# II - Towards a general mathematical model for astrophysical systems: the notion of distribution function *f* (**x,v**,t)

Any system of particles or superparticles can be described by a distribution function *f* which expresses the *probability density* to have a particle (or superparticle) with position between **x** and **x + dx** and velocity between **v** and **v + dv** at time t.
In other words it describes how phase space (**w= (x,v)**) is filled at any given time, by "real" particles or superparticles.

Next question is: how does *f* evolve in time? Or equivalently, how particles/superparticles evolve in phase-space?

First we must specify a field by which these particles interact (eg gravity or electromagnetic field, so we give them a mass or a charge),then find the dynamical equation that *f* obeys under the action of such a field.

# Uncorrelated and correlated systems

Consider a system of *N* particles subject to some mutual force
If the particles are *uncorrelated*, namely the evolution of a particle in phase space is independent from that of another particle, at any given time *t* the state of the system the two-particle distribution function can be written as:

$$f_2\,(\mathbf{r},\mathbf{v},\mathbf{r'},\mathbf{v'},t) = f_1(\mathbf{r},\mathbf{v},t)\; f_1{}'(\mathbf{r'},\mathbf{v'},t)$$

which can be extended to N-particle distribution function as:

$$f_N = f_1\,(\mathbf{r_i},\,\mathbf{v_i},\,t)\ldots f_n(\mathbf{r_n},\,\mathbf{v_n},\,t)$$

and holds when the direct interaction between particles (through whatever force field) is negligible. We will see that for gravitational systems this means the acceleration induced by *individual* particles is negligible, yet particles will still respond to the *mean gravity field*

A note on the statistical meaning of *f*.....

First define the *exact* distribution describing state of a system of N particles at time t (occupation of states in phase space) as:

$$F(\mathbf{r},\mathbf{v},t) = \sum_{i=1}^{N} \delta(\mathbf{r}-\mathbf{r}_i(t)) \cdot \delta(\mathbf{v}-\mathbf{v}_i(t)).$$

Then *f* the ensembled-average distribution function, namely the exact distribution function weighted by the probability density of states in phase space:

$$f_1(\mathbf{r},\mathbf{v},t) = \langle F(\mathbf{r},\mathbf{v},t)\rangle = \int F \cdot p \cdot \mathbf{dr}_1\, \mathbf{dr}_2 \cdots \mathbf{dr}_N\, \mathbf{dv}_1\, \mathbf{dv}_2 \cdots \mathbf{dv}_N$$

This is important to remember as it means, in practice, that *f* can be correctly evaluated only after considering many realizations of a system of *N* particles and then taking their weighted average in phase space. We will return to this when dealing with numerical models

The dynamical equations obeyed by $f$ follow from the notion of conservation of probability density in phase space (formally from Liouville's equation in statistical mechanics).

Conservation of probability in phase space can be understood less formally with the analogy with mass conservation in fluid flow. For an arbitrary volume $V$ in phase space we can define the probability P of finding a particle in V as:

$$P = \int_V \mathrm{d}^6\mathbf{w}\, f(\mathbf{w}). \qquad \mathbf{w=(r,v)}$$

While $f$ can evolve P must be conserved, which can then be expressed with a continuity equation *in phase space*:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \cdot (f\dot{\mathbf{w}}) = 0.$$

Since the collisionless systems that we will model (gravitational) will also be hamiltonian, we can use Hamilton's equations to reformulate in *p, q* coordinates:

**r,v --> q,p**
**w= (q,p)**

$$\frac{\partial}{\partial \mathbf{q}} \cdot (f\dot{\mathbf{q}}) + \frac{\partial}{\partial \mathbf{p}} \cdot (f\dot{\mathbf{p}}) = \frac{\partial}{\partial \mathbf{q}} \cdot \left(f\frac{\partial H}{\partial \mathbf{p}}\right) - \frac{\partial}{\partial \mathbf{p}} \cdot \left(f\frac{\partial H}{\partial \mathbf{q}}\right)$$
$$= \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}}$$
$$= \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}},$$

In the last line one uses the fact that $\partial^2 H/\partial\mathbf{q}\partial\mathbf{p} = \partial^2 H/\partial\mathbf{p}\partial\mathbf{q}$

Rewriting the second term in the left-hand side of the continuity equation using the above result we obtain the Vlasov equation (collisionless Boltzmann):

$$\frac{\partial f}{\partial t} + \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0, \longrightarrow \frac{df}{dt} = \mathbf{0}$$

( lagrangian derivative )

which is a partial differential equation for *f* as a function of the six phase space coordinates plus time. Question: since we said we discard particle-particle interactions what is the acceleration due to here?

The answer is apparent if we realize that individual particles will still move under the action of the mean field produced by the self-gravity of the galaxy, or the charge distribution in a plasma. In the case of gravitational system we can thus state that what we want to solve is the coupled Vlasov-Poisson system of partial differential equations (PDEs).

Let's first define the mass density of a system of N particles with equal mass *m* (just for simplicity) as:

$$\rho(\mathbf{r},t) = m \int f(\mathbf{r},\mathbf{v},t)\,\mathrm{d}\mathbf{v}.$$

Then the two equations that we want to solve together are:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \Phi}{\partial \mathbf{r}}\frac{\partial f}{\partial \mathbf{v}} = 0,$$

$$\nabla^2 \Phi = 4\pi G m \int f(\mathbf{r},\mathbf{v},t)\,\mathrm{d}\mathbf{v}.$$

with the acceleration in the Vlasov equation being given by:

$$\mathbf{a} = -\frac{\partial \Phi}{\partial \mathbf{r}}$$

# Physical regime of collisionless systems

Physically treating the system as collisionless bears meaning only as long as the potential energy of interaction (eg gravitational binding energy) between individual particles undergoing encounters is much smaller than their kinetic energy, which is equivalent to say that they move under the action of the mean field and not the mutual interaction.
This yields a characteristic distance scale *a* below which the collisionless approximation brakes down. *a* is defined by equating potential and kinetic/thermal energy and must be compared with the typical interparticle separation. So in plasma and galaxies, for example ratios between such two energies are:

$$\epsilon_p = a_p n^{1/3} \qquad \epsilon_g = a_g \sigma^{1/2}$$

$$a_p = \frac{e^2}{4\pi\epsilon_0 k_B T} \qquad a_g = \frac{Gm^2}{\frac{1}{2}mv^2}$$

If these ratios are << 1 then the collisionless approximation works well. (for galaxies we use surface number density $\sigma$ assuming a flat disk-like configuration)

If we equate gravitational and centrifugal acceleration at the edge of the disk we can eliminate v and get $a_g \sim (\sigma R)^{-1}$

----> $\epsilon_g \sim (\sigma R^2)^{-1/2}$ $\sim 10^{-5} << 1$

A more rigorous criterion deals with timescales, highlighting the notion that a system may or may not be treated as collisionless depending on the timescale over which its evolution is considered.
This is particularly useful in astrophysics where we often deal with timescales of processes and compare them to the age of the Universe (~13.7 billion years). This brings in the notion of *relaxation time* (see Springel's lecture notes pag. 8-10 for derivation):

$$t_{\rm relax} = \frac{N}{8 \ln N} t_{\rm cross}$$

where $t_{cross} \sim R/V_p$ is the typical timescale required by a particle to travel through the system on a straight trajectory and $\Lambda = b_{max}/b_{min}$ is the ratio between largest and smallest impact parameters in particle encounters
The relaxation time expresses the timescale over which the deflection of particle trajectories due to the cumulative gravitational effect of encounters with other particles becomes significant ---> *orbits of particles do not reflect anymore the action of the mean potential* $\Phi$ *for* $t > t_{relax}$.
An astrophysical system will be collisionless if its relaxation time is (much) longer than the age of the Universe. As it can be seen, this is a function of *N*, a notion that will extend to superparticle models of physical systems.

()For a galaxy N ~ $10^{11}$, $t_{cross}$ ~ ~ $10^8$ yr ~ Age of the Universe/100

()For a star cluster (eg Globulal Cluster) clearly the approximation does not work: N ~ $10^5$-$10^6$, $t_{cross}$ ~ 1 Myr ~ Age of the Universe/$10^4$ !

()For a halo of (cold) dark matter particles surrounding a typical galaxy, assuming the elusive particles have a mass of ~ 100 GeV, since astronomical measurements tell us the halo should have a mass of ~ $10^{12}$ Mo it follows that N ~ $10^{77}$, and $t_{cross}$ ~ Age of the Universe/10 (a dark halo is ~ 10 times larger than its embedded galaxy albeit characteristic velocities are comparable)
----> *a dark matter halo is the ideal case of a collisionless system.*

# Weakly Correlated and Strongly Correlated (Fluid) systems

Before we move on with discretization and numerical methods for collisionless systems it is instructive to discuss the more general cases, namely when interactions between particles cannot be neglected anymore. The numerical modeling of such systems will be discussed later in the course.

Let us start from weakly correlated systems, namely those for which a small collisional term must be introduced but the wavelength of the collisional processes is still large compared to the characteristic size of the system (or the relaxation time is still relatively long).

$$f(\mathbf{r},\mathbf{v},\mathbf{r'},\mathbf{v'},t) = f_1'(\mathbf{r},\mathbf{v},t)\, f_2'(\mathbf{r'},\mathbf{v'},t) + g(\mathbf{r},\mathbf{v},\mathbf{r'},\mathbf{v'},t)$$

This would be the case for star clusters in astrophysics or weakly collisionless plasma where electromagnetic forces between individual ions cannot be discarded.

The Vlasov equation becomes thus the generalized Boltzmann equation, which in lagrangian form is simply:

$$\frac{df}{dt} = -\int \frac{\mathbf{F}}{m} \cdot \frac{\partial}{\partial \mathbf{v}} g \, d\mathbf{x}' \, d\mathbf{v}' = \left(\frac{\partial f}{\partial t}\right)_c$$

This approximation is used a lot in astroparticle physics and Big Bang cosmology, where g is complex and contains various types of interactions between particles in the standard model of physics. It can also be used to described photons and their interaction with matter, which leads to the radiative transfer equation (we will get back to this at the end of the course)

If the interactions are happening on very short timescales, or the wavelength associated with them is much smaller than the size of the system (which can also happen in photon gases, for example when absorption/scattering is very efficient), then one can move to a moment-based approximation to reduce the dimensionality of the problem and make the calculation computationally feasible.

# N-Body models of collisionless systems

In a gravitational system one solves a simpler, alternative system of ordinary differential equations for the mean potential $\Phi$ rather than solving directly the Vlasov-Poisson equation. The resulting trajectories in phase space should coarsely sample $f$ as long as the relaxation time of the N-Body model is long compare to the duration of the simulation. However a single N-Body experiment with a computer will only represent a noisy representation of $f$ because no ensemble averaging is performed (one should run many experiments and then average out the resulting forces to get trajectories in the mean potential)

The gravitational softening $\varepsilon$ is introduced for both computational eff ciency and to enforce numerical robustness (=smoothness) of the model for the collisionless f uid. To avoid correlation between particles

Note also that the mass of the macro particle used to discretize the model drops out of the equation of motion (no self-force). The trajectory of a macro-particle should thus "trace" the trajectory of real physical particles if the mean potential is sampled well enough to be as close as possible to the actual mean potential. This means again we need large N!

## Calculating dynamics of N-Body systems

**1st -** Compute right side of equation of motion, i.e. gravitational forces --> this gives us accelerations

**2nd -** Integrate the equation of motion in time so that we update velocities and then positions of particles ---> trajectories in phase space under mean potential, equivalent to evolve the distribution function in time, in turn equivalent to solve the Vlasov-Poisson system!