The Sequence $s_n = \frac{2^{2n+1}+1}{3}$: A Complete Mathematical Analysis

[Author Name]

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Abstract

We provide a rigorous mathematical analysis of the integer sequence defined by $s_n = \frac{2^{2n+1}+1}{3}$ for $n \geq 0$. We prove that this formula is well-defined, establish a linear recurrence relation, and analyze divisibility properties. All results are presented with complete proofs.

1 Introduction and Preliminaries

Definition 1.1 (2-adic valuation)**.** *For a nonzero integer m, the 2-adic valuation $\nu_2(m)$ is the largest integer k such that 2^k divides m. We define $\nu_2(0) = \infty$.*

Definition 1.2 (2-adic absolute value)**.** *For a nonzero integer m, the 2-adic absolute value is $|m|_2 = 2^{-\nu_2(m)}$. We define $|0|_2 = 0$.*

Lemma 1.3. *For all integers $n \geq 0$, we have $2^{2n+1} \equiv 2 \pmod 3$.*

Proof. We have $2 \equiv -1 \pmod 3$, so $2^{2n+1} = 2 \cdot (2^2)^n = 2 \cdot 4^n$. Since $4 \equiv 1 \pmod 3$, we get $2^{2n+1} \equiv 2 \cdot 1^n = 2 \equiv -1 \pmod 3$. Therefore $2^{2n+1} + 1 \equiv 0 \pmod 3$. □

2 Main Results

Theorem 2.1 (Well-defined sequence)**.** *The sequence $s_n = \frac{2^{2n+1}+1}{3}$ consists of positive integers for all $n \geq 0$.*

Proof. By Lemma 1.3, 3 divides $2^{2n+1}+1$, so s_n is an integer. Since $2^{2n+1} > 0$, we have $s_n > 0$. □

Theorem 2.2 (Explicit values)**.** *The first terms of the sequence are:*

- $s_0 = 1$
- $s_1 = 3$
- $s_2 = 11$
- $s_3 = 43$
- $s_4 = 171$

Proof. Direct computation:

$$s_0 = \frac{2^1+1}{3} = \frac{3}{3} = 1 \tag{1}$$
$$s_1 = \frac{2^3+1}{3} = \frac{9}{3} = 3 \tag{2}$$
$$s_2 = \frac{2^5+1}{3} = \frac{33}{3} = 11 \tag{3}$$
$$s_3 = \frac{2^7+1}{3} = \frac{129}{3} = 43 \tag{4}$$
$$s_4 = \frac{2^9+1}{3} = \frac{513}{3} = 171$$

□

Theorem 2.3 (Recurrence relation)**.** *For all $n \geq 0$, we have $s_{n+1} = 4s_n - 1$.*

Proof. Starting from the definition:

$$s_{n+1} = \frac{2^{2(n+1)+1}+1}{3} = \frac{2^{2n+3}+1}{3}$$

We can rewrite $2^{2n+3} = 4 \cdot 2^{2n+1}$. Therefore:

$$s_{n+1} = \frac{4 \cdot 2^{2n+1}+1}{3}$$

From the definition of s_n, we have $3s_n = 2^{2n+1}+1$, which gives $2^{2n+1} = 3s_n - 1$.
Substituting:

$$s_{n+1} = \frac{4(3s_n-1)+1}{3} = \frac{12s_n-4+1}{3} = \frac{12s_n-3}{3} = 4s_n - 1$$

□

Theorem 2.4 (Modular inverse property)**.** *For all $n \geq 0$, s_n is the unique integer with $0 < s_n < 2^{2n+1}$ satisfying $3s_n \equiv 1 \pmod{2^{2n+1}}$.*

Proof. From the definition, $3s_n = 2^{2n+1}+1$, so $3s_n - 1 = 2^{2n+1}$. This means $3s_n \equiv 1 \pmod{2^{2n+1}}$.

Since $s_n = \frac{2^{2n+1}+1}{3} < \frac{2^{2n+1}+2^{2n+1}}{3} = \frac{2\cdot 2^{2n+1}}{3} < 2^{2n+1}$, we have $0 < s_n < 2^{2n+1}$.

For uniqueness: Suppose $3t \equiv 1 \pmod{2^{2n+1}}$ with $0 < t < 2^{2n+1}$. Then $3(s_n - t) \equiv 0 \pmod{2^{2n+1}}$. Since $\gcd(3, 2^{2n+1}) = 1$ (as 3 is odd and 2^{2n+1} is a power of 2), we have $s_n \equiv t \pmod{2^{2n+1}}$. Since both s_n and t lie in the interval $(0, 2^{2n+1})$, we must have $s_n = t$. □

Theorem 2.5 (Divisibility by powers of 2)**.** *For all $n \geq 0$, s_n divides $2^{4n+2} - 1$.*

Proof. We have $3s_n = 2^{2n+1}+1$, so $2^{2n+1} \equiv -1 \pmod{s_n}$. Squaring both sides: $(2^{2n+1})^2 \equiv 1 \pmod{s_n}$. Therefore $2^{4n+2} \equiv 1 \pmod{s_n}$, which means $s_n \mid (2^{4n+2}-1)$. □

Theorem 2.6 (Congruence modulo 8)**.** *We have:*

- *For $n = 0$: $s_0 = 1 \equiv 1 \pmod 8$*
- *For $n = 1$: $s_1 = 3 \equiv 3 \pmod 8$*
- *For $n \geq 2$: $s_n \equiv 3 \pmod 8$*

Proof. For $n \geq 2$, we have $2n+1 \geq 5$, so $2^{2n+1} \equiv 0 \pmod{32}$. Therefore $2^{2n+1}+1 \equiv 1 \pmod{32}$.

We need to find $s_n = \frac{2^{2n+1}+1}{3}$ modulo 8. Since $2^{2n+1}+1 \equiv 1 \pmod{32}$, we can write $2^{2n+1}+1 = 32k+1$ for some integer k.

We need $32k+1 \equiv 0 \pmod 3$, which gives $k \equiv 1 \pmod 3$. So $k = 3m+1$ for some integer m, and $2^{2n+1}+1 = 32(3m+1)+1 = 96m+33$.

Therefore $s_n = \frac{96m+33}{3} = 32m+11 \equiv 11 \equiv 3 \pmod 8$. □

3 Convergence in the 2-adic Integers

Theorem 3.1 (Cauchy sequence)**.** *The sequence $\{s_n\}$ is Cauchy with respect to the 2-adic metric.*

Proof. For $m > n \geq 0$:

$$s_m - s_n = \frac{2^{2m+1}+1}{3} - \frac{2^{2n+1}+1}{3} \tag{5}$$

$$= \frac{2^{2m+1} - 2^{2n+1}}{3} \tag{6}$$

$$= \frac{2^{2n+1}(2^{2(m-n)} - 1)}{3} \tag{7}$$

Since $2^{2(m-n)} - 1$ is odd for $m > n$ (as $2^k - 1$ is odd for all $k \geq 1$), and 3 is odd, the denominator contributes no factors of 2. Therefore $\nu_2(s_m - s_n) = 2n + 1$.

This gives $|s_m - s_n|_2 = 2^{-(2n+1)} \to 0$ as $n \to \infty$. □

4 Base-4 Representation

Theorem 4.1. *For $n \geq 1$, in base-4 notation: s_n consists of $(n-1)$ digits of 2 followed by a single digit 3.*

Proof. We proceed by induction.

Base case ($n = 1$): $s_1 = 3 = 3_4$. ✓

Inductive step: Assume s_n has $(n-1)$ twos followed by a three in base 4.

This means $s_n = 2(4^{n-1} + 4^{n-2} + \cdots + 4 + 1) + 1 = 2 \cdot \frac{4^n-1}{4-1} + 1 = \frac{2\cdot 4^n - 2 + 3}{3} = \frac{2\cdot 4^n+1}{3}$.

Now $s_{n+1} = 4s_n - 1 = 4 \cdot \frac{2\cdot 4^n+1}{3} - 1 = \frac{8\cdot 4^n + 4 - 3}{3} = \frac{8\cdot 4^n+1}{3} = \frac{2\cdot 4^{n+1}+1}{3}$.

This has the form required for $n + 1$, completing the induction. □

5 Open Questions

1. How many terms s_n are prime numbers?
2. Can we characterize all prime divisors of s_n for given n?
3. What is the growth rate of the largest prime factor of s_n?

6 Conclusion

We have provided a complete mathematical characterization of the sequence $s_n = \frac{2^{2n+1}+1}{3}$ with rigorous proofs of all stated properties. The sequence has interesting connections to modular arithmetic and the 2-adic integers.

References

[1] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 2008.

[2] N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, Springer-Verlag, 1984.