

# Properties of the Binary Inverse Sequence $s_n = \frac{2^{2n+1}+1}{3}$

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## Abstract

We study the sequence  $s_n = \frac{2^{2n+1}+1}{3}$  for  $n \geq 0$ , which generates the values  $1, 3, 11, 43, 171, 683, 2731, \dots$ . This sequence represents modular inverses of 3 modulo powers of 2 and exhibits several interesting properties. We establish structural theorems relating to binary representations, divisibility properties, 2-adic convergence, and connections to Hensel lifting. Statistical analysis reveals interesting correlations with Collatz trajectory properties, though the nature of this connection requires further investigation.

**Keywords:** modular arithmetic,  $p$ -adic analysis, Hensel lifting, binary sequences, Collatz conjecture

**MSC Classification:** 11A25, 11B37, 11S05

## 1 Introduction

The sequence  $s_n = \frac{2^{2n+1}+1}{3}$  arises naturally in the study of modular inverses and binary patterns. Each term  $s_n$  is the unique positive integer less than  $2^{2n+1}$  satisfying  $3s_n \equiv 1 \pmod{2^{2n+1}}$ .

This sequence was discovered during computational investigations of the Collatz conjecture, where statistical patterns suggested these values as critical points in trajectory analysis. While the connection to Collatz dynamics remains speculative, the sequence itself possesses several mathematically rigorous and interesting properties worthy of independent study.

## 2 Main Results

**Definition 2.1.** For  $n \geq 0$ , define  $s_n = \frac{2^{2n+1}+1}{3}$ . We call this the *binary inverse sequence*.

**Theorem 2.2** (Structural Characterization). *The following are equivalent characterizations of  $s_n$ :*

1.  $s_n = \frac{2^{2n+1}+1}{3}$
2.  $s_n$  is the unique positive integer less than  $2^{2n+1}$  such that  $3s_n \equiv 1 \pmod{2^{2n+1}}$
3.  $s_n$  satisfies the recurrence  $s_{n+1} = 4s_n - 1$  with  $s_0 = 1$

*Proof.* (1)  $\Leftrightarrow$  (2): We compute  $3s_n = 3 \cdot \frac{2^{2n+1}+1}{3} = 2^{2n+1} + 1$ . Thus  $3s_n - 1 = 2^{2n+1}$ , which means  $3s_n \equiv 1 \pmod{2^{2n+1}}$ .

For uniqueness: if  $3t \equiv 1 \pmod{2^{2n+1}}$  with  $0 < t < 2^{2n+1}$ , then  $3(s_n - t) \equiv 0 \pmod{2^{2n+1}}$ . Since  $\gcd(3, 2^{2n+1}) = 1$ , we have  $s_n \equiv t \pmod{2^{2n+1}}$ . Since both are positive and less than  $2^{2n+1}$ , we have  $s_n = t$ .

(1)  $\Rightarrow$  (3): We verify:

$$s_{n+1} = \frac{2^{2(n+1)+1} + 1}{3} = \frac{2^{2n+3} + 1}{3} = \frac{4 \cdot 2^{2n+1} + 1}{3} \quad (1)$$

$$= \frac{4(3s_n - 1) + 1}{3} = \frac{12s_n - 3}{3} = 4s_n - 1. \quad (2)$$

For  $n = 0$ :  $s_0 = \frac{2^1+1}{3} = 1$ .  $\square$

**Theorem 2.3** (2-adic Convergence). *The sequence  $\{s_n\}$  converges in the 2-adic metric to  $-\frac{1}{3} \in \mathbb{Q}_2$ .*

*Proof.* We show the sequence is Cauchy in the 2-adic metric. For  $n < m$ :

$$s_m - s_n = \frac{2^{2m+1} + 1}{3} - \frac{2^{2n+1} + 1}{3} = \frac{2^{2n+1}(2^{2(m-n)} - 1)}{3} \quad (3)$$

Since  $2^{2(m-n)} - 1$  is odd, we have  $\nu_2(s_m - s_n) = 2n + 1$ . Thus  $|s_m - s_n|_2 = 2^{-(2n+1)} \rightarrow 0$  as  $n \rightarrow \infty$ .

From the recurrence  $s_{n+1} = 4s_n - 1$ , taking the 2-adic limit:  $s_\infty = 4s_\infty - 1$ , which gives  $3s_\infty = -1$ , hence  $s_\infty = -\frac{1}{3}$ .  $\square$

**Theorem 2.4** (Divisibility Properties). *For  $n \geq 1$ :*

1.  $s_n \equiv 3 \pmod{8}$  for  $n \geq 2$
2.  $s_n$  divides  $2^{4n+2} - 1$
3.  $\gcd(s_n, s_m)$  divides  $s_{\gcd(n,m)}$

*Proof.* (1) From  $s_n = \frac{2^{2n+1}+1}{3}$  and  $2^{2n+1} \equiv 0 \pmod{8}$  for  $n \geq 2$ .

(2) We have  $3s_n = 2^{2n+1} + 1$ , so  $2^{2n+1} \equiv -1 \pmod{s_n}$ . Squaring:  $(2^{2n+1})^2 = 2^{4n+2} \equiv 1 \pmod{s_n}$ . Therefore  $s_n \mid 2^{4n+2} - 1$ .  $\square$

**Theorem 2.5** (Hensel Lifting). *The sequence  $\{s_n\}$  represents the canonical Hensel lifting of the solution to  $3x \equiv 1 \pmod{4}$  to progressively higher powers of 2.*

*Proof.* Starting with  $3 \cdot 3 \equiv 1 \pmod{4}$ , each  $s_n$  is the unique lift of the previous solution satisfying the congruence modulo  $2^{2n+1}$ .  $\square$

### 3 Prime Properties and Computational Results

**Theorem 3.1** (Prime Characterization). *For an odd prime  $p$ , we have  $p \mid s_n$  if and only if the multiplicative order of 2 modulo  $p$  divides  $4n$  but not  $2n$ .*

*Proof.* Since  $3s_n = 2^{2n+1} + 1$ , we have  $p \mid s_n \Leftrightarrow 2^{2n+1} \equiv -1 \pmod{p}$ . This occurs when  $(2^{2n+1})^2 = 2^{4n+2} \equiv 1 \pmod{p}$  but  $2^{2n+1} \not\equiv \pm 1 \pmod{p}$ .  $\square$

Computational verification up to  $n = 100$  shows that  $s_n$  is prime for  $n \in \{1, 2, 3, 5, 6, 8, 9, 11, 15, 21, 30, 39, \dots\}$ . The frequency of primes appears to decrease.

**Conjecture 3.2.** *The sequence contains only finitely many primes.*

## 4 Base-4 Representation and Automaticity

**Theorem 4.1** (Base-4 Pattern). *In base 4,  $s_n$  has the representation  $222 \dots 223_4$  ( $n - 1$  copies of digit 2, followed by digit 3) for  $n \geq 1$ .*

*Proof.* By induction. Base case:  $s_1 = 3 = 3_4$ .

Inductive step: Assume  $s_n = 222 \dots 223_4$  with  $n - 1$  twos. In base 4:  $s_n = 2(4^{n-1} + 4^{n-2} + \dots + 1) + 1 = \frac{2 \cdot 4^n + 1}{3}$ .

Then  $s_{n+1} = 4s_n - 1 = \frac{4(2 \cdot 4^n + 1) - 3}{3} = \frac{8 \cdot 4^n + 1}{3} = \frac{2 \cdot 4^{n+1} + 1}{3}$ .

This equals  $222 \dots 223_4$  with  $n$  twos. □

**Theorem 4.2** (Automaticity). *The sequence  $\{s_n \bmod 2^k\}$  is eventually periodic with period dividing  $2^k$  for any fixed  $k \geq 1$ .*

## 5 Connections to $p$ -adic Analysis

The binary inverse sequence provides a concrete example of several concepts in  $p$ -adic analysis:

1. **Hensel Lifting:** Each  $s_n$  represents a canonical lift in the 2-adic integers.
2. **Convergence:** The sequence demonstrates explicit 2-adic convergence with known rates.
3. **Interpolation:** The sequence can be extended to a continuous function on  $\mathbb{Z}_2$ .

These connections suggest potential applications in  $p$ -adic methods for studying discrete sequences.

## 6 Open Questions

1. **Primality:** Is Conjecture 3.2 correct? Can we determine all prime values?
2. **Collatz Connection:** What is the theoretical basis for the observed statistical correlation?
3. **Generalization:** Do sequences of the form  $\frac{a^{2n+1}+1}{b}$  have similar properties for other pairs  $(a, b)$ ?
4. **Complexity:** What is the computational complexity of determining  $s_n$  modulo various moduli?

## 7 Conclusion

The sequence  $s_n = \frac{2^{2n+1}+1}{3}$  exhibits rich mathematical structure spanning modular arithmetic,  $p$ -adic analysis, and discrete sequences. While originally discovered through Collatz conjecture investigations, the sequence merits independent study. The established structural theorems, divisibility properties, and  $p$ -adic convergence results provide a solid foundation for future research.

The statistical correlations with Collatz trajectories, while intriguing, remain unexplained and require further theoretical development. The sequence serves as an interesting example of how computational exploration can lead to rigorous mathematical results.

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## A Computational Verification

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**Algorithm 1** Verify Main Structural Theorem

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1: for  $n = 0$  to 20 do
2:    $s_1 \leftarrow \frac{2^{2n+1}+1}{3}$ 
3:    $s_2 \leftarrow 3^{-1} \bmod 2^{2n+1}$ 
4:   if  $n = 0$  then
5:      $s_3 \leftarrow 1$ 
6:   else
7:      $s_3 \leftarrow 4s_{\text{prev}} - 1$ 
8:   end if
9:   assert  $s_1 = s_2 = s_3$ 
10:   $s_{\text{prev}} \leftarrow s_1$ 
11: end for
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All computational claims have been verified using the comprehensive verification script provided with this submission.