Complex Analysis - 2^{st} homework

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<u>Ex. 1:</u> Let Ω be an open subset of $\mathbb{C}$ and let $\mathcal{F}$ be a family of holomorphic functions from Ω to $\mathbb{C}$.

(a) Let us show that $\mathcal{F}$ is a normal family if and only if $\mathcal{F}|_D = \{f|_D;\ f \in \mathcal{F}\}$ is a normal family for every disc $D \subset \Omega$.

From left hand side to right hand side, the statement is obvious. If every sequence of functions in $\mathcal{F}$ has a subsequence converging uniformly on all compact subsets of Ω, that of course includes all compact subsets of any given disc $D \in \Omega$.

Conversely, take a compact subset $K \subset \Omega$. By compactness, we can cover K with finitely many open discs $D_i \subset \Omega$, $i = 1, \dots, n$. Because Ω is a normal space, it is in particular Hausdorff, discs D_i can be chosen in such a way that some smaller discs $E_i \subset D_i$ also cover K (this is really needed only if we chose some D_i that shares a piece of its boundary with the boundary of Ω - this argument states that we need not choose such discs, but rather smaller, which we can later close in some slightly larger disc). Note that the sets $\overline{E_i} \subset D_i$ are compact in D_i for every $i = 1, \dots, n$. By the converse to Montel's theorem, the family $\mathcal{F}$ is equibounded on $\overline{E_i}$ by $M_i \in \mathbb{R}$. Clearly, the family $\mathcal{F}$ is equibounded on K by $\max_{i=1,\dots,n} M_i$. By Montel's theorem, $\mathcal{F}$ is a normal family.

(b) Assume now these exists a point $a \in \Omega$ such that the sequence $\{f(a);\ f \in \mathcal{F}\}$ is bounded. Suppose as well that the family $\mathcal{F}' = \{f';\ f \in \mathcal{F}\}$ is equibounded and that Ω is connected. Let us prove that $\mathcal{F}$ is a normal family.

Firstly, let us prove the statement in the case where Ω is an open disc containing a (or in the case of a disc in Ω containing a). Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{F}$. By assumption, there exists a subsequence $\{f_{j_k}\}_{k\in\mathbb{N}}$ such that the sequence $\{f'_{j_k}\}_{k\in\mathbb{N}}$ converges uniformly on compact subsets. Denote $f'_{j_k} \to g$ and $f_{j_k}(a) \to \omega$. Now, define a new function for every $z \in \Omega$

$$G(z) = \omega + \int_{[a,z]} g(t)dt,$$

where $[a, z]$ is a straight line segment from a to z. The function G is holomorphic on the disc Ω and $G' = g$, so we have

$$f_{j_k}(z) - G(z) = f_{j_k}(a) - \omega + \int_{[a,z]} (f'_{j_k}(z) - g(t))dt.$$

Next, take a compact K in Ω and a closed disc B such that $a \in B$ and $K \subset B \subset \Omega$. By convexity, $[a, z] \subset B$ for every $z \in K$ and we have

$$|f_{j_k}(z) - G(z)| \leq |f_{j_k}(a) - \omega| + \operatorname{diam}(\Omega) \cdot \sup_{t\in B} |f'_{j_k}(z) - g(t)|,$$

which goes to 0 uniformly on K.

Secondly, in order to generalize to the whole set Ω, define the set

$$A = \{z \in \Omega;\ \mathcal{F} \text{ is a normal family on a neighbourhood of } z\}.$$

The set A is clearly open, since it is defined with an open condition. It is also not empty, since $a \in A$. We will show that $\Omega \setminus A$ is also open in Ω. Assume there exists $b \in \Omega \setminus A$. Let $D \subset \Omega$ be a disc centered in b. We will show that $D \subset \Omega \setminus A$. Assume there exists a point $c \in D \cap A$. Then, $\mathcal{F}$ is normal in a neighbourhood of c, which implies the set $\{f(c);\ f \in \mathcal{F}\}$ is bounded. From the previous argument, $\mathcal{F}$ is normal in D, which is a contradiction. By connectedness of Ω, $A = \Omega$.

Finally, every point in Ω has a neighbourhood in which $\mathcal{F}$ is a normal family. Without loss of generality we can take these neighbourhoods to be discs in Ω. Now, simply apply (a).

Ex. 2: Let $n \in \mathbb{N}$ and $a_n, b_n \in \mathbb{R}$ such that $0 < b_n < a_n < n$.

(a) Let us show that there exists a polynomial p_n such that $|p_n(z)| > n$ for $z \in B_n = \mathbb{D}(o,n) \cap \{\Im(z) = b_n\}$ and $|p_n(z)| < \frac{1}{n}$ for $z \in A_n = \mathbb{D}(o,n) \cap (\{\Im(z) > a_n \text{ or } \Im(z) < 0\})$.

Observe that $\overline{A_n}$ and $\overline{B_n}$ are disjoint compact sets, so $K = \overline{A_n} \cup \overline{B_n}$ is a compact set with no holes, that is, $\mathbb{C} \setminus K$ is connected. By Runge's theorem, there exists a sequence of holomorphic polynomials approximating the holomorphic function

$$f(z) = \begin{cases} n+1;\ z \in \overline{B_n}, \\ 0;\ z \in \overline{A_n}, \end{cases}$$

that is, for every $\epsilon > 0$ there exists a polynomial $P(z)$ such that $\max_{z \in K} |f(z) - P(z)| < \epsilon$. For $\epsilon < \frac{1}{n}$ polynomial $P(z)$ satisfies our conditions.

(b) We would like to construct a sequence of polynomials that is pointwise converging to 0 on $\mathbb{C}$ such that the convergence is uniform on compact subsets of $\mathbb{C} \setminus \mathbb{R}$, but not in any neighbourhood of a real point.

We will use polynomials from (a). Take the sequence $\mathcal{P} = \{p_n\}_{n \in \mathbb{N}}$ for $a_n = \frac{1}{n}$ and $b_n < a_n$ arbitrary. The sequence $\mathcal{P}$ is obviously converging pointwise to 0 on $\mathbb{C}$, even on $\mathbb{R}$ because of continuity and by construction (consult (a), we closed the condition sets). Now, take a compact set $K \in \mathbb{C} \setminus \mathbb{R}$. We can assume K is connected, otherwise do the following on every connected component. Since K is bounded, there exists $n_0 \in \mathbb{N}$ such that $K \in \mathbb{D}(0, n_0)$. If K is located in the upper halfplane, we can also suppose that $a_{n_0} < \Im(z)$ for every $z \in K$. Now we see, that $p_n \to 0$ uniformly on K by Runge's theorem from (a). Next, take a real point $a \in \mathbb{R}$ and a neighbourhood U around it. Without loss of generality suppose $U = \mathbb{D}(a, r)$, for some $r > 0$. There exists $n_0 \in \mathbb{N}$ such that $b_{n_0} < r$. Since for every $n \in \mathbb{N}$ we have $|p_n| > n$, the sequence is unbounded, and hence convergence cannot be uniform there.

(c) We would like to constructa sequence of polynomials that is pointwise converging to 0 on $\mathbb{R}$ and to 1 on $\mathbb{C} \setminus \mathbb{R}$.

Let $n \in \mathbb{N}$ and $0 < a_n < n$. By the same argument from (a) there exists a polynomial q_n so that $|q_n| < \frac{1}{n}$ on $\mathbb{D}(0,n) \cap \{\Im(z) = 0\}$ and $1 + \frac{1}{n} > |q_n| > 1 - \frac{1}{n}$ on $\mathbb{D}(0,n) \cap \{|\Im(z)| > a_n\}$ (we approximate the constant function 0 on the first set and 1 on the second). Now, simply take the sequence of polynomials $\mathcal{Q} = \{q_n\}$ with $a_n = \frac{1}{n}$.

Ex. 3: Let $f, g, h \colon \mathbb{C} \to \mathbb{C}$ be holomorphic functions satisfying $h = e^f + e^g$.

(a) Let us prove that the equation $h(z) = 0$ has either infinitely many solutions or none at all.

We are solving the equivalent equation $e^f = -e^g$. This solution will rely heavily on the Picard's Little Theorem and the notion of entire functions. We will separate certain cases.

Firstly, if f and g are both constant, then we have a solution precisely when $f = 2\pi i k + g$ for some $k \in \mathbb{Z}$. In that case, we obviously have a solution for every $z \in \mathbb{C}$.

Secondly, if f is constant, but g is not, we have $e^{g(z)} = -e^f$. Since g is entire, it omits at most one value a. Therefore e^g omits at most 0 and e^a. But if it omits both, e^g will be constant and by extension g, which means e^g omits only $0 \neq -e^f$. That means there is a solution z_0 and by extension infinetely many of them, since g takes values $2\pi i k + g(z_0)$ for every $k \in \mathbb{Z}$.

Finally, let $f \neq g$ be non-constant entire functions. Since the function e^z has no zeros, we are solving $e^{f-g} = -1$. Since $f-g$ is an entire function, it omits at most one value a, so e^{f-g} omits at most 0 and e^a. If it omits both, we have that e^{f-g} is constant, by extension is $f-g$ constant. By the argument above, if we have a solution, we have infinitely many of them. So, suppose e^{f-g} omits merely 0. Then by entirety, we have a solution and by the argument above infinitely many of them.

(b) We will prove that the equation $e^z = p(z)$ has a solution for any non-constant polynomial p.

We will forget the zeros of p, since there can be no solutions there for any polynomial. Let us look at the following equation

$$\frac{e^z}{p(z)} = 1.$$

Let us switch the variable $\omega \longleftrightarrow \frac{1}{z}$ and limit the domain to $\mathbb{D}(0,r)$, where $r>0$ is such that no zeros of p (except maybe 0) lie in the domain. We now have the equivalent problem

$$f(z) = \frac{e^{\frac{1}{z}}}{p(\frac{1}{z})} = 1.$$

Note that f has an essential singularity in $z=0$ and is holomorphic on $\mathbb{D}(0,r)\setminus\{0\}$. By Picard's Big Theorem, f omits at most one value, and it certainly omits 0. Therefore there must exist such z_0 so that $f(z_0)=1$. Our solution is $\frac{1}{z_0}$.

Ex. 4: Let $f\colon \mathbb{D}\to\mathbb{C}$ be schlicht, that is, an injective holomorphic function with $f(0)=0$ and $f'(0)=1$. Assume $D=f(\mathbb{D})$ is a convex set and let $r\in(0,1)$ and $e^{i\theta}\in\partial\mathbb{D}$.

(a) Let us directly calculate

$$\frac{1}{2\pi i}\int_{|z|=r} f(z)\left(1+\frac{z}{2re^{i\theta}}+\frac{re^{i\theta}}{2z}\right)\frac{dz}{z} = \frac{1}{2\pi i}\left(\int_{|z|=r}\frac{f(z)}{z-0}dz + \frac{1}{2re^{i\theta}}\int_{|z|=r} f(z)dz + \int_{|z|=r} f(z)\frac{re^{i\theta}}{2z^2}\right)dz =$$
$$= \frac{1}{2\pi i}\left(2\pi i f(0) + 0 + \frac{1}{2}re^{i\theta}\cdot 2\pi i f'(0)\right) = \frac{1}{2}re^{i\theta}$$

(b) We continue from (a) by changing integral variables

$$\frac{1}{2}re^{i\theta} = \frac{1}{2\pi i}\int_{|z|=r} f(z)\left(1+\frac{z}{2re^{i\theta}}+\frac{re^{i\theta}}{2z}\right)\frac{dz}{z} = \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(re^{i\phi})\left(1+\frac{re^{i\phi}}{2re^{i\theta}}+\frac{re^{i\theta}}{2re^{i\phi}}\right)\frac{rie^{i\phi}d\phi}{re^{i\phi}} =$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{i\phi})\left(1+\frac{e^{i(\theta-\phi)}+e^{-i(\theta-\phi)}}{2}\right)d\phi = \frac{1}{\pi}\int_{-\pi}^{\pi} f(re^{i\phi})\left(1+\left(\frac{e^{i\frac{\theta-\phi}{2}}+e^{-i\frac{\theta-\phi}{2}}}{2}\right)^2-\frac{2e^0}{2}\right)d\phi =$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(re^{i\phi})\cos^2\left(\frac{\theta-\phi}{2}\right)d\phi$$

(c) Next we will show that $\frac{1}{2}re^{i\theta}\in D$, hence it will follow that $\mathbb{D}(0,\frac{1}{2})\subset D$, once we send $r\to 1$.

There are various ways to approach this, we chose a measure theoretic one. Recall, firstly, that D is convex and then observe that $\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2\left(\frac{\phi}{2}\right)d\phi = 1$. By change of variables $t \leftrightarrow \frac{\theta-\phi}{2}$, we also see the same is true in our case, that is, $\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2\left(\frac{\theta-\phi}{2}\right)d\phi = 1$.

Let $a(x)$ be a continuous mapping (it can be vector valued, in this case complex). Denote by $\Lambda = \{\int a(x)\omega(x)dx;\ \omega(x)$ is a measurable function with $\int\omega(x)dx = 1\ \}$ the set of all *continuous convex combinations* of $a(x)$ and denote by Γ the set of all convex combinations of values of $a(x)$. We will prove that these two sets are in fact the same.

The inclusion $\Gamma\subseteq\Lambda$ follows from the fact that a convex combination $\Sigma_{i=1}^k a(x_i)\omega(x_i)$ is equal to $\int a(x)\omega(x)dx$ with $\omega(x) = \Sigma_{i=1}^k\omega(x_i)\delta_{a(x_i)}$, where $\delta_{a(x_i)}$ is a Dirac delta concentrated at $a(x_i)$.

The opposite inclusion will follow from Jensen's inequality. Consider the following indicator (or characteristic) function

$$1_\Gamma(x) = \begin{cases} 0; \ x \in \Gamma \\ \infty; \ x \notin \Gamma \end{cases}$$

This function is convex and by definition $\Gamma = \{x; \ 1_\Gamma(x) = 0\}$. Now, take $\lambda = \int a(x)\omega(x)dx \in \Lambda$. By Jensen's inequality, it holds that

$$1_\Gamma(\lambda) \le \int 1_\Gamma(a(x))\omega(x)dx = \int 0 \cdot \omega(x) = 0.$$

So, $1_\Gamma(\lambda) = 0$, meaning $\lambda \in \Gamma$. With this we conclude the general proof.

Take now the measurable space $(-\pi, \pi)$ and the function $f(re^{i\phi})$. The integral $\frac{1}{\pi}\int_{-\pi}^{\pi} f(re^{i\phi}) \cos^2\left(\frac{\theta-\phi}{2}\right) d\phi$ is by our observation a continuous convex combination of the function $f(re^{i\phi})$, and hence lies in D. The conclusion follows.