Introduction to Algebraic Geometry - 1^{st} homework

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<u>Ex. 1:</u> Let A be a commutative ring with a unit element. Let us show the following statements:

(a) *the set of nilpotent elements in A forms an ideal:*

We need to merely verify that the set N of nilpotents in A is an additive subgroup in A which is closed under multiplication with elements from A. Indeed, take $a, b \in N$ and $n, m \in \mathbb{N}$ such natural numbers that $a^n = b^m = 0$. We calculate

$$\begin{aligned}(a+b)^{n+m} &= \sum_{k=0}^{n+m} \binom{n+m}{k} a^k b^{n+m-k} \\ &= \underbrace{\sum_{k=0}^{n} \binom{n+m}{k} a^k b^{n+m-k}}_{\text{div. by } b^m} + \underbrace{\sum_{k=n+1}^{n+m} \binom{n+m}{k} a^k b^{n+m-k}}_{\text{div. by } a^n} \\ &= 0\end{aligned}$$

Now, take $a \in N$ with $a^n = 0$ and $x \in A$. Since A is commutative, clearly

$$(xa)^n = x^n a^n = 0$$

Notice that the case $x = -1$ (where 1 is the unit element) proves that N is closed for additive inverses as well, and hence an ideal. This ideal is commonly called the *nilradical* of A and is in fact equal to the intersection of all prime ideals of A.[1]

(b) *the sum of a nilpotent and a unit element is always a unit:*

Let $a \in N$ with $a^n = 0$ and $u \in A^{-1}$. Since a is nilpotent, we get the following (informal) idea for the inverse

$$\frac{1}{u+a} = \frac{u^{-1}}{1+u^{-1}a} = u^{-1} \sum_{k=0}^{\infty} (-u^{-1}a)^k = u^{-1} \sum_{k=0}^{n-1} (-u^{-1}a)^k$$

Let us check that the above is indeed the element $(u+a)^{-1}$:

$$\begin{aligned}(a+u) \cdot u^{-1} \sum_{k=0}^{n-1} (-u^{-1}a)^k &= u^{-1} \sum_{k=0}^{n-1} (-u^{-1})^k a^{k+1} + \sum_{k=0}^{n-1} (-u^{-1}a)^k \\ &= \sum_{k=1}^{n-1} (-1)^{k-1} (u^{-1}a)^k + \sum_{k=0}^{n-1} (-1)^k (u^{-1}a)^k \\ &= (-u^{-1}a)^0 = 1\end{aligned}$$

since only the element at power 0 survives.

(c) *$f \in A[x]$ is nilpotent $\iff$ all its coefficients are nilpotent:*

- <u>$(\Longleftarrow)$</u>: Let $f = a_0 + a_1 x + a_m x^m \in A[x]$ with $a_0^{n_0} = \cdots = a_m^{n_m} = 0$. Then we can now use the multinomial formula to get $f^{n_0 + \cdots + n_m} = 0$ the same way we used the binomial formula in (1.a).

[1] Last year's Commutative Algebra course covered this.

- ($\implies$): Suppose now that $f^n = 0$ for some n (write f by coefficients as in the converse). We will do a sort of induction on the degree $\deg f$. If $\deg f = 0$ then the statement trivially holds, as all higher coefficients are zero. For the induction step suppose $\deg f = m$. Since

$$f^n(x) = x^{mn} a_m^n + \cdots + a_0^n = 0$$

we get in particular that a_0 is nilpotent. By (1.a), it follows that

$$f - a_0 = a_1 x + \cdots + a_m x^m = x(a_1 + a_2 x + \cdots + a_m x^{m-1})$$

is nilpotent, but this will happen precisely when $g(x) = a_1 + \cdots a_m x^{m-1}$ is nilpotent. Notice that $\deg g < \deg f$, so by the induction hypothesis, all coefficients of g are nilpotent. By construction, all coefficient of f are nilpotent and the proof is complete.

(d) *$f \in A[x]$ is a unit $\iff$ a_0 is a unit and the other coefficients are nilpotent:*

- ($\impliedby$): If a_0 is a unit and $a_1, \ldots, a_m$ are nilpotent, then by (1.c) the polynomial $g(x) = a_1 x + \cdots + a_m x^m$ is nilpotent, hence by (1.b) the polynomial $f = a_0 + g$ is a unit.
- ($\implies$): Suppose $g = f^{-1}$ with $g(x) = b_0 + \cdots + b_r x^r$. Again we prove the claim by induction on $\deg f$. If $\deg f = 0$, then the statement trivially holds, as all higher coefficients are zero. Suppose $\deg f = m$ and multiply

$$1 = fg = \sum_{k=0}^{m+r} c_k x^k$$

where

$$c_k = \sum_{i+j=k} a_i b_j$$

Clearly, $b_0 = a_0^{-1}$, so a_0 is a unit. We now compare coefficients from the other end. Since $a_m b_r = 0$, we get

$$\begin{aligned} a_{m-1} b_r + a_m b_{r-1} = 0 &\overset{a_m \cdot}{\implies} a_m^2 b_{r-1} = 0 \\ a_{m-2} b_r + a_{m-1} b_{r-1} + a_m b_{r-2} = 0 &\overset{a_m^2 \cdot}{\implies} a_m^3 b_{r-2} = 0 \\ &\cdots \\ \sum_{i+j=k} a_i b_j &\overset{a_m^{r-k} \cdot}{\implies} a_m^{r+1-k} b_k = 0 \end{aligned}$$

and hence $a_m^{r+1} b_0 = 0$, but since b_0 is a unit, a_m is nilpotent. Then by (1.b), $h(x) = f(x) - a_m x^m$ is a unit and a polynomial with $\deg h < \deg f$. By the induction hypothesis, we get that $a_1, \ldots, a_{m-1}$ are nilpotent, so the proof is complete.

(e) *$f \in A[x]$ is a zero divisor $\iff$ there exists a non-zero $a \in A$ with $af = 0$:*

- ($\impliedby$): Obvious, since $A \hookrightarrow A[x]$ via constant polynomials.
- ($\implies$): Let $fg = 0$ for non-zero polynomials f and g, and denote the coefficients of f and g as above. If $\deg f = 0$ the statement is again trivially true, as $f \in A$ via the identification from the converse. Suppose $\deg f = m$. Then clearly $a_m b_r = 0$, so $b_r f$ is a polynomial with $(b_r f) g = 0$ with $\deg b_r f < \deg f$. By the induction hypothesis, there exists a non-zero constant $a \in A$ such that $ab_r f = 0$. Then, by associativity, $a(b_r f) = (ab_r) f = 0$ and $ab_r \in A$ non-zero.

Ex. 2: Let $C = \{(x, y) \in \mathbb{A}^2;\ y^2 - x^3 = 0\}$.

- Is the map $\varphi\colon \mathbb{A}^1 \to C$, defined by $\varphi(t) = (t^2, t^3)$, an isomorphism of affine varieties?

  NO. A map ϕ is an isomorphism of affine varieties if and only if the map ϕ^* is an algebra isomorphism. In our case we have $\operatorname{im}\varphi^* = F[t^3, t^2]$, which is not isomorphic to $F[t]$. Indeed, $t \notin \operatorname{im}\varphi^*$.

  Another way to see this is by noticing that

  $$\psi(x,y) = \varphi^{-1}(x,y) = \begin{cases} \frac{y}{x};\ x \neq 0 \\ 0;\ x = 0 \end{cases}$$

  is not a morphism, since it is not a regular map. (This exercise is a typical example to show that isomorphism $\neq$ bijective morphism.)

- Is φ a homeomorphism with respect to the Zariski topology?

  YES. The map φ is clearly Zariski-continuous as a polynomial map. It is also bijective (we have the inverse above), so what remains to show is that ψ is also Zariski-continuous. Since $\psi\colon C \to \mathbb{A}^1$ and since the Zariski topology on $\mathbb{A}^1$ is precisely the finite-complements topology, it is enough to see that $\psi^{-1}(c)$ is Zariski-closed in C (or $\mathbb{A}^2$) for every point $c \in A^1$, which is obvious as points are closed in C (ψ is bijective, hence $1-1$).

Ex. 3: We want to find the irreducible components of the affine variety

$$V(x - yz, xz - y^2) \subset \mathbb{A}^3.$$

We calculate

$$\begin{aligned} V(x - yz, xz - y^2) &= V(x - yz, yz^2 - y^2) \\ &= V(x - yz, y(z^2 - y)) \\ &= V(x - yz) \cap (V(y) \cup V(z^2 - y)) \\ &= (V(x - yz) \cap V(y)) \cup (V(x - yz) \cap V(z^2 - y)) \\ &= (V(x) \cap V(y)) \cup (V(x - z^3) \cap V(y - z^2)) \\ &= V(x, y) \cup V(x - z^3, y - z^2) \end{aligned}$$

which are both clearly irreducible. The first component is the z-axis and the second component is the curve $t \mapsto (t^3, t^2, t)$. Alternatively, both associated ideals are clearly prime, since $F[x,y,z]/(x,y) \cong F[z]$ and $F[x,y,z]/(x - z^3, y - z^2) \cong F[z]$ are domains.

Ex. 4: Let us determine the radical of the ideal $I = (x^3 - y^6, xy - y^3) \lhd \mathbb{C}[x,y]$.
Let us first think of the solution informally:

$$\begin{aligned} x^3 = y^6 &\implies x = y^2 \implies y^2 - x = 0 \\ xy = y^3 &\implies y(y^2 - x) = 0 \end{aligned}$$

So we start suspecting that $(y^2 - x) = \sqrt{I}$.[2]

- (⊆): Take $f \in (y^2 - x)$, that is, $f = g \cdot (y^2 - x)$ for some $g \in \mathbb{C}[x,y]$. We calculate

  $$\begin{aligned} f^3 &= g^3 \cdot (y^2 - x)^3 \\ &= g^3 \cdot (y^6 - 3y^4x + 3y^2x^2 - x^3) \\ &= g^3 \cdot (y^6 - x^3) + 3g^3 \cdot (y^2x^2 - y^4x) \\ &= g^3 \cdot (y^6 - x^3) + 3g^3 \cdot xy \cdot (xy - y^3) \in I \end{aligned}$$

  Hence, $f \in \sqrt{I}$.

[2] Calculating this ideal in Macaulay 2 (with $\mathbb{Q}$ coefficients) also gives this solution, which is a strong indicator.

- $\underline{(\supseteq)}$: Let $f \in I$, that is, $f = a\cdot(x^3 - y^6) + b\cdot(xy - y^3)$ for some $a, b \in \mathbb{C}[x,y]$. Further, we calculate

$$\begin{aligned} f &= a\cdot(x^3 - y^6) + b\cdot(xy - y^3) \\ &= a\cdot(x - y^2)^3 + 3a\cdot xy\cdot(xy - y^3) + b\cdot(xy - y^3) \\ &= a\cdot(x - y^2)^3 + (3axy + b)\cdot xy\cdot(x - y^2) \in (y^2 - x) \end{aligned}$$

Hence, $I \subseteq (y^2 - x)$, but $(y^2 - x)$ is clearly radical, e.g. $\mathbb{C}[x,y]/(y^2 - x) \cong \mathbb{C}[y]$ is reduced. So, $\sqrt{I} \subseteq (y^2 - x)$.

Ex. 5: Let X be the union of the three coordinate axes. We will compute the generators of the ideal $I(X)$ and show that I cannot be generated by fewer than three elements.

We write

$$X = V(x,y) \cup V(y,z) \cup V(z,x)$$

So we get

$$I(X) = I(V(x,y)) \cap I(V(y,z)) \cap I(V(z,x)) = (x,y) \cap (y,z) \cap (z,x)$$

since each of the ideals is clearly radical. Denote $I_1 = (x,y)$, $I_2 = (y,z)$, and $I_3 = (z,x)$. If we take, e.g., the ideals I_1 and I_2 and canonically gather coefficients at y, we get that the remaining coefficient of an element in $I_1 \cap I_2$ has to be divided by both x and z, hence by xz. Taking all other combinations of ideals, we get that each element in $I_1 \cap I_2 \cap I_3$ can be written as a combination of elements xy, yz, and xz. Hence $I_1 \cap I_2 \cap I_3 \subseteq (xy, yz, xz)$, and clearly $(xy, yz, xz) \subseteq I_i$ for $i = 1,2,3$, so we get $I_1 \cap I_2 \cap I_3 = (xy, yz, xz)$.[3]

Now, suppose for contradiction that there exist $p, q \in F[x,y]$ such that $I = I(X) = (p,q)$. Notice, that I is a homogeneous ideal, as we have seen it can be generated by three homogeneous elements. Take the homogeneous maximal ideal $M = (x,y,z)$ and consider the quotient I/MI of F-modules, which is now an F-vector space. Then by assumption, $\overline{p}$ and $\overline{q}$ generate I/MI, so $\dim_F I/MI \leq 2$ (in the case where p and q are not linearly independent, we can get 1). However, $\overline{xy}$, $\overline{yz}$, and $\overline{xz}$ are F-linearly independent in I/MI. Indeed, take $\alpha, \beta, \gamma \in F$ and consider

$$\alpha\overline{xy} + \beta\overline{yz} + \gamma\overline{xz} = 0$$

Since we are in the quotient, we get that

$$\alpha xy + \beta yz + \gamma xz \in MI$$

However, since the product of ideals can be obtained by simply multiplying all generators, MI is generated by homogeneous monomials of degree 3, that is, take the product of generating sets $\{x,y,z\}$ and $\{xy, yz, xz\}$ (minus redundancy). Hence, no element in MI has non-zero terms of degree 2, so all α, β, γ must be zero. So $\dim_F I/MI \geq 3$, a contradiction.

Ex. 6: Let Y be a non-empty irreducible subvariety of an affine variety X and denote $U = X \setminus Y$. We assume that the coordinate ring $F[X] = \mathcal{O}_X(X)$ of X is a unique factorization domain. We will show that $\mathcal{O}_X(U) = \mathcal{O}_X(X)$ if and only if $\operatorname{codim} Y \geq 2$.

Firstly, recall that $F[X]$ is a unique factorization domain precisely when every prime ideal of codimension 1 is principal. Furthermore, every irreducible variety of codimension 1 is then defined by a single irreducible polynomial. Now, suppose $\operatorname{codim} Y < 2$. Then $Y = V(f)$, hence $U = D(f)$. Another theorem from the lectures then tells us that $\mathcal{O}_X(U) = F[X]_f$, which is not isomorphic to $F[X]$. Conversely, let $\operatorname{codim} Y \geq 2$. We want to show that every regular function on U extend to a regular function on X. If that is indeed the case, the extension is unique by a Corollary from lectures, as the extensions would match on the open set U. Write $Y = V(I)$, where I is the associated prime ideal to the irreducible subvariety Y. Then $U = \bigcup_{f\in I} D(f)$. Furthermore, there exist independent irreducible polynomials $f_1, \dots, f_r$ for some $r \geq \operatorname{codim} Y$ such that $U = \bigcup_{i=1}^r D(f_i)$ (the important part is not how

[3] We also get the same result if we calculate the intersection with Macaulay 2.

many there are per se, but rather that $r \geq 2$ and that they do not divide each other). Indeed, since $\operatorname{codim} I \geq 2$, there exist prime ideals $P_0, P_1, P_2 \lhd F[X]$ such that

$$(0) \subsetneq P_0 \subsetneq P_1 \subsetneq P_2 \subseteq I$$

Then there exist polynomials f_1 and f_2 such that $f_1 \in P_1$ but $f_1 \notin P_0$, and $f_2 \in P_2$ but $f_2 \notin P_1$. Since the ideals are prime, f_1 and f_2 must necessarily be independent, and clearly $D(f_1), D(f_2) \subseteq U$. We work in Noetherian rings so we can always find finitely many, but as we have seen, at least two. Now, take $\varphi \in \mathcal{O}_X(U)$. On $D(f_i)$ we have that $\varphi = \frac{g_i}{f_i^{k_i}}$ for some $g_i \in F[X]$ and $k_i \in \mathbb{N}_0$, where g_i is not divisible by f_i. Since $r \geq 2$, we look at intersections. On $D(f_i) \cap D(f_j)$ we have

$$\frac{g_i}{f_i^{k_i}} = \frac{g_j}{f_j^{k_j}} \implies g_i f_j^{k_j} = g_j f_i^{k_i}$$

hence $k_i = 0$ for all $i = 1, \ldots, r$ and all g_i are equal, as they pairwise match on open intersections. Since $D(f_i)$ cover U, we have found an extension of φ to X as a regular function, hence $\varphi \in \mathcal{O}_X(X)$ and the proof is complete.

Lastly, we will find a counter-example if $\mathcal{O}_X(X)$ is not a unique factorization domain. We take, e.g., the 3-dimensional affine variety $X = V(x_1x_4 - x_2x_3) \subset \mathbb{A}^4$, where the element $x_1x_4 = x_2x_3$ has two ways of factoring in $F[X] = F[x_1, x_2, x_3, x_4]/(x_1x_4 - x_2x_3)$. For Y we take the irreducible subvariety $V(x_1x_4 - x_2x_3, x_1, x_2) = V(x_1, x_2)$. Indeed, we know from tutorials that $J = (x_1, x_2)$ is a prime ideal in $F[X]$ of $\operatorname{codim} J = 1$, which is, importantly, not principal. We therefore get that $U = D(x_1) \cup D(x_2)$ and continue as above to obtain $\mathcal{O}_X(U) = \mathcal{O}_X(X)$, even though $\operatorname{codim} Y = \operatorname{codim} J = 1$.