Machine Learning HW 3

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1.

ED[Ein(Wlin)] = sig^2*(1-(d+1)/N) > 0.008
0.01*(1-9/N) > 0.008
1-9/N > 0.8
9/N < 0.2
N > 9 * 5 = 45 => N = 46

2.
a, d, e
a,e:

let v be any vector in R^n
Ht = (Xt)t *(XtX)^(-t)* (X)t = X * (XtX)^(-1) Xt = H
H * H = X * (XtX)^(-1) Xt * X * (XtX)^(-1) Xt = X * (XtX)^-1 * Xt = H => H^n = H for n >= 1
<Hv,Hv> = vt*Ht*H*v = vt*H*H*v = vt*H*v >=0 since <Hv,Hv> = ||Hv||^2 >=0
=> vt*H*v >=0 for all possible v => H is positive semi-definite

d:
H = X*(XtX)^-1*Xt
HX = X
let v be eigenvector of X such that Xv = cv for some constant c
(HX)v = H(Xv) = H(cv) = Xv = cv
let q = cv
HXv = H(Xv) = H(cv) = Hq = q => q is an eigenvector of H with corresponding eigenvalues = 1
=> For all eigenvectors v of X, we can construct an eigenvector q of H with eigenvalues =1 by mutiplying the v with its corresponding eigenvalue
=> There are d+1 eigenvalues = H are 1 (X is of R^N x (d+1))

3.
a,b,e

[[sign(wtx)!=y]] =1 >0 when wrong
=0 when right
-y*wt*x > 0 when wrong
-ywtx < 0 when right
a):
err(w) = max(0,1-ywtx), when right , count =0 , the 0 in max gaurantees's upperbound
, when wrong, count =1 , 1-ywtx = -ywtx +1 > 1 is also upperbound
b):

err(w) = (max(0,1-ywtx))^2, when right, count =0 , the 0 in max still gaurantees's upperbound
, when wrong, count =1 , 1-ywtx = -ywtx +1 > 1 => (1-ywtx)^2 > 1 still upperbound
c):
err(w) = max(0,-ywtx) when wrong, count =1 , -ywtx >0 but might be less than 1
d):
err(w) = theta(-ywtx) = 1 / (1 + e^(-ywtx)) , when wrong , count =1; 1/1+e^(-ywtx) wher e^(-ywtx) >1 => err(w) < 1/2 not upperbound
e):
err(w) = exp(-ywtx), when right , count =0; exp(negative) >= 0 is upperbound
when wrong, count =1 , exp(something >=0) >= 1 is upperbound

4.

b,d,e:
ywtx = yxtw => d(ywtx)dw = yx
ywtx continuous over w and differentiable
for err(w) of a , b , c that consists of max of two functions, we need to consider the point where the two components match
that is 1-ywtx =0
The combination of continuous functions are continuous
a). from 0's point of view , derivative = 0
from 1-ywtx's point of view, derivative = -yx need not be 0
b). from 0^2's point of view , derivative = 0
from (1-ywtx)^2's point of view, by chain rule, the derivative is
2*(1-ywtx)*(-yx) where yx is constant and 1-ywtx =0 > derivative = 0
=> differentiable
c). from -ywtx's point of view, derivative = -yx need not be 0
d). v = -ywtx
theta(s) = 1/(1+exp(-s))
err(w) = theta(-ywtx)
d err(w) / dw = d theta(v) / dv * (dv/dw)
= (v - ln(1+exp(-v))) * (yx)
e).
d exp(v) /dw = d(exp(v)) / dv * (dv/dw) = exp(v) * dv/dw =(-ywtx) * (-yx)

5.

let err(w) = max(0,-ywtx)
wt+1 = wt - d err(w) / dw
where d err(w) /dw = 0 if -ywtx < 0 , that is , when the prediciton is right
= -yx if -ywtx > 0 , that is , when prediciton is wrong and update is needed
So for SGD requires
wt+1 = wt - (-yx) = wt+yx when wrong
no update when right

This result is the same as the update in PLA .

6.7.

```
E = @(u,v) exp(u)+exp(2*v)+exp(u*v) + u^2 - 2*u*v+ 2*v^2 - 3*u-2*v
dEdu =@(u,v) exp(u)+v*exp(u*v)+2*u-2*v-3;
dEdv= @(u,v) 2*exp(2*v)+u*exp(u*v)-2*u+4*v-2;
eta = 0.01;
u = 0;
v = 0;
for i =1 : 5
fprintf('u= %d, v =%d du = %d dv = %d \n',u,v,dEdu(u,v),dEdv(u,v));
new_u = u - eta*dEdu(u,v);
new_v = v - eta*dEdv(u,v);
u = new_u;
v = new_v;
end
fprintf('u= %d, v =%d du = %d dv = %d \n',u,v,dEdu(u,v),dEdv(u,v));
```

```
u= 0, v =0 du = -2 dv = 0
u= 2.000000e-02, v =0 du = -1.939799e+00 dv = -2.000000e-02
u= 3.939799e-02, v =2.000000e-04 du = -1.881220e+00 dv = -3.779752e-02
u= 5.821018e-02, v =5.779752e-04 du = -1.824220e+00 dv = -5.358309e-02
u= 7.645238e-02, v =1.113806e-03 du = -1.768758e+00 dv = -6.753046e-02
u= 9.413996e-02, v =1.789111e-03 du = -1.714795e+00 dv = -7.979840e-02

ans =

   2.8250
```

gradient at (0,0) is (-2,0)
E(u,v) after 5 updates is 2.8250

8.

E = @(u,v) exp(u)+exp(2*v)+exp(u*v) + u^2 - 2*u*v+ 2*v^2 - 3*u-2*v

f(x) = f(x0) + f'(x0)*(x-x0) + f''(x0) *(x-x0)^2 / 2!
let x = [u+du,v+dv]
    x0 = [u,v]

E(x) = E(x0) + (x-x0)t * gradient(E)(x0) + (x-x0)t Hessian(E)(x0) (x-x0) /2!
dEdu =@(u,v) exp(u)+v*exp(u*v)+2*u-2*v-3;
dEdv= @(u,v) 2*exp(2*v)+u*exp(u*v)-2*u+4*v-2;
d2E/du^2 = @(u,v) exp(u)+v^2*exp(u*v)+2
d2E/dudv = @(u,v) exp(u*v)+v*u*exp(u*v)-2
d2E/dv^2 = @(u,v) 4*exp(2*v)+u^2*exp(u*v)+4

E(u+du,v+dv) = E(u,v) + [du,dv] * [dEdu(u,v) dEdv(u,v)]' +
[du,dv]*H(du,dv)*[du,dv]'/2
E(0,0) = 1+1+1+0-0+0-0-0 =3
dE/du(0,0) = 1+ 0+2-2-3 = -2
dE/dv(0,0) = 2+0-0+0-2 =0
d2E/du^2 = 1+0+2 =3
d2E/dudv = 1+0-2 = -1
d2E/dv^2 = 4+0+4 =8
[du dv] * [dE/du(u,v) dE,dv(u,v)] = -2*du
H(du,dv) = [3 -1 ,-1 8]
E(u+du,v+dv) = 3 - 2*du + (3*du^2-2*du*dv+8*dv^2) /2
                                                = 1.5*du^2+4*dv^2-1*du*dv-2*du+0*dv+3
=> ANS = (1.5,4,-1,-2,0,3)

9.

E(x+s) = E(x)-(s)'gradient(E)(x) + s'*Hessian(E)(x)*s /2
dE(x+s)/ds = gradient(E)(x) + Hessian(E)(x)*s = 0
        => s = -inv(Hessian(E)(x))* gradient(E)(x)
                = -inv(gradient (gradient (E))(x))*gradient(E)(x)

10.

```
E = @(u,v) exp(u)+exp(2*v)+exp(u*v) + u^2 - 2*u*v+ 2*v^2 - 3*u-2*v;
gE = @(u,v)[exp(u)+v*exp(u*v)+2*u-2*v-3;  2*exp(2*v)+u*exp(u*v)-2*u+4*v-2;
];
HE = @(u,v)[ exp(u)+v^2*exp(u*v)+2 exp(u*v)+v*u*exp(u*v)-
2;exp(u*v)+v*u*exp(u*v)-2 4*exp(2*v)+u^2*exp(u*v)+4];

x = [0 ; 0];
for i =1 : 5
        dx =  - inv(HE(x(1),x(2)))*gE(x(1),x(2));
        fprintf('u =%d , v= %d ,du = %d v = %d\n',x(1),x(2),dx(1),dx(2));
        x = x+dx;
end
E(x(1),x(2))
```

```
u =0 , v= 0 ,du = 6.956522e-01 v = 8.695652e-02
u =6.956522e-01 , v= 8.695652e-02 ,du = -8.188995e-02 v = -1.584862e-02
u =6.137622e-01 , v= 7.110790e-02 ,du = -1.949361e-03 v = -6.078377e-04
u =6.118129e-01 , v= 7.050006e-02 ,du = -1.142618e-06 v = -5.142346e-07
u =6.118117e-01 , v= 7.049955e-02 ,du = -4.467273e-13 v = -2.802453e-13

ans = 2.3608
```

11.

```
X = [ 1 1; 1 -1; -1 -1 ; -1 1; 0 0 ; 1 0]
X = X';
```

```
A = []
gen_row = @(x) [ 1 x(1) x(2) x(1)*x(2) x(1)*x(1) x(2)*x(2)]
for x = X
        A = [A ;gen_row(x)];
end

A =[

    1   1   1   1   1   1
    1   1  -1  -1   1   1
    1  -1  -1   1   1   1
    1  -1   1  -1   1   1
    1   0   0   0   0   0
    1   1   0   0   1   0
  ]

inv(A) = [


        0         0         0         0    1.0000         0
   0.2500    0.2500   -0.2500   -0.2500         0         0
   0.2500   -0.2500   -0.2500    0.2500         0         0
   0.2500   -0.2500    0.2500   -0.2500         0         0
  -0.2500   -0.2500    0.2500    0.2500   -1.0000    1.0000
   0.5000    0.5000         0         0         0   -1.0000

   ]
```

Since inv(A) exists,
for any labeling y , we can solve the equation A*w = y for w by
w = inv(A)* y => Thus the all six points can be shattered

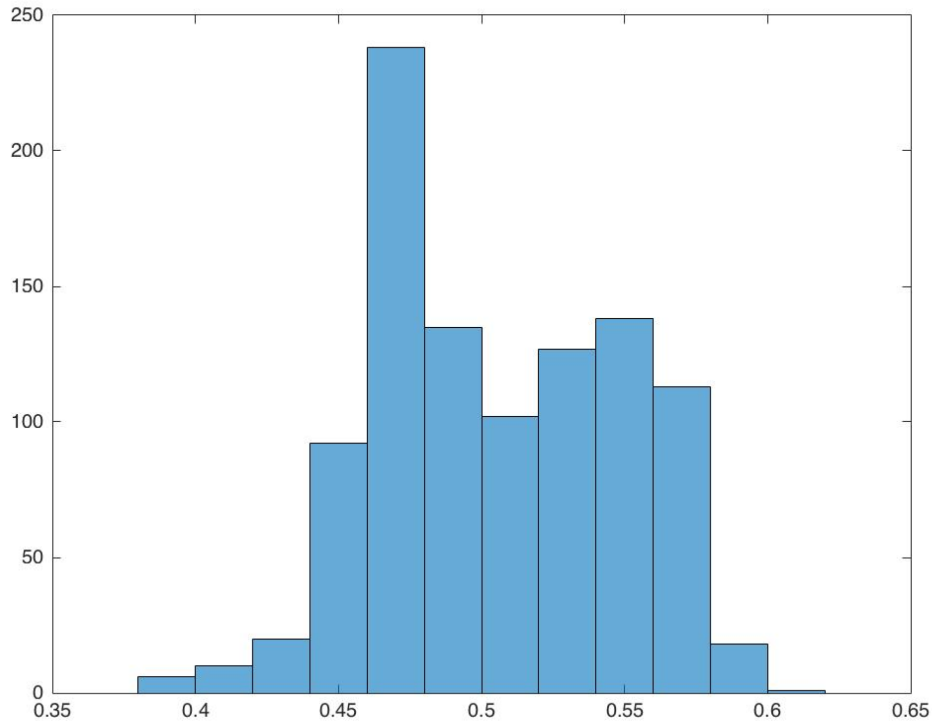
12.

Such transformation will result in N by N matrix Z where zij = 1 iff i == j => an Identity Matrix
Thus , for solving an equation Z w = y
It is equivlant to solving I w =y , where the trivial solution is take w = y
That is , for every given y, the hypothesis can shatter that labeling using weight w = y
=> The VC dimension of this hypothesis is infinity

13.

```
f = @(x)  sign(x(1)^2+x(2)^2-0.6)
error_rate_sum = 0;
error_history = [];
for t = 1:1000
        X = rand(1000,1) * 2 -1;
```

```
        Y = rand(1000,1) *2  -1;
        D = [X Y];
        Z = [];
        for i = 1:1000
                z = f(D(i,:));
                if(rand()<=0.1)
                        z = -z;
                end
                Z = [Z ;z];
        end
        D = [ones(1000,1) D];
        W = inv(D'*D)*D'*Z;
        OUT = D*W;
        err = 0;
        for i = 1:1000
                if(sign(Z(i))~=sign(OUT(i)))
                        err= err+1;
                end
        end
        error_rate_sum = error_rate_sum+err/1000;
        error_history = [error_history; err/1000];
end
histogram(error_history)
error_rate_sum/1000
```

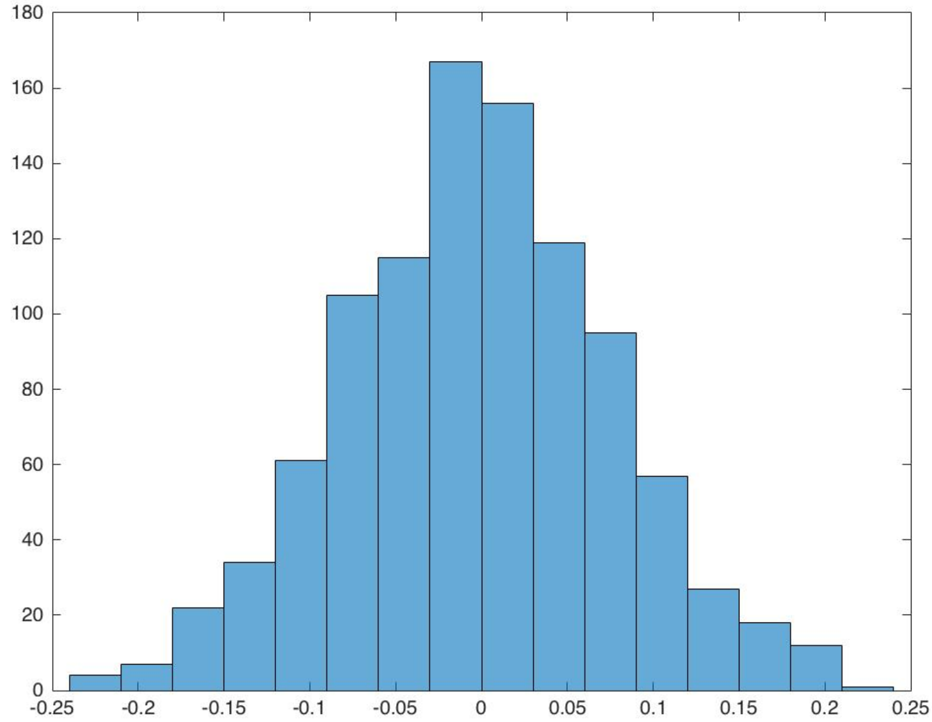


average in-sample error = 0.5047

14.

```
f = @(x)  sign(x(1)^2+x(2)^2-0.6)
g = @(x) [1 x(1) x(2) x(1)*x(2) x(1)*x(1) x(2)*x(2)]
error_rate_sum = 0;
Ws = [];
for t = 1:1000
    X = rand(1000,1) * 2 -1;
    Y = rand(1000,1) *2  -1;
    D = [X Y];
    Z = [];
    G = [];
    for i = 1:1000
        z = f(D(i,:));
        if(rand()<=0.1)
            z = -z;
        end
        Z = [Z ;z];
        G = [G; g(D(i,:))];
    end

    W = inv(G'*G)*G'*Z;
    OUT = G*W;
    err = 0;
    for i = 1:1000
        if(sign(Z(i))~=sign(OUT(i)))
            err= err+1;
        end
    end
    Ws = [Ws  W];
    error_rate_sum = error_rate_sum+err/1000;
end
error_rate_sum/1000
weight = [];
for i = 1:6
    weight = [weight ; sum(Ws(i,:))/1000 ];
end
histogram(Ws(4,:));
weight(4)
```



average w3 = -0.0025

15.

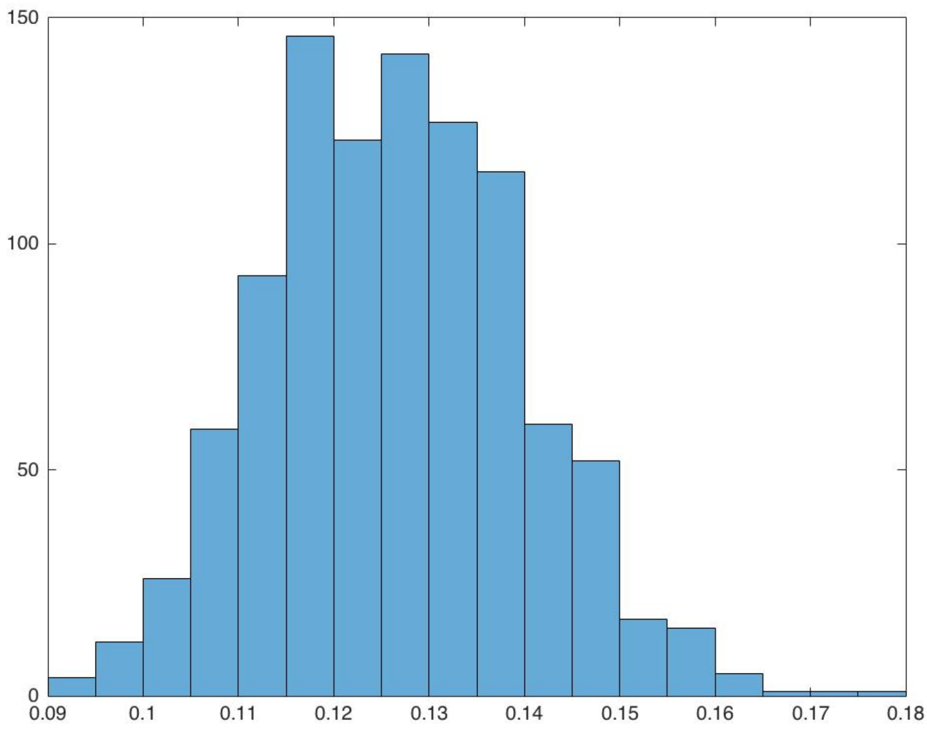
```
f = @(x)  sign(x(1)^2+x(2)^2-0.6)
g = @(x) [1 x(1) x(2) x(1)*x(2) x(1)*x(1) x(2)*x(2)]
error_rate_sum = 0;
Ws = [];
error_history = [];
for t = 1:1000
        X = rand(1000,1) * 2 -1;
        Y = rand(1000,1) *2  -1;
        Xout = rand(1000,1) * 2 -1;
        Yout = rand(1000,1) *2  -1;

        D = [X Y];
        Dout = [Xout Yout];
        Z = [];
        Zout = [];
        G = [];
        Gout = [];
        for i = 1:1000
                z = f(D(i,:));
                zout = f(Dout(i,:));
                if(rand()<=0.1)
                        z = -z;
                end
                if(rand()<=0.1)
```

```
                zout = -zout;
            end
            Zout = [Zout ; zout];
            Z = [Z ;z];
            G = [G; g(D(i,:))];
            Gout = [Gout; g(Dout(i,:))];
        end

        W = inv(G'*G)*G'*Z;
        OUT = Gout*W;
        err = 0;
        for i = 1:1000
            if(sign(Zout(i))~=sign(OUT(i)))
                err= err+1;
            end
        end
        Ws = [Ws  W];
        error_rate_sum = error_rate_sum+err/1000;
        error_history = [error_history ; err/1000];
end
error_rate_sum/1000

histogram(error_history);
```



average out of sample error = 0.1260

16.

minimizing negative log likelihood

Ein = -log (hy1(x) * hy2(x) * hy3(x) * ... hyn(x))
= -log hy1(x) + log (hy2(x)) + ... log(hyn(x))
= -sigma (n=1...N)(log exp (wynt*x)) - N * log(sigma(i=1...K)(exp(wit*x)))
= -sigma (n=1...N)(wynt*x) - N * log(simga(i=1...K)(exp(wit*x)))
= sigma (n = 1...N) (log(sigma(i=1...K)(exp(wit*xn)))-wynt*xn)

17.

d log(sigma(i=1...K)(exp(wit*xn)))-wynt*xn / dwi
= (sigma(i=1...K) (exp(wit*xn)))^-1 * xn*exp(wit*xn) - [[yn==i]]xn
= xn*(hi(xn)-[[yn==i]])

=> gradient(Ein) = sigma(n=1...N) xn*(hi(xn)-[[yn==i]])

18.

```
tt = load('hw3_train.dat');
row_size = size(tt,1);
feature_size = size(tt,2)-1;
X = tt(:,1:feature_size)';
Y =tt(:,feature_size+1);

err = @(y,w,x) (-y*x)/(1+exp(y*w'*x))
w = zeros(feature_size,1);
for t = 1:2000
        grad = zeros(feature_size,1);
         for i = 1 : row_size
                grad = grad+err(Y(i,1),w,X(:,i));
        end
        delta = grad/row_size;
        w = w-0.001*delta;
end

tt = load('hw3_test.dat');
row_size = size(tt,1);
feature_size = size(tt,2)-1;
X = tt(:,1:feature_size)';
Y =tt(:,feature_size+1);

count = 0;
for i = 1:row_size
        if(sign(w'*X(:,i))~=sign(Y(i,1)))
                count =count+1;
        end
end
w
count / row_size
```

```
w= [
        -0.0111
   0.0423
  -0.0311
   0.0166
  -0.0351
   0.0141
   0.0497
  -0.0206
   0.0263
   0.0705
   0.0209
  -0.0184
  -0.0072
   0.0476
   0.0594
   0.0628
  -0.0457
   0.0622
  -0.0146
  -0.0333
]

Eout = 0.4717
```

19.

```
tt = load('hw3_train.dat');
row_size = size(tt,1);
feature_size = size(tt,2)-1;
X = tt(:,1:feature_size)';
Y =tt(:,feature_size+1);

err = @(y,w,x) (-y*x)/(1+exp(y*w'*x))
w = zeros(feature_size,1);
for t = 1:2000
        grad = zeros(feature_size,1);
         for i = 1 : 1000
                grad = grad+err(Y(i,1),w,X(:,i));
        end
        delta = grad/1000;
        w = w-0.01*delta;
end

tt = load('hw3_test.dat');
row_size = size(tt,1);
feature_size = size(tt,2)-1;
X = tt(:,1:feature_size)';
```

```
Y =tt(:,feature_size+1);

count = 0;
for i = 1:row_size
	if(sign(w'*X(:,i))~=sign(Y(i,1)))
		count =count+1;
	end
end
w
count / row_size

w = [
  -0.1894
   0.2659
  -0.3538
   0.0407
  -0.3798
   0.0195
   0.3337
  -0.2642
   0.1347
   0.4912
   0.0870
  -0.2557
  -0.1632
   0.3004
   0.3999
   0.4319
  -0.4625
   0.4320
  -0.2081
  -0.3697
]

Eout =  0.2207
```

20.

```
tt = load('hw3_train.dat');
row_size = size(tt,1);
feature_size = size(tt,2)-1;
X = tt(:,1:feature_size)';
Y =tt(:,feature_size+1);

err = @(y,w,x) (-y*x)/(1+exp(y*w'*x))
w = zeros(feature_size,1);
random_index = randperm(row_size);

for t = 1:2000
```

```
            grad =
err(Y(random_index(1,mod(t,row_size)+1)),w,X(:,mod(t,row_size)+1));
            delta = grad;
            w = w-0.001*delta;
end

tt = load('hw3_test.dat');
row_size = size(tt,1);
feature_size = size(tt,2)-1;
X = tt(:,1:feature_size)';
Y =tt(:,feature_size+1);

count = 0;
for i = 1:row_size
            if(sign(w'*X(:,i))~=sign(Y(i,1)))
                        count =count+1;
            end
end
w
count / row_size

w =[

    0.0137
    0.0087
    0.0064
   -0.0013
    0.0094
    0.0088
    0.0051
    0.0052
    0.0072
    0.0270
    0.0333
   -0.0018
    0.0283
    0.0078
    0.0231
    0.0218
    0.0043
    0.0227
    0.0026
    0.0221
]

Eout = 0.4770
```

21.

hty = h(x1)*y1+...+h(xn)*yn = sigma(n=1..N) (h(xn)yn)

RMSE^2 = 1/N * sigma(n=1,N) (yn^2 - 2*yn*h(xn)+h(xn)^2)

N*RMSE^2 = sigma yn^2 - 2 * sigma * yn*h(xn) + sigma h(xn)^2

sigma yn*h(xn) = 1/2 * (N*RMSE^2-sigma yn^2 - sigma h(xn)^2)

since xn is known, h(xn) is known for given , we only need to know sigma yn^2 to compute hty
One way to do so is to first apply a constant hypothesis h =0
such that RMSE(h)^2 * N = simga (yn-0)^2 = sigma(yn)^2
Thus for any h, we can do two queries , one for h , one for 0 to get hty

22.

N*RMSE^2 = sigma(n= 1...N) (yn-H(xn))^2
= sigma(n= 1...N) (yn^2 - 2*yn*H(xn)+H(xn)^2)
= sigma(n= 1...N) yn^2 - 2*sigma(n=1...N) yn *H(xn) + sigma(n=1...N) H(xn)^2

yn*H(xn) = yn* sigma(k=1...K) wk * hk(xn)
= yn*w1 h1(xn) + yn*w2 * h2(xn) ... yn*wk*hk(xn)
sigma yn*H(xn)
= y1*w1*h1(x1)+y1*w2*h2(x1)... y1*wk*hk(x1)
+ y2*w1*h1(x2)+y2*w2*h2(x2)... y2*wk*hk(x2)
+ ...
+ yn*w1*h1(xn)+yn*w2*h2(xn)... yn*wk*hk(xn)
= w1 * (sigma yn*h1(xn)) + w2 * (sigma yn * h2(xn))+ ... wk*(sigma yn*hk(xn))

since xn is known, H(xn) is known , sigma H(xn)^2 is known
from problem 19, we know that we can get sigma yn^2 by one query
the only thing left to known is sigma yn*(H(xn)) , which needs to compute sigma yn*hk(xn) for all k in K
==> Total k+1 queries are needed (the sigma yn ^2 is also used to compute the value of yn * hk (xn))