# Differential Geometry - $1^{\text{st}}$ homework

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**<u>Ex. 1:</u>** Let $M$ and $N$ be smooth manifolds and let $C(M)$ denote the algebra of all continuous functions $f\colon M \to \mathbb{R}$. Given a continuous map $g\colon M \to N$, we define the map $g^*\colon C(N) \to C(M)$ by $g^* f = f \circ g$.

(i) We will show that $g\colon M \to N$ is a smooth map iff $g^*(C^\infty(N)) \subset C^\infty(M)$ holds.

- <u>$(\implies)$</u>: Suppose $g$ is a smooth map between smooth manifolds $M$ and $N$. Then $f \circ g$ is smooth as a composition of smooth maps. Indeed, we have

$$\begin{aligned}
&f \circ g \text{ smooth} \\
&\iff \text{for any chart } (U, \varphi) \text{ on } M\text{: } f \circ g \circ \varphi^{-1} \text{ smooth} \\
&\iff \text{for any } (U, \varphi) \text{ and intermediate chart } (V, \psi) \text{ on } N\text{: } f \circ \psi^{-1} \circ \psi \circ g \circ \varphi^{-1} \text{ smooth}
\end{aligned}$$

  which is now a composition of two smooth real functions, hence smooth. By intermediate chart, we simply mean such a chart $(V, \psi)$ that $\operatorname{im} g \circ \varphi^{-1} \cap V \neq \emptyset$.

- <u>$(\impliedby)$</u>: Suppose now that $f \circ g$ is a smooth map for any smooth function $f$ on $N$. Denote $m = \dim M$ and $n = \dim N$. By the above chain of equivalences, we get that $f \circ \psi^{-1} \circ \psi \circ g \circ \varphi^{-1}$ is a smooth real function $\mathbb{R}^m \to \mathbb{R}$ (for appropriate charts as above). Denote by $g_i$ the $i$-th component of the real function $\psi \circ g \circ \varphi^{-1}\colon \mathbb{R}^m \to \mathbb{R}^n$ for $i = 1, \ldots, n$. By assumption, we can choose any smooth function $f$ on $N$, so let $f_i$ be such choices of $f$ that $f_i \circ \psi^{-1}\colon \mathbb{R}^n \to \mathbb{R}$ is the projection to the $i$-th component. Then we have that $g_i = f_i \circ \psi^{-1} \circ \psi \circ g \circ \varphi^{-1}$ are smooth for all $i = 1, \ldots, n$. Hence $\psi \circ g \circ \varphi^{-1}$ is a smooth real function for all appropriate charts, so $g$ is a smooth map.

(ii) Suppose now that $g\colon M \to N$ is a homeomorphism between smooth manifolds. Let us show that $g$ is a diffeomorphism iff $g^*|_{C^\infty(N)}\colon C^\infty(N) \to C^\infty(M)$ is an isomorphism.

- <u>$(\implies)$</u>: Since function addition and multiplication are defined pointwise, $g^*$ is clearly an algebra homomorphism. By (1i), it is also well-defined. To prove injectivity, consider

$$\begin{aligned}
g^* f_1 = g^* f_2 &\iff f_1 \circ g = f_2 \circ g \\
&\iff f_1(g(x)) = f_2(g(x)) \text{ for every } x \in M \\
&\overset{g \text{ bij.}}{\iff} f_1(g(g^{-1}(y))) = f_2(g(g^{-1}(y))) \text{ for every } y \in N \\
&\iff f_1(y) = f_2(y) \text{ for every } y \in N \\
&\iff f_1 = f_2
\end{aligned}$$

  For surjectivity, take $h \in C^\infty(M)$. We want to find a map $f \in C^\infty(N)$ such that $f \circ g = h$, but since $g$ is a diffeomorphism, this is obviously the map $f = h \circ g^{-1}$. Notice that this also shows that $(g^*)^{-1} = (g^{-1})^*$.

- <u>$(\impliedby)$</u>: Since $g^*$ is an isomorphism, we get

$$\begin{aligned}
g^*(C^\infty(N)) = C^\infty(M) &\overset{(1i)}{\implies} g \text{ smooth} \\
(g^{-1})^*(C^\infty(M)) = C^\infty(N) &\overset{(1i)}{\implies} g^{-1} \text{ smooth}
\end{aligned}$$

  hence $g$ is a diffeomorphism.

**Ex. 2:** Let $g\colon M \to N$ be a smooth map between smooth manifolds.

(i) Suppose $S \subset N$ is an immersed submanifold in $N$, and suppose that $G(M) \subset S$. Let us prove that if $g$ is continuous as a map from $M$ to $S$ then $g\colon M \to S$ is smooth.

Take $p \in M$ and denote $q = g(p) \in S$. We use the hint right away: since $i\colon S \hookrightarrow N$ is an immersion, there exists a neighbourhood $U$ of $q = g(p)$ in $S$ such that $i|_U\colon U \hookrightarrow N$ is a smooth embedding. Hence, there exists a chart $(W, \psi)$ for $U$ in $N$ (that sends $q$ to 0) such that $(U \cap W, \pi \circ \psi)$ is a (flattening) submanifold chard for $U$, that is, $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ is a projection to the first $k = \dim S$ coordinates in $\mathbb{R}^n$. Denote $(V, \tilde{\psi}) = (U \cap W, \pi \circ \psi)$. Since $V = (i|_U)^{-1}(W)$ is open in $U$, which is in turn open in $S$, $V$ is also open in $S$, so $(V, \tilde{\psi})$ can be seen as a chart in $S$. We now finally use our assumption: since $g\colon M \to S$ is continuous, $V_0 = g^{-1}(V)$ is an open set in $M$ containing $p$.

Choose a smooth chart $(U_0, \varphi)$ in $M$ that is contained in $V_0$ and contains the point $p$. Then the coordinate representation of $g\colon M \to S$ with respect to charts $(U_0, \varphi)$ and $(V, \tilde{\psi})$

$$\tilde{\psi} \circ g \circ \varphi^{-1} = \pi \circ \underbrace{\psi \circ g \circ \varphi^{-1}}_{\text{smooth}}$$

is smooth. Hence $g$ is smooth on a neighbourhood of $p$ for every $p \in M$, so smooth everywhere.

(ii) Suppose $S \subset N$ is an embedded submanifold in $N$, and suppose again that $g(M) \subset S$. We will prove that $g\colon M \to S$ is smooth.

By (2i), since every embedding is an immersion, we only need to show that $g\colon M \to S$ is continuous, but this is clearly the case as $S \subset N$ now has the subspace topology.

(iii) The *lemniscate* $L$ is the image of the map $\phi\colon (-\pi, \pi) \to \mathbb{R}^2$, defined by $\phi(t) = (\sin(2t), \sin(t))$, and is an immersed, but not embedded submanifold in $\mathbb{R}^2$. Is the map $\psi\colon \mathbb{R} \to \mathbb{R}^2$, given by $\psi(t) = (\sin(2t), \sin(t))$, smooth as a map $\psi\colon \mathbb{R} \to L$?

By (2i), this will be the case precisely when $\psi$ is continuous. Let us prove that this is not the case. Concretely, we will show that $\phi^{-1} \circ \psi\colon \mathbb{R} \to (-\pi, \pi)$ is not continuous. Since $\phi^{-1}$ is continuous and the composition of continuous maps is continuous, the conclusion follows. Indeed, $\phi^{-1} \circ \psi$ is not continuous at $t = -\pi$. We calculate

$$\phi^{-1} \circ \psi(-\pi) = \phi^{-1}(\sin(-2\pi), \sin(-\pi)) = \phi^{-1}(0, 0) = 0$$

Take $U_\varepsilon = (-\varepsilon, \varepsilon) \subset (-\pi, \pi)$ a basis neighbourhood of 0. Then

$$\begin{aligned}(\phi^{-1} \circ \psi)^{-1}(-\varepsilon, \varepsilon) &= \psi^{-1}(\{(\sin(2t), \sin(t));\ t \in (-\varepsilon, \varepsilon)\}) \\ &= \{(2k\pi - \varepsilon, 2k\pi + \varepsilon);\ k \in \mathbb{Z}\} \cup \{k\pi;\ k \in \mathbb{Z}\}\end{aligned}$$

which is not an open set in $\mathbb{R}$.

**Ex. 3:** For any $n \in \mathbb{N}$, we define the unitary group of $n \times n$ matrices as

$$U(n) = \{A \in GL(n, \mathbb{C});\ A^*A = I\},$$

and also denote $\mathcal{H}_n = \{A \in \mathbb{C}^{n \times n};\ A^* = A\}$ as the vector space of $n \times n$ hermitian matrices.

(i) Let us show that $I$ is a regular value of the smooth map $\phi\colon GL(n, \mathbb{C}) \to \mathcal{H}_n$, given by $\phi(A) = A^*A$ and conclude that $U(n)$ is a smooth embedded submanifold in $GL(n, \mathbb{C})$. We will also determine its dimension, its tangent space $T_I U(n)$ at $I$, and show that $U(n)$ is path-connected.

- *I is a regular value*: By definition, $I$ is a regular value of $\phi$ precisely when for every $A \in \phi^{-1}(I)$ we have that the map

$$d\phi_A \colon T_A GL(n,\mathbb{C}) \to T_I \mathcal{H}_n$$

is surjective. We already know that $T_A GL(n,\mathbb{C}) \cong \mathbb{C}^{n\times n}$, and since $\mathcal{H}_n$ is a vector space we also get $T_I\mathcal{H}_n \cong \mathcal{H}_n$. We calculate

$$\begin{aligned} d\phi_A(X) &= \frac{d}{dt}|_{t=0}\phi(A+tX) \\ &= \frac{d}{dt}|_{t=0}\left((A^*+tX^*)(A+tX)\right) \\ &= \frac{d}{dt}|_{t=0}(A^*A + t(X^*A + A^*X) + t^2X^*X) \\ &= X^*A + A^*X. \end{aligned}$$

We now want to see that for every $Y \in \mathcal{H}_n$ there exists $X \in \mathbb{C}^{n\times n}$ such that $X^*A + A^*X = Y$. Since $Y = Y^*$ and $A^*A = I$, taking $X = \frac{1}{2}AY$ satisfies our requirement. By the Implicit Map Theorem, we can now conclude that $U(n)$ is a smooth embedded submanifold in $GL(n,\mathbb{C})$.

- *dimension:* We calculate

$$\dim U(n) = \dim GL(n,\mathbb{C}) - \dim \mathcal{H}_n = 2n^2 - n^2 = n^2.$$

- *tangent space at I*: Again, we merely calculate

$$T_I U(n) = \ker d\phi_I = \left\{X \in C^{n\times n};\ X^* + X = 0\right\},$$

that is, all skew-hermitian matrices.

- *p-connectedness*: Recall from linear algebra that unitary matrices can be diagonalized by unitary matrices, that is, for any unitary matrix $A$ there exists another unitary matrix $S$, such that

$$A = S\,\mathrm{diag}\left(e^{i\theta_1},\ldots,e^{i\theta_n}\right)S^{-1}$$

where we know that diagonal elements of diagonal unitary matrices must have absolute value 1. We thus obtain a path from $I$ to $A$ in $U(n)$ by taking

$$t \mapsto S\,\mathrm{diag}\left(e^{it\theta_1},\ldots,e^{it\theta_n}\right)S^{-1}$$

(ii) Additionally, we define the special unitary group of $n \times n$ matrices as

$$SU(n) = \{A \in GL(n,\mathbb{C});\ A^*A = I, \det A = 1\}.$$

Let us show that $SU(n)$ is a smooth embedded submanifold in $U(n)$ and then again determine its dimension, its tangent space at $I$, and show that it is path-connected. Additionally, we will show that the matrices $i\sigma_x, i\sigma_y, i\sigma_z$ form a basis for the vector space $T_I SU(2)$, where

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are the Pauli matrices.

- *smooth embedded submanifold*: Similarly as above, we define the map

$$\psi = \det \colon U(n) \to \mathbb{S}^1$$

(clearly, the determinant of any unitary matrix has absolute value 1) and prove that 1 is its regular value. By the calculation from tutorials, we get

$$d(\det)_A(X) = \det(A)\,\mathrm{tr}(A^{-1}X)$$

for every $A \in GL(n,\mathbb{C})$ and $X \in \mathbb{C}^{n\times n}$. In particular, this holds for $A \in \psi^{-1}(1)$, where we get just the trace. This map is surjective, since the trace function is surjective. As above, we conclude by the Implicit Mapping Theorem.

- *dimension*: We calculate

$$\dim SU(n) = \dim U(n) - \dim \mathbb{R} = n^2 - 1$$

- *tangent space at $I$*: We calculate

$$T_I SU(n) = \ker d(\psi)_I = \left\{X \in \mathbb{C}^{n\times n};\ X^* + X = 0,\ \operatorname{tr}(X) = 0\right\},$$

  that is, all skew-hermitian matrices with vanishing trace.
- *p-connectedness*: We would like to take the same path as in (3i), so we simply verify that all intermediate diagonal matrices are themselves already in $SU(n)$. Indeed, we calculate

$$\begin{aligned} \det \operatorname{diag}\left(e^{it\theta_1}, \dots, e^{it\theta_n}\right) &= e^{it\theta_1} \cdots e^{it\theta_n} \\ &= e^{t(i\theta_1 + \dots + i\theta_n)} \\ &= \left(e^{i\theta_1 + \dots + i\theta_n}\right)^t \\ &= \left(\det \operatorname{diag}\left(e^{i\theta_1}, \dots, e^{i\theta_n}\right)\right)^t \\ &= 1^t = 1 \end{aligned}$$

- *Paoli matrices*: Take a skew-hermitian matrix with vanishing trace

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  From zero trace we get that

$$a = -d$$

  and from skew-hermitian property we get that

$$a = -\overline{a},\ b = -\overline{c},\ c = \overline{b},\ d = -\overline{d}.$$

  The first and fourth equation tell us that $a$ can only be a pure imaginary number (and $d = -a$ as well), while the second and third equation are the same and tell us nothing more. Hence

$$A = \begin{bmatrix} \lambda i & -\overline{z} \\ z & -\lambda i \end{bmatrix}$$

  for some $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$. The vector space $T_I SU(2)$ is then indeed of dimension 3 and all $i\sigma_x, i\sigma_y, i\sigma_z$ clearly fit the above description. It is therefore enough to verify that they are linearly independent, so we take

$$\alpha \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = 0$$

  out of which it clearly follows that $\gamma = 0$, $\alpha i = \beta$, and $\alpha i = -\beta$, so $\alpha = \beta = 0$ as well.

(iii) Explain how it follows from this and Ex.2 that $U(n)$ and $SU(n)$ are Lie groups.

Recall now that $GL(n, \mathbb{C})$ is a Lie group and that by (3i), $U(n)$ is its embedded submanifold. Clearly, $U(n)$ is a subgroup of $GL(n, \mathbb{C})$ (algebraically), hence the operation functions

$$\mu\colon U(n) \times U(n) \to GL(n, \mathbb{C})$$

and

$$\iota\colon U(n) \to GL(n, \mathbb{C})$$

have their ranges restricted to $U(n)$. Since they are smooth as functions to $GL(n, \mathbb{C})$ (as restrictions of operation functions on $GL(n, \mathbb{C})$), by (2ii), they are smooth as functions to $U(n)$. Hence, $U(n)$ is also Lie group. Now repeat the same argument with $SU(n)$ being a subgroup and an embedded submanifold in $U(n)$.

(iv) Lastly, we prove that $SU(2)$ is diffeomorphic to $\mathbb{S}^3$.

Notice, that elements in $SU(2)$ are precisely of the form

$$SU(n) = \left\{ \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix} \in M_2(\mathbb{C}); |z|^2 + |w|^2 = 1 \right\}.$$

Indeed, this set is contained in $SU(n)$ by defining equations, and for any matrix in $SU(n)$, the defining equations enforce this form. Next, notice that $\mathbb{S}^3$ contained in $\mathbb{C}^2$ (or $\mathbb{R}^4$) is precisely the set

$$\mathbb{S}^3 = \left\{ (z,w) \in \mathbb{C}^2;\ |z|^2 + |w|^2 = 1 \right\}.$$

We can now define $f \colon \mathbb{S}^3 \to SU(2)$ as

$$(z,w) \mapsto \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}.$$

This functions is clearly well-defined, as the defining conditions match. It is also obvious that it is both injective, surjective, and of course continuous, since its component functions are continuous. Now view $SU(2) \subset M_2(\mathbb{C}) \cong \mathbb{R}^8$ and $\mathbb{S}^3 \subset \mathbb{R}^4$. We can view $f$ as a function $\tilde{f} \colon \mathbb{R}^4 \to \mathbb{R}^8$, and it is clear that both $\tilde{f}$ and $\tilde{f}^{-1}$ (properly restricted) are smooth, since their component functions are smooth. Since $SU(2)$ and $\mathbb{S}^3$ are submanifolds in above sets, $f$ and $f^{-1}$ must also be smooth, since $f = \tilde{f} \circ i$ where $i \colon \mathbb{S}^3 \hookrightarrow \mathbb{R}^4$ is the (smooth) inclusion function.

**Ex. 4:** Let the map $\pi \colon \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$\pi(x,y,z,t) = (2xz + 2yt, 2yz - 2xt, x^2 + y^2 - z^2 - t^2)$$

(i) We first show that $\pi$ restricts to a map $\pi|_{\mathbb{S}^3} \colon \mathbb{S}^3 \to \mathbb{S}^2$.

We assume

$$x^2 + y^2 + z^2 + t^2 = 1$$

and make a short calculation to prove that the restriction is well-defined:

$$\begin{aligned}
&(2xz + 2yt)^2 + (2yz - 2xt)^2 + ((x^2+y^2) - (z^2+t^2))^2 \\
&= 4x^2z^2 + 8xyzt + 4y^2t^2 + 4y^2z^2 - 8xyzt + 4x^2t^2 + (x^2+y^2)^2 - 2(x^2+y^2)(z^2+t^2) + (z^2+t^2)^2 \\
&= 4x^2(z^2+t^2) + 4y^2(z^2+t^2) + (x^2+y^2)^2 - 2(x^2+y^2)(z^2+t^2) + (z^2+t^2)^2 \\
&= 4(x^2+y^2)(z^2+t^2) + (x^2+y^2)^2 - 2(x^2+y^2)(z^2+t^2) + (z^2+t^2)^2 \\
&= (x^2+y^2)^2 + 2(x^2+y^2)(z^2+t^2) + (z^2+t^2)^2 \\
&= (x^2+y^2+z^2+t^2)^2 \\
&= 1
\end{aligned}$$

(ii) Let us not show that $\pi$ is a submersion on $\mathbb{R}^4 \setminus \{0\}$.

We know that $d\pi$ at any point is precisely the Jacobi matrix at that point. To prove that this map is surjective at any non-zero point, it is enough to show that this Jacobi matrix has full rank 3. We calculate

$$A = D_{(x,y,z,t)}\pi = \begin{bmatrix} 2z & 2t & 2x & 2y \\ -2t & 2z & 2y & -2x \\ 2x & 2y & -2z & -2t \end{bmatrix}$$

Denote by $A_i$ the $3 \times 3$ minor of $A$ with the $i$-th column skipped. We calculate

$$\begin{aligned}
\frac{1}{8}\det A_4 &= z(-z^2 - y^2) + t(-tz + xy) + x(-ty - xz) \\
&= -z^3 - y^2z - zt^2 + xyt - xyt - x^2z \\
&= -z(x^2 + y^2 + z^2 + t^2)
\end{aligned}$$

which by assumption vanishes precisely when $z = 0$. Notice that $z$ is the only variable missing in the skipped column 4. Indeed, for the rest $3 \times 3$ minors we get analog determinants, where we replace $z$ by the value missing in that column. Since by assumption not all $x, y, z, t$ are zero, there exists a non-zero $3 \times 3$ minor in $A$, hence $A$ has full rank and the differential map is surjective.

(iii) Let us now show that the following holds

$$T_{(x,y,z,t)}\mathbb{S}^3 = \mathrm{Lin}\left\{\begin{bmatrix} -y \\ x \\ t \\ -z \end{bmatrix}, \begin{bmatrix} -z \\ -t \\ x \\ y \end{bmatrix}, \begin{bmatrix} -t \\ z \\ -y \\ x \end{bmatrix}\right\},$$

and then use it to show that the map $\pi|_{\mathbb{S}^3} : \mathbb{S}^3 \to \mathbb{S}^2$ is a submersion.

We know that the tangent vectors to $\mathbb{S}^3 \subset \mathbb{R}^4$ are all orthogonal to the normal vector, i.e., the point vector of a point on $\mathbb{S}^3$. Since $\mathbb{S}^3$ is a 3-dimensional manifold, the vector space $T_{(x,y,z,t)}\mathbb{S}^3$ has 3 vectors in its basis. Denote the above vectors $v_1, v_2, v_3$, respectively. By the above, it is enough to show that $(x, y, z, t) \cdot v_i = 0$ for each $i = 1, 2, 3$, and that the vectors $v_i$ are linearly independent. The first claim is clearly true, e.g.

$$(x, y, z, t) \cdot (-y, x, t, -z) = -xy + xy + zt - tz = 0$$

and similarly for the other two vectors. For linear independence, take the linear combination

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$$

from which we immediately get

$$\begin{bmatrix} -y & -z & -t \\ x & -t & z \\ t & x & -y \\ -z & y & x \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

We quickly notice that the rows of this matrix are precisely columns of matrix $\frac{1}{2}A$ from (4ii) with a possible multiplication by $-1$. Denote this matrix by $B$ and let $B_i$ be its $3 \times 3$ minor with the $i$-th row removed. We calculate

$$\begin{aligned} \det B_4 &= -y(ty - xz) - z(xy + tz) - t(x^2 + t^2) \\ &= -y^2t + xyz - xyz - z^2t - x^2t - t^3 \\ &= -t(x^2 + y^2 + z^2 + t^2) \\ &= -t \end{aligned}$$

where $t$ is the only variable missing in the skipped row 4. For the other $3 \times 3$ minors we get similar results as in (4ii), always obtaining the variable missing in the skipped column. Since at least one of these is non-zero (take hemisphere charts on $\mathbb{S}^3$), the system is solvable by the unique solution $\alpha = \beta = \gamma = 0$.

To show that the restriction $\pi|_{\mathbb{S}^3}$ is a submersion, take

$$(a, b, c) = (2xz + 2yt, 2yz - 2xt, x^2 + y^2 - z^2 - t^2) \in \mathbb{S}^2$$

By (4i), the restriction $\pi|_{\mathbb{S}^3}$ is well-defined and clearly smooth. By (4ii), the differential map of the restriction is precisely the matrix $A$ at points from $\mathbb{S}^3$. Now, apply $A$ to the three basis vectors

$v_1, v_2, v_3$. We get

$$Av_1 = \begin{bmatrix} -2b \\ 2a \\ 0 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 2c \\ 0 \\ -2a \end{bmatrix}$$

$$Av_3 = \begin{bmatrix} 0 \\ -2c \\ 2b \end{bmatrix}$$

Denote the resulting vectors by $w_1, w_2, w_3$. Clearly, $(a, b, c)$ is orthogonal to each $w_i$, hence they are indeed in $T_{(a,b,c)}\mathbb{S}^2$, and since not all $a, b, c$ are zero at the same time (take hemisphere charts on $\mathbb{S}^2$), at least two of the $w_i$ must be linearly independent at any given point. The dimension of $T_{(a,b,c)}\mathbb{S}^2$ is of course 2, so the differential map is indeed surjective.