ALGTOP2: Homework

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1. naloga

1. Let K be the Klein bottle. Calculate Δ-complex homology of K with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_m$ for all $m \geq 2$.

2. Calculate $H_n(K \times S^1)$ using the Künneth formula.

3. Calculate $H_n(K \times S^1)$ from $I^3/\sim$, where the equivalence relation $\sim$ is given by

$$(0, y, z) \sim (1, y, z) \qquad (x, y, 0) \sim (x, y, 1) \qquad (x, 0, z) \sim (x, 1, 1 - z)\,.$$

4. Let $p, q \in K$ be different points in K. Calculate relative homology groups

$$H_n(K \times S^1, \{p, q\} \times S^1)\,.$$

2. naloga

Let $p : P \to X$ be a 2-sheeted covering space over X.

1. Consider the sequence

$$0 \to C_n(X;\mathbb{Z}_2) \xrightarrow{\tau} C_n(P;\mathbb{Z}_2) \xrightarrow{p_\#} C_n(X;\mathbb{Z}_2) \to 0\,. \tag{1}$$

   Here $\tau : C_n(X;\mathbb{Z}_2) \to C_n(P;\mathbb{Z}_2)$ is given by

$$\tau(\sigma) = \tilde{\sigma}_1 + \tilde{\sigma}_2 \qquad \sigma : \Delta^n \to X\,,$$

   where $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are the two distinct lifts of $\sigma : \Delta^n \to X$.

   Show that the sequence (1) is exact for all $n \geq 0$.

   Show that the exact sequences (1) form a short exact sequence of chain complexes.

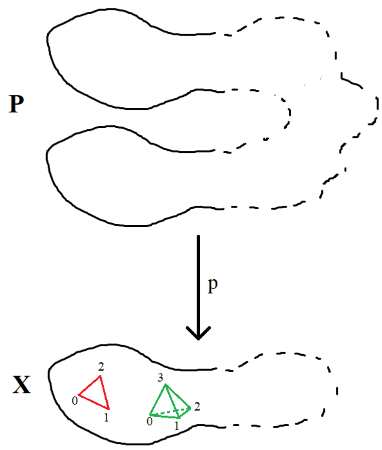
2. Derive the associated long exact sequence and describe (on the level of chains) the connecting homomorphism

$$\partial : H_n(X;\mathbb{Z}_2) \to H_{n-1}(X;\mathbb{Z}_2)\,.$$

   Use figure below to explicitly (on the level of chains) write down

$$\partial c_1 \text{ and } \partial c_2\,,$$

   where $c_1 = [01] + [02] + [12]$ and $c_2 = [012] + [013] + [023] + [123]$ are given cycles (the coefficients are $\mathbb{Z}_2$ so we can write pluses everywhere).



3. Let $n \geq 2$. Calculate $H_k(\mathbb{R}P^n;\mathbb{Z}_2)$ using the above long exact sequence for the covering projection $p : S^n \to \mathbb{R}P^n$.

   Assume $H_k(S^n;\mathbb{Z}_2)$ as known.

3. naloga

Let $X = U_0 \cup U_1 \cup \cdots \cup U_N$, where $U_0, U_1, ..., U_N \subseteq X$ are open subsets of X. Denote by $U_{ij} = U_i \cap U_j$ and suppose that

$$U_i \cap U_j \cap U_k = \emptyset$$

for pairwise different i, j, k (there are no triple or higher order intersections).

Consider the following sequence ($n \geq 0$), where we take only those i, j, where $U_{ij} \neq \emptyset$:

$$0 \to \bigoplus_{i<j} C_n(U_{ij}) \xrightarrow{\Phi} \bigoplus_i C_n(U_i) \xrightarrow{\Pi} C_n(U_0 + U_1 + ... + U_N) \to 0\,, \tag{2}$$

where Φ, Π are given by the following

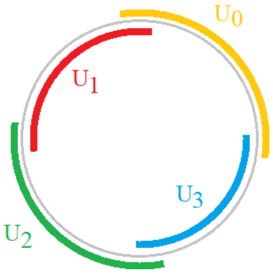
$$\Phi(0, 0, ..., 0, \alpha_{ij}, 0, ...0) = (0, ..., 0, \underbrace{\alpha_{ij}}_{i}, 0, ..., 0, \underbrace{-\alpha_{ij}}_{j}, 0, ..., 0)\,,$$

and

$$\Pi(\alpha_0, \alpha_1, ..., \alpha_N) = \alpha_0 + \alpha_1 + ... + \alpha_N\,.$$

1. Show that the sequence (2) is exact for all $n \geq 0$. Show that short exact sequences (2) form a short exact sequence of chain complexes. Derive the associated long exact sequence.

2. Using the above long exact sequence, compute the homology of S^1, by taking U_0, U_1, U_2, U_3 as on the figure below.



3. Using the above long exact sequence, compute the homology of $S^1 \vee S^2$, by taking U_0, U_1, U_2 as on the figure below.