Differential geometry: 4. homework

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Exercise 1 (Covariant derivative of tensor fields)**.** Let ∇ be a covariant derivative on TM.

i) Show that the map $\nabla\colon \mathfrak{X}(M) \times \Omega^1(M) \to \Omega^1(M)$, given by

$$(X, \omega) \mapsto (\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y),$$

determines a covariant derivative on T^*M. Then prove that in any coordinates (x^i), the 1-form $\nabla_X \omega$ is given by

$$\nabla_X \omega = \big(X(\omega_k) - X^i \omega_j \Gamma^j_{ik}\big)\,\mathrm{d}x^k.$$

ii) Explain why the map $\mathfrak{X}(M) \times \Gamma^\infty\big(T^{(k,l)}TM\big) \to \Gamma^\infty\big(T^{(k,l)}TM\big)$, given by

$$\begin{aligned}(\nabla_X A)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l) = {} & X(A(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l)) \\ & - \sum_{i=1}^{k} A(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^k, Y_1, \ldots, Y_l) \\ & - \sum_{j=1}^{l} A(\omega^1, \ldots, \omega^k, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_l),\end{aligned}$$

determines a covariant derivative on $T^{(k,l)}TM$.

Exercise 2 (Gradient, divergence, Hessian and Laplacian)**.** Let (M, g) be a Riemannian manifold. As a preparation, first solve the following two easy exercises.

a) Prove that the map $\bar{g}\colon \mathfrak{X}(M) \to \Omega^1(M)$ given by $\bar{g}(X) = g(X, \cdot)$ is an isomorphism of vector spaces, which induces an isomorphism of vector bundles TM and T^*M. Express $\bar{g}(X)$ and $\bar{g}^{-1}(\omega)$ in local coordinates, for any $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$. We call $\bar{g}$ the *musical isomorphism* and denote $X^\flat = \bar{g}(X)$, $\omega^\sharp = \bar{g}^{-1}(\omega)$.*

b) The *trace* of a $(1,1)$-tensor field $A \in \Gamma^\infty(T^{(1,1)}TM)$ is defined locally by $\mathrm{tr}(A) = A^i{}_i$, where $A = A^i{}_j \partial_i \otimes \mathrm{d}x^j$ in given local coordinates $(x^i)_i$. Show that the definition of $\mathrm{tr}(A)$ is independent of the chosen coordinates, so that $\mathrm{tr}(A)$ is a globally defined function.†

Now let ∇ denote the Levi-Civita connection on (M, g).
Advice. Parts ii), iii) and iv) are at some points easily solvable with normal coordinates.

i) The *gradient* of a function $f \in C^\infty(M)$ is defined as the vector field

$$\nabla f = (\mathrm{d}f)^\sharp, \quad \text{which may also be written as} \quad \langle \nabla f, \cdot \rangle = \mathrm{d}f.$$

Express ∇f in local coordinates; express ∇f in spherical coordinates on $\mathbb{R}^3$, for $f \in C^\infty(\mathbb{R}^3)$.

ii) Let $X \in \mathfrak{X}(M)$ be a vector field. Show that in any coordinates $(x^i)_i$ on M, there holds

$$\mathrm{tr}(\nabla X) = (\nabla_{\partial_k} X)^k = \frac{1}{\sqrt{\det[g_{ij}]}} \partial_k \left(\sqrt{\det[g_{ij}]} X^k \right),$$

(summation on both sides is implicit). It follows that if (M, g) is oriented, then $\mathrm{div}(X) = \mathrm{tr}(\nabla X)$, where the divergence is defined as in 3. homework. If M isn't oriented, then we define the divergence by $\mathrm{div}(X) = \mathrm{tr}(\nabla X)$.

Finally, express $\mathrm{div}(X)$ in spherical coordinates on $\mathbb{R}^3$, for $X \in \mathfrak{X}(\mathbb{R}^3)$.

iii) We define the *covariant Hessian* of a function $f \in C^\infty(M)$ as the covariant 2-tensor field $H(f)$, given by

$$H(f)(X, Y) = (\nabla_X \mathrm{d}f)(Y).$$

Explain why it is indeed a 2-tensor field. Then express $H(f)$ in local coordinates and prove that it is a symmetric 2-tensor field.

Hence, at any point $x \in M$, $H(f)_x$ is a symmetric quadratic form, which corresponds to a diagonalizable matrix – do its eigenvalues depend on the choice of metric g on M?

iv) The *Laplacian* of a function $f \in C^\infty(M)$ is defined as the function

$$\Delta f = \mathrm{div}(\nabla f).$$

Express Δf in local coordinates and show that $\Delta f = \mathrm{tr}(H(f)^\sharp)$, where $H(f)^\sharp$ is a (1,1)-tensor field which is locally given by $H(f)^\sharp = g^{ij} H(f)_{ik} \partial_j \otimes \mathrm{d}x^k$.

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*Similarly, we have that the vector bundles $T^{(k,l)}TM$ and $T^{(i,j)}TM$ are isomorphic whenever $k + l = i + j$, on any Riemannian manifold M.

†Similarly, we may define traces of tensor fields of higher order; these are maps $\mathrm{tr}\colon \Gamma^\infty(T^{(k,l)}TM) \to \Gamma^\infty(T^{(k-1,l-1)}TM)$, given locally by contraction of two indices (one covariant and one contravariant).

Exercise 3 (Isometries and geodesics)**.** Let M and $\tilde{M}$ be smooth manifolds.

i) Suppose $f\colon M \to \tilde{M}$ is a diffeomorphism and $\tilde{\nabla}$ is a connection on $\tilde{M}$. The *pullback of connection* $\tilde{\nabla}$ is a map $f^*\tilde{\nabla}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by

$$\left(f^*\tilde{\nabla}\right)_X Y = \left(f^{-1}\right)_* \left(\tilde{\nabla}_{f_*X}\, f_*Y\right),$$

where f_* denotes the pushforward via f. Prove that $f^*\tilde{\nabla}$ is a connection on M.

ii) With the assumptions from i), denote $\nabla = f^*\tilde{\nabla}$. Let γ be a smooth path in M and $V \in \mathfrak{X}(\gamma)$ a vector field along this path. Prove that

$$\mathrm{d}f \circ D_\gamma V = \tilde{D}_{f\circ\gamma}(\mathrm{d}f \circ V)$$

and conclude that f maps ∇-geodesics to $\tilde{\nabla}$-geodesics, i.e. if γ is a ∇-geodesic, then $f \circ \gamma$ is a $\tilde{\nabla}$-geodesic.

A smooth map $f\colon M \to \tilde{M}$ between Riemannian manifolds (M, g) and $(\tilde{M}, \tilde{g})$ is *metric-preserving* if $g = f^*\tilde{g}$. An *isometry* is a metric-preserving diffeomorphism; a *local isometry* is a metric-preserving local diffeomorphism.

iii) Suppose $f\colon M \to \tilde{M}$ is an isometry between Riemannian manifolds and $\nabla, \tilde{\nabla}$ denote the Levi–Civita connections on $M, \tilde{M}$, respectively. Show that $f^*\tilde{\nabla} = \nabla$.

Remark. This property is called *naturality of Levi–Civita connection.*

iv) Prove that an isometry of Riemannian manifolds maps geodesics to geodesics.

v) Explain why any local isometry of Riemannian manifolds also maps geodesics to geodesics.

Exercise 4 (Hyperbolic half-plane and disk)**.** Let $(\mathbb{H}, g_{\mathbb{H}})$ denote the hyperbolic half-plane, i.e.

$$\mathbb{H} = \mathbb{R} \times (0, \infty), \quad g_{\mathbb{H}} = \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2).$$

i) Calculate the Christoffel symbols of the Levi–Civita connection ∇ on M, with respect to the standard coordinates.

ii) Write down the connection 1-form ω of the Levi-Civita connection ∇ with respect to the standard coordinates. Then calculate its curvature 2-form F.

iii) Let $(\mathbb{D}, g_{\mathbb{D}})$ denote the hyperbolic disk, i.e.

$$\mathbb{D} = \mathring{D}(0,1), \quad g_{\mathbb{D}} = \frac{4}{(1 - x^2 - y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2).$$

Show that the Möbius transformation $\tau \colon \mathbb{H} \to \mathbb{D}$, given by

$$\tau(z) = \frac{1 + iz}{z + i},$$

is an isometry.

Advice. First show that for a smooth map $\tau \colon (M, g) \to (N, h)$ between Riemannian manifolds, the equality $\tau^* h = g$ is written in in matrix form (in appropriate bases) as

$$(\mathrm{d}\tau)_p{}^T \cdot h_{\tau(p)} \cdot \mathrm{d}\tau_p = g_p,$$

for any $p \in M$. Identifying the algebra of complex numbers with the appropriate subalgebra of matrices as

$$x + iy \longleftrightarrow \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

and using the fact that Cauchy–Riemann equations hold for the map τ, show that the above condition for τ to be an isometry may be rewritten as

$$\overline{\tau'(z)} \frac{4}{\left(1 - |\tau(z)|^2\right)^2} \tau'(z) = \frac{1}{\operatorname{Im}(z)^2}.$$

Finally, prove that the last equality indeed holds.

iv) Let $v = \partial_x|_{(0,0)}$ and $w = \partial_y|_{(0,0)}$ be two vectors in $(\mathbb{D}, g_{\mathbb{D}})$. Describe the (maximal) geodesics γ_v and γ_w of the space $(\mathbb{D}, g_{\mathbb{D}})$. Then describe the shape of any (maximal) geodesic of $(\mathbb{D}, g_{\mathbb{D}})$.

Hint. Use your knowledge about geodesics of $(\mathbb{H}, g_{\mathbb{H}})$ and the fact that τ is an isometry that is also a Möbius transformation. Draw some pictures!

v) Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{R})$ and let $\tau_A \colon \mathbb{H} \to \mathbb{H}$ be the Möbius transformation

$$\tau_A(z) = \frac{az + b}{cz + d}.$$

Show that τ_A is an isometry of $(\mathbb{H}, g_{\mathbb{H}})$.

Hint. Use a similar trick as in advice of part iii).