École Centrale School of Engineering, Mahindra University, Hyderabad

MA2208 (Numerical Methods), Problem Sheet–I

1. Use the 32-bit long real format to find the decimal equivalent of the following floating-point machine number.

$$0 \quad 10000010 \quad 10010000000000000000000$$

2. Using Bisection, Secant, Regula-Falsi, Newton-Raphson and fixed point methods find the smallest positive real roots of $\cos x - x\ e^x = 0$, correct to three decimal places.

3. Find the rate of convergence and the asymptotic error constant of the secant method.

4. How should the constant $\alpha \in \mathbb{R}$ be chosen to ensure the fastest possible convergence with the iterative formula

$$x_{n+1} = \frac{\alpha x_n + x_n^{-2} + 1}{\alpha + 1}.$$

5. Let $a > 0$ and $f(x) = x^2 - a$. Let x_n be the iterates in Newton's method with $x_0 > 0, x_0 \neq \sqrt{a}$. Show that x_n are strictly decreasing, $x_n \to \sqrt{a}$ and

$$\sqrt{a} - x_{n+1} = -\frac{1}{2x_n}\left(\sqrt{a} - x_n\right)^2$$

6. Let $x_n \subseteq [a,b]$ be a sequence generated by a fixed point iteration method with continuous iteration function $g(x)$. If this sequence converges to c, then show that

$$|x_{n+1} - c| \leq \frac{\lambda}{1-\lambda}|x_{n+1} - x_n|,$$

where $\lambda = \max\limits_{x\in[a,b]} |g^{'}(x)|$. (This enables us to use $|x_{n+1} - x_n|$ to decide when to stop iterating.)

7. Let $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$.

   (a) Show that $f(x)$ has a zero of multiplicity 3 at $x = \dfrac{1}{3}$.

   (b) Use modified Newton's Method to solve this equation with $x_0 = 0$. (Perform 2 iterations)

8. The system of equations $y\cos(xy) + 1 = 0,\ \ \sin(xy) + x - y = 0$ has one solution close to $x = 1, y = 2$. Calculate this solution correct to two decimal places.

Practice problems for Lab

1. Given a 32-bit floating-point representation of a number, write a program to determine the sign, exponent and mantissa of the number.

2. Using Bisection, Secant, Regula-Falsi, Newton-Raphson and fixed point methods find the smallest positive real roots of $\cos x - x\ e^x = 0$, correct to three decimal places.

3. The system of equations $y\cos(xy) + 1 = 0,\ \ \sin(xy) + x - y = 0$ has one solution close to $x = 1, y = 2$. Calculate this solution correct to two decimal places.

4. Apply Newton's method to find the approximation of the root of $x = \tan x$, starting with initial guess $x_0 = 4$ and $x_0 = 4.6$. Compare the results obtained from these two initial guesses. Does the method converge?

5. Use secant method for finding the approximations of the two zeros, one in $[-1, 0]$ and other in $[0, 1]$ to within $1e-3$ accuracy of $f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$. Use the end points of the interval as initial guesses. Observe in the later case that the computed root is outside of the interval $[0, 1]$.

Home work (* Submit solutions of these problems as an first assignment on or before February 15, 2023)

1. * Find the nearest machine number of the decimal number 20.23 in a 32-bit floating-point representation.

2. * Use Newton-Raphson method to find a root of $(a)\ f(x) = xe^{-x},\ x_0 = 2,\quad (b)\ f(x) = x^3 - x - 3, x_0 = 0,\quad (c)\ f(x) = \tan^{-1}(x),\ x_0 = 1.45.$

3. Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a)f(b) < 0$. The Bisection method generates a sequence of approximations x_n which converges to a root $c \in (a, b)$ with the property

$$|x_n - c| \leq \frac{b-a}{2^n} \to 0 \text{ as } n \to \infty$$

   Then find the rate of convergence and the asymptotic error constant.

4. * Let $a, b \in \mathbb{R}$, the equation $x^2 + a\ x + b = 0$ has two real roots α and β. Show that the iteration method

$$x_{k+1} = -(a\ x_k + b)/x_k$$

   is convergent near $x = \alpha$ if $|\alpha| > |\beta|$.

5. Using Bisection, Regula-Falsi, Secant and Newton-Raphson methods find the smallest positive real roots of (i) $x^4 - x - 10 = 0$, (ii) $x - e^{-x} = 0$ correct to three decimal places.

6. * Consider the Newton's method for finding the real root c of

$$f(x) = e^{-ax} - x,\ 0 < a \leq 1.$$

If $x_0 > 0$, then show that

$$|c - x_{n+1}| \leq \frac{1}{2}\ |c - x_n|^2$$

7. * Determine p, q, r so that the order of convergence of iterative method

$$x_{n+1} = p\ x_n + \frac{qa}{x_n^2} + \frac{ra^2}{x_n^5}$$

for finding $a^{1/3}$ becomes as high as possible. For this choice of p, q, r, indicate how the error in x_{n+1} depends on error in x_n.

8. * Determine the values of $a, b, c\ (\neq 0) \in \mathbb{R}$ so that the order of iterative method

$$x_{k+1} = x_k - a\ W_1(x_k) - b\ W_2(x_k), \text{where}\ \ W_1(x_k) = \frac{f(x_k)}{f'(x_k)},\ \ W_2(x_k) = \frac{f(x_k)}{f'(x_k + c\ W_1(x_k))}$$

for finding a simple root of the equation $f(x) = 0$ becomes as high as possible.

9. Let $c \in \mathbb{R}$ be the smallest positive root of $f(x) = 20\ x^3 - 20\ x^2 - 25\ x + 4$. Consider, $g(x) = x^3 - x^2 - \frac{x}{4} + \frac{1}{5},\ x \in [0, 1]$. Then show that $f(c) = 0 \iff g(c) = c$. If $x_0 = 0, x_{n+1} = g(x_n), n = 0, 1, 2, \ldots$. Find the smallest value of n such that $|c - x_n| \leq 10^{-3}$.

10. The equation $f(x) = 3\ x^3 + 4\ x^2 + 4\ x + 1 = 0$ has a root in the interval $(-1, 0)$. Determine an iteration function $g(x)$, such that the sequence of iterations obtained from $x_{k+1} = g(x_k), k = 0, 1, 2, ...$ converges to a root.