Differential geometry: 2. homework

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Exercise 1. Choose and solve one of the following two exercises; if you took the course Analysis on manifolds in the previous semester, solve ii).

i) Let $g \in C^\infty(M, N)$ be a smooth map between smooth manifolds and let $\pi\colon E \to N$ be an $\mathbb{F}$-vector bundle of rank k. We define the *pullback of vector bundle* E by

$$g^*E = \coprod_{p \in M} E_{g(p)} = \{(p, e) \in M \times E \mid g(p) = \pi(e)\},$$

together with the map $\pi^*\colon g^*E \to M$, $\pi^*(p, e) = p$.

a) Suppose (U, ϕ) is a local trivialization on E. Define the map

$$\begin{aligned} &\psi\colon (\pi^*)^{-1}(g^{-1}(U)) \to g^{-1}(U) \times \mathbb{F}^k \\ &\psi(p, e) = (p, \mathrm{pr}_{\mathbb{F}^k} \circ \phi(e)). \end{aligned}$$

Show that $(g^{-1}(U), \psi)$ is a local trivialization of g^*E and conclude that g^*E is indeed a vector bundle.
Hint. First show that ψ is bijective (find the inverse of ψ). Then suppose $(\tilde{U}, \tilde{\phi})$ is another trivialization on E and $\tilde{\psi}$ is defined analogously to ψ. Compute $\tilde{\psi} \circ \psi^{-1}$ using the fact that $\tilde{\phi} \circ \phi^{-1}\colon (U \cap \tilde{U}) \times \mathbb{F}^k \to (U \cap \tilde{U}) \times \mathbb{F}^k$ is given by

$$\tilde{\phi} \circ \phi^{-1}(p, \vec{v}) = (p, \tau(p)\vec{v}),$$

for some smooth map $\tau\colon U \cap \tilde{U} \to \mathrm{GL}(n, \mathbb{F})$, called the *transition map* between $\tilde{\phi}$ and ϕ. From the computed prescription of $\tilde{\psi} \circ \psi^{-1}$, read out that the transition map between $\tilde{\psi}$ and ψ is smooth.

b) Define the map $\xi\colon g^*E \to E$ by $\xi(p, e) = e$. Show that the map ξ is a smooth morphism of vector bundles, i.e. that it is smooth and the following diagram commutes:

$$\begin{tikzcd} g^*E \arrow[r, "\xi"] \arrow[d, "\pi^*"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

ii) Let $\xi\colon E \to F$ be a smooth morphism of vector bundles $\pi_E\colon E \to M$ and $\pi_F\colon F \to M$; denote $\xi_p := \xi|_{E_p}\colon E_p \to F_p$. Show that the sets

$$\ker(\xi) = \coprod_{p \in M} \ker(\xi_p), \quad \mathrm{im}(\xi) = \coprod_{p \in M} \mathrm{im}(\xi_p)$$

are vector subbundles of E and F (respectively) iff the morphism ξ has constant rank (i.e. $\dim(\mathrm{im}(\xi_p))$ is the same for all $p \in M$).

Exercise 2. A basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of the Lie group $\mathrm{SL}(2,\mathbb{R})$ is given by the matrices

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compute the expressions of the left-invariant vector fields E^L, F^L and H^L on $\mathrm{SL}(2,\mathbb{R})$, their commutators and flows.

Exercise 3. Let G be a matrix Lie group, i.e. G is a Lie group that is a closed subgroup and an embedded submanifold of $\mathrm{GL}(n, \mathbb{F})$. Denote by $\mathfrak{g} = T_I G \subset \mathbb{F}^{n \times n}$ its Lie algebra.

i) A *one-parametric subgroup of* G is a smooth homomorphism $\alpha \colon \mathbb{R} \to G$ of Lie groups, where $\mathbb{R}$ is the additive Lie group.

   a) Given $v \in \mathfrak{g}$, show that $t \mapsto \mathrm{e}^{tv}$ is a one-parametric subgroup of G.

   b) Conversely, given a one-parametric subgroup α of G, show that there holds

$$\alpha(t) = e^{t\alpha'(0)}.$$

   Hint. Consider $\frac{d}{dt}\big|_{t=s}\alpha(t)$ to show that α is an integral curve of the left-invariant vector field $\alpha'(0)^L$.

ii) (Naturality of exp) Let H be another matrix Lie group and let $\phi \colon G \to H$ be a homomorphism of Lie groups. Show that the diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{d}\phi_I"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

commutes. Use this result on conjugation $C_g \colon G \to G$, $C_g(h) = ghg^{-1}$ by an element $g \in G$.
Hint. Consider $\alpha(t) = \phi(\mathrm{e}^{tv})$ for $v \in \mathfrak{g}$.

iii) Let $\phi \colon G \to H$ be a homomorphism of matrix Lie groups. Show that there holds:

$$\phi \text{ is an immersion} \iff \ker(\phi) \text{ is discrete.}$$

Hint. For direction ($\Longrightarrow$), use that every immersion is locally an embedding. For direction ($\Longleftarrow$), first show that injectivity of $\mathrm{d}\phi_g$ is equivalent to injectivity of $\mathrm{d}\phi_I$, then assume $\dim\ker \mathrm{d}\phi_I > 0$ and prove by contradiction. In both directions you will have to utilize the fact that exp is a local diffeomorphism at 0 (since $\mathrm{d}(\exp)_0 \colon \mathfrak{g} \to \mathfrak{g}$ is the identity map, hence invertible), and naturality from ii).

Remark. The same results hold for general (non-matrix) Lie groups, with exponentiation defined abstractly as $\exp \colon \mathfrak{g} \to G$, $\exp(v) = \phi_1^{v^L}(e)$, where $\phi_t^{v^L}$ denotes the flow of the left invariant vector field v^L associated to the vector $v \in \mathfrak{g}$.

Exercise 4. Let

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

be the Pauli matrices and denote $\sigma(\vec{s}) = is^j\sigma_j$, for any $\vec{s} \in \mathbb{R}^3$. Hence $\sigma\colon \mathbb{R}^3 \to \mathrm{Lin}(i\sigma_j)_{j=1}^3$ is an isomorphism of vector spaces.

i) Show that there holds $\mathrm{e}^{t\sigma(\vec{s})} = \cos t\, I + \sin t\, \sigma(\vec{s})$, for any $\vec{s} \in S^2$.
Hint. Use the commutation and anticommutation relations

$$[\sigma_a, \sigma_b] = \sigma_a\sigma_b - \sigma_b\sigma_a = 2i\varepsilon_{ab}{}^c\sigma_c \quad \text{and} \quad \{\sigma_a, \sigma_b\} = \sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{ab}I,$$

where $\varepsilon_{ab}{}^c$ is the Levi–Civita symbol and δ_{ab} is the Kronecker delta. If you haven't met these relations yet, feel free to make sure and compute some of them.

ii) Prove that there holds $\mathrm{e}^{\sigma(\vec{s})} \in \mathrm{SU}(2)$, for any $\vec{s} \in \mathbb{R}^3$. The map $\vec{s} \mapsto \mathrm{e}^{\sigma(\vec{s})}$ is surjective (you do not need to prove this).

iii) We define the map $\pi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ by

$$\pi\left(\mathrm{e}^{\frac{\varphi}{2}\sigma(\vec{s})}\right) = R_{\vec{s},\varphi} \quad (\vec{s} \in S^2, \varphi \in \mathbb{R})$$

where the matrix $R_{\vec{s},\varphi}$ corresponds to the rotation around the axis $\vec{s}$ by an angle φ. This map is smooth and surjective (you do not need to prove this). Show that there holds

$$\ker \pi = \{I, -I\}$$

and conclude (using exercise 3) that π is a local diffeomorphism between connected Lie groups.

Remark. Since $\mathrm{SU}(2)$ is isomorphic to the Lie group S^3 of unit quaternions (in particular, diffeomorphic), it is simply connected. This shows that π is a two-sheeted universal covering projection, hence the fundamental group of $\mathrm{SO}(3)$ equals $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$.