Differential Geometry - 5$^{\text{th}}$ homework

Benjamin Benčina

June 15, 2021

<u>Ex. 1:</u> Let V be a vector space with $\dim V = n$ and let $\mathcal{B}(V^*)$ denote the set of all covariant 4-tensors $\alpha \in T^4(V^*)$ following

(a) $\alpha(w, z, x, y) = -\alpha(z, w, x, y)$,

(b) $\alpha(w, z, x, y) = -\alpha(w, z, y, x)$,

(c) $\alpha(w, z, x, y) = \alpha(x, y, w, z)$,

for all $x, y, z, w \in V$. We denote by $\mathcal{R}(V^*) \subset \mathcal{B}(V^*)$ with the additional property

(d) $\alpha(w, z, x, y) + \alpha(w, x, y, z) + \alpha(w, y, z, x) = 0$.

We call the elements of $\mathcal{R}(V^*)$ the *algebraic curvature tensors* on V.

(i) We briefly explain why $\mathcal{B}(V^*)$ and $\mathcal{R}(V^*)$ are vector subspaces of $T^4(V^*)$.

Since properties (a-c) are entirely linear, it is obvious why $\mathcal{B}(V^*)$ is a vector subspace. For example

$$f\alpha(w, z, x, y) + g\beta(w, z, x, y) = -f\alpha(z, w, x, y) - g\beta(z, w, x, y)$$

so property (a) indeed holds for $f\alpha + g\beta$. Properties (b) and (c) are analogous. Property (d) is also trivially checked for tensors in $\mathcal{R}(V^*)$

$$\begin{aligned} &f\alpha(w, z, x, y) + g\beta(w, z, x, y) + f\alpha(w, x, y, z) + g\beta(w, x, y, z) + f\alpha(w, y, z, x) + g\beta(w, y, z, x) \\ &= f(\alpha + \alpha + \alpha) + g(\beta + \beta + \beta) = 0 \end{aligned}$$

(ii) Let us determine the dimension of $\mathcal{B}(V^*)$.

By the hint we consider $S = S^2(\Lambda^2(V^*))$. We define a map $\Phi\colon S \to \mathcal{B}(V^*)$ by

$$\Phi(A)(w, z, x, y) = A(w \wedge z, x \wedge y)$$

This map is indeed well defined, since A is linear and $a \wedge b = -b \wedge a$ (property (c) is achieved since A is symmetric), and it is clearly linear, since the entire construction is as well. Let us prove that Φ is an isomorphism by constructing its inverse. We choose a basis $(b_i)_i$ for V, so the collection $\{b_i \wedge b_j;\ i < j\}$ is a basis for $\Lambda^2(V)$. The inverse Ψ of Φ is then clearly defined on the basis by

$$\Psi(B)(b_i \wedge b_j, b_k \wedge b_l) = B(b_i, b_j, b_k, b_l)$$

where $i < j$ and $k < l$.

It now follows that the dimensions of $\mathcal{B}(V^*)$ and S must match. Since $\dim \Lambda^2(V) = \binom{n}{2} = \frac{n(n-1)}{2}$, and the space of symmetric bilinear forms on a vector space of dimension m is $\frac{m(m+1)}{2}$, we obtain

$$\dim \mathcal{B}(V^*) = \dim S = \frac{\binom{n}{2}\left(\binom{n}{2} + 1\right)}{2} = \frac{n(n-1)(n^2 - n + 2)}{8}$$

(iii) Let us now prove that the restriction $\mathrm{Alt}|_{\mathcal{B}(V^*)}$ is given by

$$\mathrm{Alt}|_{\mathcal{B}(V^*)}(\alpha)(w,z,x,y) = \frac{1}{3}\left(\alpha(w,z,x,y) + \alpha(w,x,y,z) + \alpha(w,y,z,x)\right)$$

and show it is onto as a map $\mathcal{B}(V^*) \to \Lambda^4(V^*)$.

Recall that

$$\mathrm{Alt}(\alpha) = \frac{1}{24}\sum_{\sigma\in S_4} \mathrm{sign}(\sigma)\alpha^\sigma$$

but our space $\mathcal{B}(V^*)$ is equipped with the symmetries (a-c). From permutations

$$id, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)$$

we get $8\cdot\alpha(w,z,x,y)$, and similarly the other two groups of eight permutations give $8\times$ the other two terms. Hence the formula indeed holds.

The restriction is surjective, since every $\alpha \in \Lambda^4(V^*)$ satisfies (a-c) and thus lies in $\mathcal{B}(V^*)$, so $\mathrm{Alt}|_{\mathcal{B}(V^*)}(\alpha) = \alpha$.

(iv) We now determine the dimension of $\mathcal{R}(V^*)$.

Since $\mathcal{R}(V^*)$ is precisely the kernel of the above restriction, by Rank-Nullity Theorem we have

$$\dim \mathcal{R}(V^*) = \dim \mathcal{B}(V^*) - \dim \Lambda^4(V^*) = \frac{n(n-1)(n^2-n+2)}{8} - \binom{n}{4} = \frac{n^2(n^2-1)}{12}$$

<u>Ex. 2:</u> Let (M,g) be a Riemannian manifold, $p \in M$ and $V \subset T_pM$ an open neighbourhood of zero on which $\exp_p$ is a diffeomorphism onto $U \subset M$. Let $v, w \in T_pM$ be non-zero linearly independent vectors that span $\pi = \mathrm{Lin}(v,w) \leq T_pM$, and denote by $S(\pi) = \exp_p(V\cap\pi)$ the embedded Riemannian 2-submanifold of M, swept out by geodesics whose initial velocities lie in π. We define the *sectional curvature* of the plane π as

$$\sec(\pi) = \sec(v,w) = \frac{1}{2}R_{S(\pi)}(p)$$

where $R_{S(\pi)}$ denotes the intrinsic scalar curvature of $S(\pi)$.

(i) Let us show that there holds

$$\sec(v,w) = \frac{R(v,w,v,w)}{||v\wedge w||^2}$$

where $||v\wedge w||^2 = ||v||^2||w||^2 - \langle v,w\rangle^2$.

We first show that the Second Fundamental Form of $S(\pi)$ vanishes at p. Let $z \in \pi$ be an arbitrary vector and let γ_z be the g-geodesic with initial velocity z whose image lies in $S(\pi)$ for some t small enough. By the Gauss Formula from Tutorials, we get

$$0 = D_t\gamma_z' = \hat{D}_t\gamma_z' + \mathrm{II}(\gamma_z',\gamma_z')$$

where we denote by $\hat{\cdot}$ induced objects on $S(\pi)$, e.g., $\hat{g}$ is the metric induced by g on $S(\pi)$. Since the terms on the RHS are by definition orthogonal, they must both vanish. If we evaluate at $t=0$ we get $\mathrm{II}(z,z) = 0$, and since $z \in \pi = T_pS(\pi)$ was arbitrarily chosen and II is symmetric, II must be zero at p. The Gauss Equation from Tutorials then yields that at p we have

$$R(w,z,x,y) = \hat{R}(w,z,x,y)$$

We now prove the equality for the case where v, w are orthonormal. Indeed, (v, w) is then an orthonormal basis for π, and recall that the scalar curvature is locally given by $R = g^{ij} R_{ij} \overset{ONB}{=} R_{11} + R_{22}$. The sectional curvature of π must then be

$$\begin{aligned}
\sec(v, w) &= \frac{1}{2} R_{S(\pi)}(p) \\
&= \frac{1}{2}\left(\hat{R}(v, w, v, w) + \hat{R}(w, v, w, v)\right) \\
&= \hat{R}(v, w, v, w) \\
&= R(v, w, v, w)
\end{aligned}$$

which is precisely our formula, since

$$||v \wedge w||^2 = ||v||^2 ||w||^2 - \langle v, w \rangle^2 = 1 \cdot 1 - 0 = 1$$

To generalize to any basis of π we employ Gram-Schmidt. Let now (v, w) denote an arbitrary basis of π. By the Gram-Schmidt algorithm we get

$$b_1 = \frac{v}{|v|} \qquad\qquad b_2 = \frac{w - \langle w, b_1 \rangle b_1}{|w - \langle w, b_1 \rangle b_1|}$$

The above calculation then yields

$$\begin{aligned}
\sec(v, w) &= \frac{1}{2} R_{S(\pi)}(p) \\
&= R(b_1, b_2, b_1, b_2) \\
&= R\left(\frac{v}{|v|}, \frac{w - \langle w, b_1 \rangle b_1}{|w - \langle w, b_1 \rangle b_1|}, \frac{v}{|v|}, \frac{w - \langle w, b_1 \rangle b_1}{|w - \langle w, b_1 \rangle b_1|}\right) \\
&= \frac{R(v, w, v, w)}{|v|^2 |w - \langle w, b_1 \rangle b_1|^2}
\end{aligned}$$

where we use the fact that b_1 is by construction a multiple of v and hence $R(v, b_1, \cdot, \cdot) = R(\cdot, \cdot, v, b_1) = 0$. The denominator is then simplified into

$$|v|^2 |w - \langle w, b_1 \rangle b_1|^2 = |v|^2 \left(|w|^2 - 2\frac{\langle w, v \rangle^2}{|v|^2} + \frac{\langle w, v \rangle^2}{|v|^2}\right) = |v|^2 |w|^2 - \langle v, w \rangle^2 = |v \wedge w|^2$$

which proves the formula.

(ii) Suppose that R_1 and R_2 are two algebraic curvature tensors on some finite dimensional vector space V such that

$$R_1(v, w, v, w) = R_2(v, w, v, w)$$

for any $v, w \in V$. Let us prove that $R_1 = R_2$.

First of all, we can assume without loss of generality that v, w are linearly independent, otherwise the equation reads $0 = 0$. As usual in such proofs, we define $D = R_1 - R_2$. Since algebraic curvature tensors form a vector space, D is also an algebraic curvature tensor, and we have that $D(v, w, v, w) = 0$ for all $v, w \in V$. We get

$$\begin{aligned}
0 &= D(v + w, x, v + w, x) \\
&= D(v, x, v, x) + D(v, x, w, x) + D(w, x, v, x) + D(w, x, w, x) \\
&= 2D(v, x, w, x)
\end{aligned}$$

and it follows that

$$\begin{aligned}
0 &= D(v, x + u, w, x + u) \\
&= D(v, x, w, x) + D(v, x, w, u) + D(v, u, w, x) + D(v, u, w, u) \\
&= D(v, x, w, u) + D(v, u, w, x) \\
&\overset{(b)}{=} -D(v, x, u, w) - D(v, u, x, w)
\end{aligned}$$

The Bianchi Identity from Tutorials now yields

$$\begin{aligned}0 &= D(v,w,x,u) + D(w,x,v,u) + D(x,v,w,u)\\ &= D(v,w,x,u) + D(w,v,x,u) + D((v,x,w,u)\\ &= 3D(v,w,x,u)\end{aligned}$$

for all v, w, u, x. Therefore $D = 0$.

(iii) Let us prove that the following are equivalent:

(a) For any plane $\pi \le T_pM$ there holds $\sec(\pi) = C$.

(b) There holds $R(x,y)z = C \cdot (\langle y,z\rangle x - \langle z,x\rangle y)$ for any $x, y, z \in T_pM$.

(c) There holds $R(x,y)z = C \cdot (x - \langle x,y\rangle y)$ for any $x, y \in T_pM$ where y has unit length.

- $\underline{(a) \implies (b)}$: Recall that
$$R(w,z,x,y) = \langle w, R(x,y)z\rangle$$
and define
$$S(w,z,x,y) = \langle w, S(x,y)z\rangle$$
for $S(x,y)z = k(\langle y,z\rangle x - \langle z,x\rangle y)$. Let v, w be some (linearly independent) basis for π. Then by definition $R(v,w,v,w) = S(v,w,v,w)$ with the constant $k = C$, hence by (2.ii) $R = S$ By linearity of inner products
$$R(x,y)z = S(x,y)z = C(\langle y,z\rangle x - \langle z,x\rangle y)$$
- $\underline{(b) \implies (c)}$: Simply input $z = y$.
- $\underline{(c) \implies (a)}$: Assume $||w|| = 1$ and calculate
$$\begin{aligned}R(v,w,v,w) &= \langle v, R(v,w)w\rangle\\ &= \langle v, C(v - \langle v,w\rangle w)\rangle\\ &= C(||v||^2 - \langle v,w\rangle^2)\end{aligned}$$

  For a non-unit w simply replace w with $\frac{w}{||w||}$ and obtain $C||v \wedge w||^2$.

(iv) Let $v \in T_pM$ be a unit tangent vector. We will prove that there holds

$$\mathrm{Ric}(v,v) = \sum_{i=2}^{n} \sec(v, v_i)$$

where $v_2, \ldots, v_n \in T_pM$ is any completion of $v_1 = v$ to an orthonormal basis of T_pM. Furthermore, we will show that

$$R(p) = \sum_{i \neq j} \sec(v_i, v_j)$$

Let v be as above and let $(v_1, \ldots, v_n)$ be any orthonormal basis with $v_1 = v$. Then the Ricci curvature is given by

$$\mathrm{Ric}(v,v) = R^k_{1k1}(p) = \sum_{k=1}^{n} R(v_1, v_k, v_1, v_k) = \sum_{k=2}^{n} \sec(v, v_k)$$

For the scalar curvature we calculate

$$R(p) = R^i_i(p) = \sum_{i=1}^{n} \mathrm{Ric}(v_i, v_i) = \sum_{i=1,j=1}^{n} R(v_i, v_j, v_i, v_j) = \sum_{i \neq j} \sec(v_i, v_j)$$

(v) Suppose now that (x^i) are some local coordinates on M and that (M, g) has constant sectional curvature C. Let us show that there holds

$$\begin{aligned} R_{lkij} &= C(g_{li}g_{kj} - g_{lj}g_{ki}) \\ R_{ij} &= C(n-1)g_{ij} \\ R &= Cn(n-1) \end{aligned}$$

The first equality follows from (3iii.b)

$$R_{lkij} = \langle \partial_l, R(\partial_i, \partial_j)\partial_k \rangle = C\langle \partial_l, \langle \partial_j, \partial_k \rangle \partial_i - \langle \partial_k, \partial_i \rangle \partial_j \rangle = C(g_{li}g_{kj} - g_{lj}g_{ki})$$

The second equality goes similarly, since $R_{ij} = g^{kl}R_{kilj}$. The third equality of course immediately follows from the last part of (2iv)

$$R = \sum_{i \neq j} C = Cn(n-1)$$

(vi) For any $C \in \mathbb{R}$ and $n \geq 2$ we now give an example of an n-dimensional Riemannian manifold with constant sectional curvature C. For $C = 0$ we clearly have that $(\mathbb{R}^n, g)$ has sectional curvature 0 for any $n \geq 2$ with the standard Euclidean metric. Indeed, its curvature tensor is identical to zero.

For $C > 0$ we have already calculated at Tutorials that all principal curvatures of $(\mathbb{S}^n(R), g_R)$ are $-\frac{1}{R}$, which makes the sectional curvature of $\mathbb{S}^n(R)$ equal to $\frac{1}{R^2}$. This can be easily verified since for any plane π in $\mathbb{S}^n(R)$ we have that $S(\pi)$ is isomorphic to $\mathbb{S}^2(R)$ because it is spanned by two great circles.

With a similar thought process we get that $(\mathbb{H}^n(R), g_R)$ has sectional curvature $-\frac{1}{R^2}$ since we know from Tutorials that $\mathbb{H}$ has Gaussian curvature $-\frac{1}{R^2}$.

Ex. 3: Let $U \subset \mathbb{R}^2$ be an open subset and let $\vec{r}\colon U \to \mathbb{R}^3$ be a smooth embedding, so that $S = \vec{r}(U)$ is an embedded Riemannian submanifold of $\mathbb{R}^3$. Denote by $\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ the elements of S.

(i) Let us write down the metric g and the scalar second fundamental form h on S in matrix form

$$[g] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}, \quad [h] = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

in term of $\vec{r}$. We will then write down the shape operator S in $\mathbb{R}^3$ and provide a formula for the Gaussian curvature κ and the principal curvatures.

We follow the definition for $[g]$ and obtain

$$\begin{aligned} E &= \langle \vec{r}_u(u, v), \vec{r}_u(u, v) \rangle \\ F &= \langle \vec{r}_u(u, v), \vec{r}_v(u, v) \rangle \\ G &= \langle \vec{r}_v(u, v), \vec{r}_v(u, v) \rangle \end{aligned}$$

since our local frame at any point is now given by $\partial_u \vec{r}, \partial_v \vec{r}$.

For the Scalar Second Fundamental Form we use an exercise from Tutorials about the Weingarten equation. Let W be some unit field normal to S in $\mathbb{R}^3$. Then, since

$$0 = \langle \partial_u \vec{r}(u, v), W(u, v) \rangle = \langle \partial_v \vec{r}(u, v), W(u, v) \rangle$$

by differentiating

$$\begin{aligned}
0 &= \langle \partial_u \partial_u \vec{r}(u,v), W(u,v)\rangle + \langle \partial_u \vec{r}(u,v), \nabla_u W(u,v)\rangle \\
0 &= \langle \partial_u \partial_v \vec{r}(u,v), W(u,v)\rangle + \langle \partial_u \vec{r}(u,v), \nabla_u W(u,v)\rangle \\
0 &= \langle \partial_v \partial_u \vec{r}(u,v), W(u,v)\rangle + \langle \partial_v \vec{r}(u,v), \nabla_v W(u,v)\rangle \\
0 &= \langle \partial_v \partial_v \vec{r}(u,v), W(u,v)\rangle + \langle \partial_v \vec{r}(u,v), \nabla_v W(u,v)\rangle
\end{aligned}$$

The rightmost terms are by definition of the scalar fundamental form precisely its coefficients, hence

$$\begin{aligned}
L &= \langle \vec{r}_{uu}(u,v), W(u,v)\rangle \\
M &= \langle \vec{r}_{uv}(u,v), W(u,v)\rangle \\
N &= \langle \vec{r}_{vv}(u,v), W(u,v)\rangle
\end{aligned}$$

Of course finding W is easy in $\mathbb{R}^3$. We get

$$W(u,v) = \frac{\vec{r}_u(u,v) \times \vec{r}_v(u,v)}{||\vec{r}_u(u,v) \times \vec{r}_v(u,v)||}$$

where we get to choose the sign in the front, which then determines h.

The shape operator is given by the Weingarten equation

$$\begin{aligned}
sX &= -\nabla_X W \\
&= -X^j(\partial_j W^i)\partial_i \\
&= -X^u W_u^u \vec{r}_u - X^u W_u^v \vec{r}_v - X^v W_v^u \vec{r}_u - X^v W_v^v \vec{r}_v
\end{aligned}$$

Note that in the last line, upper indices denote the component, and lower indices denote differentiation. For principal curvatures we are looking for eigenvalues of s, so the local extremes of the function

$$\kappa_n(t,s) = \begin{bmatrix} t & s \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix}$$

where

$$||(t,s)||^2 = \begin{bmatrix} t & s \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = 1$$

and we know from Introduction to differential geometry that we get the Gaussian curvature

$$\kappa = \frac{LN - M^2}{EG - F^2} = \frac{\det[h]}{\det[g]}$$

(ii) Let us express the second partial derivatives $\vec{r}_{uu}$, $\vec{r}_{uv}$,$\vec{r}_{vv}$ in terms of Christoffel symbols on S and the components of the scalar second fundamental form h.

From the Gauss formula

$$\tilde{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X,Y)$$

it directly follows that by components we have

$$\begin{aligned}
\vec{r}_{uu} &= \Gamma^u_{uu}\vec{r}_u + \Gamma^v_{uu}\vec{r}_v + LW \\
\vec{r}_{uv} &= \Gamma^u_{uv}\vec{r}_u + \Gamma^v_{uv}\vec{r}_v + MW \\
\vec{r}_{vv} &= \Gamma^u_{vv}\vec{r}_u + \Gamma^v_{vv}\vec{r}_v + NW
\end{aligned}$$

since $\vec{r}_u, \vec{r}_v, W$ form a basis of $\mathbb{R}^3$.

(iii) We will now show that the components of Riemann and Ricci curvature tensors on S are given by

$$R_{lkij} = \kappa(g_{li}g_{kj} - g_{lj}g_{ki})$$
$$R_{ij} = \kappa g_{ij}$$

The only non-zero Riemann tensor coefficients are of the form R_{ijij} and in both cases the first equation translates to

$$R_{ijij} = \kappa(EG - F^2) = LN - M^2 = \det[h]$$

which holds true by the Gauss Equation. Again, the second equality follows, since $R_{ij} = g^{kl}R_{kilj}$.

(iv) We now consider a surface of revolution, parametrized by

$$\vec{r}(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $f, g\colon I \to \mathbb{R}$ are defined on an open interval I, so that $U = I \times (0, 2\pi)$ and f is positive.

We first calculate the Gaussian curvature of S for the case when $f'(u)^2 + g'(u)^2 = 1$, i.e., when the curve $u \mapsto \vec{r}(u, 0)$ is naturally parametrized. The naturality condition gives us a nice expression for the induced metric on S

$$\begin{aligned}
\vec{r}^*\tilde{g} &= d(f(u)\cos v)^2 + d(f(u)\sin v)^2 + dg(u)^2 \\
&= (f'(u)\cos v du - f(u)\sin v dv)^2 + (f'(u)\sin v du + f(u)\cos v dv)^2 + (g'(u)du)^2 \\
&= (f'(u)^2 + g'(u)^2)du^2 + f(u)^2 dv^2 \\
&= du^2 + f(u)^2 dv^2
\end{aligned}$$

from which it immediately follows that

$$E = 1, \quad F = 0, \quad G = f^2$$

Furthermore, the second derivatives are as follows

$$\begin{aligned}
\vec{r}_{uu} &= (f''(u)\cos v, f''(u)\sin v, g''(u)) \\
\vec{r}_{uv} &= (-f'(u)\sin v, f'(u)\cos v, 0) \\
\vec{r}_{vv} &= (-f(u)\cos v, -f(u)\sin v, 0)
\end{aligned}$$

and the unit normal field is

$$W = \frac{(g'(u)f(u)\cos v, -g'f\sin v, ff'(\cos^2 v + \sin^2 v))}{\sqrt{f^2(u)(f'(u)^2 + g'(u)^2)}} = (-g'(u)\cos v, -g'(u)\sin v, f'(u))$$

which yields

$$N = -g'f'' + f'g'', \quad M = 0, \quad N = fg'$$

Notice also that

$$f'^2 + g'^2 = 1 \implies 2f'f'' + 2g'g'' = 0 \implies g'g'' = -f'f''$$

The Gaussian curvature is then calculated as

$$\kappa = \frac{(f'g'' - g'f'')fg'}{f^2} = \frac{f'g'g'' - (g')^2 f''}{f} = \frac{-(f')^2 f'' - (g')^2 f''}{f} = -\frac{f''}{f}$$

We now show that any meridian is a geodesic and that a parallel is a geodesic iff $f'(u_0) = 0$. We write

$$L = \frac{1}{2}(\dot{u}^2 + f(u)^2\dot{v}^2)$$

and consider the Euler-Lagrange equations

$$\underline{u}: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) - \frac{\partial L}{\partial u} = \ddot{u} - f(u)f'(u)\dot{v} = 0$$
$$\underline{v}: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\partial L}{\partial v} = f(u)^2\ddot{v} = 0$$

It is now clear that these equations are satisfied for any path $u \mapsto \vec{r}(u, v_0)$, whereas in order to make the first equation hold for a path $v \mapsto \vec{r}(u_0, v)$, we have to eliminate the second term, but since f is positive, we must have $f'(u_0) = 0$.

(v) We parametrize the torus $T(r, R)$ by $\vec{r}\colon U \to \mathbb{R}^3$,

$$\vec{r}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$$

where $r < R$.

Let us first calculate its Gaussian curvature. We derive

$$\begin{aligned}\vec{r}_u &= (-r\sin u\cos v, -r\sin u\sin v, r\cos u)\\ \vec{r}_v &= (-(R + r\cos u)\sin v, (R + r\cos u)\cos v, 0)\end{aligned}$$

and obtain

$$\begin{aligned}E &= r^2\sin^2 u\cos^2 v + r^2\sin^2 u\sin^2 v + r^2\cos^2 u\\ &= r^2\sin^2 u + r^2\cos^2 u\\ &= r^2\\ F &= rR\sin u\sin v\cos v + r^2\sin u\sin v\cos u\cos v\\ &\quad - rR\sin u\sin v\cos v - r^2\sin u\sin v\cos u\cos v\\ &= 0\\ G &= (R + r\cos u)^2\sin^2 v + (R + r\cos u)^2\cos^2 v\\ &= (R + r\cos u)^2\end{aligned}$$

The second derivates are

$$\begin{aligned}\vec{r}_{uu} &= (-r\cos u\cos v, -r\cos u\cos v, -r\sin u)\\ \vec{r}_{uv} &= (r\sin u\sin v, -r\sin u\cos v, 0)\\ \vec{r}_{vv} &= (-(R + r\cos u)\cos v, -(R + r\cos u)\sin v, 0)\end{aligned}$$

and we see the normal is

$$W = (\cos u\cos v, \cos u\sin v, \sin u)$$

Its partial derivatives give us the shape operator

$$\begin{aligned}-s\vec{r}_u &= W_u = (-\sin u\cos v, -\sin u\sin v, \cos u)\\ -s\vec{r}_v &= W_v = (-\cos u\sin v, \cos u\cos v, 0)\end{aligned}$$

Comparing these to $\vec{r}_u, \vec{r}_v$ we obtain eigenvalues

$$s\vec{r}_u = -\frac{1}{r}\vec{r}_u, \quad s\vec{r}_v = -\frac{\cos u}{R + r\cos u}\vec{r}_v$$

Since κ is the determinant of s, we get

$$\kappa = \det\begin{bmatrix}-\frac{1}{r} & 0\\ 0 & -\frac{\cos u}{R+r\cos u}\end{bmatrix} = \frac{\cos u}{r(R + r\cos u)}$$

Finally, let us show that the integral of κ over $T(r, R)$ is zero.

$$\begin{aligned}
\int_{T(r,T)} \kappa dA &= \int_0^{2\pi} \int_0^{2\pi} \kappa \sqrt{EG - F^2} du dv \\
&= \int_0^{2\pi} \int_0^{2\pi} \frac{r \cos u (R + r \cos u)}{r(R + r \cos u)} du dv \\
&= 2\pi \int_0^{2\pi} \cos u du \\
&= 0
\end{aligned}$$