Analysis on Manifolds Homework

Deadline: 2/12/2019 at 12:00

(I) Let $SL_2(\mathbb{R})$ be the group of real two by two matrices with determinant one. Prove that it is a smooth manifold by finding explicit charts.

(II) For $n \in \mathbb{N}$, consider the map $\varphi_n : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_n(t) = \begin{cases} t \text{ for } t \leq 0 \\ t^n \text{ for } t > 0 \end{cases}$$

(a) show that φ_n is a homeomorphism for every $n \in \mathbb{N}$;

(b) determine which values of $n, m \in \mathbb{N}$ are such that $\{(\varphi_n, \mathbb{R}), (\varphi_m, \mathbb{R})\}$ is a smooth atlas for $\mathbb{R}$;

(c) denote by $\mathbb{R}_n$ the manifold $\mathbb{R}$ with atlas given by $\{\varphi_n, \mathbb{R}\}$. For which $n, m \in \mathbb{N}$ are $\mathbb{R}_n$ and $\mathbb{R}_m$ diffeomorphic to each other?

(III) Let $n \in \mathbb{N}$ be a positive natural number and

$$SO(n) = \{A \in GL_n(\mathbb{R}) : AA^T = \mathrm{Id},\ \det(A) = 1\}$$

be the group of special orthogonal matrices. Consider the map given by

$$\begin{aligned} \varphi : SO(n+1) \times \mathbb{R}^{n+1} &\to \mathbb{R}^{n+1} \\ (A, x) &\mapsto Ax \end{aligned}$$

(a) Prove that φ is a group action;

(b) prove that φ restricts to a well-defined action on the sphere $S^n \subset \mathbb{R}^{n+1}$;

(c) determine the isotropy group G of the action at the point $(1, 0, \ldots, 0) \in S^n$;

(d) * prove that the group quotient $SO(n+1)/G$ is a smooth manifold diffeomorphic to S^n.

(IV) Let M be a smooth manifold. Prove that the tangent bundle TM is an orientable manifold.

Exercises marked with $(*)$ are harder.
Use of computer is allowed, collaboration with colleagues is forbidden.
Please include and sign the following statement:
I declare that I solved the homework problems by myself.

You can send the solutions via email to *riccardo.ugolini@fmf.uni-lj.si* or place them in my mailbox at the ground floor of Jadranska 19.