# Analysis on manifolds - 2$^{\text{nd}}$ homework

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**<u>Ex. 1:</u>** Let $F\colon \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^6$ be such that $F(x_1, x_2, x_3) = (x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3)$.

(a) Let us first prove that $F$ is an immersion. We calculate the Jacobian matrix:

$$JF = \begin{bmatrix} 2x_1 & 0 & 0 \\ 0 & 2x_2 & 0 \\ 0 & 0 & 2x_3 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \end{bmatrix}.$$

For any element $(x_1, x_2, x_3)$ all three coordinates are not simultaneously equal to 0. Without loss of generality, take $x_1 \neq 0$. The rows $[2x_1, 0, 0]$, $[x_2, x_1, 0]$, and $[x_3, 0, x_1]$ are clearly independent, hence rank $JF = 3$ and $F$ is an immersion.

(b) Is $F$ an injective mapping? With the method of a sharp look we observe, that all coordinates in the image are quadratic in nature, meaning $F(x) = F(-x)$. As an example, take

$$F(1,1,1) = (1,1,1,1,1,1),$$
$$F(-1,-1,-1) = (1,1,1,1,1,1).$$

(c) To determine the fibres of $F|_{\mathbb{S}^2}$, let $(x_1, x_2, x_3) \in \mathbb{S}^2$. Then $F^{-1}(F(x_1, x_2, x_3)) = \pm(x_1, x_2, x_3)$. These are precisely the antipodal points (also seen from the argument in (b)).

Next, we would like to prove that $\widetilde{F}\colon \mathbb{RP}^2 \to \mathbb{R}^6$, defined by

$$\widetilde{F}(x_1 : x_2 : x_3) = \frac{F(x_1, x_2, x_3)}{x_1^2 + x_2^2 + x_3^2},$$

is an injective immersion. We observe, that $\widetilde{F}(tx_1 : tx_2 : tx_3) = \widetilde{F}(x_1 : x_2 : x_3) = F(x_1, x_2, x_3)$, where $(x_1, x_2, x_3) \in \mathbb{S}^2$. This can be seen both by manipulating homogenous coordinates on the left hand side, or by simply eliminating $t$ from the expression on the right hand side of the defining equation. By (a), $\widetilde{F}$ is an immersion. Since only antipodal points on the sphere have the same image with $F$ and $\mathbb{RP}^2$ identifies those points (as a qoutient of $\mathbb{S}^2$), the mapping is also injective.

**<u>Ex. 2:</u>** Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be such that $f(x, y, z) = xy + z$ and let $M$ be the zero set of $f$. Let $V = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (z + 3xy)\frac{\partial}{\partial z}$ be a vector field.

(a) To prove that $M$ is a manifold, let us simply calculate the gradient $\nabla f = (y, x, 1)$, which can never be equal to 0. Thusly, $M$ is a 2-dimensional submanifold in $\mathbb{R}^3$.

(b) In order for $V$ to restrict to a vector field on $M$, $V$ has to be tangent to $M$. In $p = (x, y, z) \in M$, we calculate

$$df_p(V) = V(f(x,y,z)) = \frac{\partial f}{\partial x}(-x) + \frac{\partial f}{\partial y}(-y) + \frac{\partial f}{\partial z}(z+3xy) = y(-x) + x(-y) + 1(2xy) = -2xy + 2xy = 0.$$

(c) Denote by $W$ the restriction of $V$ to $M$. We are searching for its fixed points. In order to find them, let us calculate the flow of $W$, that is, we will solve

$$\frac{\partial}{\partial t}\varphi_W(t,x,y,z) = W(\varphi_W(t,x,y,z)).$$

Equivalently, we have

$$\dot{x}(t)\frac{\partial}{\partial x} + \dot{y}(t)\frac{\partial}{\partial y} + \dot{z}(t)\frac{\partial}{\partial z} = -x\frac{\partial}{\partial x} + -y\frac{\partial}{\partial y} + 2xy\frac{\partial}{\partial z}.$$

We solve this in coordinates and get

$$\begin{aligned} x(t) &= Ae^{-t} \\ y(t) &= Be^{-t} \\ z(t) &= -ABe^{-2t} \end{aligned}$$

Using the initial condition $\varphi_W(0,x,y,z) = (x,y,z)$, we get

$$\begin{aligned} x(t) &= x_0e^{-t} \\ y(t) &= y_0e^{-t} \\ z(t) &= -x_0y_0e^{-2t} \end{aligned}$$

There is clearly only one fixed point, that is $(x,y,z) = (0,0,0)$.

We now ask ourselves whether this fixed point is locally stable. Take a basis of neighbourhoods around $(0,0,0)$ as $U_n = \frac{1}{n}\mathbb{B}^3 \cap M$ for $n \in \mathbb{N}$. For $t \geq 0$ and $(x,y,z) \in U_n$, we calculate

$$\max_{t\geq 0} |\varphi_W(t,x,y,z)| = |(x,y,-xy)| \implies \varphi_W(U_n) \subseteq U_n.$$

The fixed point is therefore locally stable.

**<u>Ex. 3:</u>** Let $M = \{((x,y),[v:w]) \in \mathbb{C}^2 \times \mathbb{CP}^1;\ xv = yv\}$ and let $\pi\colon M \to \mathbb{C}^2$ be such that $\pi((x,y),[v:w]) = (x,y)$.

(a) Firstly, let us prove that $M$ is a complex manifold. Take $U_w = \{((x,y),[v:w]) \in M;\ w \neq 0\}$. On $U_w \subset M$ we now have

$$((x,y),[\frac{v}{w}:1]) = ((\frac{v}{w}y,y),[\frac{v}{w}:1]) \mapsto (y,\frac{v}{w} = z) \in \mathbb{C}^2.$$

Similarly, on $U_v \subset M$ we have

$$((x,y),[1:\frac{w}{v}]) = ((x,\frac{w}{v}x),[1:\frac{w}{v}]) \mapsto (x,\frac{w}{v} = z) \in \mathbb{C}^2.$$

All that remains is to calculate the transition maps. Suppose $z \neq 0$ and calculate

$$\begin{aligned} \varphi_{wv}\colon (x,z) &\mapsto ((x,zx),[1:z]) = ((yz^{-1},y),[z^{-1}:1]) \mapsto (y,z^{-1}) = (xz,z^{-1}), \\ \varphi_{vw}\colon (y,z) &\mapsto ((zy,y),[z:1]) = ((x,xz^{-1}),[1:z^{-1}] \mapsto (x,z^{-1}) = (yz,z^{-1}), \end{aligned}$$

which are both holomrphic maps, since they are holomorphic on each component.

(b) Secondly, let us prove that the map $\pi|_{M\setminus\pi^{-1}(0,0)}$ is a biholomorphism onto $\mathbb{C}^2 \setminus \{(0,0)\}$. Without loss of generality, suppose $x \neq 0$. On $M \setminus \pi^{-1}(0,0)$ we now have

$$\frac{w}{v} = \frac{y}{x}.$$

The inverse function is now

$$(\pi|_{M\setminus\pi^{-1}(0,0)})^{-1}(x,y) = ((x,y),[\frac{y}{x}:1]),$$

which is well-defined because of homogeneity of the second coordinate pair. This map is therefore bijective. Since it is a canonical projection, it is clearly holomorphic, and by extension biholomorphic.

(c) Lastly, we will show that the map $p\colon M \to \mathbb{CP}^1$ given by $\pi((x,y),[v:w]) = [v:w]$ is a holomorphic line bundle over $\mathbb{CP}^1$.

- Suppose $v \neq 0$. We have
$$p^{-1}([v:w]) = ((x,y),[v:w]) = ((x, x\frac{w}{v}),[v:w]).$$
This is a 1-parameter family. In other words, the map $\varphi\colon \mathbb{C} \to p^{-1}([v:w])$ given by $x \mapsto ((x, x\frac{w}{v}),[v:w])$ is a vector space isomorphism (since $v,w$ are parameters, every component is linear), making $p^{-1}([v:w])$ a vector space of $\mathbb{C}$-dimension 1. The same thought process holds for the case $w \neq 0$.

- Take $U_v = \{[v:w];\ v \neq 0\}$ and $U_w = \{[v:w];\ w \neq 0\}$. Consider the map $\gamma_v\colon p^{-1}(U_v) \to U_v \times \mathbb{C}$ given by $\gamma_v((x,y),[v:w]) = ([1:\frac{w}{v}], x)$. Let us calculate the preimage
$$\gamma_v^{-1}([1:\frac{w}{v}],x) = ((x,\frac{w}{v}),[1:\frac{w}{v}]).$$
We get
$$\gamma_v \circ \gamma_v^{-1} = id,$$
and same holds for $U_w$. We got a local trivialization.

- Again, what remains is to find the transition maps and verify that they are $\mathbb{C}$-linear. We calculate
$$\gamma_{vw} = \gamma_v \circ \gamma_w^{-1}([\frac{v}{w}:1],y) = \gamma_v((\frac{w}{v}y,y),[v:w]) = ([1:\frac{w}{v}],\frac{w}{v}y),$$
and similarly for $\gamma_{wv}$. The transition maps are in fact $\mathbb{C}$-linear and the transition coefficients are
$$g_{vw} = \frac{v}{w},$$
meaning this line bundle is equivalent to the tautological bundle $\mathcal{O}(-1)$, that is, its power is $k=1$.

<u>**Ex. 4:**</u> Consider the differential 1-form $\omega = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz$ on $\mathbb{R}^3$, where $A,B,C\colon \mathbb{R}^3 \to \mathbb{R}$ are smooth functions without common zeros. To simplify, we will write functions without their arguments, bar the first time they are written.

(a) Let us first show that $\xi = \ker\omega$ is a 2-dimensional vector bundle over $\mathbb{R}^3$. Take an arbitrary derivation $V = \alpha(x,y,z)\frac{\partial}{\partial x} + \beta(x,y,z)\frac{\partial}{\partial y} + \gamma(x,y,z)\frac{\partial}{\partial z}$ and fix a point $p \in \mathbb{R}^3$. In $p$, we calculate
$$\omega_p(V) = A\alpha + B\beta + C\gamma|_p = A(p)\alpha(p) + B(p)\beta(p) + C(p)\gamma(p).$$
Since $A$, $B$, and $C$ do not have a common zero, at least one of them does not have a zero in $p$. Without loss of generality, let $C(p) \neq 0$. We can now express the coefficient $\gamma$ in the following way
$$\gamma(p) = -\frac{A(p)\alpha(p) + B(p)\beta(p)}{C(p)}.$$
For every $p \in \mathbb{R}^3$, $\ker\omega_p$ has 2 dimensions and is a vector subspace of $T_p\mathbb{R}^3$, making $\ker\omega = \bigsqcup_{p\in\mathbb{R}^3}\ker\omega_p$ a vector subbundle over $\mathbb{R}^3$.

(b) Suppose $C \equiv 1$, and $A$ and $B$ are do not depend on $z$. We will find the conditions on $A$ and $B$ such that $\xi$ is totally integrable.

Recall that by Frobenius' theorem, $\xi$ is totally integrable if and only if $\xi$ is an involutive subbundle, that is, for every point $p \in \mathbb{R}^3$ there has to exist $v_1, v_2$ vector fields in a neighbourhood of $p$, so that $v_1$ and $v_2$ span $\xi$ and $[v_1,v_2]$ is tangent to $\xi$. Note, that this theorem is independant of the choice of $v_1, v_2$, so we can take any two vector fields, that satisfy the conditions.

With that in mind, take $v_1 = \frac{\partial}{\partial x} - A\frac{\partial}{\partial z}$ and $v_2 = \frac{\partial}{\partial y} - B\frac{\partial}{\partial z}$ as they clearly span $\xi$. Let us calculate the Lie bracket, cancelling second order parts as we go. Also note that $A_z = B_z = 0$ by assumption.

$$\begin{aligned}[v_1, v_2] &= [\frac{\partial}{\partial x} - A\frac{\partial}{\partial z}, \frac{\partial}{\partial y} - B\frac{\partial}{\partial z}] = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] + [-A\frac{\partial}{\partial z}, \frac{\partial}{\partial y}] + [\frac{\partial}{\partial x}, -B\frac{\partial}{\partial z}] + [A\frac{\partial}{\partial z}, B\frac{\partial}{\partial z}] \\ &= (A_y - B_x)\frac{\partial}{\partial z}.\end{aligned}$$

Now, insert it into $\omega$:

$$\omega((A_y - B_x)\frac{\partial}{\partial z}) = A_y - B_x = 0.$$

The condition we are searching for is

$$A_y = B_x.$$

(c) Additionally assume that there exists a function $F\colon \mathbb{R}^3 \to \mathbb{R}$ such that $dF = \omega$. Let us find it's integral manifolds of $\xi$.

The only thing to notice is that on $\xi = \ker \omega$ the equation transforms into $dF = 0$. Integrating both sides we get $F = c$ for constant $c \in \mathbb{R}$. In other words, $F$ is the integral of $\xi$ (in the sense of the theory of differential equations) and the integral manifolds must have the form $F^{-1}(c)$ for every $c \in \mathbb{R}$.

(d) Concretely, let $\omega = dz + xdy$. We will find the basis of $\xi$ and show that their commutator (Lie bracket) does not belong to $\xi$.

Clearly, for the basis we can take $v_1 = \frac{\partial}{\partial x}$ and $v_2 = x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}$. Let us calculate

$$[v_1, v_2] = [\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}] = [\frac{\partial}{\partial x}, x\frac{\partial}{\partial z}] - [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = \frac{\partial}{\partial x}(x\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}.$$

Now, insert it into $\omega$

$$\omega(\frac{\partial}{\partial z}) = 1 \implies [v_1, v_2] \notin \xi.$$