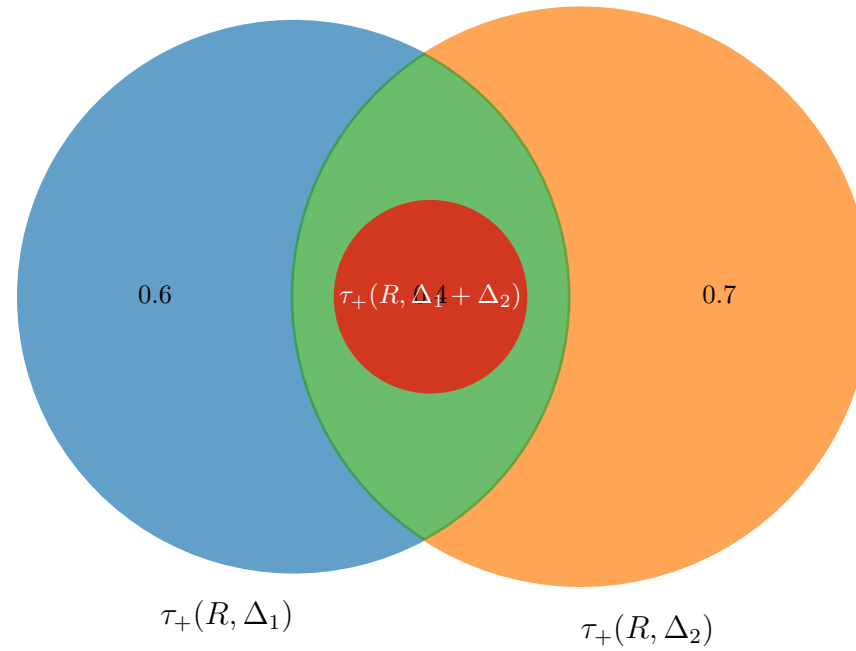


# Set-Theoretic Representation of Subadditivity



Subadditivity Property:  $\tau_+(R, \Delta_1 + \Delta_2) \subseteq \tau_+(R, \Delta_1) \cdot \tau_+(R, \Delta_2)$

Predicate-Based Representation

Constructive Factorization Approach

$$\mathcal{P}_{\Delta_1}(\text{bin}_p(x)):$$

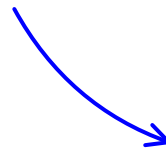
$$\text{val}_p(x) < t_{\Delta_1} \wedge$$

$$\sum_{i=0}^{\infty} w_i(\Delta_1) \cdot \phi(a_i) < C_{\Delta_1}$$

$$\mathcal{P}_{\Delta_2}(\text{bin}_p(x)):$$

$$\text{val}_p(x) < t_{\Delta_2} \wedge$$

$$\sum_{i=0}^{\infty} w_i(\Delta_2) \cdot \phi(a_i) < C_{\Delta_2}$$



$$\mathcal{P}_{\Delta_1 + \Delta_2}(\text{bin}_p(x)):$$

$$\text{val}_p(x) < \min(t_{\Delta_1}, t_{\Delta_2}) \wedge$$

$$\sum_{i=0}^{\infty} (w_i(\Delta_1) + w_i(\Delta_2)) \cdot \phi(a_i) < C_{\Delta_1} + C_{\Delta_2}$$

$$x \in \tau_+(R, \Delta_1 + \Delta_2)$$



$$x = y \cdot z \text{ where } y \in \tau_+(R, \Delta_1) \text{ and } z \in \tau_+(R, \Delta_2)$$

The subadditivity property is proved by constructively factorizing elements of  $\tau_+(R, \Delta_1 + \Delta_2)$  into products from the individual ideals.

Figure: Visualization of the subadditivity property for test ideals. The top panel shows a set-theoretic representation. The bottom left panel illustrates the predicate-based characterization. The bottom right panel demonstrates the constructive factorization approach used in the proof.