Constraint Particles

Outline

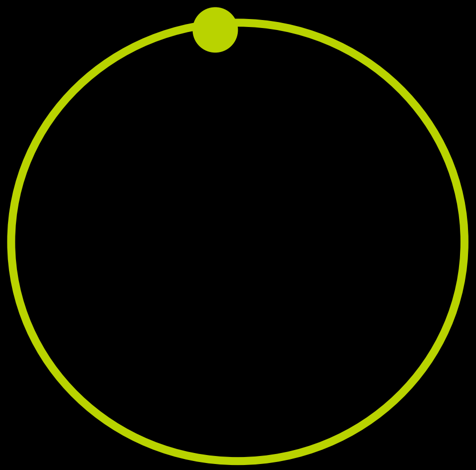
- Constraint force
  - Single implicit constraint
  - Multiple implicit constraint
- Parametric constraint

A Simple Example

- A bead on a wire

- The bead can slide freely along the wire but cannot come off it

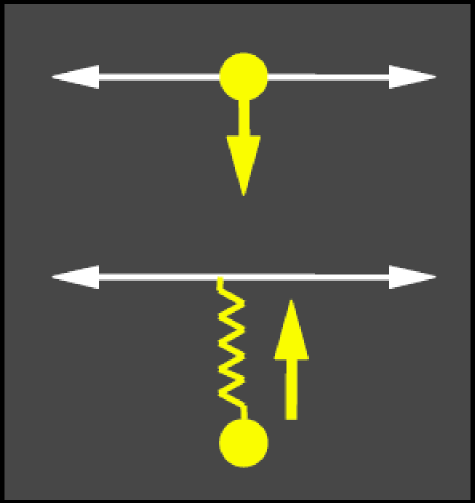
- How do we simulate the motion of the bead when arbitrary forces applied to it



Penalty Constraints

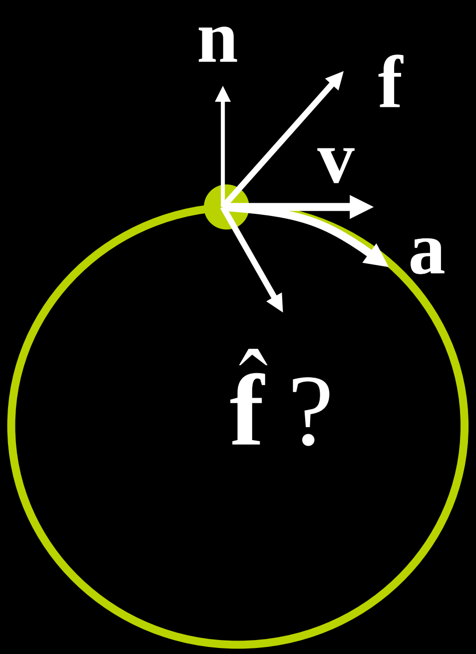
- Why not use a spring to hold the bead on the wire?

- Problems:
  - Weak spring won't do the job
  - Strong springs results in stiff systems



Constraint Force

- What is legal acceleration?
- Depends on both **n** and **v**
- Compute $\hat{\mathbf{f}}$ such that the net force $\mathbf{f} + \hat{\mathbf{f}}$ only generates legal acceleration



Constraint Force

- Need to compute constraint forces that cancel the illegal applied forces

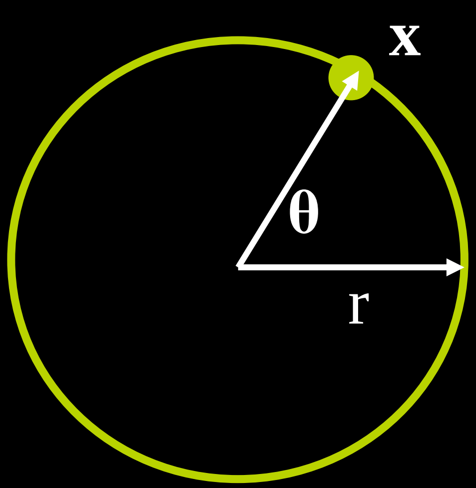
- Need to know what legal acceleration is

- Constraint force
  - Single implicit constraint
  - Multiple implicit constraint
- Parametric constraint

Constraints

- Implicit

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$



- Parametric

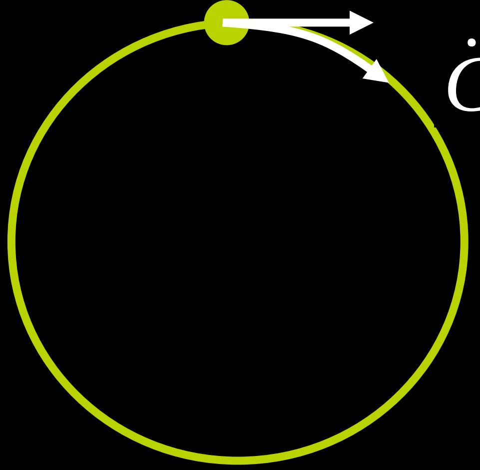
$$\mathbf{x} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

Legal Acceleration

- What is the legal position?

$$C(\mathbf{x}) = \frac{1}{2}(\mathbf{x} \cdot \mathbf{x} - 1)$$

inner product assume unit circle



- What is the legal velocity

$$\dot{C}(\mathbf{x}) = \mathbf{x} \cdot \dot{\mathbf{x}} = 0$$

- What is the legal acceleration?

$$\ddot{C}(\mathbf{x}) = \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \mathbf{x} \cdot \ddot{\mathbf{x}} = 0$$

Legal Conditions

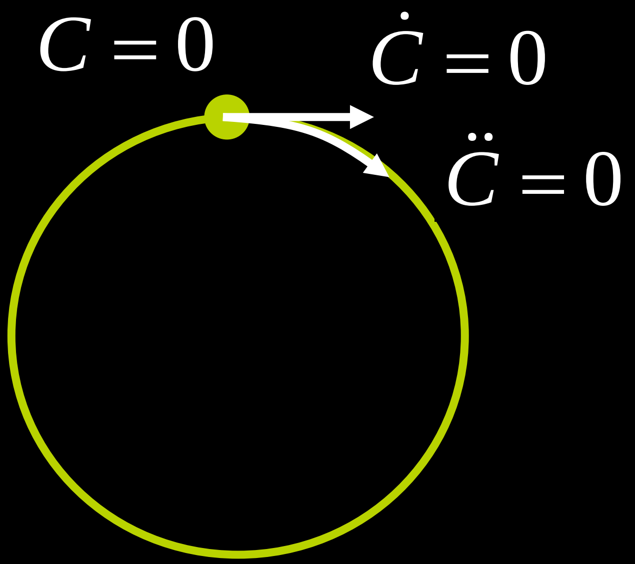
- If we start with legal position and velocity

$$C(\mathbf{x}) = 0$$

$$\dot{C}(\mathbf{x}) = 0$$

- We only need ensure the legal acceleration

$$\ddot{C}(\mathbf{x}) = 0$$



Constraint Force

- Use the legal condition to compute the constraint force $\hat{\mathbf{f}}$

$$\ddot{C}(\mathbf{x}) = \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \mathbf{x} \cdot \ddot{\mathbf{x}} = 0$$

- Substitute $\ddot{\mathbf{x}}$ with $\ddot{\mathbf{x}} = \dfrac{\mathbf{f} + \hat{\mathbf{f}}}{m}$

$$\ddot{C}(\mathbf{x}) = \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \mathbf{x} \cdot \frac{\mathbf{f} + \hat{\mathbf{f}}}{m} = 0$$

$$\hat{\mathbf{f}} \cdot \mathbf{x} = -\mathbf{f} \cdot \mathbf{x} - m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$$

Constraint Force

$$\hat{\mathbf{f}} \cdot \mathbf{x} = -\mathbf{f} \cdot \mathbf{x} - m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$$

- We have one equation and two variables
- Need one more condition to solve the constraint force

Virtual Work

- Constraint force is passive—no energy gain or loss
- Kinetic energy of the system $T = \frac{1}{2} m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$

$$\dot{T} = \frac{1}{2} m(\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}}) = m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{f} + \hat{\mathbf{f}})$$

which is virtual work done by f and $\hat{\mathbf{f}}$

- Make sure does no work for every legal velocity $\hat{\mathbf{f}} \cdot \dot{\mathbf{x}} = 0, \forall \dot{\mathbf{x}} \in \{\dot{\mathbf{x}} \mid \mathbf{x} \cdot \dot{\mathbf{x}} = 0\}$

set of legal velocity

Virtual Work

$$\hat{\mathbf{f}} \cdot \dot{\mathbf{x}} = 0, \forall \dot{\mathbf{x}} \in \{\dot{\mathbf{x}} \mid \mathbf{x} \cdot \dot{\mathbf{x}} = 0\}$$

- The virtual work condition can be rewritten as

$$\hat{\mathbf{f}} = \lambda \mathbf{x}$$

- Substituting for $\hat{\mathbf{f}}$ in $\hat{\mathbf{f}} \cdot \mathbf{x} = -\mathbf{f} \cdot \mathbf{x} - m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$

$$\lambda = \frac{-\mathbf{f} \cdot \mathbf{x} - m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}{\mathbf{x} \cdot \mathbf{x}}$$

Geometric Interpretation

- Constraint force

$$\hat{\mathbf{f}} = \lambda \mathbf{x} = \frac{-\mathbf{f} \cdot \mathbf{x} - m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}{\mathbf{x} \cdot \mathbf{x}} \mathbf{x}$$

- If the system is at rest

Normal component of the applied force

$$\hat{\mathbf{f}} = -\frac{\mathbf{f} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} \mathbf{x}$$

- Only the tangent component of the applied force is kept

- Constraint force
  - Single implicit constraint
  - Multiple implicit constraint
- Parametric constraint

Generalization to Particle System

- We know how to simulate a bead on a wire now
- Apply the same idea, we can create a constrained particle system

Constraint Particles

- Particles: each particle represents a point in the phase space
- Forces: each force affects the acceleration of certain particles
- Constraints:
  - Each is a function $C_i(x_1, x_2, \ldots)$
  - Legal state: $C_i(x_1, x_2, \ldots) = 0$, for all i
  - Constraint force: linear combination of constraint gradients $\frac{\partial \mathbf{C}_i}{\partial \mathbf{x}}, \forall i$

Motion Equation in General Case

- For a system of *n* particles

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{bmatrix} = \begin{bmatrix} \frac{1}{m_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{m_n} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

$$\Downarrow$$

$$\ddot{\mathbf{q}} = \mathbf{W}\mathbf{Q}$$

Constraint Equation in General Case

- System of constraint equation $\mathbf{C}(\mathbf{q}) = 0$

$$\dot{\mathbf{C}} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}} \frac{d\mathbf{q}}{dt} = \mathbf{J}\dot{\mathbf{q}} = 0 \qquad \ddot{\mathbf{C}} = \dot{\mathbf{J}}\dot{\mathbf{q}} + \mathbf{J}\ddot{\mathbf{q}} = 0$$

Jacobian

- To compute the legal acceleration:

Let $\ddot{\mathbf{q}} = \mathbf{W}(\mathbf{Q} + \hat{\mathbf{Q}})$ Constraint forces

we can get

$$\ddot{\mathbf{C}} = \dot{\mathbf{J}}\dot{\mathbf{q}} + \mathbf{J}\mathbf{W}(\mathbf{Q} + \hat{\mathbf{Q}}) = 0$$

$$\mathbf{J}\mathbf{W}\hat{\mathbf{Q}} = -\dot{\mathbf{J}}\dot{\mathbf{q}} - \mathbf{J}\mathbf{W}\mathbf{Q}$$

Principle of Virtual Work

- To ensure the constraint force does not produce work

$$\hat{\mathbf{Q}}\dot{\mathbf{q}} = 0, \forall \dot{\mathbf{q}} \in \{\dot{\mathbf{q}} \mid \mathbf{J}\dot{\mathbf{q}} = 0\}$$

- All vectors satisfy the condition can be expressed as

$$\hat{\mathbf{Q}} = \mathbf{J}^T \underbrace{\boldsymbol{\lambda}}_{\text{vector}}$$

Two Conditions for Constraint Forces

- Legal acceleration

$$\mathbf{J}\mathbf{W}\hat{\mathbf{Q}} = -\dot{\mathbf{J}}\dot{\mathbf{q}} - \mathbf{J}\mathbf{W}\mathbf{Q}$$

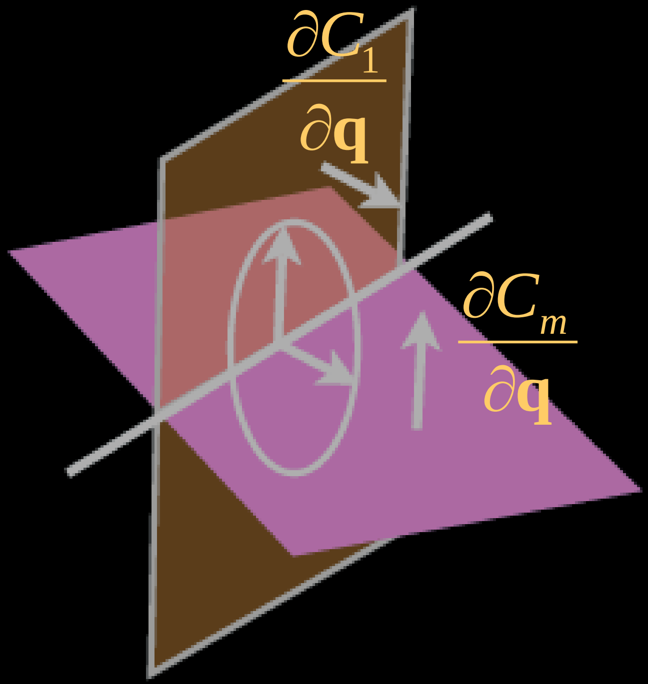
- Principle of virtual work

$$\hat{\mathbf{Q}} = \mathbf{J}^{T}\boldsymbol{\lambda}$$

- Constraint forces obtained by solving the linear system

$$\mathbf{J}\mathbf{W}\mathbf{J}^{\mathrm{T}}\boldsymbol{\lambda} = -\dot{\mathbf{J}}\dot{\mathbf{q}} - \mathbf{J}\mathbf{W}\mathbf{Q}$$

Constraint Gradients (cont.)



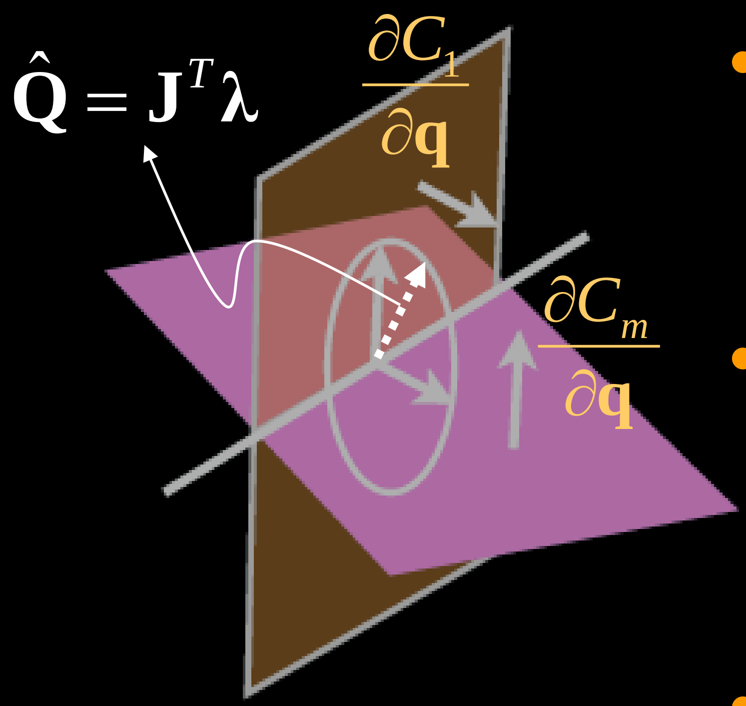
$$\hat{\mathbf{Q}} = \mathbf{J}^T \boldsymbol{\lambda}$$

Normal of C_1: $\frac{\partial C_1}{\partial \mathbf{q}}$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial C_1}{\partial q_1} & \frac{\partial C_1}{\partial q_2} & \cdots & \frac{\partial C_1}{\partial q_n} \\ \frac{\partial C_2}{\partial q_1} & \frac{\partial C_2}{\partial q_2} & \cdots & \frac{\partial C_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial C_m}{\partial q_1} & \frac{\partial C_m}{\partial q_2} & \cdots & \frac{\partial C_m}{\partial q_n} \end{bmatrix}$$

Normal of C_m: $\frac{\partial C_m}{\partial \mathbf{q}}$

Constraint Gradients (cont.)



- Legal states: the intersection of two planes
- Normal of legal states

$$\hat{\mathbf{Q}} = \mathbf{J}^T\boldsymbol{\lambda} \longrightarrow \sum_{i=1}^{m} \lambda_i \frac{\partial C_i}{\partial \mathbf{q}}$$

- Constraint forces are aligned with the normal of legal states
- Work done by constraint forces is zero

Impress your friends

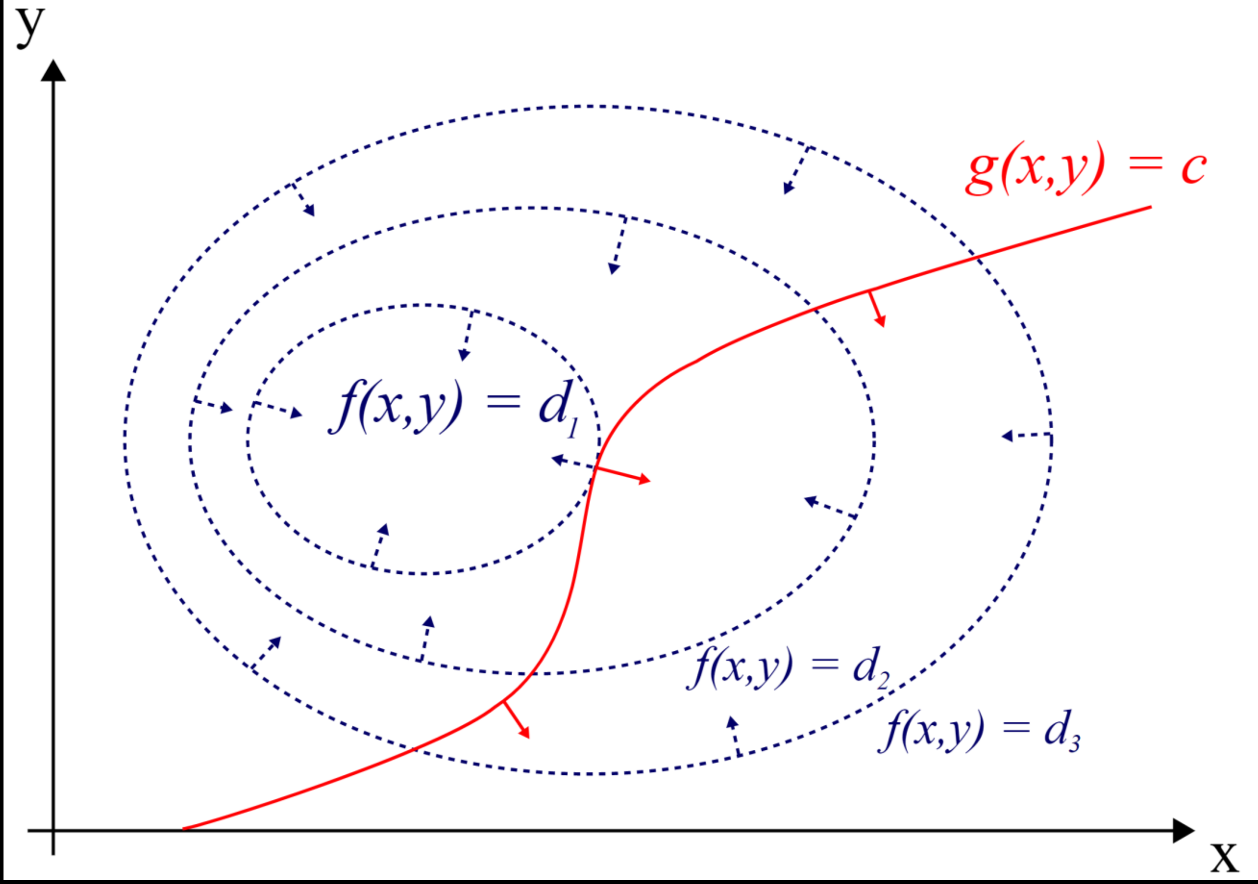
- The requirement that constraints not add or remove energy is called the Principle of Virtual Work
- The λ's are called Lagrangain Multipliers

Side Note on Lagrange Multiplier

- Finds the local maxima/minima of a function subject to equality constraints

Maximize f(x,y)

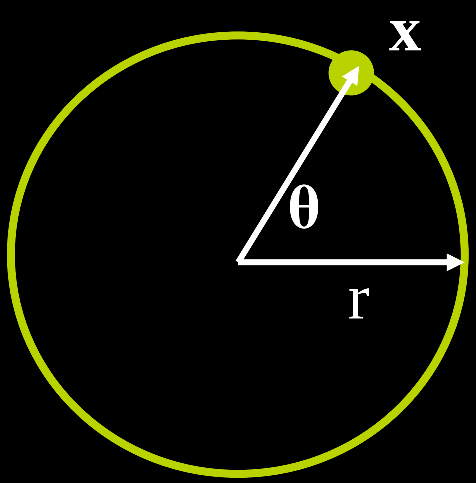
Sub. to g(x,y)=0



Constraints

- Implicit

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

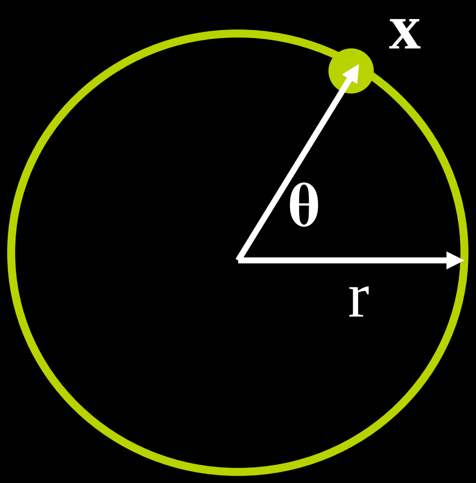


- Parametric

$$\mathbf{x} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

Parametric Constraints

- Constraint is always met exactly
- 1 degree of freedom:
  - Solve for $\ddot{\theta}$

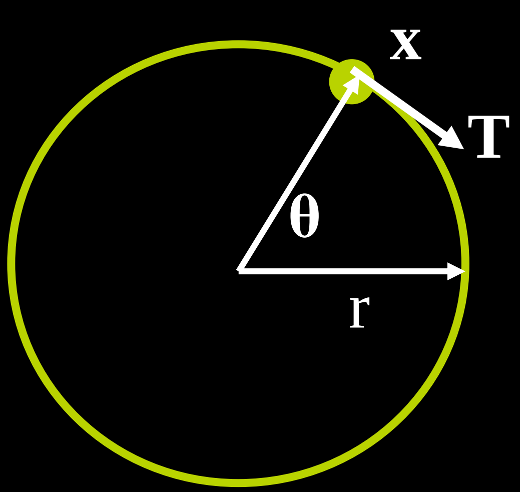


$$\mathbf{x} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

Parametric Constraints

- Constraint is always met exactly

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial \theta}\frac{d\theta}{dt} = \mathbf{T}\dot{\theta}, \quad \mathbf{T} = \frac{\partial \mathbf{x}}{\partial \theta}$$



$$\ddot{\mathbf{x}} = \dot{\mathbf{T}}\dot{\theta} + \mathbf{T}\ddot{\theta} = \frac{\mathbf{f} + \hat{\mathbf{f}}}{m}$$

Multiply both sides by T

$$\hat{\mathbf{f}} \cdot \mathbf{T} = 0$$

virtual work principle

$$\ddot{\theta} = \frac{\mathbf{T} \cdot \frac{\mathbf{f}}{m} - \mathbf{T} \cdot \dot{\mathbf{T}}\dot{\theta}}{\mathbf{T} \cdot \mathbf{T}}$$

Lagrangian Dynamics

- See Witkin & Baraff's note for generalization to a system of *n*-particles using parametric constraints

- This type of method is called Lagrangian Dynamics
  - Instead of working on unconstrained $\mathbf{q}$
  - We work on a constrained space $\mathbf{u}$ and solve for $\ddot{\mathbf{u}}$ through the parametric function $\mathbf{q}(\underset{\uparrow}{\mathbf{u}})$

Generalized coordinate

Parametric Constraints

- Advantages
  - Fewer degrees of freedom
  - Constraints are always met

- Disadvantages
  - Hard to find a parametric function that captures the desired constraints
  - Hard to combine constraints