MA 668-Numerical Analysis I
Nonlinear Equations in One Variable

Jaman Mohebujjaman

Department of Mathematics
University of Alabama at Birmingham (UAB)

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Finding Roots

We will focus on finding solutions to scalar, nonlinear equation

$$f(x) = 0, \quad x \in [a, b],$$

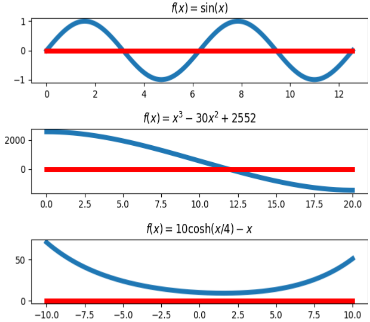
under the assumption $f \in C[a, b]$. We will denote a solution by x^*.

Example: $f(x) = x - 1$, on the interval $[a, b] = [0, 2]$, has the only one solution $x^* = 1$.

Finding Roots

Examples:

1. $f(x) = \sin(x)$, the solutions are $x^* = n\pi, \quad n \in \mathbb{Z}$. It has five roots in the interval $[a, b] = [0, 4\pi]$.
2. $f(x) = x^3 - 30x^2 + 2552$, $0 \leq x \leq 20$, has one root in the given interval.
3. $f(x) = 10\cosh\left(\frac{x}{4}\right) - x, \quad x \in [-10, 10]$. No roots.



Python code

```
import numpy as np
import matplotlib.pyplot as plt

def my_pol(x):
  return x**3-30*x**2+2552

def my_cosh(x):
  return 10*np.cosh(x/4.0)-x

t1=np.linspace(0,4*np.pi,1000)
t2=np.linspace(0,20,1000)
t3=np.linspace(-10,10,1000)
```

Python code

```
fig, axs = plt.subplots(3)
axs[0].plot(t1,np.sin(t1),t1,0*t1,'r-',linewidth=5)
axs[1].plot(t2,my_pol(t2),t2,0*t2,'r-',linewidth=5)
axs[2].plot(t3,my_cosh(t3),t3,0*t3,'r-',linewidth=5)
axs[0].set_title('$f(x)=\sin(x)$')
axs[1].set_title('$f(x)=x^3-30x^2+2552$')
axs[2].set_title('$f(x)=10\cosh(x/4)-x$')
plt.tight_layout()
plt.show()
```

Iterative Methods for Finding Roots

- Initial guess x_0:
  - Plot $f(x)$.
  - Look for locations where $f(x)$ changes sign: If $f(a) \cdot f(b) < 0$ then $f(x)$ has a root in $[a, b]$.
- Generates a sequence of iterates $\{x_1, x_2, \cdots, x_k, \cdots\}$
- Stopping criteria: $|x_n - x_{n-1}| < atol$, and/or $\frac{|x_n - x_{n-1}|}{|x_n|} < rtol$, and/or $|f(x_n)| < ftol$, where *atol*, *rtol*, and *ftol* are user-specified constants.
  - Often the relative criterion is more robust than the absolute one.
  - a favorite combination uses $\frac{|x_n - x_{n-1}|}{1+|x_n|} < tol$
- Desirable properties of root finding algorithms:
  - Efficient- requires a small number of function evaluations.
  - Robust-fails rarely, if fails it announce.
  - Require a minimal amount of additional data such as the function's derivative.
  - Requires f to satisfy only minimal smoothness properties.
  - Scalable.

Algorithm 1: Bisection Method

Given $f(x) \in C[a, b]$, with $f(a) \cdot f(b) < 0$.
Step 1: Evaluate $f(p)$ where $p = \frac{a+b}{2}$. If $|x^* - p| \leq atol$ then $x^* \approx p$, STOP.
Step 2: If $f(a) \cdot f(p) < 0$:
$b \leftarrow p$. Go to Step 1.
else if $f(a) \cdot f(p) > 0$:
$a \leftarrow p$. Go to Step 1.
else $x^* = p$ is the root, STOP.

Pros

- Most simple method
- Requires minimal assumptions on $f(x)$

Cons

- Slow in convergence
- Difficult to generalize to higher dimensions.

Bisection Method

After a total of n iterations the algorithm is guaranteed to converge if

$$|x^* - x_n| \leq \frac{b-a}{2} 2^{-n} \leq atol,$$

which gives

$$n = \left\lceil \log_2 \left(\frac{b-a}{2\ atol} \right) \right\rceil.$$

Bisection Method

Examples:

1. Apply the bisection routine, to find the root of the function

$$f(x) = \sqrt{x} - 1.1$$

   starting from the interval $[0, 2]$ with $atol = 0.1$.

2. Write a Python code of the Bisection algorithm and find the root of $f(x) = \sqrt{x} - 1.1$ with $[a, b] = [0, 2]$ and $atol = 1.e - 8$. Show the results in a table with columns as: Iteration number, p value, and $|f(p)|$ value.
3. Write a Python code of the Bisection algorithm using the "while" loop and test it in Problem 2.
4. Write a Python code of the Bisection algorithm using the "Recursive function" and test it in Problem 2.

Fixed Point Iteration Method (FPIM)

- Unlike the Bisection method, the Fixed Point Iteration method is scalable.
- Let $f(x) = 0$ be rewritten as $x = g(x)$, e.g.,

$$x(x-1) = 0 \Leftrightarrow x = x^2.$$

- Starting from an initial guess x_0, generate a sequence $\{x_1, x_2, \cdots, x_k, \cdots\}$ as

$$x_{k+1} = g(x_k)$$

so that a **fixed point** x^* can be found such that

$$x^* = g(x^*).$$

- The convergence of the above sequence depends on the choice of $g(x)$ (not all choices are good).

Fixed Point Iteration Method

Some choice of $g(x)$ can be

- $g(x) = x - f(x)$,
- $g(x) = x + 10f(x)$,
- $g(x) = x - \frac{f(x)}{f'(x)}$, assuming $f'(x)$ exists and $f'(x) \neq 0$.

Algorithm 2: Fixed Point Iteration Method

Given $f(x) \in C[a, b]$, $?g(x) \ni f(x) = 0 \Leftrightarrow x = g(x)$.
Step 1: Initial guess x_0.
Step 2: For $k = 0, 1, 2, \cdots$

$$x_{k+1} = g(x_k)$$

continue until the termination criterion is satisfied.

Fixed Point Iteration Method

Existence and uniqueness: Let $g(x) \in C[a, b]$

1. Is there a fixed point $x^* \in [a, b]$?
2. If yes, is it unique?

Fixed Point Iteration Method

Theorem (Fixed Point)

If $g \in C[a,b]$, and $g(x) \in [a,b] \;\; \forall x \in [a,b]$, then there exists a fixed point $x^* \in [a,b]$.
If, in addition, g' exists and $\exists \underbrace{\rho}_{\in \mathbb{R}^+} < 1 \ni |g'(x)| \leq \rho \;\; \forall x \in (a,b)$, then the fixed point x^* is unique in the interval.

Fixed Point Iteration Method

Convergence:

- Does the sequence $\{x_1, x_2, \cdots, x_k, \cdots\}$ with $x_{k+1} = g(x_k)$ converges to a fixed point x^*?

Fixed Point Iteration Method

Convergence:

- Does the sequence $\{x_1, x_2, \cdots, x_k, \cdots\}$ with $x_{k+1} = g(x_k)$ converges to a fixed point x^*?

Proof.

Under the above-stated assumption, we have

$$|x_{k+1} - x^*| = |g(x_k) - g(x^*)| \overset{\text{MVT}}{=} |g'(\xi)(x_k - x^*)| \leq \underbrace{\rho}_{<1} |x_k - x^*|,$$

which is a *contraction* by the factor ρ. Thus,

$$|x_{k+1} - x^*| \leq \rho|x_k - x^*| \leq \rho^2|x_{k-1} - x^*| \leq \cdots \leq \rho^{k+1}|x_0 - x^*|.$$

Now

$$\lim_{k\to\infty} |x_{k+1} - x^*| \leq \lim_{k\to\infty} \rho^{k+1}|x_0 - x^*| \implies \lim_{k\to\infty} x_{k+1} = x^*.$$

Fixed Point Iteration Method

Example: For the function $g(x) = e^{-x}$ on $[0.2, 1.2]$, find a fixed point x^* satisfying $x^* = e^{-x^*}$ with an initial guess $x_0 = 1$.

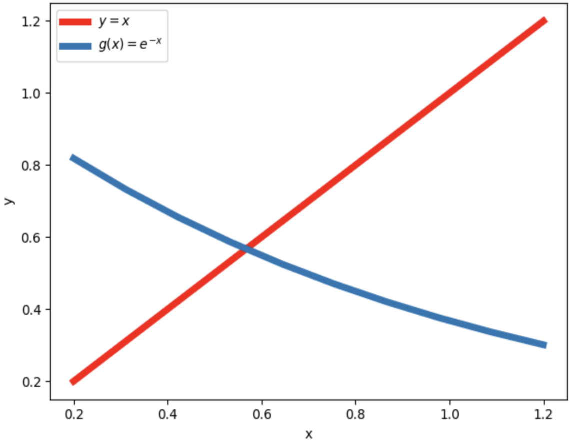


Figure: Graphs of the functions $y = x$, and $g(x) = e^{-x}$.

Fixed Point Iteration Method

Example: For the function $g(x) = e^{-x}$ on $[0.2, 1.2]$, find a fixed point x^* satisfying $x^* = e^{-x^*}$ with an initial guess $x_0 = 1$.
Solution: $g \in C[0.2, 1.2]$, monotonically decreasing, $g(0.2) \approx 0.82$, and $g(1.2) \approx 0.30$, thus $g(x) \in [0.2, 1.2] \implies \exists$ a fixed point $x^* \in [0.2, 1.2]$.

$$
\begin{aligned}
x_1 &= g(x_0) = g(1) = e^{-1} \approx 0.37 \\
x_2 &= g(x_1) = g(0.37) = e^{-0.37} \approx 0.69 \\
x_3 &= g(x_2) = g(0.69) = e^{-0.69} \approx 0.50 \\
x_4 &= g(x_3) = g(0.50) = e^{-0.50} \approx 0.61 \\
x_5 &= g(x_4) = g(0.61) = e^{-0.61} \approx 0.54 \\
x_6 &= g(x_5) = g(0.54) = e^{-0.54} \approx 0.58 \\
x_7 &= g(x_6) = g(0.58) = e^{-0.58} \approx 0.56 \\
x_8 &= g(x_7) = g(0.56) = e^{-0.56} \approx 0.57 \\
x_9 &= g(x_8) = g(0.57) = e^{-0.57} \approx 0.57.
\end{aligned}
$$

Consider $f(x) = 2\cosh(\frac{x}{4}) - x$, which has two roots at x_1^*, and x_2^*, and we can bracket them as $2 \leq x_1^* \leq 4,\ \ 8 \leq x_2^* \leq 10$.

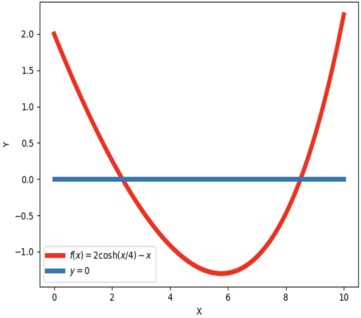


Figure: Roots of $f(x) = 2\cosh(\frac{x}{4}) - x$.

Speed of Convergence

Bisection Method: The speed of convergence with $\rho = \frac{1}{2}$. Here $a = 2$, $b = 4$, and $atol = 1e-8$.

```
Iter=  1    p= 3.0000000000  |f(p)|= 0.4106334306
Iter=  2    p= 2.5000000000  |f(p)|= 0.0964926140
Iter=  3    p= 2.2500000000  |f(p)|= 0.0748374817
Iter=  4    p= 2.3750000000  |f(p)|= 0.0119814777
.
.
.
Iter=  23   p= 2.3575508595  |f(p)|= 0.0000001338
Iter=  24   p= 2.3575509787  |f(p)|= 0.0000000517
Iter=  25   p= 2.3575510383  |f(p)|= 0.0000000107
Iter=  26   p= 2.3575510681  |f(p)|= 0.0000000098
Iter=  27   p= 2.3575510532  |f(p)|= 0.0000000005
```

We found $x_1^* = 2.35755106061697$

Speed of Convergence

Bisection Method: The speed of convergence with $\rho = \frac{1}{2}$. Here $a = 8$, $b = 10$, and $atol = 1e - 8$.

```
Iter=  1    p= 9.0000000000  |f(p)|= 0.5931350609
Iter=  2    p= 8.5000000000  |f(p)|= 0.0076695436
Iter=  3    p= 8.7500000000  |f(p)|= 0.2750998717
Iter=  4    p= 8.6250000000  |f(p)|= 0.1294401872
.
.
.
Iter=  23   p= 8.5071995258  |f(p)|= 0.0000000479
Iter=  24   p= 8.5071996450  |f(p)|= 0.0000000793
Iter=  25   p= 8.5071995854  |f(p)|= 0.0000000157
Iter=  26   p= 8.5071995556  |f(p)|= 0.0000000161
Iter=  27   p= 8.5071995705  |f(p)|= 0.0000000002
```

We found $x_2^* = 8.507199577987194$

Fixed Point Iteration Method: $g(x) = 2\cosh\left(\frac{x}{4}\right)$, $g'(x) = \frac{1}{2}\sinh(\frac{x}{4})$ near x_1^*, $|g'(x)| < 0.4$, and near x_2^*, $|g'(x)| > 1$.

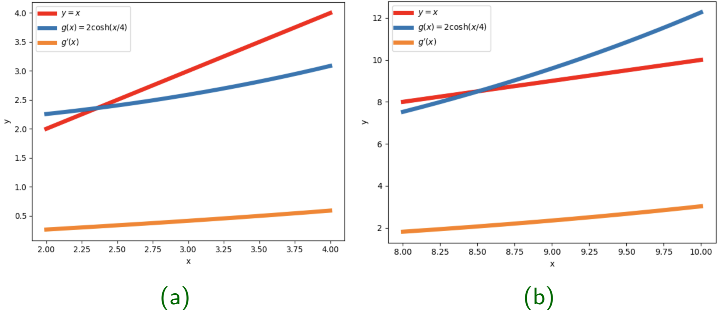


Figure: Roots of $f(x) = 2\cosh\frac{x}{4} - x$ (a) $2 \le x_1^* \le 4$, and (b) $8 \le x_2^* \le 10$.

Speed of Convergence

Fixed Point Iteration Method: Initial guess $x_0 = 2$, $atol = 1e-8$

```
n= 1    p= 2.2552519304   g(p)= 2.3263957274   |f(p)|= 0.0711437969
n= 2    p= 2.3263957274   g(p)= 2.3479003113   |f(p)|= 0.0215045840
n= 3    p= 2.3479003113   g(p)= 2.3545463562   |f(p)|= 0.0066460449
. . .
n= 9    p= 2.3575421720   g(p)= 2.3575482822   |f(p)|= 0.0000061103
n= 10   p= 2.3575482822   g(p)= 2.3575501890   |f(p)|= 0.0000019067
n= 11   p= 2.3575501890   g(p)= 2.3575507840   |f(p)|= 0.0000005950
n= 12   p= 2.3575507840   g(p)= 2.3575509697   |f(p)|= 0.0000001857
n= 13   p= 2.3575509697   g(p)= 2.3575510276   |f(p)|= 0.0000000579
n= 14   p= 2.3575510276   g(p)= 2.3575510457   |f(p)|= 0.0000000181
n= 15   p= 2.3575510457   g(p)= 2.3575510513   |f(p)|= 0.0000000056
n= 16   p= 2.3575510513   g(p)= 2.3575510531   |f(p)|= 0.0000000018
```

Speed of Convergence

Fixed Point Iteration Method: Initial guess $x_0 = 4$, $atol = 1e - 8$

```
n= 1    p= 3.0861612696   g(p)= 2.6253959730   |f(p)|= 0.4607652967
n= 2    p= 2.6253959730   g(p)= 2.4464830853   |f(p)|= 0.1789128877
n= 3    p= 2.4464830853   g(p)= 2.3858876707   |f(p)|= 0.0605954146
...
n= 10   p= 2.3575774751   g(p)= 2.3575592988   |f(p)|= 0.0000181763
n= 11   p= 2.3575592988   g(p)= 2.3575536267   |f(p)|= 0.0000056720
n= 12   p= 2.3575536267   g(p)= 2.3575518568   |f(p)|= 0.0000017700
n= 13   p= 2.3575518568   g(p)= 2.3575513044   |f(p)|= 0.0000005523
n= 14   p= 2.3575513044   g(p)= 2.3575511321   |f(p)|= 0.0000001724
n= 15   p= 2.3575511321   g(p)= 2.3575510783   |f(p)|= 0.0000000538
n= 16   p= 2.3575510783   g(p)= 2.3575510615   |f(p)|= 0.0000000168
n= 17   p= 2.3575510615   g(p)= 2.3575510563   |f(p)|= 0.0000000052
n= 18   p= 2.3575510563   g(p)= 2.3575510546   |f(p)|= 0.0000000016
```

Fixed Point Iteration Method: Initial guess $x_0 = 9$ $atol = 1e-8$

```
n= 1  p= 9.59314    g(p)= 11.09515  |f(p)|= 1.50201
n= 2  p= 11.09515   g(p)= 16.08161  |f(p)|= 4.98646
n= 3  p= 16.08161   g(p)= 55.74146  |f(p)|= 39.65986
n= 4  p= 55.74146   g(p)= 1127334.04079  |f(p)|= 1127278.29933
---------------------------------------------------------------------------
OverflowError                             Traceback (most recent call last)
```

Speed of Convergence

Fixed Point Iteration Method: Initial guess $x_0 = 8$ $atol = 1e - 8$

```
n= 1    p= 7.52439138   g(p)= 6.71312623  |f(p)|= 0.81126516
n= 2    p= 6.71312623   g(p)= 5.54303801  |f(p)|= 1.17008822
n= 3    p= 5.54303801   g(p)= 4.24799489  |f(p)|= 1.29504312
...
n= 17   p= 2.35755208   g(p)= 2.35755137  |f(p)|= 0.00000071
n= 18   p= 2.35755137   g(p)= 2.35755115  |f(p)|= 0.00000022
n= 19   p= 2.35755115   g(p)= 2.35755109  |f(p)|= 0.00000007
n= 20   p= 2.35755109   g(p)= 2.35755106  |f(p)|= 0.00000002
n= 21   p= 2.35755106   g(p)= 2.35755106  |f(p)|= 0.00000001
n= 22   p= 2.35755106   g(p)= 2.35755105  |f(p)|= 0.00000000
```

Rate of Convergence

In a fixed point iteration:

- Assume x^* is a given root and $\rho = |g'(x^*)|$ with $0 < \rho < 1$.
- x_0 is an initial guess, sufficiently close to x^*.

  $|x_k - x^*| = |g'(\xi)||x_{k-1} - x^*| \approx \rho|x_{k-1} - x^*| \approx \cdots \approx \rho^k|x_0 - x^*|$

$$\rho^{-k} \approx \frac{|x_0 - x^*|}{|x_k - x^*|}$$

$$-k\log_{10}\rho \approx \log_{10}\left(\left|\frac{x_0 - x^*}{x_k - x^*}\right|\right)$$

The rate of convergence is defined by

$$Rate = -\log_{10}\rho$$

Then, the number of iteration takes

$$k \approx \frac{\log_{10}\left(\left|\frac{x_0 - x^*}{x_k - x^*}\right|\right)}{Rate}$$

Rate of Convergence

Bisection Method: (though it is not an iterative method) $\rho = \frac{1}{2}$.

$$Rate = -\log_{10}\rho = -\log_{10}\left(\frac{1}{2}\right) \approx 0.301$$

If we want $\left|\frac{x_0 - x^*}{x_k - x^*}\right| \approx 10$, then $k = 4$. After $k = 4$, iteration, the error reduction in bisection method is a factor of $16 > 10$. For Fixed Point Iteration Method:

- For $f(x) = 2\cosh(\frac{x}{4}) - x$, $g(x) = 2\cosh(x/4)$, and $g'(x) = 0.5\sinh(x/4)$, at $x_1^* = 2.3575510513$, $\rho = g'(x_1^*) \approx 0.312$
- $Rate = -\log_{10}(0.312) \approx 0.506$
- $k = 2$.

If $\rho > 1 \implies$ Negative rate! No convergence.
If $\rho = 0 \implies$ Infinite rate $\implies$ Error reduction is faster than a constant factor.

- Given $f \in C^2[a, b]$
- Taylor series: For ξ between x, and x_k

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2}f''(\xi)(x - x_k)^2$$

$$0 = f(x_k) + f'(x_k)(x^* - x_k) + \underbrace{f''(\xi)}_{=0,\text{ if } f \text{ linear}} \frac{1}{2}(x^* - x_k)^2$$

$$x^* = x_k - \frac{f(x_k)}{f'(x_k)}$$

- If f is non-linear, and x_k is close to x^*, then

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \cdots$$

Newton's Method (NM)

Algorithm 3: Newton's Method

Given $f(x) \in C^2[a, b]$.
Step 1: Start from an initial guess x_0.
Step 2: For $k = 0, 1, 2, \cdots$, set

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

until x_{k+1} satisfies termination criteria.

Newton's Method

Example: For $f(x) = 2\cosh(\frac{x}{4}) - x$, the Newton iteration is

$$x_{k+1} = x_k - \frac{2\cosh\left(\frac{x_k}{4}\right) - x_k}{0.5\sinh\left(\frac{x_k}{4}\right) - 1}$$

```
Iter= 0    p= 8.000000000000000    |f(p)|= 0.475608617832737
Iter= 1    p= 8.584695055013547    |f(p)|= 0.084309114681618
Iter= 2    p= 8.508657714758835    |f(p)|= 0.001556683216439
Iter= 3    p= 8.507200100111358    |f(p)|= 0.000000564969103
Iter= 4    p= 8.507199570713095    |f(p)|= 0.000000000000073
Iter= 5    p= 8.507199570713027    |f(p)|= 0.000000000000002
```

Comparison

- $atol = 1e - 8$
- $x_1^* = 2.35755105$
- $x_2^* = 8.50719957$

		Number of Iterations		
x_0	Root	NM	FPIM	BSM
2	x_1^*	4	16	27
4	x_1^*	5	18	
8	x_2^*	5	DNC	27
10	x_2^*	5	DNC	

Example:

- Apply the Newton's Method, to find the root of the function

$$f(x) = \sqrt{x} - 1.1$$

starting from an initial guess $x_0 = 2$ with $atol = 0.001$.

Speed of Convergence

Assume the method converges and

$$\lim_{k \to \infty} x_k = x^*$$

then

$$\lim_{k \to \infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^q} = \rho$$

- **Linear convergence:** $0 < \rho < 1$, and $q = 1$
- **Quadratic convergence:** $0 < \rho < \infty$, and $q = 2$
- **Cubic convergence:** $0 < \rho < \infty$, and $q = 3$
- **Superlinear convergence:** $\rho = 0$, and $q = 1$

Quadratic and cubic convergence are both superlinear convergence.

Relation between Newton's Method and general FPIM

Note: x^* is a root of $f(x)$.

1. If the assumptions of the Fixed Point Theorem holds, also $g'(x^*) \neq 0$, then the method converges **linearly**.
2. In this case, the size of $|g'(x^*)|$ quantifies the *rate of convergence.*
3. If $g'(x^*) = 0$, then the method may converge faster than linearly:
   - which is the case of the Newton's method with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

   - $g'(x) =$
   - $g'(x^*) =$

Disadvantages of Newton's Method

For $k = 0, 1, 2, \cdots$, set

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

until x_{k+1} satisfies termination criteria.

1. The derivative f' need to exist, and need to evaluate it at each iteration.
2. The local nature of the method's convergence.

Secant Method

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

Algorithm 4: Secant Method

Given a scalar differential function $f(x)$:

1. Start from two initial guess x_0, and x_1.
2. For $k = 1, 2, \cdots$, set

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

until x_{k+1} satisfies termination criteria.

Secant Method

- $f(x) = 2\cosh(\frac{x}{4}) - x$
- $atol = 1e - 8$
- $x_1^* = 2.35755105$
- $x_2^* = 8.50719957$
- $x_0 = 10,\ x_1 = 8$

```
n= 1   p= 8.3471358106   |f(p)|= 0.1640947903
n= 2   p= 8.5299950386   |f(p)|= 0.0244655135
n= 3   p= 8.5062692341   |f(p)|= 0.0009926166
n= 4   p= 8.5071943070   |f(p)|= 0.0000056173
n= 5   p= 8.5071995719   |f(p)|= 0.0000000013
n= 6   p= 8.5071995707   |f(p)|= 0.0000000000
```

Comparison

- $f(x) = 2\cosh(\frac{x}{4}) - x$
- $atol = 1e - 8$
- $x_1^* = 2.35755105$
- $x_2^* = 8.50719957$
- $x_0 = 10,\ x_1 = 8$

			Number of Iterations		
x_0	Root	NM	FPIM	BSM	Secant
2	x_1^*	4	16	27	6
4	x_1^*	5	18		
8	x_2^*	5	DNC	27	7
10	x_2^*	5	DNC		

Theorem

Convergence of the Newton and Secant Methods If $f \in C^2[a, b]$ *and* $\exists\, x^* \in [a, b] \ni f(x^*) = 0 \wedge f'(x^*) \neq 0$, *then* $\exists\, \delta \ni$ *starting from with* x_0 *(also* x_1 *in the case of Secant Method) from anywhere in the neighborhood* $[x^* - \delta, x^* + \delta]$, *Newton's Method converges quadratically and Secant Method converges superlinearly.*

Multiple Root

$f(x^*) = 0 \wedge f'(x^*) \neq 0$ are the assumptions of Newton's Method and Secant Method.

- What if $f'(x^*) = 0$? This is the case of a **multiple root**.
- If $f(x) = (x - x^*)^m q(x)$ with $q(x^*) \neq 0$, then x^* is a root of multiplicity m.
- If $m > 1$, then $f'(x^*) = 0$.

Example: $f(x) = x^m,\ m > 1.\ x^* = 0.\ f'(x) =$ $\frac{f(x)}{f'(x)} =$
Newton's method: $x_{k+1} = x_k -$

$$|x_{k+1} - x^*| = \underbrace{\frac{m-1}{m}}_{\rho} |x_k - x^*| \ \text{:Linearly convergent}$$

Rate$= -\log_{10}\left(\frac{m-1}{m}\right)$, as $m \to \infty$, Rate$\to 0$ For $f(x) = x^2$, $m = 2$, then $\rho = \frac{1}{2}$.

- Globalizing Newton's method
- Convergence and roundoff errors