The corresponding physical situation is that of two gas phases (one cold and dense, the other one hot and rarefied for example). If $\mathbf{v}_R = \mathbf{v}_L = 0$ then we call this "Sod Shock tube" problem.

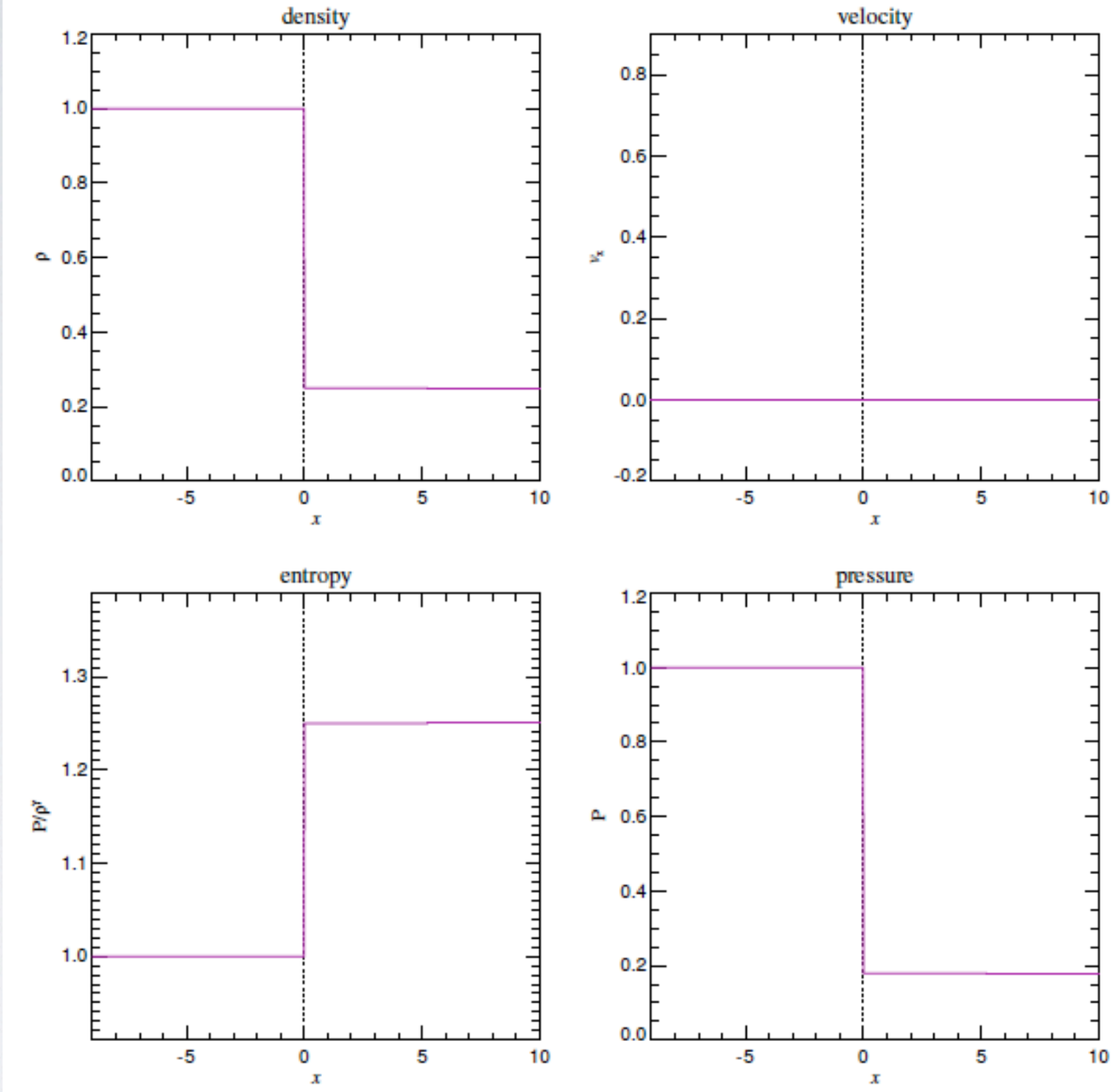
The contact wave is the solution of the flow at the interface, while the rarefaction and shock waves can exist on both sides (left *and* right) or only at one of them

Let us consider a example, a 1D Sod shock tube in which two phases are brought into contact at x=0 for t=0

Note: in astrophysics shocks are associated with supersonic motions of an object through a medium that causes a large disturbance but this need not to be the case (eg nonlinear wave-steepening in ocean also causes shocks)

Phase I

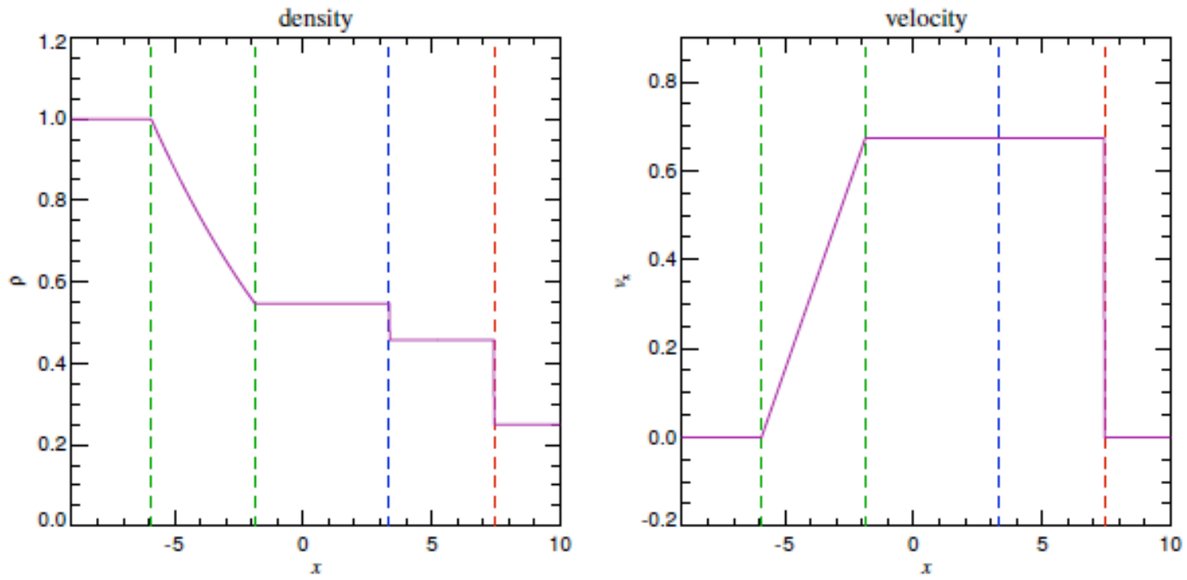
Phase II



In the next slide we can look at the solution in the variables for $t = 5$, identifying the general qualitative features associated with the existence of the three waves

Quick digression: why are we focusing on methods to solve for discontinous flows?

Because such methods, as we will see, can be generalized to any flow that is discretized on a grid. Namely, one can view the flow values at two nearby cells (eg at cell midpoint) as an example of a Riemann problem for which only two boundary values are known (i.e. as it physically is the case for flows with discontinuities such as shocks)



- *Shock:* This is a sudden compression of the fluid, associated with an irreversible conversion of kinetic energy to heat, i.e. here entropy is produced. The density, normal velocity component, pressure, and entropy all change discontinuously at a shock.
- *Contact discontinuity:* This traces the original separating plane between the two fluid phases that have been brought into contact. Pressure as well as the normal velocity are constant across a contact, but density, entropy and temperature can jump.

Red

Blue

- *Rarefaction wave:* This occurs when the gas (suddenly) expands. The rarefaction wave smoothly connects two states over a finite spatial region; there are no discontinuities in any of the fluid variables.

Green

Finite volume methods (Godunov)

In finite volume methods one considers an integral form of the Euler equations. One starts from conservation laws. These are special cases of *hyperbolic conservation laws* of the type:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = 0.$$

U is a state vector and **F** the flux vector. Think about mass conservation, namely the continuity equation, as the prototype.

We will now see how we can use Riemann solvers to devise a method for the solution of this type of equations on a grid
We can write in this form the entire set of fluid equations:

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ \rho e \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v}\mathbf{v}^T + P \\ (\rho e + P)\mathbf{v} \end{pmatrix}$$

with e being the total specific energy, $e = u + \mathbf{v}^2/2$, u being the specific thermal energy. As always the equation of state gives the closure to the system of equations.

We now describe the state of a (1D) system as an average over a finite set of cells:

$$\mathbf{U}_i = \frac{1}{V_i} \int_{\text{cell } i} \mathbf{U}(\mathbf{x})\, \mathrm{d}V.$$

The problem is now equivalent to finding an update for these cell-averaged state variables. We start by integrating the conservation law over a cell and over a finite time interval:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathrm{d}x \int_{t_n}^{t_{n+1}} \mathrm{d}t \left(\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} \right) = 0.$$

Performing finite differencing on the grid one obtains:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathrm{d}x\,[\mathbf{U}(x,t_{n+1}) - \mathbf{U}(x,t_n)] + \int_{t_n}^{t_{n+1}} \mathrm{d}t\,\left[\mathbf{F}(x_{i+\frac{1}{2}},t) - \mathbf{F}(x_{i-\frac{1}{2}},t)\right] = 0.$$

In the first term, we recognize the definition of cell average:

$$\mathbf{U}_i^{(n)} \equiv \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(x,t_n)\mathrm{d}x.$$

Using the above equation we can recast the integral form of the conservation law in a stencil immediately amenable to be treated as a Riemann problem:

$$\Delta x\left[\mathbf{U}_i^{(n+1)} - \mathbf{U}_i^{(n)}\right] + \int_{t_n}^{t_{n+1}} \mathrm{d}t\,\left[\mathbf{F}(x_{i+\frac{1}{2}},t) - \mathbf{F}(x_{i-\frac{1}{2}},t)\right] = 0.$$

At the interface the solution is independent on time and we can write:

$$\mathbf{F}(x_{i+\frac{1}{2}},t) = \mathbf{F}^{\star}_{i+\frac{1}{2}},$$

So we can rewrite into:

$$\Delta x \left[\mathbf{U}_i^{(n+1)} - \mathbf{U}_i^{(n)} \right] + \Delta t \left[\mathbf{F}^{\star}_{i+\frac{1}{2}} - \mathbf{F}^{\star}_{i-\frac{1}{2}} \right] = 0.$$

which leads to the explicit update formula:

The first term on the right gives the flux that flows from the left to the cell, the second the flux out of the cell from its right side

$$\mathbf{U}_i^{(n+1)} = \mathbf{U}_i^{(n)} + \frac{\Delta t}{\Delta x} \left[\mathbf{F}^{\star}_{i-\frac{1}{2}} - \mathbf{F}^{\star}_{i+\frac{1}{2}} \right]$$

The method of using the Riemann solution in the updating step is called *Godunov's scheme*.

Note that we have not made any approximation in the update formula. In principle this should be valid for abitrary large timesteps since the flux part is independent on time. Is that really correct?
We have implicitly treated the problem as the sum of two independent Riemann problems at cell faces. This is reasonable if the cell spacing is such that waves do not propagate faster than the timestep Δt between cell phases ---> in other words it is ok for timesteps smaller than the Courant timestep

Rea (Reconstruct-Evolve-Average) scheme
Describes a scheme based on the Godunov update idea and is made of three steps:

Reconstruct
Construct fluxes using piece-weise constant states at cell interfaces (1st order in accuracy). Think of this as a discretization of the fluid state at a given time.

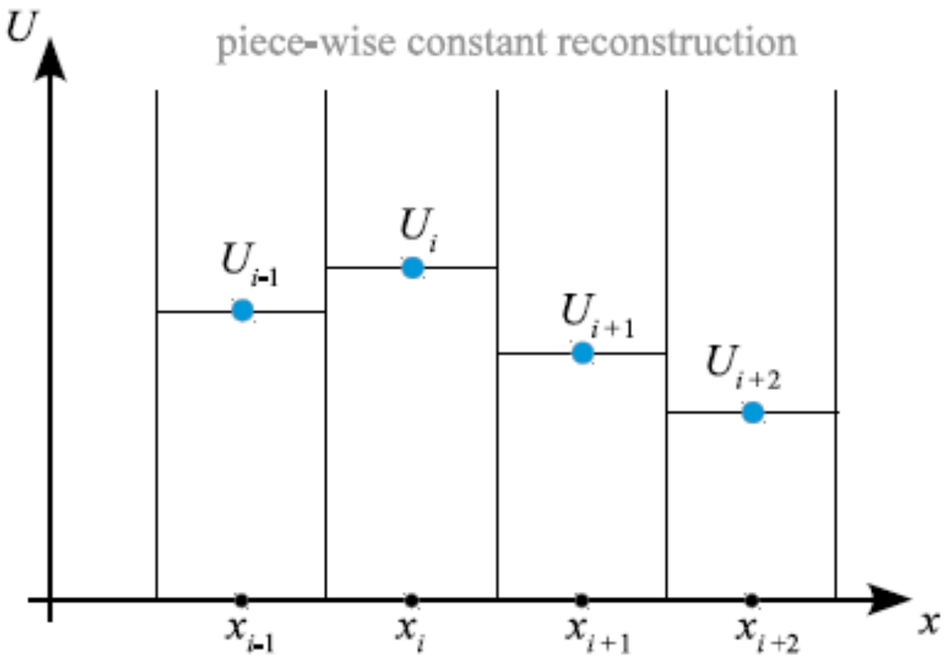


Fig. 26 Piece-wise constant states of a fluid forming the simplest possible reconstruction of its state based on a set of discrete values U_i known at spatial positions x_i.

Evolve

Evolve reconstructed state forward in time by Δt. In the Godunov method this is done by evolving many piece-wise constant initial value problems, one for each cell interface, using the Riemann solver exactly or approx. For what we have said, the solution can be treated as a superposition of the individual solutions as long as there is no interaction between waves generated at each interface, namely as long as we respect the Courant condition for the timestep

Average

Compute a smooth average of the wave structure resulting from the evolution step to compute the new starting flow states $\mathbf{U}^{n+1}$. The cycle is then repeated for all timesteps.

The core of the method is the *multiple Riemann problem* approach. The piece-wise state at cell interfaces, for which a Riemann problem has to be solved, can be formally written as:

$$\mathbf{F}^{\star} = \mathbf{F}_{\text{Riemann}}(\mathbf{U}_L, \mathbf{U}_R).$$

For ideal gas, and also a generic isothermal gas, this can be solved exactly using an iterative root-finding method for a non-linear equation (as in the Sod-shock tube test). Since we have to solve this many times in the REA scheme for computational efficiency it is customary to use approximate solvers (eg Toro 1997). The latter approach is the default in Adaptive Mesh Refinement codes that deal with large N_{cell}