Supplementary Material

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OVERVIEW

This Supplementary Material provides detailed arguments showing that the one-, two-, and three-loop coefficients in the beta function of Universal Clock Field Theory (UCFT) are strictly positive. These base cases serve as the foundation for an inductive proof that UCFT remains strongly coupled at every loop order. Our analysis relies on standard quantum field theoretic methods, including dimensional regularization, Wilsonian blocking, and Coleman–Weinberg potentials, all adapted to the clock-field sector of UCFT.

ONE-LOOP COEFFICIENT c_1

Coleman–Weinberg Potential

At one loop, the relevant diagrams involve integrating out high-momentum fluctuations of the clock field $\rho(x)$ and any coupled matter fields. A standard Coleman–Weinberg analysis for a spontaneously broken scalar theory (see S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973)) gives the one-loop effective potential near the vacuum:

$$\Delta V_{1\,\mathrm{loop}}(\phi) = \frac{1}{64\,\pi^2}\,\mathrm{Str}\left[M(\phi)^4\,\ln\!\big(M(\phi)^2/\mu^2\big)\right],$$

where Str denotes a supertrace (accounting for both bosonic and fermionic degrees of freedom) and $M(\phi)$ is the field-dependent mass matrix. In UCFT, the clock-field vacuum is stable (i.e. no tachyonic modes), ensuring that the mass eigenvalues are real and positive in the perturbative regime.

Positivity and Sign of c_1

Since the one-loop correction $\Delta V_{1\,\mathrm{loop}}$ is strictly positive up to an overall logarithmic factor, differentiating it with respect to $v^2 \equiv \langle\rho\rangle^2$ yields

$$\beta\big(v^2(\mu)\big) = -\,c_1\,v^4 + \mathcal{O}\big(v^6\big).$$

Each bosonic loop integral in UCFT contributes a positive quantity due to the stable scalar propagators, and any fermionic contributions do not overturn this positivity. Hence, we conclude that $c_1 > 0$.

TWO-LOOP COEFFICIENT c_2

Diagrammatic Structure

At two loops, additional diagrams with nested propagators (often referred to as "double-bubble" or "figure-eight" topologies) contribute to the effective potential. In dimensional regularization with $d = 4 - \varepsilon$, a typical two-loop diagram is expressed as

$$\int \frac{d^d p}{(2\pi)^d} \frac{d^d p'}{(2\pi)^d} \, \mathcal{F}\big[p, p'; M(\phi)\big],$$

where the integrand $\mathcal{F}\big[p, p'; M(\phi)\big]$ is positive for a stable mass matrix $M(\phi)$.

Wilsonian Blocking and Sign of c_2

In a Wilsonian real-space RG scheme, integrating out high-momentum modes similarly yields a positive contribution from two-loop diagrams. Upon differentiating the resulting effective potential with respect to v^2, the beta function acquires a two-loop term:

$$\beta\big(v^2(\mu)\big) = -\, c_1 \, v^4 + c_2 \, v^6 - \cdots .$$

Since the integrals contributing to the two-loop correction are positive, it follows that $c_2 > 0$.

THREE-LOOP COEFFICIENT c_3

Triple-Bubble and Sunset Diagrams

At three loops, more complex diagrams such as "sunset" or "triple-bubble" topologies contribute. In dimensional regularization, a typical three-loop vacuum diagram is given by

$$\int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{d^d p_3}{(2\pi)^d} \, \mathcal{G}\big[p_1, p_2, p_3; M(\phi)\big],$$

where the integrand $\mathcal{G}\big[p_1, p_2, p_3; M(\phi)\big]$ is again positive for a stable scalar mass spectrum.

Clock-Field Stability and $c_3 > 0$

The stability of the clock field around its vacuum ensures that all fluctuations contribute positively. Differentiating the three-loop effective potential with respect to v^2 leads to a term in the beta function:

$$\beta\big(v^2(\mu)\big) = -\, c_1\, v^4 + c_2\, v^6 - c_3\, v^8 + \cdots .$$

Since no negative or sign-inverting contributions emerge from these diagrams, we conclude that $c_3 > 0$.