2.25 a) 7, 4, 1, -2, -5, -8

-3 -3

arithmetic sequence

$$x_n = 7 + (n-1)(-3)$$

$$= 7 + (-3n+3)$$

$$x_n = -3n + 10$$

check:

$x_1 = -3(1)+10 = 7$ ✓

$x_2 = -3(2)+10 = 4$ ✓

$x_3 = -3(3)+10 = 1$ ✓

b) 0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001

geometric sequence

$$x_n = 0.1(10^{1-n})$$

$x_1 = 0.1(10^{1-1}) = 0.1$ ✓

$x_2 = 0.1(10^{1-2}) = 0.01$ ✓

$x_3 = 0.1(10^{1-3}) = 0.001$ ✓

2.25

c) $2, \sqrt{2}, 1, \ldots$

$2^{1} \quad 2^{1/2} \quad 2^{0} \quad 2^{-1/2}, \; 2^{-1}, \; 2^{-3/2}$

$n=1 \quad n=2 \quad n=3 \quad n=4$

geometric sequence w/ arithmetic sequence in powers

$2^{n} \leftarrow$ arithmetic sequence: $\overset{-1/2}{1}, \overset{-1/2}{1/2}, \overset{-1/2 = d}{0}, -1/2, \ldots$

$$x_n = c + (n-1)d$$
$$= 1 + (n-1)(-1/2)$$
$$= 1 + \frac{(-n+1)}{2} = \frac{2-n+1}{2}$$
$$= \frac{3-n}{2}$$

$$x_n = 2^{\frac{3-n}{2}}$$

Check:

$n=1$: $x_1 = 2^{\frac{3-1}{2}} = 2^{1}$ ✓

$n=2$: $x_2 = 2^{\frac{3-2}{2}} = 2^{1/2}$ ✓

$n=3$: $x_3 = 2^{\frac{3-3}{2}} = 2^{0}$ ✓

2.26) $x_1 = \sqrt{2}$ $x_n = \sqrt{2} + x_{n-1}$

$\sqrt{2}$, $\sqrt{2} + \sqrt{2}$, $\sqrt{2} + \sqrt{2} + \sqrt{2}$

$\sqrt{2}$, $2\sqrt{2}$, $3\sqrt{2}$ . . .

$x_n = n\sqrt{2}$

2.27) $x_1 = \sqrt{2}$ $x_n = \sqrt{2}\, x_{n-1}$

$\sqrt{2}$, $\sqrt{2} \cdot \sqrt{2}$, $\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2}$

$x_n = (\sqrt{2})^n$

2.2.8) $X_1 = \sqrt{2}$ Show by induction

$x_n = \sqrt{2 + x_{n-1}}$ $X_n < 2$ For all n.

Let $S \subset \mathbb{N}$ such that $\forall n \in S$, $P(n)$ = true

Basis Step: Consider $n=1$.

$X_1 < 2$

$\sqrt{2} < 2$ ✓

∴ P(1) is true and $1 \in S$ so $S \neq \emptyset$

Induction Step: consider $k \geq 1$ such that $P(k)$ is true and $k \in S$

$X_k < 2$

$X_{k+1} < 2$

$X_{k+1} = \sqrt{2 + X_k}$

$(\sqrt{2 + X_k})^2 < (2)^2$

$2 + X_k < 4$

$-2 \quad -2$

$X_k < 2$

2.2.9) $x_1 = \sqrt{2}$

$x_n = \sqrt{2 + x_{n-1}}$

Show by induction: $x_n < x_{n+1} \quad \forall n$

Let $S \subset \mathbb{N}$ such that $\forall n \in S$, $P(n) =$ true

Basis Step: Consider $n=1$

$$x_1 < x_2$$
$$\sqrt{2} < \sqrt{2+\sqrt{2}}$$
$$1.414 < 1.847$$

$\therefore$ P(1) is true and $1 \in S$ so $S \neq \emptyset$.

Induction Step: Assume $k \geq 1$ such that $P(k)$ is true and $k \in S$

hypothesis

Consider $x_k < x_{k+1}$

$x_{k+1} < x_{k+2}$

$\sqrt{2+x_k} < \sqrt{2+x_{k+1}}$

$x_1 < x_2$

$\sqrt{2} < \sqrt{2+\sqrt{2}}$

$1.414 < 1.847$

$$\left(\sqrt{2+x_k}\right)^2 < \left(\sqrt{2+x_{k+1}}\right)^2$$

$$\cancel{2} + x_k < \cancel{2} + x_{k+1}$$

$$x_k < x_{k+1} \Longleftarrow$$

Induction Step:

$$x_k < x_{k+1}$$

$$+2 \quad +2$$

$$\Leftrightarrow 2 + x_k < 2 + x_{k+1}$$

$$\Leftrightarrow \sqrt{2+x_k} < \sqrt{2+x_{k+1}}$$

Hence, $x_{k+1} < x_{k+2}$.

So,

$P(k) \Rightarrow P(k+1)$ and $S = \mathbb{N}$.

$\therefore$ By PMI For $x_1 = \sqrt{2}$, $x_n = \sqrt{2 + x_{n-1}}$, $x_n \le x_{n+1}$ $\forall n \in \mathbb{N}$