• Vzemimo $A \in \mathcal{D}$. $\mu(A^c) := \mu(X_n \setminus A) \underset{\mu(X_n)<\infty}{=} \mu(X_n) - \mu(A) =$

$$= \underset{\text{točka (a)}}{k \cdot m(X_n)} - \underset{\text{def. } \mathcal{D}}{k \cdot m(A)} = k \cdot m(X_n \setminus A) =: k \cdot m(A^c) \quad (\text{na } X_n) \checkmark$$

• Vzemimo $(A_n)_n \subset \mathcal{D}$ disj.:

$$\mu\left(\bigcup_n A_n\right) \underset{\mu \text{ mera}}{=} \sum_n \mu(A_n) \underset{\text{def. } \mathcal{D}}{=} \sum_n k \cdot m(A_n) \underset{m \text{ mera}}{=} k \cdot m\left(\bigcup_n A_n\right) \checkmark$$

c) Vzemimo sedaj $A \in \mathcal{B}_{\mathbb{R}}$. Iz (b) dobimo naraščajoče zaporedje $(A \cap X_n)_n$. Zaradi velja $A = \bigcup_{n \in \mathbb{N}} (A \cap X_n)$.
Zato uporabimo notranjo zveznost:

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) \overset{(b)}{=} \lim_{n\to\infty} m(A \cap X_n) \cdot k = k \cdot \lim_{n\to\infty} m(A \cap X_n) = k \cdot m(A).$$

3) Naj bo $m \times \xi$ ena izmed produktnih mer na $\mathcal{B}_{[0,1]\times[0,1]}$, tako $(m\times\xi)(A\times B) = m(A)\cdot\xi(B)$.
Δ = diagonala

a) $1_\Delta : [0,1]^2 \to \mathbb{R}$ Borelova merljiva: oglejmo si praslike $A \in \mathbb{R}$.

- $0,1 \in A \Rightarrow 1_\Delta^{-1}(A) = [0,1]\times[0,1]$ ✓
- $0 \in A,\ 1 \notin A \Rightarrow 1_\Delta^{-1}(A) = [0,1]^2 \setminus \Delta$ odprta v $[0,1]^2 \Rightarrow$ merljiva ✓
- $0 \notin A,\ 1 \in A \Rightarrow 1_\Delta^{-1}(A) = \Delta$ zaprta v $[0,1]^2 \Rightarrow$ komplement odprte $\Rightarrow$ merljiva ✓
- $0,1 \notin A \Rightarrow 1_\Delta^{-1}(A) = \emptyset$ ✓

b) Računajmo integrale. Označimo $I := [0,1]$.
Definiramo sekcije: $\Delta_x = \{y \in I;\ (x,y) \in \Delta\} = \{x\}$, $\Delta^y = \{x \in I;\ (x,y) \in \Delta\} = \{y\}$.
$1_{\Delta_x} : [0,1]_y \to \mathbb{R}$; $1_{\Delta_x}(y) = 1_\Delta(x,y)$ in $1_\Delta^y : [0,1]_x \to \mathbb{R}$; $1_\Delta^y(x) = 1_\Delta(x,y)$.

• $$\int_I \int_I 1_\Delta(x,y)\, dm(x)\, d\xi(y) = \int_I \left(\int_I 1_\Delta^y(x)\, dm(x)\right) d\xi(y) = \int_I \underbrace{m(\{*\})}_{0}\, d\xi(y) = 0$$

• $$\int_I \int_I 1_\Delta(x,y)\, d\xi(y)\, dm(x) = \int_I \left(\int_I 1_{\Delta_x}(y)\, d\xi(y)\right) dm(x) = \int_I \xi(\{*\})\, dm(x) = \int_I 1\, dm(x) = 1.$$

Po konstrukciji mere $m\times\xi$ imamo $(m\times\xi)(\Delta) = \inf\Big\{\sum_{n=1}^{\infty} \underbrace{(m\times\xi)([a_n,b_n)\times[c_n,d_n))}_{P};\ \Delta \subset P\Big\}$
Ampak, $(m\times\xi)(P) = \underbrace{(b_n - a_n)}_{\neq 0} \cdot \underbrace{\xi([c_n,d_n))}_{\infty} = \infty$ za vse P.

$\Rightarrow (m\times\xi)(\Delta) = \infty$.

• $$\int_{I\times I} 1_\Delta(x,y)\,(m\times\xi)(x,y) \underset{\text{def. integrala}}{=} (m\times\xi)(\Delta) = \infty.$$

c) V izreku ni izpolnjen pogoj σ-končnosti za mero števna ξ.