Methods of Proof: A.5.1, A.6.1, A.7.1, A.7.2, A.7.3, A.8.1, A.8.2, A.8.5, A.9.1, A.9.3, A.9.4

MA 403 - Advanced Calculus - Brandiece Berry

A.5.1) Show that $\sqrt{2}$ is irrational by giving an indirect proof

Proof:

Assume $\sqrt{2}$ is rational. Then by definition, $\sqrt{2} = \frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Let us also assume that $\frac{a}{b}$ is in simplest terms, that is that a and b have no common factors. The expression can be rewritten as: $2 = \frac{a^2}{b^2}$ and finally as, $2b^2 = a^2$. It follows that by definition, a is an even number and b is a multiple of a, meaning it is also an even number. But it was given that a is not a multiple of b, and here lies the contradiction $\rightarrow\leftarrow$
$\therefore$ by proof by contradiction, $\sqrt{2}$ is irrational.

This proof relies on the definition of a rational number: r is rational if r = a/b for some integers a and b, with b ne 0. We may assume that a and b have no common factors because otherwise, we would simply reduce a/b by canceling all common factors.

#proof #indirect_proof #irrational

A.6.1) Prove the following assertion by contraposition: If x is irrational, then $x + r$ is irrational for all rational numbers r.

If-Then statement: If **x is irrational** then **x+r is irrational for all rational numbers.**
The Contrapositive: If $x + r$ **is rational for all rational numbers** r, then x **is rational.**

Then there exists $\frac{a}{b}$ such that $x + r = \frac{a}{b}$. It is given that r is already a rational number, let it be given by $\frac{p}{q}$, and therefore x can be written as:
$x = \frac{a}{b} - \frac{p}{q}$
$x = \frac{aq-bp}{bq} = \frac{w}{z}$ where $a, b, p, q, w, z \in \mathbb{Z}$
$\therefore$ by proof by contrapositive, if x is irrational, $x + r$ is irrational for all rational numbers r.
■

#irrational #proof #contrapositive #if_then #algebra_proof

A.7.1) Disprove the statement: For any natural number n the equation

$$4x^2 + x - n = 0$$

has no rational root.

To disprove this statement, a counterexample is given.

Let's assume there is a rational root that satisfies this equation. Meaning, some number for n will provide a rational root for the equation. Consider one possible root:

$x = \frac{-1+\sqrt{(1)^2-4(4)(-n)}}{2(4)} = \frac{a}{b}$

With some algebra

$$\begin{aligned}
\frac{-1+\sqrt{1+16n}}{8} &= \frac{a}{b} \\
-1+\sqrt{1+16n} &= \frac{8a}{b} \\
\sqrt{1+16n} &= \frac{8a}{b}+1 \\
\sqrt{1+16n} &= \frac{8a+b}{b} \\
1+16n &= \left(\frac{8a+b}{b}\right)^2 \\
16n &= \left(\frac{8a+b}{b}\right)^2 - 1 \\
16n &= \frac{64a^2+16ab+b^2}{b^2} - 1 \\
16n &= \frac{64a^2+16ab+b^2}{b^2} - \frac{b^2}{b^2} \\
16n &= \frac{64a^2+16ab}{b^2} \\
n &= \frac{64a^2+16ab}{16b^2} \\
n &= \frac{4a^2+ab}{b^2}
\end{aligned}$$

Let a = 1 and b = 2

$$n = \frac{4(1)^2+(1)(2)}{2^2} = \frac{3}{2}$$

And so the polynomial $4x^2 + x - \frac{3}{2} = 0$ will have rational roots: $x_1 = \frac{1}{2}, x_2 = -\frac{3}{4}$.

#irrational #rational_root #algebra_proof #counterexample

A.7.2) Every prime greater than two is odd. Is the converse true?

If-Then Statement: If **a prime number is greater than two** then **it is odd**.

The Converse: If **a number is odd**, then **it is a prime number greater than two**.

Proof by Counterexample:
This is false. 15 is an odd number greater than 2 that is not prime.

#proof #counterexample #prime #odd #converse

A.7.3) State both the converse and the contrapositive of the assertion "Every differentiable function is continuous." Is there a difference between them? Are they both true?

If-Then Statement: If **a function is differentiable**, then **it is continuous.**

Converse: If **a function is continuous**, then **it is differentiable.**

This is not always true and can be disproved with a counterexample:
$f(x) = |x|$ is an example of a continuous function that is not differentiable.

Contrapositive: If **a function is not continuous**, then **it is not differentiable.**

Since the original conditional statement is true then the contrapositive is also true. A function that is not continuous is also not differentiable.

#proof #contrapositive #converse #differentiable #continuous

A.8.1)
A.8.2)
A.8.5)

A.9.1) Let $\mathbb{R}$ be as usual the set of all real numbers. Express in words what these statements mean and determine whether they are true or not. Do not give proofs; just decide on the meaning and whether you think they are valid or not.

- (a) $\forall x \in \mathbb{R}, x \geq 0$
  This statement is false because it means, "Every x that is in the set of all real numbers must be greater than or equal to zero." The set of real numbers includes numbers less than zero.

- $\exists x \in \mathbb{R}, x \geq 0$
  This statement is true because it means, "There exists a number $x \geq 0$ that resides in $\mathbb{R}$."

- $\forall x \in \mathbb{R}, x^2 \geq 0$
  This statement is true because it means "Every x within $\mathbb{R}$ has a square that is greater than or equal to zero." Even the smallest numbers or the most negative numbers within $\mathbb{R}$ will have a square greater than or equal to zero.

- $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 1$
  This statement is false because it means, "Every x and y in $\mathbb{R}$ has a sum equal 1. " There will be other sums other than 1 for all x and y in $\mathbb{R}$.

- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$
  This statement is false because it means, "Every x and some y in $\mathbb{R}$ have a sum equal 1." There will be an x in $\mathbb{R}$ where a y cannot be added to it to equal 1.

- $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 1$
  This statement is true because it means, "Some number x and every y in $\mathbb{R}$ have a sum equal 1." Any number

within $\mathbb{R}$ has another number that can be added to it and the sum will be 1.

- $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$
  This statement is true because it means "Some x and some y in $\mathbb{R}$ have a sum equal 1." There are several examples: $x = 0.2$ and $y = 0.8$ is one example.

#quantifiers #forall #exists #Reals

A.9.3) Explain what must be done in order to prove an assertion of the following form: (a) $\forall s \in S$ "statement about s" is true. (b) $\exists s \in S$ "statement about s" is true.

- (a) $\forall s \in S$ "statement about s" is true.
  In order to prove a "for all" statement is true, every element of S must hold true for the "statement about s."

- (b) $\exists s \in S$ "statement about s" is true.
  In order to prove a "there exists" statement is true, only 1 element of S must hold true for the "statement about s."

In order to disprove a "for all" statement, a "there exists" statement will be used to show that at least one element of S does not fit the "statement about s."

In order to disprove a "there exists" statement, a "for all" statement will be used to show that not one element of S fits the "statement about s."

#proof #forall #exists #sets

A.9.4) In the preceding exercise suppose that $S = \emptyset$. Could either statement be true? Must either statement be true?

The "for all" statement could be true. The statement could be $\forall s \in S, S = \emptyset$.
The "there exists" statement is always false because there is nothing to verify that "there exists" within the empty set.

No, both statements do not need to be true.

#sets #empty_set #forall #exists