Second homework UAG

Domačo nalogo oddajte preko spletne učilnice do vključno 28.5.2021. You have to hand in the solutions until 28.5.2021

1. [25 T] Is the ideal generated by $f(x,y) = x^4 + y^3 + 4y^2 + 6y + 3$ prime in $\mathbb{C}[x,y]$? Find the maximal ideals that contain this ideal. Homogenize the polynomial f and denote the resulting polynomial by f^h. Find the radical homogenous ideals that contain the ideal generated by f^h.

2. [35 T] Let $C \subset \mathbb{A}^2$ be an algebraic curve given by the equation

$$x^4 - x^2y - y^3 = 0.$$

Find all the singular points of C and show that C is rational by constructing a birational map $\mathbb{A}^1 \to C$. Show also that the corresponding projective curve $\bar{C}$ is rational by constructing a birational map

$$\mathbb{P}^1 \to \bar{C}.$$

We consider a blow up $\pi : \tilde{\mathbb{A}^2} \to \mathbb{A}^2$ at the point $(0,0)$. Denote by E the exceptional divisor. Explicitly describe $\overline{\pi^{-1}(C \setminus (0,0))} \cap E$.

3. [40T] The curve $C_k \subset \mathbb{P}^2$ is given by

$$y^2z - x(x-z)(x-kz) = 0,$$

where $k \in \mathbb{C}$. Find the coefficients k, for which holds that C_k is smooth. For smooth curves C_k show the following:

- The map $\Phi : \mathbb{P}^2 \to \mathbb{P}^2$, given by $[x,y,z] \mapsto [x,-y,z]$ has the property $\Phi(C_k) \subseteq C_k$. Show that there exist $p,q,r,s \in C_k$, da velja $\Phi(p) = p$, $\Phi(q) = q$, $\Phi(r) = r$ in $\Phi(s) = s$.
- Show that the automorphisms of $\mathbb{P}^1$ are given by

$$[t,s] \mapsto [at+bs, ct+ds],$$

with some condition on $a,b,c,d \in \mathbb{C}$ (write this condition).
- Show that every automorphism of $\mathbb{P}^1$ with more than two fixed points is equal to the identity.
- Show that C_k is not rational.

Remark: Since every smooth cubic we can write as C_k for some k, this exercise shows that a smooth cubic is not rational.