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Vector Space

* **Binary Operation:** A binary operation on a set G can be defined as follows.

let G be a non-empty set, then,

$G \times G = \{(a,b) : a \in G, b \in G\}$

If $f : G \times G \to G$ then

f is said to be binary op. on the set G

* **Group:** Let G be a non-empty set equipped with a binary operation denoted by the symbol '$*$' i.e,

$a * b \in G, \forall a, b \in G$ then,

G is a group.

If the Binary operation '$*$' satisfies the following:—

(i) Associativity:

$$(a*b)*c = a*(b*c)$$

(ii) Existence of Identity:

Their existence element, 'e'

$e \in G : a*e = a = e*a$

(iii) Existence of Inverse:

$\forall a \in G$, Their existence element 'b'

$b : a*b = e = b*a$, when $a, b, e \in G$

Note

(1) Here e is said to be identity of the group G. The inverse of an element a is denoted by a^{-1} i.e $a * a^{-1} = e$

* **Abelian or Commutative Group:** A group G is said to be Abelian or commutative if it satisfy the property of commutativity.

i.e $a*b = b*a$, $\forall a, b \in G$.

* **Field:** Suppose F is a ~~####~~ non-empty set, equipped with two Binary operation called addition & multiplication & is denoted by '+' & '.', respectively.

$\forall a, b \in F$; $a+b \in F$

$a \cdot b \in F$ then

$(F, +, \cdot)$ is called a field if the following are satisfied :—

Properties —

F_1 : $(F, +)$ is an abelian group

F_2 : $(F, -)$ is an abelian group

F_3 : Multiplication is distributive w.r.t addition.

* Note

① In a field '0' is the identity element of addition composition. The element 1 is the identity element for multiplication composition one is called as unity in the field.

In a field each non-zero element is invertible i.e posses inverse for multiplication.

The set 'Q' of all rational nos. is a field. The addition & multiplication of rational nos. being in two field composition.

The rational no. '0' element of this field is the identity & '1' element is the unity in the field.

(II) The set R of all real nos. is a field, the addition & multiplication of real nos. being the two field composition

$\because Q \subseteq R$

$\therefore$ The field of rational nos. is a sub-field of the field of real nos.

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→ Let A be any set '$*$' is said to be an internal composition of A.
if, $a * b \in A$, $\forall\ a, b \in A$
& $a * b$ is unique

→ Let V and F be any two sets. '$\cdot$' is said to be an external composition in V over F. If,

$$a \cdot v \in V \ , \forall\ a \in F, v \in V$$

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* <u>Vector ~~Sep~~ Space</u>: V(F) Let $(F, +, \cdot)$ be a field and V be any non-empty set. Then, V is a vector space over F. If

→ '+' is an internal composition in V & $(V, +)$ is abelian — {vector addi

→ '$\cdot$' is an external composition in V(F) ——————— {Scalar multiplication

→ (i) $a(v_1 + v_2) = av_1 + av_2$

(ii) $(a+b)v = av + bv$

(iii) $(ab)v = a(bv)$

(iv) $1 \cdot v = v$

$\forall\ a, b \in F$ & $v_1, v_2, v \in V$
where 1 is the unity element of field F.

* The field K can be regarded as a Vector Space over any subfield F of K.

Proof: Given K is a field and F is a subfield of K, which implies F is also a field w.r.t the binary operation defined in K.

→ Let us consider the elements of K as vectors.

Now, $\alpha_1 + \alpha_2 \in K$, $\forall\ \alpha_1, \alpha_2 \in K$

$\because$ K is a field.

$\therefore$ '+' is a interval composition in K

$\because$ By definition of field
(K, +) is abelion.

→ Let us consider the element of F as Scalars.

$\therefore a \in F \subset K, \alpha \in K$

which implies, $a.\alpha \in K$
{$\because$ K is a field}

(i) $a(\alpha_1+\alpha_2) = a\alpha_1 + a\alpha_2$

by distributive law in K.

(ii) $(a+b)\alpha = a\alpha + b\alpha$

by distributive law in K.

(iii) $(ab)\alpha = a(b\alpha)$

by associativity in K

(iv) $1.\alpha = \alpha$, $\forall$ $a, b \in F$ & $\alpha_1, \alpha_2, \in V$

{where 1 is the unity element of field

$\because$ 1 is multiplicative identity of K

$\therefore$ " " " of F

$\Rightarrow 1 \cdot \alpha = \alpha$ in F

K(F) is a vector space.

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$\boxed{\text{Note}}$ C(R) is a vector space.

But R(C) is not a vector space.

$\because$ external composition is not satisfied

Let, $a = i$, $b = 2$

but $2i \notin R$

(11)

[※ Let, V be the set of all ordered n tupple over a field F, i.e

$$V = \{(a_1, a_2, \dots a_n) : a_1, a_2, \dots a_n \in F\}$$

Let us define equality, addition & scalar multiplication of n tupple.

Let, $\alpha = (a_1, a_2, a_3, \dots, a_n)$

$\beta = (b_1, b_2, \dots, b_n)$

then, $\alpha = \beta$

$\Rightarrow a_i = b_i \quad \forall\, i = 1, 2, \dots, n$

$$\alpha + \beta = (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n)$$
$$= \{(a_1+b_1), (a_2+b_2), \dots, (a_n+b_n)\}$$

Let, $a \in F$, then $a\alpha = a(a_1, a_2, \dots, a_n)$

$$= (aa_1, aa_2, \dots, aa_n)$$

then show that $V(F)$ is vector space.

Soln Given

$\alpha = (a_1, a_2, \dots, a_n)$

$\beta = (b_1, b_2, \dots, b_n)$

$\therefore \alpha + \beta = \{(a_1+b_1), (a_2+b_2), \dots, (a_n+b_n)\}$

(iii)

$\because a_i \in F$ & $b_i \in F$

$\Rightarrow (a_i + b_i) \in F \quad \forall\, i = 1, 2, \ldots, n$

$\Rightarrow \alpha + \beta \in V$

Now, we are to show that $(V, +)$ is abel.

→ Associativity

Let $\alpha, \beta, \gamma \in V$

$\alpha + (\beta + \gamma) = (a_1, a_2, \ldots, a_n) +$

$\left[(b_1, b_2, \ldots, b_n) + (c_1, c_2, \ldots, c_n)\right]$

$= \{(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots, (a_n + b_n)$

$\because F$ is a field

$= \left[(a_1 + b_1) + (a_2 + b_2), \ldots, (a_n + b_n)\right] +$

$(c_1 + c_2 + \ldots\ldots + c_n)\Big]$

$= (\alpha + \beta) + \gamma$

$\therefore$ Associativity is satisfied.

→ Existence of Identity

We have, $0 = (0, 0, \ldots, 0) \in V$

$$\text{Let, } \alpha = (a_1, a_2, \ldots, a_n)$$

$$\text{Now, } \alpha + 0 = (a_1, a_2, \ldots, a_n) + (0, 0, \ldots, 0)$$

$$= \{(a_1 + 0), (a_2 + 0), \ldots, (a_n + 0)\}$$

$$= (a_1, a_2, \ldots, a_n)$$

$$= \alpha$$

$\because$ F is a field

$\therefore$ '0' is the additive identity V(F).

→ Existence of Inverse

~~Now, $a \ast b = e$~~

~~i.e $a \ast b = (a_1, a_2, \ldots a_n, a_n) \ast (b_1, b_2, b_n, b_n)$~~

~~$= \{(a_1 \ast b_1), (a_2 \ast b_2), \ldots, (a_n \ast b_n)\}$~~

~~$\therefore a_1 \ast b_1 = 1$~~

$$\because (a_1, a_2, \ldots, a_n) \in F$$

$$\Rightarrow (-a_1, -a_2, -a_3, \ldots, -a_n) \in F$$

$$\left[\begin{array}{l} \because F \text{ is a field} \\ \therefore a_i \in F \\ \Rightarrow -a_i \in F \end{array}\right]$$

ⓥ

$\therefore (-a_1, -a_2, \ldots, -a_n) \in V$

i.e, $-\alpha \in F$

$\Rightarrow -\alpha \in V$

Now, $\alpha + (-\alpha)$

$= \{(a_1 - a_1), (a_2 - a_2), \ldots, (a_n - a_n)\}$

$= 0$, which is identity of V.

→ Commutivity:

Let, $f, g \in F[x]$

$\therefore f + g = \{a_0 + a_1 x + a_2 x^2 + \ldots\} + \{b_0 + b_1 x + b_2 x^2 + \ldots\}$

$= \{(a_0 + b_0) + (a_1 + b_1)x + \ldots\}$ [$\because$ F is a field]

$= \{(b_0 + a_0) + (b_1 + a_1)x + \ldots\}$

$= \{b_0 + b_1 x + \ldots\} + \{a_0 + a_1 x + \ldots\}$

$= g + f$, commutivity satisfied.

Hence, $(V, +)$ is an abelian group.

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Now, for external composition

$a \cdot \alpha \in V$

Let, $a \in F$ & $\alpha \in V$ where $\alpha = (a_1, a_2, \ldots a_n)$

$a \cdot \alpha = (aa_1, aa_2, \ldots, aa_n)$

$\because a \in F$ & $a_i \in F$

$\Rightarrow aa_i \in F, \forall i = 1, \ldots, n$

[$\because$ F is a field]

$\therefore a\alpha \in V(F)$

Hence, external composition is satisfied.

Now,

(a) $a(\alpha + \beta) = a\alpha + a\beta$

$L.H.S = a(\alpha + \beta)$

$= a[(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots b_n)]$

$= a[(a_1+b_1), (a_2+b_2), \ldots, (a_n+b_n)]$

$= [(aa_1+ab_1), (aa_2+ab_2), \ldots, (aa_n+ab_n)]$

$= a\alpha + a\beta = R.H.S$

11

$[1.\alpha = \alpha]$

(d) $\because 1 \in F$ & let $\alpha \in V$

Then, $1.\alpha = 1.(a_1, a_2, \ldots, a_n)$

$= (1a_1, 1a_2, \ldots, 1a_n)$

$= (a_1, a_2, \ldots, a_n)$

$= \alpha$

(e) $a(f+g) = af + ag$

L.H.S $= a[(a_0 + a_1x + a_2x + \ldots) + (b_0 + b_1x + b_2x^2 + \ldots)]$

$= a[(a_0 + b_0) + (a_1 + b_1)x + \ldots]$

$= [(aa_0 + ab_0) + (aa_1 + ab_1)x + \ldots]$

$= [(aa_0 + aa_1x + \ldots) + (ab_0 + ab_1x + \ldots)]$

$= af + ag$

(f) $ab(\alpha) = a(b\alpha)$

L.H.S $= ab(a_0 + a_1x + a_2x^2 + \ldots)$

$= [(aba_0 + aba_1x + aba_2x^2 + \ldots)]$

$$= a\left[ba_0 + ba_1x + ba_2x^2 + \cdots\right]$$

$$= a\left[b\left(a_0 + a_1x + \cdots\right)\right]$$

$$= a(b\alpha) = R.H.S$$

$\therefore V(F)$ is a VS

* Let $F[x]$ denote the set of all polynomial in a variable x over a field F, $F[x](F)$. Then, $F[x](F)$ is a vector space w.r.t addition of ~~vectors~~ two polynomials as addition of vectors & product of polynomials by constant polynomial as scalar multiplication.

Solⁿ: Let, $F[x] = \{a_0 + a_1x + a_2x^2 + \cdots, a_i \in F\}$

$$= \left\{\sum_{i=0}^{\alpha} a_i x^i, \ a_i \in F\right\}$$

Let; $f, g \in F(x)$

$$f = a_0 + a_1x + a_2x^2 + \cdots = \sum_{i=0}^{\alpha} a_i x^i$$

$$g = b_0 + b_1x + b_2x^2 + \cdots = \sum_{i=0}^{\alpha} b_i x^i$$

$$\therefore f+g = \left\{(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots\right\}$$

$$= \sum_{i=0}^{\alpha} (a_i+b_i)x^i$$

$\because a_i \in F$ & $b_i \in F$

(iv)

$\Rightarrow (a_i + b_i)x^i \in F, \forall i = 1, \dots, \alpha$

$\Rightarrow f + g \in F$

& $f + g \in V$

→ To show, $(V, +)$ is abelian

→ Associativity

Let; $f, g, h \in V$

$\therefore f + (g+h) = (a_0 + a_1 x + a_2 x^2 + \dots) + [(b_0 + b_1 x + b_2 x^2 + \dots) + (c_0 + c_1 x + c_2 x^2 + \dots)]$

$= (a_0 + a_1 x + a_2 x^2 + \dots) + [(b_0 + c_0) + (b_1 + c_1)x + \dots]$

$= [(a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)x + \dots]$

$= [(a_0 + b_0) + (a_1 + b_1)x \dots] + (c_0 + c_1 x + \dots)$

$= (f+g) + h$

satisfied

→ Existence of identity

We have, $0 = \{0, 0, 0, 0, \dots\} \in V$

Now, $f + 0 = \{(a_0 + a_1 x + \dots) + (0, 0, 0, 0 \dots$

$= \{(a_0 + 0) + (a_1 + 0)x + \dots\}$

(V)

$= f$ [$\because$ F is a field]

$\therefore$ f' is the additive identity.

$\longrightarrow$ Existence of inverse

$\because$ $(a_0 + a_1 x_1 + a_2 x^2 + \cdots) \in F$

$\therefore$ $(-a_0 - a_1 x_1 - a_2 x^2 - \cdots) \in F$

$\boxed{\because \text{F is a field}}$

$\Rightarrow a_i \in F$ & $-a_i \in F$

Thus, $(-a_0 - a_1 x - a_2 x^2 - \cdots) \in V$

Now, $f + (-f) =$

$\{(+a_0 - a_0) + (a_1 - a_1)x + \cdots\}$

$= 0$, which is the

— identity of V.

$\longrightarrow$ Commutativity

$\because$ $f, g \in F$

& $f + g = \{(a_0 + b_0) + (a_1 + b_1)x + \cdots\}$

$= \{(b_0 + a_0) + (b_1 + a_1)x + \cdots\}$ [$\because$ F is a field]

$= g + f$ satisfied

Hence, $(V,+)$ is a abelian group.

$\rightarrow$ For ~~ex~~ternal composition,

$a \cdot f \in V$

Let, $a \in F$ & $f \in V$

$$\therefore a \cdot f = \{ a a_0 + a a_1 x + a a_2 x^2 + \cdots \}$$

$\because a \in F$ & $a_i x^i \in F$

$\Rightarrow a a_i x^i \in F \quad , \forall \ i = 1, \ldots, r$

[$\because F$ is a field]

$\therefore a \cdot f$ is satisfied

Now,

$\longrightarrow a(f+g) = af + ag$

$$\text{R.H.S} = a \left\{ (a_0 + a_1 x + a_2 x^2 + \cdots) + (b_0 + b_1 x + b_2 x^2 + \cdots) \right\}$$

$$= a \{ (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x_2 + \cdots \}$$

$$\Rightarrow \{ (a a_0 + a b_0) + (a a_1 + a b_1) x + \cdots \}$$

$$= \{ (a a_0 + a a_1 x + \cdots) + (a b_0 + a b_1 x + \cdots) \}$$

(vii)

$= a f + a g$

$= R.H.S$

$\longrightarrow (a+b)f = af + bf$

$L.H.S = (a a_0 + a a_1 x + a a_2 x^2 + \dots) + (b a_0 + b a_1 x + b a_2 x^2 \dots)$

$= [a(a_0 + a_1 x + a_2 x^2 + \dots) + b(a_0 + a_1 x_1 + a_2 x_2 + \dots)]$

$= af + bf$

$= R.H.S$

$\longrightarrow (ab)f = a(bf)$

$L.H.S = (ab)(a_0 + a_1 x + a_2 x^2 + \dots)$

$= a(ba_0 + ba_1 x + ba_2 x^2 + \dots)$

$= a\{b(a_0 + a_1 x + a_2 x^2 + \dots)\}$

$= a(bf)$

$\longrightarrow 1.f = f$

$L.H.S = 1.(a_0 + a_1 x + a_2 x^2 + \dots)$

$= (1.a_0 + 1.a_1 x + 1.a_2 x^2 + \dots) = f = R.H.S$

* Let 'V' be the set of all pairs (x,y) of real nos. and let 'R' be the field of real nos. Define $(x,y)+(x_1,y_1) = (x+x_1, 0)$

& $c(x,y) = (cx, 0)$.

Is $V(R)$ a vector space?

Soln $V = \{(x,y) : x,y \in \mathbb{R}\}$

Here, $V(R)$ is not a vector space.

We will show that additive identity doesn't exists.

Let, $(x,y) \in V(R)$

Let, if possible $(\bar{x},\bar{y})$ be the additive identity doesn't exist. Let, $(x,y) \in V(R)$

Let, if possible $(\bar{x},\bar{y})$ be the additive identity of $V(R)$.

$$\therefore (x,y) + (\bar{x},\bar{y}) = (x,y)$$

$$\Rightarrow (x+\bar{x}, 0) = (x,y) \ \forall \ x,y \in V$$

But if $y \neq 0$, there doesn't exits $(\not\exists)$ any elements such that $(x,y) + (\bar{x}+\bar{y}) = (x,y)$

$\therefore$ additive identity doesn't exists.

$\Rightarrow (V,+)$ is not abelian

$\Rightarrow V(R)$ is not a vector space.

14/07 (1)

General Properties of a Vector Space:

Let $V(F)$ be a vector space & ~~B~~
'0' be a zero vector of V then,

(i) $a \cdot 0 = 0, \ \forall a \in F$

(ii) $0 \cdot \alpha = 0, \ \forall \alpha \in V$

(iii) $a \cdot (-\alpha) = -a\alpha, \ \forall a \in F, \alpha \in V$

(iv) $a(\alpha - \beta) = a\alpha - a\beta, \ \forall a \in F \ \& \ \alpha, \beta \in V$

(v) $(-a) \cdot \alpha = -a\alpha, \ \forall a \in F, \alpha \in V$

(vi) $a\alpha = 0 \Rightarrow$ either $a = 0$ or $\alpha = 0$

---

(3)

The OREM:

The necessary and sufficient condition for a non-empty ~~set~~ subset 'W' of vector V(F). i.e be a sub-space of 'V', is that W is closed under vector addition & scalar multiplication in V.

Proof: Necessary condition

Let, 'W' be a vector sub-space of 'V'

$\Rightarrow W(F)$ is itself a vector space

Continue →

* Vector Sub-space :

Let, V(F) be a vector space

Let, $W \subseteq V$, then W is said to be a vector sub-space of V if 'W' is itself a vector space, over field F, w.r.t the same operation of vector addition and scalar multiplication.

Eg : $(C, +, \cdot)$

$\Rightarrow R \subseteq C$

$\Rightarrow (R, +, \cdot)$ ≅

$\Rightarrow$ 'W' is closed under vector addition & scalar multiplication.

Sufficient Condition :

Let, 'W' be a non-empty subset of 'V' and 'W' is closed under vector addition & scalar multiplication.

Now, we are to show that —

'W' is a vector sub-space of 'V'.

(1) Given, 'W' is closed under vector addition

$\because$ elements of 'W' are also elements of 'V'.

$\therefore$ vector addition in 'W' will be associative & communitative.

If 1 is the unity (multiplicative identity) element of 'F' then,

$-1 \in F$ [$\because$ F is a field]

Now, $-1 \in F$ & let $\alpha \in W$

$\Rightarrow -1 \cdot \alpha \in W$ [$\because$ W is closed under scalar multiplication]

$\Rightarrow (-1 \cdot \alpha) \in W$

$\Rightarrow -\alpha \in W$

$\because$ α is an arbitrary element of 'W'.

$\therefore$ additive inverse of each element of 'W' is also in 'W'.

$\Rightarrow$ additive inverse exists.

Now, $\alpha \in W$, $-\alpha \in W$

$\Rightarrow \alpha + (-\alpha) \in W$

[$\because$ W is closed under vector addition]

$\Rightarrow \alpha - \alpha \in W$

$\Rightarrow 0 \in W$

where, '0' is the zero vector of 'W'.

$\Gamma \Rightarrow$ additive identity exists

$\therefore (W,+)$ is abelian

(2) Given that 'W' is closed under scalar multiplication.

(3) (i) $(a+b)\alpha = a\alpha + b\alpha$

(ii) $a(\alpha+\beta) = a\alpha + a\beta$

(iii) $(ab)\alpha = a(b\alpha)$

(iv) $1 \cdot \alpha = \alpha$

holds in V

$\because$ V is a vector space

$\therefore$ above properties holds in 'W'

$\because W \subseteq V$

$\Rightarrow$ W is ~~subset~~ a vector space.

& W is vector sub-space of 'V'.

16/09

* The necessary & sufficient condition for a non-empty subset W of vector space $V(F)$ to be a sub space of V.

(i) $\to \alpha, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $\to a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

Soln

Necessary Condition

Let 'W' be a subspace of 'V'

Then, (i)

Let $\alpha, \beta \in W$

If 1 is the unity element of 'F'

Then, $-1 \in F$ [$\because$ F is a field]

Now, $-1 \in F$ & $\beta \in W$

$\therefore -1 \cdot \beta \in W$

[$\because$ W is a sub space of V

$\Rightarrow$ W is a Vector space itself

$\Rightarrow$ Scalar multiplication holds]

$\Rightarrow -(\beta) \in W$

$\Rightarrow -\beta \in W$

Now, $\alpha \in W$ & $-\beta \in W$

$\Rightarrow \alpha + (-\beta) \in W$

[by vector addition property in W]

$\Rightarrow \alpha - \beta \in W$

(ii)

$\because$ W is a Vector Sub-Space

$\therefore$ W is a Vector Space itself

$\therefore$ Scalar multiplication holds

$\therefore a \in F, \alpha \in W$

$\therefore a \cdot \alpha \in W$

$a\alpha \in W$

Sufficient Condition

Let 'W' be a non-empty subset of 'V'

Satisfying following properties:—

(i) $\alpha, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Now, we are to show that

W is a Vector subspace of V.

Let, $-1 \in F$ & $\beta \in W$

$\Rightarrow -\beta \in W$ [by — (ii)]

$\because \alpha \in W$ & $-\beta \in W$

$\therefore \alpha - (-\beta) \in W$

$\Rightarrow \alpha + \beta \in W$

internal composition or vector addition is satisfied.

Again, element of W are also element of V

$\therefore$ vector addition in W is associative & commutative.

[$\because W \subseteq V$, V is a Vector Space]

By condition (i), $\alpha \in W$ & $\alpha \in W$

$\therefore \alpha - \alpha \in W$

$0 \in W$

additive identity exists in W.

By condition (ii), $-1 \in F$, $\alpha \in W$

$\Rightarrow -\alpha \in W$

additive inverse exists in W

$\therefore (W, +)$ is abelian.

By condition (ii), external composition or scalar multiplication holds

(a) $a(\alpha+\beta) = a\alpha + a\beta$
(b) $(a+b)\alpha = a\alpha + b\alpha$
(c) $1 \cdot \alpha = \alpha$
(d) $(ab)\alpha = a(b\alpha)$

holds in V

∴ they are also held in W
∵ $W \subseteq V$, V is a Vector Space.

Thus, W is a Vector Space
W is a Vector Sub-Space of V.

19/09

(1)

Let V be the set of odd pairs (x, y) of real nos.
consider the set
$(x, y)+(x' + y') = (x + x', y + y')$

$c.(x, y) = (|c|x, |c|y)$

Show that V is not a vector space

Sol^n
Let us assume V a vector space
$\therefore$ V should satisfy the postulate
~~$(a+b)\alpha = a\alpha + b\alpha$~~ $\Rightarrow a(\alpha+\beta) = (a\alpha + ...$

Let, $a = 1$, $b = -1 \in F$ & $\alpha = (1, 1) \in V$

$\therefore L.H.S = (1 - 1) . (1.1)$

$= 0 . (1.1)$

$= 0$

$\therefore R.H.S = 1 . (1, 1) + (-1)(1, 1)$

$= (1, 1) + (1, 1)$

$= 2$ 2

$\because LHS \neq RHS$, which contradicts our assumption.
Hence, V(F) is not a vector space.

(1) (2)

* The necessary & sufficient for a non-empty subset W of vector space $V(F)$ to be a Sub-Space of V,

$a, b \in F$ and $\alpha, \beta \in W$

$\Rightarrow a\alpha + b\beta \in W$

Soln Necessary Condition :

Let W be a subspace of $V(F)$

$\therefore$ $W(F)$ is a itself a Vector Space.

$\alpha + a\beta$

$\in V$

$\because a \in F, \alpha \in W$

$\Rightarrow a\alpha \in W$ [scalar multiplication]

Slly, $b \in F$, $\beta \in W$

$\Rightarrow b\beta \in W$

Now, $a\alpha + b\beta \in W$ [vector addition]

Sufficient Condition :

Let W be a non empty ~~set~~ subset of V

& $a, b \in F$ & $\alpha, \beta \in W$

$\therefore a\alpha + b\beta \in W$ —— (1)

putting, $a = 1$ & $b = 1$ in (1)

We get, $\alpha + \beta \in W$, vector addition satisfied

(B)

[also associativity and commutavity is ~~obious~~ obivious.

$\because W \subseteq V$ & V is Vector Space

Now, putting $a=-1$ & $b=0$ in (1)

we get, $-\alpha \in W$

Thus, every element has it's inverse in W.

Now, $a=b=0$ putting in (1)

$\Rightarrow 0 \in W$

$\therefore$ Identity exists

Thus $(V,+)$ is abelian.

$\rightarrow$ Let $a,b \in F$ & $\alpha \in W$ & $\beta = 0$

(1) $\exists a\alpha \in W$

Thus, scalar multiplication holds.

The remaining postulates hold in W

$\therefore W(F)$ is a Vector space

Hence, $W(F)$ is a vector sub-space of V.

POCO M4 PRO 5G

(B) (4)

$\boxed{\text{Note}}$: A non-empty set W of a vector space $V(F)$ is a subspace of V if and only if for each pair of vectors α & β in W & each scalar a in F the vector — $(a\alpha + \beta)$ is again in 'W'.

⟶ Let $V(F)$ be any vector space then V itself & the zero space is always ~~sup~~ sub-space of V. These are called as improper subspaces.

(1)

20/09 Show that —
* The set W of a ordered triod $(a_1, a_2, 0)$ where $a_1, a_2 \in F$, which is a Sub-space of V_3
$F = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in F\}$

Sol^n $W(F) = \{(a_1, a_2, 0) : a_1, a_2 \in F\}$

$\because (0,0,0) \in W$

$\therefore W$ is a non-empty ~~set~~ subset of V_3

Let, $a \in F$ & $\alpha, \beta \in W$

$\therefore \alpha = (a_1, a_2, 0)$ & $\beta = (b_1, b_2, 0)$

$\therefore a\alpha + \beta = a(a_1, a_2, 0) + (b_1, b_2, 0)$

$= (aa_1, aa_2, 0) + (b_1, b_2, 0)$

$= (aa_1 + b_1, aa_2 + b_2, 0 + 0)$

$\because a, a_1, a_2, b_1, b_2, 0 \in F$

$\therefore aa_1, aa_2 \in F$

$\because F$ is a field

$\therefore a\alpha + \beta \in W$

$\therefore W$ is a Sub Space.

* Let V be the Vector Space of all polynomials in variable x over the field F.
Let W be a subset of V containing all polynomial of deg less than equal to
Then show that, W is a Sub-space of

Sol:

$V(F) = \{P(x) : P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, a_i \in$

$W = \{a_0 + a_1 x + \cdots + a_n x^n\}$

We are to show that

W is a Sub-space of V

(3)

$\because 0 \equiv 0 + 0x + 0.x^2 + \ldots + 0x^n \in W$

Let $a \in F$ & $\alpha, \beta \in W$

$\alpha = (a_0 + a_1 x + \ldots + a_n x^n)$

$\beta = (b_0 + b_1 x + \ldots + b_n x^n)$

$\therefore a\alpha + \beta = \{(aa_0 + b_0) + (aa_1 + b_1)x + \ldots + (aa_n + b_n)x^n\}$

$\because a, a_i, b_i \in F$

$\therefore aa_i \in F$ $\quad \because$ [F is a field]

$\therefore a\alpha + \beta \in W$

$\therefore W$ is a subspace of ~~W~~ V.

21/09

* If a_1, a_2 & a_3 are fixed element of a field F. Thus Sub-space the set W of all ordered Triods (x_1, x_2, x_3) of elements of F such that — $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is a Vector Sub-space of $V_3(F)$.

Sol^n $V_3(F) = \{(x_1, x_2, x_3) : x_i \in F\}$

$W = \{(x_1, x_2, x_3) : a_1x_1 + a_2x_2 + a_3x_3 = 0,\ x_i \in F\}$

We are to show that —

W is a vector Sub-space of $V_3(F)$.

Clearly, $(0,0,0) \in W$

$\therefore W$ is a non-empty subset of V_3.

Let, $a \in F$ & $\alpha, \beta \in W$

$\alpha = (x_1, x_2, x_3)$

$\beta = (y_1, y_2, y_3)$

Now, $a\alpha + \beta = (ax_1, ax_2, ax_3) + (y_1, y_2, y_3)$

$= \{(ax_1 + y_1), (ax_2 + y_2), (ax_3 + y_3)\}$

$\because a, x_1, x_2, x_3, y_1, y_2, y_3 \in F$

$\therefore ax_1, ax_2, ax_3, y_1, y_2, y_3 \in F$

i.e $ax_i \in F$ & $y_i \in F$

$\therefore (ax_i + y_i) \in F$

Again, $a_1(ax_1+y_1)+a_2(ax_2+y_2)+a_3(ax_3+y_3)$

$= a(a_1x_1+a_2x_2+a_3x_3)+(a_1y_1+a_2y_2+a_3y_3)$

$= a \cdot 0 + 0$

$= 0$

$\because \alpha \in W$ & $\beta \in W$

$\therefore a_1x_1+a_2x_2+a_3x_3 = a_1y_1+a_2y_2+a_3y_3 = 0$

~~Hen~~ Thus, $a\alpha+\beta \in W$

Hence, W is a vector sub-space of $V_3(F)$

H/W

* Prove that the set of all solutions (a,b,c) of the eqn $a+b+2c=0$ is a Sub-space of the vector space $R^n(R)$.

Soln $R^n(R) = \{(a_i, b_i, c_i) : a_i, b_i, c_i \in R\}$

$W = \{(a,b,c) : a+b+2c=0\}$

Clearly, $(0,0,0) \in W$

$\therefore W$ is a non-empty subset of R.

Let, $x \in F$ & $\alpha, \beta \in W$

Let $\alpha = (a_1, b_1, c_1)$ & $\beta = (a_2, b_2, c_2)$

Now,

$x\alpha + \beta = xa_1 + xb_1 + xc_1 + a_2 + b_2 + c_2$

$= xa_1 + a_2 + xb_1 + b_2 + xc_1 + c_2$

$\because x; a_1, b_1, c_1, a_2, b_2, c_2 \in R$

$\therefore xa_i + xb_i + xc_i \in R$ [R is a field]

$\therefore x\alpha + \beta \in R$

Now, $a + b + 2c = 0$

$L.H.S = (xa_1 + a_2) + (xb_1 + b_2) + 2(xc_1 + c_2)$

$= x(a_1 + b_1 + 2c_1) + (a_2 + b_2 + 2c_2)$

$= x \cdot 0 + 0$

$= 0 = R.H.S$

$\therefore x\alpha + \beta \in W$

Hence, W is a Sub-space of $R^n(R)$.

H/w

* Which of the following set of vectors $\alpha = (a_1, a_2, \ldots, a_n)$ in R^n are subspace of R^n ? $n \geq 3$

(a) all α such that : $a \geq 0$

(b) all α such that : $a_1 + 3a_2 = a_3$

(c) all α such that : $a_2 = a_1^2$

(d) all x such that : $a_1 \times a_2 = 0$

Sol^n (a) $R^n(R) = \{a_1, a_2, \ldots, a_n\}$

$\therefore \alpha = \{(a_1, a_2, \ldots, a_n) : a_i \in R, \ a_i \geq 0\}$

Clearly, $0 \in \alpha$

$\therefore \alpha$ is a non-empty subset of R

Let, $a \in R$ & $A, B \in \alpha$

& $A = (a_1, a_2, \ldots, a_n)$, $B = (b_1, b_2, \ldots, b_n)$

$\therefore aA + B = aa_1 + aa_2 + aa_3 + \ldots aa_n + b_1 + b_2 + b_3 \ldots b_n$
$= (aa_1 + b_1), (aa_2 + b_2) \ldots (aa_n + b_n)$

$\because a_i \in R \quad \therefore aa_i, b_i \in R$

$\therefore aa_i + b_i \in R \quad [\because R$ is a field$]$

Now, $a \geq 0$

$\Rightarrow \cancel{aa_1 + b_1 \geq 0}$

$\Rightarrow \cancel{aa_1 \geq -b_1} \Leftrightarrow a_i \geq \frac{-b_i}{a_i}$

$\because \alpha \in W$ & $\beta \in W$

$\because a_1 \geq 0$ & $b_1 \geq 0$

$\therefore a a_1 + b_1 \geq 0$

$\therefore a\alpha + \beta \in W$

$\therefore W$ is a V.S.

$\because W \subset V$

Thus, W is a VSS.

S. A

(b) $\alpha = \{(a_1, a_2, \dots, a_n) ; a_i \in R$ & $a_1 + 3a_2 = a_3\}$

Clearly, $0 \in \alpha$

$\therefore \alpha$ is a non-empty subset of R.

Let, $a \in R$ & $A, B \in \alpha$

$A = (a_1, a_2, a_3)$

$B = (b_1, b_2 + b_3)$

Now, $aA + B$

$= (a a_1 + b_1) + (a a_2 + b_2) + (a a_3 + b_3)$

$\because a, a_1, b_1 \in R \quad \therefore a a_1, b_1 \in R$

$\therefore a a_i + b_i \in R$

Now, $a_1 + 3a_2 = a_3$

L.H.S $= aa_1 + b_1 + (aa_2 + b_2)3$

$= a(a_1 + 3a_2) + (b_1 + 3b_2)$

$= aa_3 + b_3 = R.H.S$

$\therefore$ $aA + B \in \alpha$

$\therefore$ α is a sub-space of $\mathbb{R}^4$.

(c) Now, $a_2 = a_1^2$

$\cancel{R.H.S \Rightarrow (aa_1 + b_1)^2 = (aa_1)^2 + 2aa_1b_1 + b_1^2}$

$\cancel{= a^2 \cdot a_1^2 \cdot 2 \cdot a \cdot a_1 \cdot b_1 + b_1^2}$

$\cancel{= a^2 \cdot b_2 \cdot 2 \cdot a}$

$= (aa_1 + b_1)^2 - (a^2a_1^2 + 2aa_1b_1 + \cancel{b_1^2}) + aa_1^2 + \cancel{b_1^2}$

L.H.S $= aa_2 + b_2$

$= aa_1^2 + b_1^2 = \dfrac{\cancel{(aa_1)^2} + \cancel{b_1^2} + 2aa_1}{+ aa_1^2 - \cancel{(a^2a_1^2)} \mp 2aa_1}$

$= \cancel{a_1^2} + (aa_1^2 - a^2a_1^2 - 2aa_1)$

$\neq R.H.S$

$\therefore$ α is not a sub space of $\mathbb{R}$.

(c) Sol^n a_2 is rational

$\therefore a a_2 + b_2$

$\Rightarrow a \, p/q + p_2/q_2 = \frac{a p q_2 + q, p_2}{q_2 q}$

(d)

Sol^n Now, $a_1 \times a_2 = 0$

L.H.S $\Rightarrow (a a_1 + b_1)(a a_2 + b_2)$

$\cancel{= (a a_1)^2 + (b_1 + b_2) a a_1 + b_1 b_2}$

$\cancel{(a a_1)^2 + (b_1 + b_2) a a_1 + 0}$

$= (a a_1 \times a a_2) + (a a_1 \times b_2) + (b_1 \times a a_2) + (b_1 \times b_2)$

$= a a_1 b_2 + b_1 a a_2$

$= a(a_1 b_2 + a_2 b_1) \neq 0$

22/09

* Let V be Vector space of all function from $R \to R$ which of the following set of function are subspaces of function V.

(a) all f such that $f(x^2) = [f(x)]^2$

(b) all f such that $f(0) = f(1)$

(c) all f such that $f(3) = 1 - f(-5)$

(d) all f such that $f(-1) = 0$

e) f which are continous.

Sol^n given, $V(R) = \{f : R \to R\}$

(a) $W = \{f : f \in V,\ f(x^2) = [f(x)]^2\}$

Let, $a \in R$ & $f, g \in W$

$\therefore f(x^2) = [f(x)]^2$

$g(x^2) = [g(x)]^2$

Now, $af + g$

$= af(x^2) + g(x^2) \quad = a[f(x)]^2 + [g(x)]^2$

$\neq [af(x) + g(x)]^2 \quad = [(af+g)(x)]^2$

$\therefore (af+g)(x^2) \neq [(af+g)(x)]^2$

$\Rightarrow af + g \notin W$, W is not a vector spac

(b) $W = \{f : f \in V,\ f(0) = f(1)\}$

* Let C be the field of complex no. & n b a positive integer $n \geq 2$.
Let V be the vector space of all $n \times n$ matrices over C. Which of the following set of matrices 'A' in V are subspaces of

(a) all matrices A are invertible

(b) all non-invertible matrices

(c) all A such that $AB = BA$, where B is the fixed matrix.

Sol^n $V(C) = \{A = [a_{ij}]_{n \times n} : a_{ij} \in C\}$

$W(C) = \{A : A \in V \text{ and } A \text{ is invertible}\}$

Let, $A, B \in W$ & $a \in C$

$\Rightarrow A, B$ are invertible

$\Rightarrow |A| \neq 0$ & $|B| \neq 0$

Let, $a = 1 \in C$

Let, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore |A| = 1 \;, \; |B| = -1$

$\therefore |a[A] + [B]| = 0$

$\therefore |a[A] + [B]|$ is not-invertible.

i.e $|a[A]+[B]| \notin W$

W is not a vector space

(b) $W = \{A = [a_{ij}]_{n \times n} : a_{ij} \in C, A \text{ is non-invertible}\}$

Let, $A, B \in W$ & $a \in C$

$\therefore$ A & B are non-invertible.

Now, $|aA + B|$ which is clearly non-invertible

$\therefore$ $|aA + B| \in W$

$\therefore$ W is a vector sub-space of V.

(c) $W = \{A = [a_{ij}]_{n \times n} : a_{ij} \in C$ & $AB = BA$
& B is fixed$\}$

Let, $A, B \in W$ & $a \in C$

$\therefore$ $a = 1$ & $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\therefore$ $[aA + B] = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$

Now, $[aA + B]B = \begin{bmatrix} 2 & 9 \\ 2 & 8 \end{bmatrix}$

Γ & $B[aA+B] = \begin{bmatrix} 8 & 9 \\ 2 & 2 \end{bmatrix}$

$\therefore AB \neq BA$

$\Rightarrow aA+B \notin W$

$\therefore W$ is not a vector space.

Date (23/09/22)

(c) $W = \{A_{n\times n} \in V : AB = BA,$ B is fixed$\}$

$\because [0]_{n\times n} \in W$

$\therefore W$ is a non-empty subset of V

Let, $a \in V$ & $A_1, A_2 \in W$

$$\therefore \left.\begin{aligned} A_1 B &= B A_1 \\ A_2 B &= B A_2 \end{aligned}\right\} \text{—— (1)}$$

Now, $(aA_1 + A_2)B$

$= aA_1B + A_2B$

$= aBA_1 + BA_2$

$= B(aA_1 + A_2)$

$\therefore (aA_1 + A_2) \in W$, W is a VSS of V

Q. Let V be the vector space of all real $(n\times n)$ matrix. Proof that, the set W consisting of $(n\times n)$ matrix which comute with a given matrix T, of V. Show that W is vector sub-space of V.

Soln
$W = \{A_{n\times n} : A_{n\times n} \in V,\ AT = TA,\ T \text{ is fixed}\}$

$\because [0] \in W$

$\therefore W$ is a non-empty subset of V

Let, $a \in V$ & A_1 & $A_2 \in W$

$\therefore A_1 B = B A_1$ & $A_2 B = B A_2$ } — (1)

Now, $(aA_1 + A_2)B = aA_1B + A_2B$

$= aBA_1 + BA_2$

$= B(aA_1 + A_2)$

$\therefore (aA_1 + A_2) \in W$

$\therefore W$ is a vector subspace of V.

* Let V be the Vector Space of all 2×2 matrixes over the real field R. Show that subset of V consisting all 2×2 matrix A satisfying $A^2 = A$ is not a Sub-space of V.

Solⁿ

Let W be vector sub-space of V.

~~if $a \in V$ & $A_1, A_2 \in W$~~

~~Now, $(aA_1 + A_2)^2$~~ & $A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \in W$

But $A^2 = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4+9 & 6+3 \\ 6+3 & 9+1 \end{bmatrix}$

$= \begin{bmatrix} 13 & 9 \\ 9 & 10 \end{bmatrix} \neq A$

It contradicts the fact that W is a subspace of V.

We conclude that W ain't sub-space of

* Linear Span :

Let $V(F)$ be the vector-space & S be any non empty subset of V. Then the Linear span of S is the set of all linear combination of the finite set of elements of S & is denoted by $L(S)$.

Thus, we have $L(S) = \{a_1x_1 + a_2x_2 + \cdots + a_nx_n : a_i \in F, x_i \in S\}$

* $\boxed{\text{Note}}$

1 → $L(S)$ is a sub-space of V.

2 → $L(S)$ is a smallest sub-space of V containing S.

* Show that the subset $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $V_3(F)$ generates of spans the entire vector space $V_3(F)$, that is $L(S) = V_3(F)$.

Sol^n^ Let, $(a,b,c) \in V_3(F)$

Now, $(a,b,c) = a(1,0,0)\ b(0,1,0)\ ,\ c(0,0,1)$

which $\in L(S)$

$\Rightarrow V_3(F) \subseteq L(S)$, we know $L(S) \subseteq V_3(S)$

Thus, $V_3(F) = L(S)$. Hence, S generates or spans the entire Vector space $V_3(F)$.

24/09

If W_1 & W_2 are two are two sub-spaces over the vector space $V(F)$, then—

(i) $W_1 + W_2$ is a sub-space of $V(F)$.

(ii) $W_1 + W_2 = L(W_1 \cup W_2)$

Solⁿ

$W_1 + W_2 = \{\alpha \in V : \alpha = a_1 + a_2$, where $a_1 \in W_1$ & $a_2 \in W_2\}$

Let, $a, b \in F$ & $\alpha, \beta \in W_1 + W_2$

~~Let~~, $\alpha = a_1 + a_2$ & $\beta = b_1 + b_2$

$a_1, b_1 \in W_1$ & $a_2, b_2 \in W_2$

Now, $a\alpha + \beta = aa_1 + aa_2 + b_1 + b_2$

$= (aa_1 + b_1) + (aa_2 + b_2)$

$\because a \in F$ & $a_1, b_1 \in W_1$, $\therefore (aa_1 + b_1) \in W_1$

& $a \in F$ & $a_2, b_2 \in W_2$, $\therefore (aa_2 + b_2) \in W_2$

We know, W_1 & W_2 are itself sub-space of $V(F)$.

$\therefore (aa_1 + b_1) + (aa_2 + b_2) \in W_1 + W_2$

$(a\alpha + \beta) \in W_1 + W_2$

Hence, $W_1 + W_2$ is a vectors sub-space of $V($

(11) Let, $a_1 \in W_1$ & $a_2 \in W_2$
also, $0 \in W_1$ & $0 \in W_2$
$\because$ W_1 & W_2 are subspaces
[identity element exists]

So, $a_1 + 0 \in W_1 + W_2$

$\Rightarrow a_1 \in W_1 + W_2$

$\therefore W_1 \subseteq W_1 + W_2$

Slly, $W_2 \subseteq W_1 + W_2$

$\Rightarrow (W_1 \cup W_2) \subseteq (W_1 + W_2)$

If V is a vector ~~sub~~-space & $S \subseteq V$
$S \neq \phi$
We know, $L(S)$ is sub-space of $V(F)$
also, $(W_1 \cup W_2)$ is sub-~~space~~ set of $W_1 + W_2(F)$.
$\Rightarrow L(W_1 \cup W_2)$ is sub-~~space~~ set of $W_1 + W_2$

$\therefore L(W_1 \cup W_2) \subseteq (W_1 + W_2)$ —— (1)

Let, $\alpha = a_1 + a_2 \in (W_1 + W_2)$ where
$a_1 \in W_1$ & $a_2 \in W_2$
also, $a_1 \in W_1 \cup W_2$
$a_2 \in W_1 \cup W_2$

α can be written as linear comivabtion
of a_1 & a_2, $\alpha = 1.a_1 + 1.a_2$

$\therefore \alpha \in L(\omega_1 \cup \omega_2)$

$\therefore (\omega_1 + \omega_2) \subseteq L(\omega_1 \cup \omega_2)$ —— (ii)

By (i) & (ii), we get

$$(\omega_1 + \omega_2) = L(\omega_1 \cup \omega_2)$$

26/09

* A finite set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of V is said to be lineary independent. if

$a\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$, $a_i \in F$

$\Rightarrow a_i = 0, \forall i = 1, 2, \ldots, n$

* Any infinite set of vectors V is said to be independent if its every finite subset is linearly independent. otherwise, it is linearly dependent.

Q. If two ~~scalars~~ vectors are linearly dependent then prove that one of them is the scalar multiple of the other.

Soln

Let, $\{\alpha, \beta\}$ ~~are~~ be linearly dependent

There exists two scalar a, b (~~not~~ both never zero)

such that $a\alpha + b\beta = 0$,

Case I: If $a \neq 0$

$$\therefore a\alpha = -b\beta$$

$$\Rightarrow \alpha = \left(\frac{-b}{a}\right)\beta$$

$$\Rightarrow \alpha = K\beta \text{ , where } k = \frac{-b}{a}$$

Thus α is scalar multiple of β.

Case II: If $b \neq 0$

$$\therefore -a\alpha = b\beta$$

$$\Rightarrow \beta = \frac{-a}{b}\alpha$$

$$\Rightarrow \beta = k'\alpha \text{ , } k' = \frac{-a}{b}$$

$\therefore$ β is also a scalar multiple but of α.

A. Prove that every super set of a linearly dependent set of vectors is linearly dependent.

Solⁿ

Let, $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be set of linearly dependent set of vectors.

$\Rightarrow$ There exists scalars $a_1, a_2, \dots, a_n$ (not all zero) such that

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0 \quad \text{——— (i)}$$

Let, $S' = \{\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m\}$

be a superset of S, then

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n + 0.\beta_1 + 0.\beta_2 + \dots + 0.\beta_m \quad \text{——— (ii)}$$

where some coefficients are non-zero.

$\Rightarrow S'$ is linearly dependent.

Hence proved

Note

From above we conclude any subset of linearly independent is linearly independent whereas, any superset of linearly dependent is linearly dependent.

27/09

* Show that the set $\{1, x, x(1-x)\}$ is a L.I. set of vectors in the space of all polynomials over $\mathbb{R}$.

Soln

Let $a, b, c \in \mathbb{R}$

such that: $a \times 1 + b.x + c.x(1-x) = 0$

$$\Rightarrow a + bx + cx - cx^2 = 0$$

$$\Rightarrow -cx^2 + (b+c)x + a = 0$$

We have,

$-c = 0$, $b+c = 0$ & $a = 0$

$\therefore c = b = a = 0$

Hence, the given set of vectors are linearly independent.

* Verify whether the set of vector $\{2x^3+x^2+x+1,\ x^3+3x^2+x-2,\ x^3+2x^2-x+3\}$ are linearly independent or not.

Sol^n

Let $a, b, c \in R$, such that —

$a.(2x^3+x^2+x+1) + b(x^3+3x^2+x-2) + c(x^3+2x^2-x+3) = 0$

$\Rightarrow 2ax^3 + ax^2 + ax + a$
$+ bx^3 + 3bx^2 + bx - 2b$
$+ cx^3 + 2cx^2 - cx + 3c = 0$

$\Rightarrow x^3(2a+b+c) + x^2(a+3b+2c) + x(a+b-c) + (a-2b+3c) = 0$

i.e
$$\left.\begin{array}{l} 2a+b+c \\ a+3b+2c \\ a+b-c \\ a-2b+3c \end{array}\right\} = 0$$

Let us construct a matrix A,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & -1 \\ 0 & 3 & 4 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 2 \\ 0 & 3 & 4 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 0 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad R_4 \to R_4 - R_3.$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 3 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 4 \end{bmatrix} \quad R_3 \to 5R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_4 \to 3R_4 + 4R_3$$

$\because$ no. of ~~zeros~~ non-zero rows $= 3$

$\therefore$ rank$(A) = 3$

i.e no. of unknowns

$a = b = c = 0$

$\therefore$ set of vectors are linearly independent

$\boxed{\text{Note}}$ - A set of vectors which contain zero vector is always linearly dependent.

A non empty subset W of a vector space $V(F)$ is a subspace of 'V' iff for each pair of vectors α & β in 'W' & each scalar 'a' in F the vector $a\alpha+\beta$ is again in W

Sol^n

Necessary Condition :

$\because W \subseteq V(F)$,

W is a subspace

$\therefore W$ is a vector-space itself

Let, $a \in F$ & $\alpha, \beta \in W$

By scalar multiplication,

we get $a\alpha \in W$

& By vector addition, we get

$a\alpha + \beta \in W$.

Sufficient condition :

W be a non empty subset of $V(F)$

& $a \in F$ & $\alpha, \beta \in W$.

We have, $a\alpha + \beta \in W$ ——— (1)

If $a = 1$ in (1)

$\Rightarrow \alpha + \beta \in W$

$\therefore$ vector addition satisfied, also associtivity, now commutativity is obvious.

$\because a\alpha + \beta \in W$

$\therefore a\alpha \in W$ —— (11) scalar multiplication holds.

If $a = -1$ in (11)

(ii) $\Rightarrow -\alpha \in W$

Thus, every element has its inverse.

If $a = 0$ in (11)

(ii) $\Rightarrow 0 \in W$

$\therefore$ Identity exists in addition so $(V, +)$ abelian

$\therefore W(F)$ is a vector space

$\because W \subseteq V(F)$

W is vector sub-space of $V(F)$.

28/09

* Find a maximal LI sub-system of the system of vectors.

$\alpha_1 = (2, -2, 4)$ $\alpha_2 = (1, 9, 3)$

$\alpha_3 = (-2, -4, 1)$ & $\alpha_4 = (3, 7, -1)$

Soln

Let, us consider a matrix A,

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & 9 & 3 \\ -2 & -4 & 1 \\ 3 & 7 & -1 \end{bmatrix}$$ whose rows are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ respectively.

Now, we will reduce the matrix A to echelon form by applying row transformation

$$\therefore A \sim \begin{bmatrix} 1 & -1 & 2 \\ -1 & 11 & -1 \\ -2 & -4 & 1 \\ 1 & 3 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1/2 \\ R_4 \to R_4 + R_3 \\ R_2 \to -R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 10 & 1 \\ -3 & -15 & 0 \\ 1 & 3 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + R_2 \\ R_2 \to R_2 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 10 & 1 \\ 3 & 5 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_4 \to 5R_4 + R_2 \\ R_3 \to -R_3/_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 10 & 1 \\ 2 & 0 & -1 \\ 2 & 0 & 0 \end{bmatrix} \quad R_3 \to 2R_3 - R_2$$

$$\sim \begin{bmatrix} \cancel{5} & \cancel{-1} & \cancel{0} \\ & & \\ & & \end{bmatrix} \quad \cancel{R_1 \to R_1 + 2R_3}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 10 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & +1 \end{bmatrix} \quad \begin{array}{l} R_4 \to R_4 - R_3 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 10 & 1 \\ 0 & 0 & 14 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to 5R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 10 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 14R_1 \\ R_4 \to 14R_4 - R_3 \end{matrix}$$

$\therefore$ no of non-zero rows is 2

$\therefore$ Rank(A) = 2

$\therefore$ Maximal no. of LI rows vectors in matrix A is 2.

$\therefore$ vector α_1 & α_2 are LI.

So, α_1 & α_2 is a maximal LI sub-system of the given vetors

We obeserve that none of the given vectors is scalar multiple vector of another vetor

$\therefore$ any two vetors will form maximal LI sub-system.

12/10

* Let, $V(R)$ is a Vector space, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-zero vectors belong to V. Then either they are LI or some α_k, $2 \leq k \leq n$ is a linear combination of the preceding ones. i.e $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$.

Solⁿ

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent then we have nothing to prove

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are LD & there exists $a_1, a_2, a_3, \ldots, a_n \in F$ not all zero; such that

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0 \quad \text{(1)}$$

Let, $k \geq 2$ be the largest integer for which $a_k \neq 0$.

$$a_k, a_{k+1} = a_{k+2} = \ldots = a_n = 0$$

There is no harm in this assumption because at the most if $a_n \neq 0$ then $k=$

(2)

Also if,

$$a_2 = a_3 = \cdots = a_k = 0$$

then,

$$a_1 \alpha_1 = 0$$

$$\Rightarrow a_1 = 0 \quad \because \alpha_1 \neq 0$$

So, we have taken $k \geq 2$

Now (1) reduces to

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k = 0$$

$$\Rightarrow a_k\alpha_k = -a_1\alpha_1 - a_2\alpha_2 - \cdots - a_{k-1}\alpha_{k-1}$$

$$\Rightarrow \alpha_k = \left(\frac{-a_1}{a_k}\right)\alpha_1 + \left(\frac{-a_2}{a_k}\right)\alpha_2 + \cdots + \left(\frac{-a_{k-1}}{a_k}\right)\alpha_{k-1}$$

$\therefore$ α_k can be written as LC of preceding vectors.

* Basis of a Vector Space :

A subset S of a Vector space V(F) if —

(A) S consists of LI vectors

(B). S generates V(F) i.e $L(S) = V$

* **Note :**

The vector space F(x) of polynomials over the field F has no finite basis.

Date 15/10

* Existence & Invarience of a Basis :

If V(F) is a finite dimensional Vector Space then any two basis of V has the same no of elements.

There exists a basis for each finite dimensional Vectors.

* Dimension of a Vectors :

If V(F) is a finite dimensional Vector space then the no. of elements in any basis of V(F) is called its dimensions of the

Vector Space $V(F)$, & is denoted by $\dim(V)$.

Eg: $\dim(C(R)) = 2$

$C(R) = \{a+ib : a, b \in R\}$

$= \{1, 2\}$

$\boxed{\text{Note}}$

→ If F is a a field, dimension of F(F)

$d(F(F)) = 1$

Eg: $R(R) = \{a : a \in R\}$

$= \{1\}$

* Extension Theorem :

Every LI subset of a finite dimensional Vector Space $V(F)$ is either a basis of V or can be extended to form a basis.

proof

Let, $S = \{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ be a LI subset of a finite dimen- dimensional Vector space $V(F)$.

Let, $\dim(V(F)) = n$

then V has a finite basis of n vectors.

(Say) $\{\beta_1, \beta_2, \beta_3, \ldots, \beta_m\}$

Case 1:

If $n=m$, then S is a basis

Case 2:

If $n \neq m$, Let

$S_1 = \{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n, \beta_1, \ldots \beta_m\}$

clearly,

$L(S_1) = V$ &

S_1 is LD.

$\therefore$ There exists some vectors of S_1 which are LC of its preceding vectors.

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(3)

(4)

There can't be α_i's as —
α_i's can't be LD.

Let us consider β_i can be expressed as LC of its preceding vectors.

Let,
$S_2 = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_n\}$

Sub Case 1:

Obiviously, $L(S_2) = V$
If S_2 is LI, then —
S_2 will be a basis of V, containing S (extended set of S).

Sub Case 2:

If S_2 is LD, then —
repeating the above process upto the finite no. of times we will get a LI set containing S & spanning V.

(5)

This set will be a basis of V

& it will contain S,

∵ Any basis will contain n no. of elements

∴ Exactly $(n-m)$ no. of β_i's will be adjointed to S to basis

Date

* Note

(i) If V is a vector space which is span by finite set of vectors.

$\beta_1, \beta_2, ---, \beta_n$

Then, any LI set of vectors in V is finite & contains not more than ~~m~~ n vectors.

(ii) If a set S of n vectors of a finite dimensional Vector Space V(F) of dimension 'n' generates V, then S is a ~~do~~ basis of V

(iii) Each ~~element~~ subspace W of a dimension Vector Space V(F) of dimension 'm' is a finite dimensional Vector Space with dimension '$m \leq n$'.

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Also, $V = W \iff \dim(V) = \dim(W)$

(iv) Let, $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$
be a basis of finite dimensional Vector Space, $V(F)$. of dimension 'n'.

Then every element of V can be uniquely expressed as LC of vectors of S.

i.e $\forall \alpha \in V$.

We can write uniquely —

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$
(where $a_1, a_2, a_3, \dots, a_n \in F$)

(v) If W_1 & W_2 are two subspaces of a finite dimensional Vector Space $V(F)$, then —
$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

* Show that —
$\{(1,i,0), (2i,1,1), (0,1+i,1-i)\}$ is a basis for $V_3(C)$.

Sol^n
Let, $a, b, c \in C$
such that, $a(1,i,0) + b(2i,1,1) + c(0,1+i,1-i) = 0$

$\Rightarrow (a + 2bi + 0), (ai + b + c + ci), (0 + b + c - ci) = (0,0,0)$

$\Gamma \Rightarrow (a+2bi), ((b+c)+(a+c)i), ((b+c)-ci) = (0,0,0)$ Γ

$\therefore a+2bi = 0$; $(b+c)+(a+c)i = 0$; $(b+c)-ci$

$\therefore a=0$, & $b=0$, & $c=0$

Thus, given set of vector is LI.

$\therefore \dim V_3(C) = 3$

$\therefore$ the given set is the basis for $V_3(C)$.

$\star$ $C_3(R) = \{(a+ib, c+id, e+if) : a, b, c \in R\}$

$C(R) = \{(a+ib) : a, b \in R\}$

$\therefore \dim(C_3(R)) = 6$

$\dim(C(R)) = 2$ //

H/w

$\star$ Show that the vectors $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,2,1)$ & $\alpha_3 = (0,-3,2)$ form a basis for $R_3(R)$.

Sol^n

Let, $a, b, c \in C$

such that –

$a(1,0,-1) + b(1,2,1) + c(0,-3,2) = 0$

$\Rightarrow (a+b), (a + 2b + 3c), (-a + b + 2c) = (0,0,0)$

~~$\therefore$ $a+b=0$, $a+2b-3c=0$, $-a+b+2c=0$~~

~~$\Rightarrow b = -a$ $\quad \Rightarrow$ [illegible] $+2b - 3c = 0$ $\quad$ [illegible]~~

~~$\Rightarrow 2b = 2c$~~

$\Rightarrow 2b = 3c$

$\therefore$ a

$\Rightarrow (a+b), (2b-3c), (-a+b+2c) = (0,0,0)$

$a+b=0$; $\quad 2b-3c=0$; $\quad -a+b+2c=0$

$\Rightarrow a = -b$ $\quad \Rightarrow 2b = 3c$ $\quad \Rightarrow \frac{3}{2}c - a + 2c = 0$

$\Rightarrow b = 0$ $\quad \Rightarrow -2a = 3c$ $\quad \Rightarrow -3c + 2c = 0$

$\Rightarrow -a = \frac{3}{2}c$ $\quad \Rightarrow c = 0$

$\Rightarrow a = 0$

$\therefore a = 0, b = 0, c = 0$

$\therefore$ Set of vector is L.I. & $\dim(\mathbb{R}^3(R)) = 3$

Hence, It forms a basis.

* Show a finite subset W of a vector space $V(F)$ is L.D. iff some element of W can be expressed as a LC of the others.

Soln

Let $W = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be finite, subset of V

Let, W is LD

$\Rightarrow$ There exists $a_1, a_2, \ldots, a_n \in F$ not all zero such that —

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$ (1)

without loss of generality

Let, $a_r \neq 0$ for some $1 \leq r \leq n$

Then, (1) $\Rightarrow a_r\alpha_r = -a_1\alpha_1 - \cdots a_{r-1}\alpha_{r-1} - a_{r+1}\alpha_{r+1} - \cdots - a_n\alpha_n$

$$\Rightarrow \alpha_r = \left(\frac{-a_1}{a_r}\right)\alpha_1 + \cdots + \left(\frac{-a_{r-1}}{a_r}\right)\alpha_{r-1} + \left(\frac{-a_{r+1}}{a_r}\right)\alpha_{r-1} + \cdots + \left(\frac{-a_n}{a_r}\right)\alpha_r$$

$\Rightarrow \alpha_r$ can be expressed as LC of the others.

Conversely,

Let some vectors of (i) can be expressed as LC of others.

Let us consider α_1 can be written LC of $\alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n$.

i.e $\alpha_1 = b_2\alpha_2 + b_3\alpha_3 + \cdots + b_n\alpha_n$

$b_1, b_2, \ldots, b_n \in F$

$\Rightarrow \alpha_1 - b_2\alpha_2 - b_3\alpha_3 - \cdots - b_n\alpha_n = 0$ (ii)

In (ii) among the vectors, coefficient of α_1 is 1. & $1 \neq 0$

$\therefore \alpha_1, \alpha_2, \ldots, \alpha_n$ are LD.

Theorem

* Linear Transformation / Vector Space Homomorphism :

Let $U(F)$ and $V(F)$ be two Vector Space over the same field F.

A LT from U to V is a f^n.

$T: U \rightarrow V : T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

or $T(a\alpha + \beta) = aT(\alpha) + T(\beta)$

$F \quad \forall \alpha, \beta \in U, \quad a, b \in F$

(*) Let $V(F)$ be the Vector Space of $m \times n$ matrices over the field F. Let P be a fixed $m \times n$ matrix over F. & Let Q be a fixed $n \times m$ matrix over F. The correspondence.

$T: V \to V$, defined by—

$T(A) = PAQ \quad \forall A \in V$

Q order is $n \times m$

A " " $m \times n$

P " " $n \times n$

Solⁿ

Let $A \in V$, be a $m \times n$ matrix

$\Rightarrow PAQ$ is also $n \times n$ matrix

Now, Let $A, B \in V$

& $a \in F$ then

$T(aA + B) = P(aA+B)Q$

$= P(aA)Q + PBQ$

$= aPAQ + PBQ$ [$\because a$ is scalar]

$\because A, B \in V$

 BBC_kits

* Properties of Linear Transformation :

(i) $T(0) = 0$

(ii) $T(-\alpha) = -T(\alpha)$

(iii) $T(\alpha - \beta) = T(\alpha) - T(\beta)$

(iv) $T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n) =$

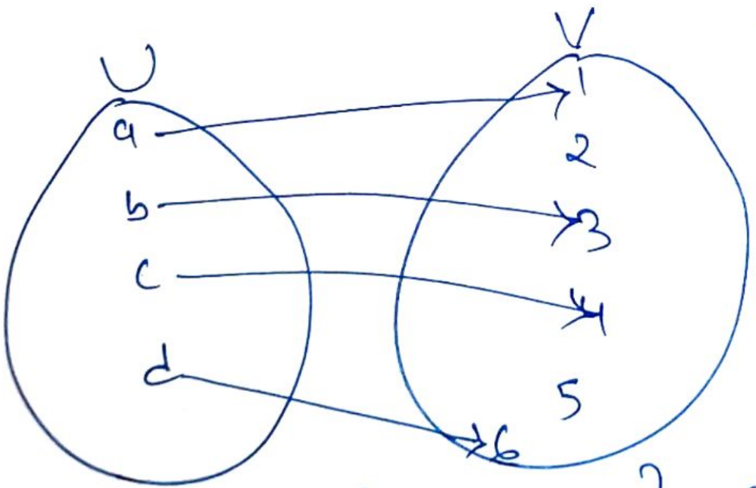
$$a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_n T(\alpha_n)$$

*** Range Space :** The range space of a Linear Transformation is denoted by $R(T)$ & defined by

$R(T) = \{\beta \in V : \beta = T(\alpha), \alpha \in V\}$

The dimension of the Range Space is said to be Rank (T).

$\text{Rank}(T) = \dim(R(T))$



$T(a) = 1$

$T(b) = 3$

$T(c) = 4$

$T(d) = 6$

$\therefore R(T) = \{1, 2, 4, 6\}$ & $\text{Rank}(T) = 4$.

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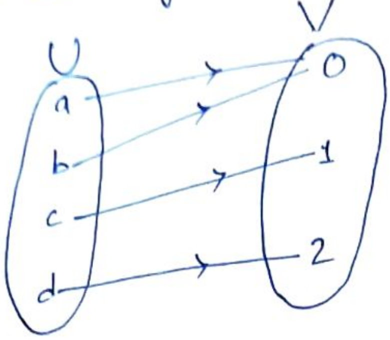
* Null Space : Null Space of a Linear Transformation T is defined as $N(T) = \{\alpha \in U : T(\alpha) = 0\}$



The dimension of Null Space is said to be Nullity.

$\text{Nullity}(T) = \dim(N(T))$.

$R(T) = \{0, 1, 2\}$, $N(T) = \{a, b\}$

$\text{Rank}(T) = 3$, $\text{Nullity}(T) = 2$

* Rank- Nullity Theorem :

Let U & V be Vector Space over the field F.
Let, $T : U \to V$ be a Linear Transformation. & U is of finite dimension then—

$\text{Rank}(T) + \text{Nullity}(T) = \dim(V)$

proof

Let $N(T)$ is the Null Space of T.
$\because$ $N(T)$ is subspace of U &
U is of finite dimension

Let, $\dim(N(T)) = k = \text{Nullity}(T)$

Let, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be the basis of $N(T)$.

$\Rightarrow S$ is Linearly Independent.

$S \subseteq N(T) \subseteq U$

$\therefore$ By extension theorem,

S can be extend to form of basis of U

Let, $\dim(U) = n$

$S' = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$

be a basis of ~~capital you~~ U.

$\therefore T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n) \in R(T)$.

We claim that,

$S'' = \{T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots T(\alpha_n)\}$

is a for $R(T)$.

$\rightarrow S''$ is Linearly Independent

Let, $C_{k+1} T(\alpha_{k+1}) + C_{k+2} T(\alpha_{k+2}) + \ldots + C_n \{T(\alpha_n) = 0$ ——— (1)

$\Rightarrow T(C_{k+1}\alpha_{k+1} + C_{k+2}\alpha_{k+2} + \ldots + C_n\alpha_n) = 0$

$\Rightarrow \therefore (C_{k+1}\alpha_{k+1} + C_{k+2}\alpha_{k+2} + \ldots + C_n\alpha_n) \in N(T)$

$\left[\because T(\alpha) = 0 \Rightarrow \alpha \in N(T)\right]$

$\because$ S is a basis of $N(T)$

$\therefore$ Any element of $N(T)$ can be written Linear Combination of elements of S.

$\therefore C_{k+1}\alpha_{k+1} + C_{k+2}\alpha_{k+2} + \cdots + C_n\alpha_n =$

$b_1\alpha_1 + b_2\alpha_2 + \cdots + b_k\alpha_k$

$\Rightarrow b_1\alpha_1 + b_2\alpha + \cdots + b_k\alpha_k + C_{k+1}\alpha_{k+1} + C_{k+2}\alpha_{k+2} + \cdots + C_n\alpha_n = 0$

$\Rightarrow b_1 = b_2 = \cdots = b_k = C_{k+1} = C_{k+2} = \cdots = C_n$

$\because S'$ is a basis of U.

$\therefore S'$ is Linearly Independent.

$\therefore S''$ is Linealy Independent.

→ S'' Spans $R(T)$:

Let $\beta \in R(T)$

$\Rightarrow$ There $\alpha \in U$: $T(\alpha) = \beta$

$\because \alpha \in U$

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

where, $a_p \in F$

$T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n)$

$\Rightarrow \beta = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_n T(a_n)$

$\Rightarrow \beta = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_k T(\alpha_k) + \dots + a_n T(a_n)$ —— (ii)

$\because S$ is a basis of $N(T)$

$\therefore \alpha_1, \alpha_2, \dots, \alpha_k \in N(T)$

$T(\alpha_1) = T(\alpha_2) = \dots = T(\alpha_k) = 0$

(ii) $\Rightarrow \beta = a_{k+1} T(\alpha_{k+1}) + \dots + a_n T(\alpha_n)$

$\because T(\alpha_{k+1}), \dots, T(\alpha_n)$ spans $R(T)$

$\therefore \dim(R(T)) = n - k$

S'' is a basis of $R(T)$

$\therefore$ Rank $(T) = n = k$

$\therefore$ Rank (T) + Nullity (T)

$= n - k + k$

$= n$

$= \dim(V)$

Show that the mapping $T: V_2(R) \longrightarrow V_3(R)$ defined as $T(a,b) = T(a+b, a-b, b)$ is a Linear Transformation.
find the range, rank, null space, Nullity of T.

Sol^n Let, $\alpha = (a_1, b_1)$ & $\beta = (a_2, b_2) \in U$
& $a \in R$

$$\text{Now, } T(a\alpha+\beta) = T\big(a(a_1,b_1) + (a_2,b_2)\big)$$

$$= T\big((aa_1, ab_1) + (a_2, b_2)\big)$$

$$= T\big((aa_1+a_2), (ab_1+b_2)\big)$$

$$= T\big(aa_1+a_2+ab_1+b_2, aa_1+a_2-ab_1-b_2, ab_1+b_2\big)$$

$$= T\big(a(a_1+b_1)+(a_2+b_2), a(a_1-b_1)+(a_2-b_2), ab_1+b_2\big)$$

$$= T\big(a(a_1+b_1), a(a_1-b_1), ab_1\big) + T\big((a_2+b_2), (a_2-b_2), b_2\big)$$

$$= aT(a_1, b_1) + T(a_2, b_2)$$

$$= aT(\alpha) + T(\beta)$$

We know that,
$\{(1,0), (0,1)\}$ is the basis of $V_2(R)$

$\therefore T\{(1,0), (0,1)\}$ span $R(T)$.

$$T(1,0) = (1,1,0)$$

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$T(0,1) = (1,-1,1)$

i.e $\{(1,1,0), (1,-1,1)\}$ spans $R(T)$.

Let, $a, b \in R$

such that —

$a(1,1,0) + b(1,-1,1) = 0$

$\Rightarrow (a, a, 0) + (b, -b, b) = 0$

$\Rightarrow (a+b, a-b, b) = (0,0,0)$

$\Rightarrow a+b = 0; \quad a-b = 0, \quad b = 0$

$\therefore a = 0$ & $b = 0$

Thus, the set $\{(1,1,0), (1,-1,1)\}$ is independent

$\therefore$ Basis of $R(T)$.

$R(T) = L\{(1,1,0), (1,-1,1)\}$

Hence, $Rank(T) = 2$

By Rank Nullity Theorem,

$Rank(T) + Nullity(T) = \dim(V_2(T))$

$\Rightarrow Nullity(T) = 2 - 2$

$= 0$

$\therefore$ Null Space $N(T) = \{(0,0)\}$

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