Differential geometry: 3. homework

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Exercise 1 (Orientation of a manifold)**.** Let M be a smooth manifold. We say that two charts (U, φ) and (V, ψ) *determine the same orientation* of M, if $\det(\mathrm{d}(\psi \circ \varphi^{-1})_{\varphi(p)}) > 0$ for any $p \in U \cap V$. A smooth atlas $\mathcal{A}$ on M is said to be *oriented* if any two charts in $\mathcal{A}$ determine the same orientation; we say that the manifold M is *orientable*, if there exists an oriented atlas on M. An *orientation* of M is a **choice** of a maximal oriented atlas; any chart from this atlas is called an *oriented chart*.

i) Show that any connected orientable manifold M admits precisely two orientations.

   Hint. For the part that M admits at most two orientations, it's enough to show for two compatible atlases $\mathcal{A} = (U_i, \varphi_i)_{i \in I}$ and $\mathcal{B} = (V_j, \psi_j)_{j \in J}$ on M which are individually oriented, that if there exists $p \in M$ and neighborhood charts $(U_i, \varphi_i) \in \mathcal{A}$, $(V_j, \psi_j) \in \mathcal{B}$ such that $\det(\mathrm{d}(\psi_j \circ \varphi_i^{-1})_{\varphi(p)}) > 0$, then $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas. (Then for any third compatible and oriented atlas $\mathcal{C}$, at least one of the atlases $\mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cup \mathcal{C}$ or $\mathcal{B} \cup \mathcal{C}$ must be oriented.) Create a separation of M with two open disjoint nonempty sets, one of which contains p.

ii) Prove that M is orientable iff it admits a *volume form*, i.e. a nowhere vanishing differential n-form $\omega \in \Omega^n(M)$, where $n = \dim M$. Thus an orientation on an orientable manifold M may be determined by a choice of a volume form (instead of by a choice of an oriented atlas); show that two volume forms ω and $\tilde{\omega}$ on an orientable manifold determine the same orientation iff there holds $\tilde{\omega} = f\omega$ for some positive smooth function $f \colon M \to (0, \infty)$.

iii) Let (M, g) be an oriented Riemannian manifold. Prove that there exists a unique volume form ω_g, called the *Riemannian volume form*, determined by the property: for any $p \in M$, if $(v_1, \ldots, v_n)$ is an (ordered) oriented orthonormal basis for T_pM, then $\omega_g(v_1, \ldots, v_n) = 1$.

   Remark. Suppose M is oriented and the atlas $\mathcal{A}$ determines its orientation. We say that an ordered basis $(v_1, \ldots, v_n)$ for T_pM is *oriented*, if the transition matrix A in the expansion

   $$v_i = A^j{}_i \partial_j|_p,$$

   has a positive determinant; here ∂_j denote the coordinate vector fields with respect to an oriented chart $(\varphi = (x^j)_j) \in \mathcal{A}$ around p. Equivalently, an ordered basis $(v_1, \ldots, v_n)$ is oriented if $\omega(v_1, \ldots, v_n) > 0$ for any volume form ω that determines the orientation of M.

Exercise 2 (Cartan's magic formula)**.** The *Lie derivative of a covariant k-tensor field* $\alpha \in \Gamma^\infty(T^kT^*M)$ with respect to a vector field $V \in \mathfrak{X}(M)$ is defined by the formula

$$\begin{aligned}(\mathcal{L}_V\alpha)(X_1,\ldots,X_k) &= V(\alpha(X_1,\ldots,X_k)) \\ &\quad - \alpha([V,X_1],X_2,\ldots,X_k) - \cdots - \alpha(X_1,\ldots,X_{k-1},[V,X_k]),\end{aligned}$$

for any vector fields $X_1,\ldots,X_k \in \mathfrak{X}(M)$.[1] Additionally, we define $\Gamma^\infty(T^0T^*M) = C^\infty(M)$ and $\mathcal{L}_V f := \mathrm{d}f(V) = Vf$.

i) Show that $\mathcal{L}_V\alpha$ is $C^\infty(M)$-multilinear in its arguments. Hence, $\mathcal{L}_V\alpha$ corresponds to a uniquely defined tensor field, denoted by the same symbol $\mathcal{L}_V\alpha$ (see Lemma 12.24 in Lee's book).

ii) Show that if $f \in C^\infty(M)$, then $\mathcal{L}_V(\mathrm{d}f) = \mathrm{d}(\mathcal{L}_V f)$.

iii) Prove that there holds

$$\mathcal{L}_V(\alpha\otimes\beta) = (\mathcal{L}_V\alpha)\otimes\beta + \alpha\otimes(\mathcal{L}_V\beta)$$

for any two k- and l- tensor fields $\alpha \in \Gamma^\infty(T^kT^*M)$, $\beta \in \Gamma^\infty(T^lT^*M)$.

Hint. It's enough to show (why?) that in any chart $(U, \varphi = (x^i)_i)$, for any simple k-tensor field $\alpha = f\,\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}$, there holds

$$\mathcal{L}_V\alpha = (\mathcal{L}_V f)\,\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k} + f\mathcal{L}_V(\mathrm{d}x^{i_1})\otimes\cdots\otimes\mathrm{d}x^{i_k} + \cdots + f\,\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathcal{L}_V(\mathrm{d}x^{i_k}).$$

iv) Briefly explain why the identity

$$\mathcal{L}_V(\omega\wedge\eta) = (\mathcal{L}_V\omega)\wedge\eta + \omega\wedge(\mathcal{L}_V\eta)$$

holds for any two differential forms $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$.

v) We define the *interior product* by a vector field V as the map $\iota_V\colon \Omega^k(M) \to \Omega^{k-1}(M)$,

$$(\iota_V\omega)(X_1,\ldots,X_{k-1}) = \omega(V,X_1,\ldots,X_{k-1}).$$

Show that there holds $\iota_V\circ\iota_V = 0$ and

$$\iota_V(\omega\wedge\eta) = (\iota_V\omega)\wedge\eta + (-1)^k\omega\wedge(\iota_V\eta)$$

for any two differential forms $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$.

vi) Prove *Cartan's magic formula* for differential forms:

$$\mathcal{L}_V\omega = \iota_V(\mathrm{d}\omega) + \mathrm{d}(\iota_V\omega),$$

which is sometimes written as $\mathcal{L}_V = \{\iota_V, \mathrm{d}\}$.

vii) Prove that the Lie derivative $\mathcal{L}_V$ and exterior derivative d commute, i.e. that there holds

$$\mathcal{L}_V(\mathrm{d}\omega) = \mathrm{d}(\mathcal{L}_V\omega),$$

for any differential form ω. Also, prove that if $\omega \in \Omega^1(M)$ is a differential 1-form, then

$$\mathrm{d}\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

for any two vector fields $X, Y \in \mathfrak{X}(M)$.

[1] The definition of $\mathcal{L}_V\alpha$ is motivated by the Leibniz rule – note that $[V,X] = \mathcal{L}_V X$ for any two vector fields $V, X \in \mathfrak{X}(M)$, so that

$$\mathcal{L}_V(\alpha(X_1,\ldots,X_k)) = (\mathcal{L}_V\alpha)(X_1,\ldots,X_k) + \alpha(\mathcal{L}_V X_1,\ldots,X_k) + \cdots + \alpha(X_1,\ldots,\mathcal{L}_V X_k).$$

Exercise 3 (Divergence)**.** Let M be an orientable smooth manifold and ω a volume form on M. We define the *divergence of a vector field* X as the smooth function $\mathrm{div}_\omega(X) \in C^\infty(M)$, defined by the equality

$$\mathcal{L}_X\omega = \mathrm{div}_\omega(X)\omega.$$

i) Let $M = \mathbb{R}^n$ and $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Calculate $\mathrm{div}_\omega(X)$ for a given vector field $X = X^i\partial_i$, where $(x^i)_i$ denote the standard coordinates.

ii) Now suppose that (M, g) is an oriented Riemannian manifold. Show that the Riemannian volume form ω_g from exercise 1. iii) is expressed as

$$\omega_g = \sqrt{\det[g_{ij}]}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

in any oriented chart $(U, \varphi = (x^i)_i)$, where $g_{ij} = g(\partial_i, \partial_j)$.

iii) Let (M, g) be an oriented Riemannian manifold and $(U, \varphi = (x^i)_i)$ an oriented chart. Show that for a vector field $X \in \mathfrak{X}(M)$, expressed in these coordinates as $X = X^i\partial_i$, there holds

$$\mathrm{div}_{\omega_g}(X) = \frac{1}{\sqrt{\det[g_{ij}]}}\partial_k\left(\sqrt{\det[g_{ij}]}X^k\right).$$

This formula is sometimes called *divergence in curvilinear coordinates.*

iv) **(Extra points)** Let M be an orientable smooth manifold, ω a volume form on M and let ϕ_t^X denote the flow of a vector field X on M. Prove that there holds

$$\mathrm{div}_\omega(X) = 0 \iff \left(\phi_t^X\right)^*\omega = \omega \text{ for any } t \text{ for which the flow is defined.}$$

Hint. Use the identity $(\mathcal{L}_X\omega)_p = \frac{d}{dt}\big|_{t=0}\big((\phi_t^X)^*\omega\big)_p$ which holds for all $p \in M$ (you do not need to prove it). One direction is then trivial, but the other is not.

Remark. The last equation is expressed in words as: the flow of the incompressible vector field preserves the volume form ω.

Exercise 4 (Introduction to symplectic geometry)**.** Consider the cotangent bundle $T^*\mathbb{R}$ of the manifold $\mathbb{R}$; this is a trivial vector bundle, with the bundle isomorphism between $\mathbb{R}^2$ and $T^*\mathbb{R}$ given by

$$(q, p) \mapsto p \, \mathrm{d}x|_q.$$

We will henceforth identify $T^*\mathbb{R}$ with $\mathbb{R}^2$ and denote the standard coordinates on $\mathbb{R}^2$ by (q, p). Suppose that ω is a differential 2-form on $\mathbb{R}^2$ given by $\omega = \mathrm{d}q \wedge \mathrm{d}p$, and let $H \in C^\infty(\mathbb{R}^2)$ be a smooth function. We define the *Hamiltonian vector field* $X_H \in \mathfrak{X}(\mathbb{R}^2)$ by the implicit identity

$$\iota_{X_H} \omega = \mathrm{d}H.$$

i) Express the vector field X_H on $\mathbb{R}^2$ in the coordinates (q, p) on $\mathbb{R}^2$, and show that the integral curves of X_H are given by the system of differential equations

$$\dot{q} = \frac{\partial H}{\partial p},$$
$$\dot{p} = -\frac{\partial H}{\partial q}.$$

Then find the flow of X_H when H is the function

$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2, \quad m, k > 0.$$

ii) Show that $X_H(H) = 0$; hence H is a constant function along the integral curves of X_H, and the level sets of H are precisely the integral curves of X_H.

iii) Compute the divergence $\mathrm{div}_\omega(X_H)$, the Lie derivative $\mathcal{L}_{X_H}\omega$, and show that there holds

$$(\phi_t^{X_H})^* \omega = \omega.$$

Remark. The last equation is expressed in words as: the flow of the Hamiltonian vector field preserves the symplectic form ω.