NEKA, 2020/21
6. DN/ 6th HW : 15. 1. 2021

(1) Poišči vse notranje automorfizme Weylove algebra $\mathcal{A}_1(= k\langle x, y \mid [y, x] = 1\rangle)$. Poišči kakšen zunanji automorfizem $\mathcal{A}_1$. (Zunanji automorfizem je automorfizem, ki ni notranji.)

(2) Določi ali so grupne algebra polenostavna (za vsako posebej). Če je algebra polenostavna, določi dekompozicijo na enostavne algebre. Če ni, določi kakšno bazo Jacobsonovega radikala.

(a) $\mathbb{Z}_3[C_4]$

(b) $\mathbb{Z}_3[C_6]$

(c) $\mathbb{Z}_3[S_3]$

(d) $\mathbb{Q}[S_3]$

(e) $\mathbb{C}[S_4]$

(C_n – ciklična grupa reda n, $\mathbb{Z}_p$ – polje s p elementi, S_n – simetrična grupa na n elementih.)

(3) Naj bo A k-algebra. Pokaži, da je A (končno dimenzionalna) centralno enostavna k-algebra natanko, tedaj ko obstaja k-algebra B, da je $A \otimes_k B$ kot k-algebra izomorfna $M_n(k)$ za nek $n \in \mathbb{N}$.

(4) Naj bosta D_1 in D_2 končno dimenzionalni k-algebri z deljenjem (ne nunjo centralni). Pokaži, da je $D_1 \otimes_k D_2$ obseg, če sta $\dim_k(D_1)$ in $\dim_k(D_2)$ tuji si števili.

(5) Naj bo R lokalen kolobar in $A = (a_{ij})_{i,j=1}^n \in M_n(R)$. Naj bo a_{ij} obrnljiv natanko tedaj, ko je $i = j$. Pokaži, da je A obrnljiva matrika.

(6) Naj bo k polje s char(k)=0, A k-algebra in $e, e', e'' \in A$ idempotenti. Naj velja $e + e' + e'' = \lambda 1$ za $\lambda \in k$. Pokaži, da je $\lambda \in \{0, 1, 2, 3, \frac{3}{2}\}$. Pokaži, da e, e', e'' komutirajo, če je $\lambda \in \{0, 1, 2, 3\}$

(7) Naj bo A k-algebra in M A-bimodul (imamo levo in desno skalarno množenje). Množico $A \times M$ opremimo z vsoto in k-skalarnim množenjem po komponetah in množenjem definirano z $(a, m)(b, n) = (ab, an + mb)$.

Pokaži, da je dobljeni objekt k-algebra. Označimo jo z A_M.

Pokaži, da je A PI-algebra natanko tedaj, ko je A_M PI algebra.

(8) Poišči polinomsko ideniteto $M_n(\mathbb{Q})$ za $n = 2, 3$ z najmanj monomi. Kaj pa za poljuben n?

Ali

Poišči polinomsko ideniteto v dveh spremenljivkah $M_n(\mathbb{Q})$ za $n = 2, 3$ z najmanjšo stopnjo. Kaj pa za poljuben n?

(9) Opiši zakaj je nekomutativna algebra pomembna (s poudarkom na temah pri predmetu).

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(1) Find all inner automorphisms of Weyl algebra $\mathcal{A}_1(= k\langle x, y \mid [y,x] = 1\rangle)$. Find an outer automorphsim of $\mathcal{A}_1$. (An outer automorphism is an automorphism that is no inner.)

(2) Decide whether a group algebra is semisimple (for each one separately). If the algebra is semisimple, find a dekompozicijo into simple algebras. If it is not, find a basis of Jacobson radical.

   (a) $\mathbb{Z}_3[C_4]$
   (b) $\mathbb{Z}_3[C_6]$
   (c) $\mathbb{Z}_3[S_3]$
   (d) $\mathbb{Q}[S_3]$
   (e) $\mathbb{C}[S_4]$

   (C_n – cyclic group of order n, $\mathbb{Z}_p$ – field with p elements, S_n – symmetric group on n elements.)

(3) Let A be a k-algebra. Show that A is a (finitely dimensional) central simple k-algebra if and only if there exists a k-algebra B, such that $A \otimes_k B$ is isomorphic as k-algebra to $M_n(k)$ for some $n \in \mathbb{N}$.

(4) Let D_1 and D_2 be finitely dimensional division k-algebras (not necessarily central). Show that $D_1 \otimes_k D_2$ is a division ring, if $\dim_k(D_1)$ and $\dim_k(D_2)$ are coprime (relatively prime).

(5) Let R be a local ring and $A = (a_{ij})_{i,j=1}^n \in M_n(R)$. Assume a_{ij} is invertible if and only if $i = j$. Show that A is invertible.

(6) Let A be a k-algebra and M A-bimodule (we have left and right scalar multiplication). We equip the set $A \times M$ with pointwise addition and scalar multiplication and multiplication defined by $(a,m)(b,n) = (ab, an + mb)$.

   Show that $A \times M$ with given operations is a k-algebra. We denote it by A_M.

   Show that A is PI-algebra if and only if A_M is PI-algebra.

(7) Let char(k)=0, A be a k-algebra and $e, e', e'' \in A$ idempotents. Assume $e+e'+e'' = \lambda 1$ for $\lambda \in k$. Show that $\lambda \in \{0, 1, 2, 3, \frac{3}{2}\}$. Show that e, e', e'' commute if $\lambda \in \{0, 1, 2, 3\}$.

(8) Find a polynomial identity of $M_n(\mathbb{Q})$ for $n = 2, 3$ with the least monomials. What about for general n?

   Or

   Find a polynomial identity in two variables of $M_n(\mathbb{Q})$ for $n = 2, 3$ with the minimal degree. What about for general n?

(9) Describe why is noncommutative algebra important (with an emphasis the topics from the course).