Algebraična topologija 2 - 2. domača naloga

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Ex. 1:

(a) Let us describe the Δ-structure of the given space X. After considering identifications we are left with four 2-simplices σ_1, σ_2, σ_3 and σ_4 (as described in the instructions of this exercise), three 1-simplices, let us denote them with $a = [01]$, $b = [12]$ and $c = [02]$, and a single 0-simplex $P = [0]$.

(b) We obtain the following chain complex

$$0 \to \mathbb{Z}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \xrightarrow{\partial_2} \mathbb{Z}(a, b, c) \xrightarrow{\partial_1} \mathbb{Z}(P) \to 0$$

Firstly, it is obvious that $\partial_1 \equiv 0$, since we only have one 0-simplex. Secondly, we immediately see

$$\begin{aligned} \partial_2(\sigma_1) &= b + c + a & \qquad \partial_2(\sigma_2) &= a + c + b \\ \partial_2(\sigma_3) &= b + c + a & \qquad \partial_2(\sigma_4) &= a + c + b \end{aligned}$$

It follows that the matrix form of ∂_2 is

$$\sigma_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

yielding

$$\begin{aligned} \ker \partial_2 &= \mathbb{Z}(\sigma_1 - \sigma_4, \sigma_2 - \sigma_4, \sigma_3 - \sigma_4) \\ \operatorname{im} \partial_2 &= \mathbb{Z}(a + b + c) \end{aligned}$$

From this we easily calculate

$$\begin{aligned} &H_0(X) = \mathbb{Z}(P)/\ker \partial_1 = \mathbb{Z}(P) \cong \mathbb{Z} \\ &H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z}(a, b, c)/\mathbb{Z}(a + b + c) \cong \mathbb{Z}(a, b, a + b + c)/\mathbb{Z}(a + b + c) \cong \mathbb{Z}(a, b) \cong \mathbb{Z}^2 \\ &H_2(X) = \ker \partial_2 = \mathbb{Z}(\sigma_1 - \sigma_4, \sigma_2 - \sigma_4, \sigma_3 - \sigma_4) \cong \mathbb{Z}^3 \end{aligned}$$

(c) By the usual procedure we obtain the following cochain complex

$$0 \leftarrow \mathbb{Z}(\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3}, \overline{\sigma_4}) \xleftarrow{d_2} \mathbb{Z}(\overline{a}, \overline{b}, \overline{c}) \xleftarrow{d_1} \mathbb{Z}(\overline{P}) \leftarrow 0$$

where by the usual notation $\overline{x}$, where $x \in C_n(X)$ generator, denotes the homomorphism in $\operatorname{Hom}(C_n(X), \mathbb{Z})$ that takes x to 1 and the other generators to 0. Since we have $d_1 = \partial_1^T \equiv 0$ and $d_2 = \partial_2^T$ (as matrices), we easily obtain

$$\begin{aligned} \ker d_2 &= \mathbb{Z}(\overline{a} - \overline{c}, \overline{b} - \overline{c}) \\ \operatorname{im} d_2 &= \mathbb{Z}(\overline{\sigma_1} + \overline{\sigma_2} + \overline{\sigma_3} + \overline{\sigma_4}) \end{aligned}$$

We finally calculate

$$\begin{aligned} &H^0(X) = \ker d_1 = \mathbb{Z}(\overline{P}) \cong \mathbb{Z} \\ &H^1(X) = \ker d_2 / \operatorname{im} d_1 = \ker d_2 = \mathbb{Z}(\overline{a} - \overline{c}, \overline{b} - \overline{c}) \cong \mathbb{Z}^2 \\ &H^2(X) = \mathbb{Z}(\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3}, \overline{\sigma_4})/\operatorname{im} d_2 \cong \mathbb{Z}(\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3}, \overline{\sigma_1} + \cdots + \overline{\sigma_4})/\mathbb{Z}(\overline{\sigma_1} + \cdots + \overline{\sigma_4}) = \mathbb{Z}(\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3}) \cong \mathbb{Z}^3 \end{aligned}$$

(d) Let us compute the cup and cap product on the (co)homology of X.

- **Cup:** We quickly see, that $\overline{P}$ is the neutral element for the cup product. We are hence only interested in the product of 1-cocycles. Looking at 1-front and back faces we quickly obtain (on the level of chains)

$$\overline{a} \cup \overline{a} = \overline{b} \cup \overline{b} = \overline{c} \cup \overline{c} = 0$$

  and

$$\overline{a} \cup \overline{c} = \overline{b} \cup \overline{c} = 0 \qquad \overline{a} \cup \overline{b} = \overline{\sigma_1} + \overline{\sigma_3} \qquad \overline{b} \cup \overline{a} = \overline{\sigma_2} + \overline{\sigma_4}$$

  which turns out to be correct once we look only at generating cocycles.

- **Cap:** Again, the cochain $\overline{P}$ quickly turns out to be a right unit. Onwards, we calculate

$$a \cap \overline{a} = P$$

  and likewise for b and c, while cap products of non-corresponding letters obviously turn out to be trivial. Similarly we get

$$\sigma_1 \cap \overline{\sigma_1} = P$$

  and likewise for σ_2, σ_3 and σ_4. Cap products $\sigma_i \cap \overline{\sigma_j}$ for $i \neq j$ again turn out to be trivial. The remaining combinations are

$$\begin{array}{cccc} \sigma_1 \cap \overline{a} = b & \sigma_2 \cap \overline{a} = 0 & \sigma_3 \cap \overline{a} = b & \sigma_4 \cap \overline{a} = 0 \\ \sigma_1 \cap \overline{b} = 0 & \sigma_2 \cap \overline{b} = a & \sigma_3 \cap \overline{b} = 0 & \sigma_4 \cap \overline{b} = a \end{array}$$

Ex. 2: Let G be a topological group and $\pi\colon E \to B$ a principal G-bundle.

(a) Let $f\colon (\mathbb{S}^n, x_0) \to (B, b_0)$ be a continuous map and $f^*E \to \mathbb{S}^n$ the pullback principal G-bundle, where $c\colon \mathbb{S}^{n-1} \to G$ is the transition function of f^*E. Let

$$\cdots \to \pi_n(G,1) \to \pi_n(E,e_0) \to \pi_n(B,b_0) \xrightarrow{\partial} \pi_{n-1}(G,1) \to \cdots$$

be the long exact sequence for the bundle E. Let us show $\partial\colon [f] \mapsto [c]$.

This will be easy to see once we recall what a pullback bundle along a continuous map is. We define $f^*E = \{(z,e) \in \mathbb{S}^n \times E;\ f(z) = \pi(e)\}$. The important part for us is that the following is a commutative diagram

$$\begin{tikzcd} f^*E \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & E \arrow[d, "\pi"] \\ \mathbb{S}^n \arrow[r, "f"] & B \end{tikzcd}$$

Moreover, denoting U_+ and U_- to be the upper and lower hemisphere of $\mathbb{S}^n$ respectively, we have $f^*E \approx U_+ \times G \sqcup U_- \times G/\sim$, where $(z,g) \sim (z, c(z)g)$ for $z \in U_+ \cap U_-$, that is, the equator.

Now that we have revised what we need, consider the following commutative ladder with exact rows

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(G,1) \arrow[r] \arrow[d, "id"] & \pi_n(f^*E,(x_0,e_0)) \arrow[r, "\mathrm{pr}_{1\#}"] \arrow[d, "\mathrm{pr}_{2\#}"] & \pi_n(\mathbb{S}^n, x_0) \arrow[r, "\partial_2"] \arrow[d, "f_\#"] & \pi_{n-1}(G,1) \arrow[r] \arrow[d, "id"] & \cdots \\ \cdots \arrow[r] & \pi_n(G,1) \arrow[r] & \pi_n(E,e_0) \arrow[r, "\pi_\#"] & \pi_n(B,b_0) \arrow[r, "\partial_1"] & \pi_{n-1}(G,1) \arrow[r] & \cdots \end{tikzcd}$$

Since the loop $[f] \in \pi_n(B, b_0)$ is actually the image $f(\mathbb{S}^n)$, we have $[f] = f_\#(1)$, where 1 is the generator of the group $\mathbb{Z} = \pi_n(\mathbb{S}^n)$. Now let us use the commutativity of the above ladder and the relation $\sim$ to get

$$\partial_1[f] = \partial_1 f_\#(1) = \partial_2(1) = [c]$$

by an exercise from tutorials.

(b) Let $p\colon U(2) \to \mathbb{S}^3$ be a principal $U(1)$-bundle, where p takes a 2×2 matrix and maps it to its first column. Let us calculate $\pi_k(U(2), Id)$ in terms of $\pi_k(U(1), Id)$ and $\pi_k(\mathbb{S}^3, (1,0))$.

We can improve on the instructions. Consider the following

$$U(1) = \{a \in \mathbb{C};\ a\bar{a} = \bar{a}a = 1\} = \{a \in \mathbb{C};\ \Re(a)^2 + \Im(a)^2 = 1\} \approx \mathbb{S}^1 \subset \mathbb{C}$$

Now consider the long exact sequence for this bundle

$$\cdots \to \pi_n(\mathbb{S}^1, 1) \to \pi_n(U(2), Id) \to \pi_n(\mathbb{S}^3, (1,0)) \to \pi_{n-1}(\mathbb{S}^1, 1) \to \cdots$$

Since we know $\pi_n(\mathbb{S}^1) \cong \mathbb{Z}$ precisely when $n = 1$ and is trivial otherwise, exactness of the above sequence gives us the following results:

- <u>$n \geq 3$:</u> $\pi_n(U(2), Id) \cong \pi_n(\mathbb{S}^3, (1,0))$,
- <u>$n = 2$:</u> Since $\pi_2(\mathbb{S}^3) \cong \pi_2(\mathbb{S}^1) \cong 0$, we get $\pi_2(U(2), (1,0)) = 0$,
- <u>$n = 1$:</u> Since $\pi_1(\mathbb{S}^3) \cong 0$ and $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, we get $\pi_1(U(2), Id) \cong \pi_1(U(1), Id) \cong \mathbb{Z}$,
- <u>$k = 0$:</u> Since both of the other sets are trivial, so is $\pi_0(U(2), Id) = \{0\}$, that is $U(2)$ is path-connected.

Is the transition map $c\colon \mathbb{S}^2 \to U(1) \approx \mathbb{S}^1$ nullhomotopic? Indeed it is; since $\pi_2(\mathbb{S}^1) = 0$, every map $\mathbb{S}^2 \to \mathbb{S}^1$ is homotopic to some constant map, which in particular holds for $c\colon \mathbb{S}^2 \to \mathbb{S}^1$.

(c) Let us use the principal $U(2)$-bundle $p\colon U(3) \to \mathbb{S}^5$ to calculate $\pi_1(U(3))$, $\pi_2(U(3))$ and $\pi_3(U(3))$.

Again we consider the long exact sequence

$$\begin{aligned} 0 &\to \pi_3(U(2)) \to \pi_3(U(3)) \to \pi_3(\mathbb{S}^5) \to \pi_2(U(2)) \to \pi_2(U(3)) \to \pi_2(\mathbb{S}^5) \to \\ &\to \pi_1(U(2)) \to \pi_1(U(3)) \to \pi_1(\mathbb{S}^5) \to 0 \end{aligned}$$

Since we know $\pi_n(\mathbb{S}^5) \cong 0$ for $n < 5$, by exactness we get:

$$\begin{aligned} \pi_3(U(3)) &\cong \pi_3(U(2)) \cong \pi_3(\mathbb{S}^3) \cong \mathbb{Z} \\ \pi_2(U(3)) &\cong \pi_2(U(2)) \cong 0 \\ \pi_1(U(3)) &\cong \pi_1(U(2)) \cong \mathbb{Z} \end{aligned}$$

(d) Let us show that the principal $SO(3)$-bundle $p\colon SO(4) \to \mathbb{S}^3$ is a trivial bundle and calculate $\pi_n(SO(4))$ in terms of $\pi_n(SO(3))$ and $\pi_n(\mathbb{S}^3)$.

From the subject Analysis on manifolds we know that $\mathbb{S}^3 \to \mathbb{RP}^3$ is a universal covering space, so by a theorem from lectures $\pi_n(\mathbb{S}^3) \cong \pi_n(\mathbb{RP}^3)$ for each $n \geq 2$. Moreover, we also know $\mathbb{RP}^3 \approx SO(3)$ and obtain

$$\pi_2(SO(3)) \cong \pi_2(\mathbb{RP}^3) \cong \pi_2(\mathbb{S}^3) = 0.$$

By the same argument from (b) it now follows that the transition map $c\colon \mathbb{S}^2 \to SO(3)$ is trivial, hence the the bundle is trivial, that is $SO(4) \approx \mathbb{S}^3 \times SO(3)$. This is easily seen either from the relation we were looking at in (a). It follows that for each $n \in \mathbb{N}$ we now have $\pi_n(SO(4)) \cong \pi_n(\mathbb{S}^3) \oplus \pi_n(SO(3))$.

<u>Ex. 3:</u> Let X be a compact orientable n-manifold, let $Y = \partial X$ and R a ring. Suppose X is an R-homology ball, that is $H_*(X;\ R) \cong H_*(\mathbb{B}^n;\ R)$.

(a) Let us compute $H_*(Y;\ R)$. We first notice the following

$$H_k(X;\ R) \cong H_k(\mathbb{B}^n;\ R) \cong (H_k(\mathbb{B}^n;\ \mathbb{Z}) \otimes R) \oplus \operatorname{Tor}(H_{k-1}(\mathbb{B}^n;\ \mathbb{Z}), R) = \begin{cases} 0; & k > 0 \\ R; & k = 0 \end{cases}$$

since the torsion parts always vanish here. Furthermore, in order for us to be able to use the Poincaré-Lefschetz duality later, we need X to also be R-orientable. By assumption X is $\mathbb{Z}$-orientable, that is the double manifold $DX = X \sqcup_{\partial X} X$ is orientable and so $H_n(DX) \cong \mathbb{Z}$. It follows that $H_{n-1}(DX)$ is torsion-free, and hence by universal coefficient theorem

$$H_n(DX;\ R) \cong (H_n(DX) \otimes R) \oplus \mathrm{Tor}(H_{n-1}(DX), R) \cong \mathbb{Z} \otimes R \cong R.$$

With that in mind we consider the long exact homology sequence

$$\cdots \to H_k(Y;\ R) \to H_k(X;\ R) \to H_k(X, Y;\ R) \to H_{k-1}(Y;\ R) \to H_{k-1}(X;\ R) \to \cdots$$

We look at parts of the sequence as follows

- $\underline{1 < k < n}$**:** We of course have $H_k(X;\ R) \cong H_{k-1}(X;\ R) \cong 0$ and by the Poincaré-Lefschetz duality we have $H_k(X, Y;\ R) \cong H^{n-k}(\mathbb{B}^n;\ R) \cong 0$, where the last equality trivially follows from the universal coefficient theorem for cohomology. From the long exact sequence above it now clearly follows that $H_{k-1}(Y;\ R) \cong 0$. Shifting the index we read $H_k(Y;\ R) \cong 0$ for all $1 \leq k < n-1$.
- $\underline{k = n}$**:** By the P-L duality we again get

$$H_{n-1}(Y;\ R) \cong H_n(X, Y;\ R) \cong H^0(\mathbb{B}^n;\ R) \cong R$$

- $\underline{k = 1}$**:** By the exactness of the above sequence we get

$$H_0(Y;\ R) \cong H_0(X;\ R) \cong R$$

- Since Y is a $(n-1)$-manifold, we need not compute the n-th homology group as we know it to be trivial, but we can mention that this fact follows from the above sequence as well.

Here we comment that the obtained results are in line with our intuition, that is if X is an R-homology n-ball, we can reasonably expect ∂X to be an R-homology $(n-1)$-sphere.

(b) Now suppose $n = 4$ and $R = \mathbb{Q}$. Let us show that the order of $H_1(Y;\ \mathbb{Z})$ is a square, denote it a^2. We will also describe a in terms of homology of X.

By the universal coefficient theorem for homology we have

$$(H_1(Y) \otimes \mathbb{Q}) \oplus \mathrm{Tor}(H_0(Y), \mathbb{Q}) \cong H_1(Y;\ \mathbb{Q}) \cong 0$$

Consequently, each of the summands must be trivial, in particular $H_1(Y) \otimes \mathbb{Q} \cong 0$ (in general, if R is a field, torsion vanishes). Note that $H_k(X)$ and $H_k(Y)$ are finitely generated Abelian groups and hence of the form $\mathbb{Z}^p \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_m}$ (free part plus torsion). Since $\mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q}$ and obviously $\mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$, we see that our group $H_1(Y;\ \mathbb{Z})$ is composed solely from the torsion part, since for all $m \in \mathbb{N}$ clearly $\mathbb{Z}_m \otimes \mathbb{Q} \cong 0$. Here we see that we nowhere used that our index is 1 except at $H_1(Y;\ \mathbb{Q}) \cong 0$. It follows that the same holds for $H_2(Y)$ and $H_3(Y)$.

On the other hand, since Y is closed and orientable, we have

$$(H_3(Y) \otimes \mathbb{Z}) \oplus \mathrm{Tor}(H_2(Y), \mathbb{Z}) \cong H_3(Y) \cong \mathbb{Z}$$

so $\mathrm{Tor}(H_2(Y), \mathbb{Z}) = 0$ and $H_2(Y)$ is torsion free. From the above it follows that $H_2(Y)$ is trivial. We can now look at the part of the long exact sequence for the pair (X, Y) that interests us:

$$0 \to H_2(X) \to H_2(X, Y) \to H_1(Y) \to H_1(X) \to H_1(X, Y) \to 0$$

since $H_0(Y) \cong H_0(X)$.

By the P-L duality and the universal coefficient theorem for cohomology we have

$$H_1(X, Y) \cong H^3(X) \cong \mathrm{Hom}(H_3(X), \mathbb{Z}) \oplus \mathrm{Ext}(H_2(X), \mathbb{Z}) \cong \mathrm{Ext}(H_2(X), \mathbb{Z}) \cong H_2(X)$$

and similarly

$$H_2(X,Y) \cong H^2(X) \cong \mathrm{Hom}(H_2(X),\mathbb{Z}) \oplus \mathrm{Ext}(H_1(X),\mathbb{Z}) \cong \mathrm{Ext}(H_1(X),\mathbb{Z}) \cong H_1(X)$$

since both groups are solely composed of the torsion part.

Finally, let $r = |H_2(X)|$. Then since by exactness of the above sequence $H_2(X) \to H_2(X,Y)$ is injective and we are dealing with finite groups, there exists $k \in \mathbb{N}$, so that $|H_2(X,Y)| = kr$. Again by exactness of the above sequence, using the fact that the arrows are homomorphisms, $|\operatorname{im} H_1(Y)| = \frac{|H_1(Y)|}{k}$. Using the two isomorphisms we obtained above and surjectivity of $H_1(X) \to H_1(X,Y)$ we get

$$r = |H_2(X)| = |H_1(X,Y)| = \frac{|H_1(X)|}{|\operatorname{im} H_1(Y)|} = k\frac{|H_2(X,Y)|}{|H_1(Y)|} = \frac{rk^2}{|H_1(Y)|}$$

It now follows that $|H_1(Y)| = k^2$, where clearly $k = \frac{|H_1(X)|}{|H_2(X)|}$ (again using one of the above isomorphisms).

<u>Ex. 4:</u> Let X be a closed, connected, orientable smooth n-manifold and let $Y \subset X$ be a smooth closed submanifold.

(a) Let us express the homology of the complement $H_*(X \setminus Y)$ in terms of the (co)homology of the pair (X,Y). We will use the fact that Y has a closed tubular neighbourhood N in X which is diffeomorphic to the unit disc bundle ($(n-m)$-dimensional) of the normal bundle of Y in X.

Let τ be the tubular neighbourhood for Y that is diffeomorphic to the unit disc bundle of the normal bundle $NY \subset TX|_Y$. Notice we can take away the zero level of τ (which is a natural representation of Y in τ) to get $\tau \setminus Y$ that now strongly deformationally retracts to the unit sphere bundle ($(n-m-1)$-dimensional) of the normal bundle $NY \subset TX|_Y$. Denote this sphere bundle $\mathbb{S}Y = \partial\tau = \mathbb{B}Y$. Now denote $\widetilde{X} = X \setminus \tau$ and notice that by assumption this bundle is a strong deformation retract of $X \setminus Y$. Using the P-L duality we get

$$H_k(\widetilde{X}) \cong H^{n-k}(\widetilde{X}, \partial\widetilde{X})$$

Now by assumption and the above arguments the right-hand-side pair is a strong deformation retract of the pair $(X \setminus Y, \tau \setminus Y)$, but since of course $\overline{Y} \subset \mathring{\tau}$, by excision and homotopy we have

$$H^{n-k}(X \setminus Y, \tau \setminus Y) \cong H^{n-k}(X,\tau) \cong H^{n-k}(X,Y).$$

It follows that for all k we have $H_k(X \setminus Y) \cong H^{n-k}(X,Y)$.

(b) When $m = n-1$ we wish to compute $H_0(X \setminus Y)$.

By (a) we have that $H_0(X \setminus Y) \cong H^n(X,Y)$. To compute this we consider the cohomology long exact sequence for the pair (X,Y)

$$\cdots \to H^{n-1}(Y) \to H^n(X,Y) \to H^n(X) \to H^n(Y) \to 0$$

Since Y is a manifold of dimension $m = n-1$ and X is orientable, this turns into

$$\cdots \to H^{n-1}(Y) \to H^n(X,Y) \to \mathbb{Z} \to 0$$

We can now say by exactness $H_0(X \setminus Y) \cong \operatorname{im} H^{n-1}(Y) \oplus \mathbb{Z}$, where the first summand depends on the number of path-components of Y and their orientability.

(c) Suppose X has the integral homology of a sphere. Let us express $H_*(X \setminus Y)$ in terms of (co)homology of Y.

By assumption $H_*(X) \cong H_*(\mathbb{S}^n)$, that is $H_k(X) \cong \mathbb{Z}$ precisely when $k = 0, n$ and trivial otherwise. We know from (a) that $H_k(X \setminus Y) = H^{n-k}(X,Y)$. To further calculate this, consider the long exact cohomology sequence for the pair (X,Y)

$$\cdots \to H^k(X,Y) \to H^k(X) \to H^k(Y) \to H^{k+1}(X,Y) \to \cdots$$

We consider parts of the sequence for calculating the desired homology groups

- $\underline{0 < k < n}$**:** Here $H^k(X) =$ so we get isomorphisms $H^k(Y) \cong H^{k+1}(X, Y)$. So now we extend (a) to $H_k(X \setminus Y) \cong H^{n-k}(X, Y) \cong H^{n-k-1}(Y)$, which is good enough for us.
- $\underline{k = 0}$**:** We are looking at the very end of the cohomology exact sequence

  $$0 \to H^{n-1}(Y) \to H^n(X, Y) \to \mathbb{Z} \to H^n(Y) \to 0$$

  If $m \leq n-1$ we necessarily have $H^n(Y) = 0$ (since homology is trivial as well) which turns the above into a split short exact sequence, so $H_0(X \setminus Y) \cong H^n(X, Y) \cong H^{n-1} \oplus \mathbb{Z}$. If $m = n$, since Y is a manifold, we have $H^n(Y) \cong \mathbb{Z}$ or 0, depending on the orientability. If $H^n(Y)$ is trivial, this is clearly the same as the previous case $m \leq n-1$, otherwise since $H^n(X) \cong \mathbb{Z}$ we get $H^n(X, Y) \cong H^{n-1}(Y)$, so $H_0(X \setminus Y) \cong H^{n-1}(Y)$.
- $\underline{k = n}$**:** Since $X \setminus Y \simeq X \setminus \tau$ and $X \setminus \tau$ is a compact orientable manifold with boundary, $H_n(X \setminus Y) = 0$.

(d) Let $K \subset \mathbb{S}^3$ be a knot, that is the image of an embedding $f\colon \mathbb{S}^1 \to \mathbb{S}^3$. Using (c) we will compute $H_1(\mathbb{S}^3 \setminus K)$.

By (c) we see $H_1(\mathbb{S}^3 \setminus K) \cong H^1(K) \cong H_1(K)$, where last isomorphism follows from the universal coefficient theorem for cohomology and the fact that all homology groups for a circle are free (concretely, either trivial or $\mathbb{Z}$). It follows that $H_1(\mathbb{S}^3 \setminus K) \cong \mathbb{Z}$ for any knot K, so this invariant does not distinguish knots.

Here we comment that in contrast the first homotopy group $\pi_1(\mathbb{S}^3 \setminus K)$ is very useful for distinguishing knots, which might have something to do with it not being Abelian by construction.

Ex. 5: Let $X = T \vee \mathbb{CP}^2$, where T is the 2-dimensional torus.

(a) Let s compute $\pi_2(X)$ where the base point x_0 is precisely at the concatenation of the two spaces. Firstly, since $p\colon \mathbb{R}^2 \to T$ is a covering space, by a theorem from lectures we have $\pi_2(T) \cong \pi_2(\mathbb{R}^2) \cong 0$. We also know that $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$, where each $\mathbb{Z}$ is generated by one distinguished circle on the torus. Secondly, consider the fiber bundle $\mathbb{S}^1 \hookrightarrow \mathbb{S}^5 \to \mathbb{CP}^2$. We obtain its long exact homotopy sequence

$$\cdots \to \pi_2(\mathbb{S}^1) \to \pi_2(\mathbb{S}^5) \to \pi_2(\mathbb{CP}^2) \to \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{S}^5) \to \pi_1(\mathbb{CP}^2) \to 0$$

that we fill in with known groups to

$$0 \to \pi_2(\mathbb{CP}^2) \to \mathbb{Z} \to 0 \to \pi_1(\mathbb{CP}^2) \to 0$$

By exactness we get $\pi_1(\mathbb{CP}^2) = 0$ and $\pi_2(\mathbb{CP}^2) \cong \mathbb{Z}$.

The idea is to look ahead and realize that while every image of S^2 will necessarily (homotopically) lie in $\mathbb{CP}^2$, every "baloon"we obtain by chaining 1-loops at the basepoint (the concatenation of the two spaces) to any such image must also lie in $\pi_2(X)$ and there are $(\mathbb{Z} \times \mathbb{Z})$-many such loops (all coming from T). We will therefore construct a universal covering space for X using $\mathbb{R}^2$ (the universal cover for T) and $\mathbb{CP}^2$.

Recall that $\mathbb{R}^2 \to T$ is a universal cover by way of the quotient projection $\mathbb{R}^2 \to \mathbb{R}^2/\sim \approx T$ where $(x, y) \sim (w, z) \iff \exists (m, n) \in \mathbb{Z} \times \mathbb{Z}\colon (w, z) = (x + m, y + n)$. The fundamental cells then become unit squares that are bound by consequtive integer points in $\mathbb{R}^2$. We construct the total space for our cover as follows. Define

$$E = \mathbb{R}^2 \sqcup_{\mathbb{Z} \times \mathbb{Z}} (\mathbb{CP}^2, x_0)$$

where at every integer point $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ we glue a copy of $\mathbb{CP}^2$ with $(m, n) \sim x_0$. By the above arguments $E \to X$ is a covering space via the quotient projection induced by the one from $\mathbb{R}^2 \to T$. Moreover, since $\mathbb{CP}^2$ and $\mathbb{R}^2$ are both simply connected, so is E, which makes $E \to X$ the universal cover over X.

Using the theorem for covering projections and the Hurewitz theorem, we get

$$\pi_2(X) \cong \pi_2(E) \cong H_2(E)$$

which is a significant improvement, since homology groups are much easier to compute. Indeed, since $\mathbb{R}^2$ is contractible, by retracting it to a point we get

$$\widetilde{E} = \bigvee_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} \mathbb{CP}^2.$$

Of course we now have

$$\pi_2(X) \cong \pi_2(E) \cong H_2(E) \cong H_2(\widetilde{E}) \cong \bigoplus_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} \mathbb{Z}$$

which is really isomorphic to $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$.

(b) We now want to describe the action $\pi_1(X) \times \pi_2(X) \to \pi_2(X)$.

We firstly calculate $\pi_1(X) \cong \pi_1(T) * \pi_1(\mathbb{CP}^2) \cong (\mathbb{Z} \oplus \mathbb{Z}) * 0 \cong \mathbb{Z} \oplus \mathbb{Z}$, where this direct sum is generated by the two distinct circles we can draw on a torus.

Let us now look at how 2-loops behave in X (by the previous point's idea we expect them to look like a 2-loop from $\mathbb{CP}^2$ with finitely many 1-loops from T attached). We'll use out previously constructed universal cover (which we know has isomorphic second homotopy group). We know from Algebraic topology 1, since $\mathbb{S}^2$ is connected and locally path-connected, and both $\mathbb{S}^2$ and E are simply connected, for each 2-loop α in Y there exists a unique lift $\bar{\alpha}$ as in the following diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ \mathbb{S}^2 \arrow[ur, "\bar{\alpha}"] \arrow[r, "\alpha"] & X \end{tikzcd}$$

Note that if α maps to $\mathbb{CP}^2 \subset X$, then $\bar{\alpha}$ maps to copies of $\mathbb{CP}^2$ upstairs. Furthermore for every 1-loop $\gamma = (m,n) \in \mathbb{Z} \oplus \mathbb{Z}$ there exists a unique lift $\bar{\gamma}$ which is necessarily homotopic to the concatenation of the line segments $(0,0) \to (m,0)$ and $(m,0) \to (m,n)$ (or appropriate translation) which is indeed a loop downstairs, since all integer points identify after projection. Consequently, the action $\pi_1(X) \times \pi_2(X) \to \pi_2(X)$ corresponds in E to changing the copy of the glued $\mathbb{CP}^2$ subspace. Concretely, denote by $\mathbb{Z}_{(m,n)}$ the (m,n)-th copy of $\mathbb{Z}$ in $\pi_2(X)$, let $\alpha = \alpha_{i_1,j_1} + \cdots + \alpha_{i_k,j_k}$ an arbitrary element of $\pi_2(X)$, where $\alpha_{i_p,j_p} \in \mathbb{Z}_{(i_p,j_p)}$ and let $\gamma = (m,n) \in \pi_1(X)$. Then the action maps as follows

$$(\gamma, \alpha) \mapsto \alpha_{i_1+m,j_1+n} + \cdots + \alpha_{i_k+m,j_k+n}$$

where $|\alpha_{i_p,j_p}| = |\alpha_{i_p+m,j_p+n}|$, it is just in a different copy of $\mathbb{Z}$ (we can think of it as translating the entire loop for the vector (m,n), but it is much more similar to a component shift morphism).