Differential geometry: 1. homework

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Exercise 1. Let M and N be two smooth manifolds and let $C(M)$ denote the algebra of all continuous functions $f\colon M \to \mathbb{R}$. Given a continuous map $g\colon M \to N$, we define the map $g^*\colon C(N) \to C(M)$ by $g^* f = f \circ g$.[1]

i) Show that $g\colon M \to N$ is a smooth map if and only if there holds $g^*(C^\infty(N)) \subset C^\infty(M)$.

ii) Suppose $g\colon M \to N$ is a homeomorphism between smooth manifolds. Show that g is a diffeomorphism if and only if $g^*|_{C^\infty(N)}\colon C^\infty(N) \to C^\infty(M)$ is an isomorphism.

Remark. Thus, the smooth structure of a given smooth manifold is (in a certain sense) encoded in the algebra $C^\infty(M)$ of all smooth functions on M. Note that categorically, $C^\infty(-)\colon \mathbf{Man}_\infty \to \mathbf{Vect}_\mathbb{R}$, which maps objects as $M \mapsto C^\infty(M)$ and morphisms as $g \mapsto g^*|_{C^\infty(M)}$, is a contravariant functor.

[1] Addition and multiplication by scalars in $C(M)$ are given pointwise; because of this, the map g^* is linear.

Exercise 2 (Restricting the codomain of a smooth map)**.** Let $g\colon M \to N$ be a smooth map between smooth manifolds.

i) Suppose $S \subset N$ is an immersed submanifold in N,[2] and suppose that $g(M) \subset S$. Prove that if g is continuous as a map from M to S, then $g\colon M \to S$ is smooth.

ii) Suppose $S \subset N$ is an embedded submanifold in N, and suppose $g(M) \subset S$. Prove that $g\colon M \to S$ is smooth.

iii) The *lemniscate* is the image $L = \mathrm{im}(\phi)$ of the map $\phi\colon (-\pi, \pi) \to \mathbb{R}^2$, $\phi(t) = (\sin(2t), \sin(t))$. As we know, L is an immersed submanifold in $\mathbb{R}^2$ that is not an embedded submanifold. Is the map $\psi\colon \mathbb{R} \to \mathbb{R}^2$, given by $\psi(t) = (\sin(2t), \sin(t))$, smooth as a map $\psi\colon \mathbb{R} \to L$? Justify your answer.

Hint. For proving i), use the fact that any immersion is locally an embedding, a corollary of the rank theorem.

[2] $S \subset N$ is an *immersed submanifold* in N, if S is a smooth manifold and the inclusion $S \hookrightarrow N$ is a smooth immersion. Note that $S \subset N$ may not have the subspace topology, so that $S \hookrightarrow N$ may not be a homeomorphism onto its image.

Exercise 3. For any $n \in \mathbb{N}$, we define the *unitary group of $n \times n$ matrices* as

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A^\dagger A = I\},$$

and also denote $\mathcal{H}_n = \{A \in \mathbb{C}^{n\times n} \mid A^\dagger = A\}$ as the vector space of $n \times n$ hermitian matrices.

i) Show that I is a regular value of the smooth map $\phi\colon \mathrm{GL}(n, \mathbb{C}) \to \mathcal{H}_n$, given by $\phi(A) = A^\dagger A$ and conclude that $\mathrm{U}(n)$ is a smooth embedded submanifold in $\mathrm{GL}(n, \mathbb{C})$. Determine its dimension, its tangent space $T_I\mathrm{U}(n)$ at I, and show that $\mathrm{U}(n)$ is path-connected.

ii) Show that $\mathrm{SU}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A^\dagger A = I, \det A = 1\}$ is a smooth embedded submanifold in $\mathrm{U}(n)$. Again, determine its dimension, its tangent space $T_I\mathrm{SU}(n)$, and show that $\mathrm{SU}(n)$ is path-connected. Additionally, show that the matrices $i\sigma_x, i\sigma_y, i\sigma_z$ form a basis for the vector space $T_I\mathrm{SU}(2)$, where

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are the *Pauli matrices.*

iii) Explain how it follows from this (and exercise 2) that $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are Lie groups.

iv) Prove that $\mathrm{SU}(2)$ is diffeomorphic to S^3.

Exercise 4. Let the map $\pi\colon \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$\pi(x, y, z, t) = (2xz + 2yt, 2yz - 2xt, x^2 + y^2 - z^2 - t^2).$$

i) Show that π restricts to a map $\pi|_{S^3}\colon S^3 \to S^2$.

ii) Show that π is a submersion on $\mathbb{R}^4 \setminus \{0\}$.

iii) Show that there holds

$$T_{(x,y,z,t)}S^3 = \mathrm{Lin}\left\{ \begin{bmatrix} -y \\ x \\ t \\ -z \end{bmatrix}, \begin{bmatrix} -z \\ -t \\ x \\ y \end{bmatrix}, \begin{bmatrix} -t \\ z \\ -y \\ x \end{bmatrix} \right\},$$

and use this to show that the map $\pi|_{S^3}\colon S^3 \to S^2$ is a submersion.

Remark. Here $\pi|_{S^3}\colon S^3 \to S^2$ corresponds to the famous Hopf fibration $S^3 \to \mathbb{C}P^1 = S^3/S^1$, given by

$$(z, w) \mapsto [z, w] = \{(e^{it}z, e^{it}w) \in S^3 \mid e^{it} \in S^1\}.$$

This is an example of a principal bundle – a notion central to gauge field theory in physics.