NEKA, 2020/21
4. DN/ 4rd HW : 4. 12. 2020
Rok za oddajo/ Due date: 23:59, 18. 12. 2020

(1) Naj bosta Q_1 in Q_2 kvaternionski F-algebri (privzamemo $\operatorname{char} F \neq 2$). Pokaži, da so naslednje trditve ekvivalentne:

(a) Obstajajo $a, b, b' \in F^{-1}$ za katere je $Q_1 \cong \left(\frac{a,b}{F}\right)$ in $Q_2 = \left(\frac{a,b'}{F}\right)$.

(b) Q_1 in Q_2 imata skupno podpolje dimenzije 2 nad F.

(c) Q_1 in Q_2 imata skupno razcepno polje dimenzije 2 nad F.

(2) Naj bo A centralno enostavna F-algebra in $\operatorname{Nrd}: A \to F$ njena reducirana norma (Glej vaje 8). Za vsak $a \in A$ definiramo levo množenje $L_a \in \operatorname{End}_F(A)$ z $L_a(x) = ax$ in (nereducirano) normo $N(a) = \det(L_a)$.

Pokaži, da je $N(a) = \operatorname{Nrd}(a)^{\deg(A)}$.

(3) Naj bo A centralno enostavna F-algebra (privzamemo $\operatorname{char} F \neq 2$). Pokaži, da je $[A] = [Q] \in \operatorname{Br}(F)$ za neko kvaternionsko algebro Q natanko tedaj, ko obstaja separabilno razcepno polje algebre A stopnje 2.

(4) Določi Brauerjevo grupo $\mathbb{C}(t)$ in $\mathbb{R}(t)$.

(5) Naj bo A centralno enostavna F-algebra. Naj bo $f: A \to A$ involucija (f je F-linearna, $f^2 = \mathrm{id}_A$ in $f(xy) = f(y)f(x)$).

S pomočjo f opiši vse involucije A.

Pokaži, da ima $M_n(A)$ involucijo.

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(1) Let Q_1 and Q_2 be quaternion F-algebras (assume $\operatorname{char} F \neq 2$). Show that TFAE:

(a) There exist $a, b, b' \in F^{-1}$ such that $Q_1 \cong \left(\frac{a,b}{F}\right)$ and $Q_2 = \left(\frac{a,b'}{F}\right)$.

(b) Q_1 and Q_2 have a common subfield of dimension 2 over F.

(c) Q_1 and Q_2 have a common splitting field of dimension 2 over F.

(2) Let A be a central simple F-algebra and $\operatorname{Nrd}: A \to F$ its reduced norm (consult ex8). For any $a \in A$ we define the left multiplication $L_a \in \operatorname{End}_F(A)$ by $L_a(x) = ax$ and the (unreduced) norm by $N(a) = \det(L_a)$.

Show that $N(a) = \operatorname{Nrd}(a)^{\deg(A)}$.

(3) Let A be a central simple F-algebra (assume $\operatorname{char} F \neq 2$). Show that $[A] = [Q] \in \operatorname{Br}(F)$ for some quaternion algebra Q if and only if A has a separable splitting field of degree 2.

(4) Determine the Brauer group of $\mathbb{C}(t)$ and $\mathbb{R}(t)$.

(5) Let A be a central simple F-algebra. Let $f\colon A \to A$ be an involution of A (f is F-linear, $f^2 = \mathrm{id}_A$ and $f(xy) = f(y)f(x)$).

Describe all involutions of A using f.

Show that $M_n(A)$ admits an involution.