \nabla \cdot \mathbf{v}^s. \tag{161}$$

Therefore,

$$\frac{D^s \rho^s}{Dt} = -\rho^s \nabla \cdot \mathbf{v}^s \tag{162}$$

Plugging into (159) gives

$$\frac{D^s p^s}{Dt} = -\rho^s \frac{\partial \widetilde{p}^s}{\partial \rho^s}\nabla \cdot \mathbf{v}^s + \frac{\partial \widetilde{p}^s}{\partial f^s}\frac{D^s f^s}{Dt}. \tag{163}$$

Therefore,

$$\frac{D^s f^s}{Dt} = \frac{1}{\dfrac{\partial \widetilde{p}^s}{\partial f^s}}\left[\frac{D^s p^s}{Dt} + \rho^s \frac{\partial \widetilde{p}^s}{\partial \rho^s}\nabla \cdot \mathbf{v}^s\right]. \tag{164}$$

Plugging this into the mass balance equation (157) for the solid phase,

$$\frac{D^s p^s}{Dt} + \rho^s \frac{\partial \widetilde{p}^s}{\partial \rho^s}\nabla \cdot \mathbf{v}^s + \frac{f^s}{K_s(p^s)}\frac{\partial \widetilde{p}^s}{\partial f^s}\frac{D^s p^s}{Dt} + f^s \frac{\partial \widetilde{p}^s}{\partial f^s}\nabla \cdot \mathbf{v}^s = 0 \tag{165}$$

or

$$\left(1 + \frac{f^s}{K_s(p^s)}\frac{\partial \widetilde{p}^s}{\partial f^s}\right)\frac{D^s p^s}{Dt} + \left(\rho^s \frac{\partial \widetilde{p}^s}{\partial \rho^s} + f^s \frac{\partial \widetilde{p}^s}{\partial f^s}\right)\nabla \cdot \mathbf{v}^s = 0 \tag{166}$$

or,

$$\frac{D^s p^s}{Dt} = -\left(\frac{\rho^s \dfrac{\partial \widetilde{p}^s}{\partial \rho^s} + f^s \dfrac{\partial \widetilde{p}^s}{\partial f^s}}{1 + \dfrac{f^s}{K_s(p^s)}\dfrac{\partial \widetilde{p}^s}{\partial f^s}}\right)\nabla \cdot \mathbf{v}^s. \tag{167}$$

Let us define the bulk modulus of the solid matrix ($\widetilde{K}^s$) using

$$\frac{D^s \widetilde{p}^s}{Dt} = \frac{\partial \widetilde{p}^s}{\partial \varepsilon_v^e}\frac{D^s \varepsilon_v^e}{Dt} =: -\widetilde{K}^s(\widetilde{p}^s)\nabla \cdot \mathbf{v}^s \tag{168}$$

In terms of the intrinsic pressure in the solid, Borja's definition leads to

$$f^s \frac{D^s p^s}{Dt} = -\widetilde{K}^s(\widetilde{p}^s)\nabla \cdot \mathbf{v}^s. \tag{169}$$

The reason for this definition is not obvious and needs re-examination but leads to the definition of the matrix bulk modulus used by Borja:

$$\widetilde{K}^s := f^s\left(\frac{\rho^s \dfrac{\partial \widetilde{p}^s}{\partial \rho^s} + f^s \dfrac{\partial \widetilde{p}^s}{\partial f^s}}{1 + \dfrac{f^s}{K_s(p^s)}\dfrac{\partial \widetilde{p}^s}{\partial f^s}}\right). \tag{170}$$

If we substitute (169) into (157), we get

$$\frac{D^s f^s}{Dt} - \frac{\widetilde{K}^s(\widetilde{p}^s)}{K_s(p^s)}\nabla \cdot \mathbf{v}^s + f^s \nabla \cdot \mathbf{v}^s = 0. \tag{171}$$

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Defining

$$B := 1 - \frac{\widetilde{K}^s(\widetilde{p}^s)}{K_s(p^s)} \tag{172}$$

we have the solid mass balance equation

$$\frac{D^s f^s}{Dt} + (B - 1 + f^s)\nabla \cdot \mathbf{v}^s = 0\,. \tag{173}$$

Water phase

For the water phase, relative to the solid phase, with $\mathbf{v}^{sw} = \mathbf{v}^s - \mathbf{v}^w$,

$$\frac{D^s}{Dt}(\langle \rho^w \rangle) - \nabla \langle \rho^w \rangle \cdot \mathbf{v}^{sw} + \langle \rho^w \rangle \nabla \cdot \mathbf{v}^w = 0\,. \tag{174}$$

Expanded,

$$\frac{D^s f^w}{Dt} + \frac{f^w}{\rho^w}\frac{D^s \rho^w}{Dt} - \left[\frac{f^w}{\rho^w}\nabla \rho^w + \nabla f^w\right] \cdot \mathbf{v}^{sw} + f^w \nabla \cdot \mathbf{v}^w = 0\,. \tag{175}$$

Once again we assume the existence of a smooth function $\rho^w = \rho^w(p^w)$ where p^w is the intrinsic water pressure and get

$$\frac{D^s f^w}{Dt} + \frac{f^w}{\rho^w}\frac{d\rho^w}{dp^w}\frac{D^s p^w}{Dt} - \left[\frac{f^w}{\rho^w}\frac{d\rho^w}{dp^w}\nabla p^w + \nabla f^w\right] \cdot \mathbf{v}^{sw} + f^w \nabla \cdot \mathbf{v}^w = 0\,. \tag{176}$$

As was done for the solid phase, the bulk modulus of water is defined as

$$K_w(p^w) = \rho^w \frac{dp^w}{d\rho^w} \tag{177}$$

which leads to

$$\frac{D^s f^w}{Dt} + \frac{f^w}{K_w(p^w)}\frac{D^s p^w}{Dt} - \left[\frac{f^w}{K_w(p^w)}\nabla p^w + \nabla f^w\right] \cdot \mathbf{v}^{sw} + f^w \nabla \cdot \mathbf{v}^w = 0\,. \tag{178}$$

Recalling from the definitions of porosity and saturation that

$$f^w = \phi S_w = (1 - f^s) S_w \tag{179}$$

and using the mass balance equation for the solid phase (157), we have

$$\frac{D^s f^w}{Dt} = -S_w \frac{D^s f^s}{Dt} + (1 - f^s)\frac{D^s S_w}{Dt} = S_w \left[\frac{f^s}{K_s(p^s)}\frac{D^s p^s}{Dt} + f^s \nabla \cdot \mathbf{v}^s\right] + (1 - f^s)\frac{D^s S_w}{Dt}\,. \tag{180}$$

Therefore the mass balance of water can be expressed as

$$(1-f^s)\frac{D^s S_w}{Dt} + \frac{f^w}{K_w(p^w)}\frac{D^s p^w}{Dt} + \frac{S_w f^s}{K_s(p^s)}\frac{D^s p^s}{Dt} + S_w f^s \nabla \cdot \mathbf{v}^s - \left[\frac{f^w}{K_w(p^w)}\nabla p^w + \nabla f^w\right] \cdot \mathbf{v}^{sw} + f^w \nabla \cdot \mathbf{v}^w = 0\,. \tag{181}$$

Recall from equation (169) that

$$f^s \frac{D^s p^s}{Dt} = -\widetilde{K}^s(\widetilde{p}^s)\nabla \cdot \mathbf{v}^s\,. \tag{182}$$

Plugging this into the mass balance equation for water gives

$$(1-f^s)\frac{D^s S_w}{Dt} + \frac{f^w}{K_w(p^w)}\frac{D^s p^w}{Dt} - \frac{S_w \widetilde{K}^s(\widetilde{p}^s)}{K_s(p^s)}\nabla \cdot \mathbf{v}^s + S_w f^s \nabla \cdot \mathbf{v}^s - \left[\frac{f^w}{K_w(p^w)}\nabla p^w + \nabla f^w\right] \cdot \mathbf{v}^{sw} + f^w \nabla \cdot \mathbf{v}^w = 0\,. \tag{183}$$

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Using the definition

$$B := 1 - \frac{\widetilde{K}^s(\widetilde{p}^s)}{K_s(p^s)} \tag{184}$$

we have

$$(1-f^s)\frac{D^sS_w}{Dt}+\frac{f^w}{K_w(p^w)}\frac{D^sp^w}{Dt}+S_w(B-1+f^s)\nabla\cdot\mathbf{v}^s-\left[\frac{f^w}{K_w(p^w)}\nabla p^w+\nabla f^w\right]\cdot\mathbf{v}^{sw}+f^w\nabla\cdot\mathbf{v}^w=0\,. \tag{185}$$

Using

$$\nabla\cdot\mathbf{v}^w=-\nabla\cdot\mathbf{v}^{sw}+\nabla\cdot\mathbf{v}^s \tag{186}$$

and

$$S_w(B-1+f^s)+f^w=S_wB-S_w+S_w(1-\phi)+S_w\phi=S_wB \tag{187}$$

we get

$$(1-f^s)\frac{D^sS_w}{Dt}+\frac{f^w}{K_w(p^w)}\frac{D^sp^w}{Dt}+BS_w\nabla\cdot\mathbf{v}^s-\left[\frac{f^w}{K_w(p^w)}\nabla p^w+\nabla f^w\right]\cdot\mathbf{v}^{sw}-f^w\nabla\cdot\mathbf{v}^{sw}=0\,. \tag{188}$$

Air phase

For the air phase, relative to the solid phase, with $\mathbf{v}^{sa}=\mathbf{v}^s-\mathbf{v}^a$,

$$\frac{D^s}{Dt}(\langle\rho^a\rangle)-\nabla\langle\rho^a\rangle\cdot\mathbf{v}^{sa}+\langle\rho^a\rangle\nabla\cdot\mathbf{v}^a=0 \tag{189}$$

or

$$\frac{D^sf^a}{Dt}+\frac{f^a}{\rho^a}\frac{D^s\rho^a}{Dt}-\left[\frac{f^a}{\rho^a}\nabla\rho^a+\nabla f^a\right]\cdot\mathbf{v}^{sa}+f^a\nabla\cdot\mathbf{v}^a=0\,. \tag{190}$$

For the air phase, with

$$\rho^a=\rho^a(p^a)\quad\text{and}\quad K_a(p^a)=\rho^a\frac{dp^a}{d\rho^a} \tag{191}$$

we get

$$\frac{D^sf^a}{Dt}+\frac{f^a}{K_a(p^a)}\frac{D^sp^a}{Dt}-\left[\frac{f^a}{K_a(p^a)}\nabla p^a+\nabla f^a\right]\cdot\mathbf{v}^{sa}+f^a\nabla\cdot\mathbf{v}^a=0\,. \tag{192}$$

From the definitions of porosity and saturation,

$$f^a=\phi S_a=(1-f^s)(1-S_w)\,. \tag{193}$$

Using the procedure used for the water phase, the mass balance of air is

$$(1-f^s)\frac{D^sS_a}{Dt}+\frac{f^a}{K_a(p^a)}\frac{D^sp^a}{Dt}+\frac{S_af^s}{K_s(p^s)}\frac{D^sp^s}{Dt}+S_af^s\nabla\cdot\mathbf{v}^s-\left[\frac{f^a}{K_a(p^a)}\nabla p^a+\nabla f^a\right]\cdot\mathbf{v}^{sa}+f^a\nabla\cdot\mathbf{v}^a=0\,. \tag{194}$$

In terms of the matrix bulk modulus,

$$(1-f^s)\frac{D^sS_a}{Dt}+\frac{f^a}{K_a(p^a)}\frac{D^sp^a}{Dt}+BS_a\nabla\cdot\mathbf{v}^s-\left[\frac{f^a}{K_a(p^a)}\nabla p^a+\nabla f^a\right]\cdot\mathbf{v}^{sa}-f^a\nabla\cdot\mathbf{v}^{sa}=0\,. \tag{195}$$

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Linear momentum balance

The linear momentum balance for the three phases in the absence of mass exchange between phases can be written as

$$\langle \rho^\alpha \rangle \frac{D^\alpha \mathbf{v}^\alpha}{Dt} = \nabla \cdot \boldsymbol{\sigma}^\alpha + \langle \rho^\alpha \rangle \mathbf{b}^\alpha + \mathbf{p}^\alpha + \mathbf{t}^\alpha \,, \qquad \sum_\alpha (\mathbf{p}^\alpha + \mathbf{t}^\alpha) = \mathbf{0} \tag{196}$$

where $\boldsymbol{\sigma}^\alpha$ is the average stress of phase α in the RVE, $\mathbf{b}^\alpha$ is the body force experience by phase α, and $\mathbf{p}^\alpha$ is the interaction force term due to the motion of the phases and $\mathbf{t}^\alpha$ is the interaction force vector due to surface tractions, between phase α and the other phases. Also note that the total stress is given by

$$\boldsymbol{\sigma} = \sum_\alpha \boldsymbol{\sigma}^\alpha \,. \tag{197}$$

Borja uses expressions for $\mathbf{p}^\alpha$ and $\mathbf{t}^\alpha$ similar (but not identical) to that derived by Haasanizadeh and Gray (Hassanizadeh and Gray, 1990) by the application of an averaged energy balance and a reduced entropy inequality to the RVE balance equations discussed in the previous appendix. These expressions are

$$\mathbf{t}^\alpha = p^\alpha \nabla f^\alpha \,, \quad \mathbf{p}^\alpha = -f^\alpha \boldsymbol{\mu}^\alpha \cdot (\mathbf{v}^\alpha - \mathbf{v}^s) \tag{198}$$

where $\boldsymbol{\mu}^\alpha$ is a material parameter related to the relative permeability of the phases.

Solid phase

The material time derivative for the solid phase is

$$\mathbf{a}^s = \frac{D^s \mathbf{v}^s}{Dt} = \frac{\partial \mathbf{v}^s}{\partial t} + \nabla \mathbf{v}^s \cdot \mathbf{v}^s \,. \tag{199}$$

The momentum equation for the solid phase is

$$\langle \rho^s \rangle \mathbf{a}^s = \nabla \cdot \boldsymbol{\sigma}^s + \langle \rho^s \rangle \mathbf{b}^s + p^s \nabla f^s \,. \tag{200}$$

Water phase

The material time derivative for the water phase is

$$\mathbf{a}^w = \frac{D^w \mathbf{v}^w}{Dt} = \frac{\partial \mathbf{v}^w}{\partial t} + \nabla \mathbf{v}^w \cdot \mathbf{v}^w \,. \tag{201}$$

If the solid phase acts as a reference, we have

$$\mathbf{a}^w = \frac{D^s \mathbf{v}^w}{Dt} - \nabla \mathbf{v}^w \cdot (\mathbf{v}^s - \mathbf{v}^w) \,. \tag{202}$$

The momentum equation for the water phase is

$$\langle \rho^w \rangle \mathbf{a}^w = \nabla \cdot \boldsymbol{\sigma}^w + \langle \rho^w \rangle \mathbf{b}^w + p^w \nabla f^w - f^w \boldsymbol{\mu}^w \cdot (\mathbf{v}^w - \mathbf{v}^s) \,. \tag{203}$$

Air phase

The material time derivative for the air phase is

$$\mathbf{a}^a = \frac{D^w \mathbf{v}^a}{Dt} = \frac{\partial \mathbf{v}^a}{\partial t} + \nabla \mathbf{v}^a \cdot \mathbf{v}^a \,. \tag{204}$$

If the solid phase acts as a reference, we have

$$\mathbf{a}^a = \frac{D^s \mathbf{v}^a}{Dt} - \nabla \mathbf{v}^a \cdot (\mathbf{v}^s - \mathbf{v}^a) \,. \tag{205}$$

The momentum equation for the air phase is

$$\langle \rho^a \rangle \mathbf{a}^a = \nabla \cdot \boldsymbol{\sigma}^a + \langle \rho^a \rangle \mathbf{b}^a + \mathbf{t}^a + p^a \nabla f^a - f^a \boldsymbol{\mu}^a \cdot (\mathbf{v}^a - \mathbf{v}^s) \,. \tag{206}$$

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Mixture

Adding the momentum equations of the three phases, and assuming that all three phases experience identical body forces, gives

$$\langle\rho^s\rangle\mathbf{a}^s + \langle\rho^w\rangle\mathbf{a}^w + \langle\rho^a\rangle\mathbf{a}^a = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b}^s \tag{207}$$

where we have used the relation

$$\rho = \sum_\alpha \langle\rho^\alpha\rangle\,. \tag{208}$$

Energy balance

The energy balance of the mixture is

$$\rho\frac{D^s e}{Dt} - \boldsymbol{\sigma}^s : \boldsymbol{d}^s - \boldsymbol{\sigma}^w : \boldsymbol{d}^w - \boldsymbol{\sigma}^a : \boldsymbol{d}^a + \nabla\cdot\mathbf{q} - \rho h = 0\,. \tag{209}$$

If the fluid stresses are isotropic, i.e.,

$$\boldsymbol{\sigma}^w = -\langle p^w\rangle\boldsymbol{I} = -f^w p^w\boldsymbol{I}\,,\quad \boldsymbol{\sigma}^a = -\langle p^a\rangle\boldsymbol{I} = -f^a p^a\boldsymbol{I} \tag{210}$$

we have

$$\rho\frac{D^s e}{Dt} - \boldsymbol{\sigma}^s : \boldsymbol{d}^s + f^w p^w\nabla\cdot\mathbf{v}^w + f^a p^a\nabla\cdot\mathbf{v}^a + \nabla\cdot\mathbf{q} - \rho h = 0\,. \tag{211}$$

Recall that the total stress is

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^s + \boldsymbol{\sigma}^w + \boldsymbol{\sigma}^a = \boldsymbol{\sigma}^s - f^w p^w\boldsymbol{I} - f^a p^a\boldsymbol{I}\,. \tag{212}$$

Therefore we can write the mixture balance equation as

$$\rho\frac{D^s e}{Dt} - \boldsymbol{\sigma} : \boldsymbol{d}^s - f^w p^w\nabla\cdot\mathbf{v}^{sw} - f^a p^a\nabla\cdot\mathbf{v}^{sa} + \nabla\cdot\mathbf{q} - \rho h = 0\,. \tag{213}$$

After multiplying by p^w and p^a, respectively, we can write (188) and (195) as

$$\begin{aligned} p^w(1-f^s)\frac{D^s S_w}{Dt} + \frac{p^w f^w}{K_w(p^w)}\frac{D^s p^w}{Dt} + Bp^w S_w\nabla\cdot\mathbf{v}^s - \\ \frac{p^w f^w}{K_w(p^w)}\nabla p^w\cdot\mathbf{v}^{sw} + p^w\nabla f^w\cdot\mathbf{v}^{sw} = f^w p^w\nabla\cdot\mathbf{v}^{sw} \end{aligned} \tag{214}$$

and

$$\begin{aligned} -p^a(1-f^s)\frac{D^s S_w}{Dt} + \frac{p^a f^a}{K_a(p^a)}\frac{D^s p^a}{Dt} + Bp^a S_a\nabla\cdot\mathbf{v}^s - \\ \frac{p^a f^a}{K_a(p^a)}\nabla p^a\cdot\mathbf{v}^{sa} + p^a\nabla f^a\cdot\mathbf{v}^{sa} = f^a p^a\nabla\cdot\mathbf{v}^{sa}\,. \end{aligned} \tag{215}$$

Adding these equations and substitution into the mixture energy balance equation, and noting that

$$\nabla\cdot\mathbf{v}^s = \boldsymbol{I} : \boldsymbol{d}^s \tag{216}$$

gives

$$\begin{aligned} \rho\frac{D^s e}{Dt} - \boldsymbol{\sigma} : \boldsymbol{d}^s - B\left[\sum_{\alpha=\{w,a\}} p^\alpha S_\alpha\right]\boldsymbol{I} : \boldsymbol{d}^s - (p^w - p^a)(1-f^s)\frac{D^s S_w}{Dt} - \sum_{\alpha=\{w,a\}}\frac{p^\alpha f^\alpha}{K^\alpha(p^\alpha)}\frac{D^s p^\alpha}{Dt} \\ + \sum_{\alpha=\{w,a\}}\left[\frac{p^\alpha f^\alpha}{K^\alpha(p^\alpha)}\nabla p^\alpha\cdot\mathbf{v}^{s\alpha} - p^\alpha\nabla f^\alpha\cdot\mathbf{v}^{s\alpha}\right] + \nabla\cdot\mathbf{q} - \rho h = 0\,. \end{aligned} \tag{217}$$

We can define the effective stress to be

$$\boldsymbol{\sigma}_{\text{eff}} := \boldsymbol{\sigma} + B\left[\sum_{\alpha=\{w,a\}} p^{\alpha} S_{\alpha}\right]\boldsymbol{I} \tag{218}$$

to get the energy balance equation

$$\begin{aligned}\rho\frac{D^s e}{Dt} - \boldsymbol{\sigma}_{\text{eff}} : \boldsymbol{d}^s - (p^w - p^a)(1-f^s)\frac{D^s S_w}{Dt} - \sum_{\alpha=\{w,a\}} \frac{p^{\alpha} f^{\alpha}}{K^{\alpha}(p^{\alpha})}\frac{D^s p^{\alpha}}{Dt} \\ + \sum_{\alpha=\{w,a\}}\left[\frac{p^{\alpha} f^{\alpha}}{K^{\alpha}(p^{\alpha})}\nabla p^{\alpha} - p^{\alpha}\nabla f^{\alpha}\right]\cdot\mathbf{v}^{s\alpha} + \nabla\cdot\mathbf{q} - \rho h = 0\,.\end{aligned} \tag{219}$$

Thus we see that the effective stress is energy conjugate to the rate of deformation of the solid. We also find that the capillary stress is energy conjugate to the saturation.

Simplified model

Assume that the water and the air phases do not move relative to the solid phase, i.e.,

$$\mathbf{v}^s = \mathbf{v}^w = \mathbf{v}^a \quad \text{and} \quad \mathbf{a}^s = \mathbf{a}^w = \mathbf{a}^a\,. \tag{220}$$

Then the mass balance equations of the three phases reduce to

$$\begin{aligned}&\frac{D^s f^s}{Dt} + (B - 1 + f^s)\nabla\cdot\mathbf{v}^s = 0\\ &(1-f^s)\frac{D^s S_w}{Dt} + \frac{f^w}{K_w(p^w)}\frac{D^s p^w}{Dt} + BS_w\nabla\cdot\mathbf{v}^s = 0\\ &-(1-f^s)\frac{D^s S_w}{Dt} + \frac{f^a}{K_a(p^a)}\frac{D^s p^a}{Dt} + B(1-S_w)\nabla\cdot\mathbf{v}^s = 0\,.\end{aligned} \tag{221}$$

The momentum balance equation for the mixture is

$$\rho\mathbf{a}^s = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b}^s \quad \text{where} \quad \rho = \sum_{\alpha}\langle\rho^{\alpha}\rangle\,. \tag{222}$$

The energy balance equation is

$$\rho\frac{D^s e}{Dt} - \boldsymbol{\sigma}_{\text{eff}} : \boldsymbol{d}^s - (p^w - p^a)(1-f^s)\frac{D^s S_w}{Dt} - \sum_{\alpha=\{w,a\}} \frac{p^{\alpha} f^{\alpha}}{K^{\alpha}(p^{\alpha})}\frac{D^s p^{\alpha}}{Dt} + \nabla\cdot\mathbf{q} - \rho h = 0 \tag{223}$$

where

$$\boldsymbol{\sigma}_{\text{eff}} = \boldsymbol{\sigma} + B[p^w S_w + p^a S_a]\boldsymbol{I} = \boldsymbol{\sigma} + B[(p^w - p^a)S_w + p^a]\boldsymbol{I} \tag{224}$$

and

$$B := 1 - \frac{\widetilde{K}^s(\widetilde{p}^s)}{K_s(p^s)}\,. \tag{225}$$

If we make the further assumption that at high strain-rates the water and air pressures are equal, $p^w = p^a$, the mass balance equation equations can be written as (with the substitution $f^s = 1 - \phi$, where ϕ is the porosity)

$$\begin{aligned}&\frac{D^s\phi}{Dt} - (B-\phi)\nabla\cdot\mathbf{v}^s = 0\\ &\left[\frac{S_w\phi}{K_w(p^w)} + \frac{(1-S_w)\phi}{K_a(p^w)}\right]\frac{D^s p^w}{Dt} + B\nabla\cdot\mathbf{v}^s = 0\,.\end{aligned} \tag{226}$$

The momentum balance equation retains the same form, while the energy balance equation becomes

$$\rho\frac{D^s e}{Dt} - \boldsymbol{\sigma}_{\text{eff}} : \boldsymbol{d}^s - p^w\left[\frac{S_w\phi}{K_w(p^w)} + \frac{(1-S_w)\phi}{K_a(p^w)}\right]\frac{D^s p^w}{Dt} + \nabla\cdot\mathbf{q} - \rho h = 0 \tag{227}$$

where

$$\boldsymbol{\sigma}_{\text{eff}} = \boldsymbol{\sigma} + Bp^w\boldsymbol{I}\,. \tag{228}$$

Partially saturated bulk modulus

In previous reports we have chosen to compute the tangent bulk modulus of the partially saturated soil using a variant of the Biot-Gassman model for fully saturated rocks that is valid for long wavelength displacements (Berryman and Milton, 1991; Berryman, 2006; Dvorkin et al., 1999). We add a few more details in this appendix.

The Biot-Gassman model is based on the Biot equations of poroelasticity and assumes small strains and long wavelength disturbances relative to the pore size. In this model, the average stress tensor ($\boldsymbol{\sigma}$) of the saturated porous medium is given by

$$\boldsymbol{\sigma} = \left[(H-2G)\text{tr}(\boldsymbol{\varepsilon}) - C\zeta\right]\boldsymbol{I} + 2G\boldsymbol{\varepsilon} \tag{229}$$

and the fluid pressure ($\overline{p^w}$) is

$$\overline{p^w} = M\zeta - C\text{tr}(\boldsymbol{\varepsilon}) \tag{230}$$

where G is the shear modulus, $\boldsymbol{\varepsilon}$ is the strain in the mixture, C is the Biot cross coefficient, and

$$C = \frac{B}{\frac{B}{K_s} + \phi\left(\frac{1}{K_w} - \frac{1}{K_s}\right)}\,,\quad H = K_d + BC + \frac{4}{3}G\,,\quad \zeta = \phi[\text{tr}(\boldsymbol{\varepsilon}) - \varepsilon_v^w]\,,\quad M = \frac{C}{B}\,,\quad B := 1 - \frac{K_d}{K_s}\,. \tag{231}$$

In the above K_d is the bulk modulus of the drained porous frame, K_s is the bulk modulus of the solid grains, ϕ is the porosity, and ε_v^w is the volumetric strain in the fluid.

Using these, the fully saturated bulk modulus for a soil (with a single material in the solid matrix) is

$$\widehat{K} = K_d + B^2 D \tag{232}$$

where

$$\frac{1}{D} = \frac{B(1-B)}{K_d} + \phi\left(\frac{1}{K_w} - \frac{1}{K_s}\right)\,. \tag{233}$$

At partial saturation, we compute the pore fluid bulk modulus using a harmonic mean (lower bound) on the air and water bulk moduli (K_a, K_w):

$$\frac{1}{K_f(\overline{p^w})} = \frac{S_w}{K_w(\overline{p^w})} + \frac{1-S_w}{K_a(\overline{p^w})} \tag{234}$$

and get the expression for the bulk modulus that is used in the Arena model

$$\widehat{K}(\bar{p}_{\text{eff}}, \overline{p^w}, \overline{\varepsilon_v^p}, \phi, S_w) = K_d(\bar{p}_{\text{eff}}) + \frac{B^2(\bar{p}_{\text{eff}})}{\frac{B(\bar{p}_{\text{eff}})}{K_s(\bar{p}_{\text{eff}})} + \phi\left(\frac{1}{K_f(\overline{p^w})} - \frac{1}{K_s(\bar{p}_{\text{eff}})}\right)} \tag{235}$$

where $\widehat{K}$ is the effective bulk modulus of the partially saturated soil, K_d is the bulk modulus of the drained soil, K_f is the bulk modulus of the pore fluid, and K_s is the bulk modulus of the solid grains. For the situation where $B \approx 1$,

$$\widehat{K}(\bar{p}_{\text{eff}}, \overline{p^w}, \overline{\varepsilon_v^p}, \phi, S_w) = K_d(\bar{p}_{\text{eff}}) + \frac{K_f(\overline{p^w})}{\phi}\,. \tag{236}$$

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Pore pressure equation

From Borja's mass balance equations we have

$$\begin{aligned}&\frac{D\phi}{Dt} - (B-\phi)\nabla\cdot\mathbf{v}^s = 0\\&\left[\frac{S_w\phi}{K_w(p^w)} + \frac{(1-S_w)\phi}{K_a(p^w)}\right]\frac{D\overline{p^w}}{Dt} + B\nabla\cdot\mathbf{v}^s = 0\,.\end{aligned} \tag{237}$$

Upon combining the two equations we get

$$\left[\frac{S_w\phi}{K_w(p^w)} + \frac{(1-S_w)\phi}{K_a(p^w)}\right]\frac{D\overline{p^w}}{Dt} + \left(\frac{B}{B-\phi}\right)\frac{D\phi}{Dt} = 0\,. \tag{238}$$

From equation (52),

$$\phi = (1-S_0)\phi_0\exp(\varepsilon_v^a - \varepsilon_v) + S_0\phi_0\exp(\varepsilon_v^w - \varepsilon_v) \tag{239}$$

Therefore, the material derivative of the porosity is

$$\frac{D\phi}{Dt} = (1-S_0)\phi_0\exp(\varepsilon_v^a - \varepsilon_v)\left(\frac{D\varepsilon_v^a}{Dt} - \frac{D\varepsilon_v}{Dt}\right) + S_0\phi_0\exp(\varepsilon_v^w - \varepsilon_v)\left(\frac{D\varepsilon_v^w}{Dt} - \frac{D\varepsilon_v}{Dt}\right) \tag{240}$$

Plugging (240) into (238) gives

$$\begin{aligned}\left[\frac{S_w\phi}{K_w(p^w)} + \frac{(1-S_w)\phi}{K_a(p^w)}\right]\frac{D\overline{p^w}}{Dt} + \left(\frac{B\phi_0}{B-\phi}\right)\Bigg[(1-S_0)\exp(\varepsilon_v^a - \varepsilon_v)\left(\frac{D\varepsilon_v^a}{Dt} - \frac{D\varepsilon_v}{Dt}\right) +\\ S_0\exp(\varepsilon_v^w - \varepsilon_v)\left(\frac{D\varepsilon_v^w}{Dt} - \frac{D\varepsilon_v}{Dt}\right)\Bigg] = 0\,.\end{aligned} \tag{241}$$

For equilibrated fluid pressures, $\overline{p^a} = \overline{p^w}$, and we can write the above using quantities that are positive in compression as

$$\begin{aligned}&\left[\frac{S_w\phi}{K_w} + \frac{(1-S_w)\phi}{K_a}\right]\frac{D\overline{p^w}}{Dt} - \left(\frac{B\phi_0}{B-\phi}\right)\left[(1-S_0)\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a})\frac{\partial\overline{\varepsilon_v^a}}{\partial\overline{p^w}} + S_0\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})\frac{\partial\overline{\varepsilon_v^w}}{\partial\overline{p^w}}\right]\frac{D\overline{p^w}}{Dt}\\&\quad = -\left(\frac{B\phi_0}{B-\phi}\right)\left[(1-S_0)\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a}) + S_0\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})\right]\frac{D\overline{\varepsilon_v}}{Dt}\end{aligned} \tag{242}$$

Noting that

$$\frac{\partial\overline{\varepsilon_v^a}}{\partial\overline{p^w}} = \frac{1}{K_a} \quad \text{and} \quad \frac{\partial\overline{\varepsilon_v^w}}{\partial\overline{p^w}} = \frac{1}{K_w} \tag{243}$$

we have

$$\begin{aligned}&\left[\frac{S_w\phi}{K_w} + \frac{(1-S_w)\phi}{K_a}\right]\frac{D\overline{p^w}}{Dt} - \left(\frac{B\phi_0}{B-\phi}\right)\left[\frac{1-S_0}{K_a}\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a}) + \frac{S_0}{K_w}\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})\right]\frac{D\overline{p^w}}{Dt}\\&\quad = -\left(\frac{B\phi_0}{B-\phi}\right)\left[(1-S_0)\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a}) + S_0\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})\right]\frac{D\overline{\varepsilon_v}}{Dt}\end{aligned} \tag{244}$$

or

$$\begin{aligned}\frac{D\overline{\varepsilon_v}}{Dt} = \frac{1}{(1-S_0)\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a}) + S_0\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})}\Bigg[-\frac{(B-\phi)\phi}{B\phi_0}\left(\frac{S_w}{K_w} + \frac{1-S_w}{K_a}\right) +\\ \frac{1-S_0}{K_a}\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a}) + \frac{S_0}{K_w}\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})\Bigg]\frac{D\overline{p^w}}{Dt}\end{aligned} \tag{245}$$

We can convert this equation into the form

$$\frac{d\overline{p^w}}{d\overline{\varepsilon_v}} = \frac{1}{\mathcal{B}} \tag{246}$$

where

$$\mathcal{B} := \frac{1}{(1-S_0)\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a}) + S_0\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})}\left[-\frac{(B-\phi)\phi}{B\phi_0}\left(\frac{S_w}{K_w} + \frac{1-S_w}{K_a}\right) + \frac{1-S_0}{K_a}\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^a}) + \frac{S_0}{K_w}\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})\right]. \tag{247}$$

We can then find the total pore pressure using

$$\overline{p^w} = \int \frac{d\overline{p^w}}{d\overline{\varepsilon_v}} \mathrm{d}\overline{\varepsilon_v}\,. \tag{248}$$

For a fully saturated medium, $S_0 = 1$, we have

$$\mathcal{B} = \frac{1}{K_w\exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})}\left[-\frac{(B-\phi)\phi}{B\phi_0} + \exp(\overline{\varepsilon_v} - \overline{\varepsilon_v^w})\right]. \tag{249}$$

Note that a medium with zero initial porosity, $\phi_0 = 0$, cannot be modeled using our approach.

Computing saturation from weight ratio

The experimental SHPB data provided by UT Dallas includes the dry sand density and the water content in the form of the ratio by weight of the water to the sand/soil. We have to convert that information into porosity, saturation, and mass density.

To do that we assume that the soil skeleton material has the density of quartz such that the mass (m) of the dry sand can be written as

$$m_{\text{drysand}} = m_{\text{air}} + m_{\text{quartz}}\,. \tag{250}$$

Expressed in terms of volume (v) and density (ρ),

$$\rho_{\text{drysand}} v_{\text{drysand}} = \rho_{\text{air}} v_{\text{air}} + \rho_{\text{quartz}} v_{\text{quartz}}\,. \tag{251}$$

If we neglect the density of air, we have

$$\rho_{\text{drysand}} v_{\text{drysand}} \approx \rho_{\text{quartz}} v_{\text{quartz}} \quad \Longrightarrow \quad \frac{\rho_{\text{drysand}}}{\rho_{\text{quartz}}} = \frac{v_{\text{quartz}}}{v_{\text{drysand}}}\,. \tag{252}$$

The porosity (ϕ) is defined as the ratio of the void volume to the total volume of the sand, i.e.,

$$\phi = \frac{v_{\text{air}}}{v_{\text{drysand}}} = 1 - \frac{v_{\text{quartz}}}{v_{\text{drysand}}} = 1 - \frac{\rho_{\text{drysand}}}{\rho_{\text{quartz}}}\,. \tag{253}$$

For partially saturated sand, the SHPB tests provide the ratio

$$\alpha = \frac{m_{\text{water}}}{m_{\text{drysand}}} = \frac{\rho_{\text{water}} v_{\text{water}}}{\rho_{\text{drysand}} v_{\text{drysand}}}\,. \tag{254}$$

The saturation is defined as

$$S_w = \frac{v_{\text{water}}}{v_{\text{air}} + v_{\text{water}}} \tag{255}$$

and the porosity of the wet soil is

$$\phi = \frac{v_{\text{air}} + v_{\text{water}}}{v_{\text{wetsand}}}\,. \tag{256}$$

Therefore, if the volume of the sand does not change due to the introduction of water,

$$S_w = \frac{1}{\phi}\frac{v_{\text{water}}}{v_{\text{wetsand}}} = \frac{1}{\phi}\frac{v_{\text{water}}}{v_{\text{drysand}}}\,. \tag{257}$$

Plugging the above relation into the expression for α gives us

$$S_w = \frac{\alpha}{\phi}\frac{\rho_{\text{drysand}}}{\rho_{\text{water}}}\,. \tag{258}$$

The mass of the wet sand is

$$m_{\text{wetsand}} = m_{\text{drysand}} + m_{\text{water}} \quad\Longrightarrow\quad \rho_{\text{wetsand}} v_{\text{wetsand}} = \rho_{\text{drysand}} v_{\text{drysand}} + \rho_{\text{water}} v_{\text{water}}\,. \tag{259}$$

Since $v_{\text{wetsand}} = v_{\text{drysand}}$, we have

$$\rho_{\text{wetsand}} = \rho_{\text{drysand}} + \rho_{\text{water}}\frac{v_{\text{water}}}{v_{\text{drysand}}} = \rho_{\text{drysand}} + \phi S_w \rho_{\text{water}}\,. \tag{260}$$

The Kayenta damage model

The Kayenta damage model uses a scalar parameter $D \in [0,1]$ to collapse the limit surface as damage increases. The parameters of the limit surface that are modified are `FSLOPE`, `PEAKI1`, `STREN`, and `YSLOPE`. Let these four parameters be

$$I_1^{\text{peak}} := \texttt{PEAKI1}\,,\quad f^{\text{slope}} := \texttt{FSLOPE}\,,\quad \sqrt{J_2^{\text{coh}}} := \texttt{STREN}\,,\quad y^{\text{slope}} := \texttt{YSLOPE}\,. \tag{261}$$

Let the input undamaged values of these parameters be

$$I_1^{\text{peak,intact}}\,,\quad f^{\text{slope,intact}}\,,\quad \sqrt{J_2^{\text{coh,intact}}}\,,\quad y^{\text{slope,intact}}\,. \tag{262}$$

Also, let the input fully damaged values of the four parameters be

$$I_1^{\text{peak,failed}}\,,\quad f^{\text{slope,failed}}\,,\quad \sqrt{J_2^{\text{coh,failed}}}\,,\quad y^{\text{slope,failed}}\,. \tag{263}$$

Then the current values of the four parameters are computed using linear interpolation between the intact and failed values:

$$\begin{aligned}
I_1^{\text{peak}} &= (1-D) I_1^{\text{peak,intact}} + D I_1^{\text{peak,failed}}\\
f^{\text{slope}} &= (1-D) f^{\text{slope,intact}} + D f^{\text{slope,failed}}\\
\sqrt{J_2^{\text{coh}}} &= (1-D)\sqrt{J_2^{\text{coh,intact}}} + D\sqrt{J_2^{\text{coh,failed}}}\\
y^{\text{slope}} &= (1-D) y^{\text{slope,intact}} + D y^{\text{slope,intact}}\,.
\end{aligned} \tag{264}$$

Also, to incorporate variability and size effects during the failure process, the parameters are scaled by the material point volume using a relation of the form

$$\begin{aligned}
I_1^{\text{peak}} &= I_1^{\text{peak,intact}}\left[\frac{v^{\text{expt}}\ln R}{v^{\text{elem}}\ln(1/2)}\right]^{1/m}\,,\quad f^{\text{slope}} = f^{\text{slope,intact}}\left[\frac{v^{\text{expt}}\ln R}{v^{\text{elem}}\ln(1/2)}\right]^{1/m}\\
\sqrt{J_2^{\text{coh}}} &= \sqrt{J_2^{\text{coh,intact}}}\left[\frac{v^{\text{expt}}\ln R}{v^{\text{elem}}\ln(1/2)}\right]^{1/m}\,,\quad y^{\text{slope}} = y^{\text{slope,intact}}\left[\frac{v^{\text{expt}}\ln R}{v^{\text{elem}}\ln(1/2)}\right]^{1/m}
\end{aligned} \tag{265}$$

where $\{I_1^{\text{peak}}, f^{\text{slope}}, \sqrt{J_2^{\text{coh}}}, y^{\text{slope}}\}^{\text{intact}}$ are mean values from experiments performed on samples that have volume v^{expt}, v^{elem} is the volume of the material point, $R \sim \mathcal{U}[0,1]$ is a uniformly distributed random number, and m is a Weibull modulus.

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At this stage we need a model for the evolution of D. Kayenta uses a time to failure model that depends on two new input parameters: the time to failure (t^{fail}) and the failure speed (f^{speed}), and defines a "coherence" parameter (C) as

$$C := 1 - D = \frac{\exp\left[-f^{\text{speed}}\left(1 - \frac{t^{\text{grow}}}{t^{\text{fail}}}\right)\right]}{1 + \exp\left[-f^{\text{speed}}\left(1 - \frac{t^{\text{grow}}}{t^{\text{fail}}}\right)\right]} \tag{266}$$

where t^{grow} is state variable that increases only when the stress state is on the limit surface. The value of C is never allowed to increase.

Implementing parameter variability

The probability density function of random variable x that is Weibull distributed can be expressed as

$$\text{We}(x) = \beta\alpha x^{\alpha-1}\exp(-\beta x^{\alpha}) \quad \text{for } x \geq 0\,. \tag{267}$$

The expression used in the C++11 standard implementation is

$$\text{We}(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1}\exp\left[-\left(\frac{x}{b}\right)^{a}\right]. \tag{268}$$

The relationship between these two expressions is

$$\alpha \equiv a \quad \text{and} \quad \beta \equiv \frac{1}{b^a} \implies b = \beta^{-1/\alpha}\,. \tag{269}$$

The shape parameter is $a = \alpha > 0$ and the scale parameter is $b > 0$. The shape parameter a is also called the **Weibull modulus** in the content of material strength distribution.

The mean of the distribution is

$$\mathbb{E}(x) = b\Gamma\left(1 + \frac{1}{a}\right) = \beta^{-1/\alpha}\Gamma\left(1 + \frac{1}{\alpha}\right) \tag{270}$$

where Γ is the gamma function. If we assume that the expected value is better represented by the median, we have

$$\mathbb{E}(x) = b\left[\ln(2)\right]^{1/a}\,. \tag{271}$$

The generate the Weibull distribution for a random variable, we typically use a transformation from a uniformly distributed random variable. To find the transformation between two probability distributions $f(y)$ and $g(x)$, we use the fundamental relation

$$f(y) = g(x)\left|\frac{dx}{dy}\right| \tag{272}$$

where the absolute value of the Jacobian of the transformation is used to make sure that probabilities sum to 1. For the special case where the distribution $g(x), x \in \mathcal{U} \sim [0,1]$ is uniform, we have

$$f(y) = \left|\frac{dx}{dy}\right|\,. \tag{273}$$

Therefore,

$$x = \int_0^y f(z)\,dz\,. \tag{274}$$

For the Weibull distribution, the right hand side is the cumulative distribution function,

$$x = \int_0^y \text{We}(z)\,dz = \int_0^y \beta\alpha z^{\alpha-1}\exp(-\beta z^{\alpha})\,dz = 1 - \exp(-\beta y^{\alpha}) = 1 - \exp\left[-\left(\frac{y}{b}\right)^{a}\right]. \tag{275}$$

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This relation can be inverted to give the transformed uniformly distributed random number between 0 and 1:

$$y = \left[-\frac{1}{\beta}\ln(1-x)\right]^{1/\alpha} = b\left[-\ln(1-x)\right]^{1/a} . \tag{276}$$

For a random variable that has the mean $\mathbb{E}(y) \approx \bar{y}$, from (270), the scale parameter is

$$b = \frac{\mathbb{E}(x)}{\Gamma\left(1+\frac{1}{a}\right)} \approx \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)} . \tag{277}$$

Therefore, the Weibull-transformed uniformly distributed random variable can be written as

$$y = \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)}\left[-\ln(1-x)\right]^{1/a} . \tag{278}$$

At this stage one typically invokes the fact that if x is uniformly distributed then so is $1-x$ and we can simplify the computation by using

$$y = \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)}\left[-\ln(x)\right]^{1/a} . \tag{279}$$

Alternatively, we can assume that the sample median is a better approximation of the expected value and use equation (271) to compute the scale parameter:

$$b = \frac{\bar{y}}{\left[\ln(2)\right]^{1/a}} . \tag{280}$$

In that case we have

$$y = \bar{y}\left[-\frac{\ln(x)}{\ln(2)}\right]^{1/a} = \bar{y}\left[\frac{\ln(x)}{\ln(1/2)}\right]^{1/a} . \tag{281}$$

The existing implementation of the Weibull generator in Uintah uses the following approach. A uniformly distributed random number x is generated. This number is used to compute the quantity

$$F = \left[-\ln(x)\right]^{1/a} \tag{282}$$

where a is the Weibull modulus. Two other quantities are computed:

$$C = \left[\frac{v_{\text{expt}}}{v_{\text{elem}}}\right]^{1/m} \quad \text{and} \quad \eta = \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)} \tag{283}$$

where v_{expt} is a reference volume, v_{elem} is the particle volume, m is an exponent, and $\bar{y}$ is the mean value of the parameter (y) that is Weibull distributed. The value of y is computed using the product of F, C, and η, giving

$$y = \left[\frac{v_{\text{expt}}}{v_{\text{elem}}}\right]^{1/m} \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)}\left[-\ln(x)\right]^{1/a} \tag{284}$$

The code typically uses $m = a$ to get

$$y = \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)}\left[-\frac{v_{\text{expt}}}{v_{\text{elem}}}\ln(x)\right]^{1/a} . \tag{285}$$

This expression is identical to equation (279) except for a size-effect factor. Note that (281) is the form used in Scott Swan's thesis (previously implemented in Uintah):

$$y = \left[\frac{v_{\text{expt}}}{v_{\text{elem}}}\right]^{1/a} \bar{y}\left[\frac{\ln x}{\ln(1/2)}\right]^{1/a} . \tag{286}$$

For our purposes, if we use the C++11 Weibull distribution generator, we can incorporate the volume scaling by just multiplying the scaling factor to the number generated, i.e.,

$$y = \left[\frac{v_{\text{expt}}}{v_{\text{elem}}}\right]^{1/m} \text{We}(\bar{y}, a, b, R) \tag{287}$$

where R is the uniformly distributed pseudorandom number in $[0,1]$ generated by the Mersenne twister algorithm. The variability produced in the yield surface is shown in Figure 1.

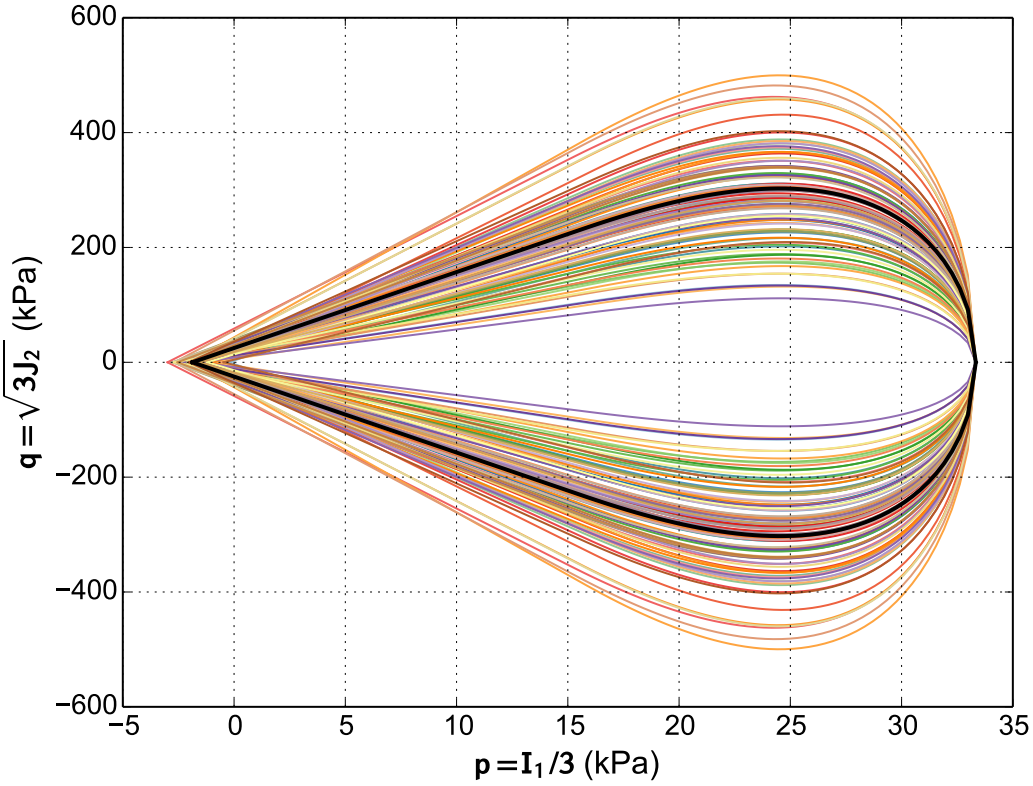


Figure 1 – *Variability in the yield surface by applying volume scaling and Weibull variability in the yield surface parameters. The median yield surface is shown by the black line. The median parameter values are* `PEAKI1` *= 1 kPa,* `FSLOPE` *= 0.453,* `YSLOPE` *= 0.31, STREN = 10 MPa. The element volume is* 10^{-6} *m*^{3} *while the reference volume is* 10^{-3} *m*^{3}*. The Weibull modulus for all the parameters is 4.*

Density-dependence model

We use the Pabst-Gregorova model (Pabst and Gregorová, 2015) to scale the bulk and shear moduli. For the hydrostatic strength we use an ad-hoc model that is calibrated based on the high density dry Mason sand SHPB tests.

Let ϕ_0 be the initial porosity of the test material and let ϕ_{ref} be the initial porosity of the reference material that was used to calibrate the bulk modulus parameters and the crush curve. Following Pabst and Gregorova, we define a modulus scaling factor K_{fac}

$$K_{\text{fac}} = \exp\left[-\frac{\phi_0}{1-\phi_0} + \frac{\phi_{\text{ref}}}{1-\phi_{\text{ref}}}\right] \tag{288}$$

such that the bulk and shear moduli of the test material is

$$K_d \leftarrow K_{\text{fac}} K_d\,, \quad \text{and} \quad G \leftarrow K_{\text{fac}} G\,. \tag{289}$$

These scaled moduli are used in the partially saturated model.

For the hydrostatic strength we use the ad-hoc density scaling model

$$p_1 \leftarrow p_1 \exp\left[\rho_{\text{fac}} K_{\text{fac}} (K_{\text{fac}} - 1)\right] \tag{290}$$

where p_1 is the parameter of the crush curve model, and ρ_{fac} is calibrated using high density data and has a value between 1 and 10.

Plots of the predicted and experimental SHPB test results for a range of initial Mason sand densities are shown in the figures below.

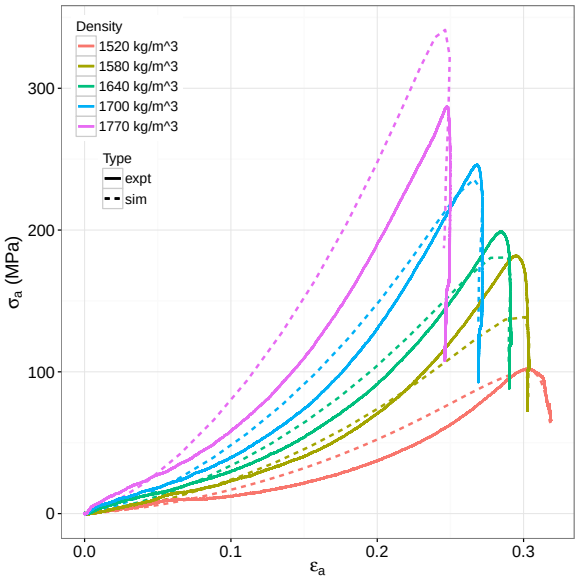
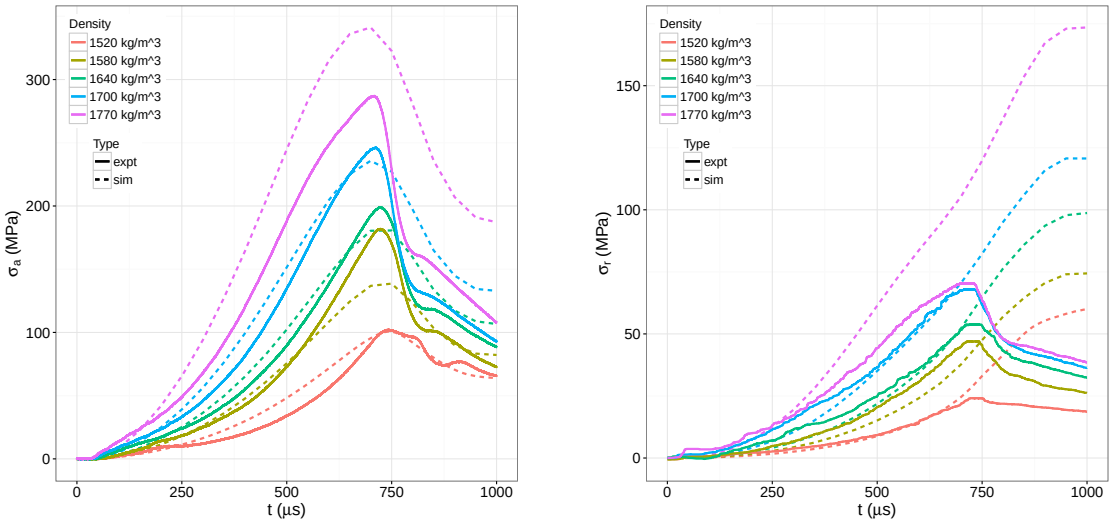


Figure 2 – *Axial stress vs. axial strain in split-Hopkinson pressure bar: dry Mason sand: Various initial densities. We have used* $\phi_{ref} = 0.42$ *and* $\rho_{fac} = 5$.

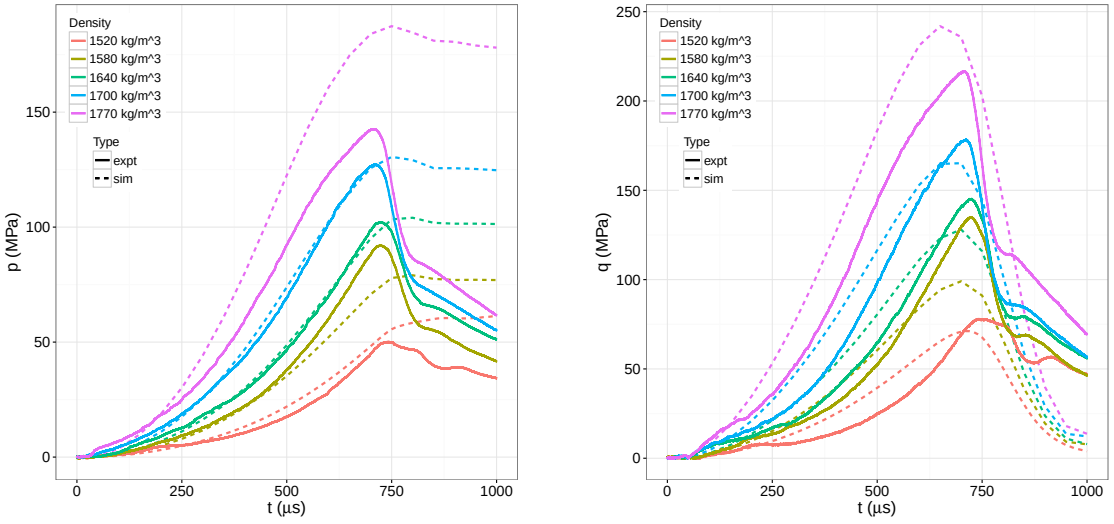
Verifying stress paths for uniaxial strain loading

Most of the simulations discussed in this report have been driven by uniaxial strain. We can check that the code is doing the right thing by comparing the slope of the loading path in p-q space for elastic states. This appendix discusses what we should expect for linearly elastic materials.



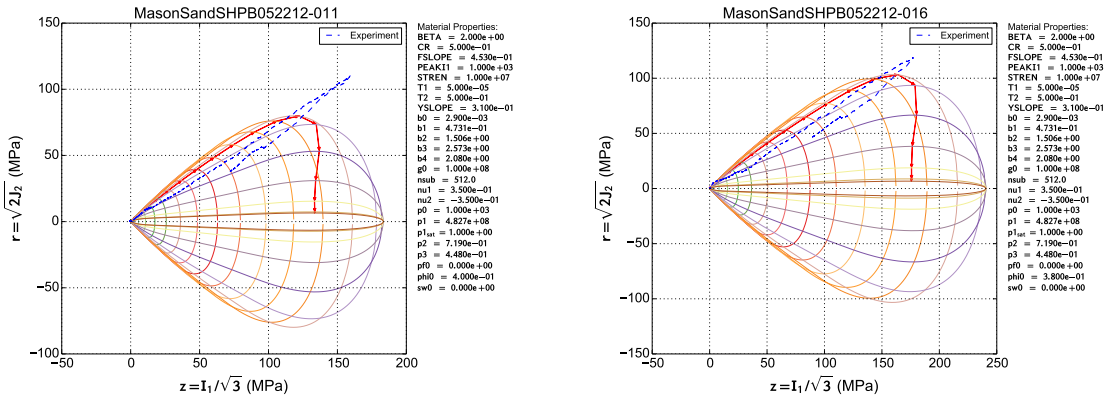
(a) Axial stress vs. time. (b) Radial stress vs. time.

Figure 3 – *Axial stress and radial stress in split-Hopkinson pressure bar: dry Mason sand: Various initial densities*



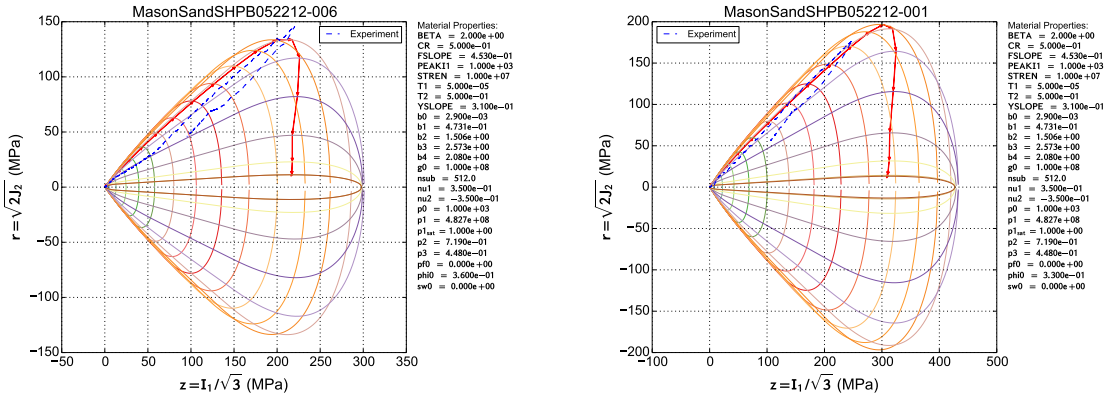
(a) p vs. time. (b) q vs. time.

Figure 4 – *Axial stress and radial stress in split-Hopkinson pressure bar: dry Mason sand: Various initial densities*



(a) 1580 kg/m³ (b) 1640 kg/m³

Figure 5 – *Plot of stress path and yield surface in z-r space. Moderate initial densities*



(a) 1700 kg/m^3 (b) 1770 kg/m^3

Figure 6 – *Plot of stress path and yield surface in z-r space. High initial densities*

From linear elasticity,

$$\boldsymbol{\sigma} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} \tag{291}$$

where

$$\lambda := K - \tfrac{2}{3}G \quad \text{and} \quad \mu := G \implies K = \lambda + \tfrac{2}{3}\mu \tag{292}$$

and K, G are the bulk and shear modulus, respectively. Therefore,

$$\begin{aligned} p &:= \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) = \left(\lambda + \tfrac{2}{3}\mu\right)\mathrm{tr}(\boldsymbol{\varepsilon}) \\ \boldsymbol{s} &:= \boldsymbol{\sigma} - p\boldsymbol{I} = \lambda\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} - \left(\lambda + \tfrac{2}{3}\mu\right)\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} = 2\mu\left[\boldsymbol{\varepsilon} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}\right]. \end{aligned} \tag{293}$$

From the second equation above, we can compute

$$q := \sqrt{3J_2} = \sqrt{\tfrac{3}{2}\boldsymbol{s}:\boldsymbol{s}} = \sqrt{\tfrac{3}{2}(2\mu)^2\left[\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} - \tfrac{1}{3}[\mathrm{tr}(\boldsymbol{\varepsilon})]^2\right]} = 2\mu\sqrt{\tfrac{3}{2}\left[\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} - \tfrac{1}{3}[\mathrm{tr}(\boldsymbol{\varepsilon})]^2\right]}. \tag{294}$$

For uniaxial strain in the 1-direction, $\mathrm{tr}(\boldsymbol{\varepsilon}) = \varepsilon_{11}$ and $\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \varepsilon_{11}^2$, and the above equations for p and q become

$$p = \left(\lambda + \tfrac{2}{3}\mu\right)\varepsilon_{11} = K\varepsilon_{11} \quad \text{and} \quad q = 2G\varepsilon_{11}. \tag{295}$$

Therefore, for uniaxial strain linear elastic deformations starting from zero strain, the slope of the loading path is

$$\frac{q}{p} = \frac{2G}{K} = \frac{2}{K}\frac{3K(1-2\nu)}{2(1+\nu)} = \frac{3(1-2\nu)}{1+\nu}. \tag{296}$$

For $\nu = 0.35$, the slope of the loading path is $\frac{2}{3}$ which can be used as a check of the algorithm.