# Complex Analysis Homework

Deadline: 20/1/2020 at 12:00

(I) Let $\Omega \subset \mathbb{C}$ be an open subset and let $\mathcal{F}$ be a family of holomorphic functions from $\Omega$ to $\mathbb{C}$.

(a) Show that $\mathcal{F}$ is a normal family if and only if $\mathcal{F}_{|D} := \{f_{|D} : f \in \mathcal{F}\}$ is a normal family for every disk $D \subset \Omega$;

(b) assume that there exists $a \in \Omega$ such that $\{f(a) : f \in \mathcal{F}\} \subset \mathbb{C}$ is bounded. Prove that $\mathcal{F}$ is a normal family if $\mathcal{F}' := \{f' : f \in \mathcal{F}\}$ is uniformly bounded.

(II) Let $n \in \mathbb{N}$ and $a_n, b_n \in \mathbb{R}$ such that $0 < b_n < a_n < n$.

(a) Show that there exists a polynomial $p_n$ such that $|p_n(z)| > n$ for $z \in \mathbb{D}(0,n)$ and $\Im(z) = b_n$ and $|p_n(z)| < \frac{1}{n}$ for $z \in \mathbb{D}(0,n)$ and $\Im(z) > a_n$ or $\Im(z) < 0$.

(b) Construct a sequence of polynomials pointwise converging to 0 and such that the convergence is uniform on compact subsets of $\mathbb{C} \setminus \mathbb{R}$, but not in any neighborhood of a real point.

(c) Costruct a sequence of polynomials pointwise converging to 0 on the real line and pointwise converging to 1 on $\mathbb{C} \setminus \mathbb{R}$.

(III) Let $f, g, h : \mathbb{C} \to \mathbb{C}$ be holomorphic functions satisfying $h = e^f + e^g$.

(a) Prove that the equation $h(z) = 0$ has either infinitely many solution or none at all.

(b) Prove that the equation $e^z = p(z)$ admits a solution for any non-constant polynomial $p$.
*Hint*: Assuming there are no solutions, prove the existence of a logarithm and use part (a).

(IV) Let $f : \mathbb{D} \to \mathbb{C}$ be an injective holomorphic function such that $f(0) = 0$ and $f'(0) = 1$ (i.e. $f \in \mathcal{S}$ is schlicht). Assume that $D = f(\mathbb{D})$ is convex and let $r \in (0,1)$ and $e^{i\theta} \in \partial\mathbb{D}$.

(a) Prove that

$$\frac{1}{2}re^{i\theta} = \frac{1}{2\pi i}\int_{|z|=r} f(z)\left(1 + \frac{z}{2re^{i\theta}} + \frac{re^{i\theta}}{2z}\right)\frac{dz}{z}.$$

(b) Using part (a), show that

$$\frac{1}{2}re^{i\theta} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(re^{i\phi})cos^2\left(\frac{\theta-\phi}{2}\right)d\phi.$$

(c) Show that $\frac{1}{2}re^{i\theta} \in D$ and conclude that $\mathbb{D}(0, \frac{1}{2}) \subset D$.
*Hint*: Use that $\int_{-\pi}^{\pi} cos^2\left(\frac{\phi}{2}\right) d\phi = \pi$ and that $D$ is convex. In particular, use the fact that given $z_0 \notin D$, there exists $z \in \mathbb{C}$ such that $\langle z, z_0\rangle > \langle z, w\rangle$ for every $w \in D$, where $\langle z, w\rangle$ denotes the standard scalar product between $w, z \in \mathbb{C}$ see as elements of $\mathbb{R}^2$.

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Use of computer is allowed, collaboration with colleagues is forbidden.
Please include and sign the following statement:
*I declare that I solved the homework problems by myself.*

You can send the solutions via email to *riccardo.ugolini@fmf.uni-lj.si* or place them in my mailbox at the ground floor of Jadranska 19.