Lecture 1

Math Foundations Team

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What is linear algebra?

- Linear algebra is the study of vectors and rules to manipulate vectors.
- Vectors are not only the familiar geometric vectors from high school (points in 2D/3D space) but any special objects which can be added together and multiplied by scalar values to produce another object of the same kind. For example, polynomials can also be treated as vectors.
- We shall deal with vectors in the space $\mathbb{R}^n$

Systems of linear equations

- Systems of linear equations form a central part of linear algebra.
- Many problems can be formulated as systems of linear equations.
- Tools of linear algebra can be used to solve such problems.
- We would use analytical techniques and numerical methods to solve the systems of linear equations.

Motivating problem

Consider the following problem.
A company produces products $N_1, N_2, \ldots N_n$ for which resources $R_1, R_2, \ldots R_m$ are required. To produce a unit of product N_i , a_{ij} units of resource R_j are needed, where $1 \leq i \leq n, 1 \leq j \leq m$. Find an optimal production plan where x_i units of product N_i are produced if a total b_j units of resource R_j are available, and no resources are left over.

Formulation

If we produce $x_1, x_2, \ldots x_n$ units of the products $N_1, N_2 \ldots N_n$ we need a total of $a_{1i}x_1 + a_{2i}x_2 + \cdots + a_{ni}x_n$ units of resource R_j. Thus we set up the equation:

$$a_{1i}x_1 + a_{2i}x_2 + \ldots + a_{ni}x_n = b_i$$

We can similarly set up the following set of linear equations in n unknowns, $x_1, x_2 \ldots x_n$.

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n &= b_1 \\ &\vdots \\ a_{1m}x_1 + a_{2m}x_2 + \ldots + a_{nm}x_n &= b_m \end{aligned}$$

A linear system has zero, one or infinitely many solutions

Example

Consider the folllowing system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_3 &= 1 \end{aligned}$$

Adding the first and second equations gives $2x_1 + 3x_3 = 5$ which contradicts the third equation. Thus there is no set of values for the variables x_1, x_2, x_3 such that the equations above are simultaneously satisfied.

Modified example

Consider a slightly modified example

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 2 \end{aligned}$$

In this case we can see from the first and third equations that $x_1 = 1$. Substituting this value of x_1 into equation (2), we get $-x_2 + 2x_3 = 1$. Adding this equation to equation (3), we get $3x_3 = 3$ which means $x_3 = 1$. Substituting $x_3 = 1$ into equation (3) shows $x_2 = 1$, so the overall solution is $x_1 = x_2 = x_3 = 1$. This is the unique solution to the problem

Infinite solutions

Now consider another modification to the original set of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_3 &= 5 \end{aligned}$$

Adding the first and second equations gives $2x_1 + 3x_3 = 5$ which is the same as the third equation. Thus the solution to the three equations is any tuple x_1, x_2, x_3 which satisfies $2x_1 + 3x_3 = 5$, and there are infinite solutions. We now express these solutions in a way whose motivation will become clear later: adding equations (1) and (2) above we get $2x_1 = 5 - 3x_3$.

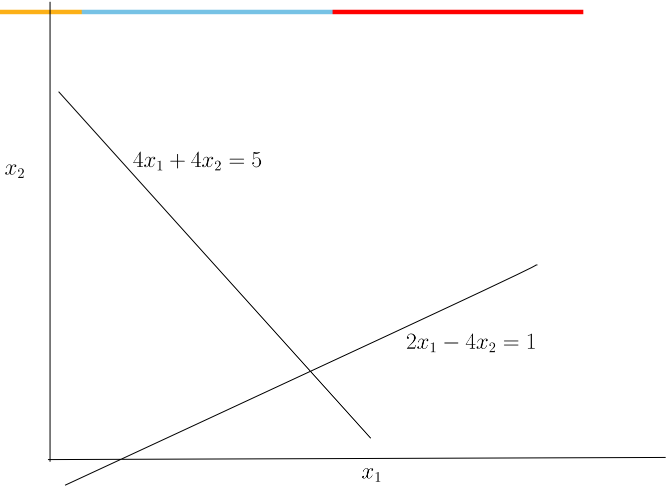
Infinite solutions

- Subtracting equation (2) from (1) we get $2x_2 - x_3 = 1$, so we can write

$$
\begin{aligned}
x_1 &= \frac{5}{2} - \frac{3}{2}x_3 \\
x_2 &= \frac{1}{2} + \frac{x_3}{2}
\end{aligned}
$$

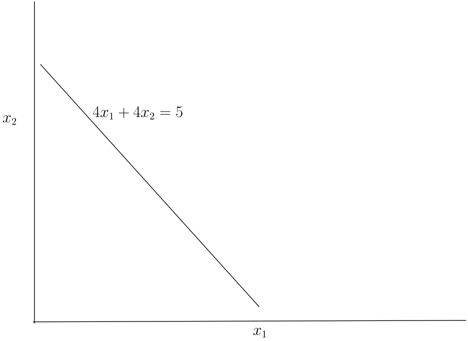
- For the previous problem we can express the set of infinite solutions in terms of the free variable x_3.
- Once x_3 is fixed, the other two variables have to take on specific values - they are known as pivot variables.
- We will show later how to identify pivot and free variables using Gaussian Elimination

Geometrical Interpretation



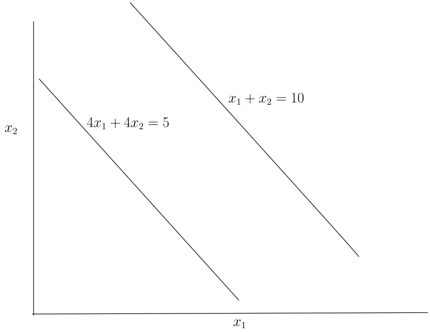
Geometrical interpretation

In the second case both constraints are the same, so there are an infinite number of solutions:



Geometrical interpretation

In the third case the constraints are mutually incompatible, so there is no assignment to x_1, x_2 which satisfies both constraints. The graph of both constraints shows a pair of parallel lines:



Higher dimensions

- In 3D each constraint is a plane.
- The intersection of two planes is a line.
- The intersection of the third plane with the first two planes will be a point on the line in case of a unique solution, or it may lead to pairs of parallel lines (constraint 1 intersection constraint 2 gives one line, constraint 1 intersection constraint 3 gives parallel line, constraint 2 intersection constraint gives parallel line) which means there is no solution.
- All three constraints or planes may intersect in the same line which means infinite solutions.

Example of system of equations

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

- This system has two equations and four unknowns, so it is underconstrained. We expect an infinity of solutions.
- Is there a special way in which to express the solutions to this system?
- Let us examine the structure of the given problem matrix.

Structure of the solution

- Looking at the previous slide we can see that a linear combination of columns of the matrix will give the right hand side.
- The ith column vector in the matrix appears in the linear combination, scaled by the corresponding x_i as below.

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 12 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

Particular solution to the example

- A closer look at the linear combination to give the right hand side shows that we can take $x_1 = 42$, $x_2 = 8$, $x_3 = 0$, $x_4 = 0$ since the first two columns are $(1, 0)^T$ and $(0, 1)^T$ respectively.
- Therefore a solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

- This solution is called the particular solution

Any other solutions possible?

- We can generate other solutions than the particular solution, by adding the vector $\mathbf{0}$ to the particular solution
- But isn't this the same as the particular solution as any vector $+\ \mathbf{0}$ is that vector itself?
- The trick is to express $\mathbf{0}$ in terms of the linear combination of some vectors.
- Describing $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4}$ as the four column vectors associated with the given matrix in the example we can see that $8\mathbf{c_1} + 2\mathbf{c_2}$ - $1\mathbf{c_3} + 0\mathbf{c_4} = \mathbf{0}$.

Any other solutions possible?

- Writing the linear combination in terms of a matrix-vector product we have

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Any vector $\lambda(8, 2, -1, 0)^T, \lambda \in R$ will also produce the $\mathbf{0}$ vector

Any other solutions possible?

- We can add the vector $(8, 2, -1, 0)^T$ to the original particular solution $(42, 8, 0, 0)^T$ to get another solution since
$$A(\begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 42 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

Any other solutions possible?

- Following the same line of reasoning as before, we can create the $\mathbf{0}$ vector by expressing the fourth column of the matrix $\boldsymbol{A}$ in terms of the first two columns - note that the first two columns appear capable of generating any two-dimensional vector!
- We can see that $-4\mathbf{c_1} + 12\mathbf{c_2} + 0\mathbf{c_3}$ - $1\mathbf{c_4} = \mathbf{0}$.
- Thus we have

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} (\lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting things together

- We obtain the following general solution as the sum of the particular solution and a linear combination of solutions to the equation $\mathbf{Ax} = \mathbf{0}$ as follows:

$$\{\mathbf{x} \in \mathbb{R}^4 : \mathbf{x} = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix}\}$$

- The general approach consisted of finding a particular solution to $\mathbf{Ax} = \mathbf{b}$, finding all solutions to $\mathbf{Ax} = \mathbf{0}$ and combining the particular and general solutions.
- Neither the particular nor general solutions are unique $\rightarrow$ why?

Algorithmic way of solving equations

- The system of equations in our example was easy to solve because of the special structure of the matrix - we could guess the solution without much difficulty.
- Can we develop an algorithmic way of solving a general system of equations?
- The answer is yes $\rightarrow$ we call the procedure Gaussian elimination

Elementary transformations

- The key idea is to take a complex looking matrix and transform it using elementary row operations to a simple looking matrix like the one we just handled, for which solutions could be obtained essentially by inspection.
- To make this work we need to preserve solutions of the original system of equations, i.e ensure that elementary transformations of the original matrix do not change its solutions.
- Do such elementary transformations exist?

What are the elementary operations?

- Exchange of rows
- Multiplying a row by a constant $\lambda \in R \backslash \{0\}$
- Adding a a row to another row
- Question $\rightarrow$ why must any multiplier to a row be non-zero?

Example to illustrate elementary operations

Consider the folliowing system where we seek all solutions for some $a \in R$.

$$\begin{aligned}
-2x_1 + 4x_2 - 2x_3 - x_4 + 4x_5 &= -3 \\
4x_1 - 8x_2 + 3x_3 - 3x_4 + x_5 &= 2 \\
x_1 - 2x_2 + x_3 - x_4 + x_5 &= 0 \\
x_1 - 2x_2 - 3x_4 + 4x_5 &= a
\end{aligned}$$

Compact representation

Let us take the preceding equations and express them compactly in matrix form:

$$\begin{bmatrix} -2 & 4 & -2 & -1 & 4| & -3 \\ 4 & -8 & 3 & -3 & 1| & 2 \\ 1 & -2 & 1 & -1 & 1| & 0 \\ 1 & -2 & 0 & -3 & 4| & a \end{bmatrix}$$

This matrix is called the augmented matrix. It is on this matrix that we will perform the elementary row operations.

Swap rows

Now swap rows 1 and 3 in the augmented matrix to get

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 1| & 0 \\ 4 & -8 & 3 & -3 & 1| & 2 \\ -2 & 4 & -2 & -1 & 4| & -3 \\ 1 & -2 & 0 & -3 & 4| & a \end{bmatrix}$$

Does this change the system of equations? No, because we are swapping **both left and right hand sides of the equality sign, so we are still dealing with the same set of equations.**

Subtract rows

$$\begin{array}{c}\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ -2 & 4 & -2 & -1 & 4 & -3 \\ 1 & -2 & 0 & -3 & 4 & a \end{array}\right]\end{array}\begin{array}{c} \\ -4R_1 \\ +2R_1 \\ +-1R_1 \end{array}$$

The notation above is used to convey that we would like to add $-4\times$ first row to the second row, $2\times$ the first row to the third row, and $-1\times$ the first row to the fourth row to get a new augmented matrix.

New Augmented Matrix

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & -1 & -2 & 3 & a \end{array}\right] \begin{array}{c} \\ \\ \\ -R_2 - R_3 \end{array}$$

Note that the augmented matrix shown is obtained by performing the operations shown on the previous slide. To get the next augmented matrix we subtract the second and third rows of this augmented matrix from the last row.

New Augmented Matrix

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & a+1 \end{array}\right] \begin{array}{c} \\ -1 \\ 1/3 \\ \\ \end{array}$$

Now multiply the second row by -1 and the third row by $\frac{1}{3}$ to get the augmented matrix in its final form, known as the row-echelon form.

Finding the particular solution

- The row echelon form makes finding a particular solution easy
- Remember that the idea is that a linear combination of the pivot columns must give the right hand side.
- In the example above this means that

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

- This looks like any regular linear combination for which we need to find the coefficients $\lambda_1, \lambda_2, \lambda_3$, so how is this really different from the original problem $\mathbf{Ax} = \mathbf{b}$?

Finding a particular solution

- The linear combination from the previous slide is easily solved.
- Start with finding the value of λ_3. We can see that the third equation establishes $\lambda_3 = 1$.
- The second equation involves only λ_2 and λ_3. Plugging the just discovered value of λ_3 into the second equation, we can find $\lambda_2 = -1$.
- Now we can plug the values of λ_2, λ_3 into the first equation to get $\lambda_1 = 2$

RREF Example

- Consider the following matrix in reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

- To find solutions for $\mathbf{Ax} = \mathbf{0}$ we need to look at non-pivot columns and note that the pivot columns are "strong enough" to generate the non-pivot columns.
- Our strategy to find solutions to $\mathbf{Ax} = \mathbf{0}$ is to find linear combinations of the pivot columns to the left of a non-pivot column to cancel out the non-pivot column, while setting all other coefficients to zero.

Example continued

- Thus we note that the second column is a non-pivot column which can be expressed as a multiple of the first column such that 3 times the first column + -1 * second column is equal to zero. This gives us our first solution.

$$\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example continued

- Similarly we note that 3 $\times$ the first column $+$ 9 $\times$ the third column $+$ -4 $\times$ the fourth column $+ -1\times$ the fifth column is equal to zero. This gives us our second solution:

$$\begin{bmatrix} 3 \\ 0 \\ 9 \\ -4 \\ -1 \end{bmatrix}$$

General solution

- If $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions to $\mathbf{Ax} = \mathbf{0}$, then any linear combination $\lambda_1\mathbf{x_1} + \lambda_\mathbf{2}\mathbf{x_2}$, $\lambda_1, \lambda_2 \in R$ is also a solution
- Thus the general solution to the problem is

$$x \in \mathbb{R}^5 : x = \lambda_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \\ 9 \\ -4 \\ -1 \end{bmatrix}, \lambda_1, \lambda_2 \in R$$

Gaussian elimination and Gauss Jordan

- Repeated use of elementary row operations done to convert a matrix to upper triangular form is called Gaussian elimination.
- Consider the system of equations $\mathbf{Ax} = \mathbf{b}$ where A is a $n \times n$ matrix.
- If A is invertible, it means that A^{-1} exists such that $AA^{-1} = A^{-1}A = I_n$
- In such a case the row-reduced echelon form of A is I_n, i.e every column is a pivot column where the pivot is 1.
- The process of converting A to I_n that we have discussed above is called the Gauss-Jordan method.

Gaussian elimination and Gauss Jordan

- In Gaussian Elimination we use multiples of the first row to eliminate the entries in the first column below the first row.
- Then we use multiples of the second row to eliminate entries in the second column below the second row and so on until we get an upper-triangular matrix.
- This Gauss Jordan process is shown diagramatically in the next slide.
- Then we take multiples of the last row to eliminate non-zero entries in the last column above the last entry, followed by multiples of the last but one row to eliminate non-zero entries in the last but one column and so on. This gives us a diagonal matrix.

Gauss Jordan diagram

$$
\begin{tikzcd}
\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \arrow[r] & \begin{bmatrix} a & b & c \\ 0 & e' & f' \\ 0 & h' & i' \end{bmatrix} \arrow[r] & \begin{bmatrix} a & b & c \\ 0 & e' & f' \\ 0 & 0 & i'' \end{bmatrix} \arrow[r] & \begin{bmatrix} a & 0 & c' \\ 0 & e' & f' \\ 0 & 0 & i'' \end{bmatrix} \arrow[d] \\
 & & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} a & 0 & 0 \\ 0 & e' & 0 \\ 0 & 0 & i'' \end{bmatrix} \arrow[l]
\end{tikzcd}
$$

Calculating inverse

- Can the Gauss Jordan procedure calculate the inverse of a matrix?
- For example, let A be a $n \times n$ matrix whose inverse A^{-1} exists. We would like to compute its inverse using Gauss Jordan procedure. Is this possible?
- Yes we can compute the inverse in the following way: we simply set up n linear systems of the form $\mathbf{Ax} = \mathbf{e_i}$, $1 \leq i \leq n$ where $\mathbf{e_i}$ is the ith canonical basis vector and find their solutions $\mathbf{x}$. Each solution vector constitutes a column in A^{-1}. Why is this true?

Calling Inverse

- Consider the linear system $\mathbf{Ax} = \mathbf{e_i}$.
- Gauss Jordan procedure will convert this sytem to the equivalent system $\mathbf{I_n x} = \mathbf{c_i}$ whose solution is $\mathbf{x} = \mathbf{c_i}$.
- On the other hand, the solution to $\mathbf{Ax} = \mathbf{e_i}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{e_i}$.
- Since the two systems are equivalent they have the same solution, so $\mathbf{x} = \mathbf{c_i} = \mathbf{A}^{-1}\mathbf{e_i}$ which means $\mathbf{c_i}$ is the ith column of $\mathbf{A}^{-1}$.
- Thus when we create the augmented matrix $[Ae_i]$, Gauss Jordan procedure will convert it into $[I_n c_i]$.
- We can solve n linear systems at once by letting the augmented matrix be $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ which will become $[I_n A^{-1}]$.