$$I = \int_0^\infty Lmn(x)Lmn(x)x^m e^{-x}dx = \frac{(n+m)!}{n!},$$

Using the Rodrigues' formula (18.121), we may write

$$I = \frac{1}{n!}\int_0^\infty L_n^m(x)\frac{\partial^n}{\partial x^n}(x^{n+m}e^{-x})dx = \frac{(-1^n)}{n!}\int_0^\infty \frac{d^n L_n^m}{dx^n}x^{n+m}e^{-x}dx,$$

where, in the second equalit y, we have integrated by parts n times and used the fact that the boundary terms all vanish. From (18.120) we see that $d^n L_m^n/dx^n = (1)^n$ Thus we have

$$I = \frac{1}{n!}\int_0^\infty x^{n+m}e^{-x}dx = \frac{(n+m)!}{n!}$$

where, in the second equality, we use the expression (18.153) defining the gamma function (see section 18.12). The above orthogonality and normalisation conditions allow us to expand any (reasonable) function in the interval $0 \leq x < \infty$ in a series of the form

$$f(x) = \sum_{n=0}^\infty a_n L_n^m(x)$$

in which the coefficients $a^n$are given by

$$a^n = \frac{n!}{(n+m)!}\int_0^\infty f(x)L_n^m(x)x^m e^{-x}dx$$

We note that it is sometimes convenient to define the *orthogonal associated Laguerre functions* $\phi_n^m(x) = x^{m/2}e^{-x/2}L_n^m(x)$, which may also be used to produce a series expansion of a function in the interval $0 \leq x < \infty$

Generating function

The generating function for the associated Laguerre polynomials is given by

$$G(x,h) = \frac{e^{-xh/(1-h)}}{(1-h)^{m+1}} = \sum_{n=0}^\infty L_n^m(x)h^n$$

This can be obtained by differentiating the generating function (18.114) for the ordinary Laguerre polynomials m times with respect to x, and using (18.119).

Use the generating function (18.123) to obtain an expression for