ECE 594 D project

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1 Pigou's network with Biases

Consider a model of flow where units of flow have a bias for each potential route.
Let this bias be characterized as a Bernoulli process such that the bias for a route i is either a_i or b_i with probability p_i.
Pigous network presents 2 routes, r_1 and r_2.
The cost on each edge is:

$$c_1(x) = x$$

$$c_2(x) = 1$$

We can denote the effect of the biases in aggregate by $B_i(x)$ meaning the contribution of a bias given a certain amount of flow.
The Nash equilibrium is then

$$\min_x \int_0^x (c_1(x) + B_1(x))dx + \int_0^{1-x} (c_2(x) + B_2(x))dx$$

Which can be minimized by setting the derivative to 0 and solving:

$$c_1(x) + B_1(x) + c_2(1-x) + B_2(1-x)$$

Then note that the bias function is unknown.
Because the biases are Bernoulli distributions, the population falls into one of four classes:

$$P(a_1 a_2) = p_1 p_2$$

$$P(a_1 b_2) = p_1(1-p_2)$$

$$P(b_1 a_2) = (1-p_1)p_2$$

$$P(b_1 b_2) = (1-p_1)(1-p_2)$$

Then we can separate the flow based on which section of the population it belongs to yielding A vector equation:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Where the bounds:

$$0 \leq x_1 \leq p_1 p_2$$

$$0 \leq x_2 \leq p_1(1 - p_2)$$

$$0 \leq x_3 \leq (1 - p_1)p_2$$

$$0 \leq x_4 \leq (1 - p_1)(1 - p_2)$$

or

$$v = \begin{bmatrix} p_1 p_2 \\ p_1(1 - p_2) \\ (1 - p_1)p_2 \\ (1 - p_1)(1 - p_2) \end{bmatrix}$$

$$0 \leq x \leq v$$

Hold.

$$B_1(x) = [a_1, a_1, b_1, b_1] \cdot x$$

$$B_1(x) = [a_2, b_2, a_2, b_2] \cdot x$$

define

$$B_1 = [a_1, a_1, b_1, b_1]$$

$$B_2 = [a_2, b_2, a_2, b_2]$$

Which yields a new cost function:

$$c_1(x) = [1, 1, 1, 1] \cdot x$$

$$c_2(x) = 1$$

The total traffic along each branch is bounded by:

$$\min_x \int_0^x (c_1(x) + B_1(x))dx + \int_0^{v-x} (c_2(x) + B_2(x))dx$$

Assuming this vector valued integral is possible (Blithely) Then taking the derivative and setting it to 0 we get:

$$c_1(x) + B_1(x) + c_2(x - v) + B_2(x - v) = 0$$

$$[1, 1, 1, 1] \cdot x + B_1 \cdot x + B_2 \cdot (x - v) - 1 = 0$$

$$[1,1,1,1]\cdot x + B_1 \cdot x - B_2 \cdot v + B_2 \cdot x - 1 = 0$$
$$(1 + B_1 + B_2)\cdot x - b_2 \cdot v - 1 = 0$$

$$b_2 \cdot v = a_2(p_1 p_2) + b_2 p_1(1-p_2) + a_2(1-p_1)p_2 + b_2(1-p_1)(1-p_2)$$
$$= a_2 p_2(p_1 + 1 - p_1) + b_2(1-p_2)(p_1 + 1 - p_1) = a_2 p_2 + b_2(1-p_2)$$

$$(1 + B_1 + B_2)\cdot x - (a_2 p_2 + b_2(1-p_2)) = 1$$

2

New theories: bias can't be joined as a function of traffic? Should be possible. A person applies their specific bias, so other people do not influence the performance of the bias: the Nash equilibria is determined by my individual bias and the current allocation of people. Then traffic is constrained by a new set of constraints: No more traffic of a type x_1, x_2, x_3, x_4 can flow on branch 1 if this inequality fails:

$$x_1 : x_1 + x_2 + x_3 + x_4 + a_1 < 1 + a2$$

$$x_2 : x_1 + x_2 + x_3 + x_4 + a_1 < 1 + b2$$
$$x_3 : x_1 + x_2 + x_3 + x_4 + b_1 < 1 + a2$$
$$x_4 : x_1 + x_2 + x_3 + x_4 + b_1 < 1 + b2$$

That is to say if the total cost of path 1 including the traffic types bias exceeds that of path 2, no more traffic of that type can flow.

$$x_1 : x_1 + x_2 + x_3 + x_4 < 1 + a_2 - a_1$$
$$x_2 : x_1 + x_2 + x_3 + x_4 < 1 + b_2 - a_1$$
$$x_3 : x_1 + x_2 + x_3 + x_4 < 1 + a_2 - b_1$$
$$x_4 : x_1 + x_2 + x_3 + x_4 < 1 + b_2 - b_1$$

Then these constraints are still subject to

$$0 \le x \le v$$

as well.
The Nash equilibria are any sets which satisfy this constraint.
The optimal Nash equilibria would be one which minimizes the global cost as well, $\min_{x\in NE} c_1(x) + c_2(1-x)$.

This might be mostly irrelevant? can't actually force the evolution of an optimal NE. Then in this model of Constraining the NE a tax function manifests in the form:

$$x_1 : c_1(x) + t(x) < c_2(1-x) + t_2(1-x) + a_2 - a_1$$

$$x_2 : c_1(x) + t(x) < c_2(1-x) + t_2(1-x) + b_2 - a_1$$

$$x_3 : c_1(x) + t(x) < c_2(1-x) + t_2(1-x) + a_2 - b_1$$

$$x_4 : c_1(x) + t(x) < c_2(1-x) + t_2(1-x) + b_2 - b_1$$

This can be characterized as $\mathcal{N}(X, \vec{c}, \vec{t}, \vec{b})$
Where:
X is the matrix of edges $\times$ bias states for all the edges of the network.
$\vec{c}$ is the vector of cost functions for each edge
$\vec{t}$ is the vector of tax functions for each edge
$\vec{b}$ is the vector of bias functions for each edge

Then the goal of a tax is to minimize this feasible region over the game until this holds:

2.1 Actual optimum

The actual optimum is given by minimizing the total cost of this function:

$$\min_{\vec{x}} \sum_{e \in G} \vec{x}_e c_e(\vec{x}_e)$$

which represents the costs paid for routing $\vec{x}_e$ traffic across an edge e times the amount of traffic routed across e
which in Pigou's network is minimizing

$$\min_x xc_1(x) + (1-x)c_2(1-x)$$

$$\min_x = x^2 + x$$

$$\Rightarrow 2x + 1 = 0$$

$$x = .5$$

$$\min_{\vec{x}} \sum_{e \in G} \vec{x}_e c_e(\vec{x}_e) = .75$$

2.2 Optimal Taxing

Then we can state the definition of an optimal tax is one where the feasible set $\mathcal{N}$ of the game (it's Nash Equilibria) is equal to the optimal solution:

$$\mathcal{N}(X, \vec{c}, \vec{t}, \vec{b}) = \min_{\vec{x}} \sum_{e \in G} \vec{x}_e c_e(\vec{x}_e)$$

Then we apply this to the Bernoulli pigou's game, simplifying the $\mathcal{N}$ given previously to:

$$\begin{bmatrix} x_1 : c_1(x) < c_2(1-x) + t_2(1-x) - t(x) + a_2 - a_1 \\ x_2 : c_1(x) < c_2(1-x) + t_2(1-x) - t(x) + b_2 - a_1 \\ x_3 : c_1(x) < c_2(1-x) + t_2(1-x) - t(x) + a_2 - b_1 \\ x_4 : c_1(x) < c_2(1-x) + t_2(1-x) - t(x) + b_2 - b_1 \end{bmatrix} = x \to .75$$

This becomes: the conditions:

$$\begin{bmatrix} x_1 : c_1(x) < c_2(1-x) + t_2(1-x) - t(x) + a_2 - a_1 \leq .75 \\ x_2 : c_1(x) < c_2(1-x) + t_2(1-x) - t(x) + b_2 - a_1 \leq .75 \\ x_3 : c_1(x) < c_2(1-x) + t_2(1-x) - t(x) + a_2 - b_1 \leq .75 \\ x_4 : c_1(x) < c_2(1-x) + t_2(1-x) - t(x) + b_2 - b_1 \leq .75 \end{bmatrix}$$

3 Generalizing

Then for a general game with more than two paths and discrete biases

4 Motivating example

Cosnider a game where

$$a_1 = a_2 = .25$$

$$b_1 = b2 = .75$$

$$p_1 = p_2 = .5$$

Which yields the constraints:

$$v = \begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix}$$

$$0 \leq x \leq v$$

and

$$x_1 : x_1 + x_2 + x_3 + x_4 < 1$$

$$x_2 : x_1 + x_2 + x_3 + x_4 < 1.5$$

$$x_3 : x_1 + x_2 + x_3 + x_4 < .5$$

$$x_4 : x_1 + x_2 + x_3 + x_4 < 1$$