4 | Matrix

→ **Sub matrix of a matrix** : Suppose A is any matrix of order $m \times n$, then a matrix obtained by ~~living~~ leaving some rows and columns from matrix A is called a sub matrix of A.

→ **Minors of a matrix** – If A is a $m \times n$ matrix then determinant of every square matrix A is called minor of the matrix A.

→ **Rank of a matrix** : A number r is said to be the rank of a matrix A, if it possesses the following two properties.

(i) There is at least one square sub matrix A of order r whose determinant is not equal to 0.

(ii) If the matrix A contains any square submatrix of order $r+1$ then the determinant of every square sub matrix of A of order r+1 should be 0.

In short the rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

~~det~~

Note: ~~det (A) = 0 ⟹ A is singular matrix~~.

$|A| = 0 \Rightarrow A$ is singular matrix.

Note:

(i) Rank of every non singular matrix of order n is n.

(ii) Rank of every non zero matrix ≥ 1

(iii) Rank of every null matrix is zero.

(iv) The rank of a matrix is $\leq r$

If r $(r+1)$-rowed minors of the matrix ~~vanished~~ vanishes

(v) The rank of a matrix is $\geq r$, if there is at least one r-rowed minor of the matrix not equal to 0.

Q. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol^n: $\rho\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 3$

$\because$ this is a non-singular matrix of order 3×3.

Echelon form of a Matrix,

A matrix A is said to be in echelon form if —

(i) Every row of A which has all it entries 0 occurs below every row which has a non zero entry.

(ii) The first non zero entry in each non zero row = 1.

(iii) The number of zeroes before the 1st non zero element in a row is less than the no. of such zeroes in the next row.

Note: The rank of a matrix in echelon form is equal to the no. of non zero rows in the matrix

Q. Find the rank of the following matrix.

(i) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\because$ The given matrix is already in the echelon form and the no. of non. zero rows in the matrix is 2.

$\therefore$ Rank of the given matrix is 2.

Note: The rank of the transpose of a matrix is same as that of the original matrix.

i.e. $\rho(A) = \rho(A^T)$

*** Determinant of any ~~squar~~ identity matrix is 0.

Elementary Matrix :

A matrix ~~of~~ obtained from a unique or identity matrix by a single elementary transformation is called an elementary matrix.

(eg).
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \longleftrightarrow R_2$$

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Q. Find rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Solⁿ, Given matrix is :-

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & 1 & -2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2/-4 \\ R_3 \to R_3/-3 \end{array}$$

Which is in the echelon form.

$\therefore$ no. of non zero rows in the echelon form of the matrix $= 3$.

~~Rank $\rho(A) = 3$~~

$\therefore \rho(A) = 3$

Reduction to normal form

Every $m \times n$ matrix of rank r can be reduced to the form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ by a chain of elementary operations

where I_r is the r-rowed identity matrix of order $r \times r$.

Q Reduce $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to canonical or normal form.

Solⁿ: Given matrix is —

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} \quad R_2 \to R_2/8$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 9/4 & -8 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 1 & -32/9 \end{bmatrix} \quad R_3 \to 9/4\, R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -32/9 \end{bmatrix} \quad C_3 \to C_3 - \frac{5}{8} C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C_4 \to C_4 + \frac{32}{9} C_3$$

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$\sim \begin{bmatrix} I_3 & O \end{bmatrix} \qquad \begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$$

Thus, the matrix A is equivalent to $\begin{bmatrix} I_3 & O \end{bmatrix}$

Hence, ~~Ran~~ $\rho(A) = 3$

Q Find the rank of the following matrix.

$$A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\cancel{\begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & d/c & 0 \end{bmatrix}}$$

$$\sim \begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & d/c & 1/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to R_2/c$$

Which is in the echelon form

$\therefore$ no. of non zero rows $= 2$

$\therefore \rho(A) = 2$

Matrix Polynomial

An expression of the form

$$F(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_m\lambda^n$$

where $A_0, A_1, \cdots A_m$ are all square matrices of the same order is called a matrix polynomial of order n.

A_m is ~~a~~ not a null matrix

Eigen values or Eigen Vectors (characteristic values & characteristic vector of a matrix)

Let $A = [a_{ij}]_{n \times n}$ be any ~~equal~~ square matrix

~~& λ be a~~

& λ be a scalar then, the matrix $[A - \lambda I]$ is called the characteristic matrix of A, where ~~xxx~~ I is the identity matrix of order $n \times n$

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Q The determinant ~~of~~ —

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_{nn}-\lambda \\ a_{n1} & a_{n2} & \cdots & \end{vmatrix}$$

which is an arbitrary polynomial in λ of degree n is called characteristic polynomial of a

Note: $|A - \lambda I| = 0$ is called characteristic equation

P.T.O.

Eigen Value

The roots of the characteristic equation are called characteristic roots or characteristic values or eigen values of the matrix A.

Eigen Vectors

If λ is a eigen values of a matrix $A_{n \times n}$. Then a non-zero vector X such ~~#~~ that $\boxed{AX = \lambda X}$ is called characteristic vector ~~of~~ or... of A corresponding to the eigen value λ.

Theorem

If X is a characteristic vector of a matrix A corresponding to the eigen value λ then ~~kX~~ kX is also a eigen vector of matrix a corresponding to the same eigen value λ, where k is any non zero scalar.

Proof: Suppose, X is a eigen vector of A ~~corres~~ corresponding to λ, then ~~$X \neq 0$~~ & ~~$AX = \lambda X$~~

$$X \neq 0$$

$$\& \; AX = \lambda X \longrightarrow (1)$$

If k is any non-zero scalar $kX \neq 0$

~~Also~~, Now we are to prove that $A(kX) = \lambda(kX)$

$$LHS = A(kX)$$

$$= k(AX) \quad \because k \text{ is a scalar.}$$

$$= k(\lambda X) \longrightarrow \text{using } (1)$$

$$= \lambda(kX) = RHS$$

Hence (kX) is an eigen vector of matrix A corresponding to the eigen value λ.

Theorem: If ~~X~~ is an eigen vector of matrix A, then X cannot correspond to more than one eigen value of A.

Proof: Let X be an eigen vector of a matrix A then let us consider two eigen values λ_1 & λ_2.

If possible let X be an eigen vector corresponding to both eigen value λ_1 & λ_2.

i.e. $AX = \lambda_1 X \longrightarrow$ (i)

$AX = \lambda_2 X \longrightarrow$ (ii)

Now from (i) & (ii)

$$\lambda_1 X = \lambda_2 X$$

$$\cancel{\Rightarrow \lambda_1 = \lambda_2}$$

$$\Rightarrow \lambda_1 X - \lambda_2 X = 0$$

$$\Rightarrow (\lambda_1 - \lambda_2) X = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0 \quad \left(\because X \text{ is an eigen vector } \therefore X \neq 0 \right)$$

$$\Rightarrow \lambda_1 = \lambda_2$$

$\therefore$ Eigen vector X cannot correspond to more than one eigen value of A.

Q Determine the ~~of~~ eigen ~~vector~~ ~~vector~~ value of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

$$|A - \lambda I| = 0$$

Sol^n: For finding eigen ~~vector~~ values we need ~~to~~ to form roots of the eq^n.

$\cancel{A - \lambda I}$

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 0-\lambda & 1 & 2 \\ 0 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{bmatrix} = 0$$

$$\Rightarrow -\lambda \begin{vmatrix} 1-\lambda & -1 \\ -1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & -1 \\ 2 & -\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & 1-\lambda \\ 2 & -1 \end{vmatrix}$$

$$\Rightarrow -\lambda(-\lambda + \lambda^2 - 1) - 1(2) + 2(2\lambda - 2) = 0$$

$$\Rightarrow +\lambda^2 - \lambda^3 + \lambda - 2 + 4\lambda - 4 = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2 + 5\lambda - 6 = 0$$

$$\Rightarrow \cancel{\lambda^3 - \lambda^2 +}\ \lambda^3 - \lambda^2 - 5\lambda + 6 = 0 \rightarrow (i)$$

Putting $\lambda = 2$ in (i) satisfies the equation

$\therefore (\lambda - 2)$ is a factor of $\lambda^3 - \lambda^2 - 5\lambda + 6$

Now, $\cancel{\Rightarrow \lambda(\lambda^2 - 5) - \lambda(}$

$\Rightarrow (\lambda - 2)(\lambda^2 + \lambda - 3) = 0$

$\Rightarrow (\lambda - 2)\left(\frac{-1 \pm \sqrt{13}}{2}\right)$

$\therefore \lambda = \frac{-1 \pm \sqrt{13}}{2}, 2$, reqd eigen value

$$
\begin{array}{r|l}
 & \lambda^2 + \lambda - 3 \\
\lambda - 2 & \lambda^3 - \lambda^2 - 5\lambda + 6 \\
 & \lambda^3 - 2\lambda^2 \\ \hline
 & \lambda^2 - 5\lambda + 6 \\
 & \lambda^2 - 2\lambda \\ \hline
 & -3\lambda + 6 \\
 & -3\lambda + 6 \\ \hline
 & 0
\end{array}
$$

$\cancel{\frac{-1 \pm \sqrt{1+12}}{2}}$

$\frac{-1 \pm \sqrt{13}}{2}$

H/W. Find the eigen values of the matrix

$$A = \begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & c & c \end{bmatrix}$$

Soln: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & h & g \\ 0 & b-\lambda & 0 \\ 0 & c & c-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)\begin{vmatrix} b-\lambda & 0 \\ c & c-\lambda \end{vmatrix} - h\begin{vmatrix} 0 & 0 \\ 0 & c-\lambda \end{vmatrix} + g\begin{vmatrix} 0 & b-\lambda \\ 0 & c \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)(b-\lambda)(c-\lambda) - 0 + 0 = 0$$

$$\Rightarrow (a-\lambda)(b-\lambda)(c-\lambda) = 0$$

$$\Rightarrow a-\lambda = 0 \text{ or } b-\lambda = 0 \text{ or } c-\lambda = 0$$

$$\Rightarrow \lambda = a \text{ or } \lambda = b \text{ or } \lambda = c$$

$\therefore$ Eigen value of matrise A are a, b, c.

Q. Determine the eigen values & eigen vectors of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol^n: ~~$A - \lambda I$~~ The given matrix is $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$$\therefore |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(2-\lambda) - (4 \times 1) = 0$$

$$\Rightarrow 10 - 5\lambda - 2\lambda + \lambda_2 - 4 = 0$$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 6) = 0$

$\Rightarrow \lambda = 1, 6$

$\therefore$ The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of matrix A corresponding to the eigen value 6 are given by the non zero sol^n

$$(A - 6I)X = 0$$

$$\Rightarrow \begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using transformation $R_2 \to R_1 + R_2$ we have

$$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -x_1 + 4x_2 = 0$

$\Rightarrow x_1 = 4x_2$

So $X = \begin{bmatrix} 4a \\ a \end{bmatrix}$ is the eigen vector of matrix A.

corresponding to $\lambda = 6$.

Hence, $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ will be a eigen vector of matrix A

corresponding to $\lambda = 6$.

Now, The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of matrix A

corresponding to the eigen value 1.

given by the non zero sol^n.

when $\lambda = 1$

$$(A - 1I)X = 0$$

$$\Rightarrow \begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\cancel{\Rightarrow \begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \left[C_2 \rightarrow C_2 - C_1 \right]}$$

$$\Rightarrow \begin{bmatrix} 4x_1 + 4x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 = -x_2$

$\therefore X = \begin{bmatrix} a \\ -a \end{bmatrix}$ is the ~~vector~~ eigen vector of matrix A corresponding to eigen value, $\lambda = 1$.

Cayley - Hamilton theorem

$$|A - \lambda I| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$\cancel{= (a_{22}-\lambda)(a_{33}-\lambda)}$$

$$= (a_{11}-\lambda)\left[a_{22}a_{33} - a_{22}\lambda - a_{33}\lambda + \lambda^2 - a_{23}a_{32}\right] - a_{12}\left[a_{21}a_{33} - a_{21}\lambda - a_{23}a_{31}\right]$$
$$+ a_{13}\left[a_{21}a_{32} - a_{31}a_{22} - a_{31}\lambda\right]$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{22}\lambda - a_{11}a_{33}\lambda + a_{11}\lambda^2 - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$
$$+ a_{12}a_{21}\lambda + a_{23}a_{31}a_{12} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} - a_{13}a_{31}\lambda$$
$$- a_{22}a_{33}\lambda + a_{22}\lambda^2 + a_{33}\lambda^2 - \lambda^3 + a_{23}a_{32}\lambda$$

$$= -\lambda^3 + (a_{11}+a_{22}+a_{33})\lambda^2 - (-a_{23}a_{32} + a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33} - a_{12}a_{21}$$
$$+ a_{13}a_{31})\lambda + (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$
$$+ a_{23}a_{31}a_{12} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22})$$

$$= -1\left[\lambda^3 - (\text{sum of the diagonal elements})\lambda^2 + (\text{sum of diagonal minors})\lambda - |A|\right]$$

or $= (-1)^n\left(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} \cdots + a_n\right)$

Then
$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = 0$$

Proof: Since the elements of $A-\lambda I$ are at most of 1^{st} degree in λ, then the elements of adjoint of $A-\lambda I$ ($adj(A-\lambda I)$) are ordinary polynomials in λ of degree $n-1$ or less.

$\therefore$ ~~adj (A)~~. adj $(A-\lambda I)$ can be written as a matrix polynomial in λ given by

$$adj(A-\lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}$$

where $B_0, B_1, \cdots, B_{n-1}$ are matrices of the order $n \times n$, whose elements are functions of ~~of~~ a_{ij}'s

Now, $(A-\lambda I)\, adj(A-\lambda I) = |A-\lambda I|\, I_n$
$(\because A\, adj(A) = |A| I_n)$

$\therefore (A-\lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1})$
$$= (-1)^n \left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n\right] I_n$$

Now, comparing the ~~compo~~ coefficients of like powers of λ on both side of the above eq^n. we get,

$$-IB_0 = (-1)^n I_n$$
$$AB_0 - IB_1 = (-1)^n a_1 I$$
$$AB_1 - IB_2 = (-1)^n a_2 I$$

$$AB_{n-1} = (-1)^n a_n I$$

~~Pre~~ Pre-multiplying these equations successively by $A^n, A^{n-1}, \ldots A, I$ ~~and~~ and adding ~~we~~ we get. $O = (-1)^n \left[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots \ldots + a_n I\right]$

$$\Rightarrow A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots \ldots + a_n I = O$$

Hence, the theorem.

Sol^n. 1

H/W

Solⁿ: 1. The given matrix is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$\therefore \quad |A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} + 6\begin{vmatrix} -6 & -4 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -6 & 7-\lambda \\ 2 & -4 \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)(21-10\lambda+\lambda^2-16) + 6(-18+6\lambda+8) + 2(24-14+2\lambda) = 0$$

$$\Rightarrow (8-\lambda)(\lambda^2-10\lambda+5) + 6(6\lambda-10) + 2(2\lambda+10) = 0$$

$$\Rightarrow 8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 36\lambda - 60 + 4\lambda + 20 = 0$$

$$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 15\lambda^2 + 45\lambda = 0$$

$$\Rightarrow \lambda^2(\lambda-3) - 15\lambda(\lambda-3) = 0$$

$$\Rightarrow (\lambda^2 - 15\lambda)(\lambda - 3) = 0$$

$$\Rightarrow \lambda(\lambda-15)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 0, 15, 3$$

But, λ cannot be zero

$\therefore \lambda = 15, 3$.

~~If X =~~

Let the eigen vector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Now,

when $\lambda = 15$

$$(A - 15I)X = 0$$

$$\Rightarrow \cancel{\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7x_1 - 6x_2 + 2x_3 \\ -6x_1 - 8x_2 - 4x_3 \\ 2x_1 - 4x_2 - 12x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 - 6x_2 + 2x_3 = 0 \quad \cancel{= 6x_1 + 8x} \longrightarrow (i)$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \longrightarrow (ii)$$

$2x_1 - 4x_2 - 12x_3 = 0 \longrightarrow$ (iii)

$\therefore$ (i) $\cancel{\ldots}\; 7x_1 = -6x_2 + 2x_3$

$\Rightarrow 2x_3 = 7x_1 + 6x_2$

$\Rightarrow x_3 = \frac{7}{2}x_1 + 3x_2 \longrightarrow$ (iv)

(ii) $\Rightarrow -6x_1 - 8x_2 - 4\left(\frac{7}{2}x_1 + 3x_2\right) = 0$

$\Rightarrow +6x_1 + 8x_2 + 14x_1 + 12x_2 = 0$

$\Rightarrow 20x_1 + 20x_2 = 0$

$\Rightarrow x_1 = -x_2$

$\cancel{\text{(iii)} \Rightarrow 2x_2 - 4x_2 - 12\left\{\frac{7}{2}(-x_2) + 3x_2\right\} = 0}$

$\cancel{\Rightarrow 6x_2 - 42x_2 + 36x_2 = 0}$

$\cancel{\Rightarrow}$

(iv) $\Rightarrow x_3 = -\frac{7}{2}x_2 + 3x_2$

$\Rightarrow 2x_3 = -x_2$

$\therefore X = \begin{bmatrix} 2a \\ -2a \\ a \end{bmatrix}$ is eigen vector of matrix A corresponding to $\lambda = 15$.

Again, when $\lambda = 3$

$$(A - 3I)X = O$$

$$\Rightarrow \begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -4 & 0 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{using } R_2 \to R_2 + R_3)$$

$$\Rightarrow \begin{bmatrix} 5x_1 - 6x_2 + 2x_3 \\ -4x_1 - 4x_3 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\cancel{\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0, \; -4x_1 - 4x_3 = 0, \; 2x_1 - 4x_2 = 0}$$

$$\Rightarrow -4x_1 - 4x_3 = 0 \longrightarrow (i)$$

$$2x_1 - 4x_2 = 0 \longrightarrow (ii)$$

(i) $\Rightarrow$ $x_1 = -x_3$

(ii) $\Rightarrow$ $x_1 = 2x_2$

$$\therefore X = \begin{bmatrix} 2a \\ a \\ -2a \end{bmatrix}$$ is the eigen vector of matrix A

corresponding to $\lambda = 3$.

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

$$= 31$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

14 + 80
20 + 88

427

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \quad \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

74 108

$$\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}} \cancel{\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}$$

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

18 1

$$= \begin{bmatrix} 31 & 32 \\ 72 & 31 \end{bmatrix}$$

Teacher's Sign / Remarks

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Q Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ and verify that it is satisfied by A.

Hence obtained A^{-1}.

Sol^n: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-(2-\lambda)+1\} + 1\{1-(2-\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)\{4+\lambda^2 - 4\lambda - 1\} + (\lambda - 1) + (\lambda - 1) = 0$$

$$\Rightarrow 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda + 2\lambda - 2 = 0$$

$$\cancel{\Rightarrow -\lambda^3 + 6\lambda^2 - 7\lambda - 12 = 0}$$

$$\cancel{\Rightarrow \lambda^3 - 6\lambda^2 + \lambda + 12 = 0}$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$\therefore$ The characteristic equation at A is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Now, we are to show that

$$A^3 - 6A^2 + 9A - 4I = 0$$

Here,

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -4 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -10-6-5 & 10+5+6 \\ -6-10-5 & 5+12+5 & -5-10-6 \\ 6+5+10 & -5-5-10 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 3A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}$$

$$- \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = 0$$

Pre multiplying the above equation by A^{-1}

we have,

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = 0$$

$$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3/4 & 1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 \\ -1/4 & 1/4 & 3/4 \end{bmatrix}$$

H/W

Q.1. Shows that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies

satisfies cayley Hamilton theorem.

H/W

Q. Obtain the characteristic equation of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

and verify that it is satisfied by A and hence find A^{-1}.

Q State Cayley Hamilton theorem. Use it to express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}^{-1}$

Sol: Firstly we will find the ~~value~~ of characteristic equation of A.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda) + 1 = 0$$

$$\Rightarrow 6 - 3\lambda - 2\lambda + \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 7 = 0$$, which is the characteristic equation of A

$\to$ (i)

By Cayley Hamilton Theorem,

The matrix A must satisfy eqn (i)

i.e. $A^2 - 5A + 7I = 0 \longrightarrow$ (ii)

From (i) we have

$$A^2 = 5A - 7I \longrightarrow \text{(iii)}$$

Now,

$$A^3 = A^2 A$$

$$\Rightarrow A^3 = (5A - 7I)A \quad (\text{using (iii)})$$

$$\Rightarrow A^3 = 5A^2 - 7A \longrightarrow \text{(iv)}$$

$$A^4 = (A^2)^2$$

$$\Rightarrow A^4 = (5A - 7I)^2$$

$$\Rightarrow A^4 = (25A^2 - 70A + 49I)$$

$$A^5 = A^3 A^2$$

$$\Rightarrow A^5 = (5A^2 - 7A)(5A - 7I)$$

$$\Rightarrow A^5 = 5(5A - 7I)(5A - 7I)$$

$$\Rightarrow A^5 = (25A - 35I)(5A - 7I)$$

$$\Rightarrow A^5 = (125A^2 - 175A - 175A + 245I)$$

$$\Rightarrow A^5 = 125A^2 - 350A + 245I$$

Now,

$$2A^5 - 3A^4 + A^2 - 4I$$

$$\Rightarrow 2(125A^2 - 350A + 245I) - 3(25A^2 - 70A + 49I) + 5A - 7I - 4I$$

$$= 250A^2 - 700A + 490I - 75A^2 + 210A - 147I$$
$$+ 5A - 7I - 4I$$

$$= 175A^2 - 485A$$

$$= 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I$$
$$= 10A^4 - 14A^3 - 3A^4 + A^2 - 4I$$
$$= 7A^4 - 14A^3 + A^2 - 4I$$
$$= 7(5A^3 - 7A^2) - 14A^3 + A^2 - 4I$$
$$= 35A^3 - 49A^2 - 14A^3 + A^2 - 4I$$
$$= 21A^3 - 48A^2 - 4I$$
$$= 21(5A^2 - 7A) - 48A^2 - 4I$$
$$= 105A^2 - 147A - 48A^2 - 4I$$
$$= 57A^2 - 147A - 4I$$
$$= 57A^2 - 147A - 4I$$
$$= 285A - 399I - 147A - 4I$$
$$= 138A - 403I$$

1 | Sets, Relation and Function

Set : A set is a well-defined collection of objects.

Subset : A set A is said to be a subset of set B if every element of A is also an element of B.

i.e. $A \subseteq B$

if $a \in A \Rightarrow a \in B$

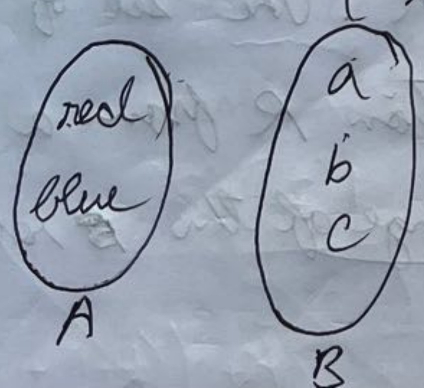
Cartesian Products of ~~a~~ sets

Let P & Q be ~~a~~ 2 non-empty sets. The cartesian product $P \times Q$ is ~~a~~ the set of all ordered pairs of elements from P & Q.

$$P \times Q = \{(a,b) : a \in P, b \in Q\}$$

(eg.) Let $A = \{\text{red}, \text{blue}\}$

$B = \{a, b, c\}$



~~Then $A \times B = \{(\text{red}, a)$~~, Then $A \times B = \{(red, a), (red, b), (red, c), (blue, a), (blue, b), (blue, c)\}$

Note: If there are p elements in ~~A~~ set A & q elements in set B. Then there will be pq elements in $A \times B$.

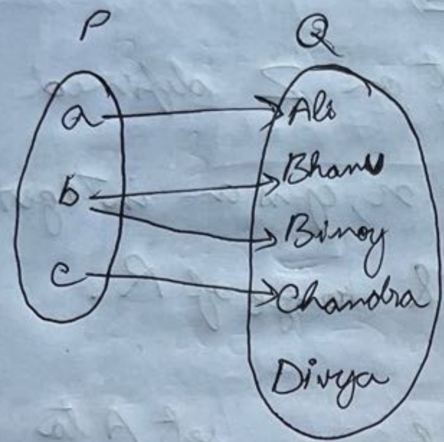
A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$.

The subset is derived by describing a relationship between the 1st element & the 2nd element of the ordered pair in a relation R, from set A to set B. The second element is called as the image of the 1st element.

The set of all 1st element of the ordered pairs in a relation R from a set A to set B is called the domain of the relation R. The set of all 2nd elements in a relation R from a set A to set B is called the range of the relation R.

· The whole set B is called the <u>co-domain</u> of the relation R.

Range $\subseteq$ Co-domain.



Let $P = \{a, b, c\}$

$Q = \{\text{Ali}, \text{Bhanu}, \text{Binoy}, \text{Chandra}, \text{divya}\}$

be two sets and consider a relation

$$R = \{(x, y) : x \text{ is the } 1^{st} \text{ letter of the name } y,\ x \in P, y \in Q\}$$

$$R = \{(a, \text{Ali}), (b, \text{Bhanu}), (b, \text{Binoy}), (c, \text{Chandra})\}$$

<u>H/W</u> ~~Let A~~

<u>Q. 1.</u> Let $A = \{1, 2, 3, 4, 5, 6\}$ define a relation R from set A to set A by

$R = \{(x, y) : y = x+1, x, y \in A\}$ depict the relation using a arrow diagram. Write down the domain, co-domain, range of R.

<u>Note</u>: If A has ~~P~~ p elements & B has q elements then the total no. of relations is 2^{pq}. ~~Let R be a relation on Z~~

<u>H/W</u>. 2) Let R be a set on $\mathbb{Z}$ defined by

$$R = \{(x, y) : x, y \text{ is an integer}; x, y \in \mathbb{Z}\}$$

Find the domain & range of R.

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<u>Function</u>: A relation f from a set A to the set B is said to be a function ~~O~~ if every element of set A has one and only one image in set B.

<u>H/W</u> 3) Let $\mathbb{N}$ be the set of natural nos. & the relation R be defined on $\mathbb{N}$ such that $R = \{x, y : y = 2x, x, y \in \mathbb{N}\}$. What is the domain, co-domain & range of R? Is this relation a function?

$$\cancel{\Rightarrow \frac{1(2)(3)}{6}}$$

$$\cancel{1 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}}$$

Principle of Mathematical Induction

Suppose there is a statement $P(n)$ involving the natural number n such that

(i) The statement is true for $n=1$,
i.e. $P(1)$ is true.

(ii) If the statement is ~~true~~ true for $n=k$, where k is some +ve integer, then the statement is also true for $n=k+1$
i.e. $P(k)$ is true $\Rightarrow P(k+1)$ is true

Then $P(n)$ is true for all natural number n.

Q For all $n \geq 1$, prove that ~~$1^2+2^2+3^2+\cdots+n^2$~~

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Sol^n, Let us consider,

$$P(n) : 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n=1$.

$$P(1) = 1^2 = \frac{1(1+1)(2+1)}{6}$$

Which is true.

Let us asume that $p(k)$ is true

i.e.

$$1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now, we are to show that $p(k+1)$ is true.

$$1^2+2^2+3^2+\cdots+k^2+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$= (k+1)\left\{\frac{k(2k+1)+6(k+1)}{6}\right\}$$

$$= (k+1)\left\{\frac{2k^2+7k+6}{6}\right\}$$

$$= \frac{(k+1)(2k^2+4k+3k+6)}{6}$$

$$= \frac{(k+1)\{2k(k+2)+3(k+2)\}}{6}$$

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

$$= \frac{(k+1)(2(k+1)+1)((k+1)+1)}{6}$$

$$= \frac{(K+1)\{(K+1)+1\}\{2(K+1)+1\}}{6}$$

$\therefore P(K)$ is true $\Rightarrow P(K+1)$ is true.

Hence, by the principle of mathematical induction the given statement is true for all $n \geq 1$.

Q ~~Soll~~ For all $n \geq 1$ prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Ans. For $n = 1$

$$P(1) = \frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{1+1}$$

Which is true.

Let us assume that $P(K)$ is true

i.e.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(K+1)} = \frac{k}{K+1}$$

Now, we are to show that $P(K+1)$ is true.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{K(K+1)} + \frac{1}{(K+1)\{(K+1)+1\}}$$

$$= \frac{K}{K+1} + \frac{1}{(K+1)\{(K+1)+1\}}$$

$$\cancel{= \frac{1}{(K+1)}\left(\frac{K\{(K+1)+1\}}{(K+1)+1}\right)}$$

$$= \frac{1}{(K+1)}\left\{\frac{K\{(K+1)+1\}+1}{(K+1)+1}\right\}$$

$$= \frac{K(K+1)+(K+2)}{(K+1)\{(K+1)+1\}}$$

$$= \frac{\cancel{(K+1)}(K+1)}{\cancel{(K+1)}\{(K+1)+1\}} = \frac{K+1}{(K+1)+1}$$

which is true.

$\therefore P(K)$ is true $\Rightarrow P(K+1)$ is true

Hence, by mathematical induction principle, $P(n)$ is true for all +ve integer n.

<u>Q</u> For every +ve integer n prove that

$$7^n - 3^n \text{ is divisible by } 4.$$

<u>Sol^n</u>: Let $P(n) : 7^n - 3^n$ is divisible 4

$P(1) = 7 - 3 = 4$ is divisible 4.

which is true.

Let us assume that $P(K)$ is true

i.e. $7^K - 3^K$ is divisible 4.

$\Rightarrow 7^K - 3^K = 4p$, for some integer 'p'.

Now, we are to show that $P(k+1)$ is true

$P(k+1): 7^{k+1} - 3^{k+1}$

$= 7^k \cdot 7 - 3^k \cdot 3$

$= (7^k - 3^k + 3^k)7 - 3^k \cdot 3$

$= (4p + 3^k)7 - 3^k \cdot 3$

$= 7 \cdot 4p + 7 \cdot 3^k - 3 \cdot 3^k$

$= 7 \cdot 4p + 3^k \cdot 4$

$= 4(7p + 3^k)$

$= 4m$, where $m = 7p + 3^k$

which is divisible by 4.

$\therefore P(k)$ is true $\Rightarrow P(k+1)$ is true.

Hence, by mathematical induction principle, $P(n)$ is true for all +ve integer n.

H/W, Ch-1, Ch-2, Ch-4.

$P(n)$

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Note: Let A be a non empty set. A relation R in A is called

(i) Reflexive : If $(a,a) \in R \ \forall a \in A$

(ii) Symmetric : If $(a,b) \in R$
$\Rightarrow (b,a) \in R \ \forall a,b \in A$

(iii) Transitive : If $(a,b),(b,c) \in R$
$\Rightarrow (a,c) \in R$ ~~$\forall a,b,c \in$~~

(iv) Anti symmetric : If $(a,b) \in R$ & $(b,a) \in R$
$\Rightarrow a = b$

Equivalence Relation —

A relation R is called an equivalence relation if it is reflescive, symetric and transitive.

Partial Order Relation —

A relation R on a set A is called a partial order relation if it is reflescive, antisymmetric and transitive.

Eg — Let 'A' be the set of all lines in a plane. Let $R \subseteq A \times A$, where $R = \{(l,m) : l \parallel m,\ l,m \in A\}$

Then we can show that R is an equivalence relation.

Sol^n: (i) Reflexive — Let $l \in L$ be any line, we know that every line is parallel to itself.

$\therefore (l, l) \in R \; \forall \, l \in A \quad \therefore R$ is reflexive

(ii) Symmetric — Let $(l, m) \in R$

$\Rightarrow l \parallel m$

$\Rightarrow m \parallel l$

$\Rightarrow (m, l) \in R$

$\therefore R$ is symmetric.

(iii) Transitive — Let $(l, m), (m, n) \in R$

$\Rightarrow l \parallel m, \; m \parallel n$

$\Rightarrow l \parallel n$

$\Rightarrow (l, n) \in R$

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

Q Let '$\mathbb{Z}$' be the set of integers, then show that the usual sign $\leq$ is a partial order relation on '$\mathbb{Z}$'.

Sol^n: (i) Reflexive — Let $a \in \mathbb{Z}$ be an integer.

$\because a \leq a \ \forall a \in \mathbb{Z}$

$\therefore \leq$ is reflexive in $\mathbb{Z}$.

(ii) <u>Anti symmetric</u> —

Let $a \leq b$ and $b \leq a$

$\Rightarrow a = b$

$\therefore \leq$ is anti-symmetric in $\mathbb{Z}$

(iii) Transitive — Let $a, b, c \in \mathbb{Z}$

Let $a \leq b$, $b \leq c$

$\Rightarrow a \leq b \leq c$

$\Rightarrow a \leq c$

$\therefore \leq$ is transitive in $\mathbb{Z}$

Hence, $\leq$ is ~~an~~ a partial order relation on $\mathbb{Z}$.

m divides n

<u>H/W</u> (4) Let $R = \{(m, n) : m, n\ \mathbb{Z},\ m \mid n\}$, where $x \mid y$ means there exist z such that $y = zx$. Then show that R is a partial order relation.

<u>Equivalence Classes</u> — Let X be a non empty set and let '$\sim$' be an equivalence relation on X for any $a \in X$, we define the quivalence class of a by $cl(a) = \{x \in X : x \sim a\}$

i.e. equivalence class of 'a' contains all those members of X which are related to 'a' under the relation '$\sim$'.

<u>Partition</u> — Let X be a non empty set. Let K = set of non-empty subsets of X such that every two distinct members of K are disjoint. Then K is called a partition of X, if X = the union of all members of K.

Eg — $X = \{1, 2, 3\}$
$K = \{\{1\}, \{2\}, \{3\}\}$

4) <u>Sol^n^</u>. Let ~~z be an~~ $z \in \mathbb{Z}$ be an integer.

~~Let m ∈ Z~~

(1) Reflexive : Let $m \in \mathbb{Z}$

Then m/m because any integer is divisible by itself.

$\therefore (m, m) \in R$

$\therefore R$ is reflexive

(ii) Antisymmetric :- Let $(m,n) \in R$

Then $m|n$

$\Rightarrow n = zm$

$\Rightarrow m = \frac{1}{z} n$

~~$\Rightarrow$ n is not divisible by n $\forall m,n \in \mathbb{Z}$~~

~~$\Rightarrow (n,m) \notin R$ when $n \neq m$~~

$\Rightarrow n|m$ only when $z=1$ i.e. $m=n$

$\therefore$ R is antisymmetric.

(iii) Transitivity :— Let $m,n,o \in \mathbb{Z}$

Let $(m,n),(n,o) \in R$

then $m|n$, $n|o$

~~$\Rightarrow m|n|o$~~

$\Rightarrow n = zm$, $o = yn$ (where $z,y \in \mathbb{Z}$)

$\Rightarrow o = y(zm)$

$\Rightarrow o = (yz)m$

$\Rightarrow o = xm$ (where $x = yz$)

$\Rightarrow m|o$

$\Rightarrow (m,o) \in R$

$\therefore$ R is transitive.

Hence R is a partial order equation.

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Combinators

Principle of Counting

In the fundamental system of counting if an event can occur 'm' in different ways following which another event can occur in n different ways, then the total no. of occurance of the event in the given order is $m \times n$.

Q How many 2 digit even no. can be formed from the digits 1, 2, 3, 4, 5, if the digits can be repeated.

$$5 \times 2 = 10$$

Q How many 3 digits nos. can be formed from the digits 1, 2, 3, 4, 5. Assuming that

(i) Repitition allowed

(ii) Repetition not allowed

(i) $5 \times 5 \times 5 = 125$

(ii) $5 \times 4 \times 3 = 60$

Q How 5 digit telephone nos. can be constructed using the digits 0-9. It ~~#~~ each no. starts with

67 and ~~no~~ digits appear more than once

$$\begin{array}{r} 42 \\ \times\ 8 \\ \hline 336 \end{array}$$

$\cancel{10 \times 9 \times 8 = 720}$

$8 \times 7 \times 6 = 336$

Permutation

A permutation is an arrangement in a definite order of a no. of objects, taken some or all at a time.

Note: (i) The no. of permutations of n different objects taken r at a time where repetition is allowed is n^r.

(ii) The no. of permutations of n objects where p objects are of same kind and rest all are different $= \dfrac{n!}{p!}$

(iii) The no. of permutations of n objects where p_1 objects are of 1^{st} kind, p_2 are of 2^{nd} kind, ——— ..., p_k are of k^{th} kind and of rest if any are of different kind is $\dfrac{n!}{p_1!\,p_2!\cdots p_k!}$

Q Find the no. of permutations of the letters of the word ~~Allahabad~~ ALLAHABAD

Sol^n: $A \rightarrow 4$

$L \rightarrow 2$

$$\frac{9!}{4!\,2!}$$

$$= \frac{5 \times \overset{3}{\cancel{6}} \times 7 \times 8 \times 9}{\cancel{2}}$$

$$= 7560$$

$$\begin{array}{r} 72 \\ \times\ 7 \\ \hline 504 \\ \times\ 15 \\ \hline 7560 \end{array}$$

Q Find the no. of different 8 letter arrangements that can be made from the letters of the word

DAUGHTER, so that

(i) All vowels occur together

(ii) All vowels do not occur together

Sol^n: (AUE) $\rightarrow$ 1 obj

$\cancel{6!}$ $6! \times 3!$

Permutation → Arrangement

Combination → ~~sel~~ selection

$$\boxed{{}^{n}C_{r} = \frac{n!}{r!(n-r)!}}$$

Q. A commettee of 3 persons is to be constituted from a group of ~~3~~ 2 man & 3 women, In how many ways this can be done. How many these commettees would constitute of one man & two women.

Sol^n: A committee of 3 person is to be constituted from a group of 2 men ~~to~~ & 3 ~~women~~ women. This can be done in ${}^{5}C_{3}$ ways

i.e. $\frac{5!}{3!(5-3)!}$

$= \frac{5 \times 4 \times \cancel{3!}}{\cancel{3!} \times 2!}$

$= 10$ ways

Again, when we constitute a committee of one man & 2 women, this can be done in

$^2C_1 \times {}^3C_2$ ways

$= \frac{2!}{1 \times 1} \times \frac{3!}{2! \times 1!}$

$=$ ~~3~~ 6 ways

∴ In 6 ways committee of one man & 2 women can be constituted.

Q - A bag contains five black & six red balls. determine the no. of ~~the~~ ways in which 2 black and 3 red balls can be constituted.

Sol^n:

$^5C_2 \times {}^6C_3$

$= \frac{5!}{2! \times 3!} \times \frac{6!}{3! \times 3!}$

$= \cancel{\frac{4 \times 5 \times 4 \times 5 \times 6}{2 \times 3 \times 2 \times 3 \times 2}}$ $\frac{4 \times 5}{2} \times \frac{4 \times 5 \times 6}{3 \times 2}$

$= \cancel{\frac{100}{3}}$ ~~ways~~ $= 10 \times 20$

$= 200$ ways.

∴ This can be done in 200 ways.

Principle of inclusion & exclusion

We know that if A and B are finite sets

then ~~cardinality~~ $|A \cup B| = |A| + |B| - |A \cap B|$

If A & B are disjoint sets,

i.e. if $A \cap B = \phi$

then, $|A \cup B| = |A| + |B|$

Suppose a set 'S' is expressed as a union

of finite no. of its subsets, say

~~S = S1~~ $S = S_1 \cup S_2 \cup \dots \cup S_n$

If $S_i \cap S_j = \phi \ \forall \ i \neq j$

then $|S| = |S_1| + |S_2| + \dots |S_n|$

Logic

~~Boolean~~

Boolean Algebra

A set B together with two binary operations '+' and '.' on B known as addition and multiplication respectively and an operation ' on B called complementation staisfying the following ~~axioms~~ axioms

(i) The operations are commutative. i.e. $a+b=b+a$, $a.b=b.a$

$$a+b=b+a$$
$$a.b=b.a \quad \forall a,b \in B$$

(ii) Each binary operation distributes over the other.

$$a+(b.c)=(a+b).(a+c) \quad \forall a,b,c \in B$$

(iii) B contains distinct identity elements '0' & '1' known as ~~zero element~~ '0' element & unit element with respect to the operation '+' & '.' respectively. i.e. $a+0=a$

$$a.1=a \quad \forall a \in B$$

(iv) For each $a \in B$ there exist an element a' such that $a+a'=1$

$a \cdot a' = 0$

Note: (i) a' is called the complement of a

(ii) $(a')'$ will be denoted by a''

(ii) The binary operations in the definition need not be written as '+' and '·'. Instead we may use the other symbols $\cup$ and $\cap$ or $\vee$ and $\wedge$.

(iii) A ~~bal~~ boolean algebra is generally denoted by B or $(B, +, \cdot, ')$ or $(B, +, \cdot, ', 0, 1)$

Example 1. Let A be a non empty set and $P(A)$ be the power set of A then

Power set

$P(A)$ is a boolean algebra under the usual operations of union, intersection and complementation in $P(A)$. The set ϕ and A are the ~~ze~~ '0' element and unit element of the boolean algebra $P(A)$

Example 2.

which ar

$B = \{1, 2,$

for any

$a \cdot b = l$

Then

LCM

$(B, +$

with

Example 2: Let B be the set of all +ve integer which are divisors of 70 . i.e. ~~$B = \{1, 2, 5, 7,$~~

$B = \{1, 2, 5, 7, 10, 14, 35, 70\}$

for any $a, b \in B$ let $a + b = LCM(a, b)$

$a \cdot b = HCF(a, b)$

$$a' = \frac{70}{a}$$

Then with the help of elementary properties of LCM and HCF, it can be verified that $(B, +, \cdot, ', 1, 70)$ is a boolean ~~algebb~~ algebra with 1 as zero element and 70 as unit element

$a + 0 = a$

Here

$a + 1 = LCM(a, 1)$
$= a$

$\therefore$ 1 is the '0' element.

$a \cdot 70 = hcf(a, 70)$

$a \cdot 70 = a$

$\therefore$ 70 is the unit element

Proposition

By a proposition in boolean algebra we mean either a statement or an algebric identity in the boolean algebra

Duel of a propotition.

By the duel of a propotition A in a boolean algebra we mean the propotition obtained from A by interchanging '+' & '·' and exchanging '0' & '1'.

The duel of the proposition $\cancel{x.(y+z)=}$

$$x \cdot (y+z) = x \cdot y + x \cdot z$$

is $$x + (y \cdot z) = (x+y) \cdot (x+z)$$ and

vice versa.

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The duel of the proposition, "0 is unique in a boolean algebra" is the proposition "1 is unique in boolean algebra".

Clearly, if proposition B is the duel of A, then A is the duel of B

~~Duality~~

Duality Principle

If a proposition A is derivable from the axioms of a boolean ~~alge~~ algebra, then the dual of a is also derivable from those axioms.

In a boolean algebra $(B, +, \cdot, ')$ the following holds,

(i) The elements 0 and 1 are unique.

(ii) Each ~~$A \in$~~ $a \in B$ has a unique complement $a' \in B$.

(iii) For each $a \in B$, $(a')' = a$

(iv) $0' = 1$ and $1' = 0$

(v) Idempotent Law:

$$a + a = a$$
$$a \cdot a = a \quad \forall a \in B$$

(vi) $a + 1 = 1$ and $a \cdot 0 = 0 \quad \forall a \in B$

(vii) ~~Absorbti~~ Absorption Law:

$$a \cdot (a+b) = a$$
$$a + (a \cdot b) = a \quad \forall a, b \in B$$

Proof:

(i) If possible let O_1 and O_2 be two zero elements of B

Then, by defination

$$a+O_1=a \longrightarrow (i)$$

$$a+O_2=a \quad \forall a\in B \longrightarrow (ii)$$

Putting $a=O_2$ in (i)

$$O_2+O_1=O_2$$

Putting $a=O_1$ in (ii)

$$O_1+O_2=O_1$$

∴ from above two equation

$$O_1=O_2$$

∴ zero element of B is unique.

By duality principle,

unique element 1 in B is unique.

(ii) Let a_1' and a_2' be two complements of a

Then by defination —

$$a+a_1'=1 \longrightarrow (i)$$

$$a+a_2'=1 \longrightarrow (ii)$$

$$a \cdot a_1' = 0 \longrightarrow (iii)$$
$$a \cdot a_2' = 0 \longrightarrow (iv)$$

$$a_1' = a_1' \cdot 1$$
$$= a_1'(a + a_2') \text{ (using (2))}$$
$$= a_1' \cdot a + a_1' \cdot a_2'$$
$$= 0 + a_1' \cdot a_2' \text{ (using (3))}$$
$$= a_1' \cdot a_2' \longrightarrow (v)$$

Similarly, we can show that

$$a_2' = a_2' \cdot a_1' \longrightarrow (vi)$$

Hence from eq^n (v) & (vi) we can say that

$a_1' = a_2'$ (By commutative property)

(iii) We know,

for each $a \in B$, there excist a unique element $a' \in B$ such that $a + a' = 1$
$a \cdot a' = 0$

$\Rightarrow a' + a = 1$ and $a' \cdot a = 0$

(By commutativity)

$\Rightarrow a$ is the complement of a'

$\therefore (a')' = a$

(iv) Let $a=1$

$$\therefore a' = a' + 0$$
$$= a' + (a \cdot a')$$
$$= a'(1+a)$$
$$= a'(a)$$
$$= a' \cdot a$$
$$= 0$$
$$\Rightarrow 1' = 0$$

1×5
2×2
4×4

Simillarly we can show that 0 complement $= 1$.

(v) Idempotent

$$a + a = (a+a) \cdot 1$$
$$= (a+a)(a+a')$$
$$= a + (a \cdot a')$$
$$= a + 0$$
$$= a$$

$$a \cdot a = a \cdot a + 0$$
$$= a \cdot a + a \cdot a'$$
$$= a(a+a')$$
$$= a \cdot 1$$
$$= a$$

(vi) $a+1 = (a+1)\cdot 1$ (by axiom 3)

$= 1\cdot(a+1)$ (by axiom 1)

$= (a+a')\cdot(a+1)$

$= a+(a'\cdot 1)$

$= a+a'$

$= 1$

By principle of duality

$a\cdot 0 = 0$

Hence proved.

(vii) $a\cdot(a+b) = a\cdot a + a\cdot b$

$= a + a\cdot b$

$= a(1+b)$

$= a$

The second part is the duel of the 1st part

i.e. $a+(a\cdot b) = a$

which can be proved by duality principle.

Theorem : In a boolean algebra $(B, +, \cdot, ')$ the following holds. For all $a, b, c \in C$

(i) if $b+a = c+a$

$b+a' = c+a'$

then $b = c$.

Also, if $b \cdot a = c \cdot a$

$b \cdot a' = c \cdot a'$

then $b = c$.

(ii) Associative Laws,

$$a + (b + c) = (a + b) + c$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(iii) ~~De morgans~~ D'Morgan's Law

$$(a + b)' = a' \cdot b'$$

$$(a \cdot b)' = a' + b'$$

(iv) $\cancel{A + B = (A' \cdot B')'}$

$$a + b = (a' \cdot b')'$$

$$a \cdot b = (a' + b')'$$

(v) $\cancel{A}$ $a + b' = 1$ iff $a + b = a$

Also, $a \cdot b' = 0$ iff $a \cdot b = a$

(vi) $a + (a' \cdot b) = a + b$

$$a \cdot (a' + b) = a \cdot b$$

Q For any Boolean algebra B, prove that

$$(a+b)(b+c)(c+a) = ab+bc+ca$$

$$\forall\ a, b, c \in C$$

An $(a+b)(b \cdot c + b \cdot a + c \cdot c + c \cdot a)$

$= (a+b)(bc + ab + c + ac)$ (by associative prop. & $c \cdot c = c$)

$= a \cdot bc + a \cdot ab + a \cdot c + ac + b \cdot bc + b \cdot ab + b \cdot c + b \cdot ac$

$= abc + ab + ac + bc$

$= a(bc + b) + ac + bc$

$= ab + ac + bc$

$= ab + bc + ca$

H/W. 1 Prove that following three expressions are equal.

(a) $(a+b)(a'+c)(b+c)$

(b) $ac + a'b + bc$

(c) $(a+b)(a'+c)$

H/W. 2. Prove that $a = 0 \Leftrightarrow b = a \cdot b' + a' \cdot b$

Two element boolean algebra:

The boolean algebra $b = 0, 1$, where operations '+', ',' and '.' are defined by following table,

+	1	0
1	1	1
0	1	0

.	1	0
1	1	0
0	0	0

	'
1	0
0	1

and is called a two element boolean algebra

Usually, this two element boolean algebra is denoted by B_2 of B

Note: (i) Any symbol such as x or y or z or x_1 or x_2 or ------, used to represent an an element of $B = \{0, 1\}$ is called a boolean variable.

(ii) Let, $x_1, x_2, \text{------} x_n$ be boolean variables, A boolean expression over B is defined as follows —

→ '0' & '1' are boolean expressions

→ $x_1; x_2, \text{---}, x_n$ are boolean expressions

→ if α is an boolean expression, then α' is also an boolean expression.

→ if α_1 & α_2 are boolean expressions then

$\alpha_1 \cdot \alpha_2$ & $\alpha_1 + \alpha_2$ are also boolean expressions.

→ only those expressions are boolean expressions on $x_1, x_2, \ldots, x_n$ which are determined by ~~rules~~ the above rules.

eg — $(x+y')$, $((x_1 \cdot x_2') + x_4)$, $((x_1 + x_2) + (x_4 + (x_3 + x_2)))$

Q Construct the truth table for the boolean expression.

$\alpha(x_1, x_2, x_3) = x_1 + x_2 \cdot x_3'$

x_1	x_2	x_3	x_3'	$x_2 \cdot x_3'$	$x_1 + x_2 \cdot x_3'$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	0	0	0
1	0	0	1	0	1
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	0	0	1

Note: Two boolean expressions $\alpha(x_1, x_2, \ldots, x_n)$

$\& \ \beta(x_1, x_2, \ldots, x_n)$

are said to be equal if they assume the same ~~value~~ values for every assignment of values (0, 1)

To the variables $x_1, x_2, \ldots, x_n$

Theorem — Let α, β, γ be boolean expressions, then the followings hold,

(i) Commutative Laws — $\alpha+\beta = \beta+\alpha$

$\alpha\cdot\beta = \beta\cdot\alpha$

(ii) Associative Laws — $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$

$\alpha\cdot(\beta\cdot\gamma) = (\alpha\cdot\beta)\cdot\gamma$

(iii) Distributive Laws — $\alpha\cdot(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma$

$\alpha+(\beta\cdot\gamma) = (\alpha+\beta)\cdot(\alpha+\gamma)$

(iv) Idempotent Laws — $\alpha+\alpha = \alpha$

$\alpha\cdot\alpha = \alpha$

(v) Identity Laws — $x+0 = x$

$x\cdot 1 = x$

(vi) Inverse Laws — $x+x' = 1$

$x\cdot x' = 0$

(vii) Dominent Laws — $x+1 = 1$

$x\cdot 0 = 0$

(viii) Absorption Laws — $x_1+(x_1\cdot x_2) = x_1$

$x_1\cdot(x_1+x_2) = x$

(ix) A

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(ix) ~~De Mor~~ D'Morgan's Law — $(x_1+x_2)' = x_1' \cdot x_2'$

$(x_1 \cdot x_2)' = x_1' + x_2'$

(x) Double Complement Law — $(x')' = x$

Dual of a Boolean Expression

Let α be a boolean expression.

A boolean expression β is said to be a dual of α if β is obtained from α by replacing each occurance of boolean sum by ~~bool~~ boolean product, Each occurance of boolean product by boolean sum, each occurance of 1 by zero, each occurance of zero by 1. If β is a dual of α, then we write β as $\boxed{\beta = \alpha^d}$

$$\cancel{\alpha = (x_1 \cdot 0) + (x_2 \cdot x_3)'}$$

$$\cancel{= x_1 \cdot 1}$$

$$\alpha = (x_1 + 1) \cdot (x_2 + x_3)'$$

$$\alpha^d = (x_1 \cdot 0) + (x_2 \cdot x_3)'$$

Boolean function

Let B be a two element boolean algebra and let $\alpha(x_1, x_2, ----, x_n)$ be a boolean expression.

Then the corresponding function.

$$f_\alpha : B^n \to B \text{ is defined by}$$

$$f_\alpha(b_1, b_2, \ldots, b_n) = \alpha(b_1, b_2, \ldots, b_n)$$

For each n tuples $(b_1, b_2, \ldots, b_n) \in B^n$

The function $f : B^n \to B$ is called as a boolean function. If there exist a boolean expression

$$\alpha : f = f_\alpha$$

Q. Find the values of $\alpha(1,0,1)$, $\alpha(0,0,1)$, $\alpha(0,1,1)$, where α is the boolean expression given by

$$\alpha(x_1, x_2, x_3) = x_1 + x_2 + x_3'$$

Sol^n:

$$\boxed{\alpha(1,0,1) =} \ 1 + 0 + 0 = 1$$

$$\alpha(0,0,1) = 0 + 0 + 0 = 0$$

$$\alpha(0,1,1) = 0 + 1 + 0 = 1$$

Q. Construct the truth table for the following expressions

(a) $x(y + x')$

(b) $xy' + y(x' + z')$

(i)

x	y	x'	$y+x'$	$x(y+x')$
0	0	1	1	0
0	1	1	1	0
1	0	0	0	0
1	1	0	1	1

(ii)

x	y	z	x'	y'	xy'	$(x'+z)$	$y(x'+z)$	$xy'+y(x'+z)$
0	0	0	1	1	0	1	0	0
0	0	1	1	1	0	1	0	0
0	1	0	1	0	0	1	1	1
0	1	1	1	0	0	1	1	1
1	0	0	0	1	1	0	0	1
1	0	1	0	1	1	1	0	1
1	1	0	0	0	0	0	0	0
1	1	1	0	0	0	1	1	1

Q. Show that the boolean expression $(x_1 . x_2)x_3$ and $x_1 . (x_2 . x_3)$ are equal

x_1	x_2	x_3	$x_1 . x_2$	$x_2 . x_3$	$(x_1 . x_2)x_3$	$x_1 . (x_2 . x_3)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

a	b	c	a'	a+b	a'+c	b+c	(a+b)(a'+c)	(a+b)(a'+c)(b+c)
0	0	0	1	0	1	0	0	0
0	0	1	1	0	1	1	0	0
0	1	0	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	0	0	0
1	0	1	0	1	1	1	1	1
1	1	0	0	1	0	1	0	0
1	1	1	0	1	1	1	1	1

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Q. By constructing

$f_\alpha : B^2 \to B$

$\alpha(x_1, x_2) = x$

Disjunctive Nor

For any

we can always

$f_\alpha(b_1, b_2,$

~~Mean~~ Mean

A boo

is said to

$x_1, x_2, \ldots$

$\tilde{x}_1 \cdot \tilde{x}_2 \cdot$

x_i or x_i

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Q. By constructing a value show that the function $f_\alpha : B^2 \rightarrow B$ is defined by the boolean expression

$$\alpha(x_1, x_2) = x_1 . x_2 + x_2$$

Disjunctive Normal Form (DNF):

For any boolean expression $\alpha(x_1, x_2, \ldots, x_n)$ we can always define a function $f_\alpha : B^n \rightarrow B$ by –

$$f_\alpha(b_1, b_2, \ldots, b_n) = \alpha(b_1, b_2, \ldots, b_n)$$

~~Mean~~ Mean term :

A boolean expression $\alpha(x_1, x_2, \ldots, x_n)$ is said to be ~~mid~~ mid term in the variables $x_1, x_2, \ldots, x_n$ if it is of the form $\tilde{x}_1 . \tilde{x}_2 . \tilde{x}_3 \ldots \tilde{x}_n$, where each $\tilde{x}_i$ is either x_i or x_i'

eg – The boolean expression xyz, $x'yz$, $x'yz'$ are the mean terms in the ~~va~~ variables x, y, z.

A boolean expression $\alpha(x_1, x_2, \ldots, x_n)$ is said to be in disjunctive normal form (DNF) or some of product form in the variables $x_1, x_2, \ldots, x_n$ if there are distinct mean

~~to~~ terms $\alpha_1, \alpha_2, \ldots, \alpha_m$ in the variable $x_1, x_2, \ldots, x_n$ such that $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m$

eg: $\alpha(x_1, x_2, x_3) = x_1 x_2 x_3 + x_1 x_2' x_3 + x_1 x_2' x_3' + x_1' x_2 x_3$ is a boolean expression in disjunctive normal form (DNF) in the variables x_1, x_2, x_3.

The boolean expressions —

$x + x'$, $x'y$, $xyz + x'yz'$ in one, two and three variables respectively are in DNF whereas, $(x+y)z$, $(xy' + xz)' + x'$ are not in DNF.

Note: A term to be a mean term all variables must be present.

Theorem: Let α be a boolean expression in the variables $x_1, x_2, \ldots, x_n$. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_m$ are mean terms in the the variables $x_1, x_2, \ldots, x_n$ such ~~the~~ that —

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_m$$

Then, for any assignment $x_1 = b_1, x_2 = b_2, \ldots$

~~Then, for any assignment~~ $\cancel{x_1 = b_1, x_2 = b_2}$

.... $x_n = b_n$ of values $b_1, b_2, \ldots, b_n \in \{0, 1\}$

$\alpha(b_1, b_2, \ldots, b_n) = 1$ ~~if and~~ iff

$\alpha_i(b_1, b_2, \ldots, b_n) = 1$ for some i, where $1 \le i \le n$