Differential geometry: 5. homework

Submit by 15. 6. 2021 via `zan.grad@fmf.uni-lj.si`

Exercise 1 (Algebraic degrees of freedom of the Riemann tensor)**.** Let V be a vector space with $\dim V = n$; let $\mathcal{B}(V^*)$ denote the set of all covariant 4-tensors $\alpha \in T^4(V^*)$ which obey the following properties:

a) $\alpha(w, z, x, y) = -\alpha(z, w, x, y)$,

b) $\alpha(w, z, x, y) = -\alpha(w, z, y, z)$,

c) $\alpha(w, z, x, y) = \alpha(x, y, w, z)$,

for any $x, y, z, w \in V$. Denote by $\mathcal{R}(V^*)$ the set of all $\alpha \in \mathcal{B}(V^*)$ with the additional property:

d) $\alpha(w, z, x, y) + \alpha(w, x, y, z) + \alpha(w, y, z, x) = 0$.

We call the elements of $\mathcal{R}(V^*)$ the *algebraic curvature tensors* on V. Your tasks in this exercise are the following.

i) Briefly explain why $\mathcal{B}(V^*)$ and $\mathcal{R}(V^*)$ are vector subspaces of $T^4(V^*)$.

ii) Determine the dimension of $\mathcal{B}(V^*)$. *Hint.* Consider $S^2(\Lambda^2(V^*))$.

iii) Prove that the restriction $\mathrm{Alt}|_{\mathcal{B}(V^*)}$ of the antisymmetrization projection $\mathrm{Alt}\colon T^4(V^*) \to \Lambda^4(V^*)$ is given by

$$\mathrm{Alt}|_{\mathcal{B}(V^*)}(\alpha)(w, z, x, y) = \frac{1}{3}(\alpha(w, z, x, y) + \alpha(w, x, y, z) + \alpha(w, y, z, x)),$$

and show that $\mathrm{Alt}|_{\mathcal{B}(V^*)}\colon \mathcal{B}(V^*) \to \Lambda^4(V^*)$ is surjective.

iv) Determine the dimension of $\mathcal{R}(V^*)$.

Exercise 2 (Sectional curvature)**.** Let (M, g) be a Riemannian manifold, $p \in M$ a point in M and let $V \subset T_pM$ be an open neighborhood of zero on which $\exp_p$ is a diffeomorphism onto some open subset $U \subset M$. Let $v, w \in T_pM$ be nonzero linearly independent vectors that span a plane $\pi = \mathrm{Lin}(v, w) \leq T_pM$, and denote by $S(\pi) := \exp_p(V \cap \pi)$ the embedded Riemannian 2-submanifold of M, swept out by geodesics whose initial velocities lie in π. The *sectional curvature* of the plane π is defined as

$$\sec(\pi) := \sec(v, w) := \frac{1}{2} R_{S(\pi)}(p),$$

where $R_{S(\pi)}$ denotes the (intrinsic) scalar curvature of $S(\pi)$, endowed with the induced metric.*

i) Show that there holds
$$\sec(v, w) = \frac{R(v, w, v, w)}{\|v \wedge w\|^2},$$
where $\|v \wedge w\|^2 := \|v\|^2 \|w\|^2 - \langle v, w\rangle^2$.

   Hint. Show that the second fundamental form of $S(\pi)$ in M vanishes at p. Prove the equality for the case when v and w are orthonormal, then use Gram–Schmidt for the general case.

ii) (Sectional curvatures contain all the information about the Riemann tensor)
   Suppose that R_1 and R_2 are two algebraic curvature tensors (see exercise 1) on some finite-dimensional vector space V, such that
$$R_1(v, w, v, w) = R_2(v, w, v, w)$$
for any $v, w \in V$. Prove that $R_1 = R_2$.

iii) (Characterizations of constant sectional curvature) Prove that the following are equivalent:
   a) For any plane $\pi \leq T_pM$, there holds $\sec(\pi) = C$.
   b) There holds $R(X, Y)Z = C(\langle Y, Z\rangle X - \langle Z, X\rangle Y)$ for any $X, Y, Z \in T_pM$.
   c) There holds $R(X, Y)Y = C(X - \langle X, Y\rangle Y)$ for any $X, Y \in T_pM$, where Y has unit length.

   Hint. For implication (a$\Rightarrow$b), define $S(W, Z, X, Y) = \langle W, S(X, Y)Z\rangle$, where $S(X, Y)Z = k(\langle Y, Z\rangle X - \langle Z, X\rangle Y)$ and use part ii) above.

iv) (Interpretation of the Ricci and scalar curvature of M)
   Let $v \in T_pM$ be a unit tangent vector. Prove that there holds
$$\mathrm{Ric}(v, v) = \sum_{i=2}^{n} \sec(v, v_i),$$
where $v_2, \dots, v_n \in T_pM$ is any completion of $v =: v_1$ to an orthonormal basis of T_pM. Furthermore, show that
$$R(p) = \sum_{i \neq j} \sec(v_i, v_j).$$

v) Suppose $(x^i)_i$ are some local coordinates on M. Show that if (M, g) has constant sectional curvature k, then there holds
$$\begin{aligned} R_{lkij} &= C(g_{li}g_{kj} - g_{lj}g_{ki}), \\ R_{ij} &= C(n-1)g_{ij}, \\ R &= Cn(n-1). \end{aligned}$$

vi) For any $C \in \mathbb{R}$ and $n \geq 2$, give an example of an n-dimensional Riemannian manifold with constant sectional curvature C. Justify your answer.

---

*The $\frac{1}{2}$ in definition of $\sec(\pi)$ is motivated by Gauss' theorema egregium.

Exercise 3 (Parametrized surfaces in $\mathbb{R}^3$)**.** Let $U \subset \mathbb{R}^2$ be an open subset and let $\vec{r}\colon U \to \mathbb{R}^3$ be a smooth embedding, so that $S := r(U)$ is an embedded Riemannian submanifold of $\mathbb{R}^3$ (with the induced metric). Hence $\vec{r}$ is a parametrisation of S – we denote by $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ the elements of S.

i) Write down the metric g and the scalar second fundamental form h on S in matrix form

$$[g] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}, \quad [h] = \begin{bmatrix} L & M \\ M & N \end{bmatrix},$$

in terms of the given parametrisation $\vec{r}$. Then write down the shape operator of S in $\mathbb{R}^3$ and provide a formula for the Gaussian curvature κ and the principal curvatures.

ii) Express the second partial derivatives $\vec{r}_{uu}, \vec{r}_{uv}, \vec{r}_{vv}$ in terms of the Christoffel symbols on S (with respect to the given parametrisation) and components of the scalar second fundamental form h.

iii) Show that the components of Riemann and Ricci curvature tensors on S are given by

$$R_{lkij} = \kappa(g_{li}g_{kj} - g_{lj}g_{ki}),$$
$$R_{ij} = \kappa g_{ij}.$$

iv) Now consider a *surface of revolution*, i.e. the surface parametrised by

$$\vec{r}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $f, g\colon I \to \mathbb{R}$ are defined on an open interval I, so that $U = I \times (0, 2\pi)$, and f is positive. Calculate the Gaussian curvature of S for the case when $f'(u)^2 + g'(u)^2 = 1$, for all $u \in I$, i.e. when the curve $u \mapsto \vec{r}(u,0)$ is naturally parametrized. Then show the following:

   a) Any meridian (i.e. a path $u \mapsto \vec{r}(u, v_0)$, where v_0 is constant) is a geodesic.

   b) A parallel (i.e. a path $v \mapsto \vec{r}(u_0, v)$, where u_0 is constant) is a geodesic iff $f'(u_0) = 0$.

v) The torus $T(r,R)$ may be parametrized by $\vec{r}\colon U \to \mathbb{R}^3$,

$$\vec{r}(u,v) = \big((R + r\cos(u))\cos v, (R + r\cos(u))\sin v, r\sin u\big)$$

where $r < R$ are some positive constants and U is an appropriate open subset of $\mathbb{R}^2$. Calculate its Gaussian curvature κ and write down the points of $T(r,R)$ at which κ is positive, negative and zero. Then show that the surface integral over $T(r,R)$ of the Gaussian curvature vanishes:

$$\int_{T(r,R)} \kappa \, \mathrm{d}A = 0.$$

Remark. The last equality is a demonstration of the *Gauss–Bonnet theorem*, which states that if S is a closed (i.e. compact with no boundary) Riemannian surface, then the total Gaussian curvature is proportional to the Euler chracteristic:

$$\int_S \kappa \, \mathrm{d}A = 2\pi\chi(S).$$