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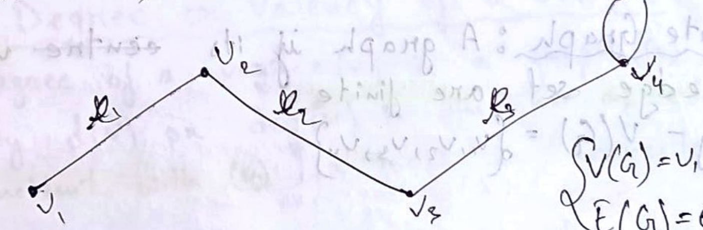
UNIT: II Graph Theory

*** A graph:** $G = (V, E)$ or $G = (V(G), E(G))$ is an ordered pair of a vertex set $V(G)$ & edge set $E(G)$ & rel^n 'G' that associates with each ~~other~~ edge by 2 vertices (not necessarily distinct) called its end points.



$V(G) = v_1, v_2, v_3, v_4$
$E(G) = e_1, e_2, e_3, e_4$

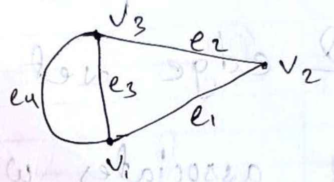
* Note

(i) Vertices are represented by points
(ii) Edges are represented by segments 'e' with end point 'u' & 'v'.
(iii) Edge is also represented as $e = uv$ or $e = vu$

(2)

* **Loop**: An edge whose end points are same.

* **Multiple Edge**: If two edge are associated with the same pair of vertices, they are called as parallel or multiple edge.



* **Simple Graph**: A graph has no loop or no multiple edges.

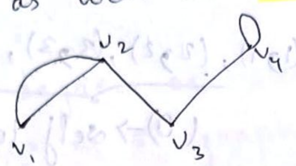
* **Finite Graph**: A graph if its ~~centre~~ vertex & edge set are finite.
  Eg – $V(G) = \{v_1, v_2, v_3, v_4\}$

* **Null graph**: A graph if the edge set,
  $E = \phi$,

* **Order & size of a graph**: The no of vertices of a graph $G=(V,E)$ is called order.
  Denoted by $|V(G)|$
  while, the no. of edges is size / cardinality
  Denoted by $|E(G)|$

* **Trivial Graph**: A graph with only a single vertex.

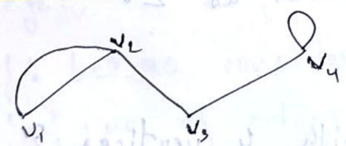
* **Pseudo Graph**: A graph which allows parrallell edges as well as self loop.



* **Multi Graph**: A graph with $\parallel^{r}$ edges but no self loop.

* Degree or Valency of a Vertex

Degree of a vertex (v) in a graph (G) denoted by $d(v)$ or $deg(v)$ is the no. of edges incident with (v_i), a loop is connected twice



$\therefore$ $d(v_1) = 2$, $d(v_2) = 3$, $d(v_3) = 2$

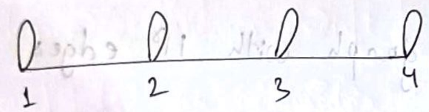
$d(v_4) = 3$ [$\because$ loop has 2 edges]

* Let us consider a graph $G(V,E)$ where V is the vertex set & E is the edge set are as follows.

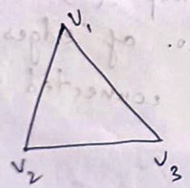
$V = \{1, 2, 3, 4\}$ & $E = \{(x,y) : |x-y| \leq 1, x, y \in V\}$

$\therefore E = \{(2,1), (3,2), (4,3), \underbrace{(1,1), (2,2), (3,3), (4,4)}\}$

$(0) \rightarrow$ self loop



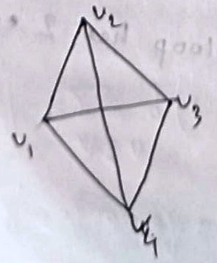
* Regular Graph: A graph whose degree of all vertices of G has same degree or valency.



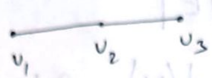
→ graph with 3 vertices

→ having degree of each vertices as 2.

* Draw a graph with 4 vertices & having degree of each as 3.



* **Pendent vertex**: In a graph, a vertex with degree '1',



→ v_1 & v_3 are pendent vertex

[*] The degree or valency of an isolated vertex in a graph is '0'.

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ODD & EVEN Vertex

* **Odd vertex**: A vertex of odd degree.
  $d(v_1) = 3$

* **Even vertex**: A vertex of even degree.
  $d(v_2) = 2$

* Prove that the sum of degrees of all vertices of a graph is even integer.

Sol^n

Proof: Let us consider a graph $G(V,E)$ & 'm' no of edges i.e cardinality of set $|E| = m$ & 'n' be no. of vertices i.e

$|V| = n$

Let $v_1, v_2, \ldots, v_n$ be the vertices of the graph,

∵ each edge contributes two degrees to the graph.

∴ the sum of degrees of all vertices (G) is twice the number of edges in G

i.e $\sum_{i=1}^{n} d(v_i) = 2m = 2|E|$

$$\boxed{\Rightarrow \sum_{i=1}^{n} d(v) = 2|E|}$$

Hence, proved

* Prove that in a graph, the no. of vertices of odd degree is always even.

Sol

Proof : Let us consider a graph $G(V,E)$ where $V = V_1 \cup V_2$ such that

V_1 = set of odd degree vertices in G &

V_2 = " " even " " in G

Now,

$$\sum_{v \in V} d(v) = \sum_{v \in V_1} d(v_1) + \sum_{v \in V_2} d(v_2)$$

$$\Rightarrow 2|E| = \sum_{v \in V_1} d(v_1) + \sum_{v \in V_2} d(v_2) \quad \text{—— (1)}$$

Again,

$$\Rightarrow 2|E| = \sum_{v \in V_1} d(v_1) + 2a$$

∵ sum of even no. is a multiple of

let $|E| = m$

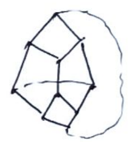
$\Rightarrow 2m = \sum_{v \in V_1} d(v_1) + 2a$

$\Rightarrow \sum_{v \in V_1} d(v_1) = 2(m-a)$

$= 2n$

$\therefore$ the no. of ~~odd~~ vertices of odd degree is even.

* How many vertices are there in a graph G with 15 edges if its each vertex ha degree 3.



Sol^n given, $|E| = 15$

$n = ?$

$deg(v) = 3$ (all same)

We have, $\sum_{i}^{n} d(v) = |E| \times 2$

$\Rightarrow n \times \cancel{3} = 2 \times \cancel{15}\,5$ $\quad n \times d(v) = 2|E|$

$\Rightarrow n = 10$, vertices

2014 (b) or.

Let $G(V,E)$ be the given graph

$\therefore |V| =$ no. of vertices

$|E| = 15$

$\sum d(v) = 3, \forall v \in V$

$\therefore$ Sum of degree of all vertices

$$\sum_{v \in V} d(v) = 2|E|$$

$$= 30$$

$\therefore$ no. of vertices $= \dfrac{\text{Total degree}}{\text{no. of } \cancel{\text{each}} \text{ vertex}}$

$= 30/3$

$= 10$ vertices

*

Soln: Let there be a graph G with $V = \{v_1, v_2, v_3, \ldots, v_n\}$ vertices &

$E = \{e_1, e_2, \ldots, e_n\}$ edges

We know that sum of degree of all vertices in a graph G, i.e twice the no. of edges

$$\Rightarrow \sum_{i=1}^{n} \deg(v_i) = 2|E| \text{ ——— (i)}$$

also,

$$\Rightarrow \sum_{i=1}^{n} \deg(v_i) = \sum_{even} \deg(v_i) + \sum_{odd} d(v_i) \text{ ——— (ii)}$$

from (i) & (ii)

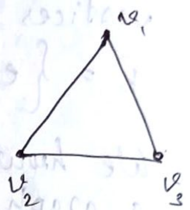
$$\Rightarrow \sum_{odd} \deg(v_i) = 2|E| - \sum_{even} d(v_i)$$

$$= \text{even}$$

Hence, the no. of vertices of odd degrees are even.

* Show that in every simple graph with $n \geq 2$ we must have atleast one pair of vertex whose degree are same.

Let $G(V,E)$ be the sample graph, with $n \geq 2$ vertices say $v_1, v_2, v_3, \ldots, v_n$



$\because$ G is a simple graph i.e no loop & no multiple edges

Maximum degree of a vertex on a simple graph is $n-1$

$\forall v_1 \in V$

$\because$ there are no loop and no multiple edges

If possible,

let $d(v_i) \neq d(v_j)$

$\forall$ $v_i \neq v_j$

& $v_i, v_j \in V$

without loss of generality, let

$d(v_k) = 0$ & $d(v_i) = n-1$

Now, $d(v_k) = 0$

$\Rightarrow v_k$ is isolated vertex

$\therefore$ v_k is not connected to any other vertex.

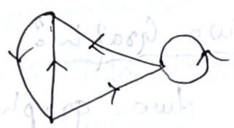
Again, $d(v) = n-1$

$\Rightarrow v_1$ is adjacent to all vertices except itself.

$\therefore$ we got a contradiction, So our assumption was wrong.

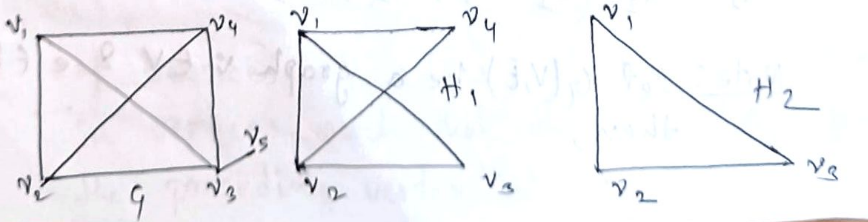
Thus, in every simple graph with $n \geq 2$ vertices must have atleast one pair of vertices whose degree are same.

→ Directed Graph: Graph ~~that~~ is made up of set of vertices connected by edges, where an edge have a direction associated with them.



* Sub Graph: Let $G(V,E)$ be a graph then it is a sub graph of

if $V(H) \subseteq V(G)$ & $E(H) \subseteq$



H_1 & H_2 are sub graph of G

* **Spanning SubGraph**: Let there be a sub graph of G then it is Spanning Sub Graph of G. If $V(H) = V(G)$

* **Union of Graph**: Let $G_1(V_1, E_1)$ & $G_2(V_2, E_2)$ be two graph then union of G_1 & G_2 is denoted by $G_1 \cup G_2$ & is defined as $G_1 \cup G_2 (V_1 \cup V_2, E_1 \cup E_2)$, where edge & vertex of $G_1 \cup G_1$.

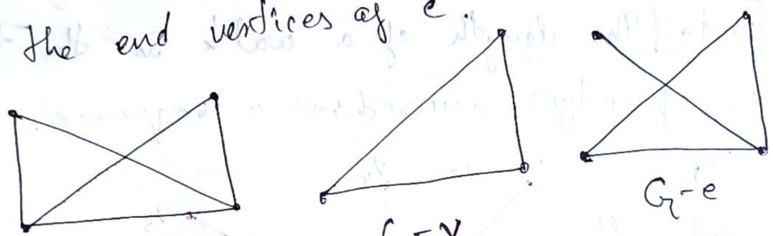
* **Intersection of two Graph**: Let $G_1(V_1, E_1)$ & $G_2(V_2, E_2)$ be two graphs. The intersection of G_1 & G_2 is denoted by $G_1 \cap G_2$ and denoted by

$G_1 \cap G_2 (V_1 \cap V_2, E_1 \cap E_2)$

Note: let $G(V,E)$ be a graph $v \in V$ & $e \in E$ then,

(a) "$G-v$" is the subgraph obtained by deleting the vertex v from G and also the edges incident with v.

(b) '$G-e$' is subgraph obtained by deleting the edge e from G but not the end vertices of e.



G G−v G−e

→ **Walk**: alternating sequence of edge & vertex (ending/starting with vertex).

Let G be a Graph then a walk of length, l in G in a finite alternating sequence then, $u e_1 u_1 e_2 u_2 e_3 \dots, u_{n-1} e_n v$

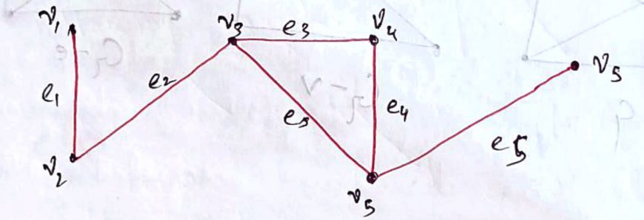
vertex edge,

of vertices of edges, starting & landing in vertices such that the edge 'e_j' joins the preceding vertex u_{j-1} & succeeding vertex : v_i, A walk is denoted by W. In any walk, a vertex or an edge may occur more than ~~1 time~~ once.

* **Trial**: A Walk in which no edge is repeated.

• A trial of length zero is called trivial trail.

$\boxed{\text{Note}}$ The length of a walk is the total no. of edges occured in a sequence.



eg. $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_3$

It is a walk as well as a trail

eg. $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_3 e_2 v_2$

$\because$ edge is repeated

$\therefore$ Not a trial but a Walk

* **Closed & Open Walk**: Let G be a graph of $u e_1 u_1 e_2 u_2 e_3 u_3 \ldots u_{n-1} e_n v$ or u-v walk is open if

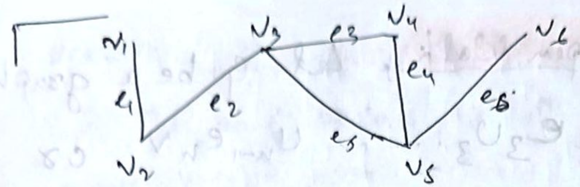
$u \neq v$

Slly, $u = v$ is a closed walk.

* **Path**: A walk having no repeated vertices and edges except the starting & ending vertices will be a path.

* **Cycle**: A non trivial closed path is called a cycle. It is called even cycle if its length is even, slly odd cycle if its length is odd.

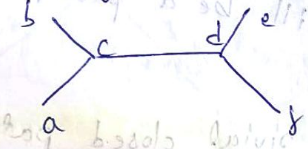
* **Eccentricity**: Let G be a graph & v be a vertex of G than the eccentricity of v is denoted by $E(v)$, it is the further distance from v to any other vertex in G



$\epsilon(v_1) = 4 \qquad \epsilon(v_4) = 3$

* shortest distance betn the point v_i & the further most point (counting the edge.

* Q/ Find the eccentricity of the vertex, a of the following graph.



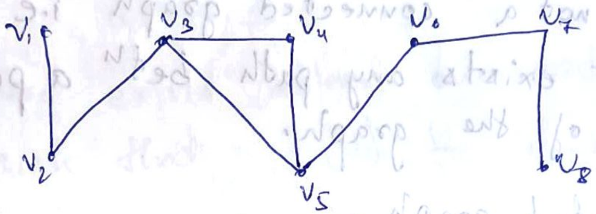
$\therefore \epsilon(a) = 3$ units

* V → Radius & diameter of a shape :

Let $G(V,E)$ be a graph then the radius of the graph G is denoted by $r(G)$ & defined by $r(G) = \min(\epsilon(v))$, $v \in V$

Again, diameter of the graph G is denoted by $d(G)$ and defined by $d(G) = \max(\epsilon(v))$ & $v \in V$,

* Find the radius & diameter.



Sol:

$d(G) = \epsilon(v_1) = 6$ (max) $\quad \epsilon(v_5) = 3$ (min)

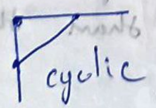
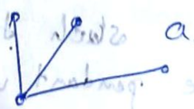
$\epsilon(v_2) = 5 \quad \epsilon(v_6) = 4$

$\epsilon(v_3) = 4 \quad \epsilon(v_7) = 5$

$\epsilon(v_4) = 4 \quad \epsilon(v_8) = 6$

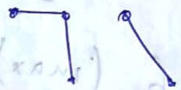
$r(G) = 3$ & $d(G) = 6$

* **Acyclic graph**: A graph having no cycle is called acyclic graph.

eg:  cyclic  acyclic

Note: A graph with one cycle is called ==unicyclic== graph.

***~~Dis~~ Connected Graph:** A graph to be disconnected if it is not a connected graph i.e there does not exists any path betn a pair of vertices of the graph.

Disconnected graph 

***Connected Graph:** A graph G is said to be connected if there is atleast one path between every pair of vertices in G.

Note A null ~~gha~~ graph having exactly one vertex is connected.

• "a connected graph"

*Q) Let G be a connected graph of order $n \geq 2$ ~~size~~ size is m such that $m < n$, then there exists atleast one pendant vertex in G.

Proof: Since G is connected, there doesn't exist any isolated vertex in G.

Then, $d(v) \geq 1$, $\forall v \in V(G)$

We claim that,

$d(v) \geq 1$

There exists atleast one vertex $v \in V(G)$ such that $d(v) = 1$

Let, any $v \in V(G)$ such that $d(v) = 1$

$\therefore \sum_{i=1}^{n} d(v_i) \geq 2n$, where n is the no. of vertices no. of vertices.

$\Rightarrow 2|E| \geq 2n$

$\Rightarrow |E| \geq n$

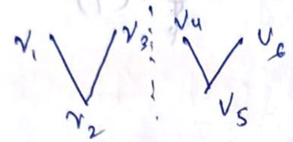
$\Rightarrow m \geq n$

But, Theorem states $m \leq n$, which is a contradiction. Hence, assumption was wrong.

$\therefore$ There exists atleast 1 pendant vertex in G

* Components of a Graph :

Every disconnected Graph consists of 2 or more connected Subgraph.



These connected **Sub-graph** are the components ~~of~~ of a Graph.

The no. of components of a Graph is generally denoted by k.

$\boxed{\text{Note}}$: Connected graph has only one element

* Theorem : A graph having exactly 2 ~~&~~ vertices of odd degree must have a path joining these two vertices.

<u>proof</u>

Let $G(V, E)$ be a graph & ~~let v_1, v_2~~ let, $v_1, v_2 \in V(G)$, such that $d(v_1)$ & $d(v_2)$ are odd remaining vertices are of degree even.

Case 1: Let G be a connected graph then there exists a path joining v_1 & v_2

Case 2: Let G be a disconnected graph then G has atleast 2 or more components

Let G' be a component of G containing the odd degree vertex v_1, Then G' itself is a connected subgraph of G.

$\therefore$ the no. of odd degree vertices is even.

$\therefore$ let, $v_1 \in |G'|$ of the graph G

$\therefore$ v_2 also must ~~be~~ lie in the component G' of the graph G.

Hence, v_1 & v_2 are 2 vertices in the same component G'.

Again, we know the every component is connected.

There exists a path joining v_1 & v_2 in G &, Hence in G.

∴ A graph having exactly two vertices of odd degree must have a path joining these two vertices.

* **Theorem:** A simple graph with 'n' vertices & 'k' no. of components can have at most $(n-k)(n-k+1)/2$ no. of edges.

Proof: Let G be a simple graph with 'n' vertices, 'k' components then 'G' has no self loop & no parallel edges.

Suppose: $G_1, G_2, \ldots, G_k$ are the components & $n_1, n_2, \ldots, n_k$ are the vertices in the components respectively.

$\therefore n_1 + n_2 + \dots + n_k$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - k$$

$$= n - k$$

$$\therefore \left\{ \sum_{i=1}^{k} (n_i - 1) \right\}^2 = (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + 2 \sum_{i,j=1}^{k} (n_i - 1)(n_j - 1) = (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + \text{sum of some non-ve terms} = (n-k)^2 \quad \left[\because n_i - 1 \geq 0 \ \forall i = 1, 2, \dots, k \right]$$

$$\Rightarrow \sum_{i=1}^{k} \left[n_i^2 + 1 - 2n_i \right] \leq (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - \sum_{i=1}^{k} 2n_i \leq (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2n \leq (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk - k + 2n \quad (\mathrm{i})$$

$\because$ G is a simple graph so the i^{th} component is also a simple graph.

$\because$ G_i contains n_i vertices & G_i is simple,

$\therefore$ The max no. of edges in the i^{th} component is

$$\frac{n_i(n_i-1)}{2}$$

$\therefore$ The max no. of edges in all K components

$$\text{is} - \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{k}\left[n_i^2 - n_i\right] \leq \frac{1}{2}\left\{n^2+k^2-2nk-k+n\right\}$$

using (i)

$$= \frac{1}{2}\left\{n(n-k) - k(n-k) + (n-k)\right\}$$

$$= \frac{1}{2}(n-k)\left\{n-k+1\right\}$$

1 * Show that the maximum no. of edges in a simple graph with n vertices is $n(n-1)/2$.

Sol^n Let maximum no. of edges incident with the vertex v_i is $n-1$

i.e $d(v_i) \leq n-1$

We know $\sum_{i=1}^{n} d(v_i) = 2|E|$

where, $|E|$ = no. of edges

$\therefore \sum_{i=1}^{n} d(v_i) \leq n(n-1) \Rightarrow 2|E| \leq n(n-1)$

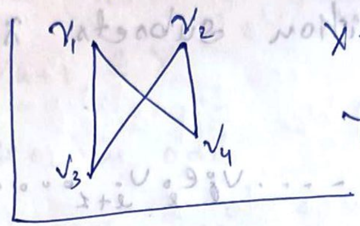
$\Rightarrow |E| \leq \frac{n(n-1)}{2}$

Hence, the maximum no. of edges in a simple graph with 'n' vertices is $\frac{n}{2}(n-1)$

* **Bipartite Graph (Bigraph):**

Let $G(V,E)$ be a simple graph & let vertex at $V(G)$ be partitioned into two disjoined independent non-empty subsets X & Y, in such a way that each edge of G has 1 end vertex in 'X' and the other in 'Y'.

Then, subsets X & Y are called Bipartition of G.



$X = \{v_1, v_2\}$

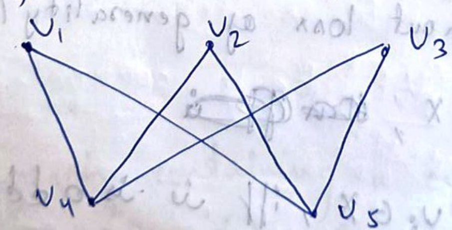
$Y = \{v_3, v_4\}$

* Complete Bipartite Graph:

A Bipartite graph G having bipartition subsets X & Y, is said to be a complete Bipartite graph. If every vertex in X, is joined to every vertex in Y.

If $|X| = m$ & $|Y| = n$

Then, complete Bipartite Graph is denoted by $K_{(m,n)}$



$X = \{v_1, v_2, v_3\}$ & $Y = \{v_4, v_5\}$

* A Bipartite doesn't contain odd cycle

Let $G(X, Y, E)$ be a Bipatite graph having Bipartition subsets X & Y and let,

$\{v_1 e_1 v_2 e_2 v_3 \ldots v_i e_i v_{i+1} \ldots v_k e_k v_{k+1}\}$

$v_1 = v_{k+1}$ be any ~~single~~ cycle of length k in the Bipartite Graph

Here, the vertices v_i & v_{k+1} are connected be e_1 &

$i = 1, 2, \ldots, k$

$\therefore$ G is Bipartite Graph

$\therefore$ $v_i \in X$ & $v_{i+k} \in Y$

$\therefore$ without loss of generality let,

$v_1 \in X$,

then, $v_i \in X$, iff i is odd ~~circle~~.

& $v_i \in Y$, iff i is even

also, $v_1 = v_{k+1}$

$\rightarrow v_{k+1} \in X$

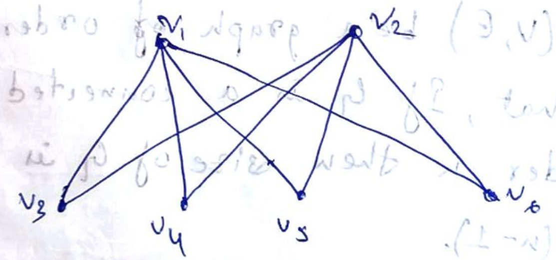
$\because k+1$ is odd

$\therefore k$ is even

$\therefore C_k$ is a even circle.

$\because C_k$ is any arbitary circle in G.

* Draw a complete Bipartite Graph on 2 & 4 vertices.



* Can a graph containing a cycle of lenght 3 be a Bipartite Graph. Justify.

Let if possible the graph 'G' is Bipartite

let the vertex set G is given by

$V(G) = \{v_1, v_2, v_3\}$

Let, X & Y be Bipartition subsets

Let, v_1 belongs to X

$\Rightarrow v_2, v_3 \in Y$ (as they ~~are~~ connected to v_1)

doen't form a cycle.

∴ Our assumption was wrong.

* Let $G(V,E)$ be a graph of order 'n'

Prove that, If G is a connected graph of order n then size of G is atleast $(n-1)$.

Sol:

Let $G(V,E)$ be a connected graph of order 'n'

If $n=1$, there is no self-loop

∴ no. of edges ≥ 0

Let, $V(G) = \{v_1, v_2, v_3, v_4, \ldots, v_n\}$

& $v_k \in V(G)$ be any arbitary vertex

Then there exist a path of non-zero

length from v_k to all v_i, where

$i = \{1, 2, 3, \ldots, n\} - k$

Hence, it follows that there is atleast

1 edge connecting v_k to all v_i except

v_k. i.e there is atleast one edge

connecting v_k to each of the ~~following~~

remaining $(n-1)$ vertex.

Thus, G has at least $(n-1)$ edges.

So the size of every connected graph

is atleast $(n-1)$ of order n.

* Let G be a disconnected graph of order n & size m. If G has exactly k components then show that $m \geq n-k$.

proof:

Let $G(V,E)$ be the given disconnected graph.

Let, $G_1, G_2, G_3, \ldots, G_k$ be its components with order $n_1, n_2, n_3, \ldots, n_k$ & size $m_1, m_2, m_3, \ldots, m_k$ respectively.

Then ~~if~~ G_i, $i = 1, 2, \ldots, k$ is connected sub graph of G.

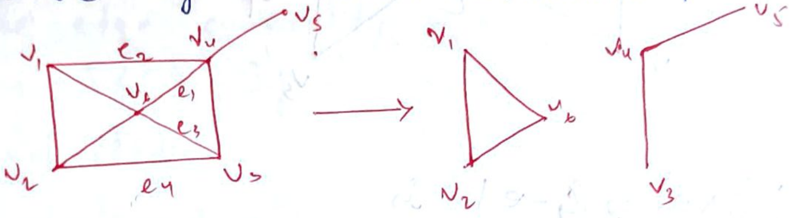
$\because$ We know that size of every connected graph of order n is atleast ____

i.e $m_i \geq n_i - 1$

also, $\sum_{i=1}^{k} m_i \geq \sum_{i=1}^{k} (n_i - 1)$

$\Rightarrow m \geq n - k$

* **Cut Shape** : Let G be a connected graph, a cut shape is a subset (C) of the set of all edges of G, whose removal from the graph G leaves the graph disconnected & removal of any proper subset of C doesn't disconnect.



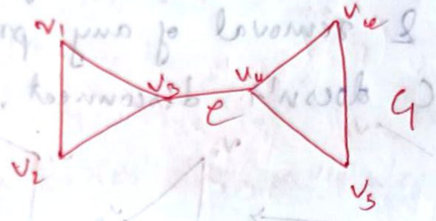
Here the Cut Shape is,

$C = \{e_1, e_2, e_3, e_4\}$

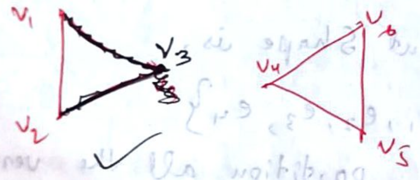
i.e if we partition all the vertices of a connected graph G, into two mutually inclusive subjects $\Rightarrow V_1$ & V_2 (which is also called as Cut of a graph. Then the Cut shape is a shape consisting of minimal no. of edge whose removal from G destroys all the path betn the ~~V_1 & V_2~~ two sets V_1 & V_2

* Cut edge / Bridge / Isthemus :

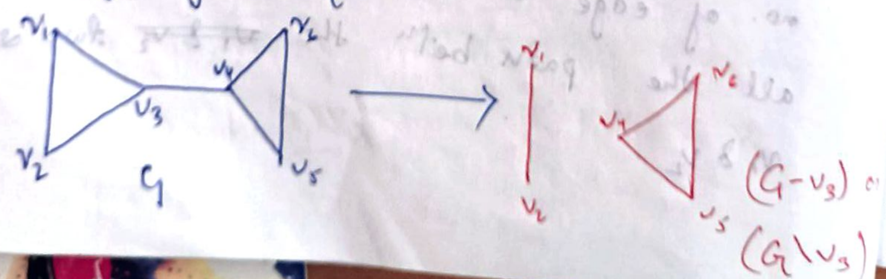
An edge e of a graph G is said to be a cut edge if the sub graph G creates more component than original in G



$\therefore (G \setminus e)$ or $(G - e)$ is



*** Cut Vertex:** A cut vertex is a vertex that when removed (with its boundary edges) from a graph creates more component than originaly in G.



*** Edge connectivity :—** A connect $G(V,E)$ is said to be k-edge connected. If it remains connected after removing fewer than k-edges from $E(G)$, or the no. of edges in the minimal cut shape (i.e cut shape with the fewest no. of edges) is called edge connectivity of v.

The edge connectivity of a graph G is denoted by $\lambda(G)$. If the graph G is k-edge connected.

*** Vertex Connectivity :—** The vertex connectivity of a connected graph G, is defined as the minimum no. of vertices whose removal from graph G gives a disconnected graph.

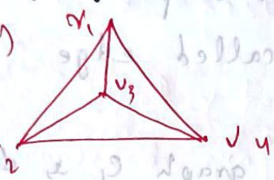
Let $G(V,E)$ be a connected a having more than k-vertices than G is said to be k-vertex connected, if it remains connected after removing fewer than k-vertices from $V(G)$. The vertex ~~contec~~ connectivity of a graph is denoted by $k(G)$.

$\lambda(G) = 2$ (edge)

$k(G) = 1$ (vertex)

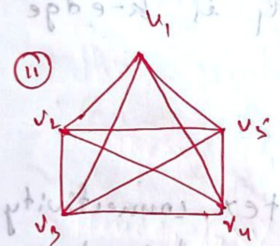
* Edge connectivity :

(i)



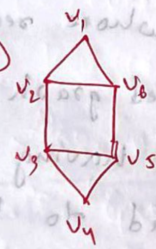
$\lambda(G) = 3$

(ii)



$\lambda(G) = 4$

(iii)



$\lambda(G) = 2$

~~Note~~ Notes

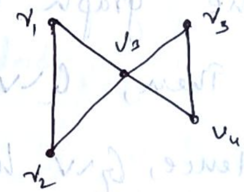
→ For any disconnected graph G:

$$\lambda(G) = k(G) = 0$$

[Coz they are not connected by any vertex or edge], Hence connectivity zero.

→ Vertex connectivity of a tree is 1.

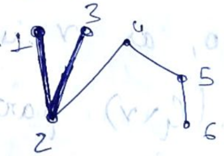
* Separable Graph :- A connected Graph G is said to be separable if its vertex connectivity is 1 (one).



Vertex connectivity = 1

i.e (v_5)

* Find the cut vertices and vertex connectivity of the following graph G.



Sol^n

Cut vertices → 2, 4, 5

vertex connectivity → 1

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*** Theorem**: A vertex v in a connected graph G is a cut shape vertex iff there exists two vertices 'x' & 'y' in G, such that every path betⁿ x & y pass through v.

Proof: Let v be any cut vertex of a connected graph G then clearly the removal of v from the graph G, disconnects the graph G.

Then, $G \setminus v$ is disconnected

Hence, $G \setminus v$ has atleast two components (∵, every disconnected G contains at least two components)

Also, as v is a cut vertex.

∴ $(G \setminus v)$ partitions the vertex into exactly two components, v_1 & v_2 (say)

Then, let us consider any two vertices x & y of $(G \setminus v)$ such that $x \in v_1$ & $y \in v_2$.

$\therefore$ Every x-y path in G contains v

Conversly, Suppose that every path bet^n x & y passes through ~~small~~ v.

Now, we are to show ~~that~~ he is a cut vertex in G.

$\because$ every path bet^n two vertices x & y in G passes through v, so there cannot be a path joining these two vertices x & y in G

Thus, $(G \setminus v)$ is disconnected.

$\therefore$ v is a cut vertex of G.

Hence, the theorem.

* The edge connectivity of a graph G can't exceed the degree of the vertex with the ~~degree of the~~ smallest degree.

Sol^n Let, the vertex v be the vertex with smallest degree in G.

Let, $\delta(v)$ be the degree of v

$\therefore$ vertex v can be separated from G be removing $\delta(v)$ edges incident on v.

$\therefore$ edge ~~contol~~ ~~connectiv~~ connectivity can't exceed the degree of the vertex with smallest degree in G.

Hence, $\lambda(G) \leq \delta(G)$

1/09/22

<u>Theorem</u>: The vertex connectivity of any graph G can never exceed the edge connectivity of G.

<u>proof</u>

Let, $\lambda(G)$ denote the edge connectivity of G

$\therefore$ there exists a cut shape C in G with $\lambda(G)$ edges.

Let cut shape C partitions the vertices of G into two subsets v_1 & v_2.

$\therefore$ By removing at most $\lambda(G)$ vertices from v_1 (or v_2) on which the edges on C. are incident, we can affect the removal of C [together with all other edges incident on these vertices] from G.

∴ If $k(G)$ is the vertex connectivity of the graph G, then —

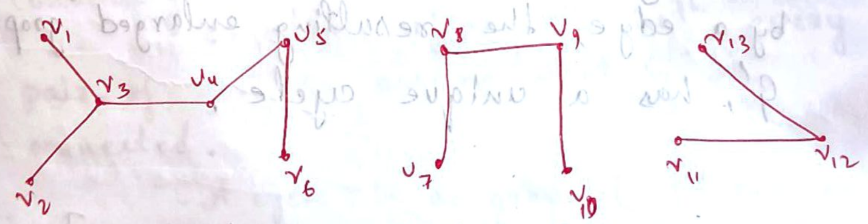
$$k(G) \leq \lambda(G)$$

$\boxed{\text{Note}}$ $k(G) \leq \lambda(G) \leq \delta(G)$

* <u>Tree</u> : An acyclic graph (also known as a forest) is a graph with no cycles.

It is a connected graph.

Thus, each component of a ~~tree~~ forest is a tree and any tree is a connected forest.



A forest containing three trees.

$\boxed{\text{Notes:}}$ The following are equivalent in a graph G with 'n' vertices.

- G is a tree
- There exists a unique path betn every where of vertices in G
- G is connected and every edge in G is bridge
- G is connected & it has $(n-1)$ edges.
- G is acyclic & it has $(n-1)$ edges
- G is acyclic & whenever any two non-adjacent vertices in G are joined by a edge, the resulting enlarged graph G', has a unique cycle.

* **Theorem:** Show that a graph is a Tree iff there exists a unique path betn every pair of vertices in the graph.

3/9/22

Soln Let, G is a tree

$\therefore$ G is connected

$\therefore$ There exists atleast one path betn every pair of vertices on G.

Let, v & w be any two vertices in G.

If possible, let there are two distinct paths betn v & w.

Then, union of these two paths will contain a cycle, which is a contradiction.

[$\because$ G is a tree]

$\therefore$ Our assumption was wrong.

Hence, there is one & only one path betn every pair of vertices in G.

Conversely,

existence of a path betn every pair of vertices assure that G is connected.

A cycle in a graph (with two or more vertices)

$\rightarrow$ There exists atleast one pair of vertices u & v such that there are two distinct path u & v between.

$\therefore$, G has one and only one path between every pair of vertices.

∴ G cannot have a cycle.

→ G is connected acyclic graph

→ G is a tree.

Hence, theorem.

* A tree with n vertices has $n-1$ edges

Sol: This theorem can be proved by the principle of induction.

Clearly, the theorem is true for $n = 1, 2, 3$.

Let, us assume the theorem is true $\forall\, n ; n < k$

Let us ~~assume~~ now consider a tree 'T' with k vertices.

In T, e_k be an edge 'n' vertices v_i & v_j.

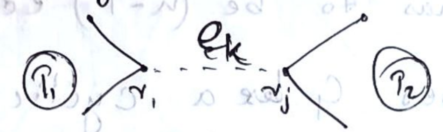
∵ T is a tree

∴ There is no other path betn v_i & v_j except e_k.

T ∴ deletion of e_k from T will disconnects the graph.

Now, $(T \setminus e_k)$ consists of exactly two components & ∵ there is no cycle

∴ each of these component is tree



Both of these T_1 & T_2 have fewer than k vertices each

∴ by the induction hypothesis,

each tree either T_1 or T_2 has one less edge then the no. of vertices in it.

∴ $(T \setminus e_k)$ consists of $k-2$ edges & k vertices

Hence, T has exactly $(n-1)$ vertices

Thus, the theorem.

<u>H/W</u>

* Any connected graph with 'n' vertices & $(n-1)$ edges is a tree.

Sol^n Let us assume a connected graph G with n vertices & $n-1$ edges

We know, the minimum no. of edges in a simple graph with 'n' vertices has to be $(n-1)$ edges.

Let, us consider G be a cyclic graph or atleast has cycle in it.

But for a cyclic graph edges have to be equal to or greater than the no. of vertices.

i.e no. of edges $\geq n$

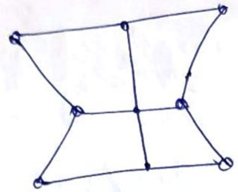
Eg: $n, n+1, n+2, \ldots$

which contradicts our consideration.

$\therefore$ the given graph is connected & not a cyclic graph,

Thus, acyclic graph & G is a tree.

Spanning Tree :- A tree 'T' is said to be a spanning tree of a connected grap G, if T is a sub-graph of G & T contains all vertices of G.



G

T