Differential Geometry - 3rd homework

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Ex. 1: Let M be a smooth manifold.

(i) Let us show that any connected orientable manifold M admits precisely two orientations.

Suppose $\mathcal{A} = (U_i, \varphi_i)_i$ and $\mathcal{B} = (V_j, \psi_j)_j$ are two compatible and oriented atlases on M. Suppose in addition that there exists a point $p \in M$ and two neighbourhood charts $(U_i, \varphi_i) \in \mathcal{A}$ and $(V_j, \psi_j) \in \mathcal{B}$ such that $\det(d(\psi_j \circ \varphi_i^{-1})_{\varphi_i(p)}) > 0$. We want to show that $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas.

Indeed, first take any other possible neighbourhood chart $(U_k, \varphi_k) \in \mathcal{A}$ that contains p. Then we have

$$\begin{aligned}\det(d(\psi_j \circ \varphi_i)_{\varphi_i(p)}) &= \det(d(\psi_j \circ \varphi_k^{-1} \circ \varphi_k \circ \varphi_i)_{\varphi_i(p)}) \\ &= \det(d(\psi_j \circ \varphi_k^{-1})_{\varphi_k(p)} \cdot d(\varphi_k \circ \varphi_i^{-1})_{\varphi_i(p)}) \\ &= \det(d(\psi_j \circ \varphi_k^{-1})_{\varphi_k(p)}) \cdot \underbrace{\det(d(\varphi_k \circ \varphi_i^{-1})_{\varphi_i(p)})}_{>0} > 0\end{aligned}$$

hence we may take any chart in $\mathcal{A}$ that contains p. Symmetrically, we may also take any chart in $\mathcal{B}$ that contains p.

Secondly, since det is a continuous map, the above determinant is greater than zero in the entire (connected) intersection of chart neighbourhoods. If it was negative in some p' in the intersection, then there would exist a point on some (compact) path between p and p' where the det would be zero, which would contradict the fact that the derivative there is actually a transition matrix between two consistently oriented local frames.

Thirdly, since M is a connected manifold, it is path-connected. Let $q \in M$ be any other point and let γ be a path between p and q. Since γ is compact, we can cover it with finitely many charts of both $\mathcal{A}$ and $\mathcal{B}$, respectively, which sequentially have non-empty intersections (otherwise we have a separation for the interval). Let those charts be (U_i, φ_i) for $i = 1, \ldots, k$ and (V_j, ψ_j) for $j = 1, \ldots, l$, where each point of γ is in at most two charts U_i and at most two charts V_j. Then we pick intersection points $x_{i,i+1} \in U_i \cap U_{i+1}$. Using the argument from step two sequentially on $p, x_{1,2}, x_{2,3}, \ldots, q$ and invariance of the choice of chart from step one, we inductively get that the determinant of the derivative must be greater than zero on any chart neighbourhood intersection of q. In other words, $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas.

What we have effectively proved is that such compatibility of orientation is an equivalence relation. Since the existence condition is binary (such a point p either exists or not), we get that we have at most two equivalence classes, i.e., possible orientations. To see that we actually get two orientations, take any oriented atlas $\mathcal{A} = (U_i, \varphi_i = (x^k))_i$ and denote $\mathcal{A}^-$ the same atlas, with the difference that instead of the x^1 coordinate, we have $-x^1$. Then $\mathcal{A}$ and $\mathcal{A}^-$ are obviously compatible but give different orientations.

Alternative solution: We form the following two sets

$$A^+ = \Big\{q \in M;\ \text{there exist } (U, \varphi) \in \mathcal{A} \text{ and } (V, \psi) \in \mathcal{B} \text{ s.t. } \det(d(\psi \circ \varphi^{-1})_{\varphi(q)}) > 0\Big\}$$

$$A^- = \Big\{q \in M;\ \text{there exist } (U, \varphi) \in \mathcal{A} \text{ and } (V, \psi) \in \mathcal{B} \text{ s.t. } \det(d(\psi \circ \varphi^{-1})_{\varphi(q)}) < 0\Big\}$$

Both are clearly open and $M = A^+ \cup A^-$. By the first step from the first solution above (invariance of neighbourhood), we also have $A^+ \cap A^- = \emptyset$. Since M is connected, and these sets define a separation for M, one of them must be empty. By assumption, $p \in A^+$, hence $M = A^+$ and $\mathcal{A} \cup \mathcal{B}$ is by definition an oriented atlas.

(ii) We will prove that M is orientable iff it admits a volume form.

- ($\Longleftarrow$): Let ω be a nowhere vanishing differential n-form on M. Take a connected open chart U and let (x^i) be local coordinates on U. Then ω has a local coordinate expression

$$\omega = f dx^1 \wedge \cdots \wedge dx^n$$

for some smooth coefficient function f. Since ω is nowhere vanishing, so must be f. We have

$$\omega(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}) = f \det I = f$$

which is nowhere zero for this local frame, and since U is connected, either always positive or always negative on U. If it is negative, we have seen in (1i) we can replace x^1 for $-x^1$ and obtain an always positive function f on U. Hence compatible charts where ω is a positive multiple of the form $dx^1 \wedge \cdots \wedge dx^n$ form an oriented atlas on M.

- ($\Longrightarrow$): Let $\mathcal{A} = (U_i, \varphi_i = (x^k))_i$ be an oriented atlas on M and let $(\phi_U)_U$ be a partition of unity subordinate to $\mathcal{A}$. Denote the orientation given by the condition in (1i) by μ_p (recall the point p). For each chart U and its local coordinates choose an n-form ω_U such that "orients" the local frame $(\frac{\partial}{\partial x^k})$, that is, for each $p \in U$ for a basis $(v_1, \dots, v_n)$ of T_pM we have

$$\omega_U(v_1, \dots, v_n) > 0 \iff (v_1, \dots, v_n) \text{ orient } U \text{ as } \mu_p$$

(we can just define ω_U as in the converse using the second step of (1i)). We now define

$$\omega = \sum_{U \in \mathcal{A}} \phi_U \omega_U$$

which is now a smooth n-form. For each $p \in M$, if the basis $(v_1, \dots, v_n) \in T_pM$ give the same orientation as μ_p then

$$(\phi_U \omega_U)(p)(v_1, \dots, v_n) \geq 0$$

for each chart neighbourhood U of p, with a strict inequality for at least one U (otherwise (1i) fails). Hence ω is nowhere vanishing.

Now suppose we have two volume forms ω and Ω on M that determine the same orientation. We then have $\omega = f dx^1 \wedge \cdots \wedge dx^n$ and $\Omega = g dx^1 \wedge \cdots \wedge dx^n$ for two positive smooth functions f, g. Hence, $\omega = \frac{f}{g}\Omega$, where $\frac{f}{g}$ is well-defined, positive, and smooth. The converse trivially follows from the above proof of the right-to-left implication.

(iii) Let (M, g) be an oriented Riemannian manifold, i.e., g is a Riemannian metric on M. Let us prove that there exists a unique volume form ω_g determined by the property: for any $p \in M$ if $(v_1, \dots, v_n)$ is an ordered oriented orthonormal basis for T_pM, then $\omega_g(v_1, \dots, v_n) = 1$.

As usual, we first prove uniqueness. Let ω be such a form (existence later). Since M is oriented, for every p in a neighbourhood chart U the basis $(v_1, \dots, v_n)$ forms an ordered oriented orthonormal local frame $(E_1, \dots, E_n)$ for U. Let $(\varepsilon^1, \dots, \varepsilon^n)$ be its dual frame. Then locally $\omega = f \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$. The defining condition for ω above then implies $f = 1$, hence $\omega = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ is unique.

To prove existence, we define ω as we have determined it to be above. Let $(F_1, \dots, F_n)$ be another ordered oriented orthonormal frame (determined by some other continuous choice of bases pointwise), and let $(f^1, \dots, f^n)$ be its dual frame. Define $\Omega = f^1 \wedge \cdots \wedge f^n$, so as above, but for the new frame. As in the remark, we can write $F_i = A_i^j E_j$ (we mean pointwise, of course) for some matrix (A_i^j) of smooth functions. Since both frames are orthonormal, we get that the matrix

$(A_i^j(p))$ is orthogonal for each p, so its determinant is either 1 or -1. By choosing a consistently oriented second frame, we get the positive determinant 1. We now apply ω to the new frame pointwise

$$\omega(F_1, \dots, F_n) = \det(\varepsilon^j(F_i)) = \det(A_i^j) = 1 = \Omega(F_1, \dots, F_n)$$

Hence, $\omega = \Omega$ (independently of frames) so defining ω_g locally as in the proof of uniqueness gives a global n-form which is nowhere vanishing, that satisfies the desired property.

Ex. 2:

(i) Let us show that $\mathcal{L}_V\alpha$ is $C^\infty(M)$-multilinear in its arguments.

For addition, this is simply a matter of elementary properties of vector and tensor fields as they are both $C^\infty(M)$-additive. To check homogeneity it is enough to check the case where we replace X_1 by fX_1, by symmetry. Indeed, we get

$$\mathcal{L}_V\alpha(fX_1, X_2, \dots, X_k) = \underbrace{V(f\alpha(X_1, \dots, X_2))}_{A_1} \underbrace{-\alpha([V, fX_1], X_2, \dots, X_k)}_{A_2} + r$$

where r is the uninteresting remainder, since α is $C^\infty(M)$-multilinear already. We will now deal with the Leibniz rule. Firstly, at A_1 we get

$$V(f\alpha(X_1, \dots, X_k)) = fV(\alpha(X_1, \dots, X_k)) + V(f)\alpha(X_1, \dots, X_k)$$

and at A_2 we get

$$-V(f)\alpha(X_1, \dots, X_k) - f\alpha([V, X_1], X_2, \dots, X_k)$$

so the unwanted parts indeed cancel out.

(ii) A quick calculation shows that for any 0-form f we have $\mathcal{L}_V(df) = d(\mathcal{L}_V f)$.

On the one hand, we have

$$\mathcal{L}_V(df)(X) = V(df(X)) - df([V, X]) = V(X(f)) - (V(X(f)) - X(V(f))) = X(V(f))$$

and on the other hand

$$(d(\mathcal{L}_V f))(X) = X(\mathcal{L}_V f) = X(V(f))$$

(iii) We will prove that there holds

$$\mathcal{L}_V(\alpha \otimes \beta) = (\mathcal{L}_V\alpha) \otimes \beta + \alpha \otimes (\mathcal{L}_V\beta)$$

for any two $k-$ and $l-$ tensor fields α and β.

Since the operator $\mathcal{L}_V$ is clearly linear in its argument, it is enough to consider simple tensor fields. Since the tensor product of two simple tensor fields is again a simple tensor field, it suffices to only consider one such simple tensor field and prove that locally we have

$$\mathcal{L}_V\alpha = (\mathcal{L}_V f)dx^{i_1} \otimes \cdots \otimes dx^{i_k} + f\mathcal{L}_V(dx^{i_1}) \otimes \cdots \otimes dx^{i_k} + \cdots + fdx^{i_1} \otimes \cdots \otimes \mathcal{L}_V(dx^{i_l})$$

where $\alpha = fdx^{i_1} \otimes \cdots \otimes dx^{i_k}$ locally.

This equality trivially holds if α is a 0-tensor field. Suppose $\alpha = fdx^i$ is a 1-tensor field. Then on the one hand

$$\begin{aligned}\mathcal{L}_V(fdx^i)(X) &= V(fdx^i(X)) - fdx^i([V, X]) = fV(X^i) + V(f)X^i - f(V(X^i) - X(V^i)) \\ &= V(f)X^i + fX(V^i)\end{aligned}$$

and on the other hand

$$(\mathcal{L}_V f)dx^i(X) + f(\mathcal{L}_V(dx^i))(X) = V(f)X^i + fdV^i(X) = V(f)X^i + fX(V^i)$$

so the equality indeed holds. For higher tensor fields we calculate similarly. By (2i) it is enough to look at coordinate vector fields, and we clearly get a non-zero result only if the vector fields $(X_{i_j})_j$ are a permutation of the coordinate vector fields $(\partial_{i_j})_j$. Without loss of generality, $\alpha = f dx^1 \otimes \cdots \otimes dx^k$ and $X_i = \partial_i$. If we calculate the left-hand side, we get

$$\mathcal{L}_V(f dx^1 \otimes \cdots \otimes dx^k)(\partial_1, \ldots, \partial_k) = V(f \det I) - \alpha([V, \partial_1], \partial_2, \ldots, \partial_k) - \cdots - \alpha(\partial_1, \ldots, [V, \partial_k])$$

notice now that

$$[V, \partial_i] = (V(\delta_{i,j}) - \partial_i(V^j))\partial_j = -\partial_i(V^j)\partial_j$$

which turns those minuses into pluses we see on the right-hand side! If we calculate the right-hand side as well (recall, $\mathcal{L}_V(dx^i)(X) = X(V^i)$), we get

$$RHS(\partial_1, \ldots, \partial_k) = V(f) \det I + f dV^1 \otimes \cdots \otimes dx^k(\partial_1, \ldots, \partial_k) + f dx^1 \otimes \cdots \otimes dV^k(\partial_1, \ldots, \partial_k)$$

which is the same as above after we calculate the determinants (we get sums at each term, same as for the left-hand side).

(iv) We now have

$$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V \omega) \wedge \eta + \omega \wedge (\mathcal{L}_V \eta)$$

for any two $k-$ and $l-$ differential forms ω and η.

Indeed, any differential form is just an alternating linear combination of tensor fields, and as we have already noticed, $\mathcal{L}_V$ is a linear operator. Concretely,

$$\omega \wedge \eta = \frac{(k+1)!}{k!l!} \mathrm{Alt}(\omega \otimes \eta)$$

(v) Let us show that there holds $\iota_V \circ \iota_V = 0$ and

$$\iota_V(\omega \wedge \eta) = (\iota_V \omega) \wedge \eta + (-1)^k \omega \wedge (\iota_V \eta)$$

for any two $k-$ and $l-$ differential forms ω and η.

The first property is straight-forwardly checked. If $k \geq 2$, then

$$\iota_V \iota_V \omega(X_1, \ldots, X_{k-2}) = \omega(V, V, X_1, \ldots, X_{k-2}) = 0$$

since ω is alternating. Otherwise this follows from the fact that for a 0-form we have $\iota_V f = 0$.

The second property is a bit messier. Again it is enough to verify the property for simple forms. More generally, we will prove that for a collection of k many simple 1-forms we have

$$\iota_V(\omega^1 \wedge \cdots \wedge \omega^k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(V) \cdot \omega^1 \wedge \cdots \hat{\omega^i} \wedge \cdots \wedge \omega^k$$

from which the property immediately follows by linearity. Take k vector fields $V, X_2, \ldots, X_k$ (consider $V = X_1$). We have to prove that

$$\begin{aligned}\iota_V(\omega^1 \wedge \cdots \wedge \omega^k)(X_2, \ldots, X_k) &= (\omega^1 \wedge \cdots \wedge \omega^k)(V, X_2, \ldots, X_k) \\ &= \sum_{i=1}^{k} (-1)^{i-1} \omega^i(V)(\omega^1 \wedge \cdots \hat{\omega^i} \wedge \cdots \omega^k)(X_2, \ldots, X_k)\end{aligned}$$

We know from Lectures that the left-hand side is the determinant of the matrix

$$A_i^j = \omega^i(X_j)$$

while the right-hand side is by the same argument precisely the expansion of $\det A$ by minors along the first column, so equal to $\det A$.

(vi) We now prove *Cartan's magic formula* for differential forms:

$$\mathcal{L}_V\omega = \iota_V(d\omega) + d(\iota_V\omega)$$

Here we use induction on k. If f is a 0-form, then clearly $\mathcal{L}_V f = V(f)$ and also

$$\iota_V(df) + d(\iota_V f) = df(V) + 0 = V(f)$$

Let now $k \geq 1$ and suppose the formula holds for all l-forms for $l < k$. Let ω be a k-form, locally written as

$$\omega = \sum_I \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Denote $u = x^{i_1}$ and $\beta = \omega_I dx^{i_2} \wedge \cdots \wedge dx^{i_k}$, so each term of ω is locally the sum of $du \wedge \beta$, where u is a smooth function. By (2ii) we have $\mathcal{L}_V(du) = d(\mathcal{L}_V u) = d(V(u))$, and by (2iv)

$$\mathcal{L}_V(du \wedge \beta) = (\mathcal{L}_V u) \wedge \beta + du \wedge (\mathcal{L}_V \beta) \overset{\text{I.H.}}{=} d(V(u)) \wedge \beta + du \wedge (\iota_V(d\beta) + d(\iota_V\beta))$$

But since d is an antiderivative, and by (2v) so is ι_V, plus $\iota_V du = du(V) = V(u)$ and $d \circ d = 0$, we compute

$$\begin{aligned}\iota_V(d(du \wedge \beta)) + d(\iota_V(du \wedge \beta)) &= \iota_V(-du \wedge d\beta) + d(Vu)\beta - du \wedge \iota_V\beta \\ &= -(Vu)d\beta + du \wedge (\iota_V(d\beta)) + d(Vu) \wedge \beta + (Vu)d\beta + d(\iota_V\beta)\end{aligned}$$

which is precisely $\mathcal{L}_V(du \wedge \beta)$ by the above. Now use linearity to extend this beyond simple forms.

(vii) We will now use this formula to prove that for any differential form ω we have

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V\omega)$$

and that for any 1-form ω and any two vector fields X, Y on M there holds

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

The first equality is a matter of simple calculation where we will merely use the formula and the fact that $d \circ d = 0$. On the one hand we have

$$\mathcal{L}_V(d\omega) = \iota_V(dd\omega) + d(\iota_V(d\omega)) = d(\iota_V(d\omega))$$

and on the other hand

$$d(\mathcal{L}_V\omega) = d(\iota_V(d\omega)) + dd(\iota_V\omega) = d(\iota_v(d\omega))$$

so we indeed have equality.

For the second equality first notice that for any 1-form ω we have

$$\mathcal{L}_V\omega(X) = V(\omega(X)) - \omega([V, X])$$

This gives us the idea to consider the difference of appropriate Lie derivatives in two different ways. Firstly, by definition, we get

$$\begin{aligned}\mathcal{L}_X\omega(Y) - \mathcal{L}_Y\omega(X) &= X(\omega(Y)) - \omega([X, Y]) - (Y(\omega(X)) - \omega([Y, X]) \\ &= X(\omega(Y)) - Y(\omega(X)) - 2\omega([X, Y])\end{aligned}$$

Secondly, by the formula, we get

$$\begin{aligned}\mathcal{L}_X\omega(Y) - \mathcal{L}_Y\omega(X) &= \iota_X(d\omega)(y) + d(\iota_X\omega)(Y) - \iota_Y(d\omega)(X) - d(\iota_Y\omega)(X) \\ &= 2d\omega(X, Y) + Y(\omega(X)) - X(\omega(Y))\end{aligned}$$

Equating both approaches, rearranging, and dividing by 2 yields the desired equality.

Ex. 3: Let M be an orientable smooth manifold and ω a volume form on M.

(i) Let $M = \mathbb{R}^n$ and $\omega = dx^1 \wedge \cdots \wedge dx^n$. We will calculate $\operatorname{div}_\omega(X)$ for a given vector field $X = X^i \partial_i$ in standard coordinates.

Firstly, we use Cartan's magic formula to get

$$\operatorname{div}_\omega(X)\omega = \mathcal{L}_X \omega = \iota_X(d\omega) + d(\iota_X \omega) = d(\iota_X \omega)$$

since all $(n+1)$-forms, in particular $d\omega$, are zero. We know from Lectures that for $\alpha = \sum_I a_I dx^I$ we have

$$d\alpha = \sum_I da_I \wedge dx^I$$

so we have to find the local expression of the $(n-1)$-form $\iota_X \omega$. For any $n-1$ vector fields $Y_1, \ldots, Y_{n-1}$ we have

$$\iota_X \omega(Y_1, \ldots, Y_{n-1}) = \omega(X, Y_1, \ldots, Y_{n-1}) = \sum_{i=1}^n X^i \omega(\partial_i, Y_1, \ldots, Y_{n-1})$$
$$= \sum_{i=1}^n X^i \det \begin{bmatrix} 0 & Y_1^1 & \cdots & Y_{n-1}^1 \\ \vdots & & & \vdots \\ 1 & Y_1^i & \cdots & Y_{n-1}^i \\ \vdots & & & \vdots \\ 0 & Y_1^n & \cdots & Y_{n-1}^n \end{bmatrix} = \sum_{i=1}^n (-1)^{i+1} X^i \det A_i$$

where A_i is the minor of the above matrix with the first column and i-th row taken out. Hence clearly

$$d(\iota_X \omega) = \sum_{i=1}^n (-1)^{i+1} \frac{\partial X^i}{\partial x^i} dx^i \wedge (dx^1 \wedge \cdots \hat{dx^i} \wedge \cdots dx^n) = \left(\sum_{i=1}^n \frac{\partial X^i}{\partial x^i} \right) \omega$$

where in the last equality we have to transpose dx^i $(i-1)$ times to the right, so the minuses cancel. We have finally got

$$\operatorname{div}_\omega(X) = \sum_{i=1}^n \frac{\partial X^i}{\partial x^i},$$

the definition we remember from Analysis 2.

(ii) Let now (M, g) be an oriented Riemannian manifold. We will show that the Riemannian volume form from (1iii) is expressed as

$$\omega_g = \sqrt{\det[g_{ij}]} dx^1 \wedge \cdots \wedge dx^n$$

in any oriented chart $(U, \varphi = (x^i))$, where $g_{ij} = g(\partial_i, \partial_j)$.

Indeed, let (U, φ) be an oriented smooth chart from the instructions. Then as in (1iii) we have $\omega_g = f dx^1 \wedge \cdots \wedge dx^n$ for some positive function f. Recall, that the condition on ω_g regards any smooth orthonormal oriented frame, so pick such a frame (E_i) defined on a neighbourhood of some point $p \in M$. Denote by (ε^i) its dual frame. As in (1iii) we again have that $\partial_i = A_i^j E_j$, so we calculate

$$f = \omega_g(\partial_1, \ldots, \partial_n) = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n(\partial_1, \ldots, \partial_n) = \det(\varepsilon^j \partial_i) = \det(A_i^j)$$

Consider now g_{ij}

$$g_{ij} = g(\partial_i, \partial_j) = (A_i^k E_k, A_j^l E_l) = A_i^k A_j^l g(E_k, E_l) = \sum_{k=1}^n A_i^k A_j^k$$

which is precisely the (i, j)-th element of the matrix $A^T A$. Hence clearly

$$\det[g_{ij}] = \det(A^T A) = \det(A)^2$$

So we finally have

$$f = \det A = \pm\sqrt{\det[g_{ij}]}$$

Since both frames $(\partial_i)_i$ and $(E_j)_j$ are oriented, the sign is positive.

(iii) Let (M, g) be an oriented Riemannian manifold and $(U, \varphi = (x^i))$ and oriented chart. Let us show that for a vector field X on M, expressed locally as $X = X^i\partial_i$, we have

$$\operatorname{div}_{\omega_g}(X) = \frac{1}{\sqrt{\det[g_{ij}]}}\partial_k\left(\sqrt{\det[g_{ij}]}X^k\right)$$

Indeed, since $\omega_g = \sqrt{\det[g_{ij}]}dx^1 \wedge \cdots \wedge dx^n$, we have that

$$\begin{aligned}\iota_X\omega_g(Y_1, \ldots, Y_{n-1} &= \omega_g(X, Y_1, \ldots, Y_{n-1}) = \sum_{i=1}^n X^i\omega_g(\partial_i, Y_1, \ldots, Y_{n-1})\\ &= \sum_{i=1}^n \sqrt{\det[g_{ij}]}X^i \det\begin{bmatrix} 0 & Y_1^1 & \cdots & Y_{n-1}^1 \\ \vdots & & & \vdots \\ 1 & Y_1^i & \cdots & Y_{n-1}^i \\ \vdots & & & \vdots \\ 0 & Y_1^n & \cdots & Y_{n-1}^n \end{bmatrix} = \sum_{i=1}^n (-1)^{i+1}\sqrt{\det[g_{ij}]}X^i \det A_i\end{aligned}$$

Now, we get

$$\begin{aligned} d(\iota_X\omega_g) &= \sum_{i=1}^n (-1)^{i+1}\frac{\partial(\sqrt{\det[g_{ij}]}X^i)}{\partial x^i}dx^i \wedge (dx^1 \wedge \cdots \hat{dx^i} \wedge \cdots dx^n)\\ &= \left(\sum_{i=1}^n \frac{\partial(\sqrt{\det[g_{ij}]}X^i)}{\partial x^i}\right)\omega = \frac{1}{\sqrt{\det[g_{ij}]}}\left(\sum_{i=1}^n \frac{\partial(\sqrt{\det[g_{ij}]}X^i)}{\partial x^i}\right)\omega_g\end{aligned}$$

which is precisely what we wanted to prove.

(iv) Let M be an orientable smooth manifold, ω a volume form on M, and let ϕ_t^X denote the flow of a vector field X on M. Let us prove that there holds

$$\operatorname{div}_\omega(X) = 0 \iff (\phi_t^X)^*\omega = \omega \text{ for any appropriate } t$$

- $(\Longrightarrow)$: This direction follows trivially from the hint. If $\operatorname{div}_\omega(X) = 0$ then by definition $\mathcal{L}_V\omega = 0$ at every point. By the hint, $(\phi_t^X)^*\omega$ is a constant function at every point. Since at $t = 0$ we have identity, we get that $(\phi_t^X)^*\omega = \omega$ for all t.
- $(\Longleftarrow)$: Suppose $\operatorname{div}_\omega(X) \neq 0$ in at least a point. Then, by continuity, there exists a neighbourhood of that point such that $\operatorname{div}_\omega(X)$ is either positive or negative. Without loss of generality, assume $\operatorname{div}_\omega(X) < 0$ in some U. Now take a compact coordinate ball $B \subset U$ and small enough $t > 0$ such that still $\phi_t^X(B) \subset U$. Then by the hint

$$\left.\frac{d}{dt}\right|_{t=0}\left((\phi_t^X)^*\omega\right)_p < \omega|_p$$

for every point $p \in B$, so $(\phi_t^X)^*\omega < \omega$ for every $p \in B$ since the flow is defined here by our choices of t. This proves the contrapositive.

Ex. 4: Suppose (q, p) are the standard coordinates on $\mathbb{R}^2 \cong T^*\mathbb{R}$. Suppose that ω is a differential 2-form on $\mathbb{R}^2$ given by $\omega = dq \wedge dp$ and let $H \in C^\infty(\mathbb{R}^2)$ be a smooth function. Define the Hamiltonian vector field X_H on $\mathbb{R}^2$ by the implicit identity $\iota_{X_H}\omega = dH$.

(i) Let us first express the vector field X_H on $\mathbb{R}^2$ in the coordinates (q,p). We write

$$X_H = a\partial_q + b\partial_p$$

for some smooth local coefficient functions. On the one hand we have

$$\iota_{a\partial_q + b\partial_p}(dq \wedge dp) = adp - bdq$$

and on the other hand we have

$$dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp$$

so by equating we immediately get

$$X_H = \frac{\partial H}{\partial p}\partial_q - \frac{\partial H}{\partial q}\partial_p$$

From this it then directly follows that the integral curves of X_H are given by the system of differential equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

Let now

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

for some positive real numbers m, k. Let us calculate the flow of X_H, and for this we need to actually calculate the integral curves of X_H. From the above formulae we get

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -kq$$

By differentiating again and inputting equations, we get

$$\ddot{q} = -\frac{k}{m}q \qquad \ddot{p} = -\frac{k}{m}p$$

hence

$$q(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right) \qquad p(t) = C\cos\left(\sqrt{\frac{k}{m}}t\right) + D\sin\left(\sqrt{\frac{k}{m}}t\right)$$

Picking a starting point $\gamma(0) = (x,y)$ we get $A = x$ and $C = y$, and then by inputting this solutions into the original differential equations, we also get $B = \frac{y}{\sqrt{km}}$ and $D = -x\sqrt{km}$. Therefore

$$\phi_t^{X_H}(x,y) = \gamma_{(x,y)}(t) = \left(x\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{y}{\sqrt{km}}\sin\left(\sqrt{\frac{k}{m}}t\right), y\cos\left(\sqrt{\frac{k}{m}}t\right) - x\sqrt{km}\sin\left(\sqrt{\frac{k}{m}}t\right)\right)$$

(ii) We will now show that $X_H(H) = 0$, indeed

$$X_H(H) = \left(\frac{\partial H}{\partial p}\partial_q - \frac{\partial H}{\partial q}\partial_p\right)(H) = \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} = 0$$

(iii) We now want to compute $\mathrm{div}_\omega(X_H)$, $\mathcal{L}_{X_H}\omega$, and show that there holds $(\phi_t^{X_H})^*\omega = \omega$. For the divergence, we simply use (3i) to get

$$\mathrm{div}_\omega(X_H) = \frac{\partial X_H^q}{\partial q} + \frac{\partial X_H^p}{\partial p} = \frac{\partial^2 H}{\partial p \partial q} - \frac{\partial^2 H}{\partial q \partial p} = 0$$

The Lie differentiation follows immediately from the definition of divergence

$$\mathcal{L}_{X_H}\omega = \mathrm{div}_\omega(X_H)\omega = 0$$

and finally, by (3iv), since the divergence is zero, there holds

$$(\phi_t^{X_H})^*\omega = \omega$$