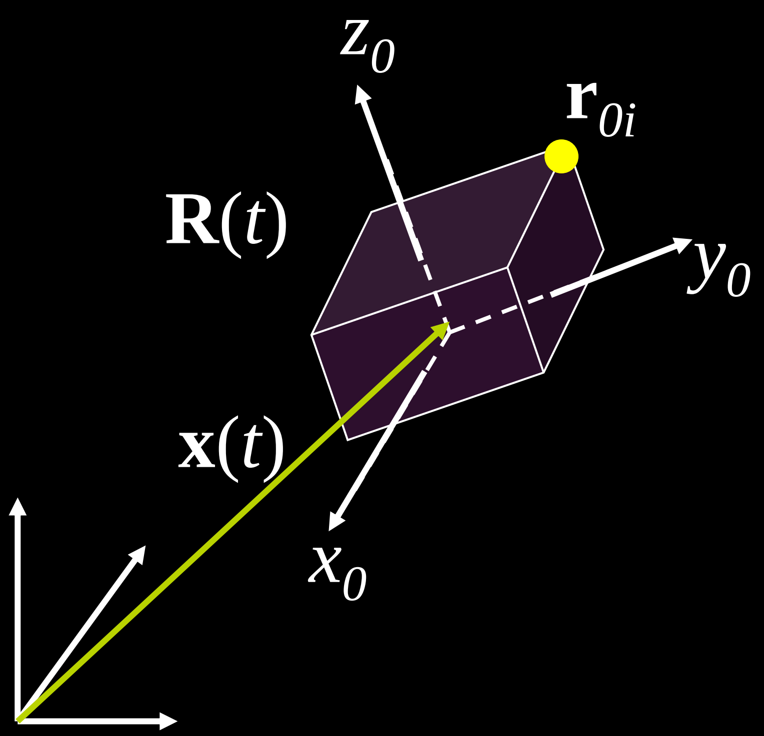
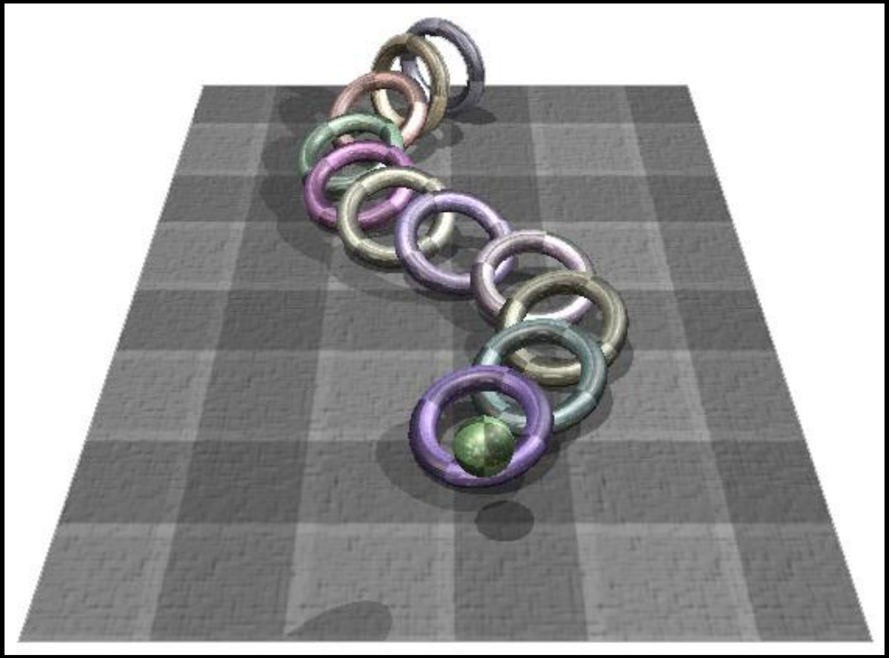
Rigid Body Dynamics



Rigid Body Simulation

- Once we consider an object with spatial extent, particle system simulation is no longer sufficient

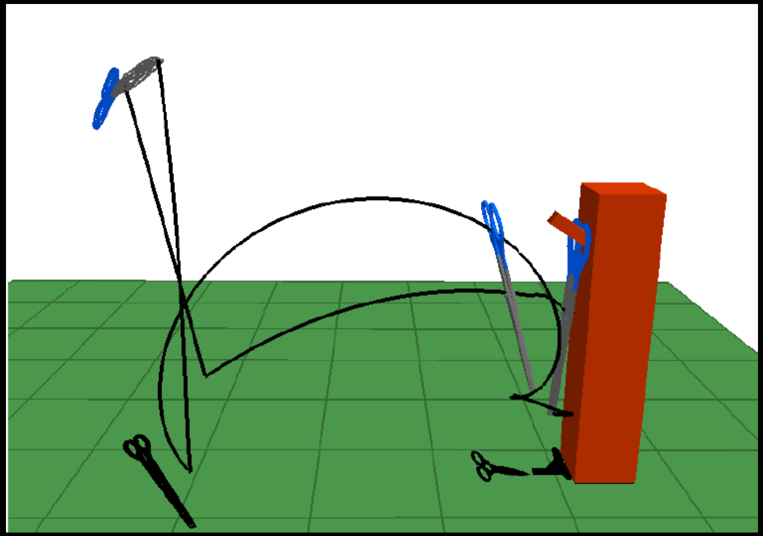


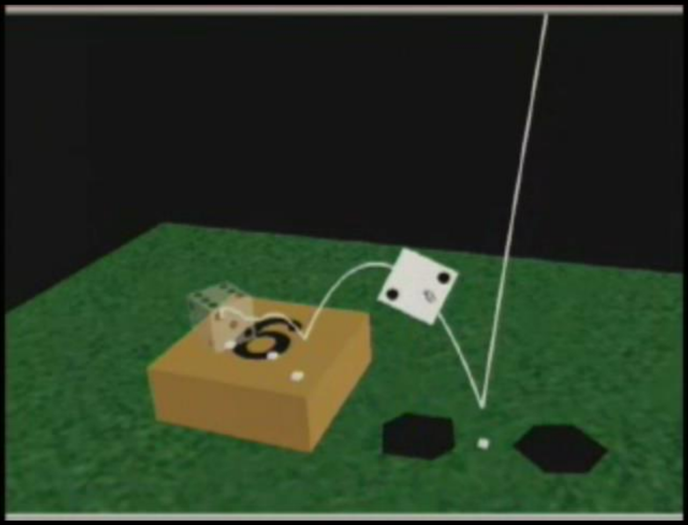
Problems

- Unconstrained system
  - No contact
- Constrained system
  - Collision detection and contact

Problems

- Computational efficiency is important!
- Controllable is desired!





Jovan Popović, Steven M. Seitz, Michael Erdmann, Zoran Popović, Andrew Witkin
"Interactive Manipulation of Rigid Body Simulations," SIGGRAPH 2000

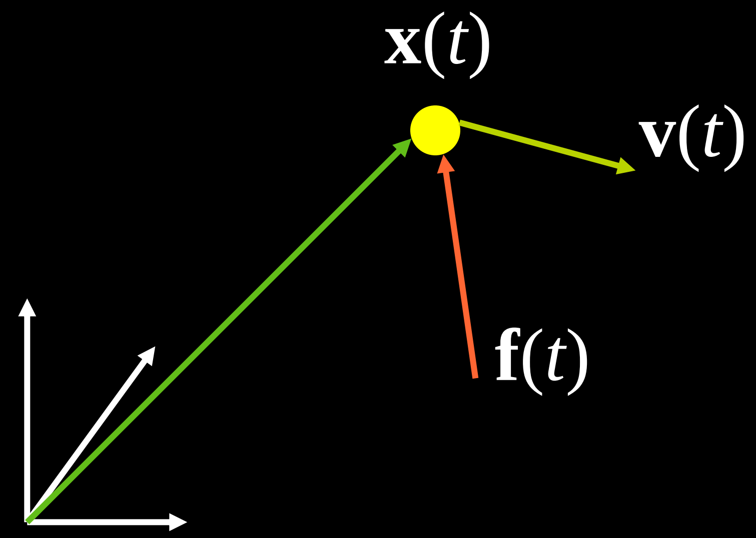
Rigid Body Concepts

Translation	Rotation
Position	Orientation
Linear velocity	Angular velocity
Mass	Inertia tensor
Linear momentum	Angular momentum
Force	Torque

Outline

- Position and orientation
- Linear and angular velocity
- Mass and inertia
- Force and torque

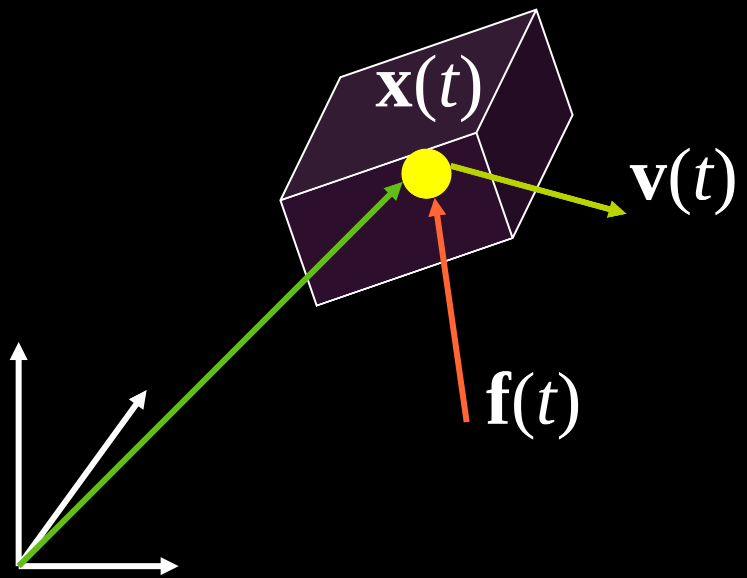
Particle State



$$\mathbf{Y}(t) = \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{pmatrix}$$

$$\frac{d}{dt}\mathbf{Y}(t) = \begin{pmatrix} \mathbf{v}(t) \\ \dfrac{\mathbf{f}(t)}{m} \end{pmatrix}$$

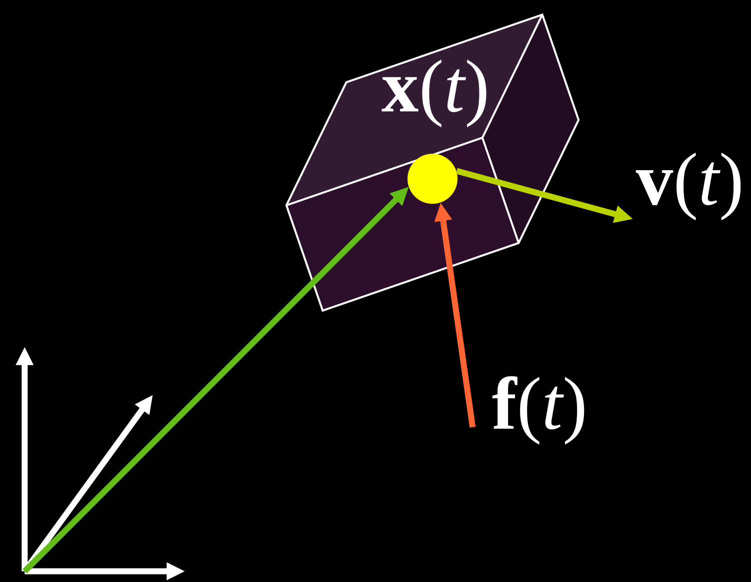
Rigid Body State



$$\mathbf{Y}(t) = \begin{pmatrix} \mathbf{x}(t) \\ ? \\ \mathbf{v}(t) \\ ? \end{pmatrix}$$

Position and Orientation

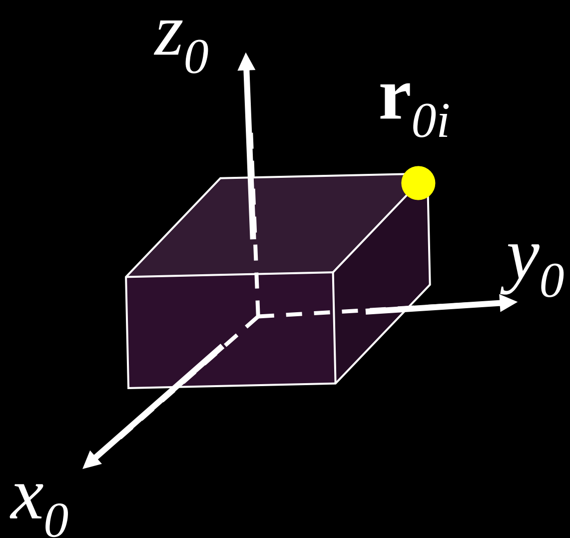
- Translation of the body
  - from the origin of the world coordinate
- Rotation of the body



$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

$$\mathbf{R}(t) = \begin{pmatrix} r_{xx}(t) & r_{yx}(t) & r_{zx}(t) \\ r_{xy}(t) & r_{yy}(t) & r_{zy}(t) \\ r_{xz}(t) & r_{yz}(t) & r_{zz}(t) \end{pmatrix}$$

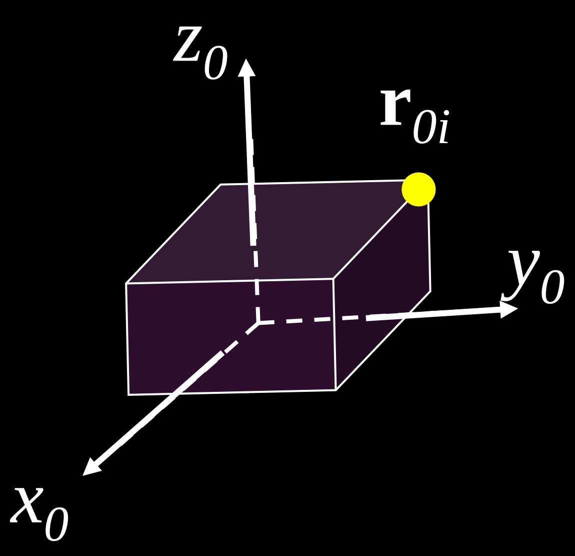
Body Space



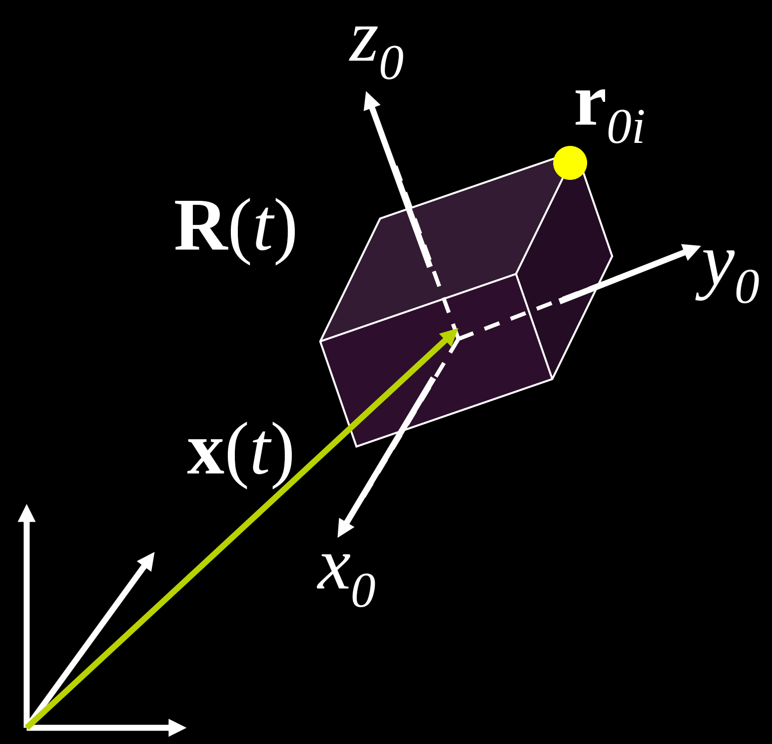
- A fixed and unchanged space where the shape of a rigid body is defined
- The geometric center of the rigid body lies at the origin of the body space
- Also called object space or local space

Position and Orientation

Body Space



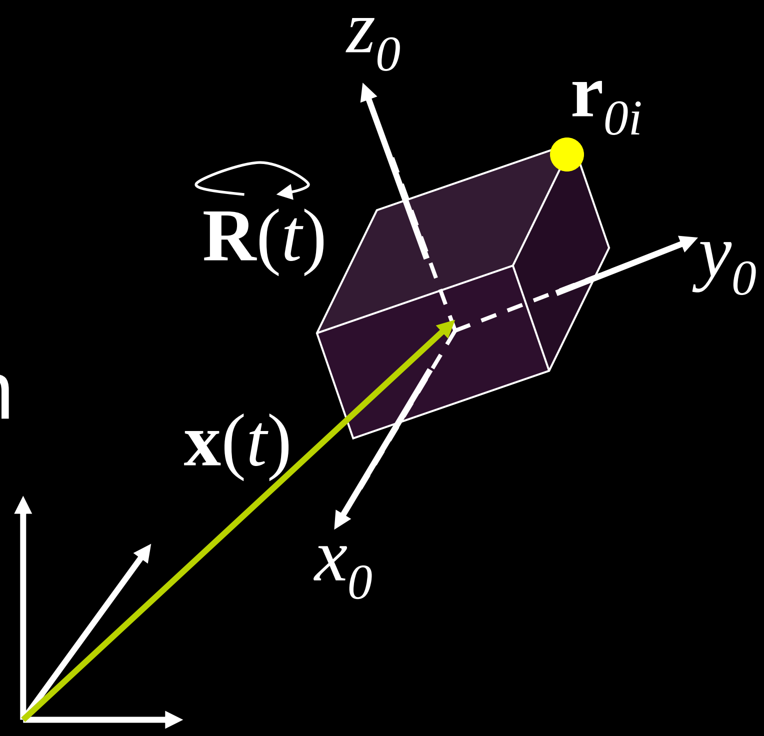
World Space



Position and Orientation

- Use $\mathbf{x}(t)$ and $\mathbf{R}(t)$ to transform the body space into world space
- The world coordinate of an arbitrary point $\mathbf{r}_{0\mathrm{i}}$ on the body

$$\mathbf{r}_{\mathbf{i}}(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_{0i}$$



Position and Orientation

- Assume the rigid body has uniform density, what is the physical meaning of $\mathbf{x}(t)$?
  - Center of mass over time

- What is the physical meaning of $\mathbf{R}(t)$?

Position and Orientation

- Consider the x-axis in body space, (1, 0, 0), what is the direction of this vector in world space at time t?

$$\mathbf{R}(t)\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}r_{xx}(t) & r_{yx}(t) & r_{zx}(t)\\ r_{xy}(t) & r_{yy}(t) & r_{zy}(t)\\ r_{xz}(t) & r_{yz}(t) & r_{zz}(t)\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}r_{xx}(t)\\ r_{xy}(t)\\ r_{xz}(t)\end{pmatrix}$$

first column of R

- $\mathbf{R}(t)$ represents the directions of x, y, z axes of the body space in world space at time t

Position and Orientation

- So $\mathbf{x}(t)$ and $\mathbf{R}(t)$ define the position and the orientation of the body at time t

- Next we needed to define how the position and orientation change over time

Linear Velocity and Angular Velocity

Linear Velocity

- Since x(t) is the position of the center of mass in world space, $\dot{\mathbf{x}}(t)$ is the velocity of the center of mass in world space

Angular Velocity

- If we fix the position of the COM in space
  - then any movement is due to the body spinning about some axis that passes through the COM
  - Otherwise, the COM would itself be moving

Angular Velocity

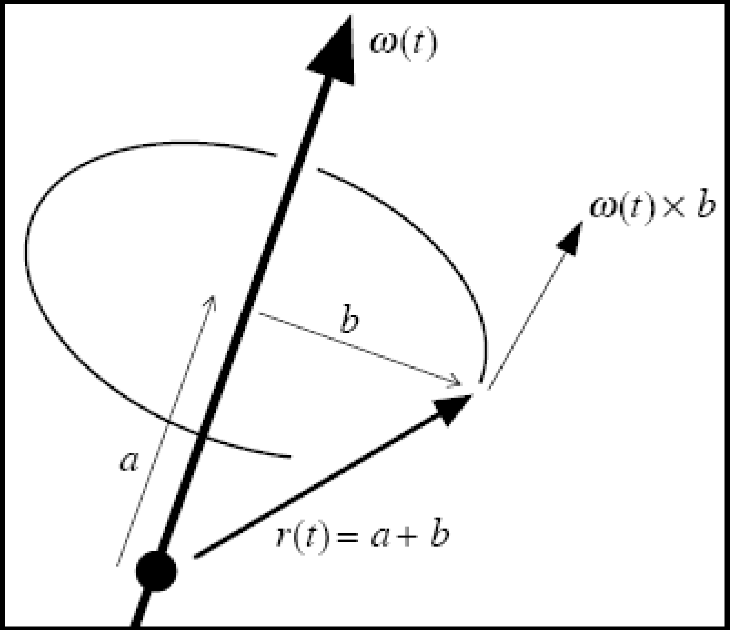
- We describe that spin as a vector $\boldsymbol{\omega}(t)$
- Linear position and velocity are related by

$$\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt}$$

- How are angular position (orientation) and velocity related?

Angular Velocity

- How are $\mathbf{R}(t)$ and $\boldsymbol{\omega}(t)$ related?
- Consider a vector r(t) at time t specified in world space, how do we represent $\dot{\mathbf{r}}(t)$ in terms of $\boldsymbol{\omega}(t)$?



$$\begin{cases} \dot{\mathbf{r}}(t) \perp \mathbf{b} \\ \dot{\mathbf{r}}(t) \perp \boldsymbol{\omega}(t) \\ |\dot{\mathbf{r}}(t)| = |\mathbf{b}||\boldsymbol{\omega}(t)| \end{cases} \Rightarrow \begin{aligned} \dot{\mathbf{r}}(t) &= \boldsymbol{\omega}(t) \times \mathbf{b} = \boldsymbol{\omega}(t) \times \mathbf{b} + \boldsymbol{\omega}(t) \times \mathbf{a} \\ &= \boldsymbol{\omega}(t) \times \mathbf{r}(t) \end{aligned}$$

Angular Velocity

- Given the physical meaning of R(t), what does each column change over time?

$$\dot{\mathbf{R}}(t) = \begin{pmatrix} \boldsymbol{\omega}(t) \times \begin{pmatrix} r_{xx}(t) \\ r_{xy}(t) \\ r_{xz}(t) \end{pmatrix} & \boldsymbol{\omega}(t) \times \begin{pmatrix} r_{yx}(t) \\ r_{yy}(t) \\ r_{yz}(t) \end{pmatrix} & \boldsymbol{\omega}(t) \times \begin{pmatrix} r_{zx}(t) \\ r_{zy}(t) \\ r_{zz}(t) \end{pmatrix} \end{pmatrix}$$

- This expression is too cumbersome, we can use a trick to simplify it

Angular Velocity

- Consider two 3 by 1 vectors: a and b, the cross product of them

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{pmatrix}$$

- Given a, we can define a matrix a* such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{a}^* \mathbf{b} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

Angular Velocity

$$\dot{\mathbf{R}}(t) = \begin{pmatrix} \boldsymbol{\omega}(t) \times \begin{pmatrix} r_{xx}(t) \\ r_{xy}(t) \\ r_{xz}(t) \end{pmatrix} & \boldsymbol{\omega}(t) \times \begin{pmatrix} r_{yx}(t) \\ r_{yy}(t) \\ r_{yz}(t) \end{pmatrix} & \boldsymbol{\omega}(t) \times \begin{pmatrix} r_{zx}(t) \\ r_{zy}(t) \\ r_{zz}(t) \end{pmatrix} \end{pmatrix}$$

$$= \boldsymbol{\omega}(t)^{*} \mathbf{R}(t)$$

$$= \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} r_{xx}(t) & r_{yx}(t) & r_{zx}(t) \\ r_{xy}(t) & r_{yy}(t) & r_{zy}(t) \\ r_{xz}(t) & r_{yz}(t) & r_{zz}(t) \end{pmatrix}$$

Perspective of Particles

- Imagine a rigid body is composed of a large number of small particles
  - the particles are indexed from 1 to N
  - each particle has a constant location $\mathbf{r}_{0\mathrm{i}}$ in body space
  - the location of i-th particle in world space at time t is

$$\mathbf{r}_{\mathbf{i}}(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_{0i}$$

Velocity of a particle

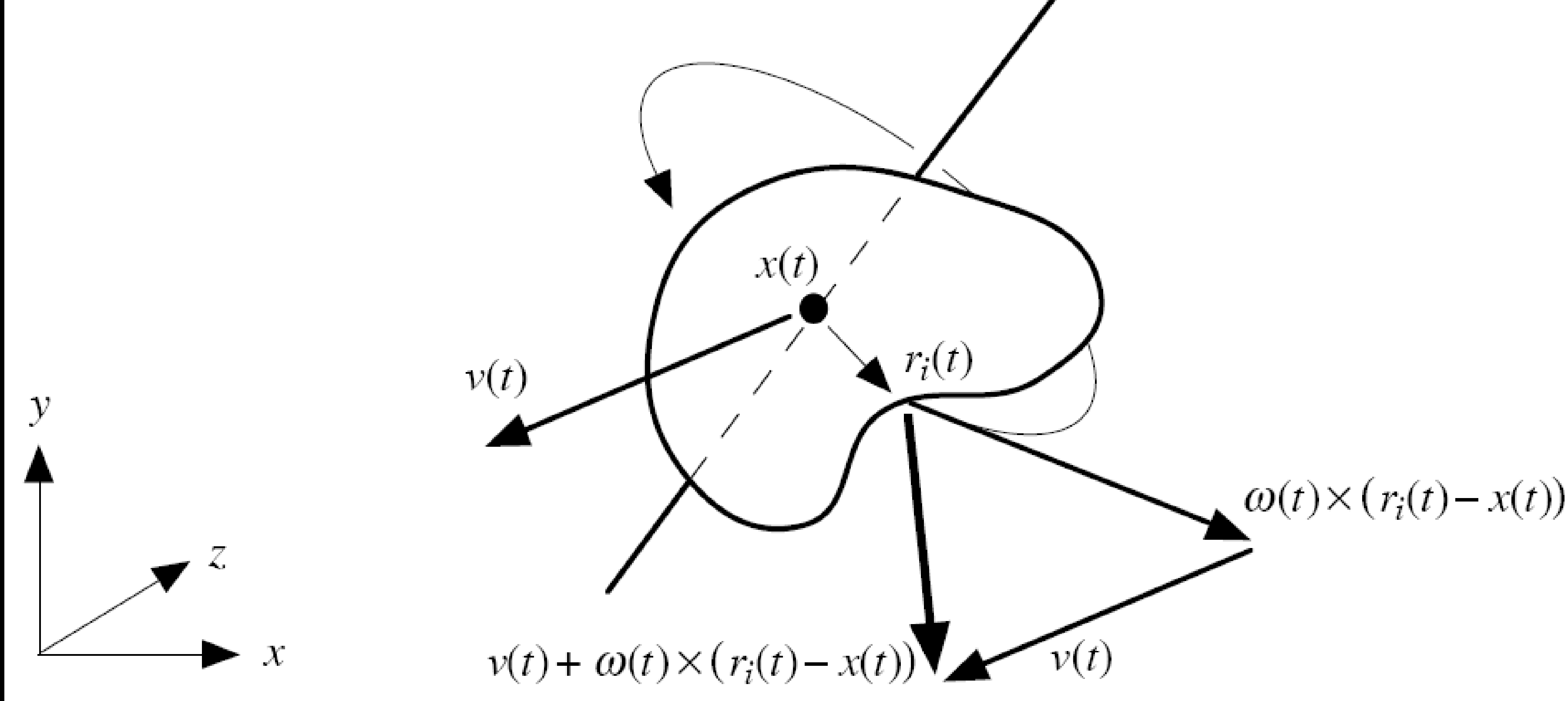
$$\frac{d}{dt}\mathbf{r_i}(t) = \frac{d}{dt}\mathbf{x}(t) + \frac{d}{dt}\mathbf{R}(t)\mathbf{r}_{0i}$$

$$\begin{aligned}
\dot{\mathbf{r}}_\mathbf{i}(t) &= \dot{\mathbf{x}}(t) + \boldsymbol{\omega}(t) * \mathbf{R}(t)\mathbf{r}_{0i} \\
&= \mathbf{v}(t) + \boldsymbol{\omega}(t) * (\mathbf{R}(t)\mathbf{r}_{0i} + \mathbf{x}(t) - \mathbf{x}(t)) \\
&= \underset{\uparrow}{\mathbf{v}(t)} + \underbrace{\boldsymbol{\omega}(t) * (\mathbf{r}_i(t) - \mathbf{x}(t))}
\end{aligned}$$

linear component angular component

Velocity of a particle

$$\dot{\mathbf{r}}_{\mathbf{i}}(t) = \mathbf{v}(t) + \boldsymbol{\omega}(t) * (\mathbf{r}_i(t) - \mathbf{x}(t))$$



- Position and orientation
- Linear and angular velocity
- Mass and inertia
- Force and torque

Mass

- The mass of the i-th particle is m_i, the mass is

$$M = \sum_{i=1}^{N} m_i$$

- The center of mass defined in world space is

$$COM = \frac{\sum_{i=1}^{N} m_i r_i(t)}{M}$$

- What about the COM in body space?

Inertia Tensor

- Inertia tensor describes how the mass of a rigid body is distributed relative to the center of mass

$$\mathbf{I}(t)=\sum_{i=1}^{N}\begin{pmatrix} m_i({r'}_{iy}^2+{r'}_{iz}^2) & -m_i r'_{ix} r'_{iy} & -m_i r'_{ix} r'_{iz} \\ -m_i r'_{iy} r'_{ix} & m_i({r'}_{ix}^2+{r'}_{iz}^2) & -m_i r'_{iy} r'_{iz} \\ -m_i r'_{iz} r'_{ix} & -m_i r'_{iz} r'_{iy} & m_i({r'}_{ix}^2+{r'}_{iy}^2) \end{pmatrix}$$

$$\mathbf{r}'_i=\mathbf{r}_i(t)-\mathbf{x}(t)$$

- I(t) depends on the orientation of a body, but not the translation

- For an actual implementation, we replace the finite sum with the integrals over a body's volume in world space

Inertia Tensor

- Inertia tensors vary in world space over time
- But are constant in body space
- Precompute the integral part in body space to save time

Derivation

$$I(t) = \sum_{i=1}^{N} m_i (r'^T_i \, r'_i \, \mathbf{I} - r'_i \, r'^T_i)$$

$$\mathbf{I}(t) = \sum_{i=1}^{N} \begin{pmatrix} m_i(r'^2_{iy} + r'^2_{iz}) & -m_i r'_{ix} r'_{iy} & -m_i r'_{ix} r'_{iz} \\ -m_i r'_{iy} r'_{ix} & m_i(r'^2_{ix} + r'^2_{iz}) & -m_i r'_{iy} r'_{iz} \\ -m_i r'_{iz} r'_{ix} & -m_i r'_{iz} r'_{iy} & m_i(r'^2_{ix} + r'^2_{iy}) \end{pmatrix}$$

$$= \sum_{i=1}^{N} m_i r'^T_i \, r'_i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - m_i \begin{pmatrix} r'^2_{ix} & r'_{ix} r'_{iy} & r'_{ix} r'_{iz} \\ r'_{iy} r'_{ix} & r'^2_{iy} & r'_{iy} r'_{iz} \\ r'_{iz} r'_{ix} & r'_{iz} r'_{iy} & r'^2_{iz} \end{pmatrix}$$

$$= \sum_{i=1}^{N} m_i (r'^T_i \, r'_i \mathbf{1} - r'_i \, r'^T_i)$$

Inertia Tensor in Body Space

- Use the facts that $\mathbf{r'_i}(t) = \mathbf{r_i}(t) - \mathbf{x}(t) = \mathbf{R}(t)\mathbf{r}_{0i}$

  $$\mathbf{R}(t)\mathbf{R}(t)^T = \mathbf{1}$$

- We can obtain

$$\mathbf{I}(t) = \sum_{i=1}^{N} m_i (r'^T_i \, r'_i \mathbf{1} - r'_i \, r'^T_i)$$

$$= \sum^{N} m_i \left[r_{0i}^T r_{0i} \mathbf{R}(t)\mathbf{R}(t)^T - \mathbf{R}(t) r_{0i} r_{0i}^T \mathbf{R}(t)^T \right]$$

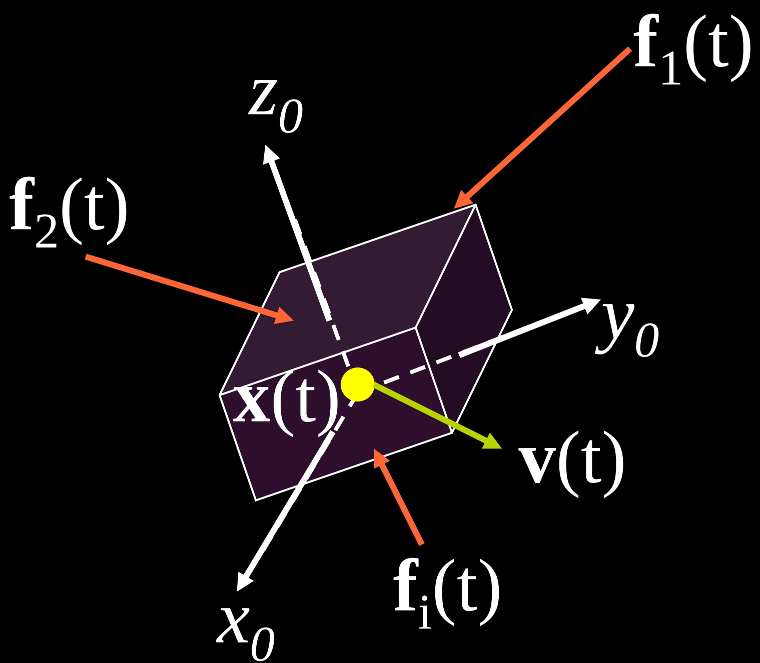
$$\mathbf{R}(t) \overbrace{r_{0i}^T r_{0i}}^{\text{scalar}} \mathbf{R}(t)^T$$

$$= \mathbf{R}(t) \sum_{i=1}^{N} m_i (r_{0i}^T r_{0i} \mathbf{1} - r_{0i} r_{0i}^T) \mathbf{R}(t)^T = \mathbf{R}(t) \mathbf{I}_{body} \mathbf{R}(t)^T$$

For details, see page G14 of Baraff and Witkin's course notes

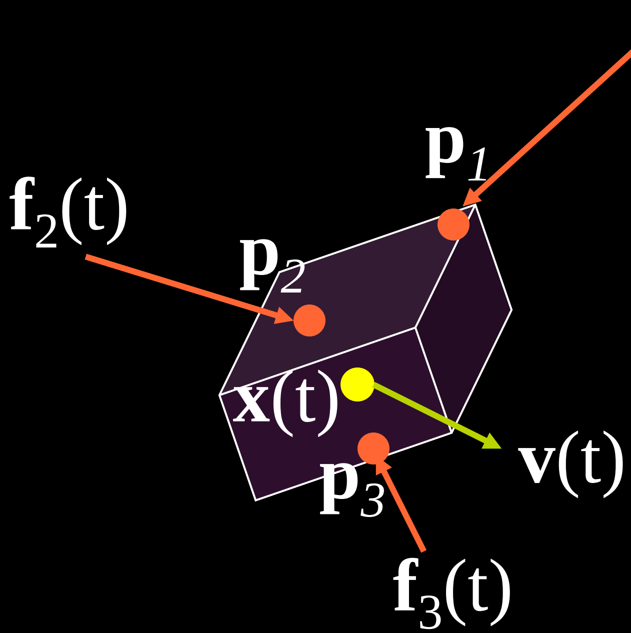
- Position and orientation
- Linear and angular velocity
- Mass and inertia
- Force and torque

Net Force



$$\mathbf{F}(t) = \sum \mathbf{f_i}(t)$$

Net Torque



$$\boldsymbol{\tau}(t) = \sum (\mathbf{p}_\mathbf{i}(t) - \mathbf{x}(t)) \times \mathbf{f}_\mathbf{i}(t)$$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{Y}(t) = \frac{d}{dt}\begin{pmatrix} \mathbf{x}(t) \\ \mathbf{R}(t) \\ M\mathbf{v}(t) \\ \mathbf{I}(t)\boldsymbol{\omega}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ \boldsymbol{\omega}(t) * \mathbf{R}(t) \\ \mathbf{F}(t) \\ \boldsymbol{\tau}(t) \end{pmatrix}$$

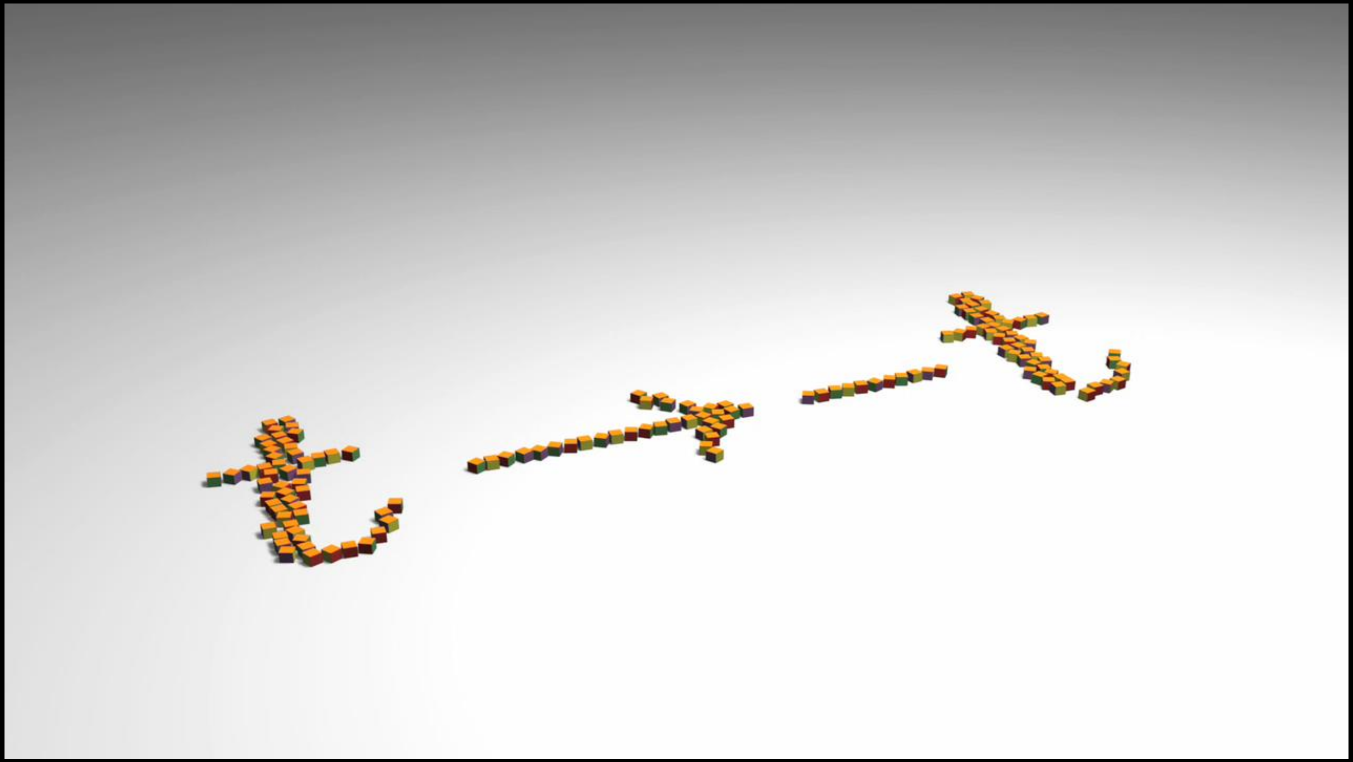
$M\mathbf{v}(t)$: Linear momentum

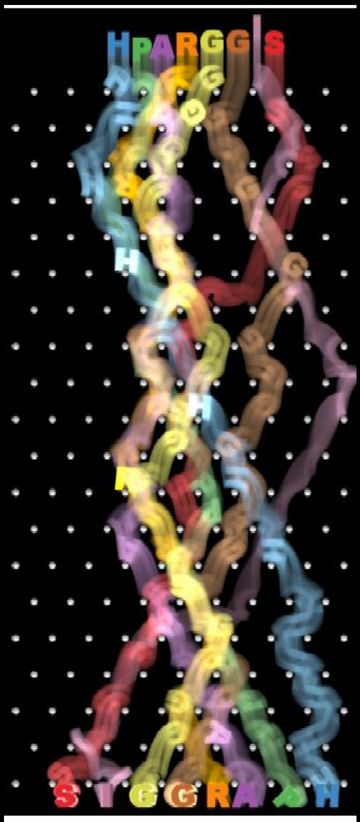
$\mathbf{I}(t)\ \boldsymbol{\omega}(t)$: Angular momentum

Some Implementation Issues

- Use quaternion to represent orientation
  - See the appendix of the course notes
- Use Green's Theorem to compute inertia tensor
  - Paper
    Brian Mirtich, "Fast and Accurate Computation of Polyhedral Mass Properties," Journal of Graphics Tools, 1996
  - C code
    http://www.cs.berkeley.edu/~jfc/mirtich/massProps.html

Control of Multibody Dynamics



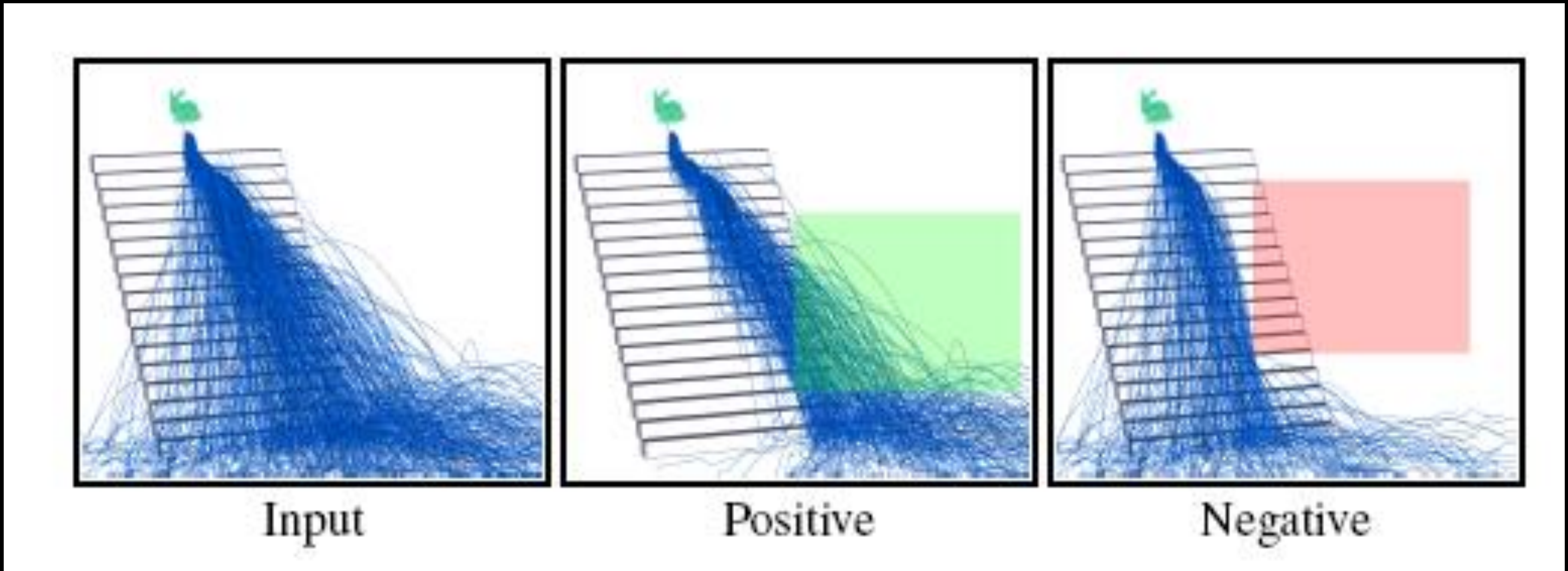


Christopher Twigg & Doug James

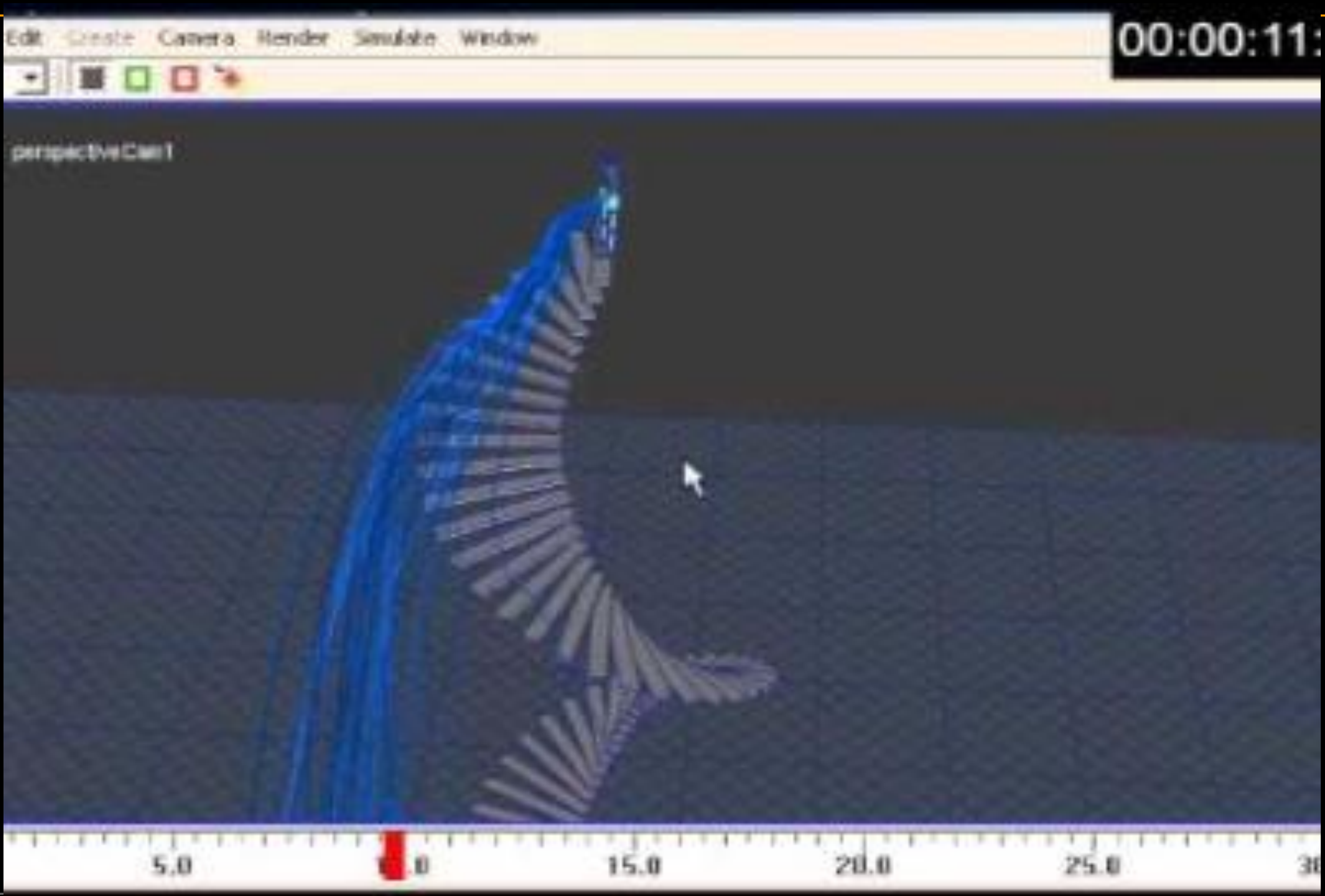
"Backwards Steps in Rigid Body Simulation," SiGGRAPH'08

"Many-Worlds Browsing for Control of Multibody Dynamics," SIGGRAPH'06

Many-Worlds Browsing



Many-Worlds Browsing



Recent Development: Rigid-IPC

Rigid-IPC (SIGGRAPH Talk)

Intersection-free Rigid Body Dynamics

Zachary Ferguson[1]
Teseo Schneider[1,4]
Timothy Langlois[5]
Denis Zorin[1]
Daniele Panozzo[1]
Minchen Li[2,3]
Francisca Gil-Ureta[1]
Chenfanfu Jiang[2,3]
Danny M. Kaufman[5]

NYU 1
UCLA 2
Penn 3
University of Victoria 4
Adobe 5

ABD: Speed-up Rigid-IPC

Affine Body Dynamics:

Fast, Stable, and Intersection-free Simulation of Stiff Materials

Lei Lan[1,2], Danny M. Kaufman[3], Minchen Li[4,5], Chenfanfu Jiang[4,5], Yin Yang[1,2,5]

1. 2. THE UNIVERSITY OF UTAH 3. Adobe 4. UCLA 5. TimeStep Inc.

Further Reference

- SIGGRAPH’19 course: Introduction to Physics-based Animation (basics, rigid & soft body, and fluids)
- CGF’14 STAR: Interactive Simulation of Rigid Body Dynamics in Computer Graphics
- SIGGRAPH’08 course: Real-time Physics
  - rigid and deformable solids, smoke and fluid simulation
- Matthias Muller’s Ten Minute Physics
- SIGGRAPH’22 Course: Contact and Friction Simulation for Computer Graphics

Multi-Physics Engines

- Houdini
- Chrono, free & open source
- MuJoCo, free & open source
- Bullet and PyBullet, free & open source