CS 573: Graduate Algorithms, Fall 2010
Homework 1

Due Friday, September 10, 2010 at 1pm.

For this and all future homeworks, groups of up to three students may submit a single, common solution. Please neatly print (or typeset) the full name and NetID on each page of your submission.

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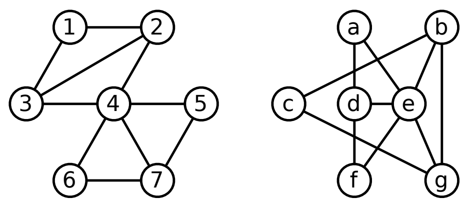
1. Two graphs are said to be ***isomorphic*** if one can be transformed into the other just by relabeling the vertices. For example, the graphs shown below are isomorphic; the left graph can be transformed into the right graph by the relabeling $(1,2,3,4,5,6,7) \mapsto (c,g,b,e,a,f,d)$.



Two isomorphic graphs.

Consider the following related decision problems:

- GraphIsomorphism: Given two graphs G and H, determine whether G and H are isomorphic.
- EvenGraphIsomorphism: Given two graphs G and H, such that every vertex in G and H has even degree, determine whether G and H are isomorphic.
- SubgraphIsomorphism: Given two graphs G and H, determine whether G is isomorphic to a subgraph of H.

(a) Describe a polynomial-time reduction from EvenGraphIsomorphism to GraphIsomorphism.

(b) Describe a polynomial-time reduction from GraphIsomorphism to EvenGraphIsomorphism.

(c) Describe a polynomial-time reduction from GraphIsomorphism to SubgraphIsomorphism.

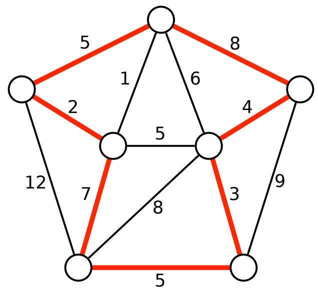
(d) Prove that SubgraphIsomorphism is NP-complete.

(e) What can you conclude about the NP-hardness of GraphIsomorphism? Justify your answer.

[Hint: These are all easy!]

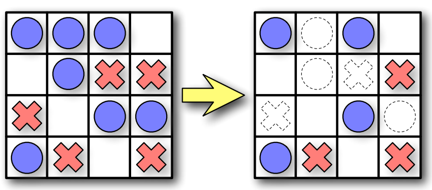
2. Suppose you are given a magic black box that can solve the 3Colorable problem ***in polynomial time***. That is, given an arbitrary graph G as input, the magic black box returns True if G has a proper 3-coloring, and returns False otherwise. Describe and analyze a ***polynomial-time*** algorithm that computes an actual proper 3-coloring of a given graph G, or correctly reports that no such coloring exists, using this magic black box as a subroutine. *[Hint: The input to the black box is a graph. Just a graph. Nothing else.]*

3. Let G be an undirected graph with weighted edges. A *heavy Hamiltonian cycle* is a cycle C that passes through each vertex of G exactly once, such that the total weight of the edges in C is at least half of the total weight of all edges in G. Prove that deciding whether a graph has a heavy Hamiltonian cycle is NP-complete.

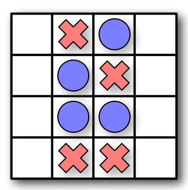


A heavy Hamiltonian cycle. The cycle has total weight 34; the graph has total weight 67.

4. Consider the following solitaire game. The puzzle consists of an $n \times m$ grid of squares, where each square may be empty, occupied by a red stone, or occupied by a blue stone. The goal of the puzzle is to remove some of the given stones so that the remaining stones satisfy two conditions: (1) every row contains at least one stone, and (2) no column contains stones of both colors. For some initial configurations of stones, reaching this goal is impossible.



A solvable puzzle and one of its many solutions.



An unsolvable puzzle.

Prove that it is NP-hard to determine, given an initial configuration of red and blue stones, whether the puzzle can be solved.

5. A boolean formula in ***exclusive-or conjunctive normal form*** (XCNF) is a conjunction (AND) of several *clauses*, each of which is the *exclusive*-or of one or more literals. For example:

$$(u \oplus v \oplus \bar{w} \oplus x) \wedge (\bar{u} \oplus \bar{w} \oplus y) \wedge (\bar{v} \oplus y) \wedge (\bar{u} \oplus \bar{v} \oplus x \oplus y) \wedge (w \oplus x) \wedge y$$

The XCNF-SAT problem asks whether a given XCNF boolean formula is satisfiable. Either describe a polynomial-time algorithm for XCNF-SAT or prove that it is NP-complete.