Mirrlees derivation with income effects

December 23, 2021

Consider separable utility,

$$U(c,z,w) = u(c) - \psi(z/w), \tag{1}$$

where w is individual wage with density $f(w)$. Define the optimal control problem,

The government maximizes

$$\max_{c(w),z(w)} \int_{w_0}^{w_1} \left[u(c(w)) - \psi\left(\frac{z(w)}{w}\right) \right] f(w)dw \tag{2}$$

subject to incentive compatibility, which we assume is fully characterized by the local FOC for truthful type reporting,

$$\left| \frac{dU(c(\tilde{w}), z(\tilde{w}), w)}{d\tilde{w}} \right|_{\tilde{w}=w} = 0 \tag{3}$$

for all w, implying

$$\frac{dU(c(w), z(w), w)}{dw} = \left| \frac{\partial U(c(w), z(w), \tilde{w})}{\partial \tilde{w}} \right|_{\tilde{w}=w} = \psi'\left(\frac{z(w)}{w}\right)\frac{z(w)}{w^2}. \tag{4}$$

The optimization is also subject to the resource constraint:

$$\int_{w_0}^{w_1} [z(w) - c(w)]\, f(w)dw.$$

Like in Mirrlees, we solve this via a Hamiltonian. Let $v(w)$ denote the utility assignment of type w, and take $v(w)$ and $z(w)$ as the planner's choice variables, with $c(w)$ defined implicitly by

$$v(w) = u(c(v(w), z(w), w))) - \psi(z(w)/w). \tag{5}$$

Then the optimal control program is

$$\max_{v(w),z(w)} \int_{w_0}^{w_1} v(w)f(w)dw \tag{6}$$

subject to the local FOC from (4)

$$v'(w) = \psi'\left(\frac{z(w)}{w}\right)\frac{z(w)}{w^2} \tag{7}$$

the monotonicity constraint $z'(w) > 0$, and the resource constraint

$$\int_{w_0}^{w_1} \left[z(w) - c(v(w), z(w), w))\right] f(w)dw. \tag{8}$$

By the implicit function theorem,

$$c'_v = \frac{1}{u'(c(v(w), z(w), w))}. \tag{9}$$

Letting $z(w)$ be the control variable and $v(w)$ the state variable, the Hamiltonian is given by

$$\mathcal{H}(v(w), z(w), \mu(w), \lambda, w) = v(w)f(w) + \lambda\left[z(w) - c(v(w), z(w), w)\right] f(w) + \mu(w)v'(w) \tag{10}$$

$$= v(w)f(w) + \lambda\left[z(w) - c(v(w), z(w), w)\right] f(w) + \mu(w)\psi'\left(\frac{z(w)}{w}\right)\frac{z(w)}{w^2}. \tag{11}$$

with λ the multiplier on the resource constraint and $\mu(w)$ the costate variable. At the optimum, $z(w)$ maximizes the Hamiltonian:

$$\frac{\partial\mathcal{H}}{\partial z(w)} = \lambda\left[1 - c'_z\right] f(w) + \mu(w)\left[\psi''\left(\frac{z(w)}{w}\right)\frac{z(w)}{w^3} + \psi'\left(\frac{z(w)}{w}\right)\frac{1}{w^2}\right] = 0, \tag{12}$$

implying

$$1 - c'_z = \frac{1}{\lambda f(w)}\left[\psi''\left(\frac{z(w)}{w}\right)\frac{z(w)}{w^3} + \psi'\left(\frac{z(w)}{w}\right)\frac{1}{w^2}\right](-\mu(w)). \tag{13}$$

Note also that

$$\frac{\partial\mathcal{H}}{\partial v(w)} = f(w)\left(1 - \lambda c'_v\right) \tag{14}$$

$$= f(w)\left(1 - \frac{\lambda}{u'(c(v(w), z(w), w))}\right). \tag{15}$$

The first-order condition for the Hamiltonian is

$$\mu'(w) = -\frac{\partial\mathcal{H}}{\partial v(w)} \tag{16}$$

$$= \left(\frac{\lambda}{u'(c(v(w), z(w), w))} - 1\right) f(w). \tag{17}$$

Integrating, and using the transversality condition $\mu(w_1) = 0$, we have

$$\mu(w) = -\int_w^{w_1} \mu'(s)ds \tag{18}$$

$$= -\int_w^{w_1} \left(\frac{\lambda}{u'(c(v(w), z(w), w))} - 1 \right) f(s)ds. \tag{19}$$

Using the other transversality condition, $\mu(w_0) = 0$, this equation pins down the multiplier on the resource constraint:

$$0 = \int_{w_0}^{w_1} \left(\frac{1}{u'(c(v(s), z(s), s))} - \frac{1}{\lambda} \right) f(s)ds \tag{20}$$

$$\Rightarrow \frac{1}{\lambda} = \int_{w_0}^{w_1} \left(\frac{1}{u'(c(v(s), z(s), s))} \right) f(s)ds \tag{21}$$

$$\lambda = \mathbb{E}\left[\frac{1}{u'_c}\right]^{-1} \tag{22}$$

Plugging (19) into (13), we can solve the Hamiltonian:

$$T'(z(w)) = 1 - c'_z$$

$$= \frac{1}{\lambda f(w) w} \left[\psi'' \left(\frac{z(w)}{w} \right) \frac{z(w)}{w^2} + \psi' \left(\frac{z(w)}{w} \right) \frac{1}{w} \right] \int_w^{w_1} \left(\frac{\lambda}{u'(c(v(w), z(w), w))} - 1 \right) f(s)ds. \tag{23}$$

Finally, consider the term in brackets. We can simplify it using the elasticity of taxable income. Start with the agent's FOC:

$$u'(c)(1 - T') - \psi' \left(\frac{z}{w} \right) \frac{1}{w} = 0. \tag{24}$$

Totally differentiating with respect to a compensated change in $1 - T'$, we have

$$\frac{dz}{d(1 - T')} = \frac{u'(c)}{\psi'' \left(\frac{z}{w} \right) \frac{1}{w^2}}, \tag{25}$$

which we can express as a compensated ETI:

$$\zeta_z^c(w) = \frac{dz}{d(1 - T')} \frac{1 - T'}{z} = \frac{u'(c)}{\psi'' \left(\frac{z}{w} \right) \frac{1}{w^2}} \cdot \frac{1 - T'}{z} \tag{26}$$

$$\Rightarrow \psi'' \left(\frac{z}{w} \right) \frac{z}{w^2} = \frac{u'(c)}{\zeta_z^c(w)} \cdot (1 - T') \tag{27}$$

Using this, and (from the FOC itself) $\psi' \left(\frac{z(w)}{w} \right) \frac{1}{w} = u'(c)(1 - T')$, the Hamiltonian solution can be written as

$$\frac{T'(z(w))}{1 - T'(z(w))} = \frac{u'(c(w))}{\lambda f(w) w} \left[1 + \frac{1}{\zeta_z^c(w)} \right] \int_w^{w_1} \left(\frac{\lambda}{u'(c(s))} - 1 \right) f(s)ds. \tag{28}$$

Finally, use the usual definition of welfare weights as a function of type at the optimum, $g(w) :=$

$\frac{u'(c(w))}{\lambda}$, we get

$$\frac{T'(z(w))}{1-T'(z(w))} = \frac{g(w)}{f(w)w}\left[1+\frac{1}{\zeta_z^c(w)}\right]\int_w^{w_1}\left(\frac{1}{g(s)}-1\right)f(s)ds \tag{29}$$

$$= \frac{1+1/\zeta_z^c(w)}{f(w)w}\int_w^{w_1}\frac{g(w)}{g(s)}\left(1-g(s)\right)f(s)ds. \tag{30}$$