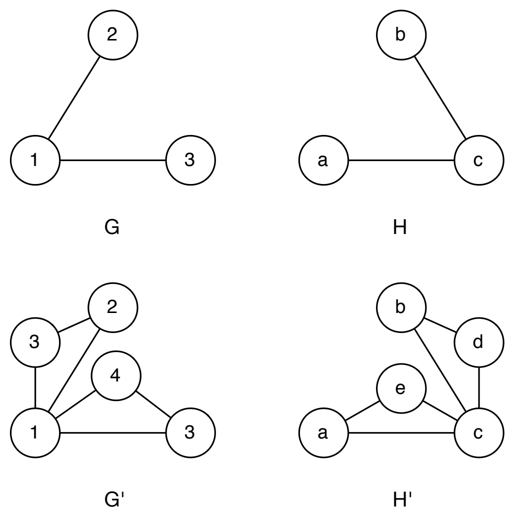
1. (•) ***I understand the course policies.***

(a) Describribe a polynomial-time reduction from EvenGraphIsomorphismto GraphIsomorphism.

Solution: EvenGraphIsomorphism is a special case of GraphIsomorphism. We can perform an identical reduction from EvenGraphIsomorphism to GraphIsomorphism. That is, given two graphs G and H in which every vertex has even degree, let $G' = G$ and $H' = H$. Then, G and H are isomorphic if and only if G' and H' are isomorphic. This copy just takes $O(n)$ time and is thus polynomial. ■

(b) Describribe a polynomial-time reduction from GraphIsomorphism to EvenGraphIsomorphism.

Solution: Given a graph G, we can transform it into an even graph by adding a dummy vertex between each pair of vertices that share an edge, and connect this vertex with the two vertices, respectively. For example, we can transform the graphs G and H as follows:



From graph G to G', vertices 3 and 4 are added between vertices 1 and 2, 1 and 3, respectively. From graph H to H', vertices d and e are added between vertices b and c, c and a, respectively. The degree of vertices are all even after the transformation. Given a relabeling of G and H, we can get a relabeling for G' and H' by appending the labeling of the added vertices to the original labeling. Given a relabeling of the even graphs, we can get a relabeling for the original graphs by ignoring the dummy vertices. Hence, G and H are isomorphic if and only if G' and H' are isomorphic. There are $O(|V|)$ edges to process and thus the reduction is polynomial-time. ■

(c) Describribe a polynomial-time reduction from GraphIsomorphism to SubgraphIsomorphism.

Solution: GraphIsomorphism is a speicial case of SubgraphIsomorphism. Given G and H, if they do not have the same number of vertices and edges, of course they are not the isomorphic. We perform an identical reduction from GraphIsomorphism to SubgraphIsomorphism. Let $G' = G$ and $H' = H$. Then, G and H are isomorphic if and only if G' and H' are isomorphic and have the same number of vertices and edges. This reduction is polynomial as shown before. ■

(d) Prove that SubgraphIsomorphism is NP-complete.

Solution: We prove SubgraphIsomorphism is NP-hard by showing a polynomial time reduction from Hamiltonian Cycle problem.

Proof: Given a graph H, we generate another graph G such that G is a cycle and has the same number of vertices as H. If G is isomorphic to a subgraph of H, then H has a hamiltonian cycle, which is just the isomorphic subgraph. Conversely, if H has a hamiltonian cycle, G will be isomorphic to the cycle, which is a subgraph of H. This transformation takes $O(n)$ time. Hence, SubgraphIsomorphism is NP-hard. □

Given a proof of subgraph isomorphism, i.e., a relabeling of a graph G to a subgraph of H, we apply the relabeling to H. Suppose we use adjacency list to represent the graph, then we need at most $O(n^2)$ time to check if the relabeling is correct. The check can be done in polynomial time and thus SubgraphIsomorphism in NP. In conclusion, SubgraphIsomorphism is NP-complete.

■

(e) What can you conclude about the NP-hardness of GraphIsomorphism? Justify your answer.

Solution: I have no idea [1]. ■

References

[1] Wiki page on graph isomorphism. http://en.wikipedia.org/wiki/Graph_isomorphism_problem#Complexity_class_GI.

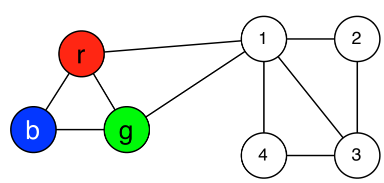
2. (•) ***I understand the course policies.***

(a) Describe and analyze a polynomial-time algorithm that computes an actual proper 3-coloring of a given graph G, or correctly reports that no such coloring exists, using this magic black box as a subroutine.

Solution: For illustration purpose we name the three color as r, g, and b. Given $G = (V, E)$, we first add a complete graph K_3 to G as a 'color gadget'. Then, the three vertices in the gadget must be colored differently using the three colors. Without loss of generality we fix their colors and use the color as their names. We denote the augmented graph as G^+. At first we use the black box to test if the original graph is 3-colorable. Return NONE if answer is NO. Next, for every vertices u in V, we try to link them with either pair of the vertices in the gadget and form a new graph, such that when performing coloring, u has to have the same color as the unselected vertex in the gadget. We then use the magic black box to test whether the new graph is 3-colorable. If YES then we output the color of the vertex that is not connected to u in the gadget, and add these two edges into G^+. Otherwise, we continue to test another pair. Let $V(G)$ denote the vertex set of G and $E(G)$ the edge set of G. The black box is refered as 3COLORABLE(G). The pseudo code is given as follows:

```
3COLORING(G):
  if 3COLORABLE(G) is NO
      return NONE
  G⁺ ← (V ∪ {r, g, b}, E ∪ {(r, g), (r, b), (g, b)})
  for each u in V
      for each v in {r, g, b}
          E′ ← E(G⁺) ∪ {(u, r), (u, g), (u, b)}/{(u, v)}
          H ← (V(G⁺), E′)
          if 3COLORABLE(H) is YES
              color[u] ← color of v
              G⁺ ← H
              break
  return color
```

An illustration is given as follows:



At first vertex 1 is connected to r and g, which means it should be colored to blue. Use the black box to test whether the new graph is 3-colorable.

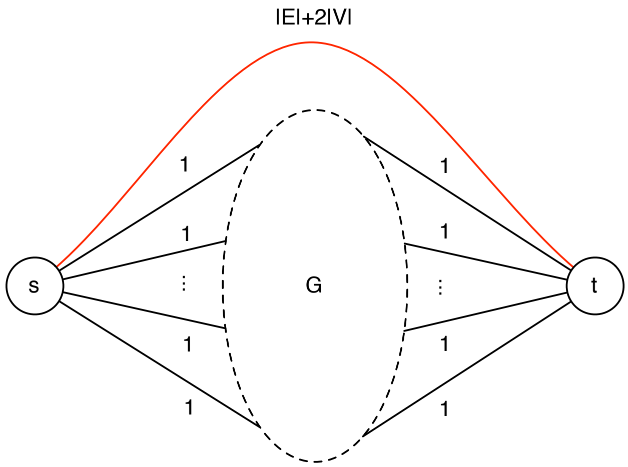
Correctness: At each step of connecting a vertex to a pair of vertices in the color gadget, the color of this particular vertex is fixed. Suppose a proof of the problem is a permutation of colors $rgbbgrrb\cdots$, essentially we are guessing the prefix of the proof, and verify it using the black box. This gurantees a correct answer.

Complexity: Let the time complexity of black box subroutine be $f(n)$. At first there is call to the subroutine and some constant-time operations. The algorithm contains a for loop that will traverse the vertex set of given graph G. Inside the for loop at most three steps will be performed, which contains a call to the subroutine and some constant time operations. The time complexity will be $O(f(n)) + \alpha + 3n(\beta + O(f(n)))$, where α and β are some constants. Hence, this algorithm is polynomial-time. ■

3. (•) ***I understand the course policies.***

(a) Prove that deciding whether a graph has a heavy Hamiltonian cycle is NP-complete.

Solution: We reduce from Hamiltonian Path problem. Given a graph $G = (V, E)$, we add two vertices s and t that connects all v in V, respectively. Also s and t is connected. Then we get a graph $G' = (V', E')$, where $V' = V \cup \{s, t\}$ and $E' = E \cup \{(s, t)\} \cup \{(s, v), (t, v) \mid v \in V\}$. We assign a weight $|E| + 2|V|$ to (s, t), and 1 to all other edges. An illustration is given as follows:



Correctness: We claim that G has a Hamiltonian Path if and only if G' has a heavy Hamiltonian Cycle. The total weight of the graph is $|E| + 2|V| + 2|V| + |E| = 4|V| + 2|E|$. Hence, the weight of (s, t) dominates the weight of the possible Heavy Hamiltonian Cycle, i.e., (s, t) must be included in a Heavy Hamiltonian Cycle. s can only be visited once, (s, t) is already used and there will be another edge (s, v) included in the cycle where $v \in E$. Similary for t. Because s and t are connected to every vertex in E respectively, the Hamiltonian Cycle can be closed if and only there is a Hamiltonian Path in E.

Complexity: We only need linear time to create two vertices and connect them with the exisiting vertices. Hence the reduction takes polynomial time.

Heavy Hamiltonian cycle problem is in NP, because we can easily verify the answer by using a depth or breadth-first search to compute the total weight, and follow the cycle to compute the weight of the cycle. This takes linear time. In conclusion, Heavy Hamiltonian cycle problem is NP-complete. ■

4. (•) ***I understand the course policies.***

(a) Prove that it is NP-hard to determine, given an initial configuration of red and blue stones, whether the puzzle can be solved.

Solution: We reduce from 3SAT to this decision problem. Given a instance of 3SAT, let n be the number of clauses, and m be the number of different variables. We designate each variable to a different column. For each clause, we create a row for it in the game, where each column in that row corresponds to a literal. In each row, a blue stone is placed if the literal is a variable, or a red stone if it is its negation. For the variables that are not included in the clause, we simply leave the corresponding squares to be empty. Finally a $n \times m$ grids of squares will be constructed. For example, given a formula $(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$, we can create an exact game setup as shown in the problem statement, where a, b, c, and d denote column 1, 2, 3 and 4 from let to right respectively.

Correctness: The conjunctive normal form of the formula ensures the first condition of the game to be satisifed, because a clause is satisfied if and only if at least one literal is set to true, which is equivalent to at least one stone is in the corresponding row. Since contradictory literals are placed in the same column, the assignment is consistent, which means no conflicting stones will be in the same column, and satisfies the second condition of the game. Hence, the boolean formula is satisfiable if and only if the puzzle is solvable. Thus, the reduction is correct.

Complexity: We only need linear time to perform this reduction, since we can allocate a column to a new variable as we go. Hence the reduction is polynomial-time.

In conclusion, this decision problem is NP-hard.

■

5. (•) ***I understand the course policies.***

(a) Either describe a polynomial-time algorithm for XCNF-SAT or prove that it is NP-complete.

Solution: XCNF-SAT can be solved in polynomial time. In order to satisfy a clause, the binary sum of the literals inside must be 1 (mod 2), since a exclusive-or clause is satisfied if and only if there is odd number of true literals. Hence, we can construct a system of linear equation over modulo 2. For each clause, an equation will be constructed. The left hand side contains the sum of the variables. If a literal is the negation of some variable x, we transformed it into $x+1$ in the equation; The right hand side is 1 (mod 2). For example, for the clause:

$$u \oplus v \oplus \bar{w} \oplus x$$

We can transform it into:

$$u + v + w + 1 + x \equiv 1 \pmod 2$$

By using Gaussian Elimination method for binary arithmetic, we can solve this sytem and obtain the satisfying assignments. It takes linear time to scan the boolean formula and construct the linear system. Given a XCNF-SAT instance that contains m clauses and n variables, Gaussian Elimination for binary arithmetic takes $O(mn^2)$ time [1]. Hence, our algorithm runs in polynomial-time.

■

References

[1] Wiki page on Gaussian elimination. http://en.wikipedia.org/wiki/Gaussian_elimination#Analysis.