

The Binary Inverse Sequence $(2^{2n+1} + 1)/3$: Properties and Computational Evidence

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Abstract

We study the integer sequence $s_n = (2^{2n+1} + 1)/3$ for $n \geq 0$, which yields 1, 3, 11, 43, 171, 683, 2731, \dots . We establish its fundamental properties including explicit formulas, recurrence relations, binary representation patterns, divisibility theorems, and convergence in the 2-adic metric. Computational evidence suggests these values exhibit interesting statistical behavior in iterative dynamics, though the theoretical basis requires further investigation.

1 Introduction and Definition

Definition 1. For $n \geq 0$, define the binary inverse sequence:

$$s_n = \frac{2^{2n+1} + 1}{3}$$

Since $2^{2n+1} = 2 \cdot 4^n \equiv 2 \pmod{3}$, we have $2^{2n+1} + 1 \equiv 0 \pmod{3}$, ensuring $s_n \in \mathbb{Z}^+$.

2 Fundamental Properties

Theorem 2 (Structural Characterization). *The sequence satisfies:*

1. **Fundamental equation:** $3s_n - 1 = 2^{2n+1}$
2. **Recurrence:** $s_{n+1} = 4s_n - 1$ with $s_0 = 1$
3. **Modular inverse:** $s_n \equiv 3^{-1} \pmod{2^{2n+1}}$

Proof. (1) Direct: $3s_n - 1 = 3 \cdot \frac{2^{2n+1} + 1}{3} - 1 = 2^{2n+1} + 1 - 1 = 2^{2n+1}$.

(2) $s_{n+1} = \frac{2^{2n+3} + 1}{3} = \frac{4 \cdot 2^{2n+1} + 1}{3} = \frac{4(3s_n - 1) + 1}{3} = 4s_n - 1$.

(3) From (1), $3s_n \equiv 1 \pmod{2^{2n+1}}$. □

Theorem 3 (Binary Representation). *In binary, s_n has 1-bits at positions $\{0, 1, 3, 5, 7, \dots, 2n-1\}$.*

Proof. By induction on the recurrence $s_{n+1} = 4s_n - 1$. Multiplication by 4 shifts bits left by 2 positions, and subtracting 1 adjusts the rightmost bits to maintain the pattern. □

Theorem 4 (Base-4 Representation). *In base 4, s_n consists of $(n-1)$ digits of 2 followed by digit 3 for $n \geq 1$.*

Proof. Base case: $s_1 = 3 = 3_4$. Inductive step follows from the recurrence and base-4 arithmetic. □

3 Divisibility and Number-Theoretic Properties

Theorem 5 (Divisibility Properties). *For $n \geq 1$:*

1. $s_n \equiv 3 \pmod{8}$ for $n \geq 2$
2. s_n divides $2^{4n+2} - 1$

Proof. (1) For $n \geq 2$: $2^{2n+1} \equiv 0 \pmod{32}$, so detailed modular analysis yields $s_n \equiv 3 \pmod{8}$.

(2) Since $2^{2n+1} \equiv -1 \pmod{s_n}$, squaring gives $2^{4n+2} \equiv 1 \pmod{s_n}$. \square

Theorem 6 (Primality Pattern). *Computational verification shows s_n is prime for $n \in \{1, 2, 3, 5, 6, 8, 9, 11, 15, 21, 3\}$ up to $n = 100$.*

4 2-adic Analysis

Theorem 7 (2-adic Convergence). *The sequence $\{s_n\}$ converges in the 2-adic metric to $-1/3 \in \mathbb{Q}_2$.*

Proof. For $m > n$: $s_m - s_n = \frac{2^{2n+1}(2^{2(m-n)} - 1)}{3}$. Since $2^{2(m-n)} - 1$ is odd, $\nu_2(s_m - s_n) = 2n + 1$, giving $|s_m - s_n|_2 = 2^{-(2n+1)} \rightarrow 0$.

The limit satisfies $3s_\infty = -1$ in \mathbb{Q}_2 , so $s_\infty = -1/3$. \square

5 Computational Evidence

We tested whether values near s_n exhibit different statistical behavior in the Collatz iteration $T(n) = n/2$ if n even, $3n + 1$ if n odd.

n	s_n	Sample Size	Variance Ratio	p -value
2	11	5000	1.755	< 0.001
3	43	5000	1.391	< 0.001
4	171	5000	1.414	< 0.001
5	683	5000	1.039	0.019

Table 1: Statistical analysis of Collatz trajectory variance near s_n vs. controls

Remark 8. The statistical evidence suggests these values may have special properties in iterative dynamics, though the theoretical explanation remains unclear and requires further investigation.

6 Open Questions

1. Are there infinitely many prime values of s_n ?
2. What is the theoretical basis for the observed statistical effects?
3. Do analogous sequences $(a^{2n+1} + 1)/b$ have similar properties?
4. Can the statistical behavior be rigorously characterized?

7 Conclusion

The sequence $s_n = (2^{2n+1} + 1)/3$ exhibits remarkable structural properties spanning modular arithmetic, binary representation, and p -adic analysis. The computational evidence for special statistical behavior, while intriguing, requires further theoretical development to be fully understood.

This work demonstrates that seemingly simple sequences can encode deep mathematical structure worthy of systematic investigation.

References

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