

BRIAN BLAIS

STATISTICAL INFERENCE FOR EVERYONE



LIFE'S MOST IMPORTANT QUESTIONS ARE, FOR THE MOST PART, NOTH-
ING BUT PROBABILITY PROBLEMS.

PIERRE-SIMON LAPLACE

STATISTICAL THINKING WILL ONE DAY BE AS NECESSARY FOR EFFICIENT
CITIZENSHIP AS THE ABILITY TO READ AND WRITE.

H.G.WELLS

STATISTICS ARE THE HEART OF DEMOCRACY.

SIMEON STRUNSKY

BRIAN BLAIS

STATISTICAL INFERENCE FOR EVERYONE

SAVE THE BROCCOLI PUBLISHING

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TYPESET WITH TUFTE-LATEX

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Proposal

I would like to propose a new introductory statistical inference textbook, which I believe takes a fresh look at a course that fits into nearly every quantitative major at universities.

Initial Motivation

My motivation for this project stems from my dissatisfaction with traditional approaches to the topic, and my belief that there is a better way. A first semester statistics course is generally divided into the following four parts:

I Basic Statistical Concepts

- Basic statistical concepts including population, parameter, sample, and statistic
- Types of data (ordinal, time-series, etc...), and sampling methodology
- Organizing the data visually or graphically - including histograms, pie graphs, box plots, and stem-and-leaf plots
- Statistical computations including mean, median, mode, standard deviation, and percentiles

II Probability

- Properties of unions, intersections, conditional probability, independence and mutual exclusivity
- Permutations and combinations
- Discrete distributions
- Continuous distributions
- Normal distribution

III One-sample Statistics

- Confidence intervals

- Sampling distributions
- Computations involving the normal distribution, t-distribution, and binomial distribution (for proportions)
- Hypothesis testing

IV Two-sample Statistics

- Two sample problems - expanding topics from Part III to two variables

Obviously, there is some variability to these topics, but as one can see from most introductory statistics textbooks, there is a consistent approach. My main concerns about the traditional approach can be summarized as follows:

- 1 Part II (probability) generally covers at least one quarter of the material in an introductory statistics course. There is a shift from data collection and analysis (Part I) to probability theory. Subsequently, Part III shifts back to a data centered approach and only a small portion of Part II generally applies in Part III. This disconnect between Parts I, II, and III, impedes the learning process. It seems to the students as if the parts are related somehow, but the connection is rarely made. The students are then left with a feeling that the course concerns two completely unrelated topics: probability and statistics.
- 2 The normal distribution is covered repetitively throughout many chapters of most introductory statistics books. The coverage is included in sections such as: empirical bell-shaped curve (Part I), normal distribution as a type of continuous distribution (Part II), sampling distributions (Part III), interval estimation (Part III), hypothesis testing (Part III), and two population testing (Part IV). There is redundant focus on the normal and t-distributions. These topics are closely related, but not handled cohesively. More importantly, there is little or no discussion of the assumptions of the normal model or how to tell what constitutes “close enough” to normal. In addition, there is generally equal consideration given to the rare practical situation in which the standard deviation is known (and knowing this does not generally alter the result much at all).
- 3 After the concept of a “statistic” is covered, there are many chapters which repeat essentially the same problem multiple times, from only slightly different perspectives. This gives the student a feeling that these are all very different problems, despite the appearances, and leads the student to approach solving problems

like a “cookbook”: just find the right recipe for the right problem. The fundamental understanding of statistical inference is undermined by this approach.

It is my view that the traditional approach detracts from student understanding, with its “cookbook” perspective, disjointed coverage of probability, and the almost exclusionary focus on the normal distribution.

A New Approach

In the field of statistical inference, there are two primary schools of thought. Each has its proponents, but it is generally accepted that on all problems covered in an introductory course, that both approaches are valid and lead to the same numerical values when applied to actual problems. Only one of these approaches is covered in a traditional course, which denies the students access to an entire field of statistical inference. The traditional approach, also called the frequentist or orthodox perspective, leads almost directly to problem (1) above. The other approach, also called Probability Theory as Logic¹, derives all statistical inference from probability theory directly. It is this approach that I hope to expose students to in an introductory course.

The probability theory approach to statistical inference has several benefits:

- 1 All of the same problems as handled traditionally can be handled with this perspective, yielding *exactly the same answers*².
- 2 Statistical inference is theoretically grounded in probability theory, which, although admittedly beyond an introductory course, avoids the “cookbook” approach, where different problems need different methods, that students take away from the traditional textbooks. Here all problems use the *same* method, derived from probability theory.
- 3 The reasoning process using the probability theory perspective is more intuitive than the orthodox perspective, especially when dealing with hypothesis testing.

For example, every statistics instructor faces the challenge of getting students to interpret p -values properly, and the logic behind setting up null-hypotheses. They have to combat the students’ initial intuition that the p -value represents the “probability that the null is true,” and many students never really obtain the proper understanding. I have even heard instructors use it this way.

¹ E. T. Jaynes. *Probability Theory: The Logic of Science*. Cambridge University Press, Cambridge, 2003. Edited by G. Larry Bretthorst

² One reason why “Probability theory as Logic” concepts are covered only in advanced courses is the misperception that they are applicable only to more advanced problems, and not applicable to problems normally found in an introductory class. The fact that this misperception exists is a strong argument for a book like this one, to dispel this misperception and to communicate both to students and instructors alike the value of this approach to basic problems.

In the Probability Theory as Logic perspective, this same calculated value is interpreted *exactly like the students' initial intuition!*

Thus, testing hypotheses, estimating parameters, and determining uncertainties are far more direct and intuitive using this approach than the traditional approach.

What I Am Proposing

This text can help solve the challenges described above, and more. By focusing on models and data, as opposed to populations and samples, this text can more cohesively bridge the topics described in Parts I, II, and III above. Probability will be introduced as a natural part of solving problems, as opposed to its standalone treatment traditionally done in today's texts.

In this text, I will use the Probability Theory as Logic approach applied to the same problems that are traditionally covered. This viewpoint can greatly enhance our understanding of statistics and can handle topics such as confidence intervals and hypothesis testing in a very intuitive manner. Statistical inference covered in this way also addresses real-life questions that are not addressed by traditional statistical methods.³

Finally, this will be a problem oriented textbook. It is imperative that the problems are cohesive with the pedagogy. I will also plan to use technology, where appropriate, to further student learning and make the textbook more interactive.

At the level targeted for this book, there is only one textbook that I know of that covers inference from the perspective proposed here, and that is Donald Berry's book *Statistics: A Bayesian Perspective*, 1996. It is my intention to modernize the approach, and include some topics that are not covered, specifically from the physical sciences and business.

³ One of the reasons why this approach is usually covered only in more advanced courses is the difficulty of the mathematics generally associated with it. Orthodox statistics makes heavy use of sampling, which is deemed more intuitive than probability distributions. It is my intention to start with low-dimensional cases, building to distributions, and to augment all concepts with numerical exercises.

1 *Introduction to Probability*

Life's most important questions are, for the most part, nothing but probability problems. - Laplace

In 1968 a jury found defendant Malcolm Ricardo Collins and his wife defendant Janet Louise Collins guilty of second degree robbery¹. The decision hinged on the testimony of bystanders, which stated that the perpetrators had been “black male, with a beard and moustache, and a caucasian female with blonde hair tied in a ponytail,” and that they escaped in a “yellow motor car.” A mathematician testified that the odds *against* this couple being innocent were one in *twelve million*, and this was enough for the jury to convict. Later, in an appeal, the California Supreme Court reversed the decision primarily because of lack of evidence, and faulty inference.

In another case, Sally Clark was convicted in 1999 of the murder of her two young sons². Again, the testimony hinged on a statistical argument - the chances of one baby dying in their bed 1 in 8500, so therefore the chances of *two* of them dying in the same way is the square of this, or 1 in 73 million. Several years later, and a public statement from the Royal Statistical Society highlighting the erroneous logic, Sally Clark was released - although she never overcame the resulting damage to her life that the conviction had caused.

We will cover these cases in more detail later, and why the inference was faulty, but I introduce the stories here for two reasons. First, is to point out that there are cases in which proper statistical inference can be a life and death matter. Second, it is to highlight the fact that such inference can run counter to one’s intuition. Part of the purpose of this book is to retrain your intuitions and your habits of intuition to avoid such failures.

We have to make decisions nearly every second of our lives, and those decisions are based on our state of knowledge. Unfortunately, we are never 100% sure of *any* information in our lives, so we are constantly forced to make decisions in the face of uncertainty. In many cases our common sense is enough to make sophisticated decisions, taking into account the uncertain nature of the situation. However, there are many times where our common sense is not enough to

¹ J. Sullivan. People v. Collins , 68 cal.2d 319, 1968. URL <http://scocal.stanford.edu/opinion/people-v-collins-22583>

² Lord Justice Kay. R vs Sally Clark, April 2003. URL <http://www.bailii.org/ew/cases/EWCA/Crim/2003/1020.html>

quantitatively resolve the level of uncertainty, and make valid inferences. It is in these cases that statistical inference is most useful.

Statistical inference refers to a field of study where we try to infer unknown properties of the world, given our observed data, in the face of uncertainty. It is a mathematical framework to quantify what our common sense says in many situations, but allows us to exceed our common sense in cases where common sense is not enough. Ignorance of proper statistical inference leads to poor decisions and wasted money. As with ignorance in any other field, ignorance of statistical inference can also allow others to manipulate you, convincing you of the truth of something that is false.

For example, in 1978 a Russian satellite deviated from its orbit and became increasingly erratic, and was going to crash into the Earth.³ This sort of event occurs from time to time, even including a recent crash of a US spy satellite in 2008.⁴ There was a local news broadcast about the impending Russian satellite crash which said something like, “the scientists had studied the trajectory of the satellite, and determined that there was only a 25% chance of it striking land, and even a much smaller chance striking a populated area.” The report was clearly designed to calm the public, and convince them that the scientists had a good handle on the situation. Unfortunately, given a little thought, one realizes that the Earth’s surface consists of about 25% land and 75% water, so *if you knew nothing about the trajectory of the satellite*, you would simply state that it had a 25% chance of striking land. Instead of communicating knowledge of the situation, the news broadcast communicated (to those who knew basic statistical inference) that either the scientists were in *complete ignorance* of the trajectory or the reporter had misinterpreted a casual statement about probabilities and didn’t realize what was implied. Either way, the intent of the message and the content of the message (to those who understood basic probability) were in direct conflict.

³ L Heaps. *Operation morning light*. Paddington, S.I, 1978. ISBN 0709203233

⁴ James Oberg. U.S. satellite shoot-down: The inside story. *IEEE Spectrum*, 2008

1.1 Models and Data

There are two main aspects of statistical inference: description of data and model analysis. In the description of data, one attempts to summarize a set of data with a smaller set of numbers. Grades in the classroom are summarized by the average, votes in a state are summarized by a percentage, etc... This smaller description of the data is useful for both practical and theoretical reasons. It is more expedient to communicate a small set of numbers than the entire data set, and it is almost always the case that the detailed properties of a set of data are not relevant to the questions that you are asking.

A model refers to a mathematical structure which is used to ap-

proximate the underlying causes of the data, and unify seemingly unrelated problems. One may have a (mathematical) model for a coin flip which ignores all of the details of the flip, the bounce, and the catch and summarizes the possible results by a single number: the chance that the coin will come up heads. You may then use that same model to describe the voting behavior of citizens during a presidential election, or to describe the radioactive decay of particles in a physics experiment. The mathematics is identical, but the interpretation of the components of the model will be different depending on the problem. Models *simplify*, by summarizing data with a small set of causes, and they are used for *inference*, allowing one to predict the outcome of subsequent events.

The goal of statistical inference is then to take data, and update our knowledge about various possible models that can describe the data. This often means deciding which of several models is the most likely. It can also entail the refinement of a single model, given the new data. All of these activities are closely related to (and perhaps identical to) the methods in science. What we are trying to do is make the best inferences from the data, improve our inferences as new data come in, and plan what data would be the most useful to improve our inferences. In a nutshell, the approach is:

$$\text{Initial Inference} + \text{New Data} \rightarrow \text{Improved Inference}$$

In order to deal with a wide variety of problems, we require a minimal amount of mathematical structure and notation, which we introduce in this chapter.

1.2 What is Probability?

Probability theory is nothing but common sense reduced to calculation. - Laplace

When you think about probability, the first things that might come to mind are coin flips ("there's a 50-50 chance of landing heads"), weather reports ("there's a 20% chance of rain today"), and political polls ("the incumbent candidate is leading the challenger 53% to 47%"). When we speak about probability, we speak about a percentage chance (0%-100%) for something to happen, although we often write the percentage as a decimal number, between 0 and 1. If the probability of an event is 0 then it is the same as saying that *you are certain that the event will never happen*. If the probability is 1 then *you are certain that it will happen*. Life is full of uncertainty, so we assign a number somewhere between 0 and 1 to describe our state of knowledge of the certainty of an event. The probability that you will get

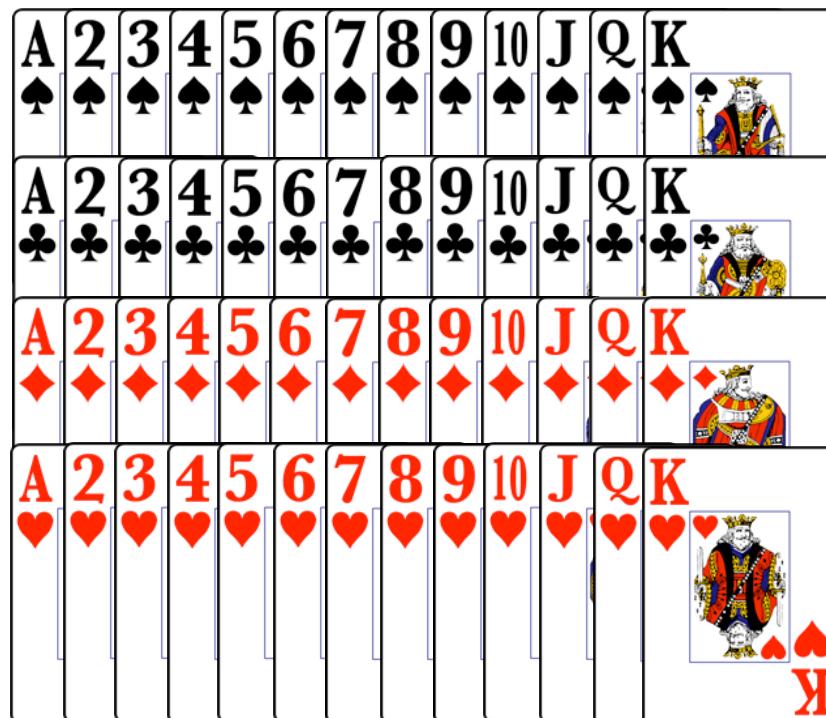
struck by lightning sometime in your life is $p = 0.0002$, or 1 out of 5000. Statistical inference is simply the inference in the presence of uncertainty. We try to make the best decisions we can, given incomplete information.

One can think of probability as a mathematical short-hand for the common sense statements we make in the presence of uncertainty. This short-hand, however, becomes a very powerful tool when our common sense is not up to the task of handling the complexity of a problem. Thus, we will start with examples that will perhaps seem simple and obvious, and move to examples where it would be a challenge for you to determine the answer without the power of statistical inference.

Let's walk through a simple set of examples to establish the notation, and some of the basic mathematical properties of probabilities.

Card Game

A simple game can be used to explore all of the facets of probability. We use a standard set of cards (Figure 1.1) as the starting point, and use this system to set up the intuition, as well as the mathematical notation and structure for approaching probability problems.



In this book, our approach is to determine, for each problem, what degree of confidence we have in all of the possible outcomes. The approach of statistical inference covered in this book is about the procedure of most rationally assigning various degrees of confidence (which we call *probability*) to the possible outcomes of some process using all the objectively available data.

Figure 1.1: Standard 52-card deck.
13 cards of each suit, labeled
Spades, Clubs, Diamonds, Hearts.

We start with what I simply call the *simple card game*⁵, which goes

⁵ In this description of the game, we do not reshuffle after each draw. The differences between this non-reshuffled version and the one with reshuffling will be explored later, but will only change some small details in the outcomes.

like:

$$\text{simple card game} \equiv \left\{ \begin{array}{l} \text{From a standard initially shuffled deck, we draw one card, note what card it is and set it aside. We then draw another card, note what card it is and set it aside. Continue until there are no more cards, noting each one along the way.} \end{array} \right. \quad (1.1)$$

There are certain principles that guide us in developing the mathematical structure of probability. We start with some common sense notions, written in English, and then write them as general principles. These principles, then, constrain our mathematics so that we can apply the ideas *quantitatively*.

When asked “what is the probability of drawing a red on the first draw?” you would generally say 50-50, or 50%, or equivalently written as a probability, $P(R_1) = 0.5$. The reason for this is that we are completely ignorant of the initial conditions of the deck (i.e. where each card is located in the deck after the initial shuffling). Given this level of (or lack of) knowledge, we could swap the colors of the two suits and we would have an equivalent state of knowledge - the problem would be identical. We will keep coming back to this concept, but in general:

Principle of Knowledge and Probability Equivalent states of knowledge must yield equivalent probability assignments.

Because of this principle, we are led to the conclusion that

$$P(R_1) = P(B_1)$$

where R_1 represents the statement “a red on the first draw” and B_1 represents “a black on the first draw.” Because these are the only two options, and they are mutually exclusive, then they must add up to 1. Thus we have

$$P(R_1) = 1 - P(B_1)$$

which leads directly to our original assignment

$$P(R_1) = P(B_1) = 0.5$$

Mutually Exclusive If I have a list of *mutually exclusive* events, then that means that only one of them could possibly be true. Example events include flipping heads or tails with a coin, rolling a 1, 2, 3, 4, 5 or 6 on dice, or drawing a red or black card from a deck of

Principle of Knowledge and Probability Equivalent states of knowledge must yield equivalent probability assignments.

Mutually Exclusive If I have a list of *mutually exclusive* events, then that means that only one of them could possibly be true. Examples include the heads and tails outcomes of coins, or the values of standard 6-sided dice. In terms of probability, this means that, for events A and B, $P(A \text{ and } B) = 0$.

cards. In terms of probability, this means that, for events A and B, $P(A \text{ and } B) = 0$.

Non Mutually Exclusive If I have a list of events that are *not mutually exclusive*, then it is possible for two or more to be true. Examples include weather with rain and clouds or holding the high and the low card in a poker game.

Now, this was a long-winded way to get to the answer we knew from the start, but that is how it must begin. We start working things out where our common sense is strong, so that we know we are proceeding correctly. We can then, confidently, apply the tools in places where our common sense is not strong.

In summary, with no more information than that there are two mutually exclusive possibilities, we assign equal probability to both. If there are only two colors of cards in equal amounts, red and black, then the probability of drawing a red is $P(R_1) = 0.5$ and the probability for a black is the same, $P(B_1) = 0.5$.

Non Mutually Exclusive If I have a list of events that are *not mutually exclusive*, then it is possible for two or more to be true. Examples include weather with rain and clouds or holding the high and the low card in a poker game.

Other Observations

If instead of just the color, we were interested in the suit (hearts, diamonds, spades, and clubs), then there would be four equal and mutually exclusive possibilities. We have a certain number of possibilities, and our state of knowledge is exactly the same if we simply swap around the labels on the cards. If we're interested in the specific card, not just the suit, the logic is the same. Thus, we have

$$P(\spadesuit) = P(\clubsuit) = P(\diamondsuit) = P(\heartsuit)$$

and for drawing one specific card from the deck,

$$P(A\spadesuit) = P(2\spadesuit) = P(3\spadesuit) = \dots = P(K\heartsuit)$$

Further, they all must add up to 1, so we get for suits

$$P(\spadesuit) + P(\clubsuit) + P(\diamondsuit) + P(\heartsuit) = 1$$

and for the specific card from the deck,

$$\underbrace{P(A\spadesuit) + P(2\spadesuit) + P(3\spadesuit) + \dots + P(K\heartsuit)}_{52 \text{ cards}} = 1$$

Putting it together, we get for the suits

$$P(\spadesuit) = P(\clubsuit) = P(\diamondsuit) = P(\heartsuit) = \frac{1}{4}$$

and for the specific card

$$P(A\spadesuit) = P(2\spadesuit) = P(3\spadesuit) = \dots = P(K\heartsuit) = \frac{1}{52}$$

Probabilities for Mutually Exclusive Events In general, for mutually exclusive events, we have

$$P(A) = \frac{\text{(number of cases favorable to A)}}{\text{(total number of equally possible cases)}} \quad (1.2)$$

Probabilities for Mutually Exclusive Events

$$P(A) = \frac{\text{(number of cases favorable to A)}}{\text{(total number of equally possible cases)}}$$

1.3 Conditional Probability

It is important to understand that probability reflects our state of knowledge about the system. As our knowledge changes, so do our probability assignments. As we gain more information, we change our probability assignments. Two people observing the same system, but with *different* information about the system, will give *different* probability assignments. All we need to make sure probability theory matches our common sense is for two people with the same state of knowledge, or the same information, to yield identical probability assignments.

Because our information about a system is so important in assigning probabilities, we introduce a way of writing it mathematically that we will use for the rest of the book. It will be good for the reader to get used to reading the mathematical short-hand in English in order to gain an understanding for what it means.

Probability Notation

In math, we choose to abbreviate long sentences in English, in order to use the economy of symbols. In this book we choose a middle-ground between mathematical succinctness and the ease of understanding English. We start with the simple card game (Equation 1.1)

We then define a new symbol, $|$, which should be read as “given.” When there is information given we call this probability *conditional* on that information. When we write the following:

$$P(\text{red on first draw}|\text{simple card game}) \quad (1.3)$$

or

$$P(R_1|\text{simple card game}) \quad (1.4)$$

this is short for

"The probability of drawing a red on the first draw, **given that** we have a standard initially shuffled deck and we follow the procedure where we draw one card, note what color it is and set it aside and continue drawing, noting, and setting aside until there are no more cards."

One can easily see that the mathematical notation is far more efficient. It is important to be able to read the notation, because it describes what we know and what we want to know.

Conditional Probability When information is given, and expressed on the right-hand side of the | sign, we say that the probability is *conditional*. $P(\text{I'm going to get wet today}|\text{raining outside})$ is an assessment of how likely it is that I will get wet *given*, or *conditional on*, the fact that it is raining outside. Clearly this number will be different if it was conditional on the fact that it is sunny outside - different states of knowledge yield different probability assignments.

When we put a comma (" ,") on the right side then we read this as "and we know that." For example, when we write the following:

$$P(\text{red on second draw}|\text{simple card game, red on first draw}) \quad (1.5)$$

or

$$P(R_2|\text{simple card game}, R_1) \quad (1.6)$$

this is short for

"The probability of drawing a red on the second draw, **given that** we have a standard initially shuffled deck and we follow the procedure where we draw one card, note what color it is and set it aside and continue drawing, noting, and setting aside until there are no more cards **and we know** that we drew a red on the first draw."

1.4 Rules of Probability

From the rule for mutually exclusive events (Equation 1.2), we assign the following probabilities for the *first draw* from this deck⁶:

- $P(10) = \frac{4}{52}$
- $P(\heartsuit) = \frac{13}{52} = \frac{1}{4}$
- $P(10\heartsuit) = \frac{1}{52}$
- $P(\text{face card}) = \frac{12}{52}$
- $P(\text{number card}) = \frac{40}{52}$

Conditional Probability When information is given, and expressed on the right-hand side of the | sign, we say that the probability is *conditional*. $P(\text{I'm going to get wet today}|\text{raining outside})$ is an assessment of how likely it is that I will get wet *given*, or *conditional on*, the fact that it is raining outside. Clearly this number will be different if it was conditional on the fact that it is sunny outside.

Causation. Imagine we have a 2-card game: a small deck with one red card and one black card, and I draw a red card first. Clearly this makes the probability of drawing red as the second card equal to zero - it can't happen. We're tempted to interpret

$$P(R_2|R_1, \text{2-card game}) = 0$$

to mean that *because we drew a red on the first draw, this causes the impossibility of drawing the red on the second* - there is only 1 red card after all, and drawing it seems to *cause* the impossibility of drawing red in the future. However, consider the following:

$$P(R_1|R_2, \text{2-card game}) = 0$$

which is, if we *knew* that the second card we drew was red, then it makes it impossible to have drawn a red card as the first card. This is just as true as the previous case, however, you can't interpret this as *causation* - the second draw didn't *cause* the first draw.

Instead, *probability statements are statements of logic, not causation*. One can use probabilities to describe causation (i.e. $P(\text{rain}|\text{clouds})$), but the statements of probability have no time component - later draws from the deck of cards act exactly the same as earlier ones.

⁶ A face card is defined to be a Jack, Queen, or King. A number card is defined to be Ace (i.e. 1) through 10.

It turns out that mathematically, the rules for *fractions of things* and of *probabilities* are the same. Thus, to gain an understanding for the rules of probability, we will calculate fractions (which are more immediately intuitive), and then summarize the same rule for probabilities.

Negation Rule

In this section I'll use the letter F for fraction, and we can determine the values simply by counting. The fraction of cards which are hearts (\heartsuit) is

$$F(\heartsuit) = \frac{13}{52} = \frac{1}{4}$$

The fraction of cards which are *not* hearts (i.e. the 3 other suits) is:

$$F(\text{not } \heartsuit) = \frac{13 \times 3}{52} = \frac{3}{4}$$

These numbers add up to one: $F(\heartsuit) + F(\text{not } \heartsuit) = 1$. We can do this with more complex statements.

$$\begin{aligned} F(\text{first card is a face card}) &= \frac{12}{52} \\ F(\text{first card is not a face card}) &= \frac{40}{52} \\ F(\text{first card is a face card}) + F(\text{first card is not a face card}) &= 1 \end{aligned}$$

EXAMPLE 1.1 What is the fraction of the first card as a jack given that we know that the first card is a face card?

We can also apply the negation rule to conditional statements, like “the first card is a jack given that we know that the first card is a face card.” Notice that there are 12 cards that are face cards, so we restrict our counts to those.

$$\begin{aligned} F(\text{jack}|\text{face card}) &= \frac{4}{12} = 1/3 \\ F(\text{not a jack}|\text{face card}) &= \frac{8}{12} = 2/3 \\ F(\text{jack}|\text{face card}) + F(\text{not a jack}|\text{face card}) &= 1 \end{aligned}$$

and they add up to one.

Negation Rule Given any information, we have

$$P(\text{statement}| \text{information}) + P(\text{not statement}| \text{information}) = 1$$

or

$$P(A|B) + P(\text{not } A|B) = 1 \quad (1.7)$$

Either-or fallacy. The negation rule, should not be taken to imply that everything is “black and white,” or “there are only two sides to every story.” It really is just a statement of logic, should be carefully considered and has some limitations. For example, the following is true,

$$P(\text{object is black}) + P(\text{object is not black}) = 1$$

However, this does not mean the same thing as

$$P(\text{object is black}) + P(\text{object is white}) \neq 1$$

“Not black” is not the equivalent of “white.” It could be red, or gray, or some other color. A common logical fallacy sometimes referred to as the “either-or fallacy” or the “fallacy of the excluded middle,” turns on this point. Some examples of these fallacies are:

- If we don't reduce public spending, our economy will collapse.
- You're either with us or you're a terrorist.
- Either modern medicine can explain how Ms. X was cured, or it is a miracle.

Negation Rule

$$P(A|B) + P(\text{not } A|B) = 1$$

Product Rule

The product rule comes from looking at the combination of events: event A *and* event B. As before, we'll work on the numbers from the fractions of the card game.

EXAMPLE 1.2 *What is the fraction of cards that are Jacks and a heart?*

This is clearly $F(J\heartsuit) = 1/52$, but we can look at it a different way that is equivalent. We note that the Jacks constitute $4/52$ of the cards, and that of those 4, only one quarter of them are hearts (one card out of the four cards). So, we can arrive at the fraction of $J\heartsuit$ by taking one quarter of the fraction of jacks. So what we have is

$$F(\text{jack and } \heartsuit) = F(\heartsuit|\text{jack}) \times F(\text{jack}) = \frac{1}{4} \times \frac{4}{52} = \frac{1}{52}$$

One can equivalently reason from the suit first: the hearts constitute $13/52$ of the cards, and that of those 13, the Jacks constitute $1/13$ of the cards. So, we can arrive at the fraction of $J\heartsuit$ by taking one thirteenth of the fraction of \heartsuit . Again, we have

$$F(\text{jack and } \heartsuit) = F(\text{jack}|\heartsuit) \times F(\heartsuit) = \frac{1}{13} \times \frac{13}{52} = \frac{1}{52}$$

In general we have

Product Rule

$$P(A \text{ and } B) = P(A|B)P(B) = P(B|A)P(A) \quad (1.8)$$

Product Rule

$$\begin{aligned} P(A \text{ and } B) &= P(A|B)P(B) \\ &= P(B|A)P(A) \end{aligned}$$

EXAMPLE 1.3 *What is the probability of drawing two Kings in a row?*

This is the same as

$$P(K_2 \text{ and } K_1)$$

From the product rule (Equation 1.8) we have

$$P(K_2 \text{ and } K_1) = P(K_2|K_1)P(K_1)$$

The second part is straight forward: $P(K_1) = 4/52$. The first part is asking the probability of drawing a second king, knowing that we have drawn a king on the first draw. Now, there are only 51 cards remaining when we do the second draw, and only 3 kings. Thus, we have $P(K_2|K_1) = 3/51$ and finally

$$\begin{aligned} P(K_2 \text{ and } K_1) &= P(K_2|K_1)P(K_1) \\ &= \frac{3}{51} \times \frac{4}{52} = \frac{1}{221} \end{aligned}$$

Independence

As a specific case of the product rule, we can change the rule of the card games such that we reshuffle the deck after each draw. In this way, the result of one draw gives you no information about other draws. In this case, the events are considered *independent*.

Independent Events Two events, A and B, are said to be independent if knowledge of one gives you no information on the other. Mathematically, this means

$$P(A|B) = P(A)$$

and

$$P(B|A) = P(B)$$

In this case, the product rule reduces to the simplified rule for independent events: the product of the individual event probabilities.

Joint Probabilities for Independent Events

$$P(\text{A and B}) = P(A) \times P(B) \quad (1.9)$$

We have already seen an example of this, when we looked at drawing the Jack of Hearts: drawing a heart gives you no information about whether it is a jack, and vice versa. Thus,

$$P(\heartsuit|\text{jack}) = P(\heartsuit)$$

EXAMPLE 1.4 *What is the probability of flipping two heads in a row?*

The probability of getting “heads” on any given coin flip is $P(H) = 0.5$. The probability of flipping two heads in a row is then simply $P(H_1) \times P(H_2) = 0.5 \times 0.5 = 0.25$, because the second flip is independent of the first. If it wasn’t, then you’d have to determine how the knowledge of the first flip influences our knowledge of the second flip, which is written as $P(H_2|H_1)$ and the full product rule (Equation 1.8) would need to be used.

Conjunction

One of the consequences of combinations of events is that the probability of two events happening, A **and** B, has to be less than (or possibly equal to) the probability of just one of them, say A, happening. The mathematical fact is seen by looking at the magnitude of the

Independent Events Two events, A and B, are said to be independent if knowledge of one gives you no information on the other. Mathematically, this means

$$P(A|B) = P(A)$$

and

$$P(B|A) = P(B)$$

Joint Probabilities for Independent Events

$$P(\text{A and B}) = P(A) \times P(B)$$

terms in the product rule

$$P(A \text{ and } B) = \underbrace{P(B|A)}_{\begin{array}{l} \text{less} \\ \text{than or} \\ \text{equal to} \\ 1 \end{array}} \times P(A) \leq P(A)$$

In other words, coincidences are less likely than either event happening individually. We intuitively know this, when we make comments like "Wow! What are the chances of that?" referring to, say, someone winning the lottery and then getting struck by a car the next day. Sometimes, however, it seems as if one's intuition does not match the conclusions of the rules of probability. One such case is called the *conjunction fallacy*.

In an interesting experiment, Tversky and Kahneman [Tversky and Kahneman, 1983] gave the following survey:

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Which is more probable?

- 1 Linda is a bank teller.
- 2 Linda is a bank teller and is active in the feminist movement.

85% chose option 2. [Tversky and Kahneman, 1974] This, they attributed, to the conjunction fallacy - mistaking the conjunction of two events as more probable than a single event. They went further and did a survey of medical internists with the following

Which is more likely: the victim of an embolism (clot in the lung) will experience partial paralysis or that the victim will experience both partial paralysis and shortness of breath?

and again, 91 percent of the doctors chose that the clot was less likely to cause the rare paralysis rather than to cause the combination of the rare paralysis and the common shortness of breath.

Even when correct, the consequence for conjunctions can be misused, or at least misidentified. Returning to our example of someone winning the lottery and then getting struck by a car the next day, rare events *occur frequently*, as long as you have enough events. There are millions of people each day playing the lottery, and millions getting struck by cars each day. We will explore this problem later in Section ??, but one immediate consequence is that winning the lottery and getting struck by a car the next day probably happens *somewhere* fairly regularly.

Combinations of Events and the English language I believe that the issue of the conjunction fallacy is more subtle than this. In English, if I were to say "Do you want steak for dinner, or steak and potatoes?" one would immediately parse this as choice between

1 steak with no potatoes

2 steak with potatoes

Although strict logic would parse this choice as

1 steak, possibly with potatoes and possibly without potatoes

2 steak, definitely with potatoes, it is common in English to have the implied negative (i.e. steak with no potatoes) when given a choice where the alternative is a conjunction (i.e. steak with potatoes).

Combinations of Events and the English language If we interpret the doctor's choice with this implied negative, we have:

1 clot with paralysis and no shortness of breath

2 clot with paralysis and shortness of breath

and the first one is much less likely, because it would be odd to have a clot and not have a very common symptom associated with it. The doctor's probability assessment is absolutely correct: both symptoms together are more likely than just one. The "fallacy" arises because the English language is sloppier than mathematical language.

Sum Rule

Now we consider the statements of the form A **or** B. For example, in the card game, what is the fraction of cards that are jacks or are hearts. By counting we get the 13 hearts and 3 more jacks that are not contained in the 13 hearts, or $F(\text{jack or } \heartsuit) = \frac{13+3}{52} = 16/52$. Now, if we tried to separate the terms, and do:

$$F(\text{jack}) + F(\heartsuit) = \frac{4}{52} + \frac{13}{52} = \frac{17}{52}$$

then we get a number that is too big! It is too big because we've double-counted the jack of hearts. Adjusting for this, by subtracting one copy of this fraction, we get

$$F(\text{jack}) + F(\heartsuit) - F(\text{jack and } \heartsuit) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = F(\text{jack or } \heartsuit)$$

In general

Sum Rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \quad (1.10)$$

Sum Rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Sum Rule for Exclusive Events If two events are *mutually exclusive* the sum rule reduces to

$$P(A \text{ or } B) = P(A) + P(B) \quad (1.11)$$

because $P(A \text{ and } B) = 0$ for such events.

So the probability of rolling a 1 or a 2 on one die is 2/6.

One more variant on the Sum Rule is where we have 3 propositions. It can be a bit tedious to write it all out, but the end result looks a lot like the original Sum Rule. All we do is break up the terms in pieces, and then apply the Sum Rule to each piece.

$$\begin{aligned} P(A \text{ or } B \text{ or } C) &= P(A \text{ or } [B \text{ or } C]) \\ &= P(A) + P(B \text{ or } C) - P(A \text{ and } [B \text{ or } C]) \\ &= P(A) + P(B) + P(C) - P(B \text{ and } C) - \\ &\quad P(A \text{ and } B \text{ or } A \text{ and } C) \\ &= P(A) + P(B) + P(C) - P(B \text{ and } C) - \\ &\quad [P(A \text{ and } B) + P(A \text{ and } C) - \\ &\quad P(A \text{ and } B \text{ and } A \text{ and } C)] \end{aligned}$$

which leads finally to

Sum Rule for Three Events

$$\begin{aligned} P(A \text{ or } B \text{ or } C) &= P(A) + P(B) + P(C) - \\ &\quad P(A \text{ and } B) - P(B \text{ and } C) - P(A \text{ and } C) + \\ &\quad P(A \text{ and } B \text{ and } C) \quad (1.12) \end{aligned}$$

Sum Rule for Three Events

$$\begin{aligned} P(A \text{ or } B \text{ or } C) &= P(A) + P(B) + P(C) - \\ &\quad P(A \text{ and } B) - \\ &\quad P(B \text{ and } C) - \\ &\quad P(A \text{ and } C) + \\ &\quad P(A \text{ and } B \text{ and } C) \end{aligned}$$

In words, when you're looking for the sum of several events, we add the probabilities (i.e. $P(A) + P(B) + P(C)$), then subtract the double counting (i.e. $P(A \text{ and } B)$) as before. Finally, we need to add back in the *triple count* (i.e. $P(A \text{ and } B \text{ and } C)$) because it was taken out too many times with the double count. The accounting here can be somewhat prone to error, but the concepts are always the same: when you add probabilities of events, say A and B , together the term $P(A)$ includes the probability of both $P(A \text{ and } B)$ and the term $P(B)$ includes the probability of both $P(A \text{ and } B)$, so you've included that probability twice and need to subtract one of them to balance the books. Likewise (although it is harder to show), the first six terms in Equation 1.12 end up subtracting one too many copies of $P(A \text{ and } B \text{ and } C)$, and we need to add one in at the end.

Marginalization

Another consequence of the sum rule and the product rule is a process called *marginalization*.

EXAMPLE 1.5 Marginalization and Card Suit

Imagine we have a number of conditional statements, like:

$$\begin{aligned} P(\text{jack}|\heartsuit) &= \frac{1}{13} \\ P(\text{jack}|\diamondsuit) &= \frac{1}{13} \\ P(\text{jack}|\spadesuit) &= \frac{1}{13} \\ P(\text{jack}|\clubsuit) &= \frac{1}{13} \end{aligned}$$

but we are interested in just the probability of drawing a jack, regardless of the suit, or in our notation

$$P(\text{jack})$$

The marginalization procedure for this problem looks like:

$$\begin{aligned} P(\text{jack}) &= \overbrace{P(\text{jack}|\heartsuit) \times P(\heartsuit) +}^{\text{all possibilities}} \\ &\quad P(\text{jack}|\diamondsuit) \times P(\diamondsuit) + \\ &\quad P(\text{jack}|\spadesuit) \times P(\spadesuit) + \\ &\quad P(\text{jack}|\clubsuit) \times P(\clubsuit) \\ &= \frac{1}{13} \times \frac{1}{4} + \frac{1}{13} \times \frac{1}{4} + \frac{1}{13} \times \frac{1}{4} + \frac{1}{13} \times \frac{1}{4} \\ &= \frac{4}{52} \end{aligned}$$

Marginalization If we have a complete set of conditional statements, like

$$\begin{aligned} P(A|B_1) \\ P(A|B_2) \\ P(A|B_3) \\ P(A|B_4) \\ \vdots \end{aligned}$$

then the *unconditional* probability is found by *marginalizing* over all possible values of the conditional events, like

$$P(A) = \overbrace{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) + \dots}^{\text{all possible } Bs} \quad (1.13)$$

Bayes' Rule

One of the most consequential rules of probability is what is known as Bayes' Rule, sometimes called Bayes' Theorem. We will use this rule throughout this book, and see its many applications. It comes as a direct result of the product rule (Equation 1.8)

$$P(A \text{ and } B) = P(A|B)P(B) = P(B|A)P(A)$$

Rearranging, we get

Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad (1.14)$$

We can verify this again with the intuitions we have in the simple card game.

EXAMPLE 1.6 What is the probability of drawing a jack, knowing that you've drawn a face card?

In terms of fractions, this should be $F(\text{jack}|\text{face card}) = 4/12 = 1/3$. Applying Bayes' Rule to the fractions we get:

$$\begin{aligned} F(\text{jack}|\text{face}) &= \frac{F(\text{face}|\text{jack}) \times F(\text{jack})}{F(\text{face})} \\ &= \frac{\frac{4}{4} \times \frac{4}{52}}{\frac{12}{52}} = \frac{4}{12} = \frac{1}{3} \end{aligned}$$

Although this calculation is true, it isn't particularly enlightening. It is nicer to cast the problem back into probability terms, rather

Marginalization If we have a complete set of conditional statements, like

$$P(A|B_1), P(A|B_2), P(A|B_3), P(A|B_4), \dots$$

then the *unconditional* probability is found by *marginalizing* over all possible values of the conditional events, like

$$P(A) = \overbrace{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots}^{\text{all possible } Bs}$$

In the 1700's Reverend Bayes proved a special case of this rule, and rediscovered in the general form by Pierre-Simon Laplace. Laplace then applied the rule in a large range of problems from geology, astronomy, medicine, and jurisprudence.

Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

than fractions, and compare the probability of drawing a jack to the probability of the same thing (i.e. drawing a jack) *given* that we know that we've drawn a face card. This is

$$\begin{aligned} P(\text{jack}) &= \frac{1}{13} \\ P(\text{jack}|\text{face card}) &= \frac{1}{3} \end{aligned}$$

This comparison highlights what Bayes' Rule represents: learning. When you are asked what the probability of drawing a jack, from the knowledge of the simple card game, you calculate the value of $1/13$. Once you learn that you drew a face card, you update your knowledge to include that information, and modify your probability assignments to reflect this. This leads to an increased chance of the card being a jack.

In a nutshell, Bayes' Rule represents learning:

Initial Belief + New Data → Improved Belief

It is used in science to infer causes from effects, and can thus be written

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause}) \times P(\text{cause})}{P(\text{effect})}$$

To infer the probability of a particular cause, given the events you observe in the world, you first have to know the probability of the cause itself (i.e. rarer causes will reduce the *prior* probability), and how likely that the cause you're looking at could have produced the effects you've observed. These two items are the $P(\text{cause})$ and $P(\text{effect}|\text{cause})$ terms, respectively. The entire calculation is scaled by $P(\text{effect})$ which is *all of the other ways that the effects could have been produced by other causes..* Thus, it is not enough to show that giving a particular medicine is followed by the symptoms disappearing to establish that the medicine was the likely cause of the symptoms disappearing. You have to calculate what other possible causes could have had those effects, such as the normal functioning of the immune system or the placebo effect. This is why carefully controlled studies are necessary, to eliminate all of the other possible causes and to determine the true cause of the effects observed.

We will spend large portions of several chapters on Bayes' Rule, to explore its long-ranging consequences.

All of learning is simply updating ones beliefs given the data. The data may be words in a book, the results of an experiment, a conversation with another person, etc... The strength of our beliefs are not often thought of in mathematical terms, but you are doing the math of probabilities whenever you are weighing the strength of your beliefs. Thus, the probabilistic rule - Bayes' rule - for updating beliefs given data is really the quantitative specification of learning. One can use it *qualitatively* as well, which is often useful in fields such as history where the data do not tend to be quantitative.

1.5 Venn Mnemonic for the Rules of Probability

It is often useful to have a picture to represent the mathematics, so that it is easier to remember the equations and to understand their meaning. It is common to use what is called a Venn Diagram to represent probabilities in an intuitive, graphical way. The idea is that probabilities are represented as the *fractional area* of simple geometric shapes. We can then find a picture representation of each of the rules of probability. We start by looking at a sample Venn Diagram, in Figure 1.2.

The fractional area of the rectangle A represents the probability $P(A)$, and can be thought of as a probability of one of the statements we've explored, such as $P(\text{Heart})$. This diagram is strictly a mnemonic, because the individual points on the diagram are not properly defined. The diagram in Figure 1.2 also represents the Negation Rule (Equation 1.7),

$$P(A) + P(\text{not } A) = 1$$

In the diagram it is easy to see that the sum of the areas inside of A (i.e. $1/4$) and outside of A (i.e. $3/4$) cover the entire area of the *Universe* of statements, and thus add up to 1.

Figure 1.3 shows the diagram which can help us remember the sum and product rules. The Sum Rule (Equation 1.10)

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

is represented in the total area occupied by the rectangles A and B , and makes up all of A (i.e. $1/4$) and the half of B sticking out (i.e. $1/8 - 1/16 = 1/16$) yielding $P(A \text{ or } B) = 5/16$. This is also the area of each added up ($1/4 + 1/8$), but subtracting the intersection ($1/16$) because otherwise it is counted twice. Adding the areas this way directly parallels the Sum Rule.

Conditional probabilities, like those that come into the Product Rule (Equation 1.8) and Bayes Rule (Equation 1.14) are a little more challenging to visualize. In Figure 1.4, $P(A|B)$ is represented by the fraction of the darker area (which was originally part of A) compared not to the *Universe* but to the area of B , and thus represents $P(A|B) = 1/2$. In a way, it is as if the conditional symbol, “|,” defines the *Universe* with which to make the comparisons. On the left of Figure 1.4, the same darker area that was originally part of B represents $P(B|A)$ making up $1/4$ of the area of A . Thus $P(B|A) = 1/4$. The Product Rule (Equation 1.8) then follows,

$$P(A \text{ and } B) = \underbrace{P(A|B)}_{1/2} \underbrace{P(B)}_{1/8} = \underbrace{P(B|A)}_{1/4} \underbrace{P(A)}_{1/4} = \frac{1}{16}$$

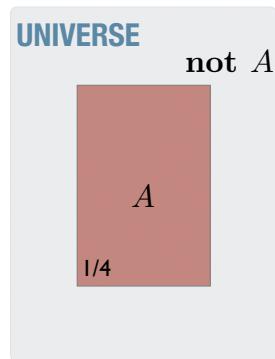


Figure 1.2: Venn diagram of a statement, A , in a *Universe* of all possible statements. It is customary to think of the area of the *Universe* to be equal to 1 so that we can treat the actual areas as fractional areas representing the probability of statements like $P(A)$. In this image, A takes up $1/4$ of the *Universe*, so that $P(A) = 1/4$. Also shown is the negation rule. $P(A) + P(\text{not } A) = 1$ or “inside” of A + “outside” of A adds up to everything.

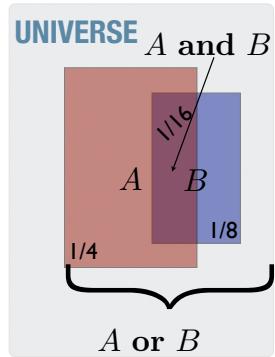


Figure 1.3: Venn diagram of the sum and product. The rectangle B takes up $1/8$ of the *Universe*, and the rectangle A takes up $1/4$ of the *Universe*. Their overlap here is $1/16$ of the *Universe*, and represents $P(A \text{ and } B)$. Their total area of $5/16$ of the *Universe* represents $P(A \text{ or } B)$.

We can further see the special case of mutually exclusive statements shown in Figure 1.5. The Sum Rule for Exclusive Events (Equation 1.11) is simply the sum of the two areas because there is no overlap

$$P(A \text{ or } B) = P(A) + P(B)$$

Further, it is straightforward to see from this diagram the following properties for mutually exclusive events

$$\begin{aligned} P(A \text{ and } B) &= 0 \\ P(A|B) &= 0 \\ P(B|A) &= 0 \end{aligned}$$

1.6 Lessons from Bayes' Rule - A First Look

Bayes' Rule is the gold standard for all statistical inference. It is a mathematical theorem, proven from fundamental principles. It structures all inference in a systematic fashion. However, it can be used without doing any calculations, as a guide to qualitative inference. Some of the lessons which are consequences of Bayes' Rule are listed here, and will be noted throughout this text in various examples.

- Confidence in a claim should scale with the evidence for that claim
- Ockham's razor, which is the philosophical idea that simpler theories are preferred, is a consequence of Bayes' Rule when comparing models of differing complexity.
- Simpler means fewer adjustable parameters
- Simpler also means that the predictions are both *specific* and not *overly plastic*. For example, a hypothesis which is consistent with the observed data, and also be consistent if the data were the opposite would be overly plastic.
- Your inference is only as good as the hypotheses (i.e. models) that you consider.
- Extraordinary claims require extraordinary evidence.⁷
- It is better to explicitly display your assumptions rather than implicitly hold them.
- It is a *good thing* to update your beliefs when you receive new information.
- Not all uncertainties are the same.

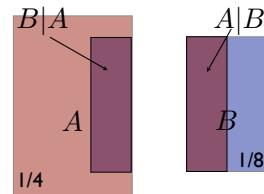


Figure 1.4: Venn diagram of conditional probabilities, $P(A|B)$ and $P(B|A)$. (Right) $P(A|B)$ is represented by the fraction of the darker area (which was originally part of A) compared not to the *Universe* but to the area of B , and thus represents $P(A|B) = 1/2$. In a way, it is as if the conditional symbol, “|,” defines the *Universe* with which to make the comparisons. (Left) Likewise, the same darker area that was originally part of B represents $P(B|A)$ which makes up $1/4$ of the area of A . Thus $P(B|A) = 1/4$.

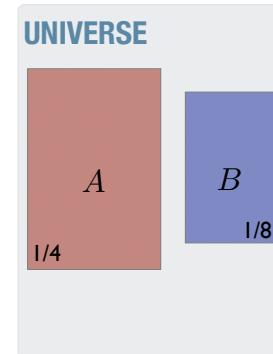


Figure 1.5: Venn diagram of mutually exclusive statements. One can see that $P(A \text{ and } B) = 0$ (the overlap is zero) and $P(A \text{ or } B) = P(A) + P(B)$ (the total area is just the sum of the two areas)

⁷ Carl Sagan. *Demon-Haunted World: Science as a Candle in the Dark*. Random House LLC, 1996

There is not a universal agreement for the translation of numerical probability values to qualitative terms in English (i.e. highly unlikely, somewhat unlikely, etc...). One rough guide is shown in Table 1.1. I will be following this convention throughout the book, but realize that the specific probability distinctions are a bit arbitrary.

term	probability
virtually impossible	$1/1,000,000$
extremely unlikely	0.01 (i.e. $1/100$)
very unlikely	0.05 (i.e. $1/20$)
unlikely	0.2 (i.e. $1/5$)
slightly unlikely	0.4 (i.e. $2/5$)
even odds	0.5 (i.e. 50-50)
slightly likely	0.6 (i.e. $3/5$)
likely	0.8 (i.e. $4/5$)
very likely	0.95 (i.e. $19/20$)
extremely likely	0.99 (i.e. $99/100$)
virtually certain	$999,999/1,000,000$

Table 1.1: Rough guide for the conversion of qualitative labels to probability values.

2 Introduction to Model Comparison

A *model*¹ as we use the term in this book is a *specific description of a possible state of nature*. This is in contrast to an *actual state of nature*, which we practically never have access to. We can never know *anything* with 100% certainty, and must therefore be open to alternate possible explanations, or models, describing our observations. For example, in medicine such models could include “I have lung cancer,” “I have pneumonia,” and “I have a cold.” In physics, models could include “the Earth moves around the Sun” and “the Sun moves around the Earth.” We can imagine many possible models that are consistent with the observed data, and our job in doing statistical inference is to determine the probabilities of our models given the data we observe. In our notation, what we are always looking for is

$$P(\text{model}|\text{data}) \tag{2.1}$$

We will explore model comparison through a series of examples.

2.1 The High/Low Deck Game

In this example we use a simple card game as a platform for discussing model comparison in general. We start with two atypical decks of cards called the High Deck and the Low Deck (Figures 2.1 on page 46 and 2.2 on page 46 respectively). The game goes as follows.

You’re handed one of the two decks, but you don’t know which. First, you draw the top card and note the value. Second, you replace the card and *reshuffle the deck*². You repeat this procedure of drawing, noting, and reshuffling for as many turns as you need. The goal is to determine which of the two decks (High or Low) you are in fact holding in your hand.

What does our intuition say?

We start by exploring our intuitions, before we do anything mathematically. Thus, we are in a position to check to see if the math is

¹ A similar term is *hypothesis*, and model comparison would then be *hypothesis testing*. We don’t choose to use that term, partly because of the colloquial use of hypothesis as a kind of “guess,” but also because hypothesis testing in some treatments focus on true/false tests of hypotheses which can lead to some significant misunderstandings. The use of models implies the possibility of multiple (i.e. more than two) models.

² Although we could make a game without replacement, which may be simpler to implement, the version of the game with reshuffling will help with an example later.

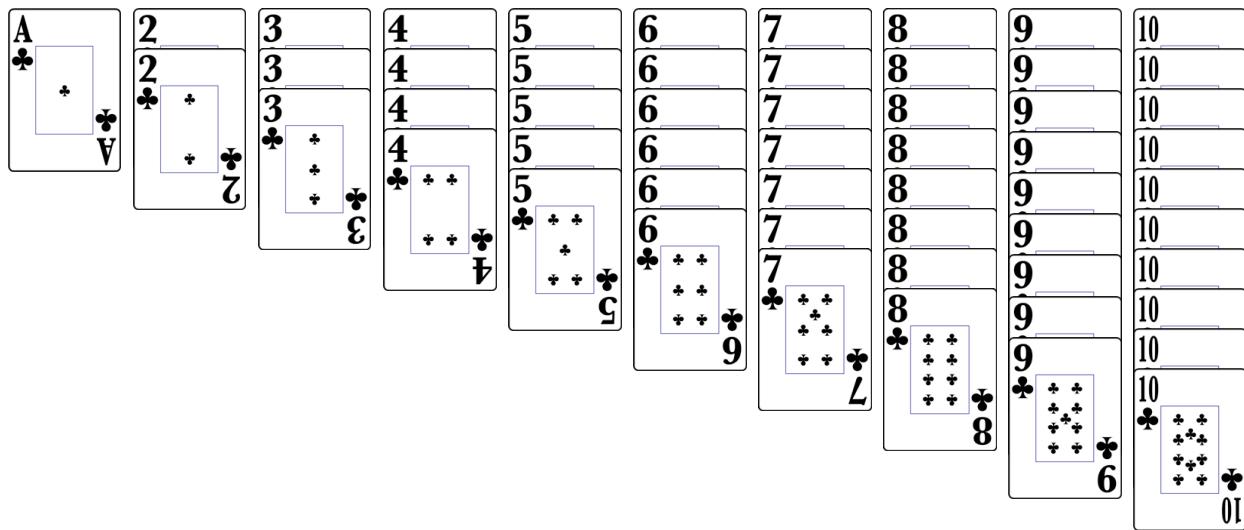


Figure 2.1: High Deck - 55 Cards with ten 10's, nine 9's, etc... down to one Ace. Aces are equivalent to the value 1.

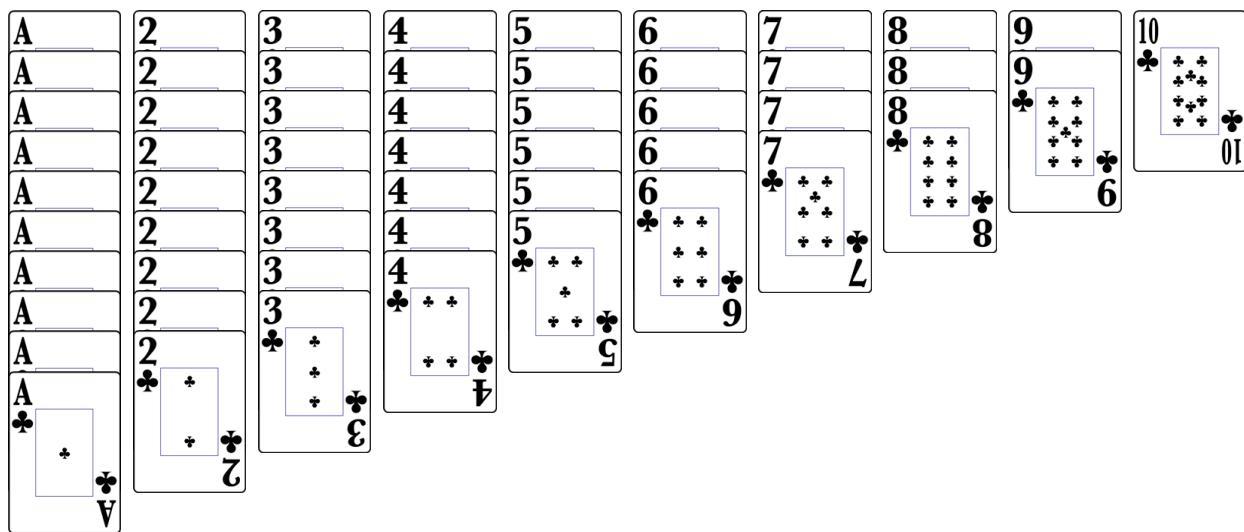


Figure 2.2: Low Deck - 55 Cards with ten Aces, two 2's, etc... up to one 10. Aces are equivalent to the value 1.

reasonable before we use the same math in areas where our intuition is not as strong. Imagine we draw only one card, and it is a 9. Intuition suggests that this constitutes reasonably strong evidence toward the belief that we're holding the High Deck. If we then (as the procedure states) place the 9 back in the deck, reshuffle and then draw a 7 we can be more strongly convinced that we are holding the High Deck. Repeating the reshuffle, and then drawing a 3 would make us a little less confident in this conclusion, but still quite certain. In this way we can sense how drawing different cards pushes our belief around, depending on how often that card comes up in the different decks.

Before the data - the prior

Before we take any data, we need to quantify our state of knowledge concerning all of the models that we are considering. In this case it is quite simple, because there are two models (High Deck and Low Deck), and we have been given no information about whether either is more common. With no such information, it is equivalent to a coin flip - we assign equal probabilities to both models *before* we see data, also known as the *prior* probabilities³.

$$\begin{aligned} P(H) &= 0.5 \\ P(L) &= 0.5 \end{aligned}$$

Surely this assessment will change *after* we see data, but that is the rest of the problem.

The "easy" question - the likelihood

Although our ultimate goal is to infer the type of deck from the cards that we draw from it, we can start looking at an easier part of this question which serves as a first step toward the more challenging, and interesting goal. That question is the following,

EXAMPLE 2.1 *What is the probability of drawing a 9, given that we know that we're holding the High Deck?*

This related question is written

$$P(\text{data} = 9|H)$$

where $\text{data} = 9$ means that we have observed (i.e. drawn) one 9. This question is “easy” in the sense that it is simply related to the properties of the High Deck: the number of 9’s and total number of cards. If you know that you have the high deck, then you know there are nine 9’s in that deck out of 55 cards, and thus we have the

³ The prior is sometimes mischaracterized as simply our guess, or some other completely subjective assessment of our knowledge. In fact in this example, and many others, we can make the *positive* case for equal probabilities given the state of our knowledge. This can be quantified with the concept of entropy, which is beyond this chapter.

probability of drawing one 9, given that we are holding the High Deck, is

$$P(\text{data} = 9|H) = \frac{9}{55}$$

We give this the name *likelihood*⁴, and is simply the probability that the data could be the result of a known model. It is also the first part of the top of Bayes' Rule, Equation 1.14 on page 39.

Applying the Bayes' recipe

Here we introduce for the first time a recipe we will follow for all model comparison examples.

Now that we have our intuition, and we have the likelihoods, we can address the math. The two models are:

$$\begin{aligned} H &\equiv \text{"We're holding the High Deck"} \\ L &\equiv \text{"We're holding the Low Deck"} \end{aligned}$$

and the initial data is

$$\text{data} \equiv \text{"We've drawn one card, and it is a 9"}$$

According to Equation 2.1 on page 45 we are looking for the two probabilities:

$$\begin{aligned} P(H|\text{data} = 9) \\ P(L|\text{data} = 9) \end{aligned}$$

which are related to the *prior* and the *likelihood* via Bayes' Rule (Equation 1.14):

$$\begin{aligned} P(H|\text{data} = 9) &= \frac{P(\text{data} = 9|H)P(H)}{P(\text{data} = 9)} \\ P(L|\text{data} = 9) &= \frac{P(\text{data} = 9|L)P(L)}{P(\text{data} = 9)} \end{aligned}$$

To calculate actual numbers, we apply the Bayes' Recipe to this problem,

1 Specify the prior probabilities for the models being considered

$$\begin{aligned} P(H) &= 0.5 \\ P(L) &= 0.5 \end{aligned}$$

2 Write the top of Bayes' Rule for all models being considered

$$\begin{aligned} P(H|\text{data} = 9) &\sim P(\text{data} = 9|H)P(H) \\ P(L|\text{data} = 9) &\sim P(\text{data} = 9|L)P(L) \end{aligned}$$

⁴ The term *likelihood* is a poorly chosen word. In English, this word is nearly synonymous with the word *probability* and thus easily leads to confusion. We could try to use a different term, like *consequent probability* or *generative likelihood* to stress the idea that the *likelihood* is the probability that the data we observe could be generated or could be a consequence of the particular model. However, we'd be going up against two centuries of continued use of the term *likelihood* and thus would probably increase confusion rather than decrease it.

where we are using the symbol \sim to denote *proportionality* or *related to*. Essentially, by calculating the top of Bayes' Rule first, the numbers are not *equal* to the final (i.e. posterior) probabilities but must be rescaled to make sure that they add up to 1. This is done in the final step. Up until that rescaling, we use the symbol \sim and think of it as *related to*.

3 Put in the likelihood and prior values

$$\begin{aligned} P(H|\text{data} = 9) &\sim \frac{9}{55} \times 0.5 = 0.082 \\ P(L|\text{data} = 9) &\sim \frac{2}{55} \times 0.5 = 0.018 \end{aligned}$$

4 Add these values for all models

$$K = 0.082 + 0.018 = 0.1$$

5 Divide each of the values by this sum, K , to get the final probabilities

$$\begin{aligned} P(H|\text{data}) &= 0.082/0.1 = 0.82 \\ P(L|\text{data}) &= 0.018/0.1 = 0.18 \end{aligned}$$

From which we can conclude that drawing a 9 does indeed constitute reasonably strong evidence toward the belief that we're holding the High Deck - the probability of us holding the High Deck, given the data, is 0.82.

Drawing the next card

So, when we draw a 7 next (after reshuffling), our intuition suggests that we'd be more confident that we're holding the High Deck. Repeating our recipe we have

The two models are:

$$\begin{aligned} H &\equiv \text{"We're holding the High Deck"} \\ L &\equiv \text{"We're holding the Low Deck"} \end{aligned}$$

and data is

$$\text{data} \equiv \text{"We've drawn one card, and it is a 9, replaced and reshuffled, and then drawn a 7"}$$

According to Equation 2.1 we are looking for the two probabilities:

$$\begin{aligned} P(H|\text{data} = 9 \text{ then a 7}) \\ P(L|\text{data} = 9 \text{ then a 7}) \end{aligned}$$

which are related to the *prior* and the *likelihood* via Bayes' Rule (Equation 1.14):

$$\begin{aligned} P(H|\text{data} = 9 \text{ then a } 7) &= \frac{P(\text{data} = 9 \text{ then a } 7|H)P(H)}{P(\text{data} = 9 \text{ then a } 7)} \\ P(L|\text{data} = 9 \text{ then a } 7) &= \frac{P(\text{data} = 9 \text{ then a } 7|L)P(L)}{P(\text{data} = 9 \text{ then a } 7)} \end{aligned}$$

To calculate actual numbers, we apply the Bayes' recipe to this problem,

- 1 Specify the prior probabilities for the models being considered

$$\begin{aligned} P(H) &= 0.5 \\ P(L) &= 0.5 \end{aligned}$$

- 2 Write the top of Bayes' Rule for all models being considered

$$\begin{aligned} P(H|\text{data} = 9 \text{ then a } 7) &\sim P(\text{data} = 9 \text{ then a } 7|H)P(H) \\ P(L|\text{data} = 9 \text{ then a } 7) &\sim P(\text{data} = 9 \text{ then a } 7|L)P(L) \end{aligned}$$

- 3 Put in the likelihood and prior values

$$\begin{aligned} P(H|\text{data} = 9 \text{ then a } 7) &\sim \frac{9}{55} \times \frac{7}{55} \times 0.5 = 0.0104 \\ P(L|\text{data} = 9 \text{ then a } 7) &\sim \frac{2}{55} \times \frac{4}{55} \times 0.5 = 0.0013 \end{aligned}$$

- 4 Add these values for all models

$$K = 0.0104 + 0.0013 = 0.0117$$

- 5 Divide each of the values by this sum, K , to get the final probabilities

$$\begin{aligned} P(H|\text{data} = 9 \text{ then a } 7) &= 0.0104/0.0117 = 0.889 \\ P(L|\text{data} = 9 \text{ then a } 7) &= 0.0013/0.0117 = 0.111 \end{aligned}$$

which again matches our intuition - we're more confident that we're holding the High Deck, now with probability 0.889 increased from 0.82 when we just observed the 9.

Prior information or not?

In the above example, we started with a prior probability of holding the High Deck at $P(H) = 0.5$, because we had no information other than that there were two possibilities. We then observed a 9, and updated the probability to 0.82, and then observed a 7, and further

updated the probability to 0.889 - making it more likely that we were holding the High Deck. One of the basic tenets of probability theory is that if there is more than one way to arrive at an answer, one should arrive at the same answer.⁵ In the above, we calculated the probability of holding the High Deck given the observed data

data \equiv "We've drawn one card, and it is a 9, replaced and reshuffled, and then drawn a 7"

and prior information

prior \equiv "We know there are only two decks."

⁵ E. T. Jaynes uses the principle that "if there is more than one way to arrive at an answer, one should arrive at the same answer" to help derive the rules of probability from first principles. Failures of this principle result in paradoxes. This principle is also applied in Section ?? for the birthday problem.

An equivalent situation is found after our first draw, after we've observed the 9, and we're about to draw our second card. In this case we have the *prior* information:

prior \equiv "We know there are only two decks, and then we draw one card and it is a 9, replace it and reshuffle."

and observed data:

data \equiv "We've drawn one card and it is a 7"

Mathematically, we apply the Bayes' recipe, but with the different prior information

1 Specify the prior probabilities for the models being considered

$$P(H, 9) = 0.82$$

$$P(L, 9) = 0.18$$

2 Write the top of Bayes' Rule for all models being considered

$$P(H|\text{data} = 9 \text{ then a } 7) \sim P(\text{data} = \{7\}|H)P(H, 9)$$

$$P(L|\text{data} = 9 \text{ then a } 7) \sim P(\text{data} = \{7\}|L)P(L, 9)$$

3 Put in the likelihood and prior values

$$P(H|\text{data} = 9 \text{ then a } 7) \sim \frac{7}{55} \times 0.82 = 0.104$$

$$P(L|\text{data} = 9 \text{ then a } 7) \sim \frac{4}{55} \times 0.18 = 0.013$$

- 4 Add these values for all models

$$K = 0.104 + 0.013 = 0.117$$

- 5 Divide each of the values by this sum, K , to get the final probabilities

$$P(H|\text{data} = 9 \text{ then a } 7) = 0.104/0.117 = 0.889$$

$$P(L|\text{data} = 9 \text{ then a } 7) = 0.013/0.117 = 0.111$$

yielding the same result.

In other words our *updated probabilities* from the first draw can be seen as our prior probabilities for the subsequent draws. Thus, Bayes' Rule describes how we update our knowledge with new evidence, or in other words, *learning*.

2.2 Multiple Hypotheses

We start this section with an example.

EXAMPLE 2.2 *What is the probability that you are holding one of either the High or the Low Deck having drawn five 9's in a row from that deck?*

We have observed the following data:

$$\text{data} \equiv \begin{cases} \text{"We've drawn one card, and it is a 9, replaced and reshuffled, redrawn and observed another 9, repeated this procedure and observed three more 9's, for a total of five 9's in a row."} \end{cases}$$

Technically, drawing 5 9's in a row should give us really strong confidence that you are drawing from the High Deck, because we would have

- 1 Specify the prior probabilities for the models being considered

$$P(H) = 0.5$$

$$P(L) = 0.5$$

- 2 Write the top of Bayes' Rule for all models being considered

$$P(H|\text{data} = 5 \text{ 9's in a row}) \sim P(\text{data} = 5 \text{ 9's in a row}|H)P(H)$$

$$P(L|\text{data} = 5 \text{ 9's in a row}) \sim P(\text{data} = 5 \text{ 9's in a row}|L)P(L)$$

- 3 Put in the likelihood and prior values

$$P(H|\text{data} = 5 \text{ 9's in a row}) \sim \underbrace{\frac{9}{55} \times \frac{9}{55} \cdots \frac{9}{55}}_{5 \text{ times}} \times P(H)$$

$$\begin{aligned}
 P(L|\text{data} = 5 \text{ 9's in a row}) &\sim \left(\frac{9}{55}\right)^5 \times 0.5 \\
 &= 0.0000587 \\
 &\sim \left(\frac{2}{55}\right)^5 \times 0.5 \\
 &= 0.0000000318
 \end{aligned}$$

4 Add these values for all models

$$K = 0.0000587 + 0.0000000318 = 0.0000587318$$

5 Divide each of the values by this sum, K , to get the final probabilities

$$\begin{aligned}
 P(H|\text{data} = 5 \text{ 9's in a row}) &= \frac{0.0000587}{0.0000587318} = 0.99946 \\
 P(L|\text{data} = 5 \text{ 9's in a row}) &= \frac{0.0000000318}{0.0000587318} = 0.00054
 \end{aligned}$$

which is *fantastically* on the side of the high deck, even though we might start getting suspicious in this situation.

EXAMPLE 2.3 What is the probability that you are holding one of either the High or the Low Deck having drawn m 9's in a row from that deck?

In general, if we look at m 9's in a row, where m could be 1, 2, 3, etc..., we can see this following the Bayes' Recipe

1 Specify the prior probabilities for the models being considered

$$\begin{aligned}
 P(H) &= 0.5 \\
 P(L) &= 0.5
 \end{aligned}$$

2 Write the top of Bayes' Rule for all models being considered

$$\begin{aligned}
 P(H|\text{data} = m \text{ 9's in a row}) &\sim P(\text{data} = m \text{ 9's in a row}|H)P(H) \\
 P(L|\text{data} = m \text{ 9's in a row}) &\sim P(\text{data} = m \text{ 9's in a row}|L)P(L)
 \end{aligned}$$

3 Put in the likelihood and prior values

$$\begin{aligned}
 P(H|\text{data} = m \text{ 9's in a row}) &\sim \underbrace{\frac{9}{55} \times \frac{9}{55} \cdots \frac{9}{55}}_{m \text{ times}} \times P(H) \\
 &\sim \left(\frac{9}{55}\right)^m \times 0.5 \\
 P(L|\text{data} = m \text{ 9's in a row}) &\sim \left(\frac{2}{55}\right)^m \times 0.5
 \end{aligned}$$

4 Add these values for all models

$$K = \left(\frac{9}{55}\right)^m \times 0.5 + \left(\frac{2}{55}\right)^m \times 0.5$$

5 Divide each of the values by this sum, K , to get the final probabilities. This step is easiest done in a table (Table 2.1), because the resulting expression is pretty messy.

m	$P(H \text{data})$	$P(L \text{data})$
1	0.81818	0.18182
2	0.95294	0.047059
3	0.98915	0.010855
4	0.99757	0.0024327
5	0.99946	0.00054163
6	0.99988	0.00012041
7	0.99997	0.000026761
8	0.99999	0.0000059470

Table 2.1: Drawing m 9's in a row, from either a High Deck or Low Deck.

It is clear from Table 2.1 that after drawing five 9's using our procedure, it should be *extraordinarily* likely that we are holding the High Deck. However, after a certain number of 9's observed, something starts to bother us. Perhaps not after five 9's, but what if the procedure were repeated and we drew ten 9's in a row? Or perhaps twenty 9's. At some point, we'd refuse to believe this is the High Deck because, although it was true that there are more 9's in the High Deck, there are many more *other* cards in the High Deck that we should see. What do we do in this case?

EXAMPLE 2.4 *What is the probability that you are holding one of either the High, Low, or Nines Deck having drawn m 9's in a row from that deck?*

The proper thing to do is to introduce a new model, say, a Nines deck. Clearly this model should have a very low prior probability, because we didn't even consider it before we saw the streak of 9's. Let's say that we assign the prior probability for the Nines deck to be a one in a million. To make all of the prior probabilities add up to 1, then the prior probabilities for the High and Low Deck must be a little less than 0.5. After that, we simply apply the Bayes' Recipe as before

1 Specify the prior probabilities for the models being considered

$$\begin{aligned} P(N) &= \frac{1}{1,000,000} = 0.000001 \\ P(H) &= 0.4999995 \\ P(L) &= 0.4999995 \end{aligned}$$

What is interesting here is that once we admit that there are many possible models we could consider, we realize that we have these models in our head all the time, or we construct them as we need them. Every model comparison is a multiple model comparison, with most of the models with very low prior probabilities that our brain naturally suppresses until needed. Mathematically, we need to unsuppress them as needed.

2 Write the top of Bayes' Rule for all models being considered

$$\begin{aligned} P(N|\text{data} = m \text{ 9's in a row}) &\sim P(\text{data} = m \text{ 9's in a row}|N)P(N) \\ P(H|\text{data} = m \text{ 9's in a row}) &\sim P(\text{data} = m \text{ 9's in a row}|H)P(H) \\ P(L|\text{data} = m \text{ 9's in a row}) &\sim P(\text{data} = m \text{ 9's in a row}|L)P(L) \end{aligned}$$

3 Put in the likelihood and prior values

$$\begin{aligned} P(N|\text{data} = m \text{ 9's in a row}) &\sim 1 \times P(N) = 0.000001 \\ P(H|\text{data} = m \text{ 9's in a row}) &\sim \underbrace{\frac{9}{55} \times \frac{9}{55} \cdots \frac{9}{55}}_{m \text{ times}} \times P(H) \\ &\sim \left(\frac{9}{55}\right)^m \times 0.4999995 \\ P(L|\text{data} = m \text{ 9's in a row}) &\sim \left(\frac{2}{55}\right)^m \times 0.4999995 \end{aligned}$$

4 Add these values for all models

$$K = 0.000001 + \left(\frac{9}{55}\right)^m \times 0.4999995 + \left(\frac{2}{55}\right)^m \times 0.4999995$$

5 Divide each of the values by this sum, K , to get the final probabilities Again, this step is easiest done in a table or, even better, a picture (Figure 2.3).

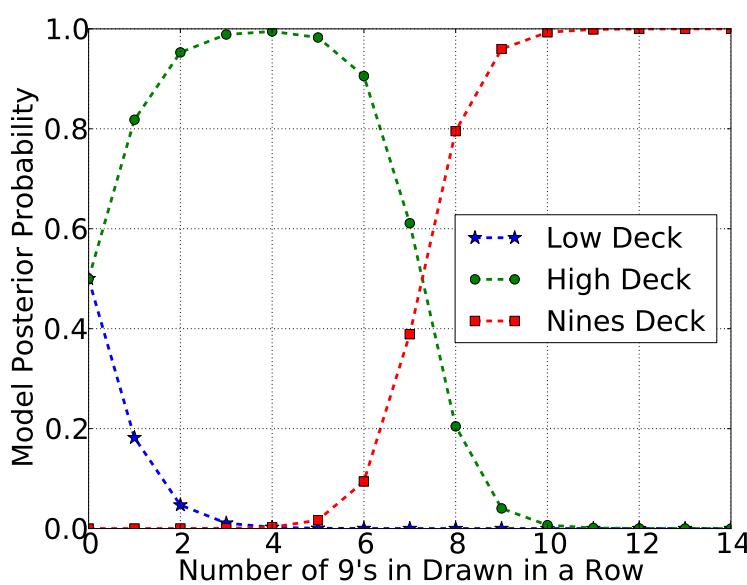


Figure 2.3: Drawing a number of 9's in a row, possibly from a High, Low, and Nines deck.

We have a clear picture here in Figure 2.3. As we initially draw 9's, our confidence that we're holding the High Deck goes up, at the expense of our confidence that we're holding the Low Deck. At a certain point (around six 9's in our example), our confidence in the High Deck starts to drop and we become more confident that something odd is happening, and our previously ignored model of the Nines deck becomes more likely. Eventually, this new model is the one in which we are the most confident.

Imagine further that if, after drawing ten 9's in a row we draw a 1. What do we do then? The likelihood for the Nines deck goes to zero instantly - the probability of drawing a 1 from a Nines deck is zero, $P(1|N) = 0$. Are we left again with the original two models, High and Low Deck? No! We would then introduce other models, perhaps something like a Mostly Nines Deck, or perhaps a High Deck with a weird shuffling procedure, or perhaps others. No matter how many models one has, the recipe is still the same. It is important to realize that in any model comparison case, there are always other models that could be brought to bear on the problem, perhaps with low prior probability. Simply showing that a model is consistent with a set of data does not insure against the possibility that another model could be better, if we could only think of it.

EXERCISE 2.1 Complete the example demonstrating the updated probabilities for the High and Low Deck, having drawn a 9, 7, and a 3. Compare with the case of drawing just the 9 and the 7, and discuss how it matches your intuition.

EXERCISE 2.2 Repeat the analysis of the sequence of 9's drawn in a row with an added hypothesis of a deck with one hundred 9's and one 8. Discuss the results. Demonstrate what happens to the probabilities for all of the hypotheses after drawing one 8, after ten 9's in a row. Discuss.

EXERCISE 2.3 I tell you that I have a coin that could have both sides heads, both sides tails, or a normal single-heads single-tails coin.

- 1 Before seeing the data, what would be a reasonable prior probability for the three hypotheses H_0 (no-heads), H_1 (one head), and H_2 (two heads)?
- 2 Would this have been different if you had simply been given a coin by a friend to flip to see who has to do the dishes? Why or why not?
- 3 Now I flip the coin once, and get a heads. Write down the likelihood of this data given each of the models. In other words, what are the values of:
 - $P(\text{data}=\text{1 heads}|H_0)$
 - $P(\text{data}=\text{1 heads}|H_1)$
 - $P(\text{data}=\text{1 heads}|H_2)$

The creative part of science is not in the calculations performed, but in the generation of new and useful models. Until we come up with a better model for our data we make do with the ones that we have, all the while being aware that a better model may come into play later. Newton's Theory of Gravity was used for over 200 years, even when there was known data that made it less likely, until it was replaced by Einstein's Theory of Gravity. Newton's Laws, however, are still used in nearly all gravitational calculations because it is "good enough" and is a lot easier to work with practically.

4 Apply Bayes' Recipe, and determine the probability of each of these three models given this data. In other words, what are the values of:

- $P(H_0|data=1\ heads)$
- $P(H_1|data=1\ heads)$
- $P(H_2|data=1\ heads)$

5 Apply this recipe for the case of observing 3 heads in a row.

3 Applications of Model Comparison

This chapter presents several applications of the model comparison concepts introduced in Chapter 2 (*Introduction to Model Comparison*).

3.1 Disease Testing

Let's imagine there is a rare, one in a million, disease that is lethal but does not have many outward symptoms at first. A new test boasts 99.9% accuracy, so you go to get tested, and receive the bad news that you test positive for the disease. Should you be devastated by the news? What is the probability that you *actually* have the disease? We are looking at two, quite different, probabilities here. In the first case, we have the claims of the test which state that *if you have the disease, the probability that the test will be positive is 0.999*, or, *if you have the disease, test will discover that fact 99.9% of the time*. In the second case we have your concern which is, *if you test positive for the test, what is the probability that you have the disease*. In our notation this is:

$$P(\text{positive test}|\text{disease}) = 0.999 \text{ (claim from test)}$$

$$P(\text{disease}|\text{positive test}) = ? \text{ (your concern)}$$

These two are related by Bayes' Rule (Equation 1.14).

The Bayes' Recipe proceeds as follows

- 1 Specify the prior probabilities for the models being considered

The models we have are simply "have the disease" and "don't have the disease". The prior probabilities for these two come from the prevalence of the disease in the population, before you get tested. Since this is a "one in a million" disease, we have

$$\begin{aligned} P(\text{disease}) &= \frac{1}{1,000,000} \\ P(\text{no disease}) &= \frac{999,999}{1,000,000} \end{aligned}$$

- 2 Write the top of Bayes' Rule for all models being considered

The top of Bayes' Rule comes down to, given the truth of the model (i.e. either with or without the disease), what is the probability of getting the data (i.e. the positive or negative test result). This is measured by how good the test is.

$$P(\text{positive test}|\text{disease}) = 0.999$$

and

$$P(\text{positive test}|\text{no disease}) = 0.001$$

So the top of Bayes' Rule looks for both models looks like:

$$\begin{aligned} P(\text{disease}|\text{positive test}) &\sim P(\text{positive test}|\text{disease}) \times P(\text{disease}) \\ &\sim 0.999 \times \frac{1}{1,000,000} = 9.99 \cdot 10^{-7} \\ P(\text{no disease}|\text{positive test}) &\sim P(\text{positive test}|\text{no disease}) \times P(\text{no disease}) \\ &\sim 0.001 \times \frac{999,999}{1,000,000} = 9.99 \cdot 10^{-4} \end{aligned}$$

3 Add these values for all models

$$K = 9.99 \cdot 10^{-7} + 9.99 \cdot 10^{-4} = 0.000999999$$

4 Divide each of the values by this sum, K , to get the final probabilities

$$\begin{aligned} P(\text{disease}|\text{positive test}) &= \frac{9.99 \cdot 10^{-7}}{0.000999999} = 0.1\% \\ P(\text{no disease}|\text{positive test}) &= 99.9\% \end{aligned}$$

Which means that, *overwhelmingly*, if you have a rare one-in-a-million disease, you are very unlikely to have it *even given a 99.9% accurate positive test for it!* This is a seriously unintuitive result, so it is helpful to visualize it in another way to build your intuition.

One way to see this result is to visualize it, as in Figure 3.1. Here, the numbers are a bit smaller - the disease is 1 out of 200 in a population of 3000, and the test is 99% accurate. This means about 15 sick people and about 2985 healthy people. If all of the sick people test positive, and 1% of the healthy people test positive due to the 99% accuracy, we would have 15 sick and 29 healthy people who all test positive. Even in this case, with much smaller numbers, we see that getting a positive test alone does not imply that it is likely you have the disease. It depends on the rarity of the disease (the more rare, the

In many medical applications, the false positive rate ($P(\text{positive test}|\text{no disease})$) is not always equal to the false negative rate ($P(\text{negative test}|\text{disease})$), so to say that a test is 99.9% accurate is actually incomplete - one needs to specify both rates of effectiveness. In this case, we are assuming that they are the same.

less likely) and the false positive rate (the number of healthy people who test positive anyway). This will vary depending on the disease and the test, but can lead to this unintuitive result, and thus can lead one to make poor medical decisions.

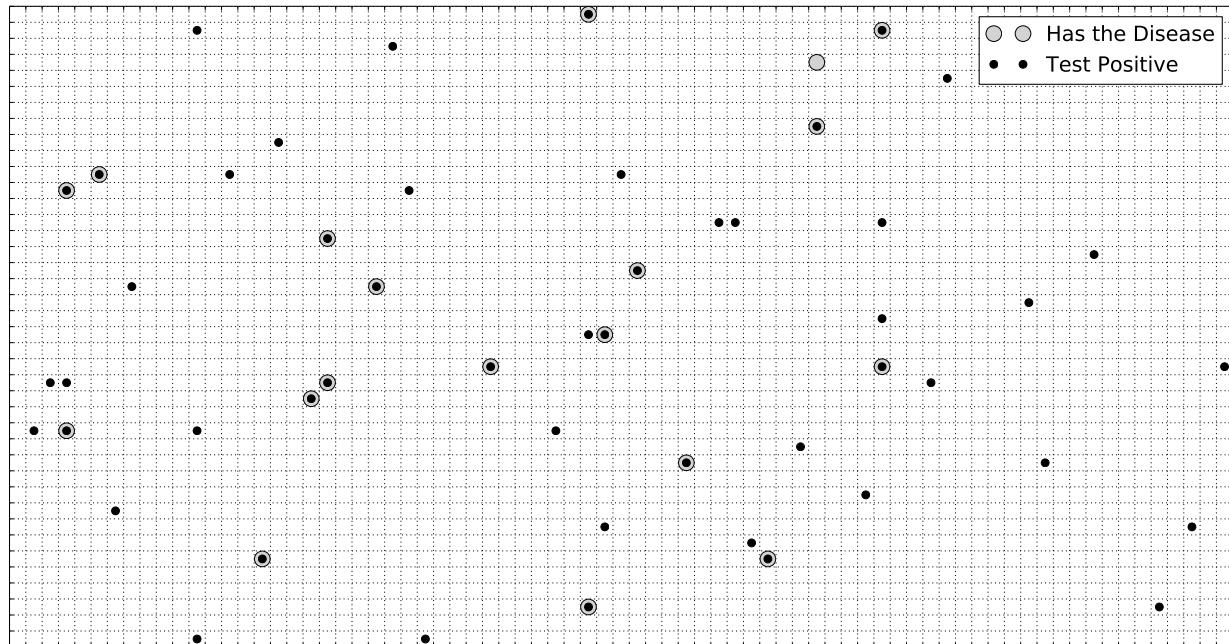


Figure 3.1: Rare disease and testing. Shown is a population of 3000 where 1 in every 200 people have the disease (large circles). A test which is 99% effective is applied to everyone in the population, and the positive test results (i.e. the test says that you have the disease) are shown as small black dots. Notice that although nearly all of those that have the disease test positive (a small black dot inside a large circle), there are many false positives (black dot in an empty square) - healthy people that test positive for the disease. Even though the test is quite good, there are many more healthy people and 1 out of 100 of them will erroneously test positive.

Consequences

This sort of disease testing has serious consequences, especially for rare diseases with tests that aren't precise. In the book "The Theory That Would Not Die: How Bayes' Rule Cracked the Enigma Code, Hunted Down Russian Submarines, and Emerged Triumphant from Two Centuries of Controversy" by Sharon McGayne there is a discussion concerning the 2009 advice from the U.S. government task force that "most women in their forties *not* to have annual mammograms." (emphasis mine) According to McGayne,

Thus the probability that a woman who tests positive has breast cancer is only 3%. She has 97 chances out of 100 to be disease free. None of this is static. Each time more research data become available, Bayes' rule should be recalculated. As far as Bayes is concerned, universal screening for a disease that affects only 4/10 of 1% of the population may subject many healthy women to needless worry and to additional treatment which in turn can cause its own medical problems. In addition, the money spent on universal screening could potentially be used for other worthwhile projects. Thus Bayes highlights the importance of improv-

ing breast cancer screening techniques and reducing the number of false positives.¹ (emphasis mine)

Thus the proper application of probability theory allows us to separate true but unintuitive things from others which only seem true and intuitive but are in fact false.

3.2 Monty Hall Problem

This problem was introduced in Section ??.

EXAMPLE 3.1 *Is it better to switch doors? - Monty Hall Problem revisited*

You may recall that we were presented with a choice of 3 doors where a car is behind one and goats behind the others. Having picked one, the host opens up a door with a goat, and offers you the opportunity to change your answer. In order to assess the probabilities, we must remember that

- 1 the host *never* opens your door
- 2 the host *always* opens a door with a goat

We'll go through a specific example, that of you choosing door 1 and the host opening door 2. The analysis proceeds in identical ways for the other possibilities. We apply the Bayes' Recipe, where the models under consideration are

- "car behind door 1"
- "car behind door 2"
- "car behind door 3"

The Bayes' Recipe proceeds as follows

- 1 Specify the prior probabilities for the models being considered

$$P(\text{car 1|you 1}) = 0.333$$

$$P(\text{car 2|you 1}) = 0.333$$

$$P(\text{car 3|you 1}) = 0.333$$

where, for example, $P(\text{car 1|you 1})$ represents the probability that the door contains the car given that you chose door 1. Since your choice of door doesn't add any information about the location of the car, all of the probabilities are equal.

- 2 Write the top of Bayes' Rule for all models being considered

$$P(\text{car 1|you 1, host 2}) \sim P(\text{host 2|you 1, car 1}) P(\text{car 1|you 1})$$

$$P(\text{car 2|you 1, host 2}) \sim P(\text{host 2|you 1, car 2}) P(\text{car 2|you 1})$$

$$P(\text{car 3|you 1, host 2}) \sim P(\text{host 2|you 1, car 3}) P(\text{car 3|you 1})$$

¹ Sharon McGayne. *The Theory That Would Not Die: How Bayes' Rule Cracked the Enigma Code, Hunted Down Russian Submarines, and Emerged Triumphant from Two Centuries of Controversy*. Yale University Press, 2011. ISBN 0300169698

3 Put in the likelihood and prior values

Due the restrictions on the host above, the host cannot open a door with a car, so $P(\text{host 2|you 1, car 2}) = 0$. In the case where you choose door 1 and the car is also behind door, the host has the freedom to choose either door 2 or door 3, so $P(\text{host 2|you 1, car 1}) = 0.5$. Where the information comes in is when the car is behind door 3 and you've chosen door 1. In that case, the host cannot open your door (door 1) or the door with the car (door 3) and *must* open door 2. Thus, $P(\text{host 2|you 1, car 3}) = 1$.

The final result of this step is

$$\begin{aligned} P(\text{car 1|you 1, host 2}) &\sim 0.5 \cdot 0.333 \\ P(\text{car 2|you 1, host 2}) &\sim 0 \cdot 0.333 \\ P(\text{car 3|you 1, host 2}) &\sim 1 \cdot 0.333 \end{aligned}$$

4 Add these values for all models

$$K = 0.5 \cdot 0.333 + 1 \cdot 0.333 = 0.5$$

5 Divide each of the values by this sum, K , to get the final probabilities

$$\begin{aligned} P(\text{car 1|you 1, host 2}) &= \frac{0.5 \cdot 0.333}{0.5} = 0.333 \\ P(\text{car 2|you 1, host 2}) &= \frac{0 \cdot 0.333}{0.5} = 0 \\ P(\text{car 3|you 1, host 2}) &= \frac{1 \cdot 0.333}{0.5} = 0.666 \end{aligned}$$

Thus, in the case, given that you choose door 1 and the host chooses 2, the probability that the car is behind door 1 (your door) is 0.333 and the other door (door 3) is 0.666. Following the same steps through the other cases, we get in summary

Your Choice	Host Choice	Probability of...		
		Car Behind 1	Car Behind 2	Car Behind 3
<i>(host can't open your door)</i>				
1	1	0.333	0	0.666
1	2	0.333	0.666	0
1	3	0	0.333	0.666
<i>(host can't open your door)</i>				
2	1	0	0.333	0.666
2	2	0.666	0.333	0
2	3	0	0.666	0.333
<i>(host can't open your door)</i>				
3	1	0.666	0	0.333
3	2	0	0.666	0.333
3	3	0.333	0.666	0

In summary, it is *always* better to switch to the remaining door, given these rules.

3.3 *Psychic Octopi*

There was a German octopus named Paul² who was claimed to be psychic during his lifetime. He was given this designation because he was supposedly able to pick the result of World Cup matches before they occurred³. His impressive results, across 2 years, shown in Figure 3.2 can be summarized as follows:

data \equiv 12 out of 14 correctly predicted

UEFA Euro 2008					
Teams	Stage	Date	Prediction	Result	Outcome
Germany vs Poland	Group stage	8 June	Germany	2–0	Correct
Croatia vs Germany	Group stage	12 June	Germany	2–1	Incorrect
Austria vs Germany	Group stage	16 June	Germany	0–1	Correct
Portugal vs Germany	Quarter-finals	19 June	Germany	2–3	Correct
Germany vs Turkey	Semi-finals	25 June	Germany	3–2	Correct
Germany vs Spain	Final	29 June	Germany ^[6]	0–1	Incorrect

2010 FIFA World Cup					
Teams	Stage	Date	Prediction	Result	Outcome
Germany vs Australia	Group stage	13 June	Germany ^[22]	4–0	Correct
Germany vs Serbia	Group stage	18 June	Serbia ^[22]	0–1	Correct
Ghana vs Germany	Group stage	23 June	Germany ^[22]	0–1	Correct
Germany vs England	Round of 16	27 June	Germany ^[23]	4–1	Correct
Argentina vs Germany	Quarter-finals	3 July	Germany ^[24]	0–4	Correct
Germany vs Spain	Semi-finals	7 July	Spain ^[25]	0–1	Correct
Uruguay vs Germany	3rd place play-off	10 July	Germany ^[26]	2–3	Correct
Netherlands vs Spain	Final	11 July	Spain ^[27]	0–1	Correct

The question we have to ask is, is this data strong evidence for a psychic octopus? In order to have a well-posed problem we need the following three components:

- 1 a set of hypotheses, or models, to compare - we need at least two, otherwise the question is meaningless
- 2 for each model, an equation denoting the *likelihood*, or in other words, how probable is the data given the particular model
- 3 a specification of the *prior* probability, or in other words, how likely was our model before we saw the data

² Paul the octopus, July 2012. URL http://en.wikipedia.org/wiki/Psychic_octopus

³ The basic procedure for Paul to make a “prediction” was for his trainers to present two food dishes, labeled with a flag representing the two countries, respectively, competing. Whichever food dish Paul chose first was his prediction for the winner of the game.

Figure 3.2: The full results of the predictions of Paul the Octopus, reproduced from en.wikipedia.org/wiki/Psychic_octopus.

Making a Well Posed Problem

We are interested in the probability of this octopus being psychic, given this data, or

$$P(\text{psychic}|\text{data})$$

which really is an example of a model comparison, or hypothesis testing. In any kind of model comparison, we need to have *multiple models* to compare to in order to proceed. The models we consider constrain the problem, and define which ideas we are willing to consider. To be specific, as a first step, let's consider the following two models

$$\begin{aligned} H &:= \{\text{Paul is psychic}\} \\ R &:= \{\text{Paul is completely random, like a coin flip}\} \end{aligned}$$

The next step is to be able to assign probabilities from these models. It is easy for the *random* hypothesis

$$\begin{aligned} P(\text{correct prediction}|R) &= 0.5 \\ P(\text{incorrect prediction}|R) &= 0.5 \end{aligned}$$

What does it mean to be psychic? What is the probability of getting a correct result if you are psychic? According to James Randi⁴ many of the psychics and dowsers claim 100% accuracy in their predictions before they are tested. However this would mean a *single* wrong answer would drive the probability of that model to zero: a perfect predictor cannot, logically, make any mistakes. For our case here, we choose to be generous to the psychic and allow for a reasonable failure rate, using 90% as our accuracy, thus

$$\begin{aligned} P(\text{correct prediction}|H) &= 0.9 \\ P(\text{incorrect prediction}|H) &= 0.1 \end{aligned}$$

Specifying the prior probability of these two models is a bit more challenging. It seems reasonable to assign a small prior probability to a psychic octopus - how many psychic octopi have you ever encountered? A small, but still quite conservative value, would be 1/100, so we have for the two models:

$$\begin{aligned} P(H) &= 1/100 \\ P(R) &= 99/100 \end{aligned}$$

The First Model Comparison

Now that we've set up the problem, we can apply the Bayes' Recipe

⁴ J. Randi. *Flim-flam!: psychics, ESP, unicorns, and other delusions*, volume 342. Prometheus Books Amherst, NY, 1982

It is possible that we could be accused of an anti-psychic bias here, especially from someone who is a true believer. Why shouldn't the prior be $P(H) = 1/2$? If you had no world experience, that is what you'd start with, but then the behavior of the first octopi that you encounter would generally lower your assignment of the probability of the next octopi being psychic. After enough world experience, updating your probability with Bayes' Rule, you'd arrive at a very small prior for Paul, the current octopus we are examining.

1 Specify the prior probabilities for the models being considered

$$\begin{aligned} P(H) &= 1/100 \\ P(R) &= 99/100 \end{aligned}$$

2 Write the top of Bayes' Rule for all models being considered

$$\begin{aligned} P(H|\text{data} = 12 \text{ out of } 14) &\sim P(\text{data} = 12 \text{ out of } 14|H)P(H) \\ P(R|\text{data} = 12 \text{ out of } 14) &\sim P(\text{data} = 12 \text{ out of } 14|R)P(R) \end{aligned}$$

where we are using the symbol \sim to denote *proportionality* or *related to*. Essentially, by calculating the top of Bayes' Rule first, the numbers are not *equal* to the final (i.e. posterior) probabilities but must be rescaled to make sure that they add up to 1. This is done in the final step. Up until that rescaling, we use the symbol \sim and think of it as *related to*.

3 Put in the likelihood and prior values

$$\begin{aligned} P(H|\text{data} = 12 \text{ out of } 14) &\sim \binom{14}{12} 0.9^{12} 0.1^{14-12} \times \frac{1}{100} \\ &= 0.00257 \\ P(R|\text{data} = 12 \text{ out of } 14) &\sim \binom{14}{12} 0.5^{12} 0.5^{14-12} \times \frac{99}{100} \\ &= 0.00549 \end{aligned}$$

4 Add these values for all models

$$K = 0.00257 + 0.00549 = 0.00806$$

5 Divide each of the values by this sum, K , to get the final probabilities

$$\begin{aligned} P(H|\text{data}) &= \frac{0.00257}{0.00806} = 0.32 \\ P(R|\text{data}) &= \frac{0.00549}{0.00806} = 0.68 \end{aligned}$$

and the psychic loses!

Furthering the Comparison

Typically, a person who is supportive of psychic phenomena would choose a prior for our psychic hypothesis (H) that would be at least as large as the prior for the random hypothesis (R). In this case, the (posterior) probability of the octopus being psychic given the data

of 12 correct out of 14 would be much higher. After “ruling out” the random octopus hypothesis, we’d be left with psychic. But is that all that is really left? No, and the analysis is easy to do.

Once presented with the success of Paul, most people instantly are suspicious of random octopus, but don’t adopt psychic octopus as the answer. Perhaps the keepers, being German, biased the data taking a little bit. Perhaps the octopus chose flags with bright yellow stripes. Notice that each of these cases still results in the same data - the octopus would have gotten 12 out of 14, but the *prior* probability of these cases should be much higher than psychic, even if lower than random. We leave it as an exercise to perform the calculation in this case, but it is directly parallel to the *Nines* deck example of Section 2.2 on page 52.

4 Introduction to Parameter Estimation

We will introduce the idea of what is called *parameter estimation* using a simple system of bent coins. This will generalize to more complex models, and form the basis for much of statistical inference.

4.1 Bent Coins

Imagine we have a series of coins bent by various amounts (Figure 4.1). If the coin is bent completely in half, then we could have the coin always flip heads (i.e. $P(\text{heads}) = 1$) or tails (i.e. $P(\text{tails}) = 1$) depending on how it is bent. If you don't bend the coin at all then we'd have a fair coin ($P(\text{heads}) = P(\text{tails}) = 0.5$). So, let's say that we have a collection of bent coins which are bent by different amounts. For convenience we will number them from 0 to 10. The Table 4.1 summarizes the probability of each coin flipping heads.

Coin Number	Probability for Flipping Heads ($P(\text{heads})$)
0	0.0
1	0.1
2	0.2
3	0.3
4	0.4
5	0.5
6	0.6
7	0.7
8	0.8
9	0.9
10	1.0

Now I have the following scenario¹, with a few questions.

Imagine I have taken a random coin from my collection, flipped it and got the following data:

T T T H T H T T T T H (i.e. 9 tails and 3 heads)

¹ From this data, which coin do I most likely have?



Figure 4.1: Bent Coin

Why do we number them from zero here? It's so that the number of the coin, say number 7, corresponds the probability that that coin flips heads, $P(\text{heads}) = 0.7$

Table 4.1: Probabilities for flipping heads given a collection of bent coins

¹ D. V. Lindley and L. D. Phillips. Inference for a bernoulli process (a bayesian view). *The American Statistician*, 30(3):112–119, 1976

- 2 Can we be *significantly confident* that this particular coin will result in more tails than heads in the future?

The way we've set up this problem is exactly like the model comparison example with the High and Low Deck (Section 2.1), except in this case we have 11 models (one for each coin). Applying the Bayes' Recipe we have

- 1 Specify the prior probabilities for the models being considered.

Given no further information, we select a *uniform* distribution for the prior (i.e. all models are initially equally probable):

$$\begin{aligned} P(M_0) &= 1/11 \\ P(M_1) &= 1/11 \\ &\vdots \\ P(M_{10}) &= 1/11. \end{aligned}$$

where M_0 is the model defined by "we're flipping coin 0," M_1 is the model defined by "we're flipping coin 1," etc...

- 2 Write the top of Bayes' Rule for all models being considered:

$$\begin{aligned} P(M_0|\text{data} = 9T, 3H) &\sim P(\text{data} = 9T, 3H|M_0)P(M_0) \\ P(M_1|\text{data} = 9T, 3H) &\sim P(\text{data} = 9T, 3H|M_1)P(M_1) \\ &\vdots \\ P(M_{10}|\text{data} = 9T, 3H) &\sim P(\text{data} = 9T, 3H|M_{10})P(M_{10}). \end{aligned}$$

- 3 Put in the likelihood and prior values. Here we are drawing from a *binomial* distribution for the likelihood:

$$\begin{aligned} P(M_0|\text{data} = 9T, 3H) &\sim \binom{12}{3} 0.0^3 \times (1 - 0.0)^9 \times 1/11 \\ P(M_1|\text{data} = 9T, 3H) &\sim \binom{12}{3} 0.1^3 \times (1 - 0.1)^9 \times 1/11 \\ &\vdots \\ P(M_{10}|\text{data} = 9T, 3H) &\sim \binom{12}{3} 1.0^3 \times (1 - 1.0)^9 \times 1/11. \end{aligned}$$

- 4 Add these values for all models: see Table 4.2.

- 5 Divide each of the values by this sum, K , to get the final probabilities: see Table 4.2.

When we are dealing with this many models, it is easier to plot the results, shown in Figure 4.2. We are now in a position to address the questions posed at the beginning of the section.

Model	$\sim P(M_i \text{data} = 9T, 3H)$	$\sim P(M_i \text{data} = 9T, 3H)/K$
M_0	0.000	0.000
M_1	0.00774	0.110
M_2	0.0214	0.306
M_3	0.0217	0.310
M_4	0.0128	0.184
M_5	0.00488	0.0696
M_6	0.00113	0.0161
M_7	0.000135	0.00192
M_8	0.00000524	0.0000748
M_9	0.000000145	0.000000208
M_{10}	0.000	0.000
$K=0.0700$		

Table 4.2: Probability for different bent-coin models, given the data=9 tails, 3 heads. The middle column is the non-normalized value from Bayes' Rule, needing to be divided by K (the sum of the middle column) to get the final column which is the actual probability.

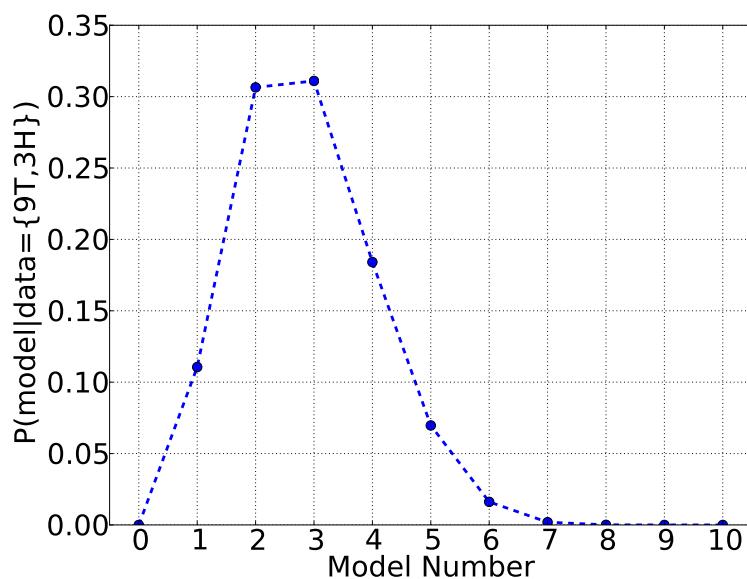


Figure 4.2: Probability for different bent-coin models, given the data=9 tails, 3 heads.

- 1 From this data, which coin do I most likely have?

The maximum probability is for coin 3, but coin 2 is a close second. Thus we can be reasonably confident that we have been flipping one of those two coins, but can't narrow our confidence any more than that.

- 2 Can we be *significantly confident* that this particular coin will result in more tails than heads in the future?

This is another way of asking for the total probability for coins less than coin 5 (the fair coin), or

$$\begin{aligned} P(\text{coin 0 or coin 1 or coin 2 or coin 3 or coin 4}) &= \\ 0.000 + 0.110 + 0.306 + 0.310 + 0.184 &= 0.912 \end{aligned}$$

which says that this coin is “likely” to “very likely” (Table 1.1 on page 43) to have a probability of yielding heads less than a fair coin, and thus yield more tails in the future.

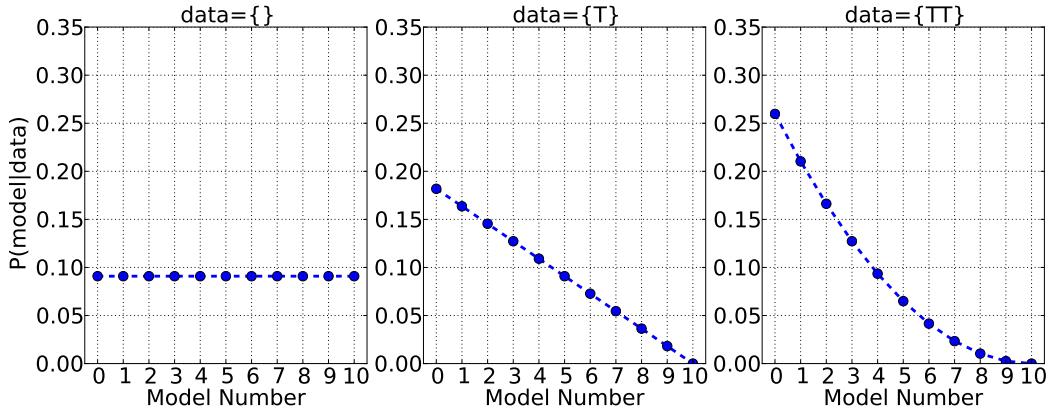
4.2 Priors versus Data

It is instructive to pause and look at this example one flip at a time, to see how the probability and thus our state of knowledge adjusts as we collect more data. In Figure 4.3 we see the result of our procedure when there is no data (i.e. our initial, prior probabilities) and when we've flipped once and then again, both times tails. The curve for no data is the same as the prior probability, and in this case all models are equally likely. When the first tails is observed, the model which states that heads are *certain* (i.e. coin 10) goes to zero probability because coin 10 *cannot* flip tails.². At this point we *know* that it is *impossible* for us to be flipping coin 10. We see also that the high-numbered coins (i.e. the ones with high probability of flipping heads) have greatly reduced probability while we've seen only tails.

As more tails are observed, the probability for the lower models is increased. As we flip more tails we become more confident in the lower-number models. Because at this point we haven't flipped any heads, the model 0 still has non-zero probability - it is still possible that we are holding a coin that cannot flip heads.

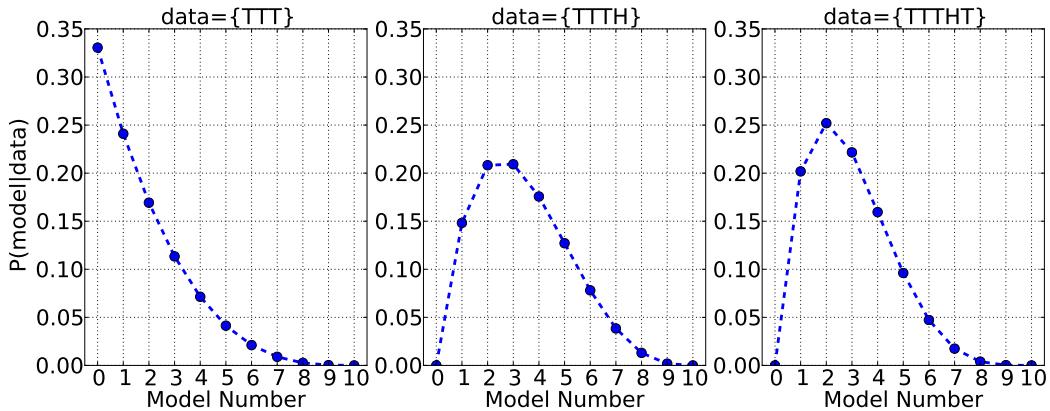
When we continue with the next few flips (Figure 4.4) we encounter our first heads on the fourth flip. At this point the model which states that heads are *impossible* (i.e. coin 0) goes to zero probability. Finally, across our entire data set (Figure 4.5) we see that the curve gets narrower, where more of the probability falls on only a few of the models and the other models become less and less likely. With only 12 data points, there is still a lot of uncertainty in which

² Notice that the only models with probability equal to zero are ones that are *logically impossible*. It's not the colloquial usage of impossible, as in “it is impossible for the Red Sox to win this year,” but in the strict usage, as in “it is impossible to flip both heads and tails at the same time.” The reason this is the case is that a statement with zero probability cannot be made possible with *any amount of data* - it is an utterly *dogmatic* statement. Thus, we reserve it only for things that are *logically impossible*.



model - several models have reasonably high probability values. We still can rule out a few models confidently (like coins 0, 6, 7, 8, 9, and 10). We are most confident in coins 2 and 3, with the most probability.

Figure 4.3: Probability for different bent-coin models, given no data (left), the first tails (middle), and the second tails (right). The curve for no data is the same as the prior probability, and in this case all models are equally likely. When the first tails is observed, the model which states that heads are *certain* (coin 10) goes to zero probability. As more tails are observed, the probability for the lower models is increased.



4.3 Moving Toward the Continuous

There is a practical problem that we face at this point, when we consider a generic bent coin. Perhaps it doesn't fit in one of the 11 models considered, falling somewhere in between, for example with

Figure 4.4: Probability for different bent-coin models, given three tails (left), the first heads (middle), and another tails (right). When the first heads is observed, the model which states that heads are *impossible* (coin 0) goes to zero probability.

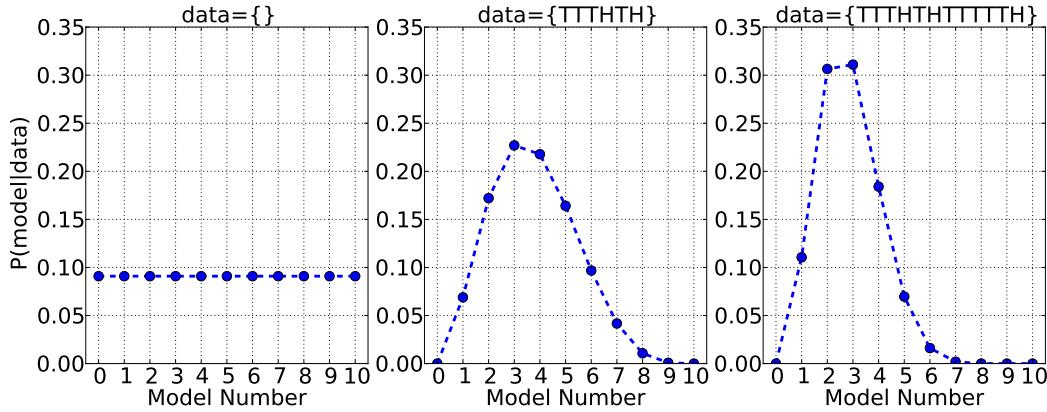


Figure 4.5: Probability for different bent-coin models, given no data (left), the first half of the data set (middle), and the entire data set of 9 tails and 3 heads (right).

$P(\text{heads}) = 0.132464$. One's first thought might be to include one thousand coins or one million coins instead of the 11 we've considered so far, so we could have coin 132464, coin 132465, coin 132466, etc... Although this can be done, we run into two problems

- 1 Because we are dealing with so many models, the probability associated with any *single* model gets very small - and gets smaller with the more models you consider
- 2 We can't practically distinguish between models such as $P(\text{heads}) = 0.132464$ and $P(\text{heads}) = 0.132465$ (the last digit is different here)

In order to solve both of these problems mathematically, we introduce the concept of a *continuous distribution*. We start by labeling the model with a *continuous* number rather than an integer. In our present case it makes sense to label the model with the probability that the coin flips heads. We'll call this label θ , and it will have a value between 0 (heads are impossible) and 1 (heads are certain) and can take on *any value* in between. Because we now have an infinite number of labels, we have two consequences:

- 1 We can't simply add up all the probabilities to get our value of K to make everything add up to 1. Instead, we look at *areas under the curve* and make sure the entire area equals 1.
- 2 Because, with distributions, *areas under the curve* (and not the values of the distribution itself) are the probabilities, we can only speak about *ranges* of values. For example, we can speak meaningfully about the probability of θ between 0.3 and 0.4 (i.e. $P(0.3 < \theta < 0.4)$). When we write down something like $P(\theta) = 1$ we're not talking about a probability of a single label but rather the magnitude of the distribution at that label, θ .

We revisit Bayes' Recipe again, using the distributions. This time we also will look at pictures of the distributions as we progress.

- 1 Specify the prior probabilities for the models being considered:

$$P(\theta) = 1.$$

- 2 Write the top of Bayes' Rule for all models being considered:

We can write one equation for all of the models labeled by θ at once as

$$P(\theta|\text{data} = 9T, 3H) \sim P(\text{data} = 9T, 3H|\theta)P(\theta).$$

- 3 Put in the likelihood and prior values.

We use the binomial model, one equation for all models, remembering that for a model labeled by θ the probability for that coin flipping heads is $P(\text{heads}) = \theta$. Thus we get the likelihood and prior values as

$$\begin{aligned} P(\theta|\text{data}) &\sim P(\text{data}|\theta) \cdot P(\theta) \\ P(\theta|\text{data} = 9T, 3H) &\sim \binom{12}{3} \theta^3 \times (1 - \theta)^9 \cdot 1 \end{aligned}$$

- 4 Find the area under this curve, and call it K .

- 5 Divide each of the values of the curve by this are, K , to get the final probabilities where the area under the curve is 1.

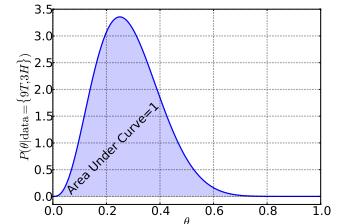
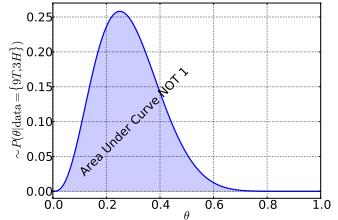
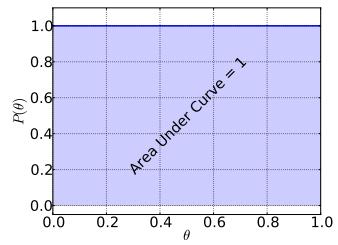
Usually these steps are done for you, for a specific data set, and you are given the final posterior distribution to use in answering any questions. However, for any particular case it is important to know what assumptions have been made in the choice of models and model parameters.

4.4 MAP and Areas

Now we revisit the questions posed in Section 4.1 on page 69 about the bent coin, this time using the distribution found above, reproduced here in Figure 4.6.

Imagine I have taken a random coin from my collection, flipped it and got the following data:

T T T H T H T T T T H (i.e. 9 tails and 3 heads)



- 1 From this data, which “coin” do I most likely have? (or in this interpretation, what is my best estimate for the probability of this coin flipping heads, denoted by θ)
- 2 Can we be *significantly confident* that this particular coin will result in more tails than heads in the future?

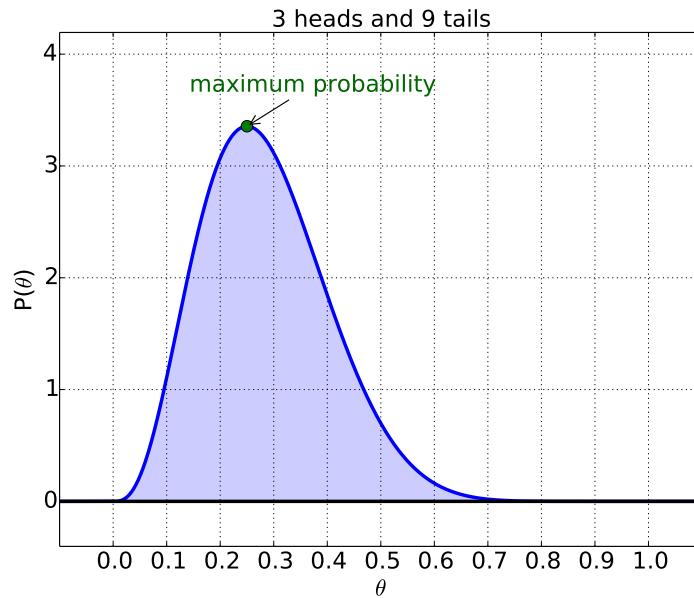


Figure 4.6: Posterior probability distribution for the θ values of the bent coin - the probability that the coin will land heads. The distribution is shown for data 3 heads and 9 tails, with a maximum at $\theta = 0.25$.

One answer to the first question can be accomplished by looking at the *maximum* of the posterior distribution, shown in Figure 4.6.³ By eye, it seems to have a maximum 0.25. In fact one can demonstrate that this distribution has a maximum at

$$\theta_{\max} = \frac{\text{number of successes}}{\text{total number of attempts}},$$

where in our example, a success is head, and an attempt is a flip.⁴ We take up this question of the best estimate of θ , given the posterior probability for θ , in more detail in Section 4.6.

The answer to the second question can be done by looking at the area under the curve from $\theta = 0$, the “all heads” coin, to $\theta = 0.5$, the “fair” coin, as shown in Figure 4.7. This area represents the probability, given the data, that the coin is skewed towards heads or, in other words, how confident are we that this is an unfair coin. Given the value of $P(\theta < 0.5) = 0.954$ we can say that this is “very likely” an unfair coin (see Table 1.1 on page 43).

³ The maximum of the posterior distribution, which represents the most likely value of a quantity, is often referred to as the *MAP estimate*. It is also commonly referred to as the *mode* of the distribution.

⁴ This distribution, given how common it is, is given the name *Beta distribution*. There are a handful of common distributions that are given names for convenience. We’ve already seen the *uniform distribution*, and there will be others.

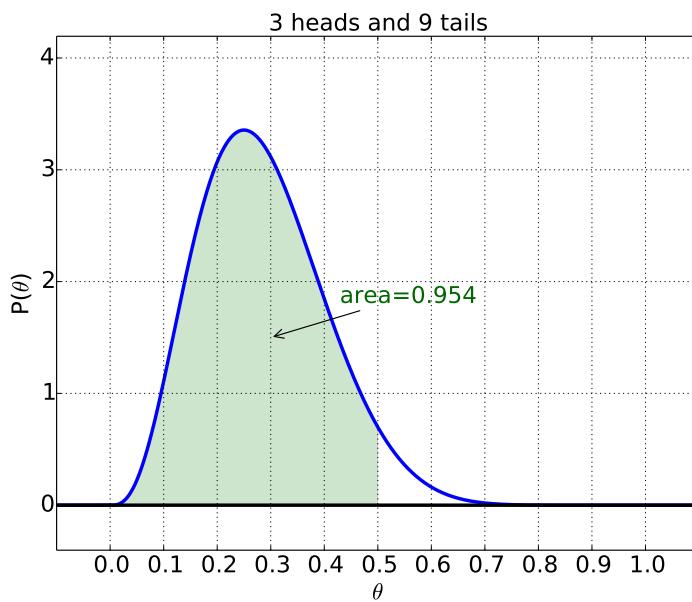


Figure 4.7: Posterior probability distribution for the θ values of the bent coin - the probability that the coin will land heads. The distribution is shown for data 3 heads and 9 tails. The area under the curve from $\theta = 0$ (the “all heads” coin) to $\theta = 0.5$ (the “fair” coin) is 0.954.

4.5 Quartiles

Given that we are dealing most often with continuous distributions, and thus need to look at areas under the curve from one point to another, it is useful to make a table for a distribution of these areas. Typically we look at the values of the parameter at which we have a given area under the curve from the minimum possible value of the parameter up to that value. For example, we might be interested in the value of θ (i.e. how skewed the coin is) such that we have an area of 50% from 0 up to θ , shown in Figure 4.8. This point (called the *median*) represents the point where we would be just as confident (given our data) that the coin is *more* skewed than this as *less* skewed.

A table of these values for a distribution can be very useful. For example, consider the table and plot shown in Figure 4.9. Shown are the various points where the area under the curve up to those points is specified. For example, the area under the curve from $\theta = 0$ up to $\theta = 0.11$ is 5%. This means, given the data of 3 heads and 9 tails, there is a probability $P = 5\%$ of the coin having less than $\theta = 0.11$, or an extreme skew towards tails.

Quartiles The term *quartiles* refers to the values of the parameter which result in an area of 25%, 50%, or 75%, or one, two, or three quarters of the area.

When we wish to refer to a non-quarter percentage, then we'll call

Quartiles The term *quartiles* refers to the values of the parameter which result in an area of 25%, 50%, or 75%, or one, two, or three quarters of the area.

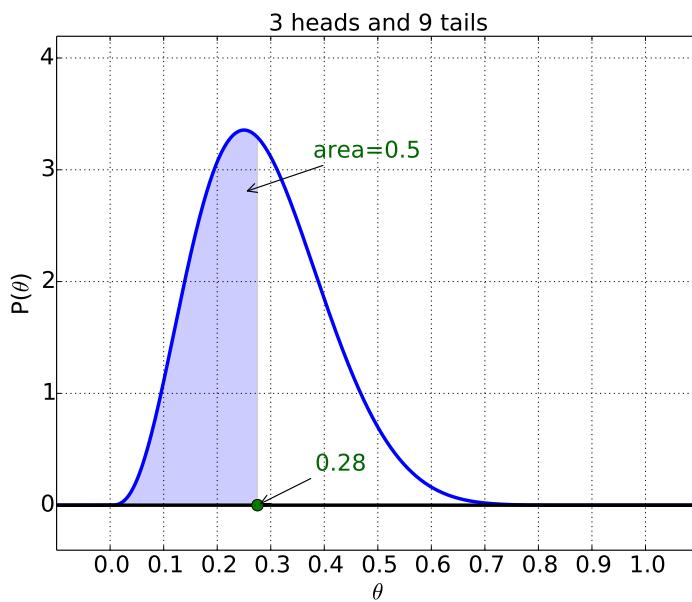


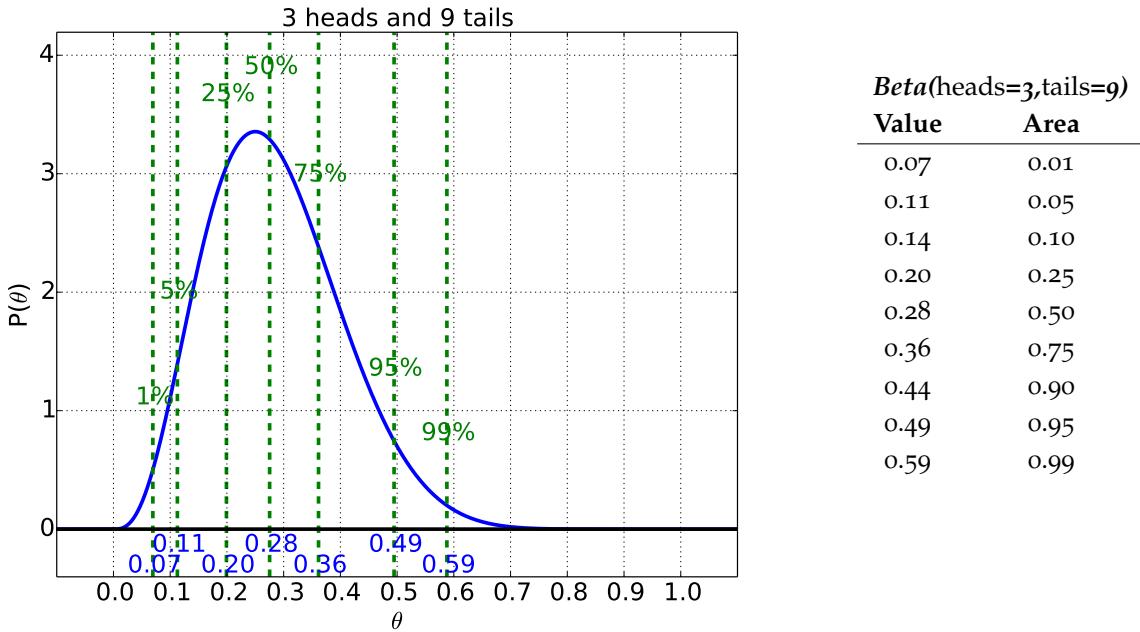
Figure 4.8: Posterior probability distribution for the θ values of the bent coin - the probability that the coin will land heads. The distribution is shown for data 3 heads and 9 tails. The area under the curve from $\theta = 0$ (the “all heads” coin) to $\theta = 0.28$ is 0.5 - half the area. This represents the *median* of the distribution.

it a *percentile*.

Percentiles The term *percentile* refers to the value of the parameter which result in a particulare area under the curve.

For example, we can say from Figure 4.9 that the 99% percentile is 0.59. Thus, it is extremely unlikely to have the coin skewed towards heads more than $\theta = 0.59$ given the observation that we flipped 3 heads and 9 tails with this coin.

Percentiles The term *percentile* refers to the value of the parameter which result in a particulare area under the curve.



4.6 Best Estimates

Perhaps surprisingly, there is not a single answer to the best estimate for θ given the poster distribution, for example the one shown in Figure 4.9. There are several plausible measures, each with their own advantages. Any specific estimate of a parameter (e.g. θ) is denoted with a hat (e.g. $\hat{\theta}$) in the descriptions that follow.

The Mode Also known as the *maximum a-posteriori probability* (MAP) estimate, the *mode* is the maximum of the posterior probability. In the case of a Beta distribution with h successes in N trials, we have

$$\hat{\theta}_{\text{mode}} = \frac{h}{N}$$

The Mean Also known as the *expected value* or *average value*, the *mean* of a distribution of a parameter θ is defined to be the sum of all of the possible values of θ times the posterior probability of θ ,

$$\hat{\theta}_{\text{mean}} = \sum_{\theta} \theta \times P(\theta|\text{data})$$

It is one measure of the *middle* of the distribution. In the special case of a Beta distribution with h successes in N trials, we have

$$\hat{\theta}_{\text{mean}} = \frac{h+1}{N+2}$$

Figure 4.9: Posterior probability distribution for the θ values of the bent coin - the probability that the coin will land heads. The distribution is shown for data 3 heads and 9 tails. The various quartiles are shown in the plot, and summarized in the accompanying table.

The Mode Also known as the *maximum a-posteriori probability* (MAP) estimate, the *mode* is the maximum of the posterior probability.

The Mean Also known as the *expected value*, the *mean* of a distribution of a parameter θ is defined to be the sum of all of the possible values of θ times the posterior probability of θ , as in

$$\hat{\theta}_{\text{mean}} = \sum_{\theta} \theta \times P(\theta|\text{data})$$

It is one measure of the *middle* of the distribution.

Intuitively this is the same as the MAP of the Beta distribution, with one more success and one more failure than actually observed. Further, for the Beta distribution, the mean value $\hat{\theta}_{\text{mean}}$ represents the predictive probability of a successful event on the *next* observation.

The Median Also known as the 50%-percentile, the median represents the middle of the distribution such that the probability of the parameter below the median equal to the probability of the parameter above the median.

$$P(\theta \leq \hat{\theta}_{\text{median}} | \text{data}) = P(\theta \geq \hat{\theta}_{\text{median}} | \text{data}) = 0.5$$

"Assume 2 successes and 2 failures" median approximation For the Beta distribution there is no simple form for the median, but a decent approximation which we will use is given by⁵

$$\hat{\theta}_{\text{median}} \approx \frac{h+2}{N+4}$$

Intuitively this is the same as the MAP of the Beta distribution, with two more successes and two more failures than actually observed, and is thus referred to as the "Assume 2 successes and 2 failures" median approximation.

Although each of these has their advantages, most notably ease of computation (especially for the mode and the mean), we will typically use the median of the distribution as the best estimate for the following reasons:

- 1 the median is intuitive as literally the middle of the distribution
- 2 the median is not as sensitive to distributions that are highly asymmetric

In most practical examples it may not make much difference, and for some distributions (such as the Normal distribution described in Chapter 5 (*Priors, Likelihoods, and Posteriors*)) there is not difference - the mean *is* the median which is also the mode.

EXAMPLE 4.1 *What is the best estimate of the probability of a bent coin flipping heads, given the observation of 9 tails and 3 heads?*

If we take the best estimate to be the median, then we have from the "assuming 2 successes and 2 failures" method,

$$\begin{aligned}\hat{\theta}_{\text{median}} &\approx \frac{h+2}{N+4} \\ &= \frac{5}{16} = 0.313\end{aligned}$$

The Median Also known as the 50%-percentile, the median represents the middle of the distribution such that the probability of the parameter below the median equal to the probability of the parameter above the median.

$$\begin{aligned}P(\theta \leq \hat{\theta}_{\text{median}} | \text{data}) &= 0.5 \\ P(\theta \geq \hat{\theta}_{\text{median}} | \text{data}) &= 0.5\end{aligned}$$

"Assume 2 successes and 2 failures" median approximation For the Beta distribution there is no simple form for the median, but a decent approximation which we will use is given by

$$\hat{\theta}_{\text{median}} \approx \frac{h+2}{N+4}$$

Intuitively this is the same as the MAP of the Beta distribution, with two more successes and two more failures than actually observed, and is thus referred to as the "Assume 2 successes and 2 failures" median approximation.

⁵ Alan Agresti and Brian Caffo. Simple and effective confidence intervals for proportions and differences of proportions result from adding two successes and two failures. *The American Statistician*, 54 (4):280–288, 2000

Notice that the maximum probability was at the somewhat lower value

$$\hat{\theta}_{\text{mode}} = \frac{h}{N} = \frac{3}{12} = 0.25$$

One reason why the median is a better estimate in this case is because, as shown in Figure 4.9, there is more probability (i.e. area under the curve) to the right of the maximum than to the left, so the best estimate should be greater than the one given by the mode.

4.7 Uncertainty in the Best Estimates

To quantify the uncertainty in the best estimates, we need a value which represents the *width* of the distribution. Looking at Figure 4.10 we'd like to provide a quick way of saying that the range of probable values lies somewhere between $\theta = 0.2$ and $\theta = 0.5$ - anything outside of this contributes only a small amount to the probability, or in other words, we are most confident that our best estimate of θ lies between those 0.2 and 0.5. Depending on the application, the symmetry of the distribution, and other practical factors one may see a few potential measures of the *width* of the distribution.

Inter-Quantile Range The Inter-Quantile Range (ICR) is the range between the 25% and 75% quartiles, and represents 50% of the probability.

In Figure 4.10, the Inter-Quantile Range range is [0.29,0.40].

95% Credible Interval (CI) The 95% Credible Interval (CI) is the range between the 2.5% and 97.5% quantiles, and thus represents 95% of the probability. According to Table 1.1 on page 43, it is "very likely" that our best estimate lies in this range.

In Figure 4.10, the 95% Credible Interval is nearly [0.2,0.5].

Standard Deviation The standard deviation is a measure of the half-width of a distribution, most commonly used specifically with reference to the particular *Normal* distribution. This will be defined more precisely in Section 5.2 on page 88), and will thus not be defined in general here.

An approximate value for the standard deviation for the Beta distribution is

$$\sigma \approx \sqrt{\hat{\theta}(1 - \hat{\theta})/N}$$

From Figure 4.10, and using the median as the best estimate, $\hat{\theta}$, we get

$$\sigma \approx \sqrt{0.34(1 - 0.34)/30} = 0.09$$

Inter-Quantile Range The Inter-Quantile Range (ICR) is the range between the 25% and 75% quartiles, and represents 50% of the probability.

95% Credible Interval (CI) The 95% Credible Interval (CI) is the range between the 2.5% and 97.5% quantiles, and thus represents 95% of the probability. According to Table 1.1 on page 43, it is "very likely" that our best estimate lies in this range.

Standard Deviation The standard deviation is a measure of the half-width of a distribution, most commonly used specifically with reference to the particular *Normal* distribution. This will be defined more precisely in Section 5.2 on page 88), and will thus not be defined in general here.

Standard Deviation to Uncertainty To convert this number to an uncertainty, it is a mathematical consequence that about 65% of the area is within 1 value of σ , 95% of the area is within 2 values of σ , and 99% of the area within 3 values.

So, of for the approximate 95% CI for the case shown in Figure 4.10 is

$$[0.34 - 2 \cdot 0.09, 0.34 + 2 \cdot 0.09] = [0.16, 0.52]$$

a bit more conservative range (larger uncertainty) than is given by the direct method of quantiles, but it much easier to calculate.

Standard Deviation to Uncertainty

To convert this number to an uncertainty, it is a mathematical consequence that about 65% of the area is within 1 value of σ , 95% of the area is within 2 values of σ , and 99% of the area within 3 values.

4.8 Marginalization

In Section 1.4 we introduced the concept of marginalization, and in Section ?? we performed a discrete example of this. In that section it was seen as simply a consequence of the sum and product rules. It was a way of taking a probability that depended on several factors, and eliminating all but the single factor we're interested in. If we have a *continuous* distribution, this process involves calculus and we will not cover it in detail, but it is the same process. In the case of the distribution above, we have a distribution over a single variable, like $\text{Beta}(\theta|h, t)$. Imagine that we have a distribution that depends on *two* parameters,

$$\text{MyDist}(\theta, \xi)$$

which specifies the probability of an event given each combination of the parameters, θ and ξ . We'd have to do a three-dimensional plot to visualize this. Many times, however, we want just the probability of one of the single parameters. In those cases we will write

$$P(\theta) \sim [\text{MyDist}(\theta, \xi)]_{\text{marginalize over } \xi}$$

where we are “summing” over all the values of the other parameters, leaving the details to the mathematicians, and simply using the result.

Likewise we can *marginalize* the parameter θ to get the distribution of the other variable.

$$P(\xi) \sim [\text{MyDist}(\theta, \xi)]_{\text{marginalize over } \theta}$$

This becomes important in Chapter 5 and Chapter 6.

4.9 Exercises

EXERCISE 4.1 Given the posterior shown in Figure 4.10 for 10 heads and 20 tails, answer the following:

- 1 The most likely estimate for the parameter θ . What does this mean?
- 2 Is it likely that this is a fair coin?
- 3 What is $P(0 \leq \theta \leq 0.3)$ approximately?
- 4 What is $P(0.2 \leq \theta \leq 0.35)$ approximately?
- 5 What is the median value? What are the quartiles?

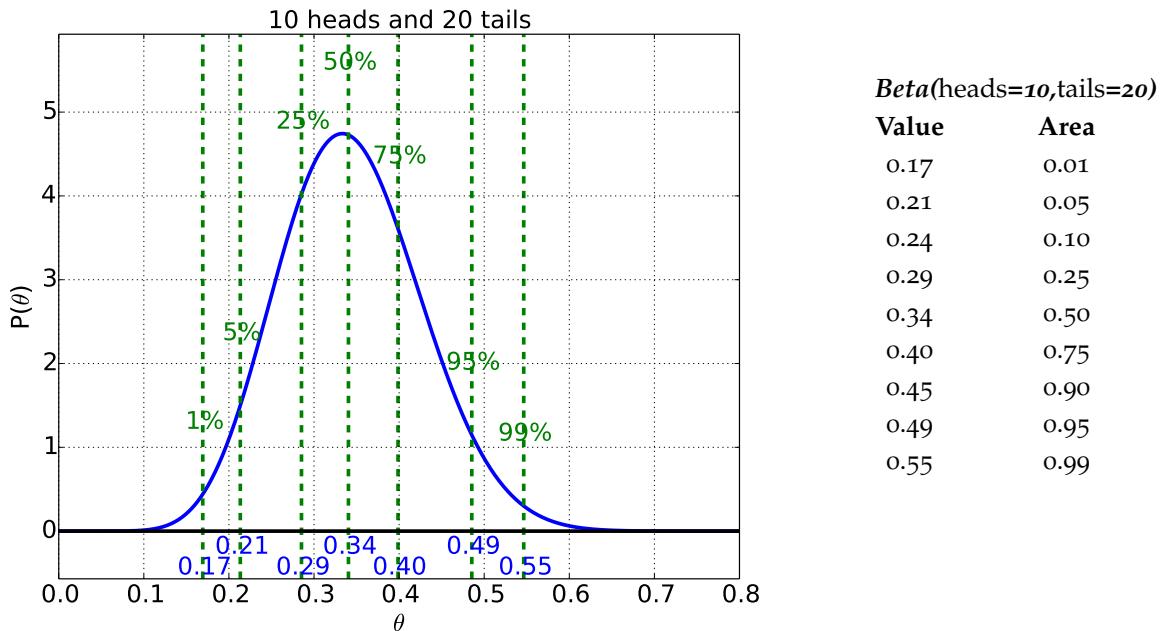


Figure 4.10: Posterior probability distribution for the θ values of the bent coin - the probability that the coin will land heads. The distribution is shown for data 10 heads and 20 tails. The various quartiles are shown in the plot, and

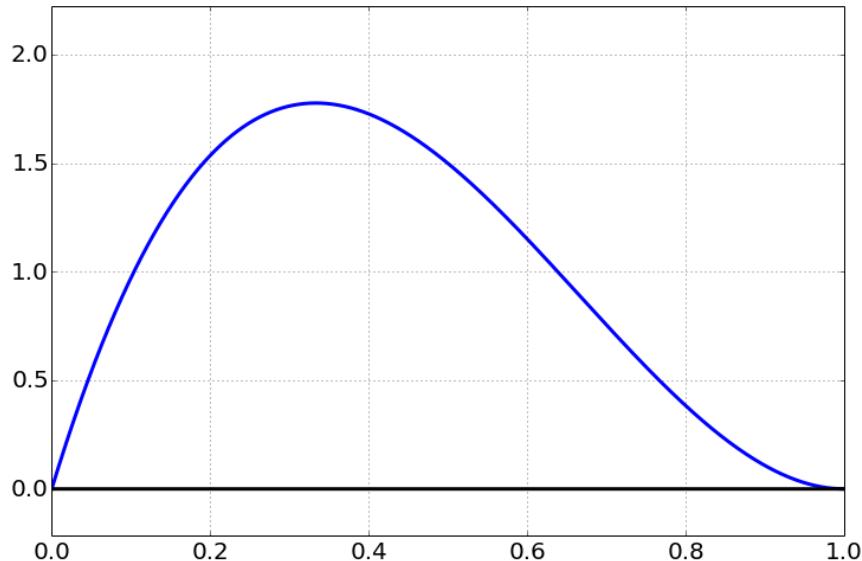
4.10 Computer Examples

```
from sie import *
```

Beta Distribution Example

3 heads and 9 tails Plot a beta distribution with 3 heads and 9 tails...

```
dist=beta(h=1,N=3)
distplot(dist,xlim=[0,1],show_quartiles=False)
```



The median of this distribution...

```
dist.median()
```

0.27527583248615201

the 95

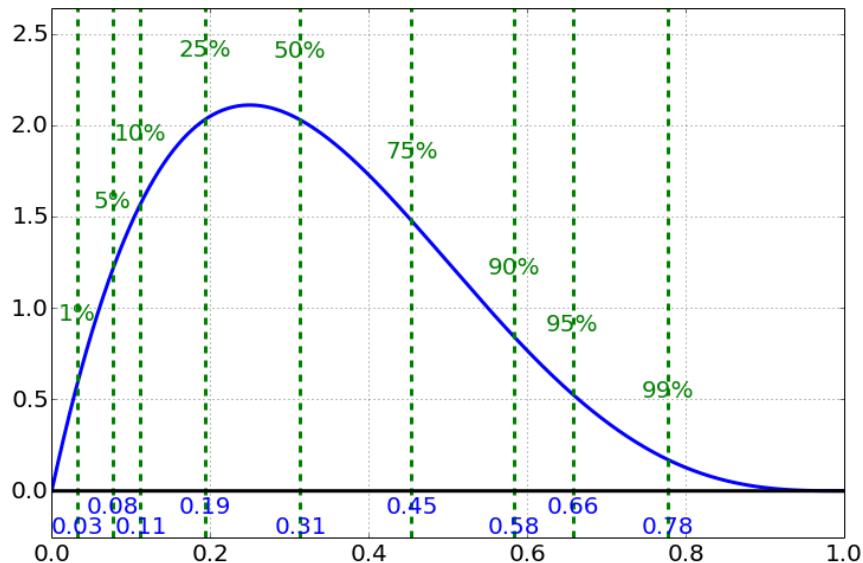
```
credible_interval(dist)
```

(0.067585986488542985, 0.38572756813238962, 0.80587955031675662)

1 heads and 3 tails This should be about the same fraction as the previous example, but broader

```
dist=beta(h=1,N=4)
distplot(dist,xlim=[0,1])
```

<matplotlib.figure.Figure at 0x108768cd0>



```
credible_interval(dist)
```

```
(0.052744950526316919, 0.31381017045569742, 0.71641793611808946)
```


5 Priors, Likelihoods, and Posteriors

5.1 Binomial and Beta Distributions

In Chapter 4 ([Introduction to Parameter Estimation](#) on page 69) we estimated the chance, θ , that a bent coin would come up heads by combining a *uniform prior* for θ (i.e. all possible values are a-priori equally likely) and a *binomial* likelihood (i.e. given θ , what is the probability of the data). This resulted in a *Beta* distribution for the posterior probability for θ .

Notice what the procedure of Bayes' Recipe is and how the Bayesian inference works here.

- 1 Specify the prior probabilities for the models being considered

We want to estimate a quantity (which we label as θ), but begin with absolutely no knowledge of its value - we have a *uniform* prior probability.

- 2 Write the top of Bayes' Rule for all models being considered

We construct a model for how different possible values of θ influence the outcome - a model we call the *likelihood*. In the case of the bent coin, the likelihood model is a *binomial* model, and describes the probability of flipping heads or tails given how bent the coin is (i.e. given θ).

- 3 Put in the likelihood and prior values

- 4 Add these values for all models

- 5 Divide each of the values by this sum, K , to get the final probabilities

Once we observe data, we can combine the prior and the *model* or *likelihood* using the Bayes' recipe, and obtain the *posterior* distribution for the unknown value, θ , giving us the probability for each value, now updated with our new observations.

The last couple of steps of the recipe, for simple cases, is done by the mathematicians so we don't have to manually add and divide as we did in the previous chapters. In the case of the coin flips we get:

$$\underbrace{\text{Beta}(\theta|\text{data})}_{\text{posterior probability}} \sim \overbrace{\text{Binomial}(\text{data}|\theta) \times \text{Uniform}(\theta)}^{\text{likelihood}}$$

From this *Beta* distribution, we can get the most likely values (i.e. maximum probability value) for the unknown quantity of interest, θ , our *uncertainty* in this quantity (i.e. the *width* of the *Beta* distribution) consistent with the known data. In other words, the posterior probability summarizes all of our knowledge about the parameter of interest given the data.

5.2 The Normal Distribution - Properties

The *Normal* distribution, also referred to as the *Gaussian* distribution,¹ is by far the most commonly occurring distribution in all of statistical inference, so it requires some special attention.

The Shape

The shape of the Normal distribution is sometimes described as *bell-shaped*, as shown in Figure 5.1, and is thus referred to as the bell-curve (although there are several other mathematical functions which are bell-shaped). The function is referred to as $\text{Normal}(\mu, \sigma)$ where μ and σ are parameters of the model. (see Appendix B.1 on page 141 for a review of greek letters)

¹ The distribution is named after Carl Friedrich Gauss who introduced it in 1809. However, it has been called in the past the Gauss-Laplacian distribution, due to the fact that Pierre Simone de Laplace was the first to apply it to real problems, and proved a number of very useful properties of it.

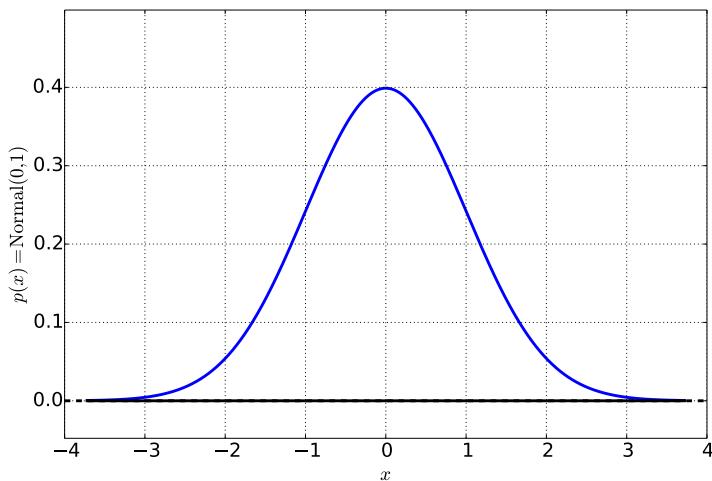


Figure 5.1: The Normal Distribution.

The location parameter, μ

The location parameter (see Figure 5.2) is the value of x for which the Normal distribution has a maximum probability. In a real sense, it is the *middle* of the distribution, and the best estimate of x . For the Normal distribution the location parameter, μ , is at once the mean, median and mode of the distribution.

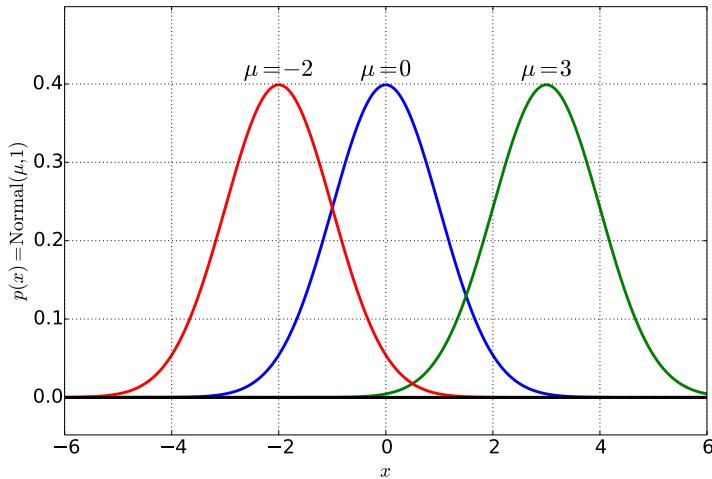


Figure 5.2: The Normal distribution with different location parameters, μ .

The deviation parameter, σ

As shown in Figure 5.3 the deviation parameter, σ , is a measure of how *spread out* the distribution is. As the width increases, the height goes down to keep the area under the curve constant (at 1). As a result, more of the probability sits at *larger* values of x as σ gets *larger*.

Three useful properties of σ for the Normal distribution are the following:

- 1 the Normal distribution value at the maximum (i.e. at $x = \mu$) is around 2.7 times larger than the value one- σ away from the maximum (at $x = \mu - \sigma$ and $x = \mu + \sigma$)
- 2 the total probability between these two points is 65%. This is typically written, $\mu \pm \sigma$.
- 3 95% of the distribution lies between $\mu - 2\sigma$ and $\mu + 2\sigma$ (see Figure 5.3)

For example, writing 5 ± 2 typically implies a Normal distribution with mean $\mu = 5$ and deviation $\sigma = 2$. One is 65% certain that the

range of the estimated value is between 3 and 7, and 95% certain that the range is between 1 and 9 (i.e. mean minus two deviations and mean plus two deviations).

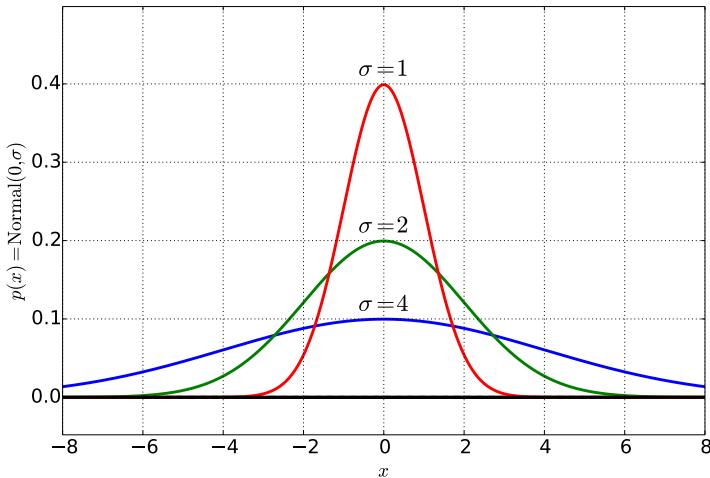


Figure 5.3: The Normal distribution with different deviation parameters, σ .

Summarizing the Distribution

We can specify the Normal distribution with just the two parameters, μ and σ - the location and deviation parameters, respectively. However, due to its symmetry, we can summarize this distribution for all cases by looking at a single special case called the *standard Normal distribution*.

The Standard Normal Distribution is the Normal distribution in the special case where $\mu = 0$ (the distribution is centered at $x = 0$) and $\sigma = 1$ (the distribution has a spread of 1).

For any Normal distribution, the area within 1σ is 0.68, within 2σ is 0.95, and 3σ is 0.99. These locations are the most prevalently used in any kind of statistical testing, and thus we will see them many times.

Moving from a General Normal to the Standard Normal and Back

In order to use the table of percentiles for the standard Normal distribution, we need to be able to translate from the Normal to the standard Normal and back again. Luckily, it is a simple process, and is one of the main reasons for using the Normal distribution - other distributions are not so easily manipulated.

The Standard Normal Distribution

The Normal distribution in the special case where $\mu = 0$ (the distribution is centered at $x = 0$) and $\sigma = 1$ (the distribution has a spread of 1).

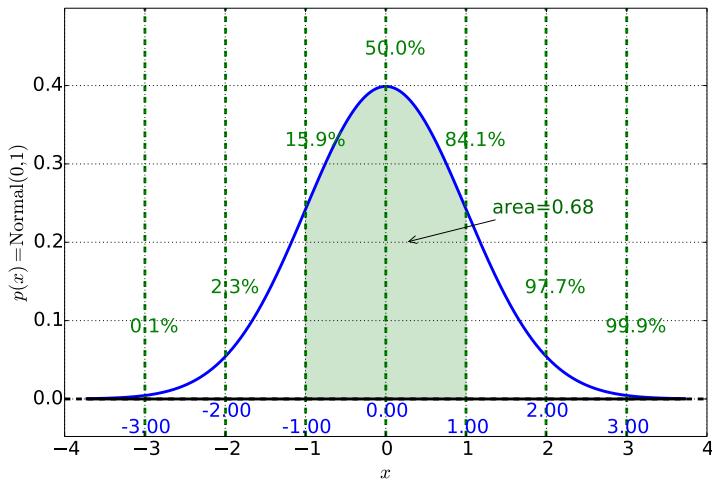


Figure 5.4: The Standard Normal Distribution (the Normal distribution in the special case where $\mu = 0$ and $\sigma = 1$). The percentiles shown are for positions $1-\sigma$ away from the center, $2-\sigma$ away, and $3-\sigma$ away. The area within $1-\sigma$ is 0.68, within $2-\sigma$ is 0.95, and $3-\sigma$ is 0.99. These locations are the most prevalently used in any kind of statistical testing, and thus we will see them many times.

To facilitate this translation, we will use the variable x for the Normal distribution and z for the standard Normal. So now, we need to have a recipe for translating x to z (or vice versa), given μ and σ . These recipes are:

- 1 x to z : subtract x by μ , and divide by σ
- 2 z to x : multiply z by σ and add μ

EXAMPLE 5.1 Given a Normal distribution with a mean of $\mu = 150$ and a $\sigma = 20$, what is the most likely value?

The most likely value is the peak of the probability distribution, $\hat{x} = \mu = 150$.

EXAMPLE 5.2 Given a Normal distribution with a mean of $\mu = 150$ and $\sigma = 30$, what is the probability $P(x > 170)$

To use the tables in Section D.3 on page 154, we first need to translate everything to the standard Normal values.

$$x = 170 \Rightarrow z = \frac{x - 150}{30} = 0.67$$

From the table in Section D.3 on page 154, the area to the left of $z = 0.67$ is 0.7486. Because we are asked the probability greater than $x = 170$ we need to have the area to the right of the curve, or

$$P(x > 170) = 1 - 0.7486 = 0.2514$$

or about $1/4$. In other words, with a mean $\mu = 150$ and deviation $\sigma = 20$, we'd expect about a quarter of the time that the value of the variable would be greater than 170. Or, given our uncertainty of a specific value, we'd assign a probability of around 25% to it being larger than 170.

EXERCISE 5.1 *Given a Normal distribution, with parameters $\mu = 10$ and $\sigma = 2$, determine the following probabilities:*

- 1 $P(x < 12)$
- 2 $P(6 < x < 14)$
- 3 $P(2 < x < 12)$

EXERCISE 5.2 *Given a Normal distribution, with parameters $\mu = 2$ and $\sigma = 10$, answer the following questions (see Table 1.1 on page 43 for reference):*

- 1 *Make a qualitative plot of the distribution to help you with the other parts of the question*
- 2 *Is likely that $x > 0$?*
- 3 *Above which value of x is it very unlikely to observe?*
- 4 *Below which value of x is it extremely unlikely to observe?*

EXERCISE 5.3 *Given a Normal distribution, with parameters $\mu = 2$ and $\sigma = 0.5$, answer the following questions (see Table 1.1 on page 43 for reference):*

- 1 *Make a qualitative plot of the distribution to help you with the other parts of the question*
- 2 *Is likely that $x > 0$?*
- 3 *Above which value of x is it very unlikely to observe?*
- 4 *Below which value of x is it extremely unlikely to observe?*

Sum and Differences

One more convenient property of the Normal distribution is that sums and differences of variables that individually have Normal distributions also have Normal distributions, although each with a different mean and deviation parameter. The relationships are summarized as follows.

Sum of two Normally distributed variables If we have two variables, x and y , which have Normal distributions

$$\begin{aligned} P(x) &= \text{Normal}(\mu_x, \sigma_x) \\ P(y) &= \text{Normal}(\mu_y, \sigma_y) \end{aligned}$$

then their sum, $x + y$, has a mean the sum of the two, $\mu_x + \mu_y$ and a deviation $\sqrt{\sigma_x^2 + \sigma_y^2}$.

One way to remember this is that the new *squared* deviation parameter is the sum of the two old ones,

$$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2$$

Differences between two Normally distributed variables For differences, $x - y$, we have a new mean of $\mu_x - \mu_y$ and deviation parameter again $\sqrt{\sigma_x^2 + \sigma_y^2}$. Note the “+” sign in the new σ , which keeps the new σ positive which is must be by definition.

If we are asked for the distribution of a quantity with an added constant, like

$$z = x + \text{constant}$$

then the probability of z is just the same as that of x (i.e. Normal distribution with the same deviation), with the location parameter moved by the constant

$$P(z) = \text{Normal}(\mu_x + \text{constant}, \sigma_x)$$

EXAMPLE 5.3 We have two Normal distributions $P(x) = \text{Normal}(\mu = 8, \sigma = 2)$ and $P(y) = \text{Normal}(\mu = 20, \sigma = 7)$. What is the distribution for $z = y - x$?

The distribution $P(z)$ is also a Normal distribution, with mean $\mu_z = 20 - 8 = 12$ and deviation $\sigma_z = \sqrt{7^2 + 2^2} = 7.3$.

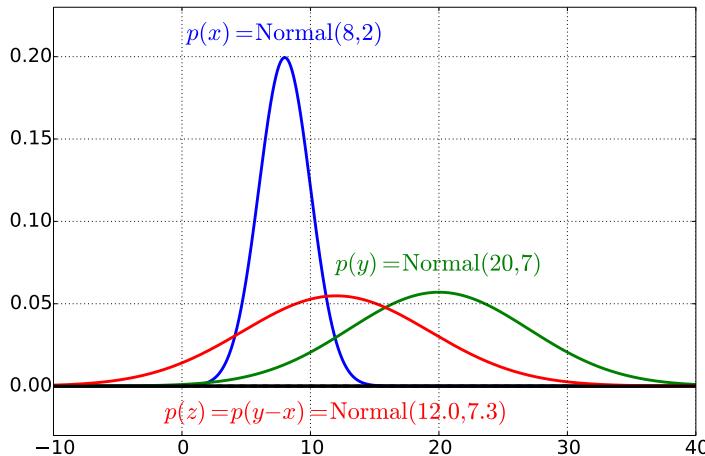
Sum of two Normally distributed variables If we have two Normally distributed variables, x and y , we have

$$\begin{aligned} P(x) &= \text{Normal}(\mu_x, \sigma_x) \\ P(y) &= \text{Normal}(\mu_y, \sigma_y) \\ P(x + y) &= \text{Normal}(\mu_x + \mu_y, \sqrt{\sigma_x^2 + \sigma_y^2}) \end{aligned}$$

Differences between two Normally distributed variables

$$P(x - y) = \text{Normal}(\mu_x - \mu_y, \sqrt{\sigma_x^2 + \sigma_y^2})$$

(Note the “+” sign in the new σ .)



Significance

There is a term used in the literature called *statistical significance*.² Roughly it means a value that is *very unlikely* to be zero (see Table 1.1 on page 43), or in other words, the value of zero is not within the 95% percentile. This is within 2 standard deviations of the value, so the following estimated values are *not statistically significant*:

- 5 ± 3 - the two-deviation range is $[-1, 11]$ contains the value 0
- 7 ± 4
- -3 ± 2

but the following *are statistically significant*:

- 5 ± 2 - the two-deviation range is $[1, 9]$ does not contain the value 0
- 7 ± 3
- -3 ± 1

Statistical significance, at the *very unlikely* level (i.e. 95% percentile) is often used as a rough guideline to publish a positive effect.

An unintuitive consequence One consequence of this is that two studies that seem different may not be statistically different. The following example is from Gelman and Hill's book on Data Analysis.³ Say, we have two measurements with means and standard deviations:

- 25 ± 10
- 10 ± 10

² Although the word "significant" occurs in the term "statistically significant", it does not imply that the result itself is important - it may be a small, uninteresting effect, but credibly non-zero. Perhaps a term like "statistically detectable" would be better, but we are unfortunately bound to the historical use of the term.

³ A. Gelman, J. Hill, and Ebooks Corporation. *Data analysis using regression and multilevel/hierarchical models*, volume 625. Cambridge University Press Cambridge, UK:, 2007

number of deviations away from zero	term	probability
1σ	slightly likely/likely	0.7 (i.e. 7/10)
2σ	very likely	0.95 (i.e. 19/20)
3σ	extremely likely	0.01 (i.e. 1/100)
$> 4\sigma$	virtually certain	$> 999,999/1,000,000$

Table 5.1: Rough guide for the conversion of deviations away from zero and the qualitative labels for probability values for being a *significant* deviation.

Further, let us suppose that we are interested in whether the measurements are zero or not. The first measurement shows a significant effect (the two-deviation range is [5,45] does not contain zero), while the second one doesn't (the two-deviation range is [-10,30] contains zero). The *difference* between them is

$$(25 - 10) \pm \sqrt{10^2 + 10^2} = 15 \pm 14$$

which is *not* significant. One should be careful comparing the magnitudes and uncertainties of measurements!

5.3 The Normal Distribution - Estimating From Data

Estimating the mean, μ , knowing the deviation, σ

Typically one is provided with a series of measurements of a quantity, and we want to *estimate* the value of that quantity, and have a description of our *uncertainty* in the estimate. In Chapter 6 (*Applications of Parameter Estimation and Inference* on page 107) we go through a number of detailed examples of this process. Here, we simply summarize the result. We are given:

- 1 A series of N measurements, $\text{data}=\{x_1, x_2, x_3, \dots, x_N\}$
- 2 The real deviation, σ
- 3 We are modeling the data as a true value, μ , with uncertainty with a likelihood from the Normal distribution with known deviation, σ , as in $\text{Normal}(0, \sigma)$. Further, we assume independence between the measurements.

Since in this case we are given σ , we wish then to estimate the parameter μ . The result will be a probability distribution over μ , with a best (i.e. most probable) value and an uncertainty in that value. The result is that the distribution of μ is also a Normal distribution,

$$P(\mu | \text{data}, \sigma) = \text{Normal}(\bar{x}, \sigma / \sqrt{N})$$

where the center value (and thus the most probable value of μ) is given by the sample mean of the data.

In scientific applications, this notation is often shortened to $\mu = \bar{x} \pm \sigma / \sqrt{N}$, so it is clear what is the best estimate of μ (i.e. \bar{x}) and what is the uncertainty in that estimate (i.e. σ / \sqrt{N}).

Sample Mean The *sample mean* of a set of N samples, x_1, x_2, \dots, x_N is given by

$$\bar{x} \equiv \frac{x_1 + x_2 + x_3 + \dots + x_N}{N}$$

The uncertainty in μ is given by σ/\sqrt{N} . As a consequence, larger N (i.e. more data points), makes us more confident in the particular estimate for μ .

Estimate of location parameter μ given N samples and σ , the known deviation In summary, the best estimate for the location parameter μ in the Normal distribution given a set of N samples, x_1, x_2, \dots, x_N is given by

$$\hat{\mu} = \frac{x_1 + x_2 + x_3 + \dots + x_N}{N} \pm \sigma/\sqrt{N}$$

EXAMPLE 5.4 Estimating the True Length of an Object

Say we have an object, and 5 measurements of its length from the same ruler but from different people,

$$5.1[\text{cm}], 4.9[\text{cm}], 4.7[\text{cm}], 4.9[\text{cm}], 5.0[\text{cm}]$$

Say that we further know that the uncertainty (given this ruler) of one measurement has $\sigma = 0.5[\text{cm}]$. What is the best estimate of the length? The best estimate should be given by the sample mean of these 5 samples,

$$\begin{aligned}\hat{\mu} &= \frac{x_1 + x_2 + \dots + x_N}{N} \\ &= \frac{5.1[\text{cm}] + 4.9[\text{cm}] + 4.7[\text{cm}] + 4.9[\text{cm}] + 5.0[\text{cm}]}{5} = 4.92[\text{cm}]\end{aligned}$$

with uncertainty related to the known uncertainty of a single measurement,

$$\begin{aligned}\hat{\sigma} &= \frac{\sigma}{\sqrt{N}} \\ &= \frac{0.5[\text{cm}]}{\sqrt{5}} = 0.223[\text{cm}]\end{aligned}$$

yielding a final best estimate of

$$\hat{\mu} = 4.92[\text{cm}] \pm 0.223[\text{cm}]$$

or (with 2σ range),

$$\hat{\mu} = 4.92[\text{cm}], 95\% \text{ CI} = [4.474[\text{cm}], 5.366[\text{cm}]]$$

Sample Mean The *sample mean* of a set of N samples, x_1, x_2, \dots, x_N is given by

$$\bar{x} \equiv \frac{x_1 + x_2 + x_3 + \dots + x_N}{N}$$

Estimate of location parameter μ given N samples and σ , the known deviation The best estimate for the location parameter μ in the Normal distribution given a set of N samples, x_1, x_2, \dots, x_N is given by

$$\hat{\mu} = \frac{x_1 + x_2 + \dots + x_N}{N} \pm \sigma/\sqrt{N}$$

In real measurements, there is always the problem of bias or *systematic* uncertainties, where the uncertainty does not follow a Normal distribution. We will not consider this issue here.

The 95% credible interval (CI) is really at the 1.96σ level, yielding $[4.481[\text{cm}], 5.358[\text{cm}]]$. We will almost always approximate it as 2σ by hand, but the computer will generate the true 95% credible interval when requested.

Estimating the mean, μ , not knowing the deviation, σ

If we are not so fortunate to be given the deviation, as in the previous case, then this parameter too must be estimated from the data. As a first step we can estimate the deviation with the *sample* deviation.

Sample Deviation The sample deviation of a set of N samples, x_1, x_2, \dots, x_N is given by

$$S \equiv \sqrt{\frac{1}{N-1} ((x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \dots + (x_N - \bar{x})^2)}$$

Approximate estimate of location parameter μ and deviation σ given N samples The posterior probability for μ and σ given a set of N samples, x_1, x_2, \dots, x_N can be approximated with

$$\begin{aligned} P(\mu|\text{data}) &\sim \text{Normal}(\bar{x}, S/\sqrt{N}) \\ P(\sigma|\text{data}) &\sim \text{Normal}\left(S, S^2/\sqrt{(N-1)/3}\right) \end{aligned}$$

which works well if we have many ($N > 30$) data points.

With a smaller data set, the value of S as an estimate for the deviation becomes too small. When the estimate for σ is too small, then the result is claiming *more confidence* in the estimate of the mean, μ , than is warranted. This discrepancy depends on the number of data points, and thus it makes sense that the proper distribution should depend on the number of data points, in addition to the sample mean and deviation. The proper, although less convenient, result is that the posterior probability for μ takes the form of the Student's t distribution,

Estimate of location parameter μ given N samples and unknown σ The posterior probability for μ takes the form of the Student's t distribution,

$$P(\mu|\text{data}) = \text{Student}_{\text{dof}=N-1}(\bar{x}, S/\sqrt{N})$$

This distribution requires *three* numbers to specify, referred to as the mean (μ), deviation (σ) and the *degrees of freedom* (dof). The degrees of freedom is defined in this case to be the number of data points less one, $N - 1$.

EXAMPLE 5.5 Estimating the True Length of an Object...Again

Say we have an object, and 5 measurements of its length from the same ruler but from different people,

$$5.1[\text{cm}], 4.9[\text{cm}], 4.7[\text{cm}], 4.9[\text{cm}], 5.0[\text{cm}]$$

Sample Deviation The sample deviation of a set of N samples, x_1, x_2, \dots, x_N is given by

$$S \equiv \sqrt{\frac{1}{N-1} ((x_1 - \bar{x})^2 + \dots + (x_N - \bar{x})^2)}$$

Approximate estimate of location parameter μ and deviation σ given N samples The posterior probability for μ and σ given a set of N samples, x_1, x_2, \dots, x_N can be approximated with

$$\begin{aligned} P(\mu|\text{data}) &\sim \text{Normal}(\bar{x}, S/\sqrt{N}) \\ P(\sigma|\text{data}) &\sim \text{Normal}(S, \\ &\quad S^2/\sqrt{(N-1)/3}) \end{aligned}$$

which works well if we have many ($N > 30$) data points.

Because the uncertainty in the mean depends explicitly on the number of data points, it goes beyond the level of this chapter to give a form for the posterior probability distribution for the deviation, σ .

Estimate of location parameter μ given N samples and unknown σ The posterior probability for μ takes the form of the Student's t distribution,

$$P(\mu|\text{data}) = \text{Student}_{\text{dof}=N-1}(\bar{x}, S/\sqrt{N})$$

This distribution requires *three* numbers to specify, referred to as the mean (μ), deviation (σ) and the *degrees of freedom* (dof). The degrees of freedom is defined in this case to be the number of data points less one, $N - 1$.

Unlike earlier, let's say that we don't know the uncertainty (given this ruler) of one measurement. What is the best estimate of the length?

Again, the best estimate should be given by the sample mean of these 5 samples,

$$\begin{aligned}\hat{\mu} &= \frac{x_1 + x_2 + \cdots + x_N}{N} \\ &= \frac{5.1[\text{cm}] + 4.9[\text{cm}] + 4.7[\text{cm}] + 4.9[\text{cm}] + 5.0[\text{cm}]}{5} = 4.92[\text{cm}]\end{aligned}$$

with uncertainty related to the sample deviation

$$\begin{aligned}s^2 &= \frac{1}{N-1} \left((x_1 - \bar{x})^2 + \cdots + (x_N - \bar{x})^2 \right) \\ &= \frac{1}{5-1} \left((5.1[\text{cm}] - 4.92[\text{cm}])^2 + (4.9[\text{cm}] - 4.92[\text{cm}])^2 + (4.7[\text{cm}] - 4.92[\text{cm}])^2 + \right. \\ &\quad \left. (4.9[\text{cm}] - 4.92[\text{cm}])^2 + (5.0[\text{cm}] - 4.92[\text{cm}])^2 \right) \\ &= 0.024[\text{cm}]^2 \\ s &= \sqrt{0.024[\text{cm}]^2} = 0.155[\text{cm}] \\ \frac{s}{\sqrt{N}} &= \frac{0.155[\text{cm}]}{\sqrt{5}} = 0.069[\text{cm}]\end{aligned}$$

Looking at Table D.2 on page 152 with "Degrees of Freedom" equal to 4, we find that the 95% credible interval for μ (between areas 0.025 and 0.975) falls $\pm 2.776 \cdot S / \sqrt{N}$, thus we have

$$\begin{aligned}\hat{\mu} &= 4.92[\text{cm}], 95\% \text{ CI} = [4.92[\text{cm}] - 2.776 \cdot 0.069[\text{cm}], 4.92[\text{cm}] + 2.776 \cdot 0.069[\text{cm}]] \\ &= 4.92[\text{cm}], 95\% \text{ CI} = [4.73[\text{cm}], 5.11[\text{cm}]]\end{aligned}$$

Although much of this is easier with the computer, it is instructive to go through simple examples by hand.

5.4 Normal Approximation

The Normal distribution is useful for many reasons: its simple shape, the fact that there are only two parameters which describe it, and the ease with which one can compare the general Normal distribution to the single standard Normal. Further, it can be used as an approximation for several other distributions, under certain limits.

The Beta Distribution

We first saw the beta distribution as the posterior description in a bent-coin parameter estimation problem (see Section 4.3 on page 73 in Chapter 4 (*Introduction to Parameter Estimation*)). The Normal approximation occurs when the number of flips gets large, compared

to how likely the coin flips heads. For notation, we will write the frequency of heads as

$$f \equiv \frac{h}{N}$$

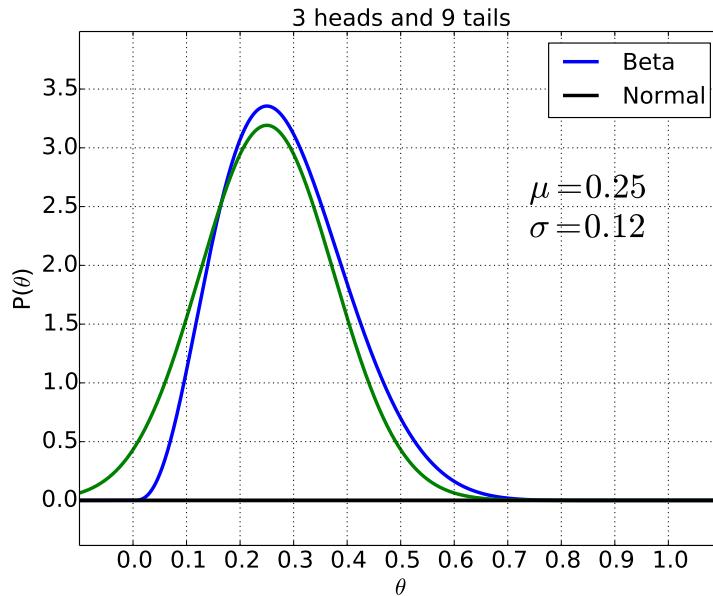
Normal Approximation to the Beta Distribution The Normal Approximation to the Beta Distribution , for large number of flips (N) of which a fraction $f \equiv h/N$ are successful is given by

$$\text{Beta}(h, N) \sim \text{Normal} \left(\mu = f, \sigma = \sqrt{f(1-f)/N} \right)$$

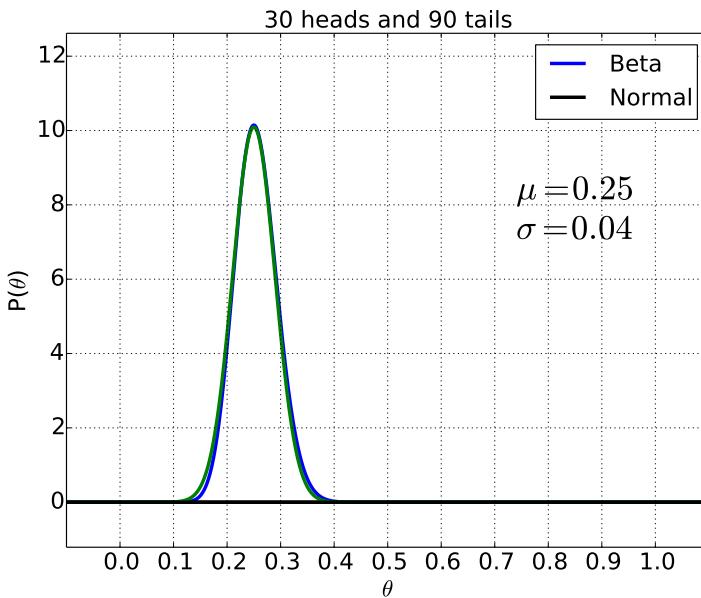
Normal Approximation to the Beta Distribution The Normal Approximation to the Beta Distribution , for large number of flips (N) of which a fraction $f \equiv h/N$ are successful is given by

$$\begin{aligned} \text{Beta}(h, N) &\sim \text{Normal} (\mu = f, \\ &\quad \sigma = \sqrt{f(1-f)/N}) \end{aligned}$$

To see how close this approximation can be, observe the following two cases:



With ten times as many flips, we have



and the curves are so close as to be nearly identical! There still is a (small) probability for getting a negative θ , which is problematic in theory but not typically in practice. To use the properties of the Normal distribution here to quantify our uncertainty about the bent coin. Given 30 heads and 90 tails, the best estimate for θ (i.e. the top of the curve) is 0.25. Our uncertainty is quantified by the width of the distribution, given by σ . Thus, we can be confident to a 95% degree for θ within 2σ , or between 0.17 and 0.33 ($0.25 - 2 \cdot 0.04$ and $0.25 + 2 \cdot 0.04$, respectively).

This is an *approximation*, and as such will certainly give seriously incorrect answers under certain circumstances. For example, in this case, the Normal approximation predicts that there is around a 1.8% chance that the bent coin might have a *negative* θ , or probability of flipping heads (look at the Normal curve to the left of $\theta = 0$)! The beta distribution is zero for any value below zero or over one, and thus will never lead to such absurd answers.

The Binomial Distribution

Similarly, with the (discrete) binomial distribution (see Equation ??) we have the Normal approximation.

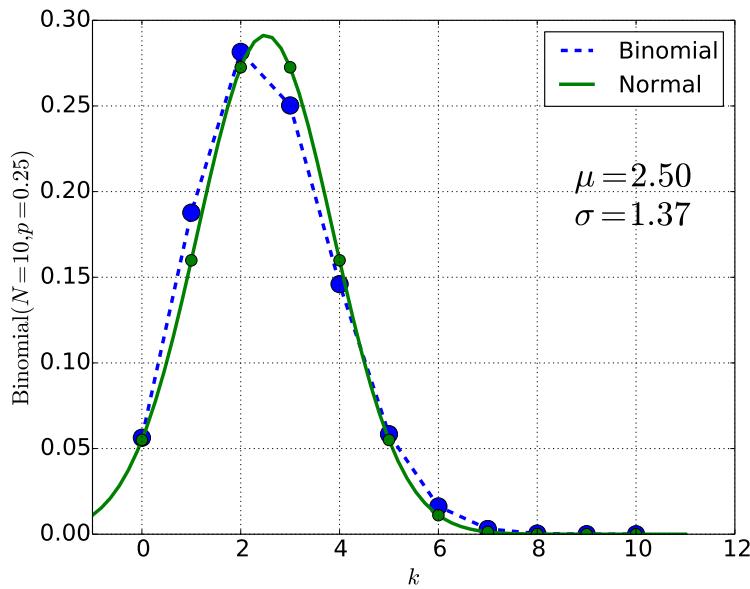
Normal Approximation to the Discrete Binomial Distribution

$$\text{Binomial}(N, p) \approx \text{Normal}(\mu = N \cdot p, \sigma = \sqrt{N \cdot p(1 - p)})$$

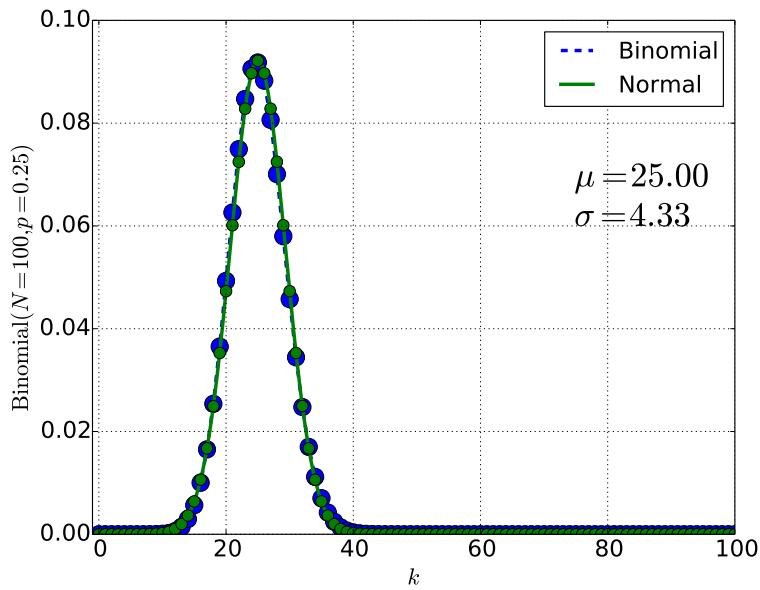
with examples

Normal Approximation to the Discrete Binomial Distribution

$$\begin{aligned} \text{Binomial}(N, p) &= \text{Normal}(\mu = N \cdot p, \\ &\quad \sigma = \sqrt{N \cdot p(1 - p)}) \end{aligned}$$



and



The Student's t Distribution

For smallish data sets, $5 < N < 30$, we can replace the estimate of the mean from the Student's *t* distribution to a Normal distribution with an increased estimate for the deviation. It then becomes practical to use the more convenient *z*-score to estimate credible intervals rather

than the full t tables. The approximation in this domain looks like⁴

Normal Approximation to the Student's t Distribution For smallish data sets, $5 < N < 30$,

$$\text{Student}_{\text{dof}=N-1}(\bar{x}, S/\sqrt{N}) \sim \text{Normal}(\bar{x}, Sk/\sqrt{N})$$

$$k \equiv 1 + \frac{20}{N^2}$$

EXAMPLE 5.6 Estimating the True Length of an Object... Yet Again

Say we have an object, and 5 measurements of its length from the same ruler but from different people,

$$5.1[\text{cm}], 4.9[\text{cm}], 4.7[\text{cm}], 4.9[\text{cm}], 5.0[\text{cm}]$$

Unlike earlier, let's say that we don't know the uncertainty (given this ruler) of one measurement. What is the best estimate of the length?

Again, the best estimate should be given by the sample mean of these 5 samples,

$$\hat{\mu} = \frac{x_1 + x_2 + \cdots + x_N}{N}$$

$$= \frac{5.1[\text{cm}] + 4.9[\text{cm}] + 4.7[\text{cm}] + 4.9[\text{cm}] + 5.0[\text{cm}]}{5} = 4.92[\text{cm}]$$

with uncertainty related to the adjusted sample deviation,

$$S^2 = \frac{1}{N-1} ((x_1 - \bar{x})^2 + \cdots + (x_N - \bar{x})^2)$$

$$= \frac{1}{5-1} ((5.1[\text{cm}] - 4.92[\text{cm}])^2 + (4.9[\text{cm}] - 4.92[\text{cm}])^2 + (4.7[\text{cm}] - 4.92[\text{cm}])^2 + (4.9[\text{cm}] - 4.92[\text{cm}])^2 + (5.0[\text{cm}] - 4.92[\text{cm}])^2)$$

$$= 0.024[\text{cm}]^2$$

$$S = \sqrt{0.024[\text{cm}]^2} = 0.155[\text{cm}]$$

$$\frac{S}{\sqrt{N}} = \frac{0.155[\text{cm}]}{\sqrt{5}} = 0.069[\text{cm}]$$

$$k = 1 + \frac{20}{5^2} = 1.8$$

$$k \cdot \frac{S}{\sqrt{N}} = 1.8 \cdot 0.069[\text{cm}] = 0.124[\text{cm}]$$

yielding a final best estimate of

$$\hat{\mu} = 4.92[\text{cm}] \pm 0.124[\text{cm}]$$

or (with 2σ range),

$$4.92[\text{cm}], 95\% \text{ CI} = [4.672[\text{cm}], 5.168[\text{cm}]]$$

Compare this range to the one shown in Example 5.5 on page 97. The one here has a slightly larger range, which is a bit more conservative than is needed, but the calculation is quite a bit easier.

⁴ D. Berry. *Statistics: A Bayesian Perspective*. Duxbury, 1996

Normal Approximation to the Student's t Distribution For smallish data sets, $5 < N < 30$,

$$\text{Student}_{\text{dof}=N-1}(\bar{x}, S/\sqrt{N}) \sim \text{Normal}(\bar{x}, k \cdot S/\sqrt{N})$$

$$k \equiv 1 + \frac{20}{N^2}$$

5.5 Summary

It is useful to see all of these results stemming from the same Bayes' Recipe, applied to different models of the data and (possibly) different prior probabilities. As we have stated, many of the simple cases have been worked out by the mathematicians, so we don't need to do the work of deriving them. It will be our task to understand their properties, to be able to apply them to real problems, and to understand their consequences. One of the immediate observations that we make is the prevalence of the *Normal* distribution, justifying our detailed exploration of it in this chapter.

1 Proportions

Parameter of Interest: θ , the chances of a single event

Applications: coin flips, voting percentages, success in sports, performance on tests

Form of the data: h successes in N total events

Model of the data:

$$\text{data} = \begin{cases} \text{success} & , \text{with probability } \theta \\ \text{failure} & , \text{otherwise (i.e. with probability } 1 - \theta) \end{cases}$$

Posterior Probability:

$$\underbrace{\text{Beta}(\theta|\text{data})}_{\text{posterior probability}} \sim \overbrace{\text{Binomial}(\text{data}|\theta)}^{\text{likelihood}} \times \underbrace{\text{Uniform}(\theta)}_{\text{prior probability}}$$

2 Magnitude with Known Deviation

Parameter of Interest: μ , the true magnitude of a quantity, given the deviation, labeled by σ , from the central value

Applications: percentages with large samples, scientific measurements such as weight and size of objects, time scales of events

Form of the data: N total data points, labeled x_1, x_2, \dots, x_N , and given known σ

Model of the data:

$$\text{data} = \mu + \text{uncertainty with probability Normal}(\mu = 0, \text{known } \sigma)$$

Posterior Probability:

$$\underbrace{\text{Normal}(\mu_2|\text{data}, \sigma)}_{\text{posterior probability}} \sim \overbrace{\text{Normal}(\text{data}|\mu, \sigma)}^{\text{likelihood}} \times \underbrace{\text{Uniform}(\mu)}_{\text{prior probability}}$$

3 Magnitude with Unknown Deviation

Parameter of Interest: μ , the true magnitude of a quantity, and the unknown deviation, labeled by σ , from the central value

Applications: scientific measurements with small samples (less than around 30), such as weight and size of objects, time scales of a small number of events

Form of the data: N total data points, labeled x_1, x_2, \dots, x_N

Model of the data:

$$\text{data} = \mu + \text{uncertainty with probability } \text{Normal}(\mu = 0, \sigma)$$

Posterior Probability:

$$\underbrace{P(\mu, \sigma | \text{data})}_{\text{posterior probability}} \sim \overbrace{\text{Normal}(\text{data} | \mu, \sigma)}^{\text{likelihood}} \times \underbrace{\text{Uniform}(\mu) \cdot \text{Uniform}(\log \sigma)}_{\text{prior probability}}$$

$$\underbrace{\text{Student-T}(\mu | \text{data})}_{\text{posterior probability}} \sim [P(\mu, \sigma | \text{data})]_{\text{marginalized over } \sigma}$$

$$\underbrace{F(\sigma | \text{data})}_{\text{posterior probability}} \sim [P(\mu, \sigma | \text{data})]_{\text{marginalized over } \mu}$$

5.6 Computer Examples

```
from sie import *
```

Estimating Lengths

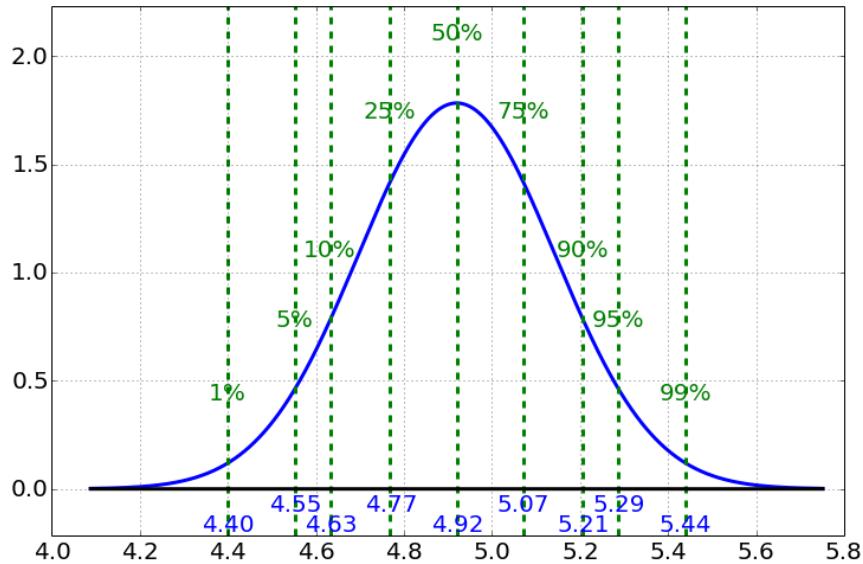
Known deviation, σ

```
x=[5.1, 4.9, 4.7, 4.9, 5.0]
sigma=0.5
```

```
mu=sample_mean(x)
N=len(x)
```

```
dist=normal(mu, sigma/sqrt(N))
distplot(dist)
```

```
<matplotlib.figure.Figure at 0x10713c710>
```



```
credible_interval(dist)
```

```
(4.4817387297117088, 4.919999999999999, 5.358261270288291)
```

Unknown σ

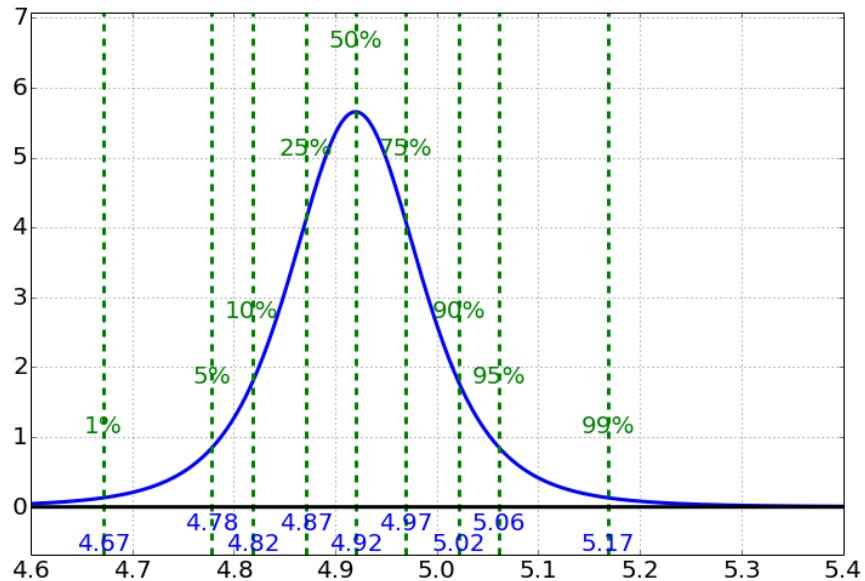
```
mu=sample_mean(x)
s=sample_deviation(x)
print mu,s
```

```
4.92 0.148323969742
```

```
dist=t dist(N-1,mu,s/sqrt(N))
```

```
distplot(dist,xlim=[4.6,5.4])
```

```
<matplotlib.figure.Figure at 0x1085b5c50>
```



```
credible_interval(dist)
```

```
(4.7358314667008017, 4.9199999999999999, 5.1041685332991982)
```

6 Applications of Parameter Estimation and Inference

6.1 Normal Model - Inference about Means

EXAMPLE 6.1 *Iris petal lengths - Best estimate*

1.4	1.4	1.3	1.5	1.4
1.7	1.4	1.5	1.4	1.5
1.5	1.6	1.4	1.1	1.2
1.5	1.3	1.4	1.7	1.5
1.7	1.5	1.0	1.7	1.9
1.6	1.6	1.5	1.4	1.6
1.6	1.5	1.5	1.4	1.5
1.2	1.3	1.5	1.3	1.5
1.3	1.3	1.3	1.6	1.9
1.4	1.6	1.4	1.5	1.4

Table 6.1: Iris petal lengths, in centimeters, for Iris type *Setosa*.

Table 6.1 shows data for the lengths (in centimeters) of the petals of one species of Iris flower¹. If we want to estimate the “true” length of the petal for this species, given all of these examples, we would apply the following model of the data:

$$\text{data} = \text{true value} + \text{Normal}(\text{mean}=0, \text{known } \sigma)$$

or equivalently

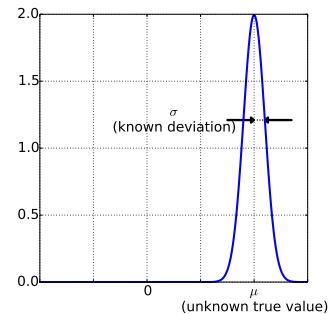
$$\text{data} = \text{Normal}(\text{mean}=\text{true value}, \text{known } \sigma)$$

The resulting distribution for the “true value”, μ , is also a *Normal* distribution (Section 5.3),

$$P(\mu | \text{data}, \sigma) = \text{Normal}(\bar{x}, \sigma / \sqrt{N})$$

where the best estimate of the true value, μ is the sample mean, \bar{x} , and the uncertainty is related to the sample deviation (which we’re

¹ K. Bache and M. Lichman. UCI machine learning repository, 2013. URL <http://archive.ics.uci.edu/ml>



going to take as the “known” deviation, $\sigma \sim 0.174$) in this case. Thus,

$$\begin{aligned}\hat{\mu} &= \bar{x} = \frac{1.4 + 1.4 + 1.3 + 1.5 + \cdots + 1.6 + 1.4 + 1.5 + 1.4}{50} \\ &= 1.464\end{aligned}$$

and the full answer, with uncertainty, is

$$\begin{aligned}\hat{\mu} &= 1.464[\text{cm}] \pm \frac{0.174}{\sqrt{50}}[\text{cm}] \\ &= 1.464[\text{cm}] \pm 0.025[\text{cm}]\end{aligned}$$

EXAMPLE 6.2 Iris petal lengths - A different species?

Imagine we have a single observation of another iris with petal length 2.5 [cm]. Is this likely to be the same type as the *Setosa* type above? As outlined in Section 5.2, we get the best estimate for the difference as:

$$\mu_{\text{diff}} = 2.5 - 1.464 = 1.036$$

with uncertainty the same as the uncertainty of the *Setosa* type, so the final estimate with uncertainty is:

$$1.036[\text{cm}] \pm 0.025[\text{cm}]$$

which is

$$\frac{1.036[\text{cm}]}{0.025[\text{cm}]} = 41 \text{ deviations away from zero!}$$

which makes it *virtually certain* to be a different type (see Table 5.1).

6.2 Normal Model Again - Inference about Means and Deviations

<i>Setosa</i>	1.4	1.4	1.3	1.5	1.4
<i>Virginica</i>	6.0	5.1	5.9	5.6	5.8
<i>Versicolor</i>	4.7	4.5	4.9	4.0	4.6

Table 6.2: Subset of iris petal lengths, in centimeters, for iris types *Virginica*, *Setosa*, and *Versicolor*.

EXAMPLE 6.3 Iris petal lengths - Significantly different?

Shown in Table 6.2 is a very small subset of the full iris petal-length data. Are the types *Virginica* and *Versicolor* longer than the type *Setosa*? Is the *Virginica* longer than *Versicolor*? For each of these, we need to specify the model, determine the best estimate for the parameters of the model, and then compare the distributions.

The model we will use is the simple Normal model,

```
data = Normal(mean=true value, unknown σ)
```

which is the same as the previous example, except that the deviation, σ , is unknown. In addition to being unknown, there are so few data points that the deviation can't be well approximated with the sample deviation.

The resulting distribution for the "true value", μ , is a *Student-t* distribution (Section 5.3),

$$P(\mu|\text{data}) = \text{Student}_{\text{dof}=N-1}(\bar{x}, S/\sqrt{N})$$

The best estimates for the true length-values of each type is given by their sample means,

$$\begin{aligned}\hat{\mu}_{\text{setosa}} &= \frac{1.4 + 1.4 + 1.3 + 1.5 + 1.4}{5} = 1.40 \\ \hat{\mu}_{\text{virginica}} &= \frac{6.0 + 5.1 + 5.9 + 5.6 + 5.8}{5} = 5.68 \\ \hat{\mu}_{\text{versicolor}} &= \frac{4.7 + 4.5 + 4.9 + 4.0 + 4.6}{5} = 4.54\end{aligned}$$

and the sample deviations for each is given by

$$\begin{aligned}S_{\text{setosa}} &= \sqrt{\frac{1}{5-1} \cdot ((1.4 - 1.40)^2 + (1.4 - 1.40)^2 + (1.3 - 1.40)^2 + (1.5 - 1.40)^2 + (1.4 - 1.40)^2)} \\ &= 0.07 \\ S_{\text{virginica}} &= \sqrt{\frac{1}{5-1} \cdot ((6.0 - 5.68)^2 + (5.1 - 5.68)^2 + (5.9 - 5.68)^2 + (5.6 - 5.68)^2 + (5.8 - 5.68)^2)} \\ &= 0.36 \\ S_{\text{versicolor}} &= \sqrt{\frac{1}{5-1} \cdot ((4.7 - 4.54)^2 + (4.5 - 4.54)^2 + (4.9 - 4.54)^2 + (4.0 - 4.54)^2 + (4.6 - 4.54)^2)} \\ &= 0.34\end{aligned}$$

The posterior probability distributions, shown in Figure 6.1, have the following form:

$$\begin{aligned}P(\mu_{\text{setosa}}|\text{data}) &= \text{Student}_{\text{dof}=4}(1.40, 0.07/\sqrt{5}) \\ P(\mu_{\text{virginica}}|\text{data}) &= \text{Student}_{\text{dof}=4}(5.68, 0.36/\sqrt{5}) \\ P(\mu_{\text{versicolor}}|\text{data}) &= \text{Student}_{\text{dof}=4}(4.64, 0.34/\sqrt{5})\end{aligned}$$

It is clear from the picture that they are very well separated, but we can quantify this by looking at the probability that the difference between their means is greater than zero.

The probability of their difference approximately takes the form of a Student's *t* distribution, with the same center and deviation shown for the Normal in Section 5.2. Here we do the calculation between the closest two iris types, *Virginica* and *Versicolor*:

$$\begin{aligned}\mu_{\text{diff}} &= 5.68 - 4.64 = 1.04 \\ \sigma_{\text{diff}} &= \sqrt{\frac{0.36^2}{5} + \frac{0.34^2}{5}} = 0.22\end{aligned}$$

This approximation is called Welch's method. The exact analysis is beyond this book, but numerically one can calculate it and it doesn't differ from this approximate analysis in any significant way. Essentially you calculate $P(\mu_{\text{versicolor}} > \mu_{\text{virginica}}|\text{data})$ by adding up the $P(\mu_{\text{versicolor}}|\text{data}) \times P(\mu_{\text{virginica}}|\text{data})$ for all possible lengths where *versicolor* is longer than *virginica*.

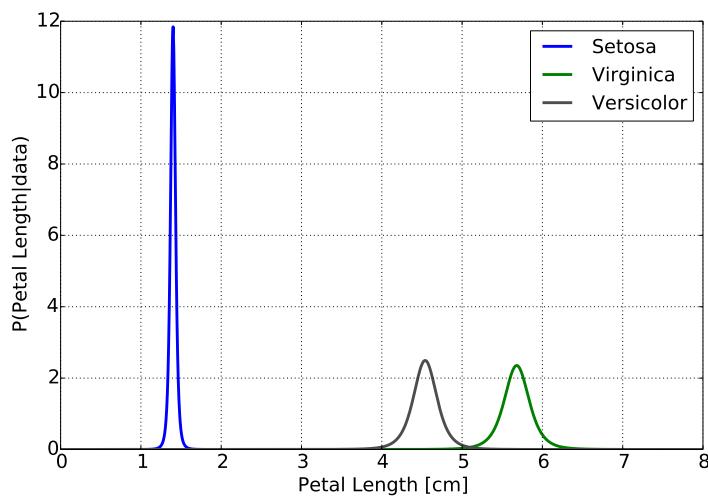


Figure 6.1: Probability distributions for the subset of iris petal lengths. Each distribution follows a Student's t form.

The degrees of freedom used for this Student's t distribution is approximately the smallest one from the two samples, or in this case (since both samples have the same number of data points), $\text{dof}=4$. The resulting posterior probability distribution for the difference of means is shown in Figure 6.2.

We observe that the difference of the means is over 4 times the deviation away from zero, so even with 4 degrees of freedom, this is significant at the 99% level. We can be highly certain that these two species have different petal lengths, and that the difference observed is not just a product of the random sample.

EXAMPLE 6.4 Ball Bearing Sizes

Here's a data set, measuring the size of ball bearings² from two different production lines.

First line [microns]										
1.18	1.42	0.69	0.88	1.62	1.09	1.53	1.02	1.19	1.32	
Second line [microns]										
1.72	1.62	1.69	0.79	1.79	0.77	1.44	1.29	1.96	0.99	

We can ask questions such as:

- What is our best estimate of the size of a ball bearing, given one of the production lines?
- Is it reasonable to believe that there is a difference in the size produced between the two lines?

² David J Hand, Fergus Daly, K McConway, D Lunn, and E Ostrowski. *A handbook of small data sets*, volume 1. CRC Press, 2011

Table 6.3: Production lines are produce a ball bearing with a diameter of approximately 1 micron. Ten ball bearings were randomly picked from the production line (i.e. the *First line*) at one time, and then again for a different production line (i.e. the *Second line*). Romano, A. (1977) *Applied Statistics for Science and Industry*.

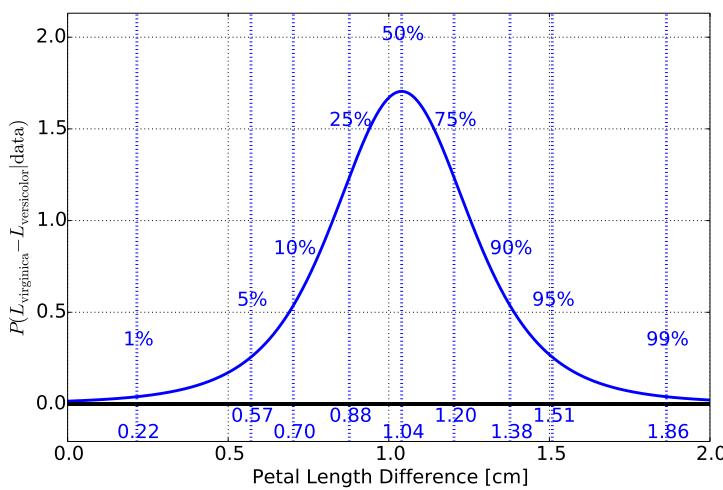


Figure 6.2: Probability distributions for the difference between iris petal lengths for the closest two iris types, *Virginica* and *Versicolor*. The distribution follows a Student-t form, and clearly shows significant probability (greater than 99%) for being greater than zero.

EXAMPLE 6.5 What is the best estimate (and uncertainty) for each of the two production lines of ball bearings?

Using the normal approximation to the Student-T distribution (Section 5.4), we have the best estimates of the two lines as

$$\begin{aligned}\mu_1 &= \frac{1.180000 + 1.420000 + 0.690000 + \dots + 1.190000 + 1.320000}{10} \\ &= 1.194\end{aligned}$$

$$\begin{aligned}\mu_2 &= \frac{1.720000 + 1.620000 + 1.690000 + \dots + 1.960000 + 0.990000}{10} \\ &= 1.406\end{aligned}$$

and their uncertainties calculated by first calculating the sample deviations

$$\begin{aligned}s_1 &= \sqrt{\frac{1}{10-1} \cdot ((1.18 - 1.194)^2 + (1.42 - 1.194)^2 + \dots + (1.19 - 1.194)^2 + (1.32 - 1.194)^2)} \\ &= 0.289 \\ s_2 &= \sqrt{\frac{1}{10-1} \cdot ((1.72 - 1.406)^2 + (1.62 - 1.406)^2 + \dots + (1.96 - 1.406)^2 + (0.99 - 1.406)^2)} \\ &= 0.428\end{aligned}$$

and then scaling the deviations by the number of data points

$$\begin{aligned}\sigma_1 &= s_1 / \sqrt{10} = 0.092 \\ \sigma_2 &= s_2 / \sqrt{10} = 0.135\end{aligned}$$

yielding the best estimates and uncertainties for the two production lines

- Production line 1: 1.194 [microns] \pm 0.092[microns]
- Production line 2: 1.406 [microns] \pm 0.135[microns]

or looking at the 95% CI for each line

This is just the $\pm 2 \cdot \sigma$ range

- Production line 1: 1.01[microns] - 1.378[microns]
- Production line 2: 1.136[microns] - 1.676[microns]

Roughly, given that these intervals overlap, there is not strong evidence that there is a difference between the two lines.

EXAMPLE 6.6 *Is it reasonable to believe that there is a difference in the size produced between the two lines?*

Using the best estimate of the difference, we get

$$\delta_{12} = \mu_2 - \mu_1 = 0.212$$

with the uncertainty in the difference from the individual uncertainties,

$$\sigma_{12} = \sqrt{\sigma_1^2 + \sigma_2^2} = 0.163$$

So the $2 \cdot \sigma$ uncertainty range for the difference,

$$[0.212 - 2 \cdot 0.163, 0.212 + 2 \cdot 0.163] = [-0.114, 0.538]$$

includes the value zero, which we interpret as a statement that the difference is *not statistically significant*. In other words, it is *not reasonable* to believe that there is a difference in the size produced between the two lines.

6.3 Beta Model - Inference About Proportions

EXAMPLE 6.7 *The Sunrise Problem*

The sunrise problem, as first stated by Laplace, is “What is the probability that the sun will rise tomorrow?” We’ll start with the assumption that initially one has never seen a sunrise, and then observe a year of sunrises each morning with no morning without one. Thus we have the form of the data as h successes (days with a sunrise) in N total days. Our model of the data is specified as before with a binomial distribution, resulting in the posterior Beta, as described in Section 4.6.

After a only 10 years of watching sunrises, and no failures of a sunrise, the best estimate for the probability of a sunrise is

$$\begin{aligned}\hat{\theta}_{\text{median}} &\approx \frac{h+2}{N+4} \\ &= \frac{3650+2}{3650+4} = 0.9995\end{aligned}$$

making it virtually certain for a sunrise.

EXAMPLE 6.8 *Cancer Rates*

This example is from Donald Berry's Statistics textbook³:

pp 192: A study (Murphy and Abbey, Cancer in Families, 1959) addressed the question of whether cancer runs in families. The investigator identified 200 women with breast cancer and another 200 women without breast cancer and asked them whether their mothers had had breast cancer. Of the 400 women in the two groups combined, 10 of the mothers had had breast cancer. If there is no genetic connection, then about half of these 10 would come from each group.

³ D. Berry. *Statistics: A Bayesian Perspective*. Duxbury, 1996

The data is that 7 of the daughters had cancer and 3 did not. Is there strong evidence of a connection?

The proper way, assuming total initial ignorance, is to use the Beta distribution:

$$P(\theta_{\text{cancer}} | \text{data}) = \text{Beta}(h = 7, N = 10)$$

which has a median of $\hat{\theta}_{\text{cancer}} = 0.68$, but a 95% credible interval of $\hat{\theta}_{\text{cancer}} = 0.39$ up to $\hat{\theta}_{\text{cancer}} = 0.89$. This means there is not strong evidence of an effect.

EXAMPLE 6.9 *Cancer Rates - Normal Approximation*

We can estimate the the Beta distribution median and credible intervals with a Normal distribution, by using the "assuming 2 successes and 2 failures" method.

$$\begin{aligned}\hat{\theta}_{\text{cancer}} &= \frac{h+2}{N+4} \\ &= \frac{7+2}{10+4} = 0.643\end{aligned}$$

and

$$\begin{aligned}\sigma &= \sqrt{\hat{\theta}_{\text{cancer}}(1-\hat{\theta}_{\text{cancer}})/(N+4)} \\ &= \sqrt{0.643(1-0.643)/(10+4)} \\ &= 0.128\end{aligned}$$

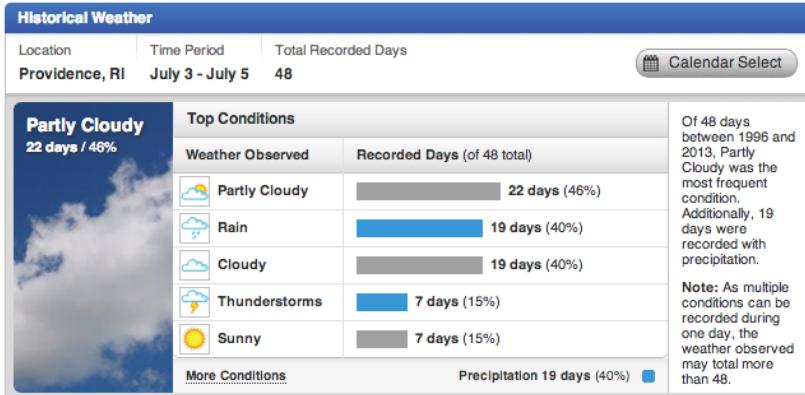
So the approximate 95% credible interval is

$$\hat{\theta}_{\text{cancer}} \pm 2\sigma$$

which is between 0.387 and 0.899, again with the same conclusion of no strong evidence of an effect.

EXAMPLE 6.10 *Will it rain on the 4th of July?*

In the United States, the 4th of July is Independence Day, and is known for parades. The oldest continuously running parade is in Bristol, RI, and it runs rain or shine. Is it likely to rain on the parade? Climate data from nearby Providence is here from wunderground.com:



We can estimate the Beta distribution median and credible intervals with a Normal distribution, by using the “assuming 2 successes and 2 failures” method.

$$\begin{aligned}\hat{\theta}_{\text{rain}} &= \frac{h+2}{N+4} \\ &= \frac{19+2}{48+4} = 0.404\end{aligned}$$

around 40%, less than an even chance (50%) of rain, but

$$\begin{aligned}\sigma &= \sqrt{\hat{\theta}_{\text{rain}}(1-\hat{\theta}_{\text{rain}})/(N+4)} \\ &= \sqrt{0.404(1-0.404)/(48+4)} \\ &= 0.068\end{aligned}$$

So the approximate 95% credible interval is

$$\hat{\theta}_{\text{rain}} \pm 2\sigma$$

which is between 0.268 and 0.540. This is not strong evidence against a purely fair and random “coin flip” for rain on the 4th of July.

EXAMPLE 6.11 Hot Hand Reexamined

In Tversky and Gilovich⁴ we have the following data for Larry Bird free throws in basketball:

- Given each of 53 missed shots, Larry Bird successfully shot 48 of the *next* attempt.
- Given each of 285 successful shots, Larry Bird successfully shot 251 of the *next* attempt.

⁴ A. Tversky and T. Gilovich. The cold facts about the “hot hand” in basketball. *Anthology of statistics in sports*, 16:169, 2005

This data alone almost suggests an anti-hot-hand (where you're less likely to make a successful attempt following a successful shot). However, we can demonstrate that these numbers are not in fact statistically different. Given the relatively large number of attempts (greater than 30) we can use the Normal approximation to estimate the two probabilities of success:

$$\begin{aligned}\theta_{\text{after a miss}} &= \frac{48+2}{53+4} = 0.877 \\ \theta_{\text{after a success}} &= \frac{251+2}{285+4} = 0.875\end{aligned}$$

and the uncertainty,

$$\begin{aligned}\sigma_{\text{after a miss}} &= \sqrt{0.877(1-0.877)/(53+4)} = 0.044 \\ \sigma_{\text{after a success}} &= \sqrt{0.875(1-0.875)/(285+4)} = 0.019\end{aligned}$$

making the 95% credible intervals for probability of a Larry Bird successful attempt

$$\begin{aligned}95\%CI_{\text{after a miss}} &= 0.877 \pm 2 \cdot 0.044 = [0.789, 0.965] \\ 95\%CI_{\text{after a success}} &= 0.875 \pm 2 \cdot 0.019 = [0.837, 0.913]\end{aligned}$$

Notice that the intervals overlap, so there is no significant evidence for a difference in Larry Bird's success following another success or following a miss. Thus, there is no significant evidence for a hot hand, or an anti-hot hand.

6.4 Model Construction

In practice, we either don't know what the optimum model we need is, or the needs of the model change as we obtain more data.

We start with the data in Table 6.4 for the mass of pennies of various years (shown graphically in Figure 6.3)⁵:

We are going to ignore the measurement uncertainties in these individual measurements, because they are quite small.

EXAMPLE 6.12 Mass of the Penny, Model 1 - One True Value

If we assume a model that states that there is a "true" value and the variation from this "true" value caused by some unknown process, but with known magnitude, σ ,

$$\text{data} = \text{true value} + \text{Normal}(\text{mean}=0, \text{known } \sigma)$$

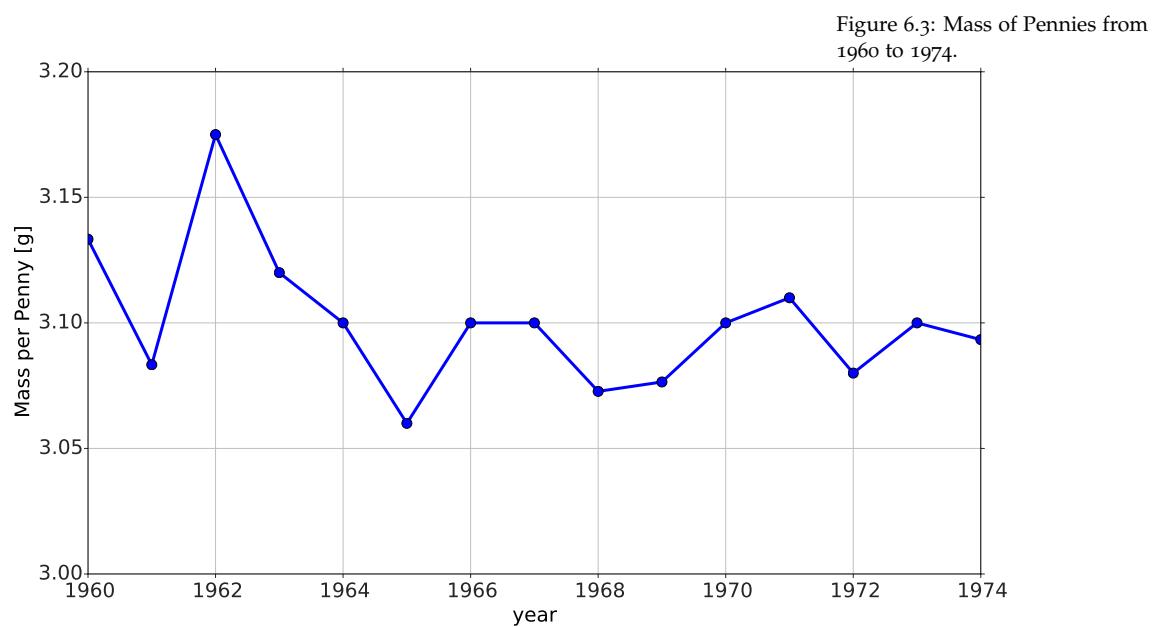
or equivalently

$$\text{data} = \text{Normal}(\text{mean}=\text{true value}, \text{known } \sigma)$$

⁵ this data was extracted from student measurements during a physics lab

Year	Mass [g]
1960	3.133
1961	3.083
1962	3.175
1963	3.120
1964	3.100
1965	3.060
1966	3.100
1967	3.100
1968	3.073
1969	3.076
1970	3.100
1971	3.110
1972	3.080
1973	3.100
1974	3.093

Table 6.4: Mass of Pennies from 1960 to 1974.



we can get the best estimate and uncertainty in that estimate from the following procedure, Using the normal approximation to the Student-T distribution (Section 5.4):

$$\hat{\mu} = \bar{x} \pm k \cdot S / \sqrt{N}$$

where the symbols in this equation are

- 1 the number of data points, N .
- 2 the best estimate for the true value, $\hat{\mu}$, is given by the *sample mean*, \bar{x} :

$$\begin{aligned}\bar{x} &= \frac{x_1 + x_2 + \cdots + x_N}{N} \\ &= \frac{3.133g + 3.083g + \cdots + 3.093g}{15} \\ &= 3.100g\end{aligned}$$

- 3 The uncertainty is directly related to the *sample standard deviation*, S :

$$\begin{aligned}S &= \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_N - \bar{x})^2}{N - 1}} \\ &= \sqrt{\frac{(3.133g - 3.100g)^2 + (3.083 - 3.100g)^2 + \cdots + (3.093 - 3.100g)^2}{14}} \\ &= 0.0278g\end{aligned}$$

- 4 The scale factor, k , adjusts for the small number of data points - there is more uncertainty in our estimate when there are fewer data points:

$$\begin{aligned}k &= 1 + \frac{20}{N^2} \\ &= 1 + \frac{20}{15^2} \\ &= 1.0889\end{aligned}$$

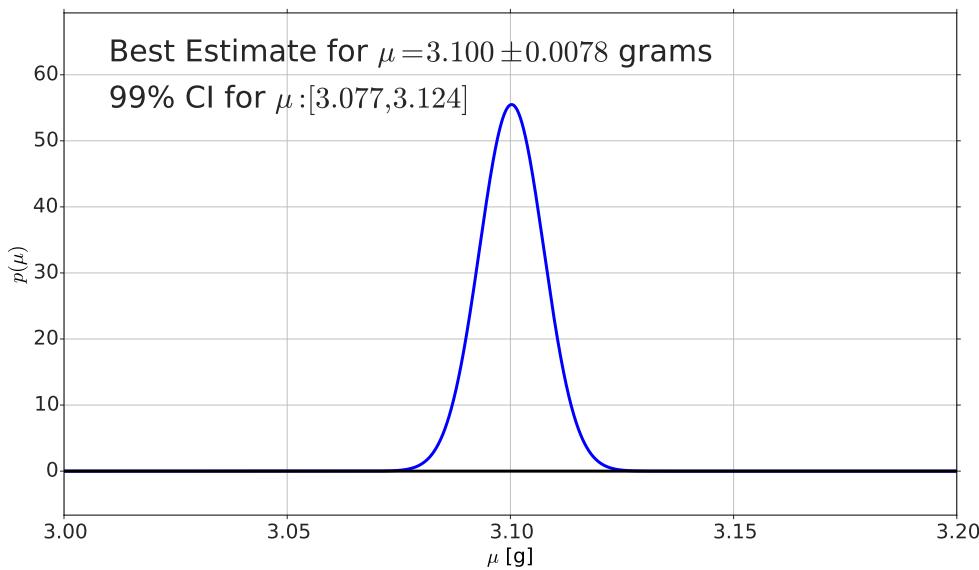
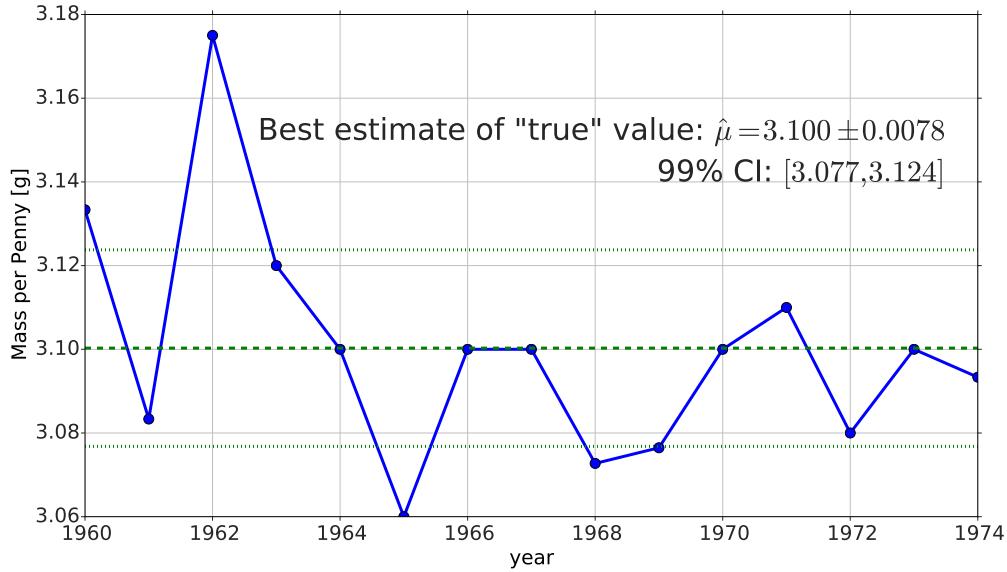
Finally, we have the best estimate and uncertainty for the pennies in this dataset:

$$\begin{aligned}\hat{\mu} &= \bar{x} \pm k \cdot S / \sqrt{N} \\ &= 3.100g \pm 1.0889 \cdot 0.0278g / \sqrt{15} \\ &= 3.100g \pm 0.0078g\end{aligned}$$

or, as a 99% credible range (3 times the uncertainty written above), we have, (see also Figure 6.4)

$$\begin{aligned}99\% \text{ CI for } \mu &= 3.100g \pm 3 \times 0.0078g \\ &= [3.077g, 3.124g]\end{aligned}$$

Figure 6.4: Mass of Pennies from 1960 to 1974, with best estimates and 99% CI (i.e. 3σ) uncertainty.



EXAMPLE 6.13 Mass of the Penny, Model 1 - One True Value with More Data

Now we collect the additional data with more recent pennies shown in Table 6.5. We can follow the same procedure, assuming our original model of one “true” value, to get the best estimate and uncertainty for this model, combining the two data sets.

Year	Mass [g]
1989	2.516
1990	2.500
1991	2.500
1992	2.500
1993	2.503
1994	2.500
1995	2.497
1996	2.500
1997	2.494
1998	2.512
1999	2.521
2000	2.499
2001	2.523
2002	2.518
2003	2.520

Table 6.5: Mass of Pennies from 1989 to 2003.

$$\hat{\mu} = \bar{x} \pm k \cdot S / \sqrt{N}$$

where the symbols in this equation are

- 1 the number of data points, $N = 30$.
- 2 the best estimate for the true value, $\hat{\mu}$, is given by the *sample mean*, \bar{x} :

$$\begin{aligned}\bar{x} &= \frac{3.133\text{g} + 3.083\text{g} + \dots + 2.520\text{g}}{30} \\ &= 2.804\text{g}\end{aligned}$$

- 3 The *sample standard deviation*, S :

$$\begin{aligned}S &= \sqrt{\frac{(3.133\text{g} - 2.804\text{g})^2 + (3.083\text{g} - 2.804\text{g})^2 + \dots + (2.520\text{g} - 2.804\text{g})^2}{29}} \\ &= 0.3024\text{g}\end{aligned}$$

- 4 The scale factor, k , adjusting for the small number of data points:

$$\begin{aligned}k &= 1 + \frac{20}{30^2} \\ &= 1.0222\end{aligned}$$

Finally, we have the best estimate and uncertainty for the pennies in this full dataset:

$$\begin{aligned}\hat{\mu} &= \bar{x} \pm k \cdot S / \sqrt{N} \\ &= 2.804\text{g} \pm 1.0222 \cdot 0.3024\text{g} / \sqrt{30} \\ &= 2.804\text{g} \pm 0.0564\text{g}\end{aligned}$$

or, as a 99% credible range (3 times the uncertainty written above), we have,

$$\begin{aligned}99\% \text{ CI for } \mu &= 2.804\text{g} \pm 3 \times 0.0564\text{g} \\ &= [2.634\text{g}, 2.973\text{g}]\end{aligned}$$

There are several things one should notice:

- 1 The scale factor, k , is less for 30 data points than it is for 15 data points. This is because the adjustment for small number of data points gets less relevant as we obtain more data. This is what we expect.
- 2 Despite there being twice as much data, our uncertainty *increased*. This is unusual, if our model is correct - more data should *sharpen* the estimates. Although it is possible that adding more data increased the system variability somehow, it is more likely that some assumption of our model is incorrect. This becomes obvious when we look at the result graphically, shown in Figure 6.5.

This should highlight a few things:

- 1 Always look at your data graphically. What you might miss looking at a table of numbers, you'll catch with a picture.
- 2 Assume your model is wrong, and outline other possible models ahead of time and explore them. The most obvious improvement in this problem is to notice that we are dealing with two separate "true" values, possibly caused by a change in the manufacturing materials.

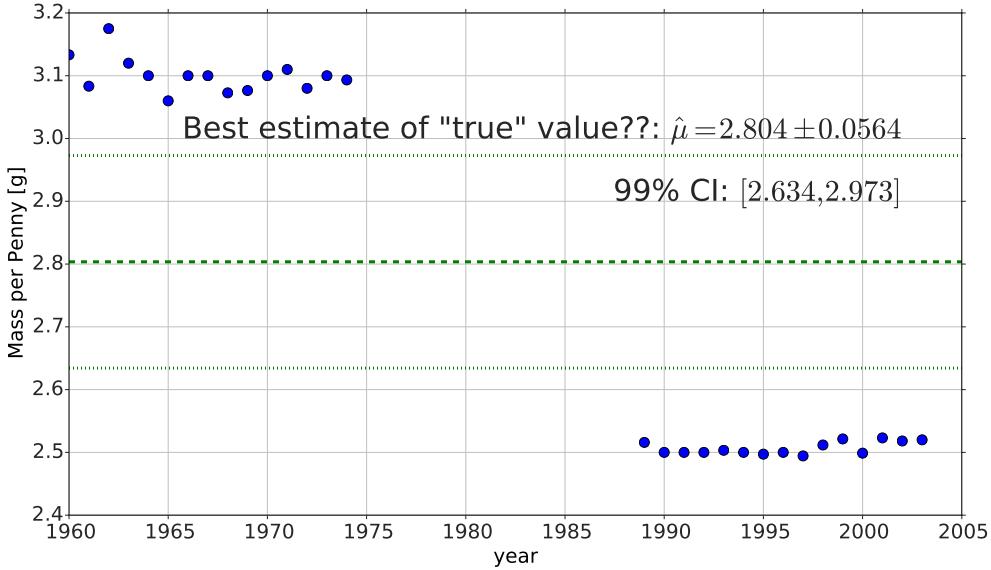
EXAMPLE 6.14 Mass of the Penny, Model 2 - Two True Values

In this model, we assume there are two true values:

- μ_1 - before 1975
- μ_2 - after 1988

There are two roughly equivalent ways of telling whether there is a significant difference.

Figure 6.5: Mass of Pennies from 1960 to 2003, with best estimates and 99% CI (i.e. 3σ) uncertainty.



Overlapping Intervals The first is the easiest to do mathematically, and yields a nice picture: obtain the best estimates for μ_1 and μ_2 , and see if their 99% credible intervals overlap. From this analysis (identical to the previous examples, however we leave the details of the calculation to the student), we get (see Figure 6.6):

- Best estimate for μ_1

$$\hat{\mu}_1 = 3.100 \pm 0.0078$$

with 99% CI: [3.077, 3.124].

- Best estimate for μ_2

$$\hat{\mu}_2 = 2.507 \pm 0.0029$$

with 99% CI: [2.498, 2.516]

where the 99% credible intervals (CI) clearly do not overlap, thus there is a statistically significant difference between them.

Is the Difference Zero? The proper way is to estimate the quantity $\mu_1 - \mu_2$ and test to see if it is greater than zero, as shown in Section 5.2 on page 92. The estimate of this quantity, which we'll call $\delta_{12} = \mu_1 - \mu_2$ is related to the means and uncertainties of two data

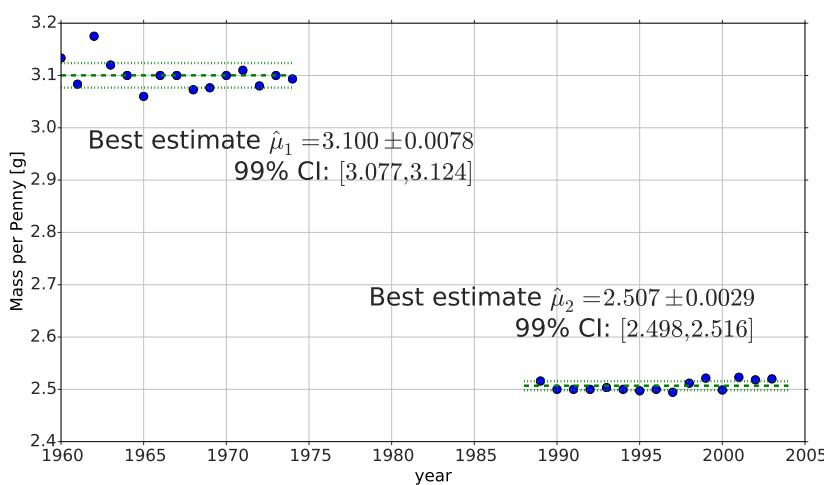


Figure 6.6: Mass of Pennies from 1960 to 2003, with best estimates for the two true values and their 99% CI (i.e. 3σ) uncertainty plotted. There is clearly no overlap in their credible intervals, thus there is a statistically significant difference between them.

sets

$$\begin{aligned}\hat{\delta}_{12} &= \bar{x}_1 - \bar{x}_2 \pm \sigma_{12} \\ \sigma_{12} &= \sqrt{\sigma_1^2 + \sigma_2^2} \\ \sigma_1 &= k_1 S_1 / \sqrt{N_1} \text{ (uncertainty from data set 1)} \\ \sigma_2 &= k_2 S_2 / \sqrt{N_2} \text{ (uncertainty from data set 2)}\end{aligned}$$

where the sample standard deviations, S_1 and S_2 , and the scale factors, k_1 and k_2 were calculated earlier. This leads to, for this data set,

$$\hat{\delta}_{12} = 0.593g \pm 0.008g$$

with the 99% credible interval [0.568g, 0.618g], the distribution shown in Figure 6.7. Again, the estimated quantities are clearly different statistically: the value of zero is *well* outside of the 99% credible interval for δ_{12} .

6.5 Computer Examples

```
from sie import *
```

Iris Example

```
data=load_data('data/iris.csv')
```

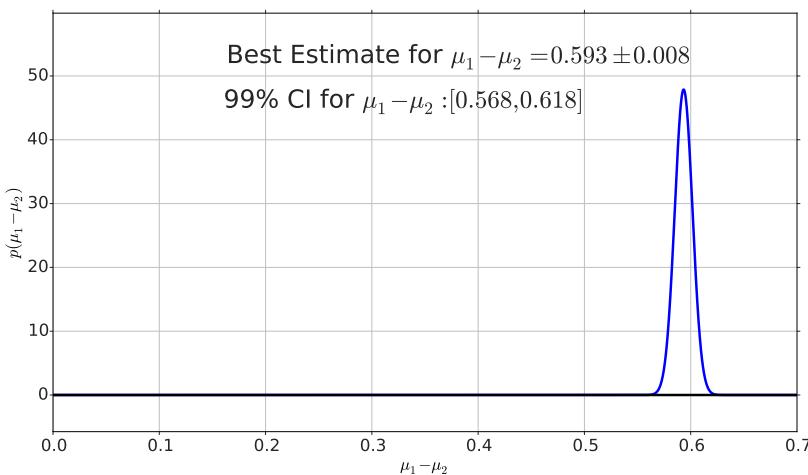


Figure 6.7: Difference in the estimated values of the pre- and post 1975 pennies, $\mu_1 - \mu_2$. The value zero is clearly outside of the 99% interval of the difference, thus there is a statistically significant difference between the two values μ_1 and μ_2 .

```
x_sertosa=data[data['class']=='Iris-setosa']['petal length [cm]']
x_virginica=data[data['class']=='Iris-virginica']['petal length [cm]']
x_versicolor=data[data['class']=='Iris-versicolor']['petal length [cm]']
```

```
print x_sertosa[:10] # print the first 10
```

```
0    1.4
1    1.4
2    1.3
3    1.5
4    1.4
5    1.7
6    1.4
7    1.5
8    1.4
9    1.5
Name: petal length [cm], dtype: float64
```

```
x=x_sertosa
mu=sample_mean(x)
N=len(x)
sigma=sample_deviation(x)/sqrt(N)
t_sertosa=t dist(N,mu,sigma)

print "total number of data points:",N
print "best estimate:",mu
```

```
print "uncertainty:",sigma
```

total number of data points: 50
best estimate: 1.464
uncertainty: 0.0245381834898

```
x=x_versicolor
mu=sample_mean(x)
N=len(x)
sigma=sample_deviation(x)/sqrt(N)
t_versicolor=tdist(N,mu,sigma)

print "total number of data points:",N
print "best estimate:",mu
print "uncertainty:",sigma
```

total number of data points: 50
best estimate: 4.26
uncertainty: 0.0664554477121

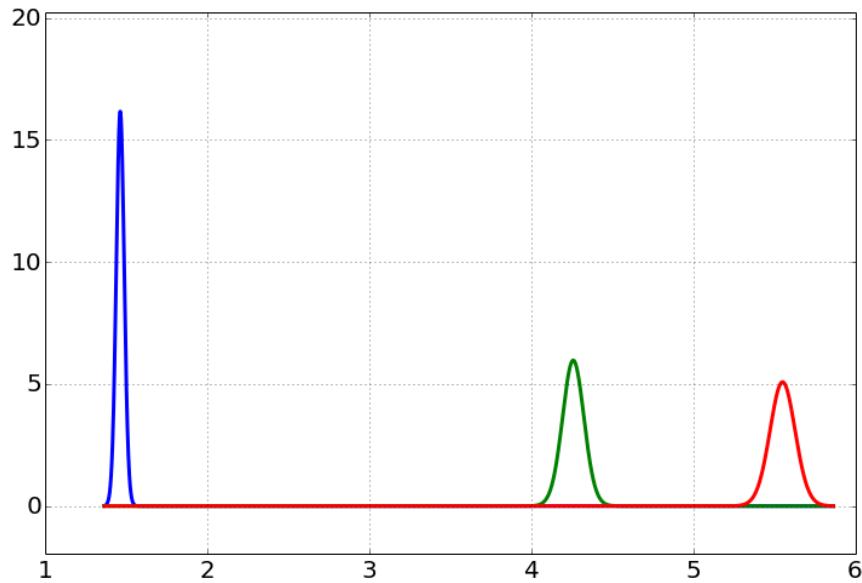
```
x=x_virginica
mu=sample_mean(x)
N=len(x)
sigma=sample_deviation(x)/sqrt(N)
t_virginica=tdist(N,mu,sigma)

print "total number of data points:",N
print "best estimate:",mu
print "uncertainty:",sigma
```

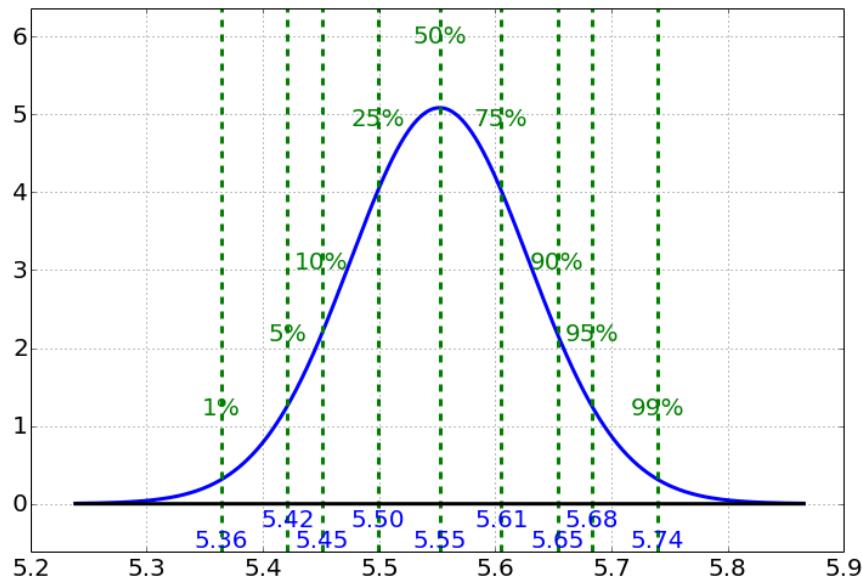
total number of data points: 50
best estimate: 5.552
uncertainty: 0.078049696361

```
distplot2([t_sertosa,t_versicolor,t_virginica],show_quartiles=False)
```

<matplotlib.figure.Figure at 0x1058d9690>



```
distplot(t_virginica)
```



```
credible_interval(t_versicolor)
```

(4.1265203051077082, 4.259999999999998, 4.3934796948922914)

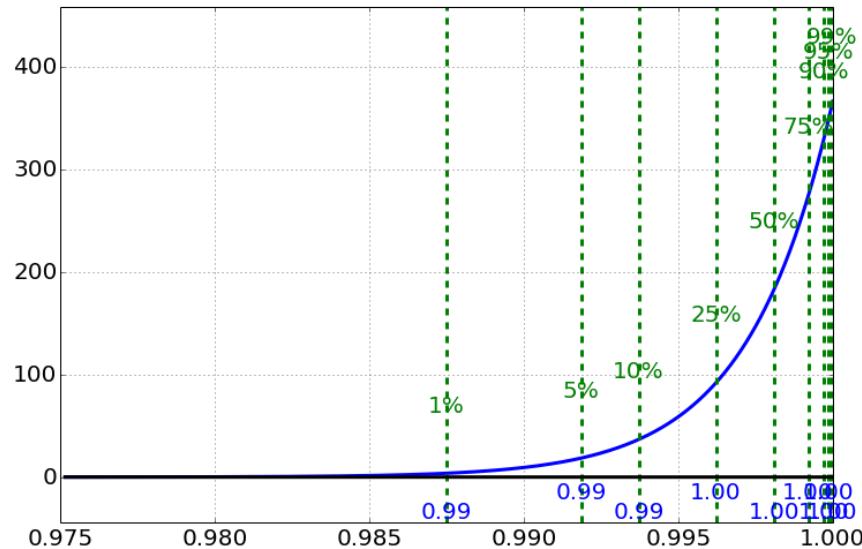
```
credible_interval(t_virginica)
```

(5.3952325713636533, 5.5519999999999996, 5.7087674286363459)

Sunrise

```
dist=beta(h=365,N=365)
```

```
distplot(dist)
```



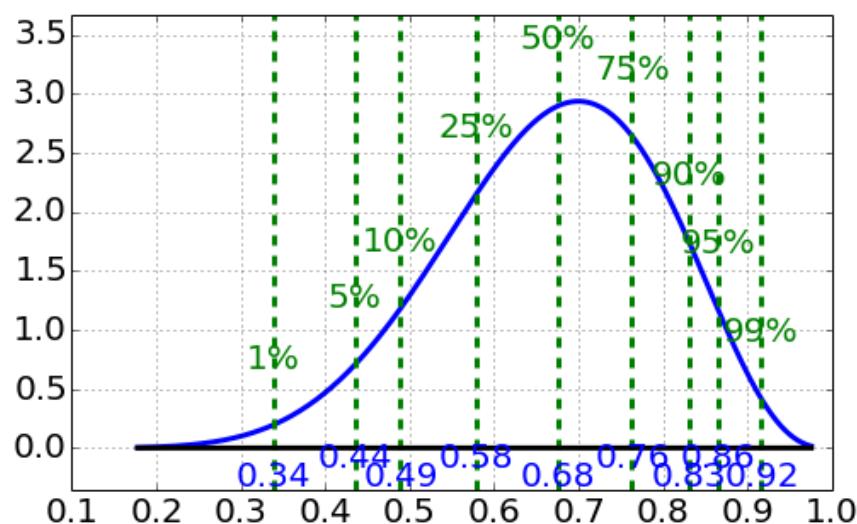
```
credible_interval(dist)
```

```
(0.98997171634278669, 0.99810794743679487, 0.99993082805373457)
```

Cancer Example

```
dist=beta(h=7,N=10)
```

```
distplot(dist , figsize =(8 , 5))
```



```
credible_interval(dist)
```

```
(0.39025744042757882, 0.67619553741481253, 0.89073655618090186)
```

Essentially no evidence of any effect over 50 percent.

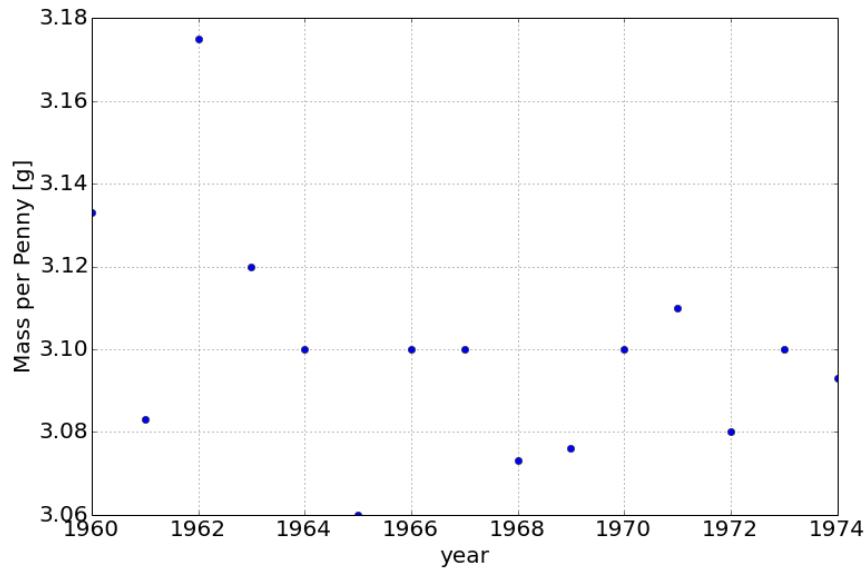
Pennies

```
data1=load_data('data/pennies1.csv')
print data1
year,mass=data1['Year'],data1['Mass [g]']
```

	Year	Mass [g]
0	1960	3.133
1	1961	3.083
2	1962	3.175
3	1963	3.120
4	1964	3.100
5	1965	3.060
6	1966	3.100
7	1967	3.100
8	1968	3.073
9	1969	3.076
10	1970	3.100
11	1971	3.110
12	1972	3.080
13	1973	3.100
14	1974	3.093

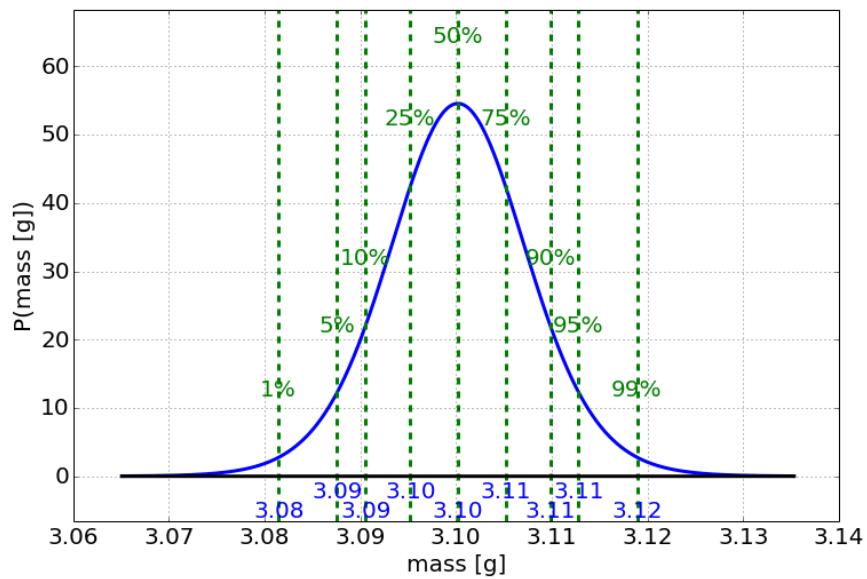
```
plot(year,mass,'o')
xlabel('year')
ylabel('Mass per Penny [g]')
```

```
<matplotlib.text.Text at 0x1087c2d90>
```



```
x=mass
mu=sample_mean(x)
N=len(x)
sigma=sample_deviation(x)/sqrt(N)
t_penny1=t dist(N,mu,sigma)

distplot(t_penny1,label='mass [g]')
```

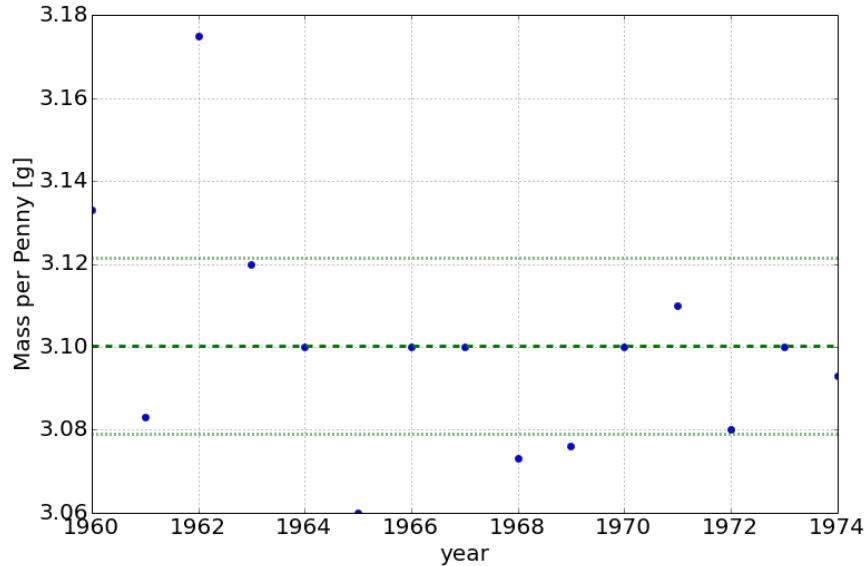


```
CI=credible_interval(t_penny1,percentage=99)
print CI
```

```
(3.0790129206702002, 3.1002000000000001, 3.1213870793298)
```

```
plot(year, mass, 'o')
credible_interval_plot(t_penny1, percentage=99)
xlabel('year')
ylabel('Mass per Penny [g]')
```

<matplotlib.text.Text at 0x1087fcf10>



Do the 2 datasets

```
data2=load_data('data/pennies2.csv')
print data2
```

```
year1, mass1=year, mass
year2, mass2=data2[ 'Year' ], data2[ 'Mass [g]' ]
```

	Year	Mass [g]
0	1989	2.516
1	1990	2.500
2	1991	2.500
3	1992	2.500
4	1993	2.503
5	1994	2.500
6	1995	2.497
7	1996	2.500
8	1997	2.494
9	1998	2.512
10	1999	2.521
11	2000	2.499
12	2001	2.523
13	2002	2.518

14 2003 2.520

```

x=mass1
mu=sample_mean(x)
N=len(x)
sigma=sample_deviation(x)/sqrt(N)
t_penny1=t dist(N,mu,sigma)

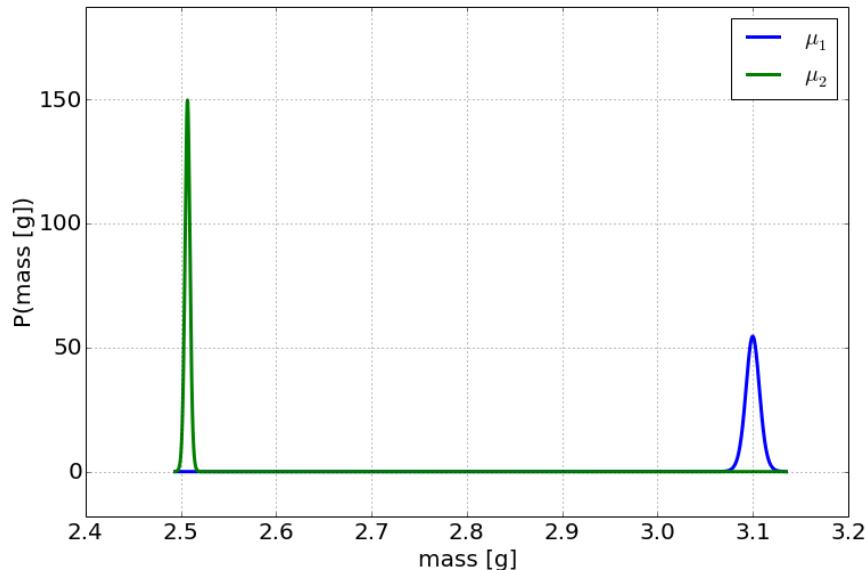
x=mass2
mu=sample_mean(x)
N=len(x)
sigma=sample_deviation(x)/sqrt(N)
t_penny2=t dist(N,mu,sigma)

distplot2([t_penny1,t_penny2],show_quartiles=False,label='mass [g]')
legend([r'$\mu_1$',r'$\mu_2$'])

```

<matplotlib.figure.Figure at 0x1087d3f10>

<matplotlib.legend.Legend at 0x1088198d0>

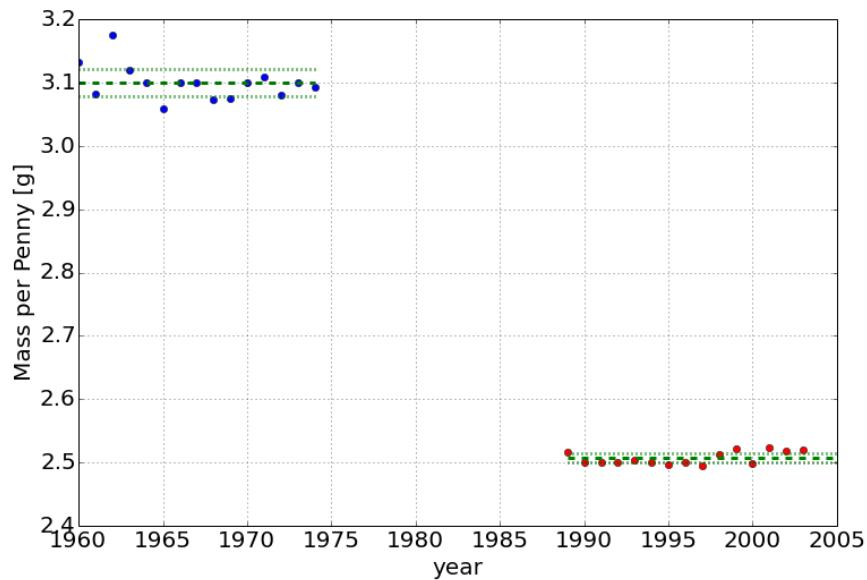


```

plot(year1, mass1, 'o')
credible_interval_plot(t_penny1, percentage=99)
plot(year2, mass2, 'ro')
credible_interval_plot(t_penny2, percentage=99, xlim=[1989,2005])
xlabel('year')
ylabel('Mass per Penny [g]')

```

<matplotlib.text.Text at 0x10907e310>



Distribution of the difference, normal approximation

```

N1=len(mass1)
N2=len(mass2)

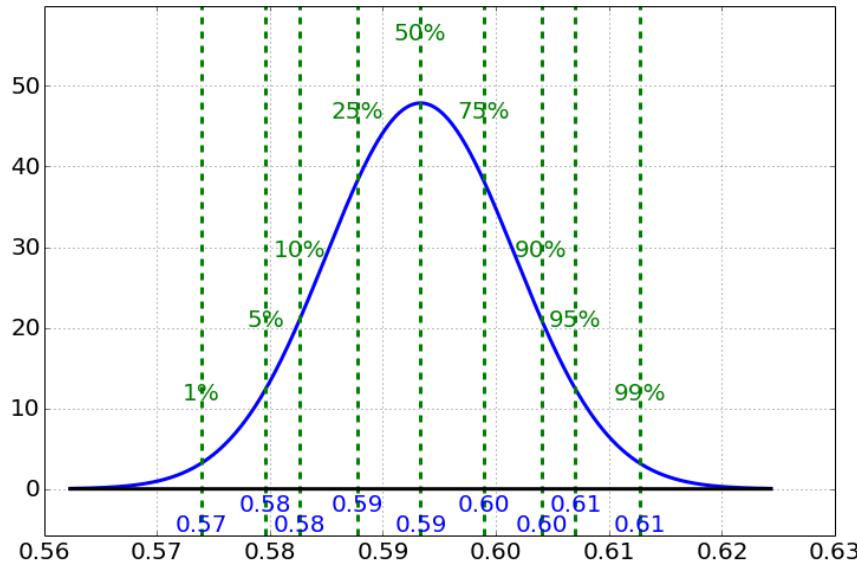
mu1=sample_mean(mass1)
mu2=sample_mean(mass2)

sigma1=(1+20.0/N1**2)*sample_deviation(mass1)/sqrt(N1)
sigma2=(1+20.0/N2**2)*sample_deviation(mass2)/sqrt(N1)

delta_12=mu1-mu2
sigma_delta12=sqrt(sigma1**2+sigma2**2)

dist_delta=normal(delta_12,sigma_delta12)
distplot(dist_delta)

```



clearly larger than zero at well over the 99

Ball Bearing Sizes

```
data1=[1.18,1.42,0.69,0.88,1.62,1.09,1.53,1.02,1.19,1.32]
data2=[1.72,1.62,1.69,0.79,1.79,0.77,1.44,1.29,1.96,0.99]
N1=len(data1)
N2=len(data2)
```

```
mu1=sample_mean(data1)
mu2=sample_mean(data2)
print mu1,mu2
```

1.194 1.406

```
S1=sample_deviation(data1)
S2=sample_deviation(data2)
print S1,S2
```

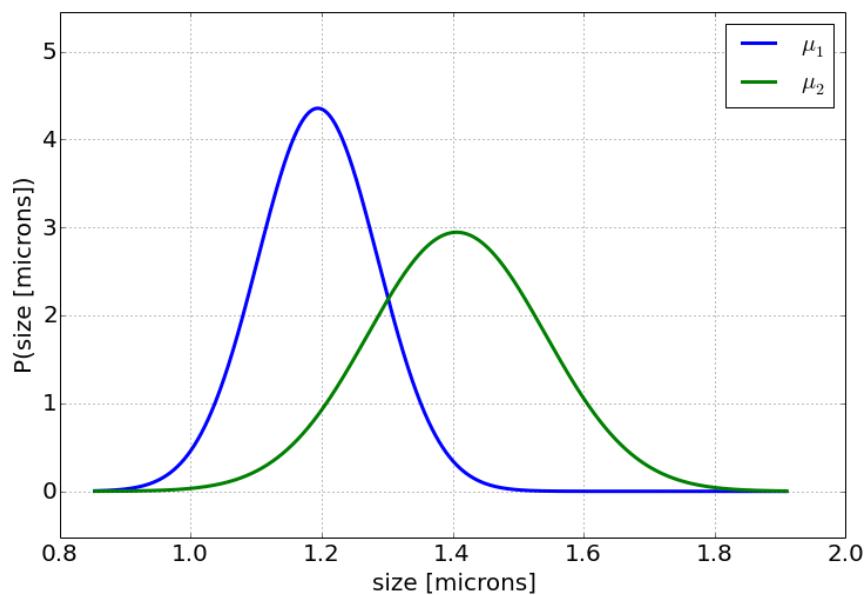
0.289681817786 0.428309337849

```
sigma1=S1/sqrt(N1)
sigma2=S2/sqrt(N2)
print sigma1,sigma2
```

0.091605434094 0.135443305072

```
dist1=normal(mu1,sigma1)
dist2=normal(mu2,sigma2)
distplot2([dist1,dist2],show_quartiles=False,label='size [microns]')
legend([r'$\mu_1$',r'$\mu_2$'])
```

```
<matplotlib.figure.Figure at 0x105d61390>
<matplotlib.legend.Legend at 0x108ca1ad0>
```



7 Concluding Thoughts

7.1 Where have we come?

We have tried in this book to present a particular picture of the world: everything is probability. We started with basic definitions and applications, and followed the consequences of the rules of probability to examine more complex problems. It is our hope that the reader sees that all of the analysis stems from a *single* perspective. In this way, one can approach *any* problem of inference in a unified way, applying the recipe we've used throughout:

- 1 Propose a model for the data you observe (which could be as simple as "there is an unknown true value for the observations")
- 2 Specify your prior knowledge of the parameters in the model, in the form of a prior probability (which is often as simple as "I don't know anything about the parameters, so all possible values are equally likely")
- 3 Specify how likely your data would be if your model were true, which is the likelihood part of Bayes' rule
- 4 Apply the rules of probability, namely Bayes' rule, to determine the posterior probability for the parameters in the model
- 5 Use the properties of probability functions to calculate answers to specific questions, for example "is it likely that this number is greater than zero?" or "are these two measurements different?"

Although I haven't covered all possible examples, and there are additions and clarifications still planned, this approach can be used for all new problems one faces. The only steps that can be daunting, at times, is the mathematical consequences and even there we have seen that the judicious use of approximations can go a long way.

7.2 *Where are we going?*

Topics I'd love to add, and will when I have the chance, include (in no particular order),

- Measurement in Science
- Linear Regression and Correlation
- Two-sample inferences
- Classification
- Model Building in Science
- Analysis of Social Science Data
- Inference for Deviation Parameters
- Experimental Design
- Computer simulations (e.g. MCMC)

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Appendix A

Computational Analysis

The book is written with an accompanying software package, written in Python. As of this writing the recommended distribution for installing python is the Anaconda distribution, available here:

<https://store.continuum.io/cshop/anaconda/>

It is

- Free
- Easy to Use
- Easy to Extend
- Very Powerful

The accompanying software for the book can be obtained from the book website, <http://web.bryant.edu/~bblais/statistical-inference-for-everyone-sie.html>

Appendix B

Notation and Standards

B.1 Useful Greek Letters

α	Alpha	slope of a line	π	Pi	Represents the constant $3.1415\cdots$, the ratio of the circumference to the diameter of a circle
β	Beta	slope of a line, intercept	Π	Pi	A product of a series of numbers
γ	Gamma		ρ	Rho	
Γ	Gamma		σ	Sigma	The standard width parameter of the normal distribution
δ	Delta	A small change in a variable	Σ	Sigma	A sum of a series of numbers
Δ	Delta	A change in a variable	τ	Tau	
ϵ	Epsilon		ϕ	Phi	
ζ	Zeta		Φ	Phi	
η	Eta		χ	Chi	A distribution related to the sum of normally distributed variables
θ	Theta	The parameters in a binomial/beta distribution	ψ	Psi	
Θ	Theta		Ψ	Psi	
κ	Kappa		ω	Omega	
λ	Lambda	the mean in a poisson distribution	Ω	Omega	
Λ	Lambda				
μ	Mu	the mean in a normal distribution (pronounced "mew")			
ν	Nu	(pronounced "new")			
ξ	Xi				
Ξ	Xi				

B.2 Some Math Notation

Variables

A set of values, labeled with subscripts...

$$\begin{aligned}x_1 &= 1 \\x_2 &= 5\end{aligned}$$

$$\begin{aligned}x_3 &= -3 \\x_4 &= 2 \\x_5 &= 8\end{aligned}$$

referred collectively as x_i .

Sums

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1 + 5 + (-3) + 2 + 8 = 13$$

is equivalent to

$$\sum_{i=1}^5 x_i = 1 + 5 + (-3) + 2 + 8 = 13$$

Products

$$x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 = 1 \cdot 5 \cdot (-3) \cdot 2 \cdot 8 = -240$$

is equivalent to

$$\prod_{i=1}^5 x_i = 1 \cdot 5 \cdot (-3) \cdot 2 \cdot 8 = -240$$

Sample Mean

The sample mean of a set of numbers is defined as...

$$\bar{x} \equiv \frac{x_1 + x_2 + \cdots + x_N}{N}$$

In the example above

$$\bar{x} \equiv \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = 2\frac{3}{5}$$

It can also be written

$$\bar{x} \equiv \frac{\sum_{i=1}^N x_i}{N}$$

or

$$\bar{x} \equiv \frac{\sum_i x_i}{N}$$

Sample Standard Deviation

$$s^2 \equiv \frac{1}{N-1} \sum_{i=1}^N (x - \bar{x})^2$$

$$s \equiv \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x - \bar{x})^2}$$

Estimates

Any specific estimate of a parameter, such as θ , is denoted with a hat, such as $\hat{\theta}$.

Although the justification for the $N - 1$ part is beyond this book, one easy way to remember it is that the sample distribution of a set of numbers is an estimate for the σ parameter of the normal distribution, representing the *spread* of the data. You can think of the $N - 1$ part as a check to keep you from doing the crazy thing of estimating a spread with only 1 data point!

Factorials

Factorials are defined as

$$N! = 1 \cdot 2 \cdot 3 \cdots (N-1) \cdot N$$

for example

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

The N -choose- k notation is a shorthand for the factorials that arise in binomial and Beta distributions.

$$\binom{N}{k} \equiv \frac{N!}{k!(N-k)!}$$

B.3 Qualitative labels to probability values

Rough guide for the conversion of qualitative labels to probability values used throughout the book.

term	probability
virtually impossible	1/1,000,000
extremely unlikely	0.01 (i.e. 1/100)
very unlikely	0.05 (i.e. 1/20)
unlikely	0.2 (i.e. 1/5)
slightly unlikely	0.4 (i.e. 2/5)
even odds	0.5 (i.e. 50-50)
slightly likely	0.6 (i.e. 3/5)
likely	0.8 (i.e. 4/5)
very likely	0.95 (i.e. 19/20)
extremely likely	0.99 (i.e. 99/100)
virtually certain	999,999/1,000,000

Appendix C

Common Distributions and Their Properties

This chapter is a reference for the standard distributions encountered in statistical inference. Although you are encouraged to read this chapter through, it can also be read out-of-order to look at a specific distribution.

C.1 Discrete and Continuous

Some distributions apply to a discrete (i.e. countable) number of possibilities while others apply to continuous values. In the case of discrete variables, the probability is given by the actual value of the distribution, so it makes sense to speak of the probability of an individual label, $P(\text{coin1})$. In the case of continuous variables, the probability is given by the area under the distribution, so it makes sense only to speak of the probability if a range of labels, $P(0.2 < \theta < 0.3)$.

C.2 Uniform

Discrete

Discrete uniform distribution The discrete uniform distribution is defined to be a *constant* value for all possibilities. Mathematically this is written

$$p(x_i) = \frac{1}{N}$$

where N is the total number of possibilities, labeled x_1 to x_N . The picture of the distribution is shown in Figure C.1

Continuous

Continuous uniform distribution The continuous uniform distribution is defined to be a *constant* between a minimum and maximum

Discrete uniform distribution The discrete uniform distribution is defined to be a *constant* value for all possibilities. Mathematically this is written

$$p(x_i) = \frac{1}{N}$$

where N is the total number of possibilities, labeled x_1 to x_N .

Continuous uniform distribution The continuous uniform distribution is defined to be a *constant* between a minimum and maximum value, and zero everywhere else. Mathematically this is written

$$p(x) = \frac{1}{\max - \min} \text{ for } \min < x < \max$$

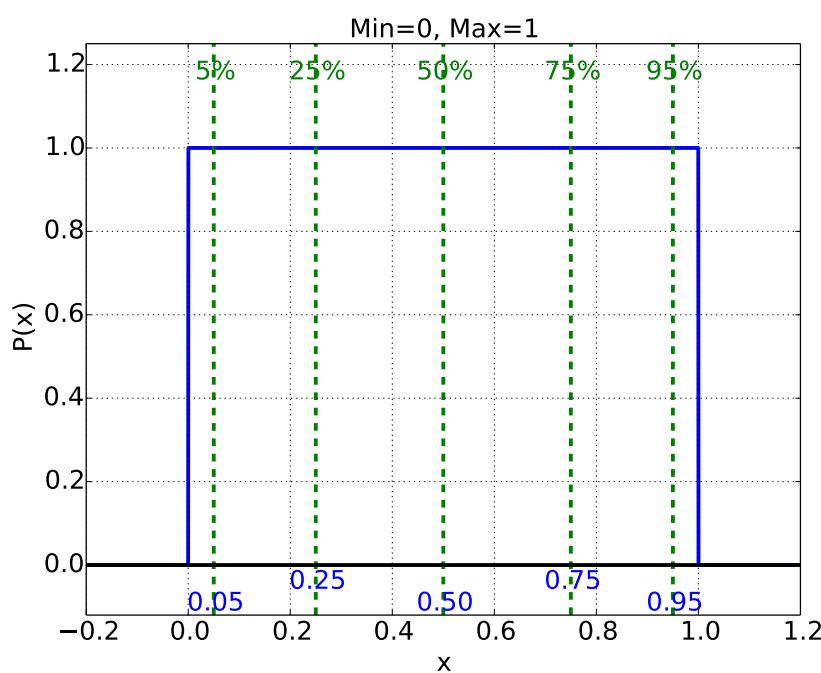


Figure C.1: Discrete uniform distribution for values 1 to 6. The value for each is $p(x_i) = 1/6$.

value, and zero everywhere else. Mathematically this is written

$$p(x) = \frac{1}{\max - \min} \text{ for } \min < x < \max$$

. The picture of the distribution is shown in Figure C.2.

EXAMPLE C.1 You call a plumber, and they say that they can come anytime in the next 4 hours. The probability of them arriving at any particular time can be represented with a uniform distribution. What is the probability that they arrive in the first 20 minutes of the second hour?

In order to ask questions about total probability from a *continuous* distribution you take the *area under the curve* between the relevant values. In this case it'd be the area under the curve from the time $t = 2\text{hr}$ and $t = 2\text{hr} + 20\text{minutes} = 2.333\text{hr}$, as shown in Figure C.3. The area under the curve is just the area of the shaded region between times $t = 2\text{hr}$ and $t = 2.333\text{hr}$, or just the area of a rectangle - $A = \text{base} \times \text{height}$. The *base* of the rectangle is the length of time, or

$$\text{base} = 0.333\text{hr}$$

The height of the rectangle is given by the constant value of the uniform distribution, or

$$\text{height} = \frac{1}{\max - \min} = \frac{1}{4\text{hr} - 0\text{hr}} = 0.25 \frac{1}{\text{hr}}$$

The reason for the particular constant value for the uniform distribution, $1/(\max - \min)$, is simply that the area of the entire rectangle must be 1, which means that there is a 100% chance of the values falling between the minimum and maximum values.

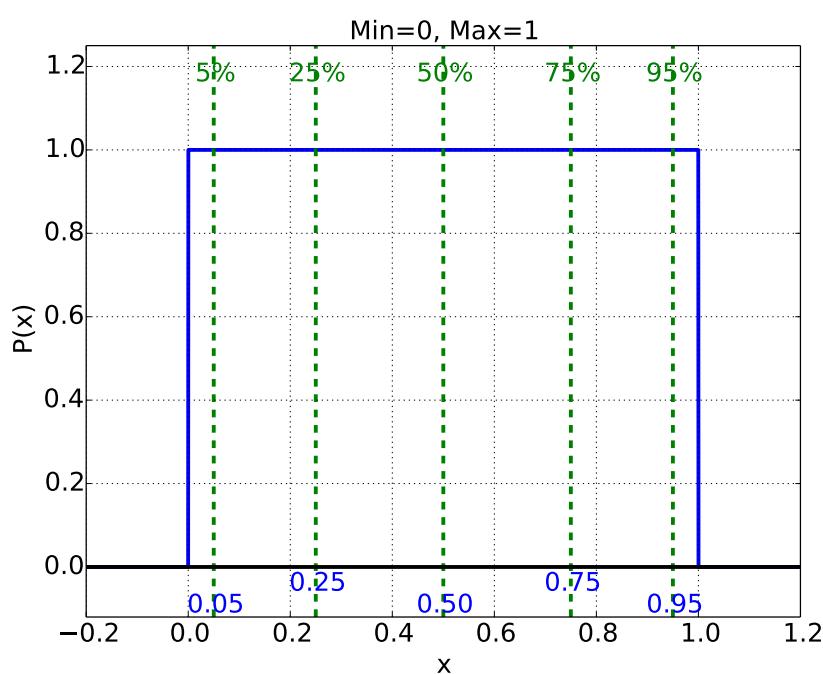


Figure C.2: Continuous uniform distribution between values 0 and 1

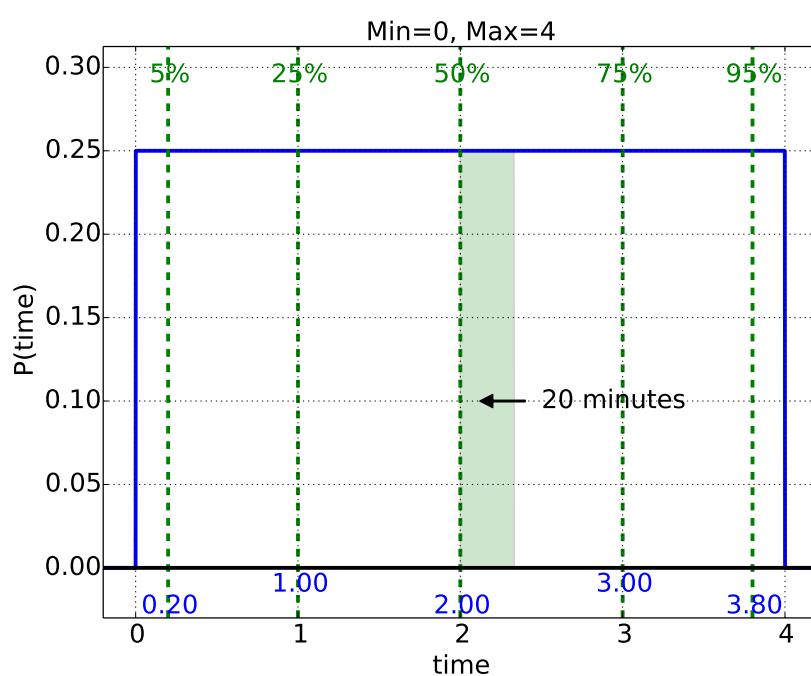


Figure C.3: Continuous uniform distribution for the plumber example (Example C.1).

So the total probability of the plumber coming in the first 20 minutes of the second hour is

$$P(2 < t < 2.25) = (0.333\text{hr}) \times \left(0.25 \frac{1}{\text{hr}}\right) = 0.0833$$

C.3 Binomial

Binomial distribution The discrete binomial distribution is defined to be the probability of achieving h successes in a given N events where each event has a given θ probability of success.

$$P(h|N,\theta) = \binom{h}{N} \theta^h (1-\theta)^{N-h}$$

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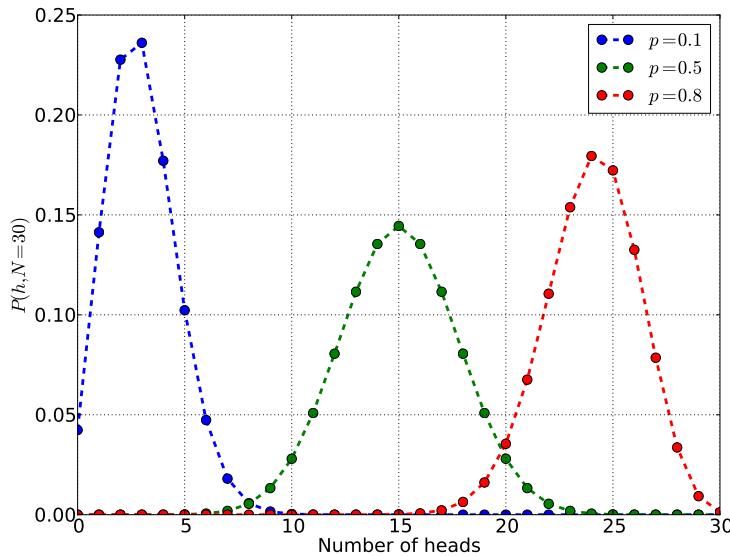


Figure C.4: Probability of getting h heads in 30 flips given a possible unfair coin. One coin has $p = 0.1$, where the maximum is for 3 heads (or 1/10 of the 30 flips), but 2 heads is nearly as likely. Another has $p = 0.5$, and is the fair coin considered earlier with a maximum at 15 heads (or 1/2 of the 30 flips). Finally, another coin shown as $p = 0.8$ where 24 heads (or 8/10 of the 30 flips) is maximum.

Although it may look like a Beta, the binomial distribution is used to find the best estimate for the number of successes, h , given the number of events, N , and the probability of the success of a single event, θ .

C.4 Beta

Beta distribution The continuous Beta distribution is the posterior probability distribution for the parameter θ , where one has observed

Beta distribution The continuous Beta distribution is the posterior probability distribution for the parameter θ , where one has observed h successes in a given N events, and each event is assumed to have a θ probability of success.

$$P(\theta|h, N) = (N+1) \cdot \binom{N}{h} \theta^h (1-\theta)^{N-h}$$

h successes in a given N events, and each event is assumed to have a θ probability of success.

$$P(\theta|h, N) = (N+1) \cdot \binom{N}{h} \theta^h (1-\theta)^{N-h}$$

Although it may look like a binomial, the Beta distribution is used to find the best estimate for the parameter θ where the number of successes and events, h and N are given.

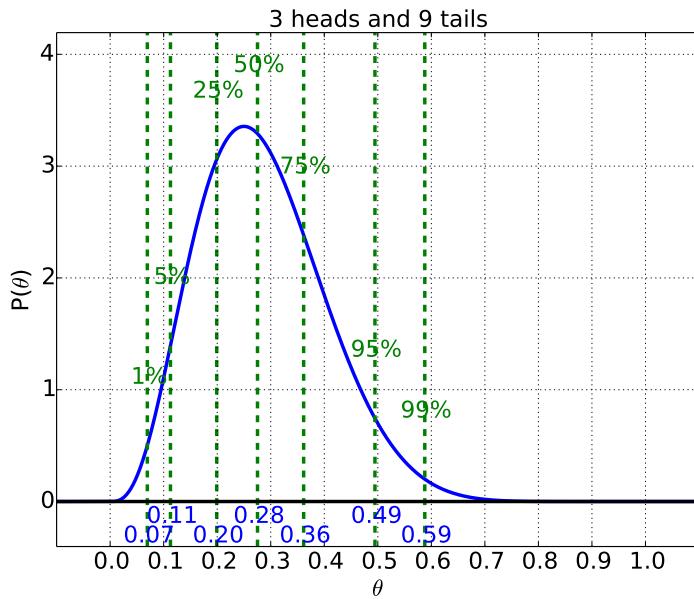


Figure C.5: Posterior probability distribution for the θ values of the bent coin - the probability that the coin will land heads. The distribution is shown for data 3 heads and 9 tails. The various quartiles are shown in the plot.

C.5 Normal (Gaussian)

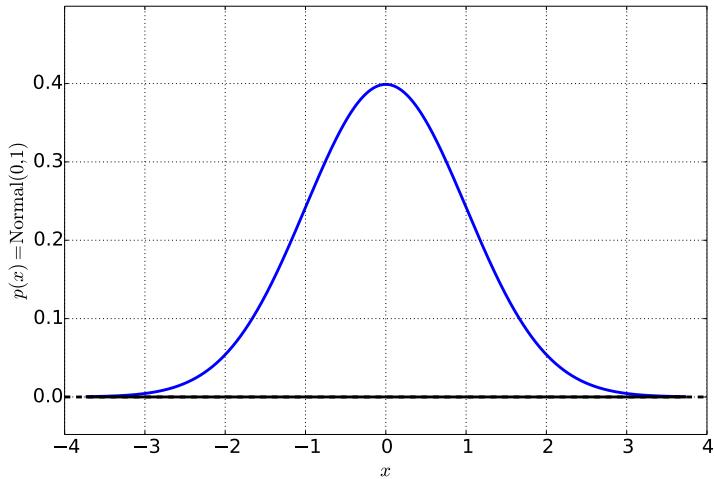
Normal distribution The Normal distribution is the most common distribution found in all of statistical inference. It is the best prior distribution to use, when all you know is that your data has a constant true value and some constant variation around that true value. It is the posterior probability distribution for the unknown true value given N samples and the known deviation, σ . It is also the approximate form for nearly every distribution when you have many samples. The mathematical form for the normal, or Gaussian, is

$$\text{Normal}(\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

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$$\text{Normal}(\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Figure C.6: The normal distribution.



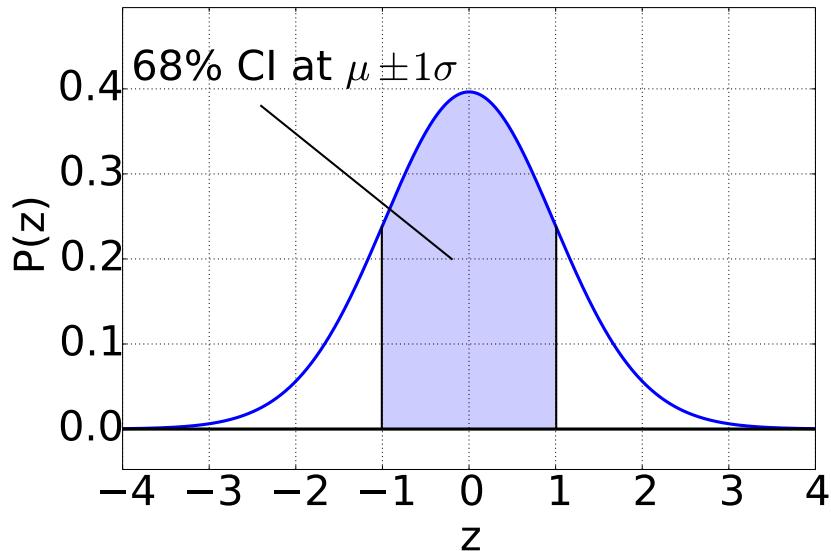
Three useful properties of σ for the normal distribution are the following:

- 1 the normal distribution value at the maximum (i.e. at $x = \mu$) is around 2.7 times larger than the value one- σ away from the maximum (at $x = \mu - \sigma$ and $x = \mu + \sigma$)
- 2 the total probability between these two points is 65%.
- 3 95% of the distribution lies between $\mu - 2\sigma$ and $\mu + 2\sigma$ (see Figure 5.3)

Appendix D

Tables

D.1 Credible Intervals for Standard Normal Distribution



Credible Interval	$\pm z$	Approximately
50.0%	0.6745σ	
68.0%	0.9945σ	1σ
90.0%	1.6449σ	
95.0%	1.9600σ	2σ
99.0%	2.5758σ	
99.8%	3.0902σ	3σ
99.995%	4.0556σ	4σ

EXAMPLE D.1 Usage of the Credible Interval Table for the Normal Distribution

Given a set of 10 samples with sample mean $\bar{x} = 5.2$ and known deviation $\sigma = 0.3$, the best estimate for the mean parameter μ , representing the true value of the data, is the sample mean, $\hat{\mu} = 5.2$ with

uncertainty σ/\sqrt{N} or $0.3/\sqrt{10} = 0.095$. Some of the credible intervals for this estimate then are the following

- 68% - $[5.2 - 0.9945 \cdot 0.095, 5.2 + 0.9945 \cdot 0.095] = [5.11, 5.29]$
- 95% - $[5.2 - 1.9600 \cdot 0.095, 5.2 + 1.9600 \cdot 0.095] = [5.01, 5.39]$
- 99.8% - $[5.2 - 3.0902 \cdot 0.095, 5.2 + 3.0902 \cdot 0.095] = [4.91, 5.49]$

or approximately

- 68% - $[5.2 - 1 \cdot 0.095, 5.2 + 1 \cdot 0.095] = [5.11, 5.29]$
- 95% - $[5.2 - 2 \cdot 0.095, 5.2 + 2 \cdot 0.095] = [5.01, 5.39]$
- 99.8% - $[5.2 - 3 \cdot 0.095, 5.2 + 3 \cdot 0.095] = [4.91, 5.49]$

D.2 Credible Intervals for Student's t Distribution

Credible Interval	Degrees of Freedom							
	1	2	3	4	5	6	7	8
50.0%	1.000σ	0.816σ	0.765σ	0.741σ	0.727σ	0.718σ	0.711σ	0.706σ
68.0%	1.819σ	1.312σ	1.189σ	1.134σ	1.104σ	1.084σ	1.070σ	1.060σ
90.0%	6.314σ	2.920σ	2.353σ	2.132σ	2.015σ	1.943σ	1.895σ	1.860σ
95.0%	12.706σ	4.303σ	3.182σ	2.776σ	2.571σ	2.447σ	2.365σ	2.306σ
99.0%	63.657σ	9.925σ	5.841σ	4.604σ	4.032σ	3.707σ	3.499σ	3.355σ
99.8%	318.309σ	22.327σ	10.215σ	7.173σ	5.893σ	5.208σ	4.785σ	4.501σ
99.995%	12732.395σ	141.416σ	35.298σ	18.522σ	12.893σ	10.261σ	8.783σ	7.851σ

Credible Interval	Degrees of Freedom							
	9	10	11	12	13	14	15	16
50.0%	0.703σ	0.700σ	0.697σ	0.695σ	0.694σ	0.692σ	0.691σ	0.690σ
68.0%	1.053σ	1.046σ	1.041σ	1.037σ	1.034σ	1.031σ	1.029σ	1.026σ
90.0%	1.833σ	1.812σ	1.796σ	1.782σ	1.771σ	1.761σ	1.753σ	1.746σ
95.0%	2.262σ	2.228σ	2.201σ	2.179σ	2.160σ	2.145σ	2.131σ	2.120σ
99.0%	3.250σ	3.169σ	3.106σ	3.055σ	3.012σ	2.977σ	2.947σ	2.921σ
99.8%	4.297σ	4.144σ	4.025σ	3.930σ	3.852σ	3.787σ	3.733σ	3.686σ
99.995%	7.215σ	6.757σ	6.412σ	6.143σ	5.928σ	5.753σ	5.607σ	5.484σ

Credible Interval	Degrees of Freedom							
	17	18	19	20	21	22	23	24
50.0%	0.689 σ	0.688 σ	0.688 σ	0.687 σ	0.686 σ	0.686 σ	0.685 σ	0.685 σ
68.0%	1.024 σ	1.023 σ	1.021 σ	1.020 σ	1.019 σ	1.017 σ	1.016 σ	1.015 σ
90.0%	1.740 σ	1.734 σ	1.729 σ	1.725 σ	1.721 σ	1.717 σ	1.714 σ	1.711 σ
95.0%	2.110 σ	2.101 σ	2.093 σ	2.086 σ	2.080 σ	2.074 σ	2.069 σ	2.064 σ
99.0%	2.898 σ	2.878 σ	2.861 σ	2.845 σ	2.831 σ	2.819 σ	2.807 σ	2.797 σ
99.8%	3.646 σ	3.610 σ	3.579 σ	3.552 σ	3.527 σ	3.505 σ	3.485 σ	3.467 σ
99.995%	5.379 σ	5.288 σ	5.209 σ	5.139 σ	5.077 σ	5.022 σ	4.972 σ	4.927 σ

Credible Interval	Degrees of Freedom							
	25	26	27	28	29	30	31	32
50.0%	0.684 σ	0.684 σ	0.684 σ	0.683 σ	0.683 σ	0.683 σ	0.682 σ	0.682 σ
68.0%	1.015 σ	1.014 σ	1.013 σ	1.012 σ	1.012 σ	1.011 σ	1.011 σ	1.010 σ
90.0%	1.708 σ	1.706 σ	1.703 σ	1.701 σ	1.699 σ	1.697 σ	1.696 σ	1.694 σ
95.0%	2.060 σ	2.056 σ	2.052 σ	2.048 σ	2.045 σ	2.042 σ	2.040 σ	2.037 σ
99.0%	2.787 σ	2.779 σ	2.771 σ	2.763 σ	2.756 σ	2.750 σ	2.744 σ	2.738 σ
99.8%	3.450 σ	3.435 σ	3.421 σ	3.408 σ	3.396 σ	3.385 σ	3.375 σ	3.365 σ
99.995%	4.887 σ	4.849 σ	4.816 σ	4.784 σ	4.756 σ	4.729 σ	4.705 σ	4.682 σ

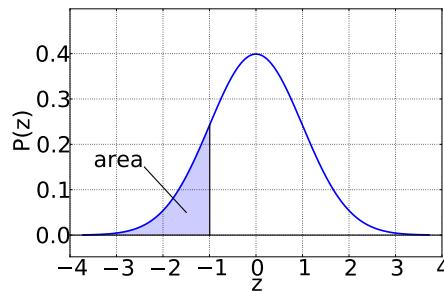
Credible Interval	Degrees of Freedom							
	33	34	35	36	37	38	39	40
50.0%	0.682 σ	0.682 σ	0.682 σ	0.681 σ				
68.0%	1.010 σ	1.009 σ	1.009 σ	1.008 σ	1.008 σ	1.008 σ	1.007 σ	1.007 σ
90.0%	1.692 σ	1.691 σ	1.690 σ	1.688 σ	1.687 σ	1.686 σ	1.685 σ	1.684 σ
95.0%	2.035 σ	2.032 σ	2.030 σ	2.028 σ	2.026 σ	2.024 σ	2.023 σ	2.021 σ
99.0%	2.733 σ	2.728 σ	2.724 σ	2.719 σ	2.715 σ	2.712 σ	2.708 σ	2.704 σ
99.8%	3.356 σ	3.348 σ	3.340 σ	3.333 σ	3.326 σ	3.319 σ	3.313 σ	3.307 σ
99.995%	4.660 σ	4.640 σ	4.622 σ	4.604 σ	4.588 σ	4.572 σ	4.558 σ	4.544 σ

EXAMPLE D.2 Usage of the Credible Interval Table for the Student's t Distribution

Given a set of 10 samples (9 degrees of freedom) with sample mean $\bar{x} = 5.2$ and sample deviation $s = 0.3$, the best estimate for the mean parameter μ , representing the true value of the data, is the sample mean, $\hat{\mu} = 5.2$ with uncertainty s/\sqrt{N} or $0.3/\sqrt{10} = 0.095$. Some of the credible intervals for this estimate then are the following

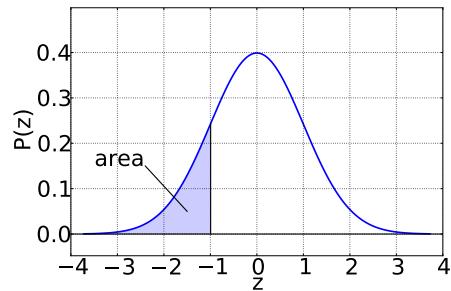
- 68% - $[5.2 - 1.053 \cdot 0.095, 5.2 + 1.053 \cdot 0.095] = [5.09, 5.3]$
- 95% - $[5.2 - 2.262 \cdot 0.095, 5.2 + 2.262 \cdot 0.095] = [4.99, 5.41]$
- 99.8% - $[5.2 - 4.297 \cdot 0.095, 5.2 + 4.297 \cdot 0.095] = [4.79, 5.61]$

D.3 Cumulative Standard Normal Distribution

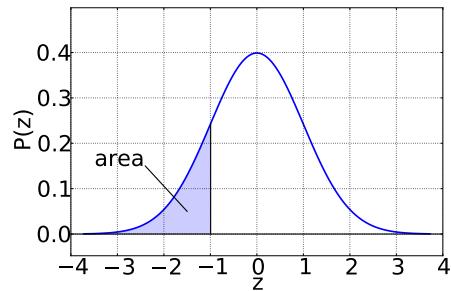


z	Area on Left								
-3.70	0.0001	-3.40	0.0003	-3.10	0.0010	-2.80	0.0026	-2.50	0.0062
-3.69	0.0001	-3.39	0.0003	-3.09	0.0010	-2.79	0.0026	-2.49	0.0064
-3.68	0.0001	-3.38	0.0004	-3.08	0.0010	-2.78	0.0027	-2.48	0.0066
-3.67	0.0001	-3.37	0.0004	-3.07	0.0011	-2.77	0.0028	-2.47	0.0068
-3.66	0.0001	-3.36	0.0004	-3.06	0.0011	-2.76	0.0029	-2.46	0.0069
-3.65	0.0001	-3.35	0.0004	-3.05	0.0011	-2.75	0.0030	-2.45	0.0071
-3.64	0.0001	-3.34	0.0004	-3.04	0.0012	-2.74	0.0031	-2.44	0.0073
-3.63	0.0001	-3.33	0.0004	-3.03	0.0012	-2.73	0.0032	-2.43	0.0075
-3.62	0.0001	-3.32	0.0005	-3.02	0.0013	-2.72	0.0033	-2.42	0.0078
-3.61	0.0002	-3.31	0.0005	-3.01	0.0013	-2.71	0.0034	-2.41	0.0080
-3.60	0.0002	-3.30	0.0005	-3.00	0.0013	-2.70	0.0035	-2.40	0.0082
-3.59	0.0002	-3.29	0.0005	-2.99	0.0014	-2.69	0.0036	-2.39	0.0084
-3.58	0.0002	-3.28	0.0005	-2.98	0.0014	-2.68	0.0037	-2.38	0.0087
-3.57	0.0002	-3.27	0.0005	-2.97	0.0015	-2.67	0.0038	-2.37	0.0089
-3.56	0.0002	-3.26	0.0006	-2.96	0.0015	-2.66	0.0039	-2.36	0.0091
-3.55	0.0002	-3.25	0.0006	-2.95	0.0016	-2.65	0.0040	-2.35	0.0094
-3.54	0.0002	-3.24	0.0006	-2.94	0.0016	-2.64	0.0041	-2.34	0.0096
-3.53	0.0002	-3.23	0.0006	-2.93	0.0017	-2.63	0.0043	-2.33	0.0099
-3.52	0.0002	-3.22	0.0006	-2.92	0.0018	-2.62	0.0044	-2.32	0.0102
-3.51	0.0002	-3.21	0.0007	-2.91	0.0018	-2.61	0.0045	-2.31	0.0104
-3.50	0.0002	-3.20	0.0007	-2.90	0.0019	-2.60	0.0047	-2.30	0.0107
-3.49	0.0002	-3.19	0.0007	-2.89	0.0019	-2.59	0.0048	-2.29	0.0110
-3.48	0.0003	-3.18	0.0007	-2.88	0.0020	-2.58	0.0049	-2.28	0.0113
-3.47	0.0003	-3.17	0.0008	-2.87	0.0021	-2.57	0.0051	-2.27	0.0116
-3.46	0.0003	-3.16	0.0008	-2.86	0.0021	-2.56	0.0052	-2.26	0.0119
-3.45	0.0003	-3.15	0.0008	-2.85	0.0022	-2.55	0.0054	-2.25	0.0122
-3.44	0.0003	-3.14	0.0008	-2.84	0.0023	-2.54	0.0055	-2.24	0.0125
-3.43	0.0003	-3.13	0.0009	-2.83	0.0023	-2.53	0.0057	-2.23	0.0129
-3.42	0.0003	-3.12	0.0009	-2.82	0.0024	-2.52	0.0059	-2.22	0.0132
-3.41	0.0003	-3.11	0.0009	-2.81	0.0025	-2.51	0.0060	-2.21	0.0136
-3.40	0.0003	-3.10	0.0010	-2.80	0.0026	-2.50	0.0062	-2.20	0.0139

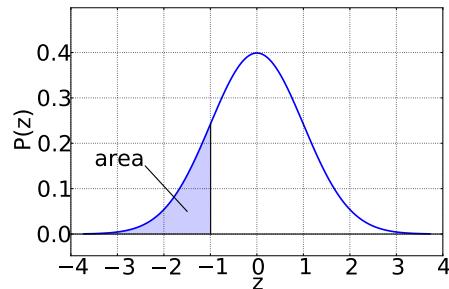
Cumulative Normal Distribution (cont.)



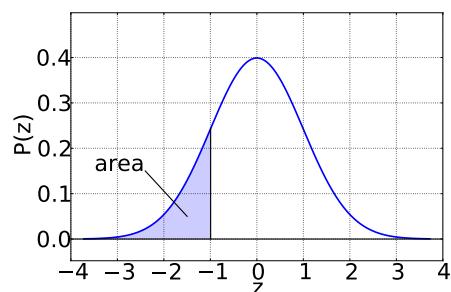
z	Area on Left								
-2.20	0.0139	-1.90	0.0287	-1.60	0.0548	-1.30	0.0968	-1.00	0.1587
-2.19	0.0143	-1.89	0.0294	-1.59	0.0559	-1.29	0.0985	-0.99	0.1611
-2.18	0.0146	-1.88	0.0301	-1.58	0.0571	-1.28	0.1003	-0.98	0.1635
-2.17	0.0150	-1.87	0.0307	-1.57	0.0582	-1.27	0.1020	-0.97	0.1660
-2.16	0.0154	-1.86	0.0314	-1.56	0.0594	-1.26	0.1038	-0.96	0.1685
-2.15	0.0158	-1.85	0.0322	-1.55	0.0606	-1.25	0.1056	-0.95	0.1711
-2.14	0.0162	-1.84	0.0329	-1.54	0.0618	-1.24	0.1075	-0.94	0.1736
-2.13	0.0166	-1.83	0.0336	-1.53	0.0630	-1.23	0.1093	-0.93	0.1762
-2.12	0.0170	-1.82	0.0344	-1.52	0.0643	-1.22	0.1112	-0.92	0.1788
-2.11	0.0174	-1.81	0.0351	-1.51	0.0655	-1.21	0.1131	-0.91	0.1814
-2.10	0.0179	-1.80	0.0359	-1.50	0.0668	-1.20	0.1151	-0.90	0.1841
-2.09	0.0183	-1.79	0.0367	-1.49	0.0681	-1.19	0.1170	-0.89	0.1867
-2.08	0.0188	-1.78	0.0375	-1.48	0.0694	-1.18	0.1190	-0.88	0.1894
-2.07	0.0192	-1.77	0.0384	-1.47	0.0708	-1.17	0.1210	-0.87	0.1922
-2.06	0.0197	-1.76	0.0392	-1.46	0.0721	-1.16	0.1230	-0.86	0.1949
-2.05	0.0202	-1.75	0.0401	-1.45	0.0735	-1.15	0.1251	-0.85	0.1977
-2.04	0.0207	-1.74	0.0409	-1.44	0.0749	-1.14	0.1271	-0.84	0.2005
-2.03	0.0212	-1.73	0.0418	-1.43	0.0764	-1.13	0.1292	-0.83	0.2033
-2.02	0.0217	-1.72	0.0427	-1.42	0.0778	-1.12	0.1314	-0.82	0.2061
-2.01	0.0222	-1.71	0.0436	-1.41	0.0793	-1.11	0.1335	-0.81	0.2090
-2.00	0.0228	-1.70	0.0446	-1.40	0.0808	-1.10	0.1357	-0.80	0.2119
-1.99	0.0233	-1.69	0.0455	-1.39	0.0823	-1.09	0.1379	-0.79	0.2148
-1.98	0.0239	-1.68	0.0465	-1.38	0.0838	-1.08	0.1401	-0.78	0.2177
-1.97	0.0244	-1.67	0.0475	-1.37	0.0853	-1.07	0.1423	-0.77	0.2206
-1.96	0.0250	-1.66	0.0485	-1.36	0.0869	-1.06	0.1446	-0.76	0.2236
-1.95	0.0256	-1.65	0.0495	-1.35	0.0885	-1.05	0.1469	-0.75	0.2266
-1.94	0.0262	-1.64	0.0505	-1.34	0.0901	-1.04	0.1492	-0.74	0.2296
-1.93	0.0268	-1.63	0.0516	-1.33	0.0918	-1.03	0.1515	-0.73	0.2327
-1.92	0.0274	-1.62	0.0526	-1.32	0.0934	-1.02	0.1539	-0.72	0.2358
-1.91	0.0281	-1.61	0.0537	-1.31	0.0951	-1.01	0.1562	-0.71	0.2389
-1.90	0.0287	-1.60	0.0548	-1.30	0.0968	-1.00	0.1587	-0.70	0.2420

Cumulative Normal Distribution (cont.)

z	Area on Left								
-0.70	0.2420	-0.40	0.3446	-0.10	0.4602	0.20	0.5793	0.50	0.6915
-0.69	0.2451	-0.39	0.3483	-0.09	0.4641	0.21	0.5832	0.51	0.6950
-0.68	0.2483	-0.38	0.3520	-0.08	0.4681	0.22	0.5871	0.52	0.6985
-0.67	0.2514	-0.37	0.3557	-0.07	0.4721	0.23	0.5910	0.53	0.7019
-0.66	0.2546	-0.36	0.3594	-0.06	0.4761	0.24	0.5948	0.54	0.7054
-0.65	0.2578	-0.35	0.3632	-0.05	0.4801	0.25	0.5987	0.55	0.7088
-0.64	0.2611	-0.34	0.3669	-0.04	0.4840	0.26	0.6026	0.56	0.7123
-0.63	0.2643	-0.33	0.3707	-0.03	0.4880	0.27	0.6064	0.57	0.7157
-0.62	0.2676	-0.32	0.3745	-0.02	0.4920	0.28	0.6103	0.58	0.7190
-0.61	0.2709	-0.31	0.3783	-0.01	0.4960	0.29	0.6141	0.59	0.7224
-0.60	0.2743	-0.30	0.3821	0.00	0.5000	0.30	0.6179	0.60	0.7257
-0.59	0.2776	-0.29	0.3859	0.01	0.5040	0.31	0.6217	0.61	0.7291
-0.58	0.2810	-0.28	0.3897	0.02	0.5080	0.32	0.6255	0.62	0.7324
-0.57	0.2843	-0.27	0.3936	0.03	0.5120	0.33	0.6293	0.63	0.7357
-0.56	0.2877	-0.26	0.3974	0.04	0.5160	0.34	0.6331	0.64	0.7389
-0.55	0.2912	-0.25	0.4013	0.05	0.5199	0.35	0.6368	0.65	0.7422
-0.54	0.2946	-0.24	0.4052	0.06	0.5239	0.36	0.6406	0.66	0.7454
-0.53	0.2981	-0.23	0.4090	0.07	0.5279	0.37	0.6443	0.67	0.7486
-0.52	0.3015	-0.22	0.4129	0.08	0.5319	0.38	0.6480	0.68	0.7517
-0.51	0.3050	-0.21	0.4168	0.09	0.5359	0.39	0.6517	0.69	0.7549
-0.50	0.3085	-0.20	0.4207	0.10	0.5398	0.40	0.6554	0.70	0.7580
-0.49	0.3121	-0.19	0.4247	0.11	0.5438	0.41	0.6591	0.71	0.7611
-0.48	0.3156	-0.18	0.4286	0.12	0.5478	0.42	0.6628	0.72	0.7642
-0.47	0.3192	-0.17	0.4325	0.13	0.5517	0.43	0.6664	0.73	0.7673
-0.46	0.3228	-0.16	0.4364	0.14	0.5557	0.44	0.6700	0.74	0.7704
-0.45	0.3264	-0.15	0.4404	0.15	0.5596	0.45	0.6736	0.75	0.7734
-0.44	0.3300	-0.14	0.4443	0.16	0.5636	0.46	0.6772	0.76	0.7764
-0.43	0.3336	-0.13	0.4483	0.17	0.5675	0.47	0.6808	0.77	0.7794
-0.42	0.3372	-0.12	0.4522	0.18	0.5714	0.48	0.6844	0.78	0.7823
-0.41	0.3409	-0.11	0.4562	0.19	0.5753	0.49	0.6879	0.79	0.7852
-0.40	0.3446	-0.10	0.4602	0.20	0.5793	0.50	0.6915	0.80	0.7881

Cumulative Normal Distribution (cont.)

z	Area on Left								
0.80	0.7881	1.10	0.8643	1.40	0.9192	1.70	0.9554	2.00	0.9772
0.81	0.7910	1.11	0.8665	1.41	0.9207	1.71	0.9564	2.01	0.9778
0.82	0.7939	1.12	0.8686	1.42	0.9222	1.72	0.9573	2.02	0.9783
0.83	0.7967	1.13	0.8708	1.43	0.9236	1.73	0.9582	2.03	0.9788
0.84	0.7995	1.14	0.8729	1.44	0.9251	1.74	0.9591	2.04	0.9793
0.85	0.8023	1.15	0.8749	1.45	0.9265	1.75	0.9599	2.05	0.9798
0.86	0.8051	1.16	0.8770	1.46	0.9279	1.76	0.9608	2.06	0.9803
0.87	0.8078	1.17	0.8790	1.47	0.9292	1.77	0.9616	2.07	0.9808
0.88	0.8106	1.18	0.8810	1.48	0.9306	1.78	0.9625	2.08	0.9812
0.89	0.8133	1.19	0.8830	1.49	0.9319	1.79	0.9633	2.09	0.9817
0.90	0.8159	1.20	0.8849	1.50	0.9332	1.80	0.9641	2.10	0.9821
0.91	0.8186	1.21	0.8869	1.51	0.9345	1.81	0.9649	2.11	0.9826
0.92	0.8212	1.22	0.8888	1.52	0.9357	1.82	0.9656	2.12	0.9830
0.93	0.8238	1.23	0.8907	1.53	0.9370	1.83	0.9664	2.13	0.9834
0.94	0.8264	1.24	0.8925	1.54	0.9382	1.84	0.9671	2.14	0.9838
0.95	0.8289	1.25	0.8944	1.55	0.9394	1.85	0.9678	2.15	0.9842
0.96	0.8315	1.26	0.8962	1.56	0.9406	1.86	0.9686	2.16	0.9846
0.97	0.8340	1.27	0.8980	1.57	0.9418	1.87	0.9693	2.17	0.9850
0.98	0.8365	1.28	0.8997	1.58	0.9429	1.88	0.9699	2.18	0.9854
0.99	0.8389	1.29	0.9015	1.59	0.9441	1.89	0.9706	2.19	0.9857
1.00	0.8413	1.30	0.9032	1.60	0.9452	1.90	0.9713	2.20	0.9861
1.01	0.8438	1.31	0.9049	1.61	0.9463	1.91	0.9719	2.21	0.9864
1.02	0.8461	1.32	0.9066	1.62	0.9474	1.92	0.9726	2.22	0.9868
1.03	0.8485	1.33	0.9082	1.63	0.9484	1.93	0.9732	2.23	0.9871
1.04	0.8508	1.34	0.9099	1.64	0.9495	1.94	0.9738	2.24	0.9875
1.05	0.8531	1.35	0.9115	1.65	0.9505	1.95	0.9744	2.25	0.9878
1.06	0.8554	1.36	0.9131	1.66	0.9515	1.96	0.9750	2.26	0.9881
1.07	0.8577	1.37	0.9147	1.67	0.9525	1.97	0.9756	2.27	0.9884
1.08	0.8599	1.38	0.9162	1.68	0.9535	1.98	0.9761	2.28	0.9887
1.09	0.8621	1.39	0.9177	1.69	0.9545	1.99	0.9767	2.29	0.9890
1.10	0.8643	1.40	0.9192	1.70	0.9554	2.00	0.9772	2.30	0.9893

Cumulative Normal Distribution (cont.)

z	Area on Left								
2.30	0.9893	2.60	0.9953	2.90	0.9981	3.20	0.9993	3.50	0.9998
2.31	0.9896	2.61	0.9955	2.91	0.9982	3.21	0.9993	3.51	0.9998
2.32	0.9898	2.62	0.9956	2.92	0.9982	3.22	0.9994	3.52	0.9998
2.33	0.9901	2.63	0.9957	2.93	0.9983	3.23	0.9994	3.53	0.9998
2.34	0.9904	2.64	0.9959	2.94	0.9984	3.24	0.9994	3.54	0.9998
2.35	0.9906	2.65	0.9960	2.95	0.9984	3.25	0.9994	3.55	0.9998
2.36	0.9909	2.66	0.9961	2.96	0.9985	3.26	0.9994	3.56	0.9998
2.37	0.9911	2.67	0.9962	2.97	0.9985	3.27	0.9995	3.57	0.9998
2.38	0.9913	2.68	0.9963	2.98	0.9986	3.28	0.9995	3.58	0.9998
2.39	0.9916	2.69	0.9964	2.99	0.9986	3.29	0.9995	3.59	0.9998
2.40	0.9918	2.70	0.9965	3.00	0.9987	3.30	0.9995	3.60	0.9998
2.41	0.9920	2.71	0.9966	3.01	0.9987	3.31	0.9995	3.61	0.9998
2.42	0.9922	2.72	0.9967	3.02	0.9987	3.32	0.9995	3.62	0.9999
2.43	0.9925	2.73	0.9968	3.03	0.9988	3.33	0.9996	3.63	0.9999
2.44	0.9927	2.74	0.9969	3.04	0.9988	3.34	0.9996	3.64	0.9999
2.45	0.9929	2.75	0.9970	3.05	0.9989	3.35	0.9996	3.65	0.9999
2.46	0.9931	2.76	0.9971	3.06	0.9989	3.36	0.9996	3.66	0.9999
2.47	0.9932	2.77	0.9972	3.07	0.9989	3.37	0.9996	3.67	0.9999
2.48	0.9934	2.78	0.9973	3.08	0.9990	3.38	0.9996	3.68	0.9999
2.49	0.9936	2.79	0.9974	3.09	0.9990	3.39	0.9997	3.69	0.9999
2.50	0.9938	2.80	0.9974	3.10	0.9990	3.40	0.9997	3.70	0.9999
2.51	0.9940	2.81	0.9975	3.11	0.9991	3.41	0.9997	3.71	0.9999
2.52	0.9941	2.82	0.9976	3.12	0.9991	3.42	0.9997	3.72	0.9999
2.53	0.9943	2.83	0.9977	3.13	0.9991	3.43	0.9997	3.73	0.9999
2.54	0.9945	2.84	0.9977	3.14	0.9992	3.44	0.9997	3.74	0.9999
2.55	0.9946	2.85	0.9978	3.15	0.9992	3.45	0.9997	3.75	0.9999
2.56	0.9948	2.86	0.9979	3.16	0.9992	3.46	0.9997	3.76	0.9999
2.57	0.9949	2.87	0.9979	3.17	0.9992	3.47	0.9997	3.77	0.9999
2.58	0.9951	2.88	0.9980	3.18	0.9993	3.48	0.9997	3.78	0.9999
2.59	0.9952	2.89	0.9981	3.19	0.9993	3.49	0.9998	3.79	0.9999
2.60	0.9953	2.90	0.9981	3.20	0.9993	3.50	0.9998	3.80	0.9999