Noncommutative algebra - 2$^{\text{nd}}$ homework

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Ex. 1: Let R be a primitive ring and L a minimal left ideal of R. Let us show that every faithful simple R-module is isomorphic to L.

Let M be a faithful simple R-module, that exists by primitivity of R. Then by faithfulness of M, the product LM is non-zero. Choose $m \in M$ such that $Lm \neq 0$. By simplicity of M, $Lm = M$. Consider the homomorphism $\varphi\colon L \to M$ defined by $a \mapsto am$. The map φ is surjective by the above. It is also injective, since $\varphi \neq 0$ and $\ker\varphi \leq L$ and therefore $\ker\varphi = 0$. This is the isomorphism we seek.

Ex. 2: Let R be a ring and V a faithful simple R-module (R is therefore primitive). By Schur's lemma, V is a left vector space over the division ring $S = \mathrm{End}_R(V)$. We define the rank of an element $r \in R$ by

$$\operatorname{rank} r = \dim_S(rV)$$

We will show that $r \in R$ has finite rank $\iff$ r is a sum of elements of rank 1.

- ($\Longleftarrow$): This direction follows from basic linear algebra. Suppose that $r = \Sigma_{i\in I} r_i$ is a finite sum (say, $|I| = n$) with $\operatorname{rank} r_i = 1$ for all i where r_i is non-zero. Then we have

$$\operatorname{rank} r = \dim_S(rV) = \dim_S\left(\Sigma_{i\in I} r_i V\right) \leq \Sigma_{i\in I}\dim_S(r_iV) = \Sigma_{i\in I} 1 = n < \infty$$

- ($\Longrightarrow$): By the Jacobson Density Theorem, since R is primitive, R is a dense ring of linear transformations of the S-vector space V. In other words, for any finite linearly independent set $\{v_1, \dots, v_n\}$ and a finite set $\{w_1, \dots, w_n\}$ there exists $r \in R$ such that $rv_i = w_i$ for all $i = 1, \dots, n$.

  We now continue with a proof by induction on the rank of r. The base case is trivial (if $\operatorname{rank} r = 1$, r is itself a one-term sum of elements of rank 1), so we proceed with the step case. Assume the implication holds for all $r \in R$ with $\operatorname{rank} r \leq n-1$ and suppose for concrete $r \in R$ that $\operatorname{rank} r = \dim_S(rV) = n < \infty$. There exists a (linearly independent) basis for rV: $\{rv_1, \dots, rv_n\}$. By the density of R, there exists $r' \in R$ mapping the basis to the set $\{0, rv_2, \dots, rv_n\}$ as described above. Clearly now $\operatorname{rank} r'r = n-1$ and we can write $r = (r - r'r) + r'r$. Since by the induction hypothesis $r'r$ can be written as a sum of elements of rank 1, we now need to prove only that $r - r'r$ has rank 1, which is easy, since clearly $(r - r'r)V = \mathrm{Lin}\{rv_1\}$ (r' acts as a projection).

Ex. 3: Let U be an $\mathbb{R}$-vector space. Let us show that $\Sigma_{i=1}^n u_i \otimes u_i = 0 \in U \otimes_{\mathbb{R}} U \iff u_i = 0$ for all $i = 1, \dots, n$.

- ($\Longleftarrow$): Since $u_i = 0$ implies $u_i \otimes u_i = 0$, this implication holds.
- ($\Longrightarrow$): Let $\{e_i\}_{i\in I}$ be a basis for U. Then $\{e_i \otimes e_j\}_{i,j\in I}$ is a basis for $U \otimes_{\mathbb{R}} U$. We can write $u_i = \Sigma_{j\in I} a_j e_j$, so we get $u_i \otimes u_i = \Sigma_{j,k\in I} a_j a_k e_j \otimes e_k$ where in both sums only finitely many coefficients are non-zero. We thus get $\Sigma_{i=1}^n u_i \otimes u_i = \Sigma_{j,k\in I} b_{jk} e_j \otimes e_k$ where b_{jk} are sums of degree 2 products of coefficients of u_i, and only finitely many coefficients of the sum are non-zero. Since this sum is by assumption equal to zero, all its coefficients must be equal to zero. Now observe coefficients at basis vectors of the form $e_p \otimes e_p$. We see they are formed precisely by the sums of squares of appropriate coefficients of vectors u_i (those that contain e_p in basis decomposition) and that each coefficient from all of vectors u_i is present in some term. In particular, we have sums of non-negative real numbers that are equal to zero, so all these numbers (squares of coefficients of vectors u_i) must be zero. It follows all u_i are equal to zero.

Ex. 4: Let F be a field with char $F \neq 2$ and $a, b \in F^*$. Let $Q = \left(\frac{a,b}{F}\right)$ be a quaternion algebra with basis $\{1, i, j, k\}$. Define the norm $N \colon Q \to F$ by $N(x + yi + zj + wk) = x^2 - ay^2 - bz^2 + abw^2$. We show the following statements are equivalent

(a) Q is not a division algebra;

(b) $Q \cong M_2(F)$;

(c) The above norm N has a non-trivial zero;

(d) We have $b \in N_a := N_{F(\sqrt{a})/F}(F(\sqrt{a})) = \{x^2 - ay^2;\ x, y \in F\}$.

Let us first prove the equivalence $(a) \iff (c)$, since we will need it later.

- $(a) \iff (c)$: We prove a more precise statement: an element $q \in Q$ is invertible $\iff N(q) \neq 0$. If $qq' = 1$ for some $q' \in Q$, then $N(q)N(q') = N(1) = 1$, so $N(q) \neq 0$. Conversely, suppose $N(q) \neq 0 \in F$. Since elements from F commute with all elements of Q (recall that Q is central simple), the expression

$$N(q) = q\overline{q} = \overline{q}q$$

gives us

$$q\frac{1}{N(q)}\overline{q} = \frac{1}{N(q)}q\overline{q} = 1$$

In other words, $q^{-1} = \frac{\overline{q}}{N(q)}$ is a two-sided multiplicative inverse of q.

We will later prove that $(a) \implies (d) \implies (b) \implies (a)$. First however, we will show some other basic properties. Recall from tutorials that $\left(\frac{1,b}{F}\right) \cong M_2(F)$. We also have $\left(\frac{a,b}{F}\right) \cong \left(\frac{b,a}{F}\right)$ by the basis transformation $(i, j) \mapsto (j, i)$, and $\left(\frac{ac^2,b}{F}\right) \cong \left(\frac{a,b}{F}\right)$ by the basis transformation $(i, j) \mapsto (ic^{-1}, j)$ for $c \neq 0$.

This clearly shows that if a is a square (that is, $\sqrt{a} \in F$) we also have $\left(\frac{a,b}{F}\right) \cong M_2(F)$. If a is a square it also easily follows that $N_a = F$ and therefore $b \in N_a$. Indeed, write $a = c^2$. Then

$$N_a = \{x^2 - c^2y^2\} = \{(x - cy)(x + cy)\} = \{x'y'\}$$

where in the last equality we merely change variables. For a special case of, say, $y' = 1$, we get the entirety of $F \subseteq N_a$. The reverse inclusion is always true.

In the following proofs we can therefore assume, that a is not a square, since it will already imply what we want.

- $(a) \implies (d)$: Let Q not be a division algebra, so there exists a non-zero non-invertible $q \in Q$. By the above, $N(q) = 0$ and we have

$$x^2 - ay^2 - bz^2 + abw^2 = 0 \implies x^2 - ay^2 = b(z^2 - aw^2)$$

Since a is not a square, we must have $z^2 - aw^2 \neq 0$. Indeed, if $z^2 - aw^2 = 0$ then $w = 0$ (if $w \neq 0$ we can solve for a and see that a is a square). It follows that also $z = 0$ and we get in the above equation that $x^2 - ay^2 = 0$, so also $x = y = 0$ since a is not a square. It follows that $q = 0$, which is a contradiction. We can now solve the first equation for b and get

$$b = \frac{x^2 - ay^2}{z^2 - aw^2} \in N_a$$

- $(d) \implies (b)$: Write $b = x_0^2 - ay_0^2$. Q then has a different quaternion basis $\{1, i, x_0j + y_0k, i(x_0j + y_0k)\}$ (the fourth element is actually $x_0k + y_0aj$, since $i^2 = a$ and $k = ij$). Indeed, we get this basis by transforming (j, k) as basis columns with the matrix

$$\begin{bmatrix} x_0 & ay_0 \\ y_0 & x_0 \end{bmatrix}$$

that has determinant $b \neq 0$. We also have $(x_0 j + y_0 k)^2 = b^2$. Since this is a different basis for the same quaternion algebra, it follows that

$$\left(\frac{a, b}{F}\right) \cong \left(\frac{a, b^2}{F}\right) \cong \left(\frac{a, 1}{F}\right) \cong M_2(F)$$

- <u>$(b) \implies (a)$</u>: Clearly $M_2(F)$ is not a division algebra.