1.6.2)

Does Domain help w/ these?

	Upperbound	max	sup	Lowerbound	min	inf
$\mathbb{N}$	None	None	∞	$-2, 0, -1, 1$	1	1
$\mathbb{Z}$	None	None	∞	None	None	$-\infty$
$\mathbb{Q}$	None	None	∞	None	None	$-\infty$
$\mathbb{R}$	None	None	∞	None	None	$-\infty$
$\{-3,2,5,7\}$	7, 9, 10	7	7	$-3, -5, -10$	-3	-3
$\{x : x^2 < 2\}$	2, 5, 7			$-5, -3, -2$		
$\{x : x^2 - x - 1 < 0\}$	$\frac{1+\sqrt{5}}{2}$			$\frac{1-\sqrt{5}}{2}$		
$\{1/n : n \in \mathbb{N}\}$	None	1	1	$-2, 0, -1, 1$	None	∞
$\{\sqrt[n]{n} : n \in \mathbb{N}\}$	None	None	∞	$-1, 0, 1, 2$	2	2

$\Leftarrow$ pg 35 #1-3

$x^2 < 2 \Leftarrow (-2, 2)$

$0^2 < 2 \quad -1^2 < 2$

$1^2 < 2$

$(-0.61\ldots, 1.618\ldots)$

$x^2 - x - 1 < 0 \Leftarrow \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$

$\frac{+1 \pm \sqrt{1^2 - 4(-1)(1)}}{2(1)}$

$= \frac{1 \pm \sqrt{1^2 + 4}}{2}$

$= \frac{1 \pm \sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

$\sqrt[2]{2}, \sqrt[3]{3}, \sqrt[4]{4} \ldots \rightarrow \sqrt[100]{100} = 1.047, \sqrt[200]{200} = 1.02 \longrightarrow 1$ as $x \to \infty$

$1.41\ldots \quad 1.44, \quad 1.41 \qquad \sqrt[1000]{1000} = 1.006\ldots$

$n^{\frac{1}{n}}$

$n > 1$ b/c no 1^{st} root

$\hookrightarrow n \geq 2$ b/c $n \in \mathbb{N}$

1.6.2 con't

1.6.3) Under what conditions does $\sup E = \max E$?

IF set E has a maximum, then its supremum is that maximum. Some sets, however, don't have a maximum.

1.6.4) Show for every nonempty finite set E that $\sup E = \max E$.

Let $P(N)$: $E_n = \{x_1, x_2, \ldots x_n\}$ where $x_i \in \mathbb{R}$. Show $\max E_n = \sup E_n \ \forall n \in \mathbb{N}$

Show $P(N)$ is true $\forall n \in \mathbb{N}$

Proof) Induction

Let $S \subset \mathbb{N}$ be such that $\forall n \in S$, $P(n) = \text{true}$

BASIS STEP

$n=1$ consider $E_1 = \{x_1\}$. Clearly $x_1 = \max E$. Since $x_1 \leq x_1$, AND $x_1 \in E_1 \Rightarrow \max$ of E_1 $\therefore$ $P(1) = \text{true}$ and $1 \in S$ so $S \neq \emptyset$.

INDUCTION STEP

Let $k \geq 1$ be such that $P(k) = \text{true}$ $(k \in S)$. We know for

Use this $\Rightarrow$ $E_k = \{x_1 \ldots x_k\}$ $\max E_k = \sup E_k$

to show this

Consider $E_{k+1} = \{x_1 \ldots x_k, x_{k+1}\}$

$= \{x_1 \ldots x_k\} \cup \{x_{k+1}\}$ $\Leftarrow$!!

Let $\max E_{k+1} = \{\max\{x_1 \ldots x_k\}, |x_{k+1}|\}$

Let $\max E_{k+1} = \{\max\{x_1 \ldots x_k\}, x_{k+1}\}$

Where positive!

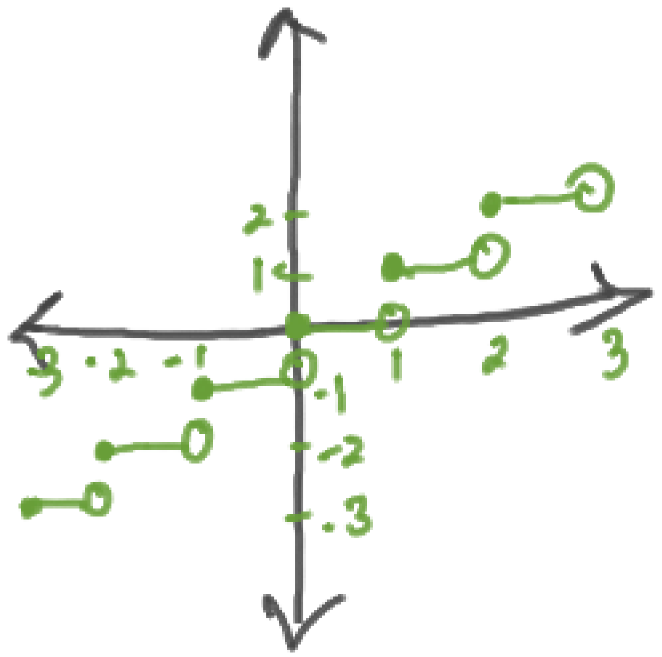
Then $P(k) \Rightarrow P(k+1)$ and $k+1 \in S$, thus $S = \mathbb{N}$

by the principle of mathematical induction (PMI) For

$$E_n = \{x_1 \cdots x_n\} \text{ where } x_i \in \mathbb{R}, \ \max E_n = \sup E_n$$

is true $\forall n \in \mathbb{N}$ ∎

1.6.5) For every $x \in \mathbb{R}$ define $[x] = \max\{n \in \mathbb{Z} : n \le x\}$ called the greatest integer fxn. Show that it is well defined & sketch a graph.



$\max\{n \in \mathbb{Z} : n \le x\}$

The greatest integer fxn results in the integer, n closest to the given real #, x.

Let $A = \{n \in \mathbb{Z} : x \ge n\}$. Then $A \ne \emptyset$. There is $n \in \mathbb{N}$ s.t. $n > -x$, b/c $\mathbb{N}$ is unbounded. But then $-n \in A$.

Let $\alpha = \sup A$. Then there is $n \in A$ such that $\alpha - 1 < n \le \alpha$. But then $\alpha < n+1 < \alpha + 1 \le x + 1$ and so $x \ge n$ and $n+1 \notin A$, that is $n \le x \le n+1$.

If $m \in A$ then $m \le n$ b/c $m > n$ implies $m \ge n+1 > x$. If $m < n$ then $m + 1 \le n \le x$ and m cannot be a solution.

Prove: $2^n \geq n \quad \forall n \in \mathbb{N}$ w/ induction

BASIS STEP:

Let $S \subset \mathbb{N}$ such that $\forall n \in S$, $P(n)$ is true.

Consider $n=1$

$$2^n \geq n$$

$2^1 \geq 1 \Rightarrow 2 \geq 1$ ✓ true so S is nonempty. $\therefore P(1)$ is true and $1 \in S$.

INDUCTION STEP:

Let $k \geq 1$ be such that $P(k)$ is true ($k \in S$)

$k \geq 1$ $\qquad 2^k \geq k$

We know $2^k \geq k$

$P(k+1) \Rightarrow \quad 2^{k+1}$

$= 2^k \cdot 2$

1.6.6) Let A be a set of $\mathbb{R}$. $B = \{-x : x \in A\}$. Find a relation btwn a) $\max A$ and $\min B$ and btwn b) $\min A$ and $\max B$.

$A = \{1, 2, 3\}$

$\max A = 3$

$\sup A = 3$

$\min A = 1$

$\inf = 1$

$B = \{-3, -2, -1\}$

$\max B = -1$

a) $\max B = -\min A$

b) $\min B = -\max A$

1.6.16) Let E be a set of $\mathbb{R}$.

$\Longrightarrow$ (If x is not an upper band of E, then $\exists e \in E$ such that $x < e$)

Contradiction (use the negation of the conclusion)

Assume x is not an upper band of E and $\forall e \in E$ $x \geq e$

Since x is greater than all elements of E, by definition of upper band $e \leq x$ $\forall e \in E$. Therefore x is an upper band.

$\times$

Assumption gives contradiction.

$\Longleftarrow$ (If $\exists e \in E$ such that $x < e$, then x is not an upper band of E)

Assume $\exists e \in E$ such that $x < e$ and x is an upper band of E

But that would mean

$e \leq x$ $\forall e \in E$ $\times$

1.6.17) Let A be a set of $\mathbb{R}$. Show that a real # x is a sup of A iff $a \leq x$ for all $a \in A$ and for every positive # ϵ there is an element $a' \in A$ such that $x - \epsilon < a'$.

If x is the sup A, then $\forall a \in A \quad a \leq x$ and $\forall \epsilon > 0, \exists a' \in A$ such that $x - \epsilon < a'$

Assume $x = \sup A$. By def. $x \geq a \quad \forall a \in A$.

Let $\epsilon > 0$. Assume $\forall a' \in A$ (negation) $x - \epsilon \geq a'$. This means $x - \epsilon$ is an upper band. Since $x - \epsilon < x$. Since $x = \sup A$ is the lowest upper band, $x - \epsilon$ is smaller then x $\times$

$\therefore x - \epsilon < a' \quad \forall a' \in A$ ∎