I understand the course policies.

1. (a) Solve the recurrences.

   Solution:

$$\begin{aligned} A(n) &= \Theta(4^n) \\ B(n) &= \Theta(n^3) \\ C(n) &= \Theta(n^2) \\ D(n) &= \Theta(n^{\log_3 2}) \\ E(n) &= \Theta(3^{n/2}) \end{aligned}$$

■

   (b) Sort the functions from asymptotically smallest to asymptotically largest, indicating ties if any.

   Solution:

$$\begin{aligned} \sqrt{\lg n} &\ll \lg n \\ &\equiv \lg\sqrt{n} \\ &\ll 7^{\sqrt{\lg n}} \\ &\ll \sqrt{n} \\ &\equiv \sqrt{\lg(7^n)} \\ &\equiv \lg(7^{\sqrt{n}}) \\ &\ll 7^{\lg\sqrt{n}} \\ &\equiv \sqrt{7^{\lg n}} \\ &\ll \lg\sqrt{7^n} \\ &\equiv \lg(7^n) \\ &\equiv n \\ &\ll 7^{\lg n} \\ &\ll 7^{\sqrt{n}} \\ &\ll \sqrt{7^n} \\ &\ll 7^n \end{aligned}$$

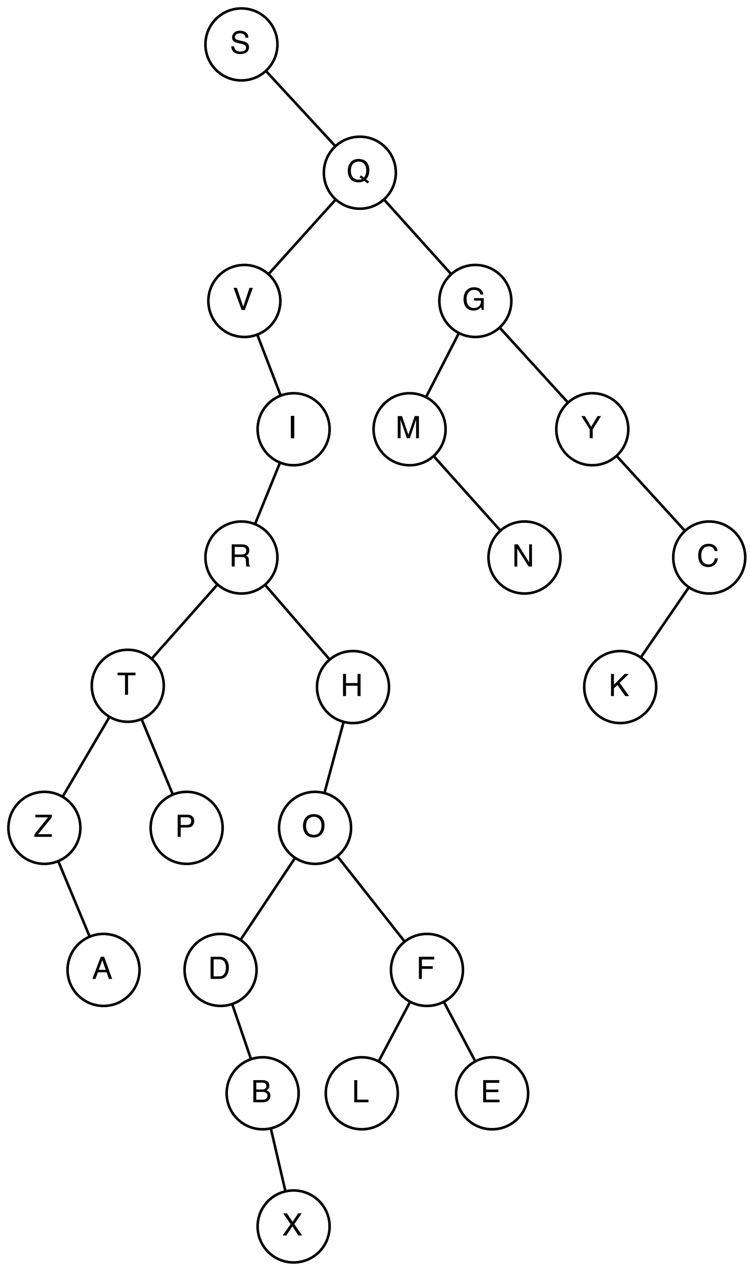
■

2. (a) List the nodes in Prof. della Giungla's tree in the order visited by a *preorder* traversal.

Solution: `S Q V I R T Z A P H O D B X F L E G M N Y C K` ■

(b) Draw Prof. della Giungla's tree.

Solution:



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3. Describe a data structure that stores a set of n points in the plane and supports the queries HIGHEST-TORIGHT and RIGHTMOSTABOVE.

Solution: We will first give a definition as follows:

Definition 1. *A point $p = (p_x, p_y)$ is dominated by a point $q = (q_x, q_y)$ if and only if $p_x < q_x$ and $p_y < q_y$.*

For each point p in S, we remove it from S if it is dominated by some point q. After we remove all such points, we get a set S' of these remaining points. Any two points in S' will not dominate each other. Then, we sort these points according to their x-coordinates and store them in an array.

To make a query HIGHESTTORIGHT(ℓ), we use a binary search to find an element $p = (i, j)$ in the array such that $l \leqslant i$ and p is closest to l in x-direction. Return this point for the query. If no such point can be found, i.e., l is greater than the x-coordinate of the last point in the array, return NONE.

Similary, to make a query RIGHTMOSTABOVE(ℓ), use binary search to find an element $p = (i, j)$ such that $l \leqslant j$ and p is closest to l in y-direciton.

Correctness: We first prove that a point p that is dominated by some other point q cannot be the output of HIGHESTTORIGHT. Otherwise, q will be a better point than p since q is higher than p and has a greater x-coordinate. Similarly this is true for RIGHTMOSTABOVE. In other words, all of the possible return values of the algorithm will be in S'.

Next, we prove the correctness of the algorithm. For HIGHESTTORIGHT, the point p we find that is closest to l in x-coordinate must have the highest y-coordinate. Assume that it is not true and there is a point q that has the highest y-coordinate and whose x-coordinate is greater than or equal to l, but it is not closest to l in x-coordinate. Then, q will then dominate any point p that has smaller x-coordinate than q. Hence, q will be the point that is closest to l in x-coordinate. This causes contradiction. Similarly, we can prove the correctness of our algorithm for RIGHTMOSTABOVE. □

Analysis: The data structure used here is an array that has at most n elements in the worst case that no point dominates others, and thus use $O(n)$ space. The query uses only a binary search and thus runs in $O(\log n)$ time. ■

4. Prove that for any arithmetic expression tree, there is an equivalent arithmetic expression tree in normal form.

Solution: We prove the statement by induction.

Proof: Let n be an arbitrary non-negative integer. Assume that for all arithmetic expression trees with depth $l < n$, there is an equivalent arithmetic expression tree in normal form.

- If $n = 0$, it is trivially true.
- If $n \geqslant 1$, for any arithmetic expression tree with depth n, there are two subcases to consider:
  - If the root node is a $+$-node, the inductive hypothesis shows that both subtrees of this node has a arithmetic expression tree in normal form. So this tree has an equivalent arithmetic expression tree in normal form because the root node is a $+$-node and will not introduce a $\times$-node parent.
  - If the root node is a $\times$-node, we first observe that we can distribute the $\times$-node as shown in Figure 1. The heights of the subtrees A, B, C and D are at most $n-2$. After distribution, the heights of the subtrees $A - \times - C$, $A - \times - D$, $B - \times - C$ and $B - \times - D$ are $n-1$. By inductive hypothesis they have equivalent arithmetic expression trees in their normal forms. Furthermore, according to the definition of normal form, all $+$-nodes in the normal form of an arithmetic expression tree will be in the higher level of the tree than the $\times$-nodes. Hence, we can transform these four subtrees to their normal form such that all the $+$-nodes are moved above the $\times$-nodes. Then, the resulting tree is also in normal form since all the $+$-nodes are still above the $\times$-nodes.

In conclusion, the statement holds for all cases. □

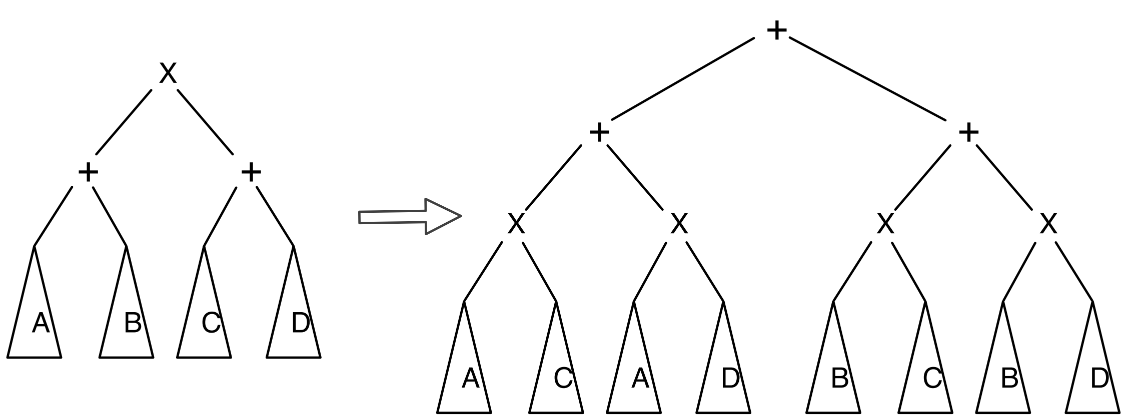

Figure 1. Distributing $\times$-node

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5. (a) What is the *exact* expected number of cards that Professor Jay hurls into the watermelon?
   (b) For each of the statements below, give the *exact* probability that the statement is true of the ***first*** pair of cards Professor Jay turns over.
      i. Both cards are threes.
      ii. One card is a three, and the other card is a club.
      iii. If (at least) one card is a heart, then (at least) one card is a diamond.
      iv. The card from the red deck has higher rank than the card from the blue deck.
   (c) For each of the statements below, give the *exact* probability that the statement is true of the ***last*** pair of cards Professor Jay turns over.
      i. Both cards are threes.
      ii. One card is a three, and the other card is a club.
      iii. If (at least) one card is a heart, then (at least) one card is a diamond.
      iv. The card from the red deck has higher rank than the card from the blue deck.

Solution: (a) $1751/52$

(b) i. $1/169$
   ii. $1/26$
   iii. $11/16$
   iv. $6/13$

(c) i. $7/103$
   ii. $31/103$
   iii. $77/103$
   iv. $48/103$

■

Credit: I had discussion with Leslie Hwang, Yuelin Du and Pei-Ci Wu when working on this homework. Thanks to Mr. Qiang Ma for providing insightful suggestion to the proof of problem 3.