TEORIJA MERE – DOMAČA NALOGA

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1) $n \in \mathbb{N}$, $a_n \in [0,\infty)$, μ_n mera na σ-alg. $\mathcal{A}$. $\mu = \sum_n a_n \mu_n$ mera na $\mathcal{A}$:

- $\mu(\emptyset) = \sum_{n=1}^{\infty} a_n \mu_n(\emptyset) \underset{\substack{\uparrow \\ \mu_n \text{ mera}}}{=} \sum_{n=1}^{\infty} a_n \cdot 0 = 0$ ✓

- $\mu(\bigcup_n A_n) = \sum_m a_m \mu_m(\bigcup_n A_n) \underset{\substack{\uparrow \\ \mu_m \text{ mera}}}{=} \sum_m a_m \sum_n \mu_m(A_n) = \sum_m \sum_n a_m \mu_m(A_n) \overset{\text{TONELLI ZA VRSTE}}{=}$

$= \sum_n \sum_m a_m \mu_m(A_n) = \sum_n \mu(A_n)$ ✓

2) Naj bo μ translacijsko invariantna mera na $\mathcal{B}_{\mathbb{R}}$ z $\mu([0,1)) = M < \infty$.

a) $a, b \in \mathbb{R}$; $b-a \in \mathbb{Q}_+ \Rightarrow \exists n \in \mathbb{N}: n(b-a) \in \mathbb{N}$.

Po eni strani: $\mu(n[a,b)) \overset{\text{TRANS. INV.}}{=} \mu(n[0,b-a)) = \mu([0, n(b-a))) \overset{\text{ADITIVNOST}}{=} n\cdot(b-a)\cdot M = n \cdot m([a,b))\cdot M$

Po drugi pa: $\mu(n[a,b)) = n \cdot \mu([a,b))$.

Pokrajšamo n in dobimo $M = k$: $\mu([a,b)) = k \cdot m([a,b))$.

b) Dokažimo najprej za poljubne intervale $[a,b)$; $b-a \in \mathbb{R}_+$.
Ker je $\mathbb{Q}_+$ gosta v $\mathbb{R}_+$, obstaja zaporedje racionalnih števil $(a_n)_{n\in\mathbb{N}}$, da je $\lim_{n\to\infty} a_n = b-a$. Upoštevamo notranjo zveznost mere:

$$\mu([a,b)) \underset{\substack{\uparrow \\ \text{TRANS. INV}}}{=} \mu([0,b-a)) = \mu(\bigcup_n [0,a_n)) \underset{\substack{\uparrow \\ \text{NOTRANJA ZVEZNOST}}}{=} \lim_{n\to\infty} \mu([0,a_n)) \overset{(a)}{=}$$

$$= \lim_{n\to\infty} k \cdot a_n = k \cdot \lim_{n\to\infty} a_n = k\cdot(b-a) = k \cdot m([a,b))$$ ✓

Vzemimo sedaj poljubno omejeno Borelovo množico B. Po omejenosti je vsebovana v nekem intervalu $[-n, n) =: X_n$. Uporabili bomo idejo iz naloge 27) iz vaj. Naj bo torej $\mathcal{D} = \{A \in \mathcal{B}_{\mathbb{R}}|_{X_n};\ \mu(A) = k\cdot m(A)\}$ in $\Pi = \{[a,b)\ ;\ [a,b) \subseteq X_n\}$. Zaradi zaprtosti za preseke je Π očitno π-sistem. Če pokažemo, da je $\mathcal{D}$ λ-sistem, ki vsebuje Π in $\sigma(\Pi) = \mathcal{B}_{\mathbb{R}}|_{X_n}$, bo po $\lambda(\Pi) = \sigma(\Pi)$ sledilo $B \in \mathcal{D}$.

Na začetku te točke smo v bistvu pokazali $\Pi \subset \mathcal{D}$. Iz predavanj še vemo, da Π generira $\mathcal{B}_{\mathbb{R}}|_{X_n}$, torej $\mathcal{B}_{\mathbb{R}}|_{X_n} = \sigma(\Pi)$.
Ali je $\mathcal{D}$ λ-sistem? Po (a) je $X_n \in \mathcal{D}$.

Cont.

• Vzemimo $A \in \mathcal{D}$. $\mu(A^c) := \mu(X_n \setminus A) \underset{\mu(X_n)<\infty}{=} \mu(X_n) - \mu(A) =$

$$= k \cdot \underset{\substack{\uparrow \\ \text{točka (a)}}}{m(X_n)} - k \cdot \underset{\substack{\uparrow \\ \text{def. } \mathcal{D}}}{m(A)} = k \cdot m(X_n \setminus A) =: k \cdot m(A^c) \quad (\text{na } X_n) \checkmark$$

• Vzemimo $(A_n)_n \subset \mathcal{D}$ disj.:

$$\mu\left(\bigcup_n A_n\right) \underset{\substack{\uparrow \\ \mu \text{ mera}}}{=} \sum_n \mu(A_n) \underset{\substack{\uparrow \\ \text{def. } \mathcal{D}}}{=} \sum_n k \cdot m(A_n) \underset{\substack{\uparrow \\ m \text{ mera}}}{=} k \cdot m\left(\bigcup_n A_n\right) \checkmark$$

c) Vzemimo sedaj $A \in \mathcal{B}_{\mathbb{R}}$. Iz (b) dobimo naraščajoče zaporedje $(A \cap X_n)_n$. Zaveda velja $A = \bigcup_{n \in \mathbb{N}} (A \cap X_n)$.

Zato uporabimo notranjo zveznost:

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) \overset{(b)}{=} \lim_{n\to\infty} m(A \cap X_n) \cdot k = k \cdot \lim_{n \to\infty} m(A \cap X_n) = k \cdot m(A).$$

3) Naj bo $m \times \xi$ ena izmed produktnih mer na $\mathcal{B}_{[0,1]\times[0,1]}$, tako, da $(m\times\xi)(A\times B) = m(A)\cdot\xi(B)$.

Δ = diagonala

a) $1_\Delta : [0,1]^2 \to \mathbb{R}$ Borelovo merljiva: ogledamo si praslike $A \in \mathbb{R}$.

• $0,1 \in A \Rightarrow 1_\Delta^{-1}(A) = [0,1]\times[0,1]$ ✓

• $0 \in A,\ 1 \notin A \Rightarrow 1_\Delta^{-1}(A) = [0,1]^2 \setminus \Delta$ odprta v $[0,1]^2 \Rightarrow$ merljiva ✓

• $0 \notin A,\ 1 \in A \Rightarrow 1_\Delta^{-1}(A) = \Delta$ zaprta v $[0,1]^2 \Rightarrow$ komplement odprte $\Rightarrow$ merljiva ✓

• $0,1 \notin A \Rightarrow 1_\Delta^{-1}(A) = \emptyset$ ✓

b) Računajmo integrale. Označimo $I := [0,1]$.

Definiramo sekcije: $\Delta_x = \{y \in I;\ (x,y) \in \Delta\} = \{x\}$, $\Delta^y = \{x \in I;\ (x,y) \in \Delta\} = \{y\}$.

$1_{\Delta_x} : [0,1]_y \to \mathbb{R}$; $1_{\Delta_x}(y) = 1_\Delta(x,y)$ in $1_\Delta^y : [0,1]_x \to \mathbb{R}$; $1_\Delta^y(x) = 1_\Delta(x,y)$.

• $$\int_I \int_I 1_\Delta(x,y)\, dm(x)\, d\xi(y) = \int_I \left(\int_I 1_\Delta^y(x)\, dm(x)\right) d\xi(y) = \int_I \underset{\substack{\shortparallel \\ 0}}{m(\{y\})}\, d\xi(y) = 0$$

• $$\int_I \int_I 1_\Delta(x,y)\, d\xi(y)\, dm(x) = \int_I \left(\int_I 1_{\Delta_x}(y)\, d\xi(y)\right) dm(x) = \int_I \xi(\{x\})\, dm(x) = \int_I 1\, dm(x) = 1.$$

Po konstrukciji mere $m\times\xi$ imamo $(m\times\xi)(\Delta) = \inf\{\sum_{n=1}^\infty \underbrace{(m\times\xi)([a_n,b_n]\times[c_n,d_n])}_{P};\ \Delta \subset P\}$

Ampak, $(m\times\xi)(P) = \underbrace{(b_n - a_n)}_{\neq 0} \cdot \underbrace{\xi([c_n,d_n])}_{\infty} = \infty$ za vse P.

$\Rightarrow (m\times\xi)(\Delta) = \infty$.

• $$\int_{I\times I} 1_\Delta(x,y)\, (m\times\xi)(x,y) \underset{\substack{\uparrow \\ \text{def. integrala}}}{=} (m\times\xi)(\Delta) = \infty.$$

c) V izreku ni izpolnjen pogoj σ-končnosti za mero števna ξ.

4) $\lambda(A) = \sum_{n\in A} \frac{e^{in^2}}{3^{|n|}}$ kompleksna mera, $\mu(A) = \sum_{n\in A\cap 2\mathbb{Z}} \frac{1}{(n+1)^2}$ pozitivna mera.

• $|\lambda|$:

$|\lambda|(A) = \sup\{\sum_{n=1}^{\infty} |\lambda(A_n)| \, ; (A_n)_n \text{ particija za } A\} \underset{\text{NAJVEČ, ČE RAZBIJEMO PO TOČKAH}}{\overset{\text{CBS}}{=}} \sum_{n\in A} \left|\frac{e^{in^2}}{3^{|n|}}\right| = \sum_{n\in A}\frac{1}{3^{|n|}}$

• Lebesguov razcep:

$\mu(A)=0 \Rightarrow A\subseteq \mathbb{Z}\setminus 2\mathbb{Z}$ LIHA ŠTEVILA

Dobimo: $\lambda(A) = \sum_{n\in A}\frac{e^{in^2}}{3^{|n|}} = \underbrace{\sum_{n\in A\cap 2\mathbb{Z}} \frac{e^{in^2}}{3^{|n|}}}_{\lambda_a} + \underbrace{\sum_{n\in A\cap(\mathbb{Z}\setminus 2\mathbb{Z})} \frac{e^{in^2}}{3^{|n|}}}_{\lambda_s}$

$\lambda_a \ll \mu$ po definiciji, $\lambda_s \perp \mu$, ker λ_s skoncentrirana na $\mathbb{Z}\setminus 2\mathbb{Z}$.

• Radon-Nikodymov odvod: $\exists f\in L^1(\mu)$, da je $\lambda_a(A) = \int_A f\,d\mu$.

Poskusimo $\frac{d\lambda_a}{d\mu} = \sum_{n\in A\cap 2\mathbb{Z}} \frac{e^{in^2}(n+1)^2}{3^{|n|}}$

$\int_A \sum_{n\in A\cap 2\mathbb{Z}} \frac{e^{in^2}(n+1)^2}{3^{|n|}}\,d\mu \underset{\text{FUBINI}}{=} \sum_{n\in A\cap 2\mathbb{Z}} \frac{e^{in^2}(n+1)^2}{3^{|n|}\underbrace{(n+1)^2}_{\mu(\{*\})}} = \lambda_a(A)$

• $|\lambda|(\mathbb{Z}) = \sum_{n\in\mathbb{Z}} \frac{1}{3^{|n|}} = 2\sum_{n=0}^{\infty} \frac{1}{3^n} - \underbrace{1}_{\frac{1}{3^0}} = 2\frac{1}{1-\frac{1}{3}} - 1 = \frac{2}{\frac{2}{3}} - 1 = 2$

5) $f:[0,1]^3 \to [0,\infty]$, $f(x,y,z) = \begin{cases}\frac{1}{\sqrt{|y-z|}}, & \text{če } y\neq z\\ \infty, & \text{sicer}\end{cases}$ merljiva, ker zvezna.

$f\in L^1(m_3)$:

$\int_{[0,1]^3}|f|\,dm_3 = \int_{\{y\neq z\}}|f|\,dm_3 + \underbrace{\int_{\{y=z\}} |f|\,dm_3}_{\text{MERA 0}} =$

$= \int_{\{y\neq z\}} \frac{1}{\sqrt{|y-z|}}\,dm_3 \overset{\text{TONELLI}}{=} \int_{[0,1]_x} dm \int_{[0,1]_y} dm \int_{[0,y]\cup[y,1]_z} \frac{1}{\sqrt{|y-z|}}\,dm \overset{R=L}{=} \underbrace{\int_0^1 dx\int_0^1 dy \int_0^y \frac{dz}{\sqrt{y-z}}}_{I_1} + \underbrace{\int_0^1 dx \int_0^1 dy\int_y^1 \frac{dz}{\sqrt{z-y}}}_{I_2} =$

$I_1 = 1\cdot\int_0^1 dy \, \frac{-1}{\frac{1}{2}}\sqrt{y-z}\Big|_{z=0}^{z=y} = 2\int_0^1\sqrt{y}\,dy = 2\frac{1}{\frac{3}{2}} y^{\frac{3}{2}}\Big|_{y=0}^{y=1} = \frac{4}{3}$

$I_2 = 1\cdot\int_0^1 dy \, \frac{1}{\frac{1}{2}}\sqrt{z-y}\Big|_{z=y}^{z=1} = 2\cdot\int_0^1\sqrt{1-y}\,dy = 2\cdot\frac{-1}{\frac{3}{2}}(1-y)^{\frac{3}{2}}\Big|_{y=0}^{y=1} = \frac{4}{3}$

$\Rightarrow \int_{[0,1]^3}|f|\,dm_3 = \frac{8}{3} < \infty$ ✓

6) Na $([0,1], \mathcal{B}_{[0,1]}, m)$ definiramo zaporedje funkcij
$f_1 = 1_{[0,1]}$, $f_2 = \sqrt{2}\, 1_{[0,\frac{1}{2}]}$, $f_3 = \sqrt{2}\, 1_{[\frac{1}{2},1]}$, ...,
$f_{2^n+k} = 2^{\frac{n}{2}} 1_{[\frac{k}{2^n}, \frac{k+1}{2^n}]}$ za $n \in \mathbb{N}$, $0 \le k \le 2^{n-1}$ (POPRAVLJENA VERZIJA)

Kako $(f_n)_n$ konvergira? Če konv, ka k funkciji $0 \to$ stekališče.

- po točkah: NE. $\forall x \in [0,1]\ \forall n \in \mathbb{N}\ \exists m > n : f_m(x) > 1$, saj zaporedoma pokrivamo interval $[0,1]$ v celoti.

$\Rightarrow$ NE enakomerno

- skoraj povsod: NE, isto kot po točkah.

$\Rightarrow$ NE skoraj enakomerno.

- po meri: $\varepsilon > 0$. ~~...~~ $m = 2^n + k$
~~...~~ $\{x \in [0,1];\ |f_m(x)| \ge \varepsilon\} = \begin{cases} [\frac{k}{2^n}, \frac{k+1}{2^n}] & ; \text{če } \varepsilon \le 2^{\frac{n}{2}} \\ \emptyset & ; \text{če } \varepsilon > 2^{\frac{n}{2}}. \end{cases}$

$\Rightarrow \mu(\{x \in [0,1];\ |f_m(x)| \ge \varepsilon\}) \le \frac{1}{2^n} \xrightarrow{n \to \infty} 0.$

~~...~~ Po meri konvergira k funkciji 0.