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**ALGTOP2: Homework**

Ime in priimek

Vpisna številka

## 1. naloga

Let

$$\sigma_1 = [012] \quad \sigma_2 = [0'1'2'] \quad \sigma_3 = [ABC] \quad \sigma_4 = [A'B'C']$$

be four standard 2-simplices and let $X$ be the quotient space of the disjoint union of $\sigma_i$, where we identify the faces of the simplices in the following way:

$$[01] \sim [1'2'] \sim [AB] \sim [B'C']$$

$$[12] \sim [0'1'] \sim [BC] \sim [A'B']$$

$$[02] \sim [0'2'] \sim [AC] \sim [A'C']$$

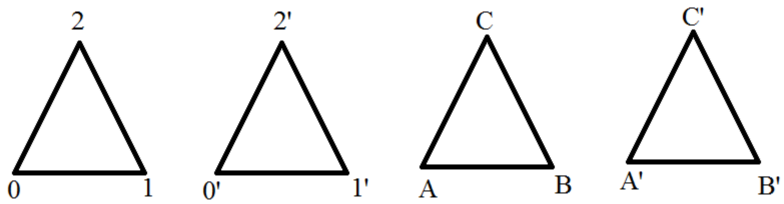
(Here $[01] \sim [1'2']$ means that $[01]$ and $[1'2']$ are identified by an affine map that maps $0 \mapsto 1'$ and $1 \mapsto 2'$.)

**a)** Describe the $\Delta$-complex structure of $X$.

**b)** Compute the homology of $X$ (with $\mathbb{Z}$ coefficients).

**c)** Compute the cohomology of $X$ (with $\mathbb{Z}$ coefficients).

**d)** Compute the cup and the cap product.



**2. naloga**

Let $G$ be a topological group and $\pi : E \to B$ a principal $G$-bundle.

**a)** Let $f : (S^n, x_0) \to (B, b_0)$ be a continuous map and

$$\begin{array}{c} f^*E \\ \downarrow \\ S^n \end{array}$$

the pullback principal $G$-bundle. Suppose $f^*E$ has transition function

$$c : S^{n-1} \to G\,.$$

Let

$$... \to \pi_n(G, 1) \to \pi_n(E, e_0) \to \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(G, 1) \to ...$$

be the long exact sequence for the bundle $E$. Show that

$$\partial : [f] \mapsto [c]\,.$$

**b)** Let $p : U(2) \to S^3$, where

$$p\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ c \end{bmatrix}$$

be the principal $U(1)$-bundle. Calculate $\pi_k(U(2), \mathrm{Id})$ in terms of $\pi_k(U(1), \mathrm{Id})$ and $\pi_k(S^3, (1, 0))$. Is the transition map $c : S^2 \to U(1)$ of this bundle nullhomotopic?

**c)** Use $U(2)$-principal bundle $p : U(3) \to S^5$ to calculate

$$\pi_1(U(3)) \qquad \pi_2(U(3)) \qquad \pi_3(U(3))\,.$$

**d)** Show that the principal $SO(3)$-bundle

$$p : SO(4) \to S^3$$

is a trivial bundle and calculate $\pi_n(SO(4))$ in terms of $\pi_n(SO(3))$ and $\pi_n(S^3)$.

**3. naloga**

Let $X$ be a compact orientable $n$-manifold, $Y = \partial X$, and $R$ a ring. Suppose that $X$ is an $R$-homology ball, i. e., $H_*(X; R) \cong H_*(B^n; R)$.

(a) Compute $H_*(Y; R)$.

(b) Suppose $n = 4$ and $R = \mathbb{Q}$. Show that the order of $H_1(Y; \mathbb{Z})$ is a square, $a^2$. Describe the number $a$ in terms of homology of $X$.

**4. naloga**

Let $X$ be a closed, connected, orientable smooth $n$-manifold and let $Y \subset X$ be a smooth closed $m$-submanifold.

(a) Express homology of the complement $H_*(X \setminus Y; \mathbb{Z})$ in terms of (co)homology of the pair $(X, Y)$.
Hint: Use that fact that $Y$ has a closed tubular neighborhood $N$ in $X$ which is diffeomorphic to the unit disk bundle of the normal bundle of $Y$ in $X$.

(b) Compute $H_0(X \setminus Y; \mathbb{Z})$ when $m = n - 1$.

(c) Suppose $X$ has the integral homology of a sphere. Express $H_*(X \setminus Y; \mathbb{Z})$ in terms of (co)homology of $Y$.

(d) Let $K \subset S^3$ be a knot, i. e., the image of an embedding $f \colon S^1 \to S^3$. Compute $H_1(S^3 \setminus K; \mathbb{Z})$ using (c). How good is this invariant at distinguishing knots?

**5. naloga**

Let $X = T \vee \mathbb{C}P^2$, where $T$ denotes the 2-dimensional torus.

(a) Compute $\pi_2(X)$.

(b) Describe the action of $\pi_1(X)$ on $\pi_2(X)$.