Complex Analysis Homework

Deadline: 2/12/2019 at 12:00

(I) Let $\Omega \subset \mathbb{C}$ be a connected open subset and $f : \Omega \to \mathbb{C}$ a holomorphic function. Prove that if $(\Re f(z))^2 + (\Im f(z))^2 = 3$ for all $z \in \Omega$, then f is constant.

(II) Let $\{w_1, \ldots, w_n\} \subset \partial\mathbb{D}$ be distinct complex numbers of unitary norm. Prove that there exists $z_0 \in \partial\mathbb{D}$ such that $|z_0 - w_1||z_0 - w_2| \ldots |z_0 - w_n| = 1$.
Hint: consider the function $(z - w_1) \ldots (z - w_n)$.

(III) Let $f : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ be a continuous function which is holomorphic on the unit disk $\mathbb{D}$ and such that $f(\partial\mathbb{D}) \subset \partial\mathbb{D}$. Prove that there exists a unique holomorphic extension of f to $\mathbb{C}$ with finitely many points removed and determine it.

(IV) Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that there exists $a \in \mathbb{D} \setminus \{0\}$ satisfying $f(a) = f(-a) = 0$.

(a) Determine $\varphi_a \in \operatorname{Aut}(\mathbb{D})$ such that $\varphi_a(a) = 0$;

(b) show that $g(z) = \frac{f(z)}{\varphi_a(z)\varphi_{-a}(z)}$ is a well-defined holomorphic map from $\mathbb{D}$ to $\mathbb{D}$;

(c) prove that $|f(0)| \leq |a|^2$;

(d) what can you conclude if $|f(0)| = |a|^2$?

Use of computer is allowed, collaboration with colleagues is forbidden.
Please include and sign the following statement:
I declare that I solved the homework problems by myself.

You can send the solutions via email to *riccardo.ugolini@fmf.uni-lj.si* or place them in my mailbox at the ground floor of Jadranska 19.