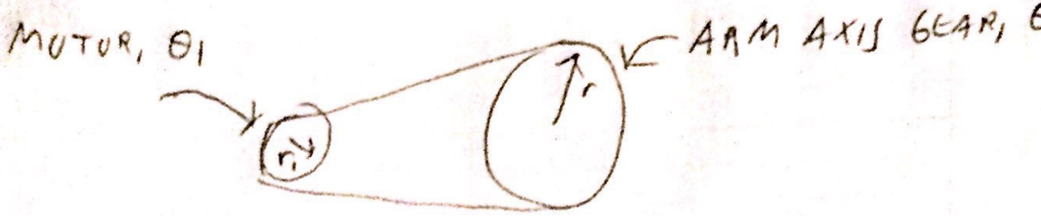
1.) OUR PEN PLOTTER MAKES USE OF POLAR COORDINATES. HOWEVER, THE POLAR COORDINATES ARE CONTROLLED BY THE ROTATION OF TWO MOTORS, EACH CONTROLLING A DEGREE OF FREEDOM. THUS, WE NEED TO CONVERT OUR POLAR COORDINATES INTO MOTOR ANGLES.

$\theta \rightarrow$ MOTOR 1 ANGLE:

SINCE WE HAVE A GEAR REDUCTION BETWEEN THE MOTOR AND ARM AXIS, $\theta \neq \theta_1$. WE DERIVE THE RELATION BELOW:



SINCE ALL PARTS OF THE PULLEY HAVE THE SAME VELOCITY,

$$v_1 = v$$

$$r_1\dot{\theta}_1 = r\dot{\theta}$$

$$\dot{\theta}_1 = (r/r_1)\dot{\theta}$$

(INTEGRATE)

$$\theta_1 = (r/r_1)\,\theta$$

$$\underline{\theta_1 = g\theta} \quad \left(\text{OR } \theta = \tfrac{1}{g}\theta_1\right)$$

WHERE g IS THE GEAR RATIO

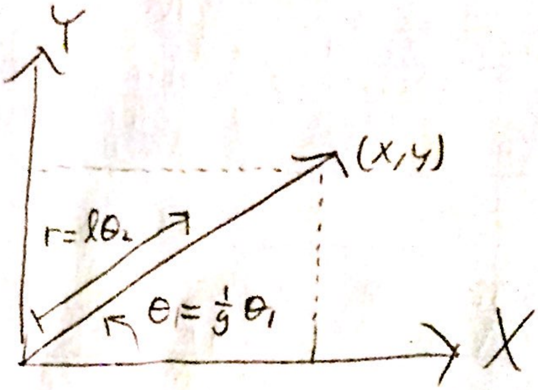
$r \rightarrow$ MOTOR 2 ANGLE:

A LEADSCREW IS ATTACHED TO MOTOR 2 SUCH THAT ROTATING THE SCREW ADVANCES THE PEN. ONE FULL ROTATION OF THE MOTOR ADVANCES THE PEN BY THE SCREW'S PITCH IN THE r DIRECTION. THUS,

$$\underline{r = l\theta_2}$$

WHERE l IS THE PITCH OF THE SCREW IN UNITS OF mm/rad.

A DIAGRAM OF THE SYSTEM IS SHOWN BELOW:



2. TO DEFINE THE FORWARD KINEMATICS, WE CAN WRITE

$$\underline{x} = \underline{f}(\underline{\theta})$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l\theta_2\cos(\frac{1}{g}\theta_1) \\ l\theta_2\sin(\frac{1}{g}\theta_1) \end{bmatrix}}$$

3. TO PREPARE FOR INVERSE KINEMATICS, WE COMPUTE THE JACOBIAN.

$$\frac{\partial \underline{f}}{\partial \underline{\theta}} = \begin{bmatrix} \partial f_1/\partial\theta_1 & \partial f_1/\partial\theta_2 \\ \partial f_2/\partial\theta_1 & \partial f_2/\partial\theta_2 \end{bmatrix}$$

$$\boxed{\frac{\partial \underline{f}}{\partial \underline{\theta}} = \begin{bmatrix} -\frac{l}{g}\theta_2\sin(\frac{1}{g}\theta_1) & l\cos(\frac{1}{g}\theta_1) \\ \frac{l}{g}\theta_2\cos(\frac{1}{g}\theta_1) & l\sin(\frac{1}{g}\theta_1) \end{bmatrix}}$$

4. WE NOW WANT THE VELOCITY OF THE END-EFFECTOR.

$$\dot{\underline{x}} = \frac{d}{dt}\underline{f}(\underline{\theta})$$

$$\dot{\underline{x}} = \frac{\partial \underline{f}}{\partial \underline{\theta}}\dot{\underline{\theta}}$$

$$\dot{\underline{x}} = \begin{bmatrix} -\frac{l}{g}\theta_2\sin(\frac{1}{g}\theta_1) & l\cos(\frac{1}{g}\theta_1) \\ \frac{l}{g}\theta_2\cos(\frac{1}{g}\theta_1) & l\sin(\frac{1}{g}\theta_1) \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\boxed{\dot{\underline{x}} = \begin{bmatrix} -\frac{l}{g}\dot{\theta}_1\theta_2\sin(\frac{1}{g}\theta_1) + l\dot{\theta}_2\cos(\frac{1}{g}\theta_1) \\ \frac{l}{g}\dot{\theta}_1\theta_2\cos(\frac{1}{g}\theta_1) + l\dot{\theta}_2\sin(\frac{1}{g}\theta_1) \end{bmatrix}}$$

5. WE NOW WRITE THE MATRIX EQUATION RESPONSIBLE FOR CARRYING OUT THE NEWTON-RAPHSON ALGORITHM.

FIRST, DEFINE

$$\underline{g}(\underline{\theta}) = \underline{X} - \underline{f}(\underline{\theta})$$

$$\underline{g}(\underline{\theta}) = \begin{bmatrix} x - l\,\theta_2 \cos(\frac{1}{g}\theta_1) \\ y - l\,\theta_2 \sin(\frac{1}{g}\theta_1) \end{bmatrix}$$

WE WILL ALSO WANT $d\underline{g}/d\underline{\theta}$. SINCE THE POINT OF THE ALGORITHM IS TO FIND THE θ_1 & θ_2 THAT GIVE $\underline{X}$, THEN $\underline{X}$ MUST BE CONSTANT FOR A RUN OF THE ALGORITHM. THUS,

$$\frac{d\underline{g}}{d\underline{\theta}} = -\frac{d\underline{f}}{d\underline{\theta}}$$

$$\frac{d\underline{g}}{d\underline{\theta}} = \begin{bmatrix} \frac{l}{g}\theta_2 \sin(\frac{1}{g}\theta_1) & -l\cos(\frac{1}{g}\theta_1) \\ -\frac{l}{g}\theta_2\cos(\frac{1}{g}\theta_1) & -l\sin(\frac{1}{g}\theta_1) \end{bmatrix}$$

THE ALGORITHM WORKS AS FOLLOWS:

$$\underline{\theta}_{n+1} = \underline{\theta}_n - \left(\frac{d\underline{g}(\underline{\theta}_n)}{d\underline{\theta}}\right)^{-1} \underline{g}(\underline{\theta}_n)$$

$$\boxed{\begin{bmatrix} \theta_{1,n+1} \\ \theta_{2,n+1} \end{bmatrix} = \begin{bmatrix} \theta_{1,n} \\ \theta_{2,n} \end{bmatrix} - \left(\frac{d\underline{g}(\underline{\theta}_n)}{d\underline{\theta}}\right)^{-1} \begin{bmatrix} x - l\,\theta_{2,n}\cos(\frac{1}{g}\theta_{1,n}) \\ y - l\,\theta_{2,n}\sin(\frac{1}{g}\theta_{1,n}) \end{bmatrix}}$$