Teorija Izračunljivosti 2020–21 Homework 2
Computability on $\mathbb{R}$

Homework released: December 23, 2020

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This homework is a paper-and-pencil exercise. Questions 1 and 2 concern computability using ordinary Turing machines. Questions 3 and 4 address real number computation using Type 2 Turing machines exploiting the Cauchy representation of $\mathbb{R}$ defined in Lecture 12.

If a question asks you to show that a function is computable, you should provide an *informal description* of a TM/T2M algorithm. Do not attempt to explicitly define an actual machine.

The homework must be completed and submitted by 23:59 (CET) on Sunday 17th January 2021. Submission instructions will be announced by email and posted on the course webpage.

Question 1 [1 mark]

Given a finite alphabet Σ, give a definition in terms of ordinary Turing machines of what it means for a partial function $f\colon \Sigma^* \times \mathbb{N} \rightharpoonup \Sigma^*$ to be *computable*.

Question 2 [4 marks]

Let q_d be the representation of dyadic rationals by words over the alphabet Σ_b, as defined in Lecture 12. Show that there exists a computable partial function $f\colon \Sigma_b^* \times \mathbb{N} \rightharpoonup \Sigma_b^*$ that satisfies:

1. $\mathsf{dom}(f) = \{u \in \mathsf{dom}(q_d) \mid q_d(u) \neq 0\} \times \mathbb{N}$ and
2. for any $(u, n) \in \mathsf{dom}(f)$, $\left|q_d(f(u, n)) - q_d(u)^{-1}\right| \leq 2^{-n}$.

(You may assume without proof that addition and multiplication of dyadic rationals are computable as functions on words over Σ_b .)

Question 3 [4 marks]

Prove that the partial function $r\colon \mathbb{R} \rightharpoonup \mathbb{R}$ below is computable.

$$r(x) \;\simeq\; \begin{cases} x^{-1} & \text{if } x \neq 0 \\ \uparrow & \text{if } x = 0 \end{cases}$$

Question 4 [1 mark]

Does there exist any total computable function $r'\colon \mathbb{R} \to \mathbb{R}$ such that $r'(x) = x^{-1}$ for all $x \neq 0$?