Homework problems for

Introduction to Harmonic Analysis

FMF, winter semester 2020-21

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INSTRUCTIONS:

Choose and mark appropriately the problems that you intend to solve.

Your grade is attained as follows:

- The initial grade is 2.
- Every correctly solved problem increases the grade by 1.
- Every incorrectly solved problem decreases the grade by 0˙33.
- A lacking or incomplete solution is worth 0 points.
- The final result is rounded upwards (and capped at 10).
- A solution counts as correct if it is flawless, or if it only contains minor mistakes in the calculations, which do not affect the course of the proof.
- The course coordinator may require the student to defend their solutions.

Solve the problems on your own. The solutions must be written very precisely, clearly and legibly. The use of any available literature is permitted, assuming it is cited appropriately, while theorems and facts that have not been proved during the lectures or recitations must be proved as part of the solution. The theorems you know from other courses are an exception to this rule, but they must be stated clearly.

Submit your solutions by 31 July 2021, either in my personal mailbox or by e-mail (`dejan.govc@fmf.uni-lj.si`), if they are in electronic form.

FINAL GRADE: To attain a positive final grade in this course, it is also necessary to pass an oral exam with the lecturer, doc. dr. Dragičević. There will be only one final grade, to which roughly 60% will be contributed by the homework and 40% by the oral exam.

1. Prove that for all $f, g \in L^2(\mathbb{T})$ we have

$$\|f * g\|_{L^2(\mathbb{T})}^2 \le \|f * f\|_{L^2(\mathbb{T})} \|g * g\|_{L^2(\mathbb{T})}.$$

2. Suppose a continuous function $f : \mathbb{T} \to \mathbb{C}$ is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^\alpha},$$

where $\alpha > 1$. Prove that f is smooth (i.e. C^∞) on $\mathbb{T} \setminus \{0\}$.

3. Let $a, b \in \mathbb{N}$ and $1 \le b \le a-1$. The function $f : \mathbb{T} \to \mathbb{C}$ is defined by the values

$$f\left(\frac{k}{a}\right) = e^{\frac{2\pi i k b}{a}}, \qquad k = 0, 1, \ldots, a,$$

and by linear interpolation everywhere else, i.e. if $t \in \left[\frac{k-1}{a}, \frac{k}{a}\right]$, we have:

$$f(t) = (at - k + 1) f\left(\frac{k}{a}\right) + (k - at) f\left(\frac{k-1}{a}\right).$$

(If e.g. $a \ge 3$ and $b = 1$, the function is a parameterization of a regular a-gon in the complex plan.) Determine its Fourier series. Use your result to calculate the value of

$$\sum_{k \in \mathbb{Z}} \frac{1}{(ak+b)^2}.$$

4. A trigonometric polynomial $f : \mathbb{R} \to \mathbb{C}$ is given by

$$f(x) = \sum_{k=1}^{n} \alpha_k e^{i \vartheta_k x},$$

where $\vartheta_1, \ldots, \vartheta_k$ are real **(!)** numbers and $\alpha_1, \ldots, \alpha_k$ are complex numbers. Suppose that

$$f(m) = 0$$

for all $m \in \mathbb{Z}$. Does it follow that f is periodic?

5. Let $f \in L^1(\mathbb{T})$. Suppose that

$$|f(x) - f(0)| \le \frac{1}{(\log |x|)^2}, \qquad 0 \ne x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Does it follow that

$$\lim_{n \to \infty} S_n f(0) = f(0)?$$

Either prove the claim or find a counterexample.

6. Show that

$$\sum_{n=2}^{\infty} \frac{\sin(2\pi n x)}{\log n}$$

converges for all $x \in \mathbb{T}$, and yet it is not the Fourier series of a function in $L^1(\mathbb{T})$.

7. Let $z \in \mathbb{R} \setminus \mathbb{Z}$. The function $f_z : \mathbb{T} \to \mathbb{C}$ is defined by

$$f_z(x) = \cos(2\pi x z), \qquad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Find its Fourier series expansion. Using this, show that every $z \in \mathbb{R} \setminus \mathbb{Z}$ satisfies

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2},$$

and use this to derive the Euler sine product formula:

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Show that this formula is valid for all $z \in \mathbb{R}$.

8. Is there a continuous function $f : \mathbb{T} \to \mathbb{R}$ such that $\lim_{n\to\infty} |S_n f(0)| = \infty$?

9. Let $f \in L^1(\mathbb{T})$ be a function, such that $\hat{f}(j) = 0$ for all $|j| < n$. Prove that for all $p \in [1, \infty]$ we have

$$\|f''\|_p \geq C n^2 \|f\|_p,$$

where $C > 0$ is a constant independent of f, p and n.

10. Let $f \in H^k(\mathbb{T})$. In recitations we showed that then there is a function $v \in L^2(\mathbb{T})$, such that for all $n \in \mathbb{N}$,

$$\int_{\mathbb{T}} f e_n^{(k)} \mathrm{d}x = \int_{\mathbb{T}} v e_n \mathrm{d}x.$$

From this, we concluded that

$$\int_{\mathbb{T}} f \varphi^{(k)} \mathrm{d}x = \int_{\mathbb{T}} v \varphi \mathrm{d}x.$$

holds for every function $\varphi \in C^\infty(\mathbb{T})$, but we have not really justified this. Fill in **all** the missing details in this implication.

11. Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function. Do there exist $C > 0$ and $\alpha > 0$, such that

$$\left|\hat{f}(n)\right| < \frac{C}{n^\alpha}, \qquad n \in \mathbb{Z}?$$

If they do, prove it, if not, find a counterexample.

12. Let $c : \mathbb{T} \to \mathbb{R}$ be the triangle wave, defined by $c(x) = 1 - 4|x|$ if $|x| \leq \frac{1}{2}$ and by its 1-periodic extension otherwise. Then, c can be expanded into a cosine series as follows:

$$c(x) = \sum_{k=1}^{\infty} \frac{8\cos\left((2k-1)2\pi x\right)}{(2k-1)^2\pi^2}.$$

Show that "conversely" there is also an expansion of the form

$$\cos(2\pi x) = \sum_{k=1}^{\infty} \alpha_k c(kx)$$

and derive and prove a general formula for the coefficients α_k.

13. Suppose a series with complex terms

$$\sum_{n=0}^{\infty} a_n$$

converges in the sense of Abel, where the sequence $(a_n)_n$ satisfies

$$\lim_{n\to\infty} n a_n = 0.$$

Show that the series must then also converge in the usual sense.

14. Given functions $f, g : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{1}{\cosh \pi x}, \qquad g(x) = \frac{x}{\sinh \pi x},$$

calculate the Fourier transforms $\hat{f}$ and $\hat{g}$. Justify all the steps.

Definition. *A partition* of the interval $[0, 1]$ is a set of points $x_0, \ldots, x_n \subseteq [0, 1]$,

$$0 = x_0 < x_1 < \ldots < x_n = 1.$$

The total variation of a function $f : \mathbb{T} \to \mathbb{C}$ is defined by

$$V(f; \mathbb{T}) := \sup_P \sum_{i=1}^{n} \left|f(x_i) - f(x_{i-1})\right|.$$

where the supremum is taken over all partitions P of the interval $[0,1]$. The function $f : \mathbb{T} \to \mathbb{C}$ is *of bounded variation*, if

$$V(f;\mathbb{T}) < \infty.$$

15. Let $\mathrm{BV}(\mathbb{T})$ be the set of all functions $f : \mathbb{T} \to \mathbb{C}$ of bounded variation. Prove that $\mathrm{BV}(\mathbb{T})$ is a vector space with respect to the usual operations of addition and multiplication by a scalar and that

$$\|f\|_{\mathrm{BV}(\mathbb{T})} := \sup_{y\in\mathbb{T}} |f(y)| + V(f;\mathbb{T})$$

is a norm on $\mathrm{BV}(\mathbb{T})$ which turns $\mathrm{BV}(\mathbb{T})$ into a Banach. Also prove that for each $n \in \mathbb{N}$ and every $x \in \mathbb{T}$ we have

$$|S_n f(x)| \leq \|f\|_{\mathrm{BV}(\mathbb{T})}.$$

16. To a continuous function $f : \mathbb{T} \to \mathbb{C}$ for every $N \in \mathbb{N}$ associate its *discrete Fourier coeffiecients*

$$a_N(n) = \frac{1}{N}\sum_{k=1}^{N} f\left(e^{\frac{2\pi ik}{N}}\right) e^{\frac{-2\pi ikn}{N}}$$

as well as its usual Fourier coefficients

$$a(n) = \int_{\mathbb{T}} f(x) e^{-2\pi inx} \mathrm{d}x.$$

Prove that

$$\lim_{N\to\infty} a_N(n) = a(n).$$

Remark. Let $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. To every finite sequence $F : \mathbb{Z}_n \to \mathbb{C}$ we can associate its *finite Fourier series* given by

$$F(k) = \sum_{n=0}^{N-1} \alpha(n) e^{\frac{2\pi ikn}{N}},$$

where the *Fourier coefficients* are given by

$$\alpha(n) = \frac{1}{N}\sum_{k=0}^{N-1} F(k) e^{\frac{-2\pi ikn}{N}}.$$