Analysis on Manifolds Homework

Deadline: 20/1/2020 at 12:00

(I) Let $F : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^6$ be such that

$$F(x_1, x_2, x_3) = (x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3).$$

(a) Prove that F is an immersion.

(b) Is F injective? Justify your answer.

(c) Determine the fibers of $F|_{S^2}$, then prove that

$$F(x_1 : x_2 : x_3) = \frac{(x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3)}{x_1^2 + x_2^2 + x_3^2}$$

defines a injective immersion from $\mathbb{RP}^2$ to $\mathbb{R}^6$.

(II) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be such that $f(x, y, z) = xy + z$ and let M be the zero set of f. Consider the vector field $V = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (z + 3xy)\frac{\partial}{\partial z}$

(i) Prove that M is a manifold.

(ii) Show that V restricts to a vector field on M (i.e. a section of the tangent bundle of M).

(iii) Denote by W the restriction of V to M. Does W have fixed points? Are they locally stable?

(III) Let $M = \{((x, y), [v : w]) \in \mathbb{C}^2 \times \mathbb{CP}^1 : xw = yv\}$ and let $\pi : M \to \mathbb{C}^2$ be such that $\pi((x, y), [v : w]) = (x, y)$.

(a) Prove that M is a complex manifold.

(b) Prove that $\pi|_{M\setminus\pi^{-1}\{(0,0)\}}$ is a biholomorphism onto $\mathbb{C}^2 \setminus \{(0, 0)\}$.

(c) Show that the map $p : M \to \mathbb{CP}^1$ given by $\pi((x, y), [v : w]) = [v : w]$ is a holomorphic line bundle over $\mathbb{CP}^1$, hence equivalent to a (possibly negative) power of the tautological bundle $\mathcal{O}(-1)^k = \mathcal{O}(-k)$ for $k \in \mathbb{Z}$. Which one?

(IV) Consider the differential 1-form $\omega = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz$ on $\mathbb{R}^3$, where $A, B, C : \mathbb{R}^3 \to \mathbb{R}$ are smooth functions without common zeros.

(a) Show that $\xi = \ker \omega$ is a 2-dimensional vector bundle over $\mathbb{R}^3$.

(b) Suppose $C \equiv 1,\ A = A(x, y)$ and $B = B(x, y)$ (i.e. A and B do not depend on z). Find conditions on A and B such that ξ is integrable.

(c) Additionally assume that there exists $F : \mathbb{R}^3 \to \mathbb{R}$ such that $dF = \omega$, then determine the integral manifolds of ξ.

(d) Let $\omega = dz + xdy$. Find a basis of ξ and show that their commutator does not belong to ξ.

Use of computer is allowed, collaboration with colleagues is forbidden.
Please include and sign the following statement:
I declare that I solved the homework problems by myself.

You can send the solutions via email to *riccardo.ugolini@fmf.uni-lj.si* or place them in my mailbox at the ground floor of Jadranska 19.