Prva domača naloga UAG

Domačo nalogo oddajte preko spletne učilnice do 19.4.2021

1. [20 T] Let A be a commutative ring with unit. Show that:
   (a) the set of nilpotent elements in A forms an ideal. Which ideal is it?
   (b) the sum of a nilpotent element and a unit is always a unit,
   (c) $f \in A[x]$ is nilpotent $\iff$ all coeficients are nilponent,
   (d) $f \in A[x]$ is a unit $\iff$ the x^0 coefficient is a unit and the others are nilpotent,
   (e) $f \in A[x]$ is a zerodivisor $\iff$ there is an $a \in A \setminus \{0\}$ with $a \cdot f = 0$.
2. [10 T]
   Let $C := \{(x, y) \in \mathbb{A}^2 \mid y^2 - x^3 = 0\}$. Is the map $\varphi : \mathbb{A}^1 \to C$, $\varphi(t) = (t^2, t^3)$ an isomorphism of affine varieties? Is φ homeomorphism with respect to the Zariski topology?
3. [10 T]
   Find the irreducible components of the affine variety
   $$V(x_1 - x_2x_3, x_1x_3 - x_2^2) \subset \mathbb{A}^3.$$
4. [10 T]
   Determine the radical of the ideal $(x^3 - y^6, xy - y^3) \subset \mathbb{C}[x, y]$.
5. [10 T]
   Let X be the union of the three coordinate axes. Compute generators for the ideal $I(X)$ and show that $I(X)$ cannot be generated by fewer than three elements.
6. [20 T]
   Let Y be a non-empty irreducible subvariety of an affine variety X and set $U = X \setminus Y$. Assume that the coordinate ring $\mathcal{O}_X(X)$ of X is a unique factorization domain. Show that $\mathcal{O}_X(U) = \mathcal{O}_X(X)$ if and only if $\operatorname{codim} Y \geq 2$. Find a counter-example if $\mathcal{O}_X(X)$ is not a unique factorization domain.