Brandiece Berry - Advanced Calculus Final Exam - SPR 2022

Problem 1
Show $\mathbb{Q}$ is a field with the field axioms.
$\mathbb{Q}$ is the set of rational numbers of the form $\frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.

Field Axioms:
A1 - *closure under addition and commutative property of addition*:
For any $a, b \in \mathbb{R}$ there is a number $a + b \in \mathbb{R}$ and $a + b = b + a$.

A1 closure) Let $x, y \in \mathbb{Q}$. By definition of the set of rational numbers,

$$x = \frac{a}{b} \quad y = \frac{c}{d}$$

where $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$. It is given that both $\mathbb{Z}$ and $\mathbb{N}$ are closed under addition and multiplication. It follows that:

$$x + y = \frac{a}{b} + \frac{c}{d}$$

and with some algebra

$$\frac{ad + cb}{bd}$$

It follows that $ad, cb,$ and $ad + cb \in \mathbb{Z}$ and $bd \in \mathbb{N}$ and therefore $\frac{ad+cb}{bd} \in \mathbb{Q}$, by the definition of rational numbers.
The addition of rational numbers creates a rational number, so $\mathbb{Q}$ is closed under addition.

A1 Commutative)
Let $x, y \in \mathbb{Q}$. By definition of the set of rational numbers, $x = \frac{a}{b} \quad y = \frac{c}{d}$
where $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$. It is given that both $\mathbb{Z}$ and $\mathbb{N}$ are closed under addition and multiplication. It follows that:

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

and

$$y + x = \frac{c}{d} + \frac{a}{b} = \frac{cb + ad}{bd}$$

To verify that $x + y = y + x$,

$$\begin{gathered}
\frac{ad + cb}{bd} = \frac{cb + ad}{bd} \\
\frac{ad}{bd} + \frac{cb}{bd} = \frac{cb}{bd} + \frac{ad}{bd} \\
\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b} \\
x + y = y + x
\end{gathered}$$

A2 - *associative property of addition* :
For any $a, b, c \in \mathbb{R}$ the identity $(a+b)+c = a+(b+c)$ is true.

Let $x, y, z \in \mathbb{Q}$, consider $(x+y)+z$ and given that $x = \frac{a}{b}$, $y = \frac{c}{d}$, and $z = \frac{f}{g}$

$$(x+y)+z = \left(\frac{a}{b}+\frac{c}{d}\right)+\frac{f}{g}$$

Using the fact that

$$\frac{a}{b}+\frac{c}{d} = \frac{ad+cb}{bd}$$

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{f}{g} = \left(\frac{ad+cb}{bd}\right)+\frac{f}{g} = \frac{g(ad+cb)+f(bd)}{bdg} = \frac{adg+cbg+fbd}{bdg}$$

Next, consider $x+(y+z)$

$$x+(y+z) = \frac{a}{b}+\left(\frac{c}{d}+\frac{f}{g}\right) = \frac{a}{b}+\left(\frac{cg+df}{dg}\right) = \frac{b(cg+df)+a(dg)}{bdg} = \frac{bcg+bdf+adg}{bdg}$$

Verify that $(x+y)+z = x+(y+z)$
Setting the two expressions equal to each other, simplifying, and utilizing A1:

$$\frac{adg+cbg+fbd}{bdg} = \frac{bcg+bdf+adg}{bdg}$$
$$\frac{adg}{bdg}+\frac{cbg}{bdg}+\frac{fbd}{bdg} = \frac{cgb}{bdg}+\frac{bdf}{bdg}+\frac{adg}{bdg}$$
$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{f}{g} = \frac{a}{b}+\left(\frac{c}{d}+\frac{f}{g}\right)$$

It follows that $(x+y)+z = x+(y+z)$.

A3 - *existence of a zero element*:
There is a unique number $0 \in \mathbb{R}$ so that, for all $a \in \mathbb{R}$, $a+0 = 0+a = a$.

Consider $x \in \mathbb{Q}$, where $x = \frac{a}{b}$, and $a \in \mathbb{Z}$ and $b \in \mathbb{N}$

$$x+0 = \frac{a}{b}+\frac{0}{1} = \frac{a}{b}+\frac{0}{1}\cdot\frac{b}{b} = \frac{a+0b}{b} = \frac{a}{b}$$

A4 - *existence of a negative element* :
For any number $a \in \mathbb{R}$ there is a corresponding number denoted by $-a$ with the property that $a+(-a) = 0$.

Consider $x \in \mathbb{Q}$, where $x = \frac{a}{b}$, and $a \in \mathbb{Z}$ and $b \in \mathbb{N}$

$$x+(-x) = \frac{a}{b}+(-\frac{a}{b}) = \frac{a}{b}$$

Since $b \in \mathbb{N}$ it cannot be negative and so it follows

$$\frac{a}{b}+(-\frac{a}{b}) = \frac{a}{b}+(\frac{-a}{b}) = \frac{a-a}{b} = \frac{0}{b} = 0$$

M1 - *closure under multiplication and commutative property of multiplication*:
For any $a, b \in \mathbb{R}$ there is a number $ab \in \mathbb{R}$ and $ab = ba$.
M1 Closure) Let $x, y \in \mathbb{Q}$. By definition of the set of rational numbers,

$$x = \frac{a}{b} \quad y = \frac{c}{d}$$

where $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$. It is given that both $\mathbb{Z}$ and $\mathbb{N}$ are closed under addition and multiplication. It follows that:

$$xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

It follows that $ac \in \mathbb{Z}$ and $bd \in \mathbb{N}$ and therefore $\frac{ac}{bd} \in \mathbb{Q}$ by the definition of rational numbers.
The product of rational numbers creates another rational number, so $\mathbb{Q}$ is closed under multiplication

M1 Commutative Property) Consider $x \cdot y$, given that $x = \frac{a}{b}$ and $y = \frac{c}{d}$

$$x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d} \cdot \frac{a}{b} = y \cdot x$$

M2 - *associative property of multiplication*:
For any $a, b, c \in \mathbb{R}$ the identity $(ab)c = a(bc)$ is true.
Let $x, y, z \in \mathbb{Q}$, consider$(xy)z$, given that $x = \frac{a}{b}$, $y = \frac{c}{d}$, and $z = \frac{f}{g}$. Because of M1, $xy = \frac{ac}{bd} \in \mathbb{Q}$ and it is given that $\frac{f}{g} \in \mathbb{Q}$.
It follows that

$$(xy)z = \left(\frac{a}{b} \cdot \frac{c}{d}\right)\frac{f}{g} = \frac{acf}{bdg} = \frac{a}{b}\left(\frac{cf}{dg}\right) = \frac{a}{b}\left(\frac{c}{d} \cdot \frac{f}{g}\right) = x(yz)$$

M3 - *identity property of multiplication*:
There is a unique number $1 \in \mathbb{R}$ so that $a1 = 1a = a$ for all $a \in \mathbb{R}$.
Consider $x \in \mathbb{Q}$, where $x = \frac{a}{b}$, and $a \in \mathbb{Z}$ and $b \in \mathbb{N}$

$$x \cdot 1 = 1x = 1\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right) \cdot 1 = \frac{a}{b}$$

M4 - *inverse property of multiplication*:
For any number $a \in \mathbb{R}, a \neq 0$, there is a corresponding number denoted a^{-1} with the property that $aa^{-1} = 1$.
Consider $x \in \mathbb{Q}$, where $x = \frac{a}{b}$and $a \in \mathbb{Z}$ and $b \in \mathbb{N}$

$$x \cdot x^{-1} = \frac{a}{b}\left(\frac{a}{b}\right)^{-1}$$

By the rules of exponents

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

It follows that

$$\frac{a}{b}\left(\frac{a}{b}\right)^{-1} = \frac{a}{b}\left(\frac{b}{a}\right) = \frac{ab}{ba}$$

By M1 $ab = ba$ so

$$\frac{ab}{ba} = \frac{ba}{ba} = \frac{b}{b} \cdot \frac{a}{a} = 1 \cdot 1 = 1$$

AM1 - *distributive property*:
For any $a, b, c \in \mathbb{R}$ the identity $(a + b)c = ac + bc$ is true.
Let $x, y, z \in \mathbb{Q}$, given that $x = \frac{a}{b}$, $y = \frac{c}{d}$, and $z = \frac{f}{g}$. It follows that:

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

Consider $(x + y)z$

$$\begin{aligned}
(x + y)z &= \left(\frac{ad + cb}{bd}\right)\frac{f}{g} \\
&= \frac{f(ad + cb)}{bdg} \\
&= \frac{adf + cbf}{bdg} \\
&= \frac{adf}{bdg} + \frac{cbf}{bdg}
\end{aligned}$$

Next, consider $xz + yz$

$$\begin{gathered}
\frac{a}{b} \cdot \frac{f}{g} + \frac{c}{d} \cdot \frac{f}{g} \\
\frac{af}{bg} + \frac{cf}{dg} \\
\frac{d(af)}{bdg} + \frac{b(cf)}{bdg} \\
\frac{adf}{bdg} + \frac{cbf}{bdg}
\end{gathered}$$

It follows that $(x + y)z = xz + yz$.

Problem 2
Show that

$$|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$$

For any numbers $x_1, x_2, \ldots, x_n$
Proof:
Let $S \subset \mathbb{N}$ such that $\forall n \in S$, $P(n)$ is true.
Basis Step: Consider $n = 1$.
$|x_1| \leq |x_1|$ ✓

$\therefore P(1)$ is true and $1 \in S$ so S is not empty.
Induction Step: Assume $k \geq 1$ such that $P(k)$ is true and $k \in S$

$$|x_1 + x_2 + \ldots + x_k| \leq |x_1| + |x_2| + \ldots + |x_k|$$

It follows that:

$$|x_1 + x_2 + \ldots + x_k| \leq |(x_1 + x_2 + \ldots + x_{k-1}) + x_k|$$

By the Triangle Inequality, $|x + y| \leq |x| + |y|$,

$$|(x_1 + x_2 + \ldots + x_{k-1}) + x_k| \leq |x_1 + x_2 + \ldots + x_{k-1}| + |x_k|$$

using the inductive hypothesis

$$|x_1 + x_2 + \ldots + x_{k-1}| \leq |x_1| + |x_2| + \ldots + |x_{k-1}|$$

Problem 3
Consider the sequence defined recursively by

$$x_1 = \sqrt{2}, \quad x_n = \sqrt{2 + x_{n-1}}$$

Show by induction that $x_n \leq x_{n+1}$ for all n.
Proof:
Let $S \subset \mathbb{N}$ such that $\forall n \in S$, $P(n)$ is true.
Basis Step: Consider $n = 1$

$$\begin{aligned} x_1 &< x_2 \\ \sqrt{2} &< \sqrt{2 + \sqrt{2}} \\ 1.414 &< 1.847 \ \checkmark \end{aligned}$$

$\therefore P(1)$ is true and $1 \in S$ so S is not empty.
Induction Step: Assume $k \geq 1$ such that $P(k)$ is true and $k \in S$
Consider $x_k < x_{k+1}$, with some algebra

$$\begin{aligned} x_k &< x_{k+1} \\ +2 &\quad +2 \\ 2 + x_k &< 2 + x_{k+1} \\ \sqrt{2 + x_k} &< \sqrt{2 + x_{k+1}} \\ \text{Hence,} \quad x_{k+1} &< x_{k+2} \end{aligned}$$

So, $P(k) \Rightarrow P(k+1)$ and $S = \mathbb{N}$.

$\therefore$ By PMI for $x_1 = \sqrt{2}, x_n = \sqrt{2 + x_{n-1}} \quad \forall n \in \mathbb{N}, x_n \leq x_{n+1}$.
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Problem 4
If $\{s_n\}$ is a sequence of positive number converging to 0, show that $\{\sqrt{s_n}\}$ also converges to zero.

Let $\epsilon > 0$. Since s_n is convergent, we can find an $N \in \mathbb{N}$ such that $\forall n > N$,

$$|s_n| < \epsilon^2$$

Since $s_n > 0, |s_n| = s_n$
Therefore $s_n < \epsilon^2$
With some algebra

$$\begin{aligned} s_n &< \epsilon^2 \\ \sqrt{s_n} &< \sqrt{\epsilon^2} \\ |\sqrt{s_n} - 0| &< \epsilon \end{aligned}$$

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Problem 5

Which statements are true?

1. If $\{s_n\}$ and $\{t_n\}$ are both divergent then so is $\{s_n + t_n\}$. True
2. If $\{s_n\}$ and $\{t_n\}$ are both divergent then so is $\{s_n t_n\}$. True
3. If $\{s_n\}$ and $\{s_n + t_n\}$ are both convergent then so is $\{t_n\}$. False
4. If $\{s_n\}$ and $\{s_n t_n\}$ are both convergent then so is $\{t_n\}$. True
5. If $\{s_n\}$ convergent then so too is $\{\frac{1}{s_n}\}$. True
6. If $\{s_n\}$ convergent then so too is $\{(s_n)^2\}$. True
7. If $\{(s_n)^2\}$convergent then so too is $\{s_n\}$. False