NEKA, 2020/21
2. DN/ 2nd HW : 6. 11. 2020

Rok za oddajo/ Due date: 23:59, 20. 11. 2020

(1) Naj bo R primitiven kolobar in $L \subset R$ minimalni levi ideal (Če je J levi ideal in $J \subsetneq L$, potem je $J = 0$). Pokaži, da je vsak zvest enostaven R-modul izomorfen L.

(2) Naj bo R kolobar in V zvest enostaven R-modul. Potem je V levi vektorski prostor nad obsegom $S = \mathrm{End}_R(V)$. *Rang* elementa $r \in R$ je $\operatorname{rang} r = \dim(rV)$. Pokaži, da je ima r končen rang natanko tedaj, ko je vsota elementov z rangom 1.

(3) Naj bo U realen vektorski prostor. Pokaži, da je $\sum_{i=1}^n u_i \otimes u_i = 0 \in U \otimes_{\mathbb{R}} U$ natanko tedaj, ko za vsak $i = 1, \ldots, n$ velja $u_i = 0$.

(4) (2 točki) Naj bo F polje s $\operatorname{char} F \neq 2$ in $a, b \in F \backslash \{0\}$. Naj bo $Q = \left(\frac{a,b}{F}\right)$ kvaternionska algebra (glej vaje 5) z bazo $\{1, i, j, k\}$.

S predpisom $N(x+yi+zj+wk) = x^2 - ay^2 - bz^2 + abw^2$ definiramo normo $N\colon Q \to F$.

Pokaži, da so naslednje trditve ekvivalentne:

(a) Q ni obseg.
(b) $Q \cong M_2(F)$
(c) Norma $N\colon Q \to F$ ima netrivialno ničlo.
(d) $b \in N_{F(\sqrt{a})/F}(F(\sqrt{a}))$

(Loči primera $\sqrt{a} \in F$ in $\sqrt{a} \notin F$. Norma $N_{F(\sqrt{a})/F}\colon F(\sqrt{a}) \to F$ je multiplikativna.)

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(1) Let R be a primitive ring and L a minimal left ideal of R (If J is a left ideal and $J \subsetneq L$, then $J = 0$). Show that every faithful simple R-module is isomorphic to L.

(2) Let R be a ring and V a faithful simple R-module. Then V is a left vector space over the division ring $S = \mathrm{End}_R(V)$. The *rank* of an element $r \in R$ is defined to be $\operatorname{rank} r = \dim_S(rV)$. Show that $r \in R$ has a finite rank if and only if r is a sum of elements of rank 1.

(3) Let U be a real vector space. Show that $\sum_{i=1}^n u_i \otimes u_i = 0 \in U \otimes_{\mathbb{R}} U$ if and only if for every $i = 1, \ldots, n$ we have $u_i = 0$.

(4) (2 points) Let F be a field with $\operatorname{char} F \neq 2$ and $a, b \in F \backslash \{0\}$. Let $Q = \left(\frac{a,b}{F}\right)$ be quaternion algebra (consult Ex5) with basis $\{1, i, j, k\}$.

By $N(x + yi + zj + wk) = x^2 - ay^2 - bz^2 + abw^2$ we define norm $N\colon Q \to F$.

Show that the following are equivalent

(a) Q is not a division algebra.
(b) $Q \cong M_2(F)$

(c) Norm $N\colon Q \to F$ has a nontrivial zero.

(d) $b \in N_{F(\sqrt{a})/F}(F(\sqrt{a}))$

(Consider the cases $\sqrt{a} \in F$ and $\sqrt{a} \notin F$. The norm $N_{F(\sqrt{a})/F}\colon F(\sqrt{a}) \to F$ is multiplicative.)