Notation and Vocabulary

1. **supremum:** $s = \sup A$ ="least upper bound" all values that come after s will be greater than or equal to s
2. **infimum:** $i = \inf A$ = "greatest lower bound" all values that come before i will be less than or equal to i
3. **union:** $A \cup B$, **intersection:** $A \cap B$
4. **Intersection of n sets:** $\bigcap_{n=1}^{\infty} A_n$. **Union of n sets:** $\bigcup_{n=1}^{\infty} A_n$
5. **empty set:** $\emptyset$, *"The set is empty"*$= \emptyset$, *"The set is nonempty"*$\neq \emptyset$
6. **complement of A:** $A^c = \{x \in \mathbb{R} \mid x \notin A\}$
7. **subset:** $A \subseteq B$
8. **"A less B":** $A \setminus B = A \cap B^c$
9. **DeMorgan's Laws:** $(A \cap B)^c = A^c \cup B^c$ and likewise $(A \cup B)^c = A^c \cap B^c$
10. **Equality:** two real numbers $a, b \in \mathbb{R}$ are **equal** iff $\forall \epsilon > 0,\ |a - b| < \epsilon$
11. **Bounded:** a set $A \subseteq \mathbb{R}$ is bounded:
12. *above* if $\exists b \in \mathbb{R}$ s.t. $\forall a \in A,\ b \geq a$.
13. *below* for some $l \leq a,\ \forall a \in A$
    Sets: $\exists M > 0$ s.t. $|a| \leq M, \ \ \forall a \in A$
    Sequential: $\exists M > 0$ s.t. $|x_n| \leq M, \ \ \forall n \in \mathbb{N}$
14. **Axiom of Completeness:** If $A \neq \emptyset$, and is bounded above, then $\exists s = \sup A,\ \text{s.t.}\ \ s \geq a, \ \ \forall a \in A$
15. **Lemma to AOC:** *decides what is NOT an upper bound of A*. $s = \sup A$ is the supremum of A iff $\forall \epsilon > 0, \ \ \exists a \in A$ s.t. $s - \epsilon < a$
16. **Triangle Inequality:** $|a + b| \leq |a| + |b|, \Rightarrow |a - b| \leq |a - c| + |c - b|$
17. **Maximum:** Some $M \in A$, s.t. $M \geq a, \forall a \in A$, **Minimum:** Some $m \in A$ s.t. $m \leq a, \forall a \in A$
18. **Nested Intervals:** given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} | a_n \leq x \leq b_n\}$, assuming $I_n \supseteq I_{n+1}$, a nested sequence of intervals $I_1 \supseteq I_2 \supseteq I_3 \ldots \neq \emptyset$
19. **function:** $f : A \to B$
20. **One-to-One:** if $a_1 \neq a_2$ in A $\Rightarrow f(a_1) \neq f(a_2)$ in B. **Onto:** given any $b \in B,\ \exists a \in A$ s.t. $f(a) = b$
21. **Cardinality:** $A \sim B$, $|\mathbb{N}|$
22. **Countable indicators:**
    1. If $A \subseteq B$ and B is countable, then A is countable or finite.
    2. $|\mathbb{Q}|$ is countable, $|\mathbb{R}|$ is uncountable

23. **ϵ-neighborhood:** $V_\epsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \epsilon\}$
= an interval centered at a, with radius ϵ: $(a - \epsilon, a + \epsilon)$
24. **convergence:** $(a_n) \to a$ if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t. whenever $n \geq N \Rightarrow |a_n - a| < \epsilon$
    1. *Functional Version:* $\lim_{n\to\infty} a_n = a$
    2. *Sequential Version:* $\exists a_N$ in the sequence (a_n) after which all terms of the sequence land $(a_n) \in V_\epsilon(a)$
25. **Sequence:** (x_n), **Subsequence:** (x_{n_k})
26. **Increasing Sequence:** $a_n \leq a_{n+1}, \quad \forall n \in \mathbb{N}$. **Decreasing Sequence:** $a_n \geq a_{n+1}, \quad \forall n \in \mathbb{N}$
27. **Cauchy sequence, $(\mathbf{a_n})$:** $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. whenever $m, n \geq N \Rightarrow |a_n - a_m| < \epsilon$
28. **Convergence indicators:**
    1. Monotone and bounded $\Rightarrow$ convergence.
    2. BWT - every bounded sequence contains a convergent subsequence.
    3. If a sequence converges, its subsequences converge to the same limit.
    4. Cauchy sequence $\Leftrightarrow$ sequence convergence, $(a_n) \to a$.
29. **Limit point iff:** $\exists a_n \in A$ s.t. $x = \lim a_n, a_n \neq x, \forall n \in \mathbb{N}$...***isolated point** if not a limit point, for some* $a \in A$
30. **Set of all limit points:** A'
31. **Closure of a set:** $\overline{A} = A \cup A'$
32. **Open set, O:** $\forall a \in O, \ \exists V_\epsilon(a) \subseteq O$ *a set whose supremum is not contained within the set.*
33. **Closed set, F, iff:** contains all limit points $\overline{F} = F \cup F'$ *every Cauchy sequence,* (a_n) *s.t.* $|a_n - a_m| < \epsilon$ *contained in F has* $\lim(a_n) \in F$
34. **Open/Closed indicators:**
    1. A set A is ***open*** if A^c is ***closed***, likewise a set B is ***closed*** if B^c is ***open***.
    2. If F is a finite collection of closed sets, then $\bigcup_{n=1}^{k} F_n$ is closed
    3. IVT
35. **Compact, K:** $\forall (k_n) \in K, \ \exists (k_{n_k}) \to k \in K$ *all sequences in K will have a subsequence,* k_{n_k}*, that converges to an element of K*
36. **Compact indicators:**
    1. K is compact if *closed* (contains all limit points) and *bounded*
    2. Let $f : A \to \mathbb{R}$ be *continuous* on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.
37. **Nested Sets:** assuming $K_n \supseteq K_{n+1}$, a nested sequence of intervals $K_1 \supseteq K_2 \supseteq K_3 \ldots \neq \emptyset$
38. **Cantor Set:** $C = \bigcap_{n=0}^{\infty} C_n$ where C_n is a set of 2^n closed intervals with length $\dfrac{1}{3^n}$.

39. **Limit of a function:** $\lim_{x \to c} f(x) = L$ provided that $\forall \epsilon > 0,\ \exists \delta > 0$ s.t. whenever $0 < |x - c| < \delta$ and $x \in A \Rightarrow |f(x) - L| < \epsilon$
    1. *Topological Version:* Let c be a limit point of the domain $f : A \to \mathbb{R}$. We say $\lim_{x \to c} f(x) = L$ provided that every ϵ-neighborhood around L, $V_\epsilon(L)$, $\exists$ a δ-neighborhood around c, $V_\delta(c)$, with the property that $\forall x \in V_\delta(c)$ where $x \neq c, x \in A$, it follows that $f(x) \in V_\epsilon(L)$
    2. *Sequential Criterion:* Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:

$$\lim_{x \to c} f(x) = L \equiv \left(\forall (x_n) \subseteq A \text{ s.t. } x_n \neq c \text{ and } x_n \to c \Rightarrow f(x_n) \to L\right)$$

40. **Algebraic Limit Theorem:**
    1. *Function version:* Let f and g be functions defined on domain $A \subseteq \mathbb{R}$ and assume $\lim_{x \to c} f(x) = L,\ \lim_{x \to c} g(x) = M$ for some limit point c of A.
        1. $\lim kf(x) = kL,\ \forall k \in \mathbb{R}$
        2. $\lim[f(x) + g(x)] = L + M$
        3. $\lim[f(x)g(x)] = LM$
        4. $\lim \dfrac{f(x)}{g(x)} = \dfrac{L}{M},\quad M \neq 0$
    2. *Sequential version:* Let $\lim a_n = a$ and $\lim b_n = b$.
        1. If $a_n > 0,\ \forall n \in \mathbb{N}$, then $a \geq 0$.
        2. If $a_n \leq b_n,\ \forall n \in \mathbb{N}$ then $a \leq b$.
        3. If $\exists c \in \mathbb{R}$ s.t. $c \leq b_n,\ \forall n \in \mathbb{N}$, then $c \leq b$. Similarly if $a_n \leq c,\ \forall n \in \mathbb{N}$, then $a \leq c$.
    3. *Continuity version:* Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at some $c \in A$ then,
        1. $kf(x)$ is continuous @ c $\forall k \in \mathbb{R}$
        2. $f(x) + g(x)$ is continuous @ c
        3. $f(x)g(x)$ is continuous @ c
        4. $\dfrac{f(x)}{g(x)}$ is continuous @ c, $g(x) \neq 0$
41. **Divergence:** Let $f : A \to \mathbb{R}$, and $c \in A'$. $\exists (x_n), (y_n) \in A$ with $x_n \neq c$ and $y_n \neq c$. If $\lim x_n = \lim y_n = c$ but $\lim f(x_n) \neq \lim f(y_n) \Rightarrow \lim f(x) = \text{DNE}$.
42. **Continuity:** A function is **continuous** if $\forall \epsilon > 0,\ \exists \delta > 0$ s.t whenever $|x - c| < \delta$, and $x \in A, \Rightarrow |f(x) - f(y)| < \epsilon$. Point c must be in the domain of f and $f(c) = L$ or $\lim_{x \to c} f(x) = f(c)$
43. **Continuity indicators:** *Continuous iff* -
    1. $\forall \epsilon > 0,\ \exists \delta > 0$ s.t. $|x - c| < \delta$ (and $x \in A$) $\Rightarrow |f(x) - f(c)| < \epsilon$

2. $\forall V_\epsilon(f(c)),\ \exists V_\delta(c)$ with the property that $x \in V_\delta(c) \Rightarrow f(x) \in V_\epsilon(f(c))$
3. $\forall (x_n) \to c$ (w/ $x_n \in A$), $\Rightarrow f(x_n) \to f(c)$
4. $lim_{x\to c} f(x) = f(c)$

44. **Discontinuity Indicator:** $f : A \to \mathbb{R}$ and $c \in A$, and $c \in A'$. If $\exists (x_n) \subseteq A$ where $(x_n) \to c$ but $f(x_n) \nrightarrow f(c)$. f is not continuous at $x = c$.
45. **Extreme Value Theorem:** $f : K \to \mathbb{R}$ is continuous on a compact set, $\exists x_o, x_1 \in K$ s.t. $f(x_o) \le f(x) \le f(x_1)$
46. **Uniform Continuity:** $f : A \to \mathbb{R}$ is **uniformly continuous on A** if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y \in A$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
47. **Uniform Continuity Indicators:**
    1. Given some $\epsilon > 0$ and $c \in A$ we can find a $\delta > 0$ s.t. if $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$
    2. $\forall \epsilon > 0$, the same $\delta > 0$ can be used for all points $c \in A$
    3. *Compact Sets:* A function that is continuous on a compact set K is uniformly continuous on K.
48. **Absence of Uniform Continuity:** fails uniform continuity iff $\exists$ a specific $\epsilon_o > 0$ and two sequences $(x_n), (y_n) \in A$ satisfying $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \ge \epsilon_o$
49. **Intermediate Value Theorem:** $f : [a, b] \to \mathbb{R}$ be continuous. Let $L \in \mathbb{R}$ satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$ then $\exists c \in (a, b)$ where $f(c) = L$
50. **IVT Indicator:** if $x < y \in [a, b]$ and $\forall L$ between $f(x)$ and $f(y)$ it is always possible to find a point $c \in (x, y)$ where $f(c) = L$
51. **Connected Sets:** A set that is not disconnected is called a **connected** set
    1. $A, B \subseteq R$ are **separated** if $A \cap B, A \cap B = \emptyset$.
    2. A set $E \subseteq R$ is **disconnected** if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.
52. **Preservation of Connected Sets:** A continuous function $f : G \to \mathbb{R}$, if $E \subseteq G$ is connected, $f(E)$ is connected as well.

Problems

1. Does $\lim_{x\to 1} \dfrac{|x-1|}{x-1}$
    1. 4.2.5 Use Definition 4.2.1 to supply a proper proof for the following limit statements.
        1. (a) $\lim_{x\to 2}(3x + 4) = 10$
        2. (b) $\lim_{x\to 0} x^3 = 0$.
        3. (c) limx→2(x2 + x − 1) = 5.
        4. (d) limx→3 1/x = 1/3

2. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. We define the set $Z = \{x \in \mathbb{R} \mid h(x) > 0\}$. Is Z closed?
3. Recall that if two sets A, B are separated, *it means that neither contains a limit point of the other* $\Rightarrow A \cap \overline{B} = \emptyset$ *and* $\overline{A} \cap B = \emptyset$. If $A \cap B = \emptyset$ and $A, B \neq \emptyset$, let U and V be open sets s.t. $U \cap V = \emptyset$ and $A \subseteq U$ and $B \subseteq V$. Under these conditions, are A and B separated?
4. Given 2 sets A, B let us define distance between them as $d(A, B) = \inf\{|x - y| : x \in A,\ y \in B\}$. If $A = (0, 1)$ and $B = (1, 2)$. Given that $A \cap B = \emptyset$, and A and B are compact is $d(A, B) > 0$?
5. Let $A \subseteq \mathbb{R}$ show that A' is always closed.
6. 2.5.2 Decide whether the following propositions are true or false, providing a short justification for each conclusion.
    1. (a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.
    2. (b) If (x_n) contains a divergent subsequence, then (x_n) diverges.
    3. (c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.
    4. (d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges
7. Give an example of 2 sets A, B where $A \cap B = \emptyset$, $\sup A = \sup B$ and $\sup A \notin A, \sup B \notin B$
8. If $A_1 \cup A_2 \cup A_3 ... \cup A_n$ is countable is $\bigcup_{n=1}^{\infty} A_n$ countable? If $A_1 \times A_2 \times A_3 ... \times A_n$ is countable is $\prod_{n=1}^{\infty} A_n$ countable?

HMWK 10

4.4.3

Show that $f(x) = \dfrac{1}{x^2}$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

4.4.5

Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).

4.4.9

(Lipschitz Functions). A function f : A → R is called Lipschitz if there exists a bound M >0 such that $\left| \dfrac{f(x) - f(y)}{x - y} \right| \leq M$ for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to R$ is Lipschitz, then it is uniformly continuous on A.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

4.5.1
Show how the Intermediate Value Theorem follows as a corollary to Theorem 4.5.2.

HMWK 9

4.2.6
Decide if the following claims are true or false, and give short justifications for each conclusion.
(a) If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then any smaller positive δ will also suffice.
(b) If $\lim_{x \to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.
(c) If $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} 3[f(x) - 2]^2 = 3(L - 2)^2$.
(d) If $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f.

4.3.5
Show using Definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

4.3.9
Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

HMWK 8

3.3.4
Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.
(a) $K \cap F$
(b) $\overline{F^c \cup K^c}$
(c) $K \setminus F = \{x \in K : x \notin F\}$
(d) $\overline{K \cap F^c}$

HMWK 7

3.2.1
(a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be finite get used?

(b) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \dots\}$ whose intersection $\bigcap_{n=1}^{\infty} O_n$ is closed, not empty and not all of $\mathbb{R}$.

3.2.3

Decide whether the following sets are open, closed, or neither.

If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbb{Q}$.

(b) $\mathbb{N}$.

(c) $\{x \in \mathbb{R} : x \neq 0\}$.

(d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$.

(e) $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbb{N}\}$.

3.2.13

Prove that the only sets that are both open and closed are $\mathbb{R}$ and the empty set $\emptyset$.

HMWK 6

2.6.5

Consider the following (invented) definition: A sequence (s_n) is pseudo-Cauchy if, for all $\epsilon > 0$, there exists an $\mathbb{N}$ such that if $n \geq \mathbb{N}$, then $|s_{n+1} - s_n| < \epsilon$.

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

HMWK 5

2.2.2

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \dfrac{2n+1}{5n+4} = \dfrac{2}{5}$.

(b) $\lim \dfrac{2n^2}{n^3+3} = 0$.

(c) $\lim \dfrac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

2.3.1

Let $x_n \geq 0 \ \forall n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

2.3.2 b

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\dfrac{2x_n - 1}{3}\right) \to 1$

(b) $(\dfrac{1}{x_n}) \to \dfrac{1}{2}$.

2.3.8 b

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

2.4.1

(a) Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \dfrac{1}{4 - x_n}$ converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

HMWK 4

$|P(\mathbb{N})| = |\mathbb{R}|$

HMWK 3

1.4.3

Prove that $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

1.4.4

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

1.5.4

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1–1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1–1 onto function between the two sets.

1.5.6

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

HMWK 2

1.3.1 a

(a) Write a formal definition in the style of Definition 1.3.2 for the infimum or greatest lower bound of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds

1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

1.3.5

As in Example 1.3.7, let $A \subseteq R$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

> sup(c+A) = c + supA
>
> Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$.
> Define the set $c + A$ by
>
> $$c + A = \{c + a : a \in A\}$$
>
> then $\sup(c + A) = c + \sup A$.
>
> *If A is a set that is nonempty and bounded above is added to some number c, the supremum of that set generated by {c+A} is the same as* ***the sum of the supremum of A and c****.*
>
> #proof
>
> Let $s = \sup A$, we see that $a \leq s$ for all $a \in A$, which implies $c + a \leq c + s$ for all $a \in A$. Thus $c + s$ is an upper bound for $c + A$ and part 1 is verified for c to be a least upper bound of A. For part 2 of the definition, let b be an arbitrary upper bound for $c + A$, i.e. $c + a \leq b, \forall a \in A$. This is equivalent to $a \leq b - c, \forall a \in A$, from which we conclude that $b - c$ is an upper bound for A. Because s is the least upper bound of A, we can say

> $s \leq b - c$, which can be rewritten as $c + s \leq b$. This verifies part 2 of the definition and we can conclude $\sup(c + A) = c + \sup A$

1.3.8 Compute, without proofs, the suprema and infima (if they
exist) of the following sets:
(a) $\{\dfrac{m}{n} : m, n \in \mathbb{N}$ with $m < n\}$.
(b) $\{(-1)^{\frac{m}{n}} : m, n \in \mathbb{N}\}$.
(c) $\{\frac{n}{(3n+1)} : n \in \mathbb{N}\}$.
(d) $\{\frac{m}{(m+n)} : m, n \in \mathbb{N}\}$.

1.3.9
(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.
(b) Give an example to show that this is not always the case if we only assume
$\sup A \leq \sup B$.

Sets and functions

$f(x) = \sin(\frac{1}{x})$
$f(x) = \dfrac{1}{x}$
$f(x) = x$
$f(x) = \sqrt{x}$
$f(x) = |x|$

$(x_n) = \dfrac{1}{n}$
$(x_n) = (-1)^n$
$(x_n) = 2^n$

$(\dfrac{-1}{n}, \dfrac{1}{\pi n})$

Techniques