SPH Notes

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Abstract

These are my notes for Fluid Mechanics Simulation using SPH

1 Kernel

The following is how the kernel function is abstracted

$$w_h^{(k)}(\tilde{\mathbf{r}}) = \frac{f^k(q)}{h^{d+k}}$$

$$q = \frac{|\tilde{\mathbf{r}}|}{h}$$

$$\nabla w_h^{(k)}(\tilde{\mathbf{r}}) = w_h^{(k)}(\tilde{\mathbf{r}}) \frac{\tilde{\mathbf{r}}}{|\tilde{\mathbf{r}}|}$$

$$\tilde{\mathbf{r}}_{ab} = \mathbf{r_a} - \mathbf{r_b}$$

The function f here is up to choice, and determines the coefficients. I will always default to Wendland kernel, as it is the most regular one from the literature I have been reading.

$$f_{W,d}(q) = a_{W,d} \begin{cases} \left(1 - \frac{q}{2}\right)^4 (1+2q) & 0 \geq \text{q} \geq 2 \\ 0 & 2 < \text{q} \end{cases}$$

$$f'_{W,d}(q) = a_{W,d} \begin{cases} \left(1 - \frac{q}{2}\right)^3 (-5q) & 0 \geq q \geq 2 \\ 0 & 2 < q \end{cases}$$

$$\alpha_{W,2} = \frac{7}{4\pi}$$

$$\alpha_{W,2} = \frac{21}{16\pi}$$

2 Solid Wall Particles

Walls can be made with solid particles. Paper I'm following uses twice as dense particles for the edges. The type of force used is the Lennard-Jones potential;

$$\mathbf{F}_{a,b} = \begin{cases} D\left[\left(\frac{r_0}{r_{ab}}\right)^{n_1} - \left(\frac{r_0}{r_{ab}}\right)^{n_2}\right] & r_{ab} < r_0 \\ 0 & r_0 \leq r_{ab} \end{cases}$$

Which has 4 parameters. ($n_1 = 12$, $n_2 = 4$, D, r_0)

3 Integration Scheme

In terms of numerical simulations, a symplectic scheme seems to be the best. I will usually default to the Störmer-Verlet scheme

$$\mathbf{r}_a^{m+1/2} = \mathbf{r}_a^m + \mathbf{u}_a^m \frac{\delta t}{2}$$

$$\mathbf{u}_a^{m+1} = \mathbf{r}_a^m + \frac{1}{m_a}\mathbf{F}_a^{m+1/2}\delta t$$

$$\mathbf{r}_a^{m+1} = \mathbf{r}_a^{m+1/2} + \mathbf{u}_a^{m+1} \frac{\delta t}{2}$$

4 Interpolators

There are multiple interpolation operators, and different ones can be used. All these formulas refer to the volume; which is

$$V_a = \frac{m_a}{\rho_a}$$

Field interpolator is simple;

$$J\left\{A_a\right\} := \left\{\sum_b V_b A_b w_{ab}\right\} \approx \left\{A_a\right\}$$

There are specific operators, gradient and divergence for tensors.

$$\mathbf{G}\left\{A_a\right\} := \left\{\sum_b V_b A_b w'_{ab} \mathbf{e}_{ab}\right\} \approx \left\{\nabla A_a\right\}$$

$$D\left\{\mathbf{A}_a\right\} := \left\{\sum_b V_b \mathbf{A}_b \cdot w'_{ab} \mathbf{e}_{ab}\right\} \approx \left\{\nabla \cdot \mathbf{A}_a\right\}$$

4.1 Variants of Interpolation Operators

There are operations that leave the differential operators invariant, if applied on the continuous fields. New types of gradient fields can be constructed out of these.

$$\mathbf{G}_a^k\left\{A_i\right\} := \left\{\sum_b V_b \frac{\rho_b^{2k} A_a + \rho_a^{2k} A_b}{\left(\rho_a \rho_b\right)^k} w'_{ab} \mathbf{e}_{ab}\right\} \approx \left\{\nabla A_a\right\}$$

$$\widetilde{\mathbf{G}}_a^k\left\{A_i\right\} := -\frac{1}{\rho_a^{2k}}\left\{\sum_b V_b(\rho_a\rho_b)^k\left(A_a - A_b\right)w_{ab}^{'}\mathbf{e}_{ab}\right\} \approx \left\{\nabla A_a\right\}$$

And the new divergence operators are;

$$D_i^k\left\{\mathbf{A}_a\right\} := \left\{\sum_b V_b\frac{\rho_b^{2k}\mathbf{A}_a + \rho_a^{2k}\mathbf{A}_b}{(\rho_a\rho_b)^k}\cdot w_{ab}^{'}\mathbf{e}_{ab}\right\} \approx \left\{\nabla\cdot\mathbf{A}_a\right\}$$

$$\widetilde{D}_i^k\left\{\mathbf{A}_a\right\} := -\frac{1}{\rho_a^{2k}}\left\{\sum_b V_b(\rho_a\rho_b)^k\left(\mathbf{A}_a - \mathbf{A}_b\right)\cdot w_{ab}^{'}\mathbf{e}_{ab}\right\} \approx \left\{\nabla\cdot\mathbf{A}_a\right\}$$

Depending on their symmetry properties, some are more appropriate for others. Do note that D^k and $-\widetilde{\mathbf{G}}^k$ are adjoint. (Likewise, $\widetilde{D}^k$ is adjoint with $-\mathbf{G}^k$) So the tilde operator should be used along with the non-tilde variant.

4.2 Renormalization

Since particles are discreet, the kernel sum is not unity. (Expeccially not around boundaries)

$$\gamma_a := J_a\left\{1\right\} = \sum_b V_b w_{ab}$$

$$\gamma_a^{'} := \frac{\partial\gamma_a}{\partial\mathbf{r}_a} = \sum_b V_b w_{ab}^{'}\mathbf{e}_{\mathbf{ab}} = \mathbf{G}_a\left\{1\right\}$$

We can correct for this, and define a new *renormalized* operator.

$$J_a^{\gamma}\left\{A_i\right\} := \frac{1}{\gamma_a}\sum_b V_b w_{ab}$$

Which is nice since $J_a^{\gamma}\left\{1\right\} = 1$. There are also corresponding corrected operators.

$$\mathbf{G}_a^{\gamma}\left\{A_i\right\} := \frac{1}{\gamma_a}\left(\mathbf{G}_a\left\{A_i\right\} - J_a^{\gamma}\left\{A_i\right\}\mathbf{G}_a\left\{1\right\}\right)$$

$$D_a^{\gamma}\left\{\mathbf{A}_i\right\} := \frac{1}{\gamma_a}\left(D_a\left\{A_i\right\} - J_a^{\gamma}\left\{A_i\right\}D_a\left\{\mathbf{1}\right\}\right)$$

The k-operators lose symmetry, hence they cannot preserve flux. They are dealt with differently.

5 Weakly Compressible Scheme

Violeau's Lid-Driven Cavity Flow details the use of weakly compressible scheme. For density evolution the operator $\widetilde{D}^k$ is used to ensure no flux between comoving points.

$$\frac{d\rho_a}{dt} = \frac{1}{\rho_a^{2k-1}}\sum_b V_b(\rho_a\rho_b)\mathbf{u}_{ab}\cdot w_{ab}^{'}\mathbf{e}_{ab}$$

Since $\widetilde{D}^k$ is used in the density equation, for interparticle force $\mathbf{G}^k$ will be used

$$\mathbf{F}_{b\to a}^{int} = -\mathbf{F}_{b\to a}^{int} = -m_a m_b \frac{\rho_b^{2k} p_a + \rho_a^{2k} p_b}{\left(\rho_a \rho_b\right)^{k+1}} w_{ab}^{'} \mathbf{e}_{ab}$$

Pressure is obtained from the Taut equation

$$p_a = \frac{\rho_0 c_0^2}{\gamma} \left[\left(\frac{\rho_a}{\rho_0} \right)^{\gamma} + C \right]$$

The dissipative forces are

$$\mathbf{F}_{b\to a}^{diss} = -\mathbf{F}_{a\to b}^{diss} = 2\left(d+2\right) \bar{\mu}_{ab} V_a V_b \frac{\mathbf{u}_{ab} \cdot \mathbf{e}_{ab}}{r_{ab}} w_{ab}^{'} \mathbf{e}_{ab}$$

Obviously, the update equations become clear

$$m_a \frac{d\mathbf{u}_a}{dt} = \sum_b \left(\mathbf{F}_{b\to a}^{int} + \mathbf{F}_{b\to a}^{diss} \right) + \mathbf{F}_a^{ext}$$