Introduction to Algebraic Geometry - 2^{nd} homework

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Ex. 1:

(i) Let $R = \mathbb{C}[x, y]$ and $f(x, y) = x^4 + y^3 + 4y^2 + 6y + 3$ a polynomial in R. Let us check that $J = (f)$ is a prime ideal.

Indeed, write $\mathbb{C}[x, y] = \mathbb{C}[y][x]$ and

$$f(x, y) = 1 \cdot x^4 + (y^3 + 4y^2 + 6y + 3) \cdot 1 = 1 \cdot x^4 + (y + 1)(y^2 + 3y + 3) \cdot 1$$

so if we take $P = (y + 1) \lhd \mathbb{C}[y]$ we see that, of course, $a_4 = 1 \notin P$, $a_0 \in P$ and clearly $a_0 \notin P^2$. Hence, by Eisenstein's criterion, f is irreducible and hence prime, since $\mathbb{C}[y]$ and $\mathbb{C}[x, y]$ are unique factorization domains.

(ii) Let us now find all maximal ideals that contain J.

Since the above equation defines a curve, maximal ideals containing J are precisely those given by points on the curve. That is, for any $b \in \mathbb{C}$ we get 4 maximal ideals $M_i = (x - a_i, y - b)$ for $i = 1, \dots, 4$, where

$$\begin{aligned} a_1 &= \sqrt[4]{-b^3 - 4b^2 - 6b - 3} \\ a_2 &= -\sqrt[4]{-b^3 - 4b^2 - 6b - 3} \\ a_3 &= i\sqrt[4]{-b^3 - 4b^2 - 6b - 3} \\ a_4 &= -i\sqrt[4]{-b^3 - 4b^2 - 6b - 3} \end{aligned}$$

some of which may be the same.

(iii) We homogenize the above polynomial with respect to coordinates $[x, y, z]$ in $\mathbb{P}^2$.

$$f^h(x, y, z) = x^4 + y^3 z + 4y^2 z^2 + 6yz^3 + 3z^4$$

This is now a homogeneous polynomial of degree 4 (we added the new coordinate at the end instead of the beginning, which is equivalent).

(iv) Let us find the homogeneous ideals that contain $J' = (f^h)$.

We know that J' defines a curve in $\mathbb{P}^2$ (represented by a conic surface in $\mathbb{A}^3$. By the Projective Nullstelensatz, radical homogeneous ideals that contain J' are precisely given by projective subvarieties of the above projective curve, and these will in turn be given by finite intersections of maximal ideals M_p of polynomials vanishing at a point p on the above curve (given by a line in $\mathbb{A}^3$ through p and the origin). Note that these ideals are not actually maximal, since they are all properly contained in the irrelevant ideal I_0, they are merely maximal in their respective local rings at p.

To find all of these point ideals for every p on the curve, we consider the affine covering. In U_2 where $z = 1$, we immediately see that the points on the curve are $[a, b, 1]$ for (a, b) zeros of f. From the procedure from tutorials we get point ideals

$$M_{[a,b,1]} = (x - az, y - bz)$$

In U_1 where $y = 1$, we get a similar result. We are searching for solutions of

$$x^4 + z + 4z^2 + 6z^3 + 3z^4 = 0$$

which are as above given by

$$\begin{aligned} a_1 &= \sqrt[4]{-c(1 + 4c + 6c^2 + 3c^3)} \\ a_2 &= -\sqrt[4]{-c(1 + 4c + 6c^2 + 3c^3)} \\ a_3 &= i\sqrt[4]{-c(1 + 4c + 6c^2 + 3c^3)} \\ a_4 &= -i\sqrt[4]{-c(1 + 4c + 6c^2 + 3c^3)} \end{aligned}$$

for any $c \in \mathbb{C}$, where some solutions may be the same. We then similarly get point ideals

$$M_{[a,1,c]} = (x - ay, z - by)$$

In U_0 where $x = 1$, we get the equation

$$1 + y^3 z + 4y^2 z^2 + 6yz^3 + 3z^4 = 0$$

and denote by (b, c) its solutions.[1] We then obtain point ideals

$$M_{[1,b,c]} = (y - bx, z - cx)$$

This last step is luckily not necessary, since the only possible zero not covered by U_1 and U_2 is $[1, 0, 0]$, which is not even on the curve.

Lastly, these are all contained in I_0 which is also a radical ideal.

Ex. 2: Let $C \subset \mathbb{A}^2$ be an algebraic curve given by the equation

$$x^4 - x^2 y - y^3 = 0$$

(i) Let us find all the singular points of C.

We are solving the system of equations

$$F(p) = \frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = 0$$

In our case, we are solving

$$\begin{aligned} x^4 - x^2 y - y^3 &= 0 \\ 2x^3 - xy &= 0 \\ x^2 + 3y^2 &= 0 \end{aligned}$$

If $x = 0$ it immediately follows that $y = 0$. If $x \neq 0$, the second equation yields $y = 2x^2$ and hence

$$x^4 - 2x^4 - 8x^6 = 0 \implies x^4(1 + 8x^2) = 0 \overset{x \neq 0}{\implies} x^2 = -\frac{1}{8}$$

We then have $y = -\frac{1}{4}$ and inputting into the third equation

$$-\frac{1}{8} + \frac{3}{16} = \frac{1}{16} \neq 0$$

We therefore have the one singularity $p = (0, 0)$.

[1] I solved the equation in `Mathematica`.

(ii) We show that C is rational by parametrizing it. Indeed, let us use the usual method of lines through the origin. Write $y = tx$ and calculate

$$x^4 - tx^3 - t^3x^3 = 0 \implies x^4 - (t+t^3)x^3 = 0 \implies x^3\left(x - (t+t^3)\right) = 0$$

Since without loss of generality $x \neq 0$, we get the parametrization

$$x = t + t^3, \qquad y = t^2 + t^4$$

The birational map $\mathbb{A}^1 \to C$ is then given by

$$t \mapsto (t + t^3, t^2 + t^4)$$

For the corresponding projective curve $\overline{C}$ we simply homogenize and get the birational map $\mathbb{P}^1 \to \overline{C}$ as

$$[t,s] \mapsto [ts^3 + t^3s, t^2s^2 + t^4, s^4]$$

(iii) Now consider the blow-up $\pi\colon \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$ at the singular point $p = (0,0)$. Denote by E its exceptional line. Let us explicitly describe $\overline{\pi^{-1}(C \setminus (0,0))} \cap E$.

We know that

$$\pi^{-1}(C) = \left\{((x,y),[t,s]);\ x^4 - x^2y - y^3 = 0,\ ty = sx\right\}$$

Now take the lowest degree terms and calculate the tangents at p as

$$x^2y + y^3 = y(x^2+y^2) = y(y+ix)(y-ix) = 0$$

hence the tangents at p are precisely

$$y = 0, \qquad y = ix, \qquad y = -ix.$$

From the $ty = sx$ we then get that the intersection contains 3 points, namely

$$\overline{\pi^{-1}(C \setminus (0,0))} \cap E = \{[1,0],[1,i],[1,-i]\}\,.$$

Ex. 3: The curve $C_k \subset \mathbb{P}^2$ is given by

$$y^2z - x(x-z)(x-kz) = 0$$

for a parameter $k \in \mathbb{C}$.

(i) We first find all of the values of k such that C_k is smooth, that is, without singularities.

We are solving the system of equations

$$\begin{aligned} y^2z - x(x-z)(x-kz) &= 0 \\ (x-z)(x-kz) + x(x-kz) + x(x-z) &= 0 \\ yz &= 0 \\ y^2 + x(x-kz) + kx(x-z) &= 0 \end{aligned}$$

Note that this curve clearly has a singularity in $(0,0,0)$, but this does not interest us, since we are looking at it in the projective space. We follow the third equation and suppose $z = 0$. Then from the first equation $x = 0$ and from the fourth $y = 0$, so this yields nothing. Now suppose $y = 0$. If $x = z$, then $x = kz$, hence $k = 1$. If $x \neq z$, then $kz = (2-k)x$. If $k \neq 0$, then it follows that either $k = 1$ or $x = z = 0$, both leading nowhere. The only case to check is $k = 0$, which gives us a solution for all $z \in \mathbb{C}$. We have therefore obtained that C_0 has a singularity in $[0,0,1]$ and C_1 has a singularity in $[1,0,1]$. For all other values of k the curve C_k is smooth in $\mathbb{P}^2$.

From now on assume that C_k is smooth, that is, $k \neq 0,1$.

(ii) Let the map $\Phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ be given by $[x, y, z] \mapsto [x, -y, z]$. Notice that this map has the property $\Phi(C_k) \subseteq C_k$, since C_k is purely quadratic in y. Let us show that there exist $p, q, r, s \in C_k$ such that they are all fixed by Φ.

By (3.i) we immediately get three points

$$p = [0, 0, 1], \qquad q = [1, 0, 1], \qquad r = [k, 0, 1],$$

which are clearly fixed by Φ. But the projective transformation Φ is of course equivalently given by $[x, y, z] \mapsto [-x, y, -z]$, which yields the fourth point

$$s = [0, 1, 0]$$

also on the curve C_k.

(iii) We will now prove that the automorphisms of $\mathbb{P}^1$ are given by

$$[t, s] \mapsto [at + bs, ct + ds]$$

and write down the condition on $a, b, c, d \in \mathbb{C}$.

We recall that $\mathbb{P}^1$ is actually the Riemann sphere $\mathbb{CP}^1$. Indeed, denote $\infty := [0, 1]$, then identify $[t, s] \longleftrightarrow [1, \frac{s}{t}]$. Automorphisms of $\mathbb{P}^1$ must then correspond to automorphisms of $\mathbb{CP}^1$, which are precisely non-degenerate Möbius transformations, that is, functions of the form

$$z \mapsto \frac{cz + d}{az + b}$$

where $ad - bc \neq 0$. Since $z \longleftrightarrow \frac{s}{t}$, multiplying by t yields the desired result.

If this feels like cheating, we can approach the problem more algebraically. As above denote $\infty := [0, 1]$ and consider the embedding $\mathbb{C} = U_0 \to \mathbb{P}^1$ via $s \mapsto [1, s]$. Suppose f is an automorphism of $\mathbb{P}^1$ and assume without loss of generality that $f(\infty) = \infty$. Indeed, once we have our automorphisms, if $f(\infty) = a = [1, a]$, consider the automorphism $g\colon [t, s] \mapsto [t - as, s]$. Then $(g \circ f)(\infty) = \infty$, so the assumption really preserves generality. Consider now the restriction $f_0 = f|_{U_0}$. Since f is an automorphism and hence bijective, f_0 is a polynomial bijection $\mathbb{C} \to \mathbb{C}$. But by the Fundamental Theorem of Algebra, f_0 must have degree 1, say $f_0(s) = \alpha + \beta s$. Repeat this for $f_1 = f|_{U_1}$ and obtain $f_1(t) = \gamma t + \delta$. To go back to the projective setting, we need to merely homogenize the above polynomials with appropriate variables and we obtain that f must be of the form

$$[t, s] \mapsto [at + bs, ct + ds]$$

The relation on the complex coefficients is also obtained locally. If we restrict the above to U_1 and without loss of generality assume $s = 1$, then we obtain that

$$[t, 1] \mapsto \frac{at + b}{ct + d}$$

must be an automorphism of $\mathbb{C}$, so we get the condition from above $ad - bc \neq 0$.

(iv) Next we show that every automorphism of $\mathbb{P}^1$ with more than 2 fixed points is equal to the identity.

One solution is again from Complex Analysis. Since automorphisms of $\mathbb{P}^1$ correspond to automorphisms of $\mathbb{CP}^1$, we get that

$$\frac{az + b}{cz + d} = z \iff az + b = cz^2 + dz \iff cz^2 + (d - a)z - b = 0$$

which is a degree 2 polynomial and hence has at most 2 different zeros.

Algebraically, we know that $[t, s]$ is a fixed point of an automorphism f of $\mathbb{P}^1$, if

$$\begin{aligned} at + bs &= \lambda t \\ ct + ds &= \lambda s \end{aligned}$$

From Linear Algebra, we know that λ must be some eigenvalue of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and the solutions must be its eigenvectors. Therefore we can have at most 2 projective solutions, unless $f = id$. Furthermore, since by (3.iii) this matrix is non-degenerate, we always get precisely 2 solutions.

(v) Lastly, let us show that C_k is not a rational curve.

This is of course just a consequence of (3.ii-iv). Indeed, C_k is by definition rational if there exists a birational map

$$\varphi \colon \mathbb{P}^1 \to C_k$$

which is equivalent to them containing open dense φ-isomorphic subsets. But this is a problem, since Φ from (3.ii) induces an automorphism on C_k with 4 fixed points. Let $U \subseteq \mathbb{P}^1$ and $V \subseteq C_k$ be some φ-isomorphic dense open sets and consider

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"] & U \arrow[d, "\varphi"] \\ V \arrow[r, "\Phi"] & V \end{tikzcd}$$

where f is such a function that makes the above diagram commute. By density and since compositions of morphisms are again morphisms, f is induced by some automorphism on $\mathbb{P}^1$ (also denoted by f). But then f has more than 2 fixed points, hence $f = id$, but $\Phi \neq id$, a contradiction. Thus, the curve C_k cannot be rational.

Alternatively, we can recall a theorem from Tutorials, stating that any birational equivalence between smooth curves is an isomorphism. Then we clearly get the induced automorphism f given by

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "f"] \arrow[d, "\varphi"] & \mathbb{P}^1 \arrow[d, "\varphi"] \\ C_k \arrow[r, "\Phi"] & C_k \end{tikzcd}$$

which clearly has 4 fixed points, implying $f = id$, contradicting $\Phi \neq id$.