MA 4/568-Numerical Analysis I
Numerical Solution of Ordinary Differential Equations

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Initial value Ordinary Differential Equations (ODEs)

$y(t)$?

$$\frac{dy}{dt} = f(t, y), \;\; a \leq t \leq b, \;\; y(a) = c.$$

- $f(t, y)$: Given.
- $\frac{dy}{dt} = y^{'}$.
- $y(t) = c + \int_a^t f(s, y(s))ds$

Example: $y^{'} = -y + t$, $t \geq 0$, together with $y(0) = c$.

ODE system

$$\boldsymbol{y}' = \frac{d\boldsymbol{y}}{dt} = \boldsymbol{f}(t, \boldsymbol{y}), \;\; a \le t \le b, \; \boldsymbol{y}(a) = \boldsymbol{c}.$$

Example: Autonomous ODE system

$$\frac{d^2\theta}{dt^2} = \theta'' = -g\sin(\theta).$$

We can rewrite as $y_1(t) = \theta(t)$, and $y_2(t) = \theta'(t)$, then $\boldsymbol{c} = \begin{pmatrix} \theta(0) \\ \theta'(0) \end{pmatrix}$.

Euler's method (forward Euler)

- $y' = f(t, y), \ a \le t \le b, \ y(a) = c.$
- $t_0 = a, \ t_i = a + ih, \ \ i = 0, 1, \cdots, N,$
- $h = \frac{b-a}{N}.$

$$y'(t_i) = \frac{y(t_{i+1}) - y(t_i)}{h} - \frac{h}{2} y''(\xi_i)$$

Then, **Forward Euler method:**

$$\begin{aligned} y_0 &= c, \\ y_{i+1} &= y_i + hf(t_i, y_i), \ i = 0, 1, \cdots, N-1 \end{aligned}$$

Example: $y' = y, \; y(0) = 1.$

Explicit vs. Implicit Methods

Forward Euler method:

$$\begin{aligned} y_0 &= c, \\ y_{i+1} &= y_i + hf(t_i, y_i),\ i = 0, 1, \cdots, N-1 \end{aligned}$$

Backward Euler method:

$$\begin{aligned} y_0 &= c, \\ y_{i+1} &= y_i + hf(t_{i+1}, y_{i+1}),\ i = 0, 1, \cdots, N-1 \end{aligned}$$

Forward Euler Convergence

Let $f(t, y)$ have bounded partial derivatives in a region $\mathcal{D} = \{a \leq t \leq b,\ |y| < \infty\}$. Note that this implies Lipschitz continuity in y: $\exists$ a constant $L \ni \forall\ (t, y),\ (t, \hat{y}) \in \mathcal{D}$, we have

$$|f(t, y) - f(t, \hat{y})| \leq L|y - \hat{y}|.$$

Then Euler's method converges and its global error decreases linearly in h. Moreover, assuming further that

$$|y^{''}(t)| \leq M, \ \ a \leq t \leq b,$$

the global error satisfies

$$|e_i| \leq \frac{Mh}{2L}\left(e^{L(t_i - a)} - 1\right), \ \ i = 0, 1, 2, \cdots, N.$$

Absolute stability and stiffness (Forward Euler Method)

- Consider a simple ODE

$$y^{'} = \lambda y$$

  - Absolute Stability: $h \leq \frac{2}{|\lambda|}$
- For the initial value ODE

$$y^{'} = -1000(y - \cos(t)) - \sin(t), \quad y(0) = 1$$

  the exact solution is $y(t) = \cos(t)$. For the following step-sizes
  - $h = 0.1$
  - $h \leq \frac{1}{500}$
  - $h = 0.0005\pi$
  - $h = 0.001\pi$

  compute the numerical solution at $x = \frac{\pi}{2}$. What happens? Why?
- **Stiff**: An ODE where stability considerations make us take a much smaller step-size for the forward Euler method than accuracy considerations would otherwise dictate, is called **stiff**.

Absolute stability (Backward Euler Method)

- Consider a simple ODE

$$y' = \lambda y$$

  - Absolute Stability: No restriction on h if $\lambda < 0$.

- For the initial value ODE

$$y' = -1000(y - \cos(t)) - \sin(t), \;\; y(0) = 1$$

  the exact solution is $y(t) = \cos(t)$. For the following step-sizes
  - $h = 0.1$
  - $h \leq \frac{1}{500}$
  - $h = 0.0005\pi$
  - $h = 0.001\pi$

  compute the numerical solution at $x = \frac{\pi}{2}$.

Forward Euler with general ODE system

$$\boldsymbol{y}' = \frac{d\boldsymbol{y}}{dt} = \boldsymbol{f}(\boldsymbol{y}) = A\boldsymbol{y}.$$

- $A \in \mathbb{R}^{m \times m}$, eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ and diagonalizable
- Say T: transformation matrix such that $T^{-1}AT$: diagonal
- Then the unknown $\boldsymbol{x} = T^{-1}\boldsymbol{y}$ decouples to give scalar ODEs

$$x_j' = \lambda_j x_j, \;\; j = 1, 2, \cdots, m$$

- Absolute stability requirement:

$$|1 + h\lambda_j| \leq 1, \;\; j = 1, 2, \cdots, m$$

Runge-Kutta Methods

Motivations:

- Euler's method is only first order accurate