Noncommutative Algebra, 2nd homework

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<u>Ex. 1:</u> Let A be a central simple k-algebra. Assume that for all $x, y, z, w \in A$ we have

$$[x, y][z, w] + [z, w][x, y] \in k$$

Let us show that $\deg A = 1$ or 2.

We consider the following map

$$f(x, y, z, w, v) = [[x, y][z, w] + [z, w][x, y], v]$$

Since A is central, it is clear that $f(x, y, z, w, v) = 0$ for all $x, y, z, w, v \in A$[1]. For now assume also that $n = \deg A < \infty$. We can always map $A \to \overline{k} \otimes A \cong M_n(\overline{k})$ with $x \mapsto 1 \otimes x$, so any function, that is zero everywhere on $M_n(\overline{k})$, will also be zero everywhere on A. This allows us to look for counterexamples in matrix algebras over algebraically closed fields. Since we can also always embed M_n into M_{n+1} by just padding matrices with zeros below and to the right, it is enough to find a counterexample for $n = 3$, which is what we will do now. We know that for basis unit matrices we have $E_{ij}E_{kl} = \delta_{j,k}E_{il}$ so we take the following matrices:

$$x = E_{11}, \quad y = E_{12}, \quad z = E_{22}, \quad w = E_{23}, \quad v = E_{33}$$

We quickly see that $f(x, y, z, w, v) = E_{13} \neq 0$, since the only non-zero product in the commutator is $xyzwv$. This now excludes all finite degrees except for 1 and 2.

<u>Note:</u> If one takes finite dimension as part of the definition of central simple algebras, we conclude the proof here. In this case, take the following as an alternative solution. Otherwise, assume A has infinite degree and continue as follows.

What remains to be seen is that the infinite degree is also not a possibility. Indeed, since A is simple, it is primitive as a ring. By the Jacobson Density Theorem, A acts as a ring of linear transformations on a vector space V over a division ring D. Now suppose there exist linearly independent vectors $u_0, \ldots, u_5 \in V$ (which must be the case if A has infinite degree and is not the case if A has degree 1 or 2). By density of A there exist $a_1, \ldots, a_5 \in A$ such that $a_i u_j = \delta_{i,j} u_{j-1}$. Then clearly $a_1 \cdots a_5 u_n = u_0$ while all other permutations of actions result in zero (notice the similarity with the above counterexample). We now have $f(a_1, \ldots, a_5) u_n = u_0 \neq 0$ (only the first of the terms of the above commutator gives us the correct order), but this is a contradiction with the fact that f is by assumption zero on A. Hence A must have finite degree. Note that this already proves that $\deg A < 3$, but the counterexample is much more illustrative.

<u>Ex. 2:</u> Let A be a simple $\mathbb{R}$-algebra of odd dimension n. We will show that $A \cong M_m(\mathbb{R})$ for some odd m.

By applying Wedderburn's Structure Theorem and Frobenius' Theorem consecutively, it is clear that any finite dimensional simple $\mathbb{R}$-algebra is isomorphic to either $M_m(\mathbb{R})$, $M_m(\mathbb{C})$ or $M_m(\mathbb{H})$ for some $m \in \mathbb{N}$. Indeed, since A is simple, we have $A \cong M_m(D)$, where D is a real division algebra, clearly $\mathbb{R} \subseteq Z(D)$, so $D \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. But we know A has odd $\mathbb{R}$-dimension, while 2 divides both $\dim_{\mathbb{R}} M_m(\mathbb{C})$ and $\dim_{\mathbb{R}} M_m(\mathbb{H})$ simply because $\dim_{\mathbb{R}} \mathbb{C} = 2$ and $\dim_{\mathbb{R}} \mathbb{H} = 4$. It follows that $A \cong M_m(\mathbb{R})$. Furthermore, since x^2 is odd precisely when x is odd, by dimension counting we get $n = m^2$ and consequently

[1] Such a function is often called a (multilinear) polynomial identity, which makes A a so called PI-ring.

$m = \sqrt{n}$, which is an odd number.

Ex. 3: Let A, B, C be finite dimensional central simple k-algebras such that $A \otimes B \cong A \otimes C$. Let us show that $B \cong C$.

Consider $\mathrm{Br}(k)$ the Brauer group of k. We have the following equivalence

$$A \otimes B \cong A \otimes C \iff [A \otimes B] = [A \otimes C] \quad \& \quad [A \otimes B : k] = [A \otimes C : k]$$

Since $[X \otimes Y] = [X] \cdot [Y]$ in $\mathrm{Br}(k)$ and $[X \otimes Y : k] = [X : k] \cdot [Y : k]$, we get by cancellation law for group multiplication that $[B] = [C]$ and $[B : k] = [C : k]$. By the analogue to the above equivalence we have $B \cong C$.

Ex. 4: Let A be a central simple k-algebra of degree n. We will show that A is split $\iff$ A contains a subalgebra $S \cong k^n$.

As usual, we prove the equivalence as two implications separately.

- ($\implies$): Let A be a split central simple k-algebra of degree n. By definition this means that $A \cong M_n(k)$. Then for instance $D = \mathrm{Diag}_n(k) \leq M_n(k)$ is a subalgebra and $D \cong k^n$.

- ($\impliedby$): Recall that $n = \sqrt{\dim_k(A)}$ as a vector space. Suppose $R \leq A$ such that $R \cong k^n$. Notice that R is a commutative k-subalgebra of dimension n in A. By Wedderburn's Structure Theorem, $A \cong M_m(D)$, where D is a division algebra that can be viewed so as to contain k. Recall now from linear algebra, that any commutative matrix subalgebra is isomorphic to a diagonal subalgebra of appropriate dimension (via a conjugation isomorphism). From this it follows, that $m \geq n$, hence $m = n$ (intuitively, we keep in mind the equation $m^2[D : k] = n^2$). This means that $D \cong k$, but now by definition, A is Brauer equivalent to k; in other words, A is split.

Ex. 5: Let S be a subalgebra of algebra A. Let us show the following

(a) If S is commutative, than so is $C_A(C_A(S))$.

(b) If $S = C_A(U)$ for some subset $U \subset A$, then $C_A(C_A(S)) = S$.

Recall that for any set $U \subseteq A$ we have $U \subseteq C_A(C_A(U))$, and notice that for $U \subseteq V \subseteq A$ we have $C_A(V) \subseteq C_A(U)$ (since these elements have to commute with a "larger" set, so there is "fewer" of them).

Take now the subalgebra $S \leq A$. Since S is commutative, clearly $S \subseteq C_A(S)$. Now using the above remarks we have

$$S \subseteq C_A(C_A(S)) \subseteq C_A(S)$$

Since $C_A(C_A(S))$ by definition commutes with all elements of $C_A(S)$, clearly $C_A(C_A(S))$ is commutative by the above inclusion chain, which proves (a).

Now suppose $S = C_A(U)$ for some set $U \subseteq A$. Of course we have $U \subseteq C_A(C_A(U))$, so we get

$$C_A(C_A(C_A(U))) \subseteq C_A(U)$$

and therefore

$$C_A(C_A(S)) \subseteq S$$

Since also $S \subseteq C_A(C_A(S))$, this proves (b).