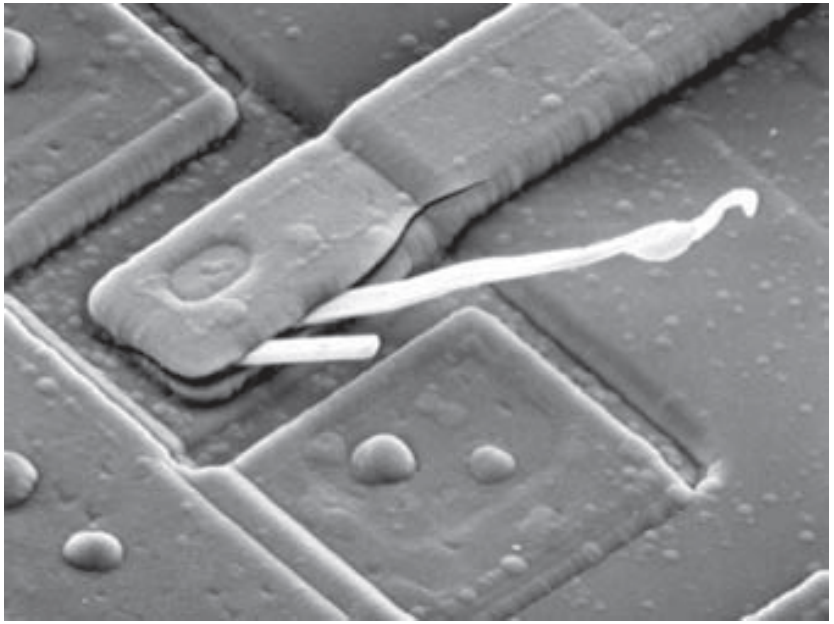
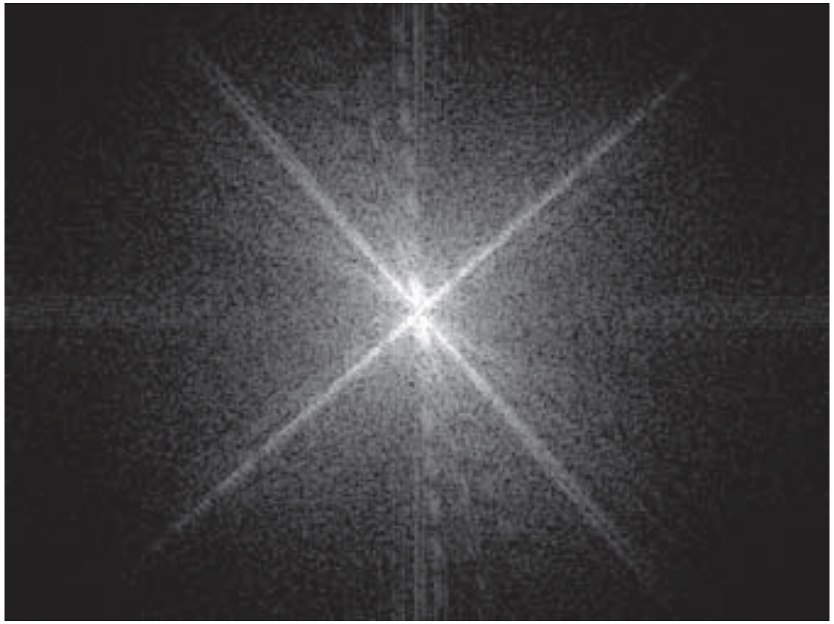
4.7 The Basics of Filtering in the Frequency Domain

- Additional Characteristics of the Frequency Domain

  The spectrum provides some useful guidelines as to the gross intensity characteristics of the image



SEM image of a damaged integrated circuit



Fourier spectrum of it

4.7 The Basics of Filtering in the Frequency Domain

- Frequency Domain Filtering Fundamentals

filtered (output) image

filter

DFT of the input image

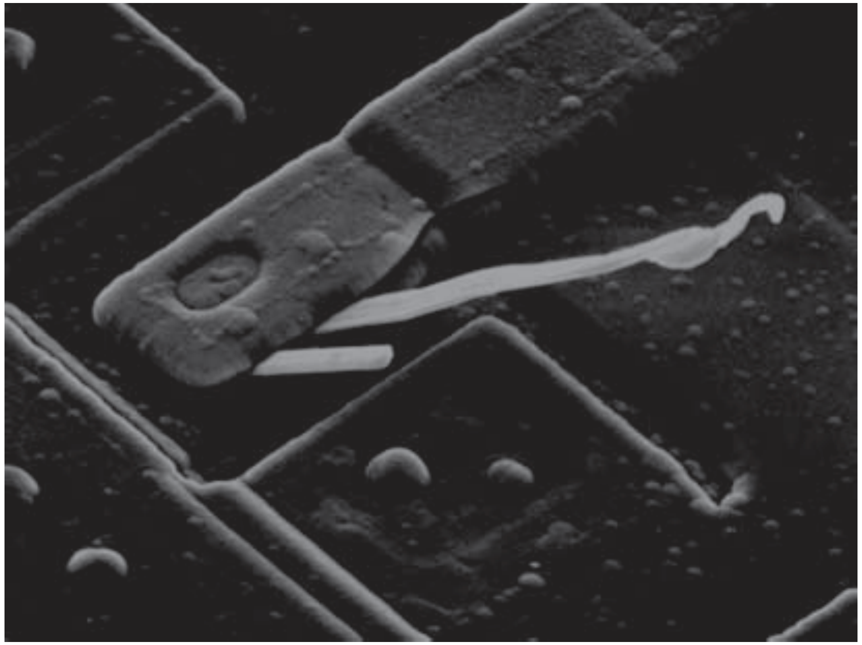
$$g(x,y) = \text{Real}\left\{\Im^{-1}\left[H(u,v)F(u,v)\right]\right\}$$

elementwise multiplication

4.7 The Basics of Filtering in the Frequency Domain

- Frequency Domain Filtering Fundamentals

$$g(x, y) = \text{Real}\left\{\Im^{-1}\left[H(u, v)F(u, v)\right]\right\}$$



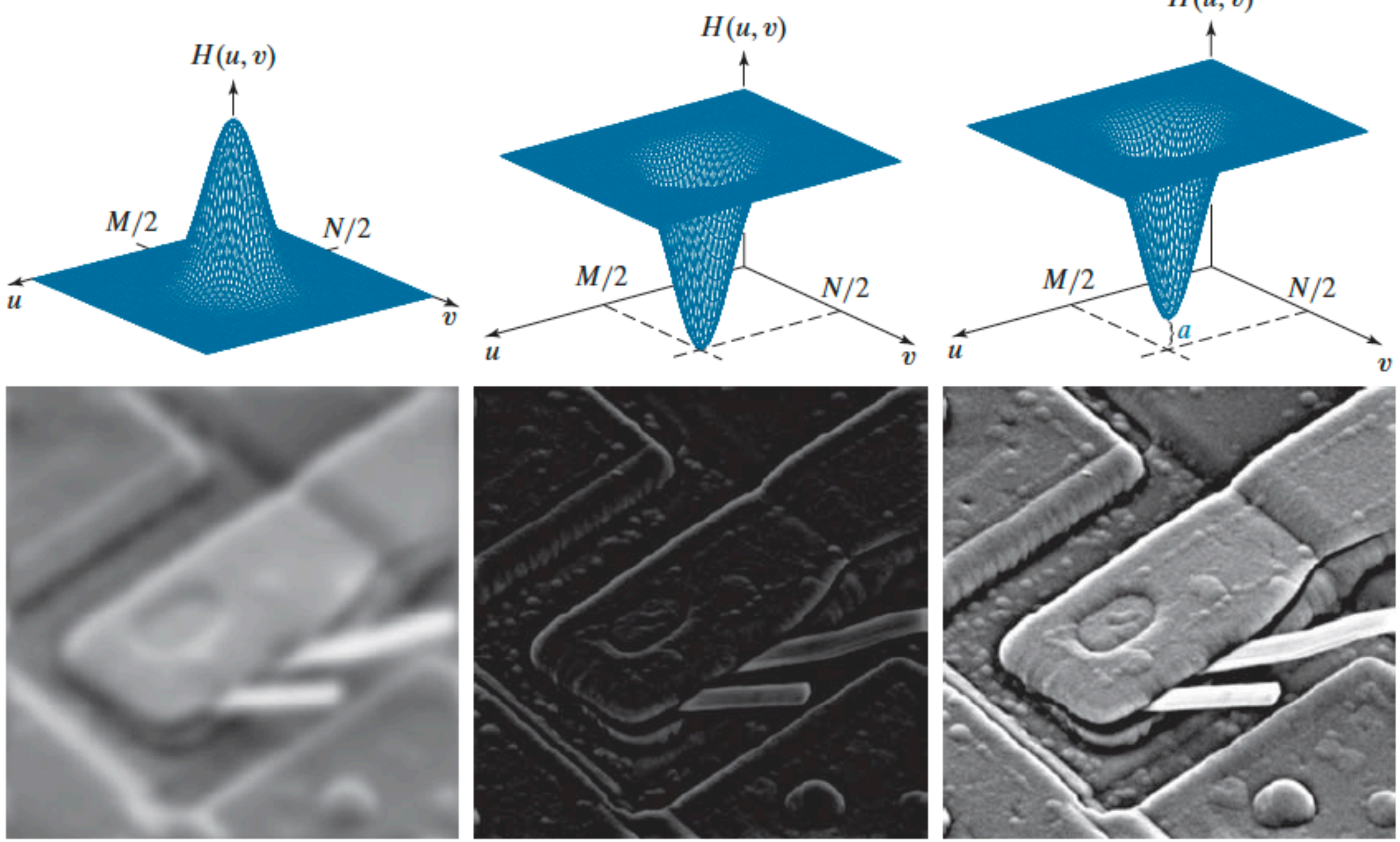
Example:
H(*u*, *v*) is 0 at the center of the (centered) transform, and 1’s elsewhere

4.7 The Basics of Filtering in the Frequency Domain

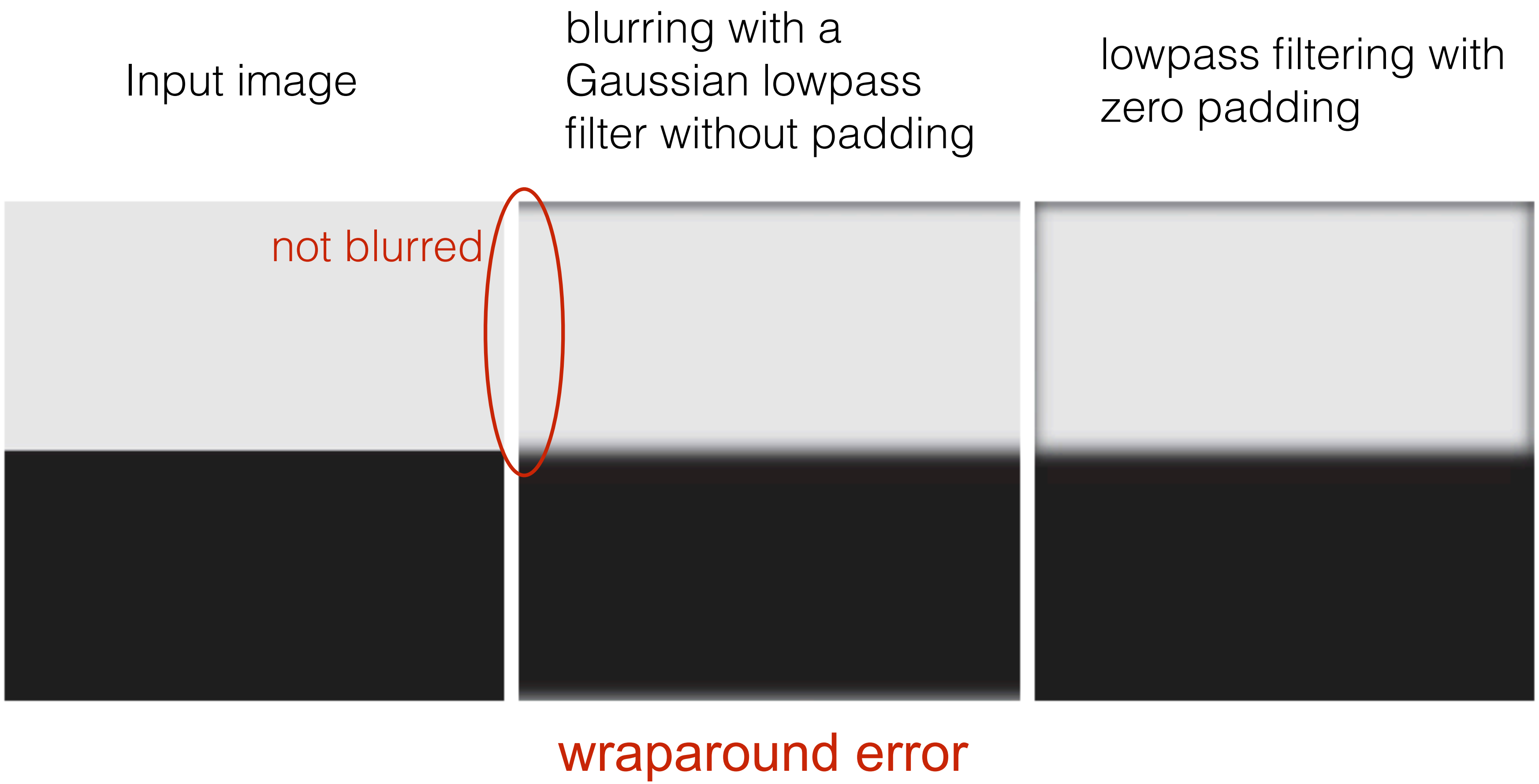
lowpass filter

highpass filter

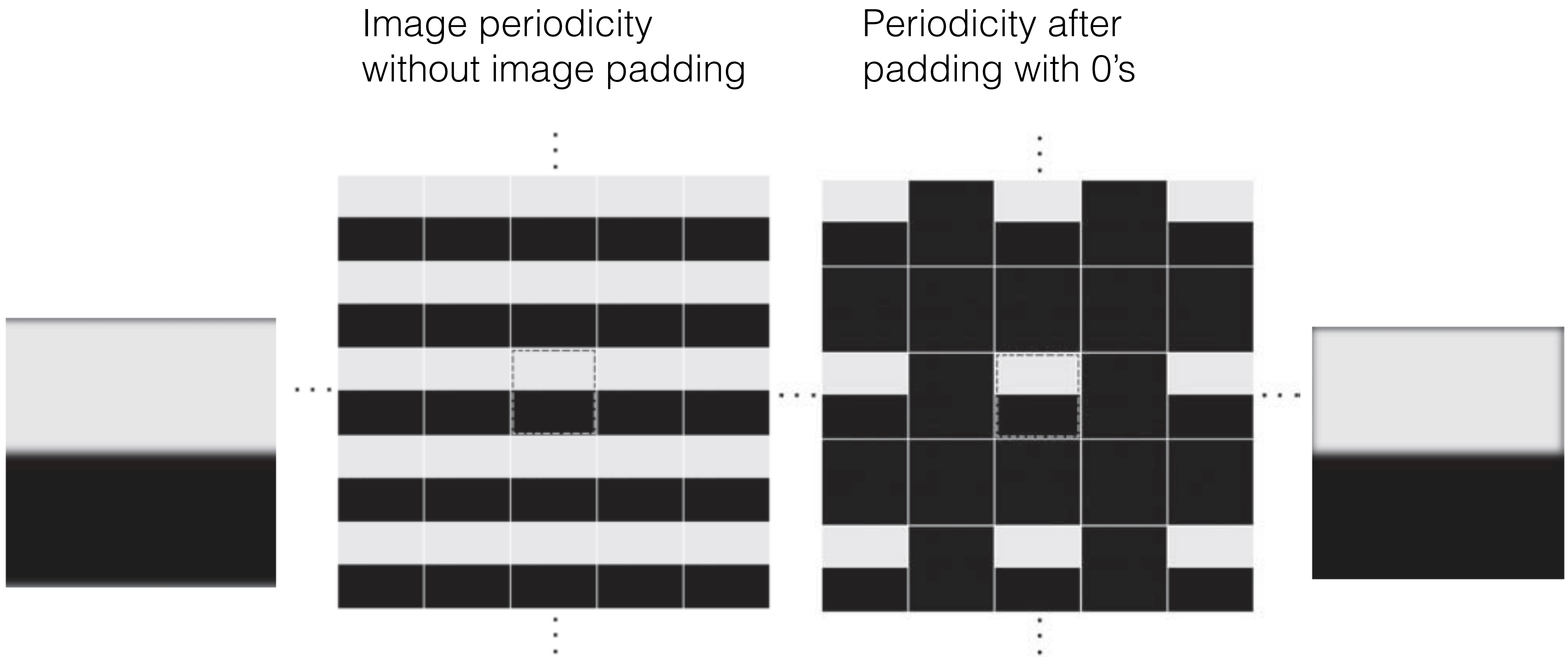
offset highpass filter
to avoid zero *F(0,0)*



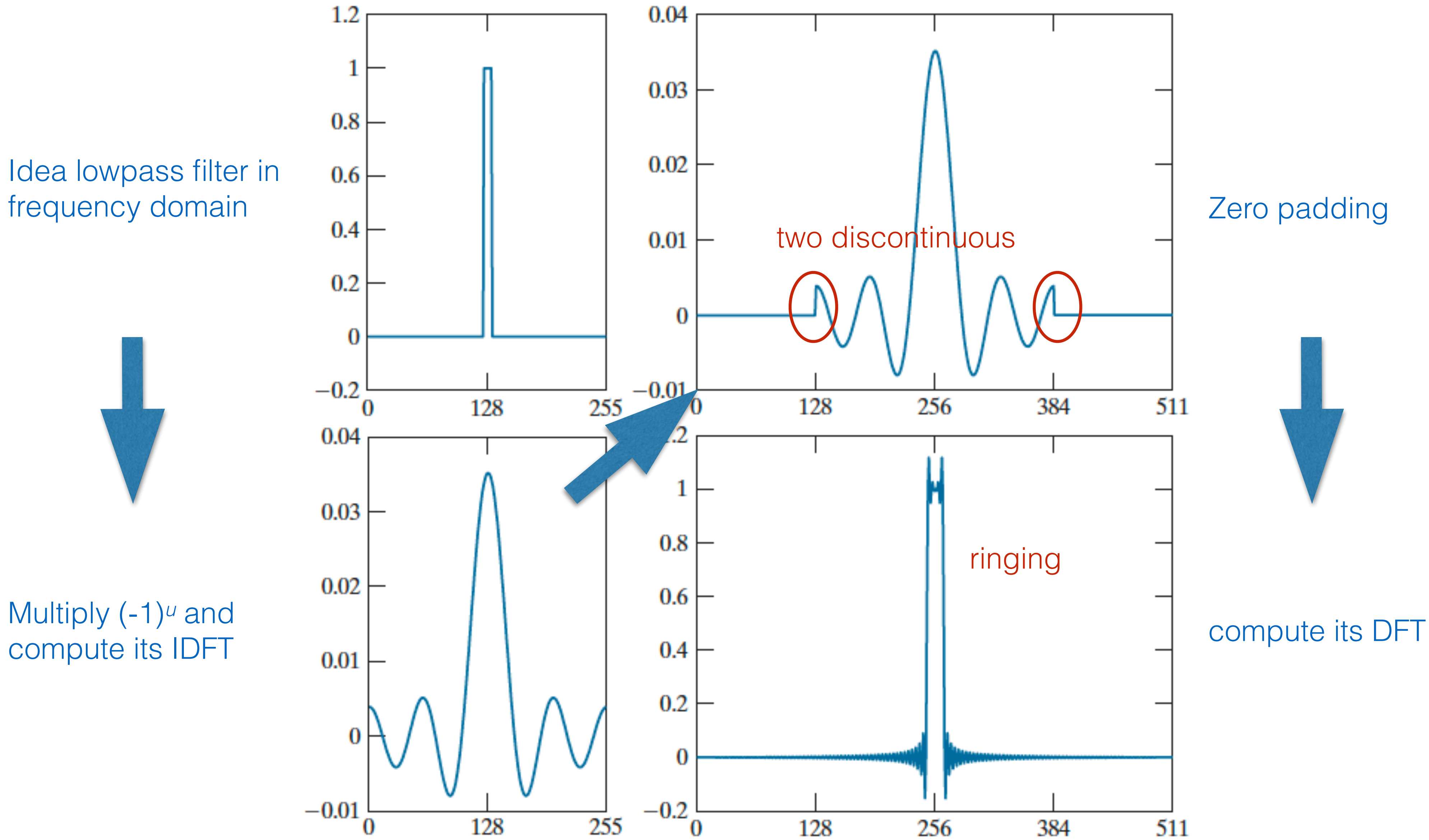
4.7 The Basics of Filtering in the Frequency Domain



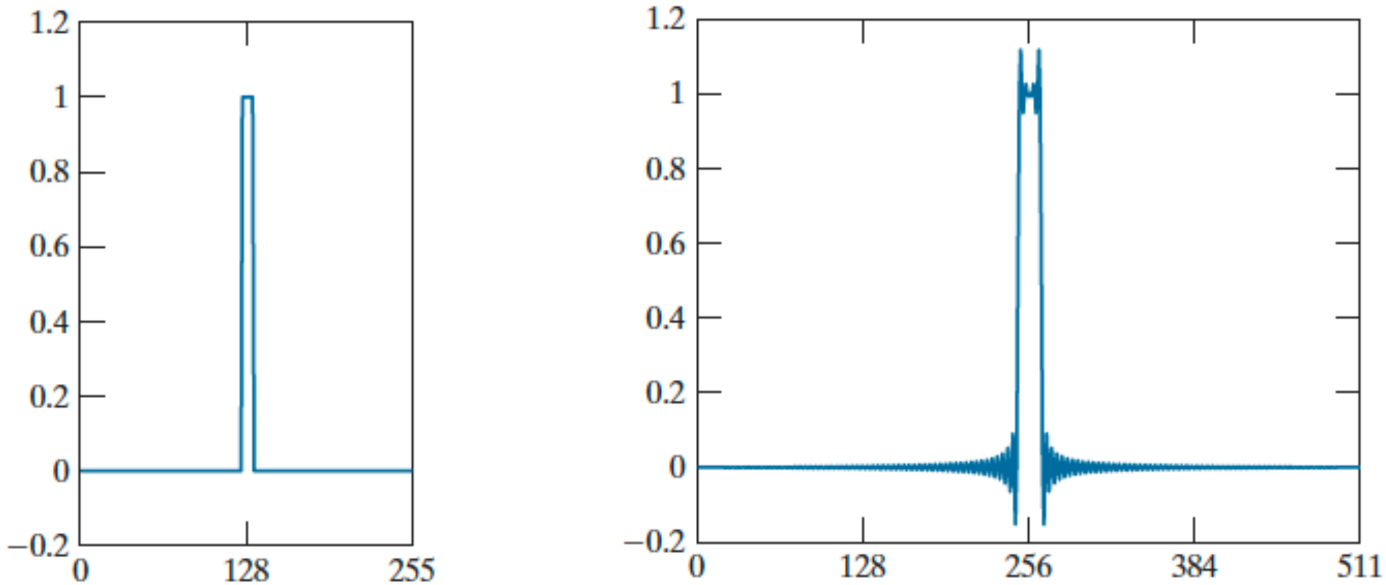
4.7 The Basics of Filtering in the Frequency Domain



4.7 The Basics of Filtering in the Frequency Domain



4.7 The Basics of Filtering in the Frequency Domain



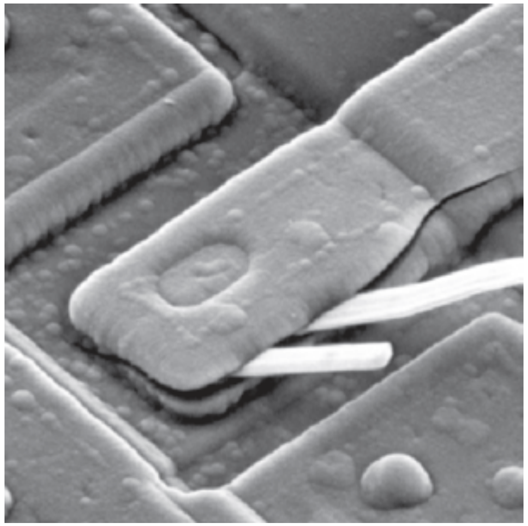
- We cannot pad the spatial representation of a frequency domain transfer function in order to avoid wraparound error
- An alternative is to pad images and then create the desired filter transfer function directly in the frequency domain

4.7 The Basics of Filtering in the Frequency Domain

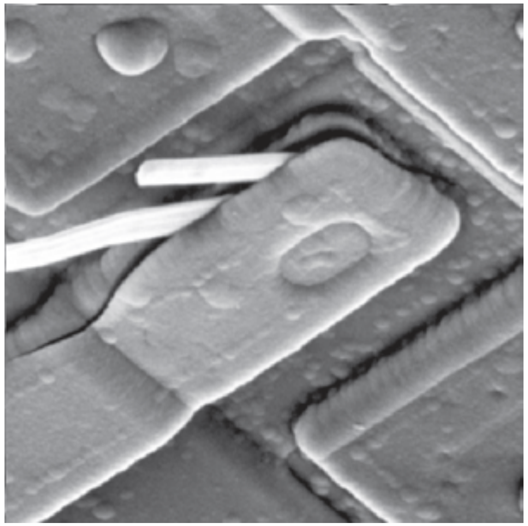
- **zero-phase-shift filters**
  - Filters that affect the real and imaginary parts equally, and thus have no effect on the phase angle, i.e. $H(u, v)$ is real

$$F(u,v) = R(u,v) + jI(u,v)$$
$$g(x,y) = \Im^{-1}\left[H(u,v)R(u,v) + jH(u,v)I(u,v)\right]$$

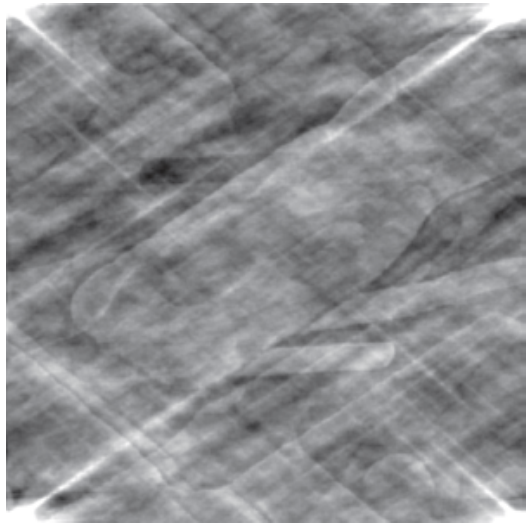
$$\boxed{\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]}$$



Original image



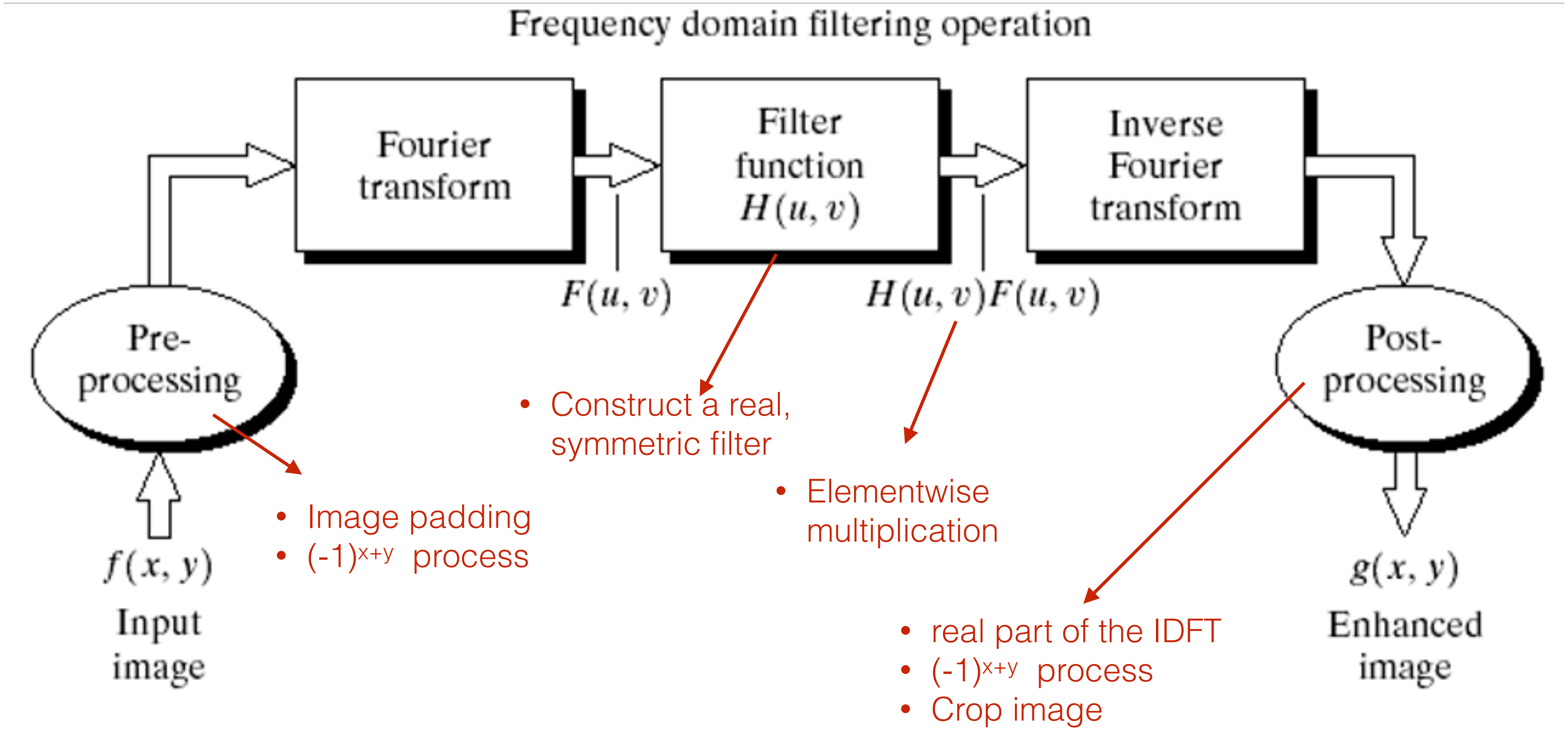
multiplying the phase angle array by −1



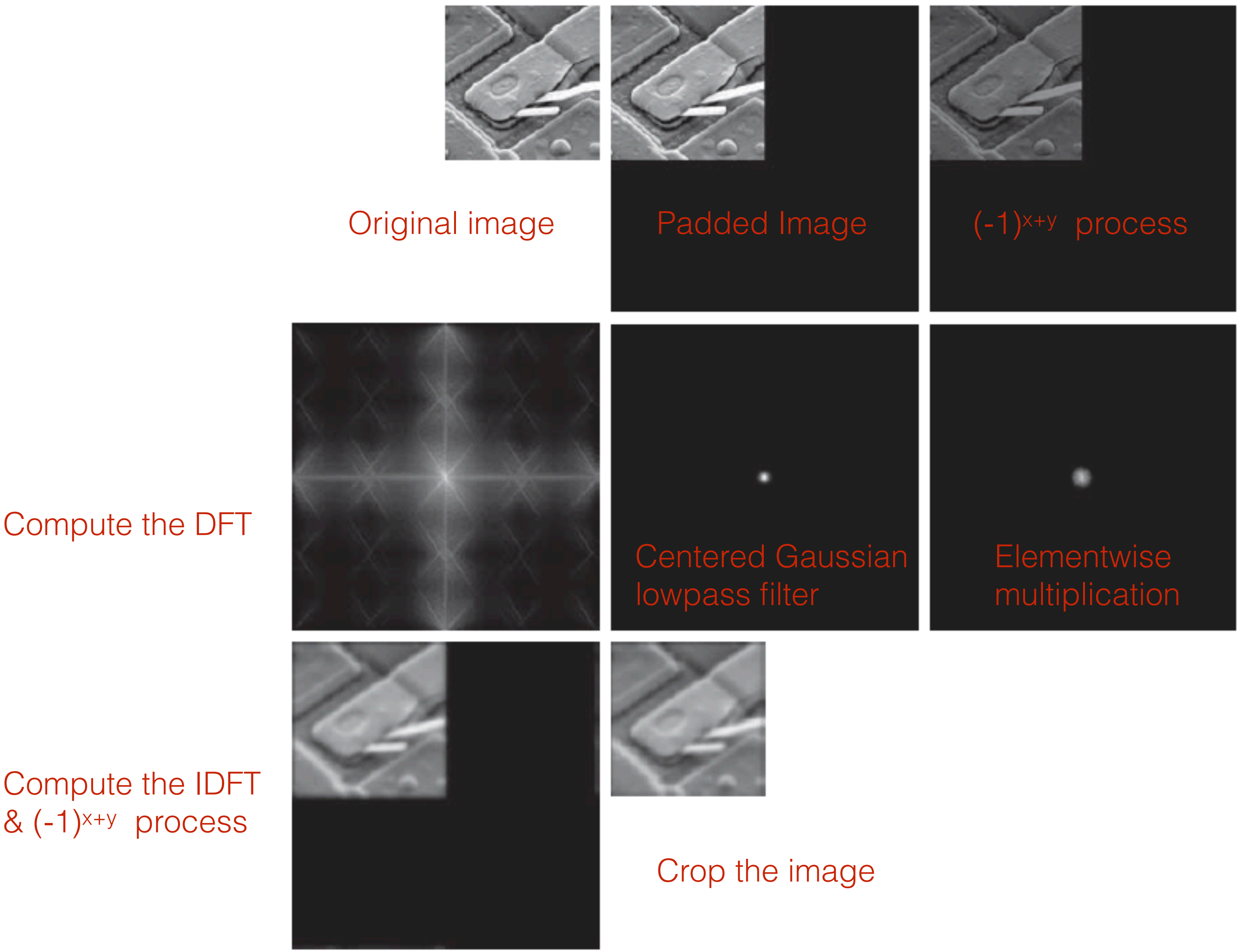
multiplying the phase angle array by 0.25

4.7 The Basics of Filtering in the Frequency Domain

- Summary of Steps for Filtering in the Frequency Domain



4.7 The Basics of Filtering in the Frequency Domain

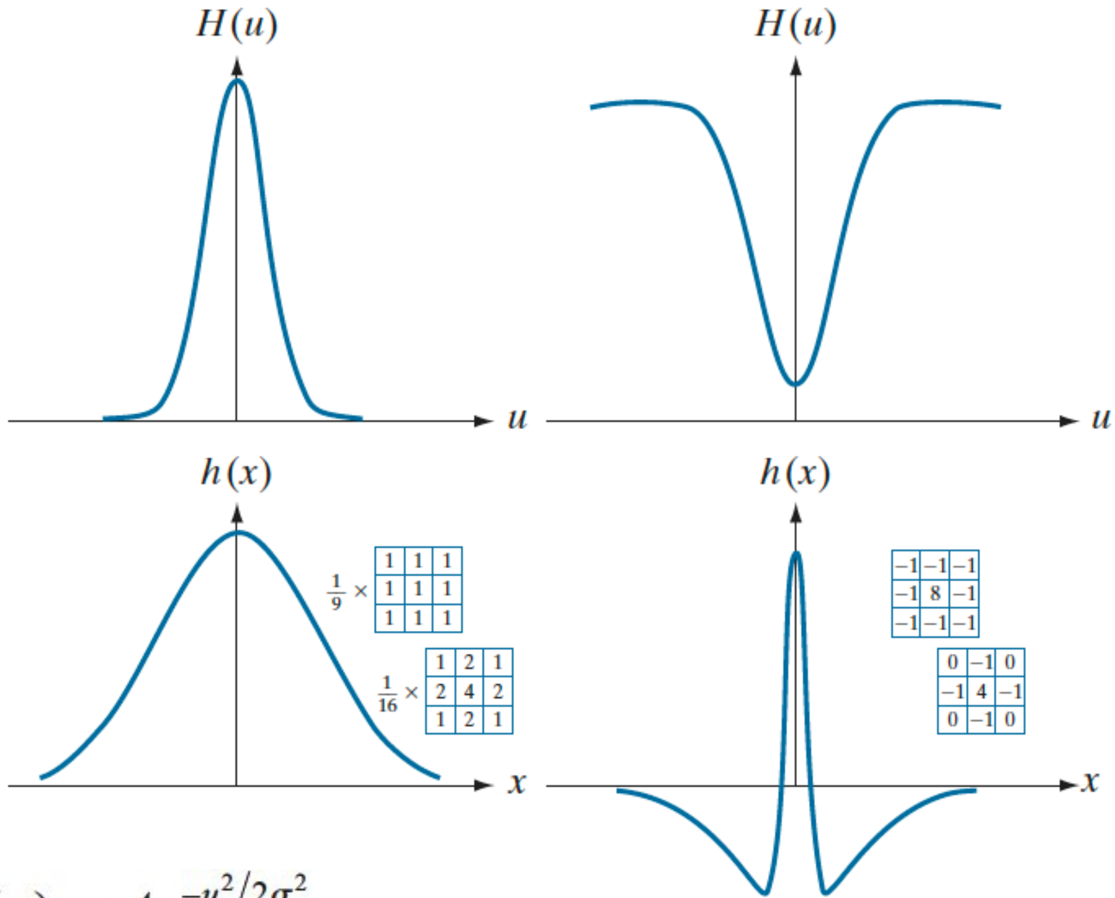


4.7 The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

Frequency domain

Spatial domain



Highness filter with Difference of Gaussian

Gaussian filter

$$H(u) = Ae^{-u^2/2\sigma^2}$$

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2x^2}$$

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

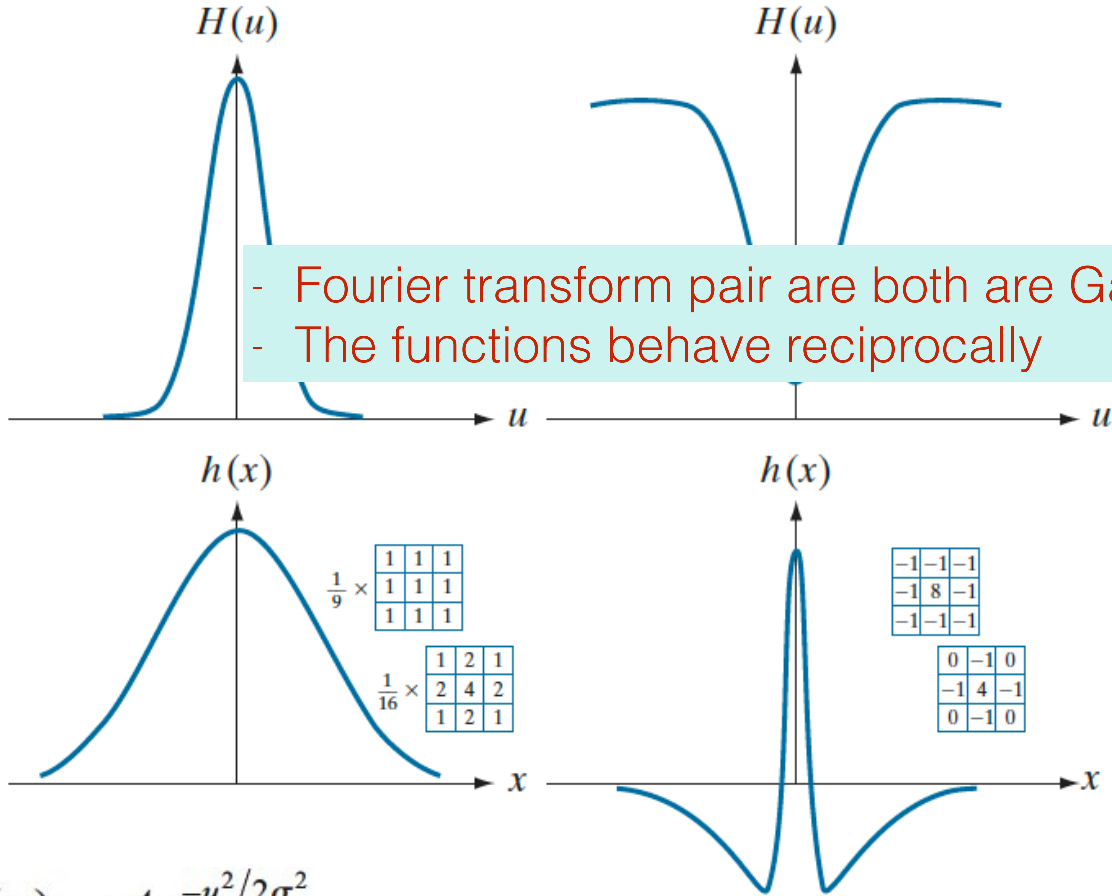
$$h(x) = \sqrt{2\pi}\sigma_1 A e^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 B e^{-2\pi^2\sigma_2^2x^2}$$

4.7 The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

Frequency domain

Spatial domain



- Fourier transform pair are both are Gaussian and real
- The functions behave reciprocally

Highness filter with Difference of Gaussian

Gaussian filter

$$H(u) = Ae^{-u^2/2\sigma^2}$$

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2x^2}$$

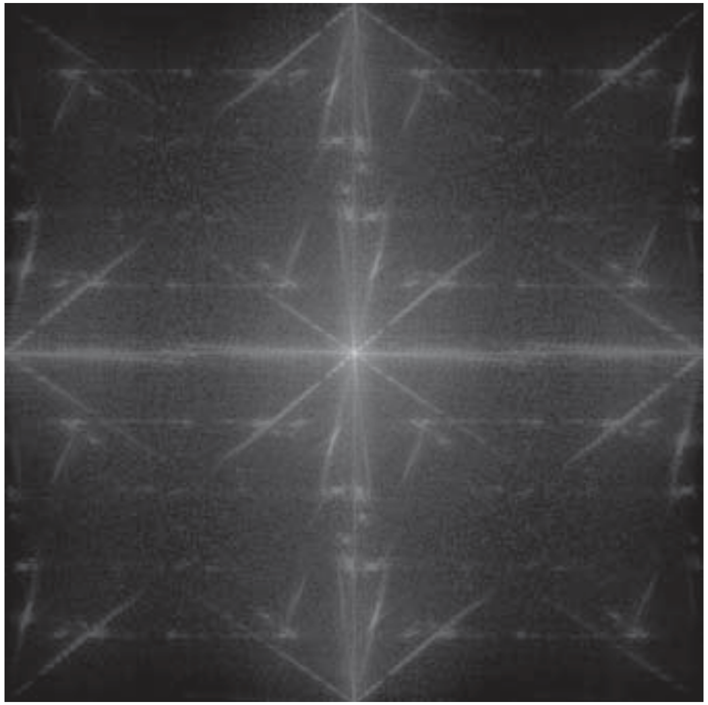
$$H(u) = A e^{-u^2/2\sigma_1^2} - B e^{-u^2/2\sigma_2^2}$$

$$h(x) = \sqrt{2\pi}\sigma_1 A e^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 B e^{-2\pi^2\sigma_2^2x^2}$$

4.7 The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

Original image



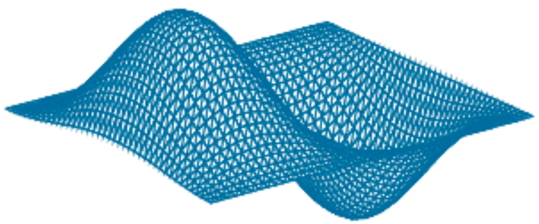
Its Fourier spectrum

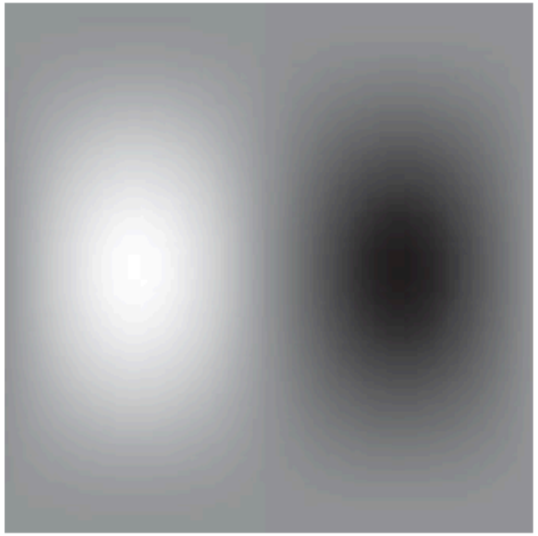
4.7 The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

Sobel kernel and its corresponding frequency domain filter

−1	0	1
−2	0	2
−1	0	1





frequency domain filter shown as an image

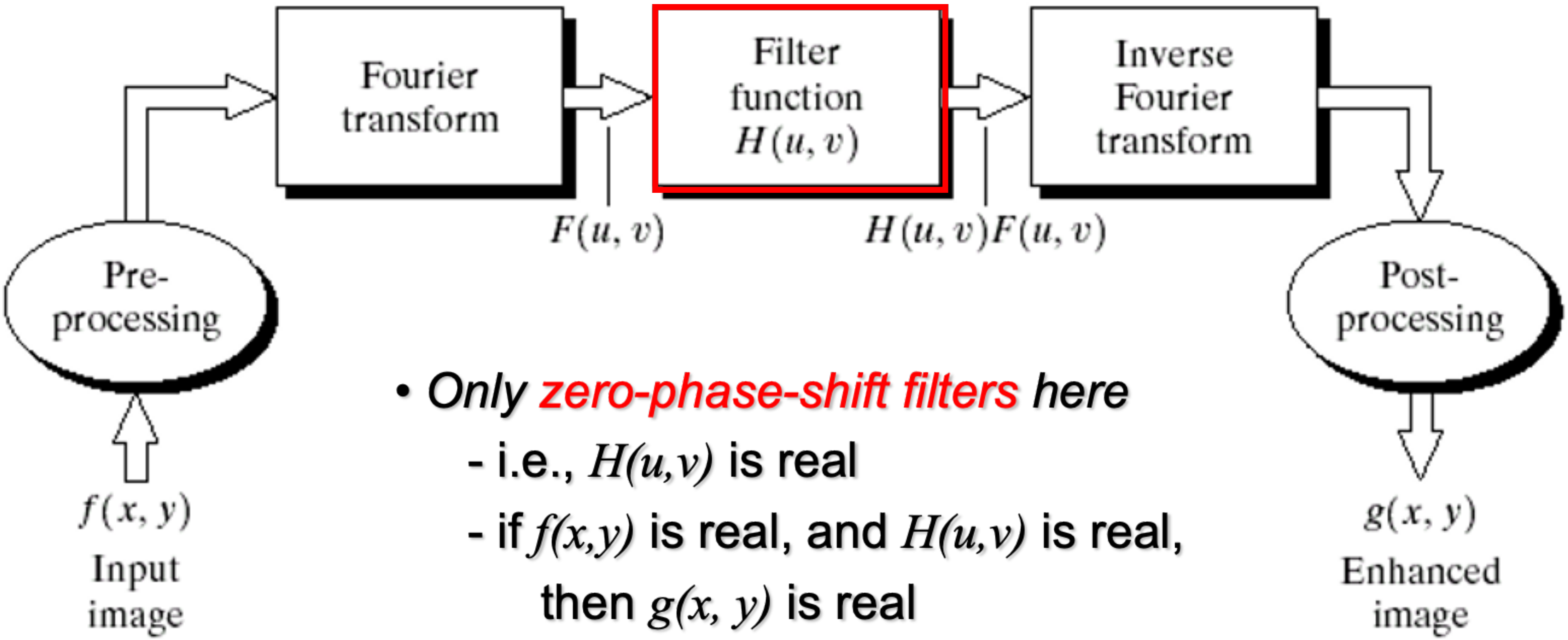
filtering in frequency domain

filtering in spatial domain

4.8 Image Smoothing Using Lowpass Frequency Domain Filters

$$\boxed{g(x,y) = \text{Real}\left\{\Im^{-1}\left[H(u,v)F(u,v)\right]\right\}}$$

Frequency domain filtering operation



- *Only* ***zero-phase-shift filters*** *here*
  - i.e., $H(u,v)$ is real
  - if $f(x,y)$ is real, and $H(u,v)$ is real, then $g(x, y)$ is real

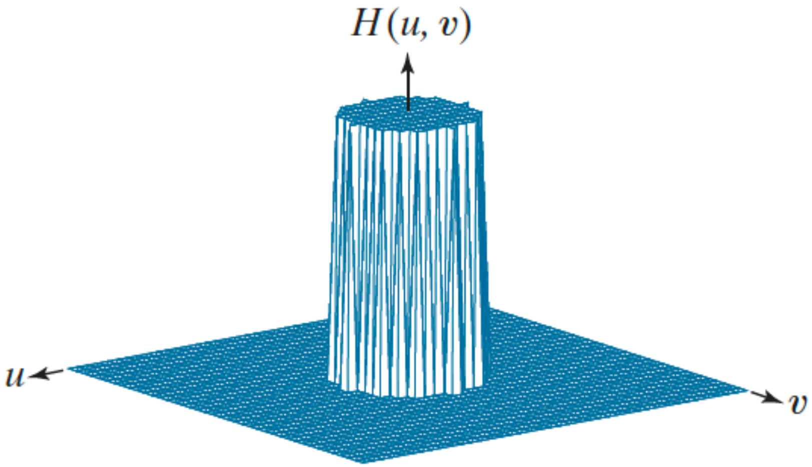
credit of this slide: Y. P. Hung

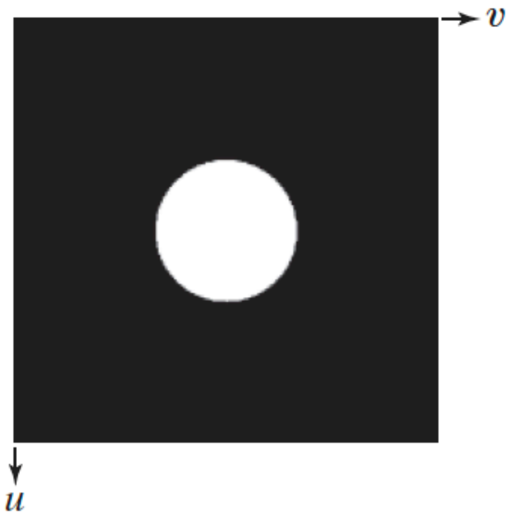
4.8 Image Smoothing Using Lowpass Frequency Domain Filters

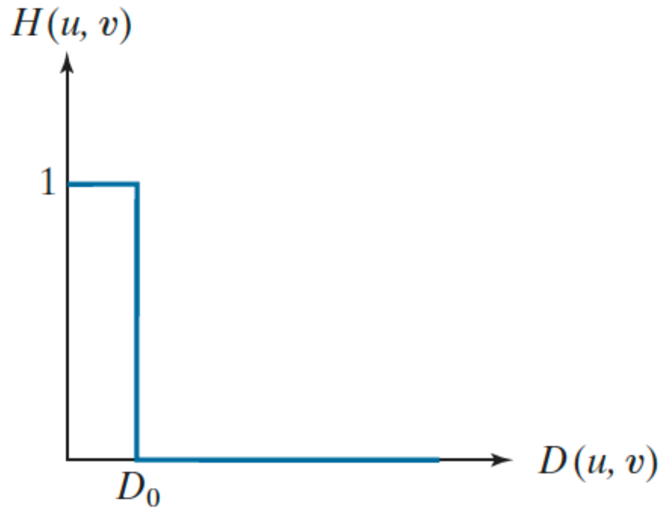
- Ideal Lowpass Filter (ILPF)

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

where $D(u,v) = \left[(u - P/2)^2 + (v - Q/2)^2 \right]^{1/2}$





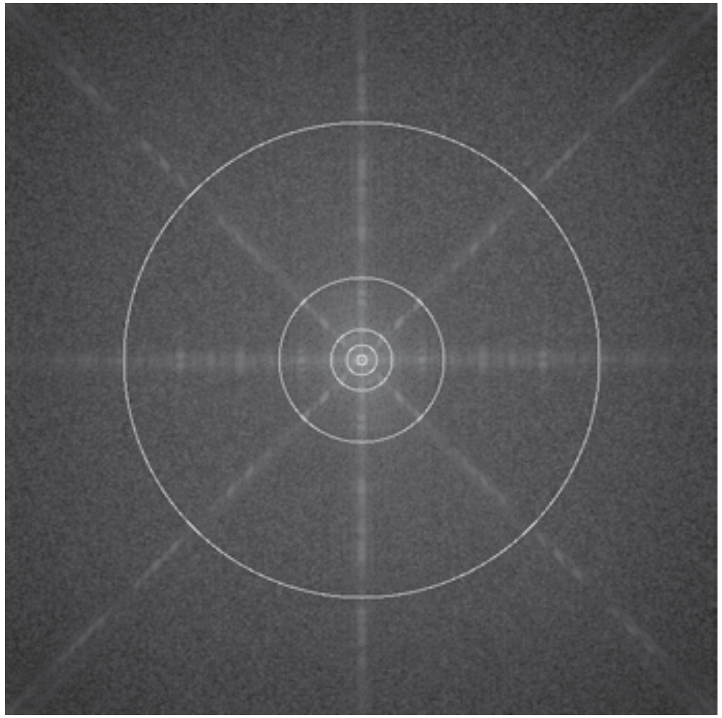


cutoff frequency

4.8 Image Smoothing Using Lowpass Frequency Domain Filters

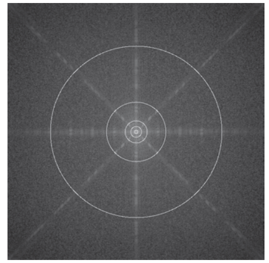
- Ideal Lowpass Filter (ILPF)
  - cutoff frequency loci

688 x 688 image



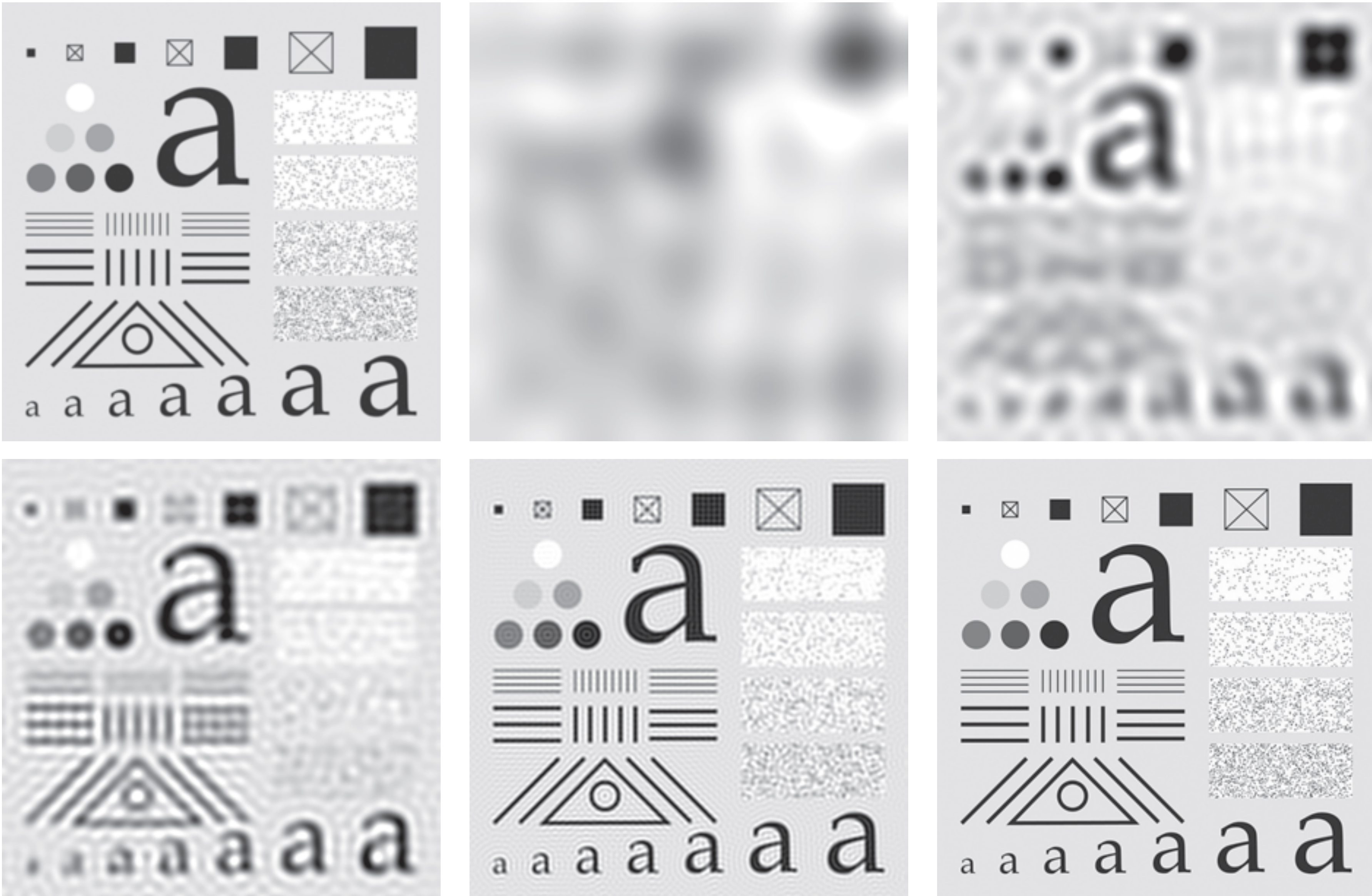
- Circles have radii of 10, 30, 60, 160, and 460 pixels.
- The radii enclose 86.9, 92.8, 95.1, 97.6, and 99.4% of the padded image power.

• Ideal Lowpass Filter (ILPF)



Example of ILPF

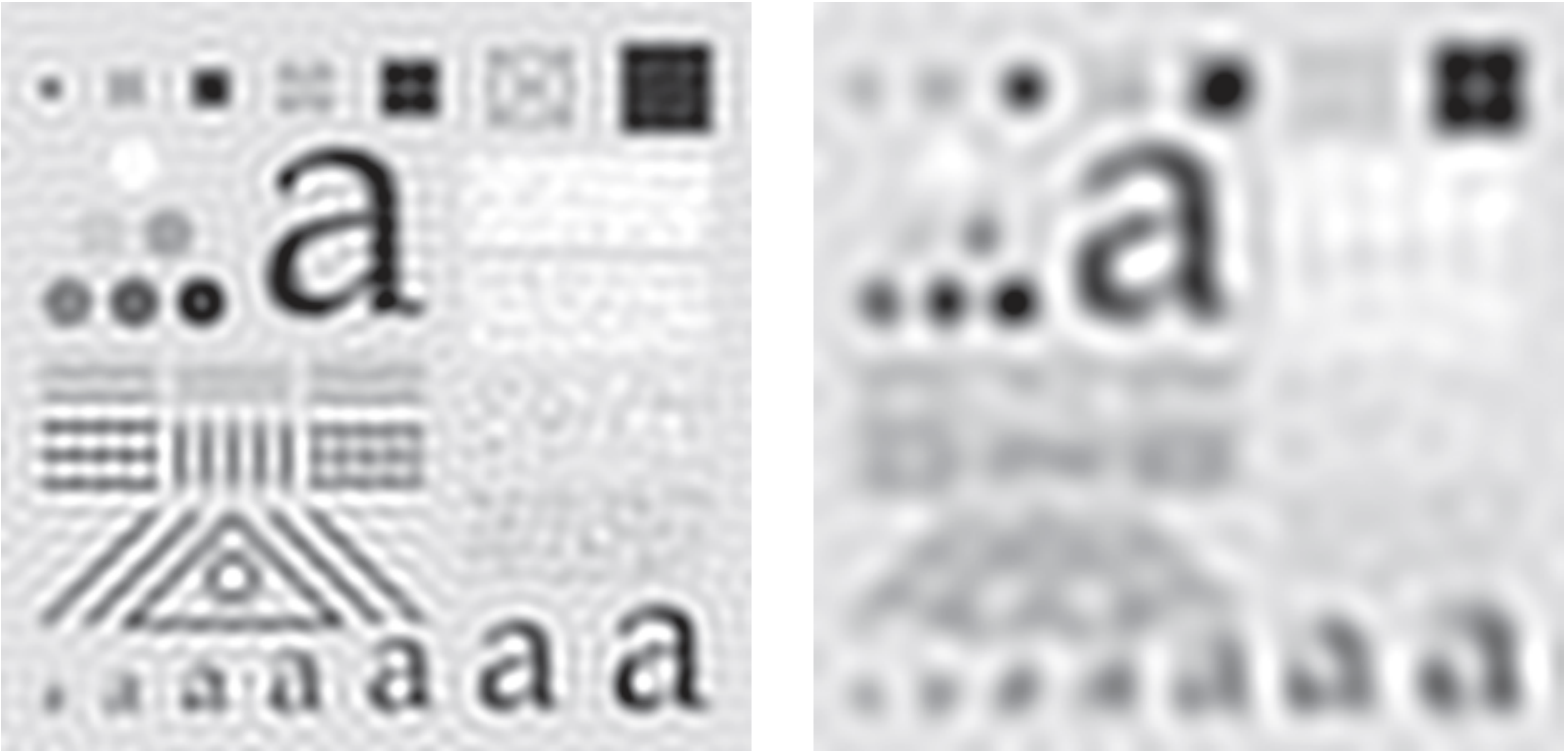
radius 10 radius 30



radius 60 radius 160 radius 460

- Ideal Lowpass Filter (ILPF)

Example of ILPF



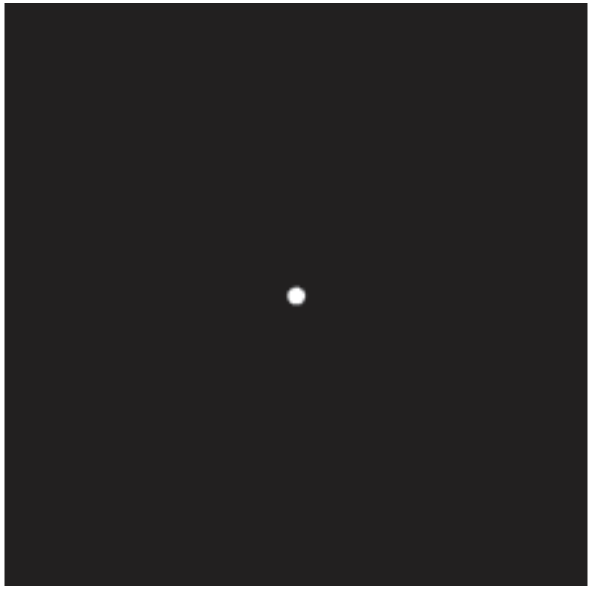
ILPF will produce *ringing effects*

- Ideal Lowpass Filter (ILPF)

Example of ILPF

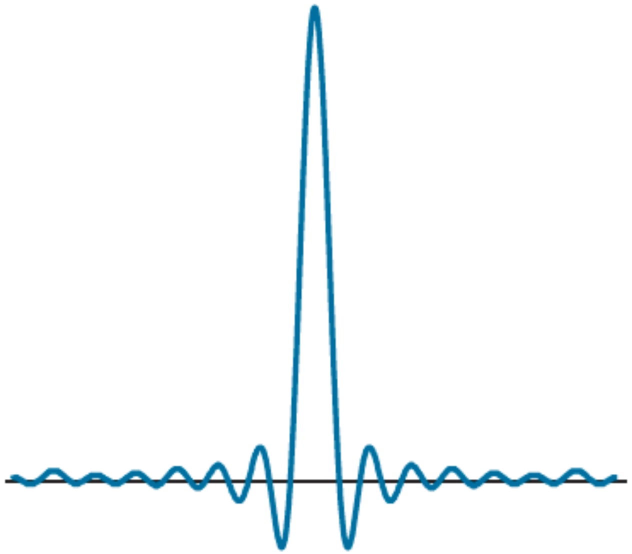
ILPF will produce *ringing effects*

- ILPF in the spatial domain is a sinc function



Frequency domain
ILPF transfer function

Corresponding spatial
domain kernel function



Intensity profile of a
horizontal line

4.8 Image Smoothing Using Lowpass Frequency Domain Filters

- Gaussian Lowpass Filter (GLPF)

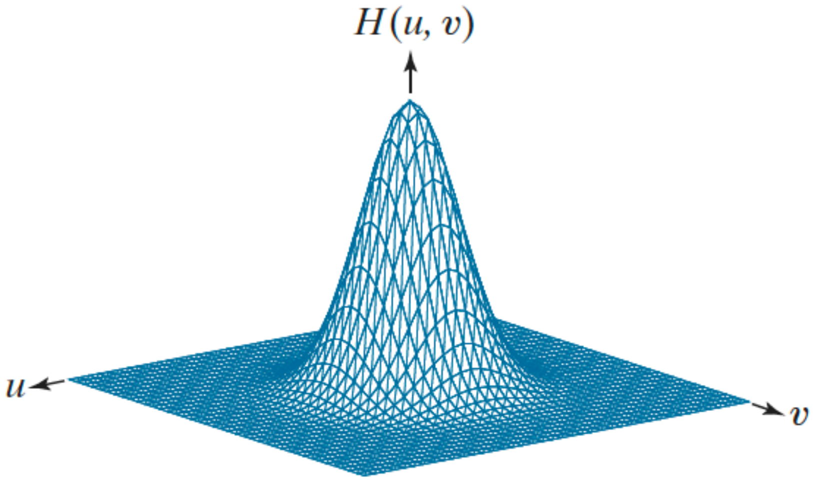
$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

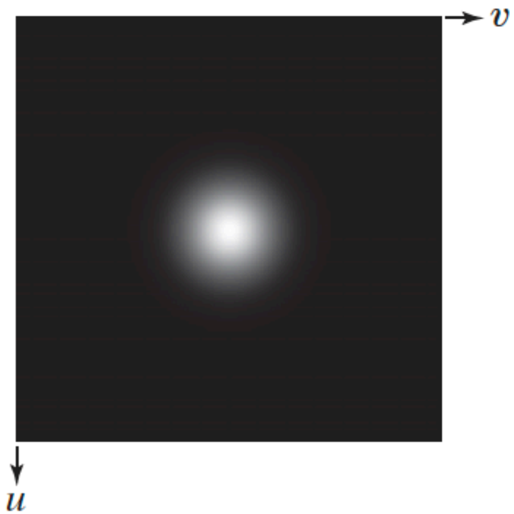
Let

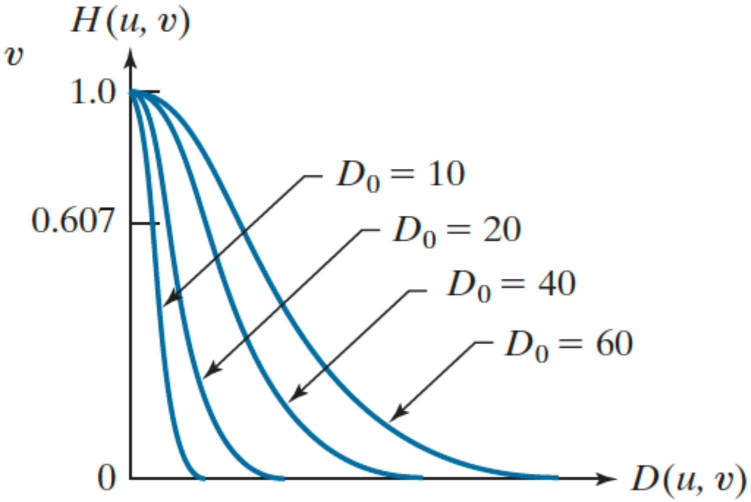
$\sigma = D_0$

cutoff frequency

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$







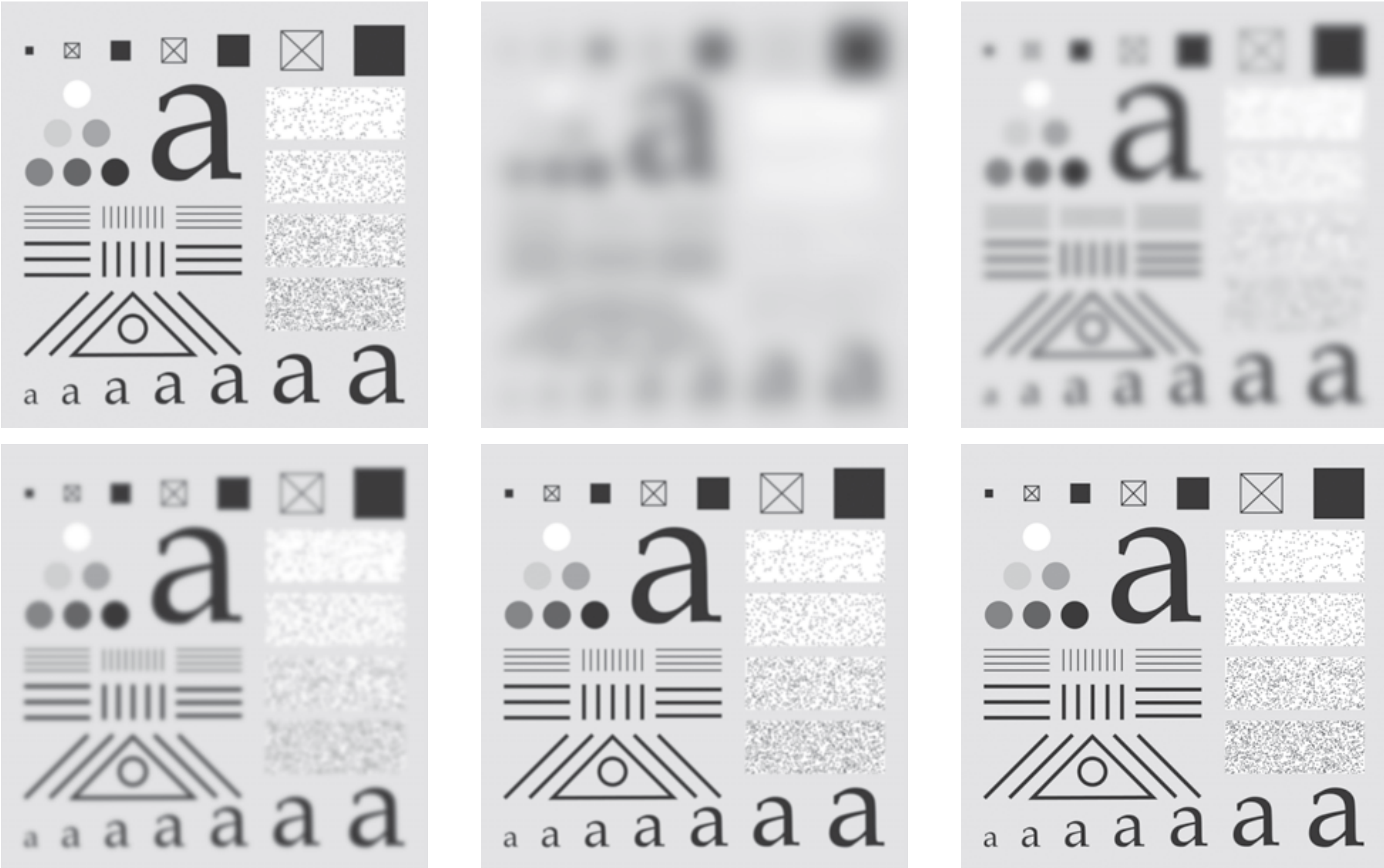
- Gaussian Lowpass Filter (GLPF)

Less smoothing than the ILPF, but no ringing!

Example of GLPF

Gaussian D_0=10

Gaussian D_0=30



Gaussian D_0=60

Gaussian D_0=160

Gaussian D_0=460

4.8 Image Smoothing Using Lowpass Frequency Domain Filters

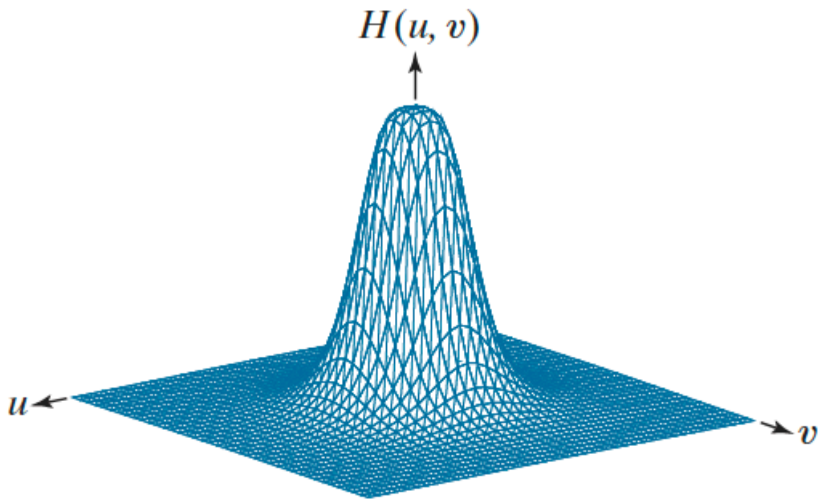
- Butterworth Lowpass Filter (BLPF)

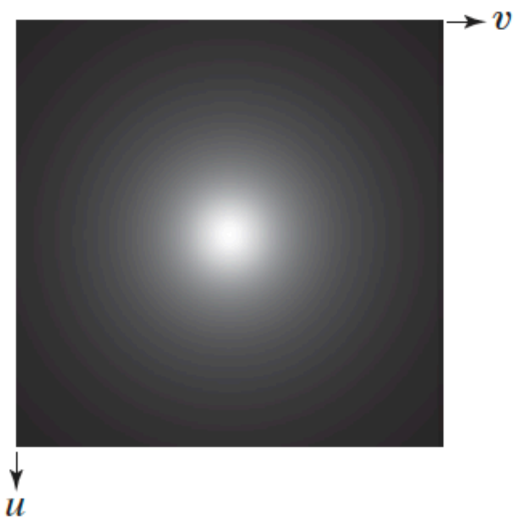
$$H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{2n}}$$

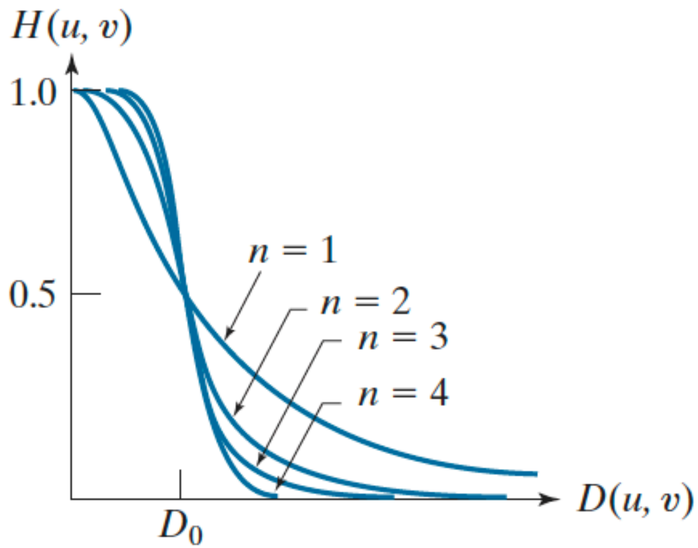
order

cutoff frequency

more control of the transition between low and high frequencies







- Butterworth Lowpass Filter (BLPF)

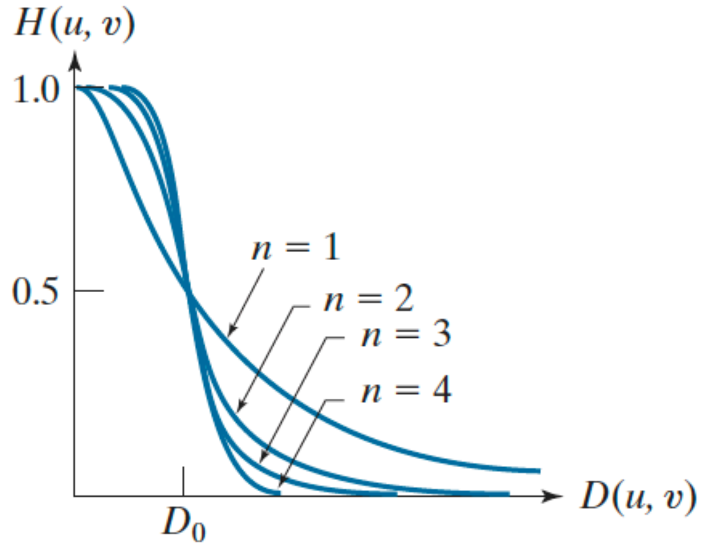
Example of BLPF (n = 2.25)

blurring are between the results obtained with using ILPFs and GLPFs

D_0=10 D_0=30

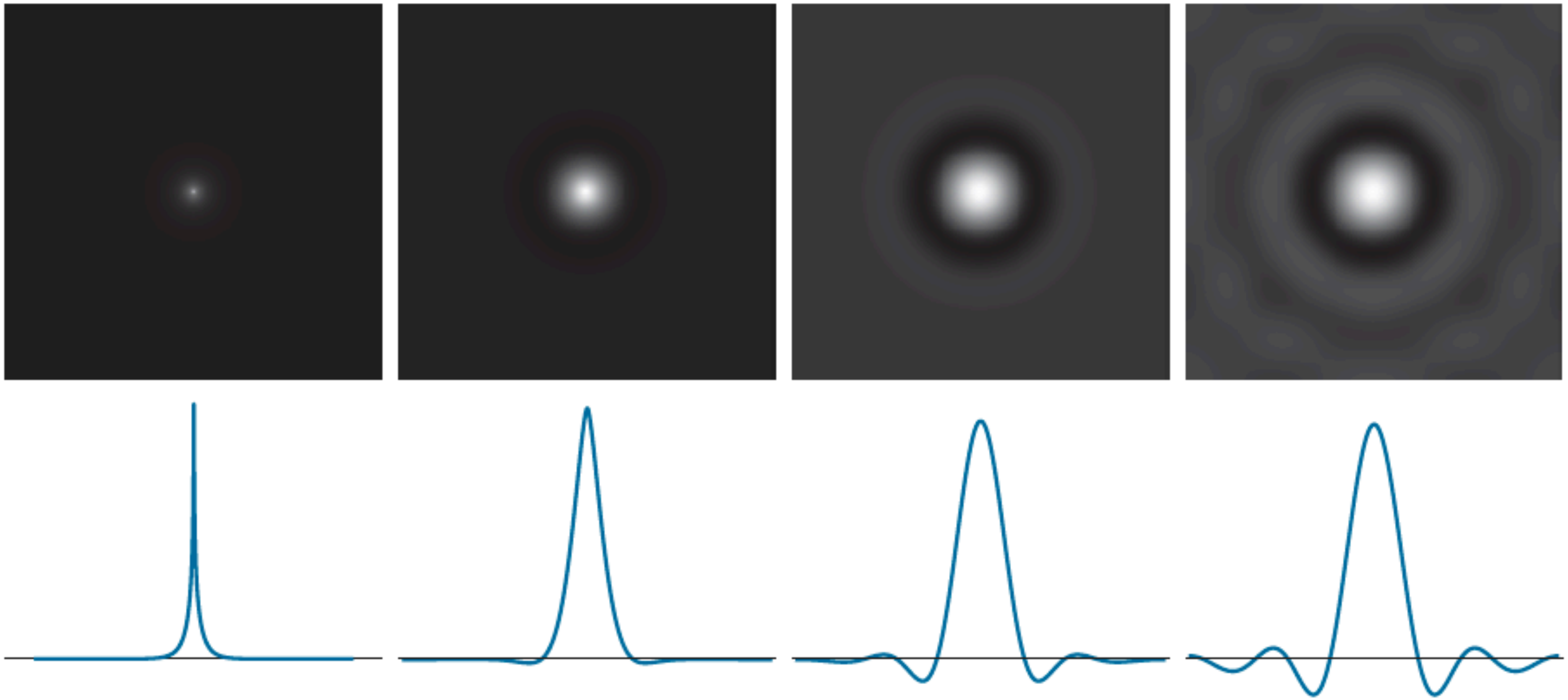
D_0=60 D_0=160 D_0=460

• Butterworth Lowpass Filter (BLPF)



larger *n* will produce ringing effect

Spatial kernels corresponding to BLPF transfer functions (cutoff frequency = 5)



n=1 n=2 n=5 n=20

4.8 Image Smoothing Using Lowpass Frequency Domain Filters

- **Comparison of Lowpass Filter**

Cutoff frequency = 30

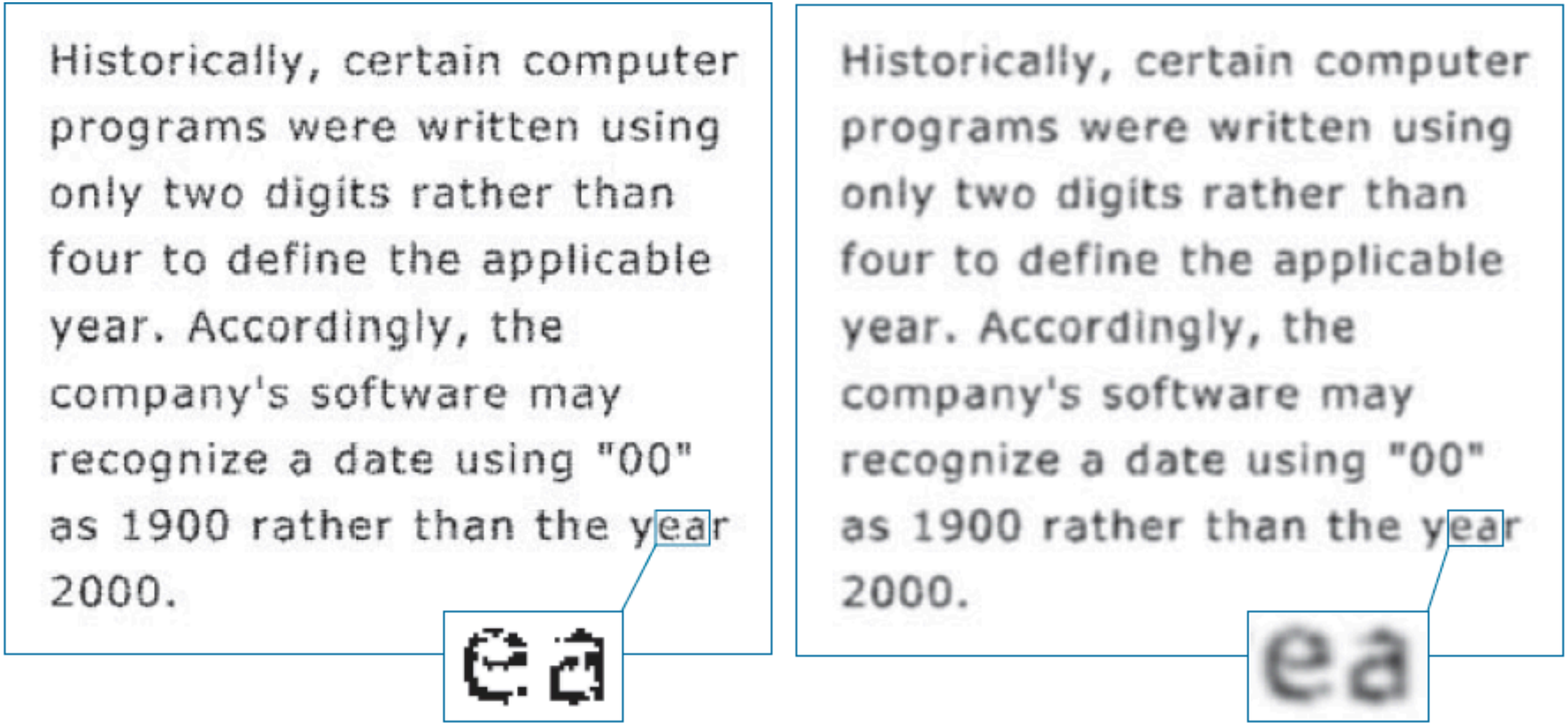
ILPF

GLPF

BLPF
$n = 2.25$

- More Examples of Lowpass Filter

GLPF is used to connect broken text



• More Examples of Lowpass Filter

GLPF is used to reduce fine skin lines

Original Image

D_0=150

D_0=130

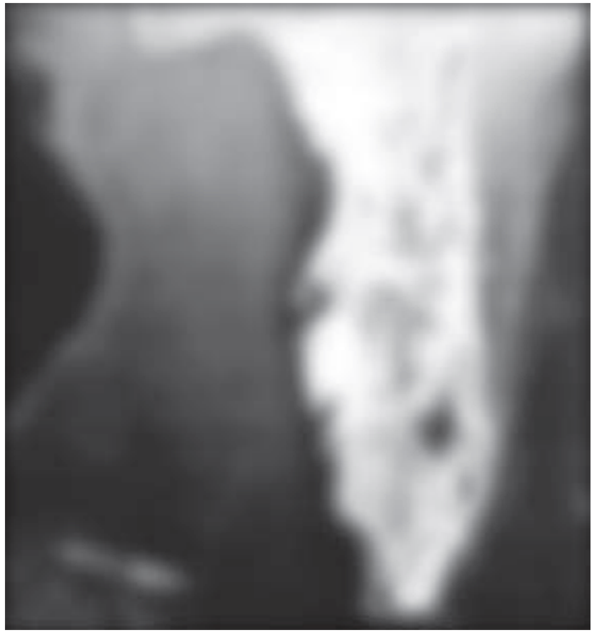
- More Examples of Lowpass Filter

GLPF is used to reduce horizontal scan line

- satellite image shows part of the Gulf of Mexico (dark) and Florida (light)

Original Image

D_0=50



D_0=20

4.9 Image Sharpening Using Highpass Filters

- Highpass Filters from Lowpass Filters

$$H_{HP}(u, v) = 1 - H_{LP}(u, v)$$

- Ideal Highpass Filter (IHPF)
- Gaussian Highpass Filter (GHPF)
- Butterworth Highpass Filter (BHPF)

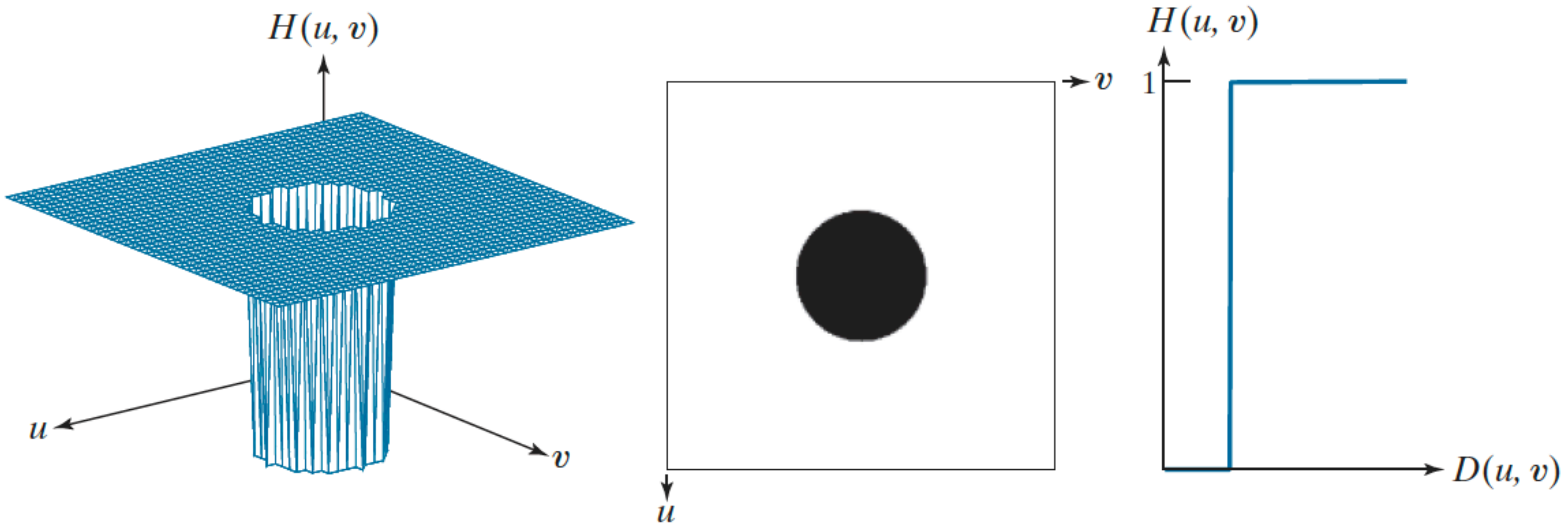
4.9 Image Sharpening Using Highpass Filters

- Ideal Highpass Filter (IHPF)

$$H_{HP}(u, v) = 1 - H_{LP}(u, v)$$

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

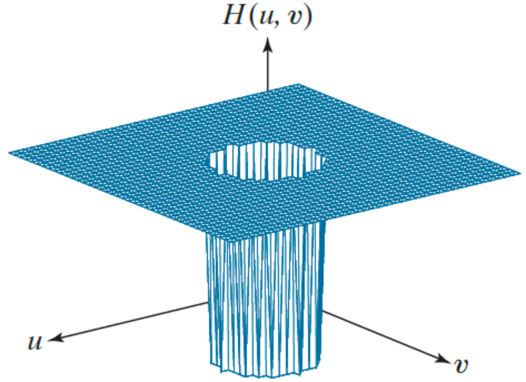
cutoff frequency



4.9 Image Sharpening Using Highpass Filters

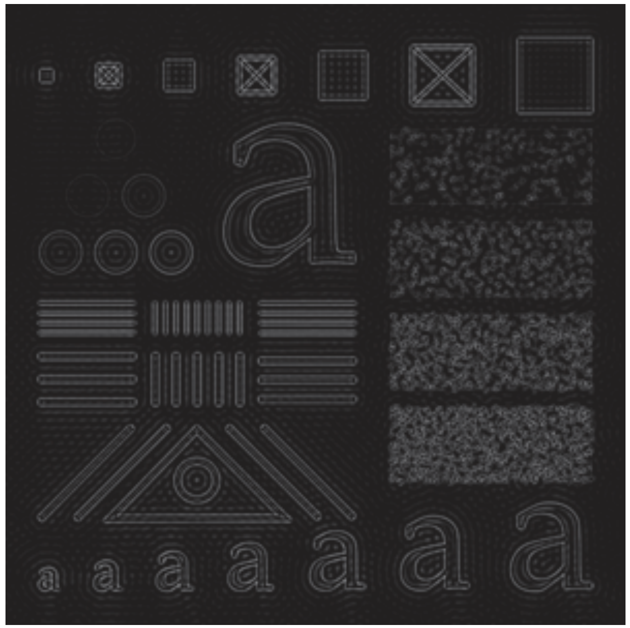
- **Ideal Highpass Filter (IHPF)**

  Example of IHPF



the negative values are displayed at 0 (black)

D_0=60



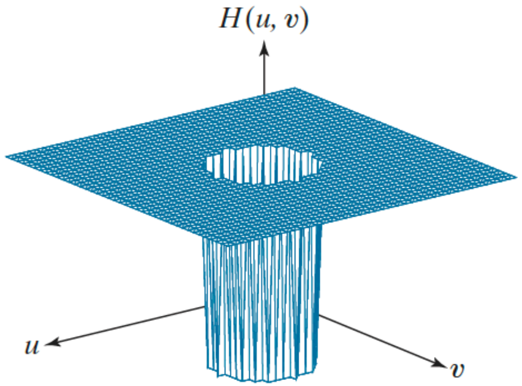
D_0=160

4.9 Image Sharpening Using Highpass Filters

- **Ideal Highpass Filter (IHPF)**

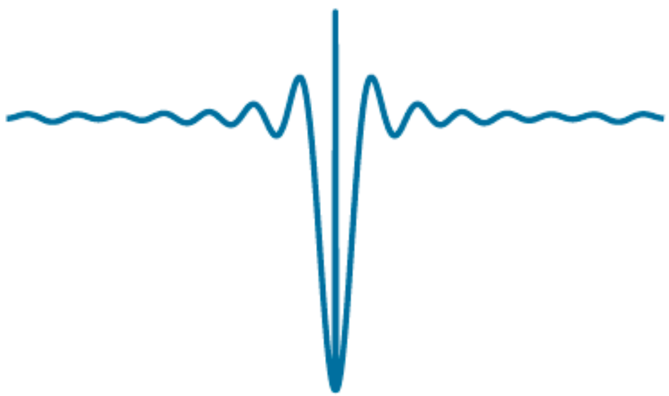
Example of IHPF

$$\begin{aligned} h_{\mathrm{HP}}(x,y) &= \Im^{-1}\left[H_{\mathrm{HP}}(u,v)\right] \\ &= \Im^{-1}\left[1 - H_{\mathrm{LP}}(u,v)\right] \\ &= \delta(x,y) - h_{\mathrm{LP}}(x,y) \end{aligned}$$



ringing effects

IHPF Corresponding spatial kernel



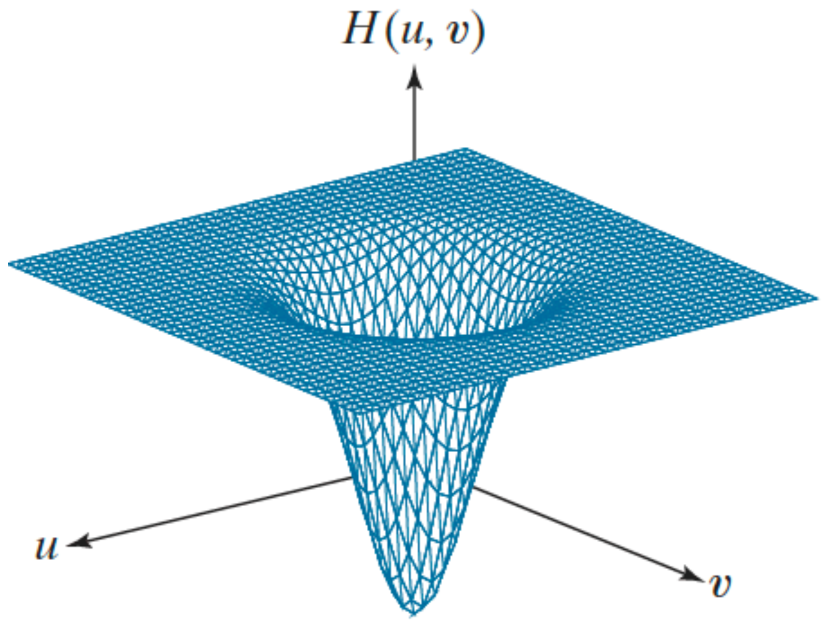
Horizontal intensity profile

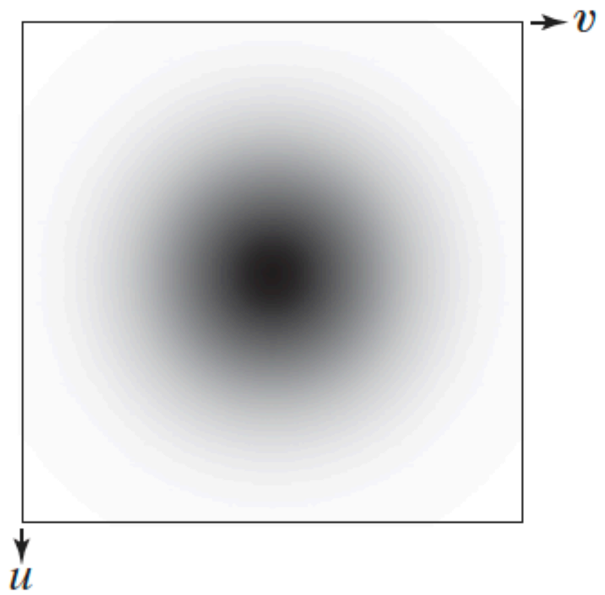
4.9 Image Sharpening Using Highpass Filters

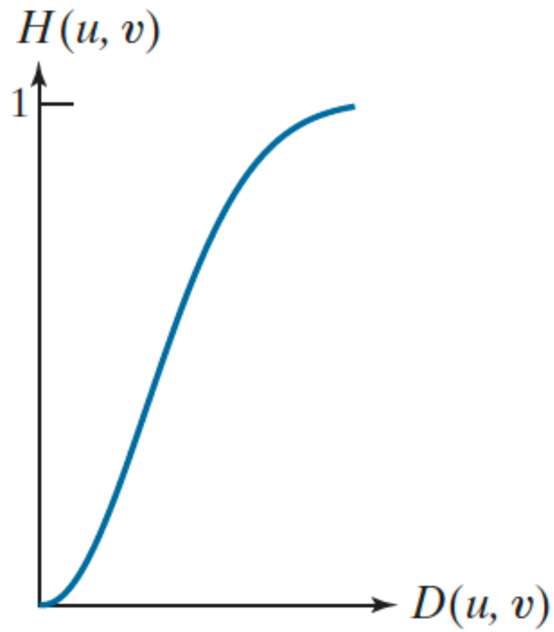
- Gaussian Highpass Filter (GHPF) $H_{HP}(u, v) = 1 - H_{LP}(u, v)$

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

cutoff frequency



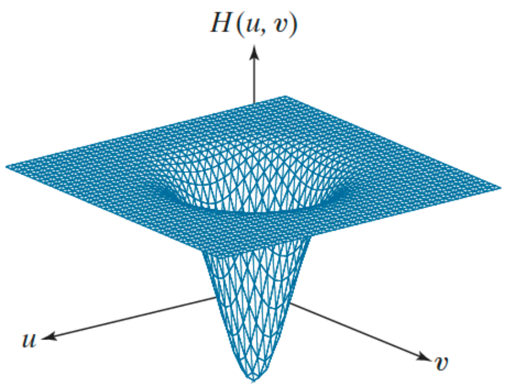




- # Gaussian Highpass Filter (GHPF)

Example of GHPF

GHPF Corresponding spatial kernel and its profile



D_0=60

D_0=160

4.9 Image Sharpening Using Highpass Filters

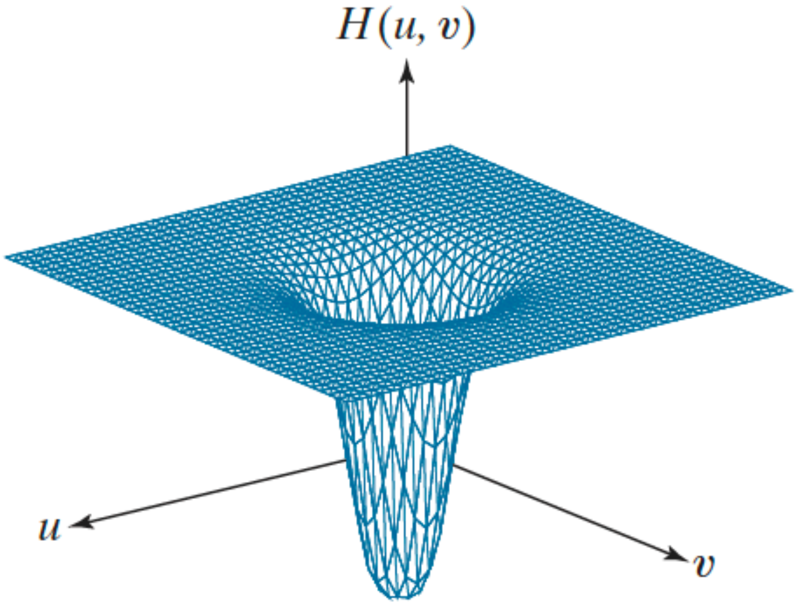
- Butterworth Highpass Filter (GHPF)

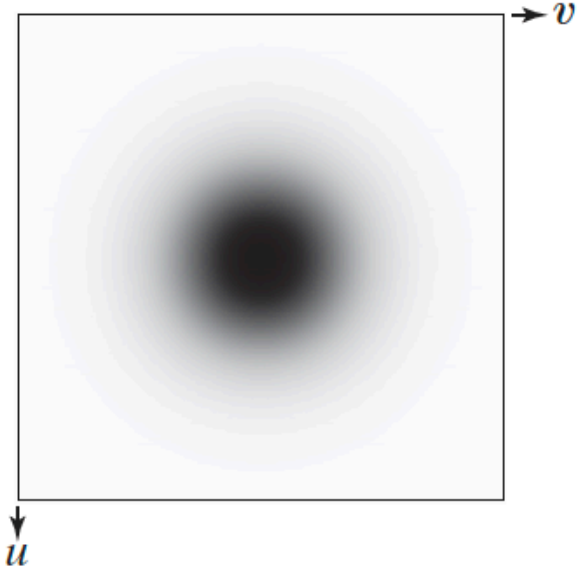
$$H_{HP}(u, v) = 1 - H_{LP}(u, v)$$

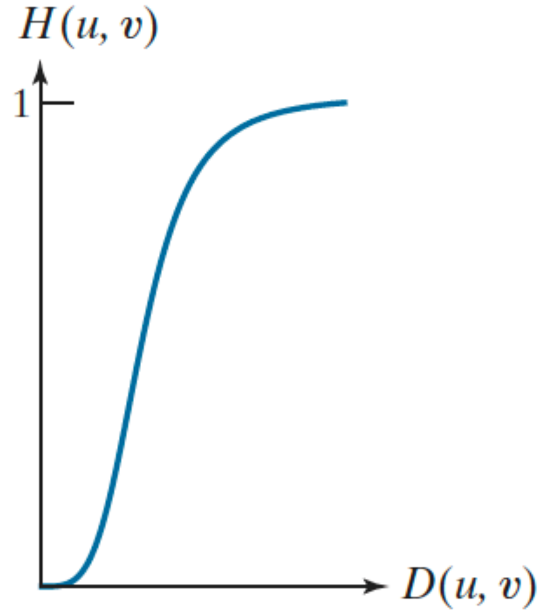
$$H(u,v) = \frac{1}{1 + [D_0 / D(u,v)]^{2n}}$$

order

cutoff frequency

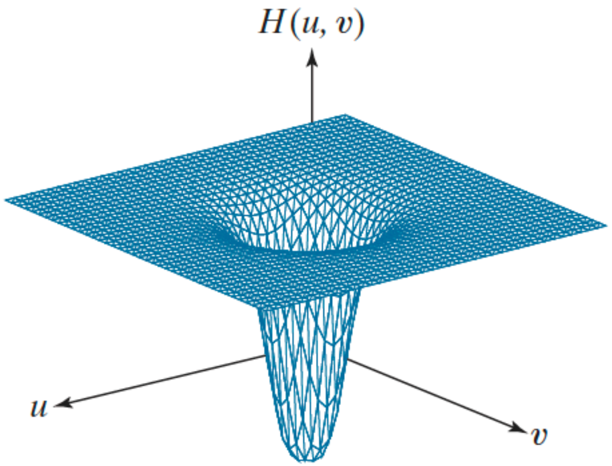




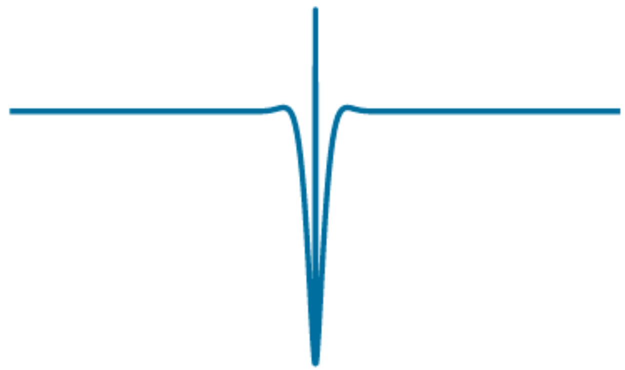


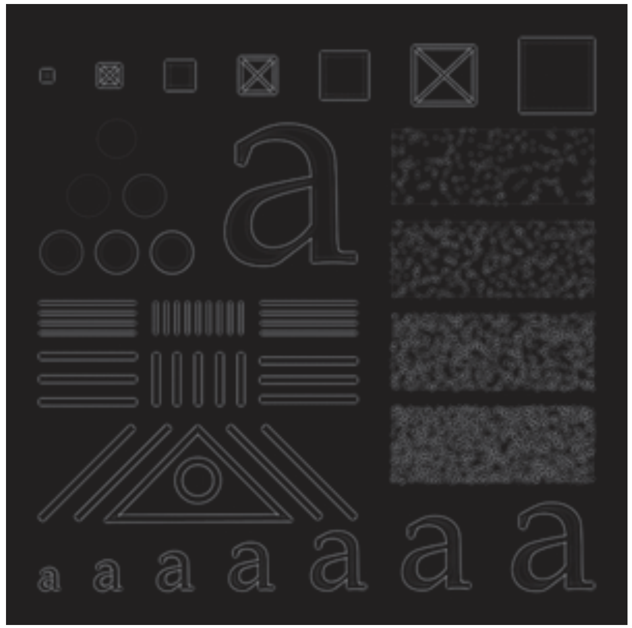
• Butterworth Highpass Filter (BHPF)

Example of BHPF



BHPF Corresponding spatial kernel and its profile





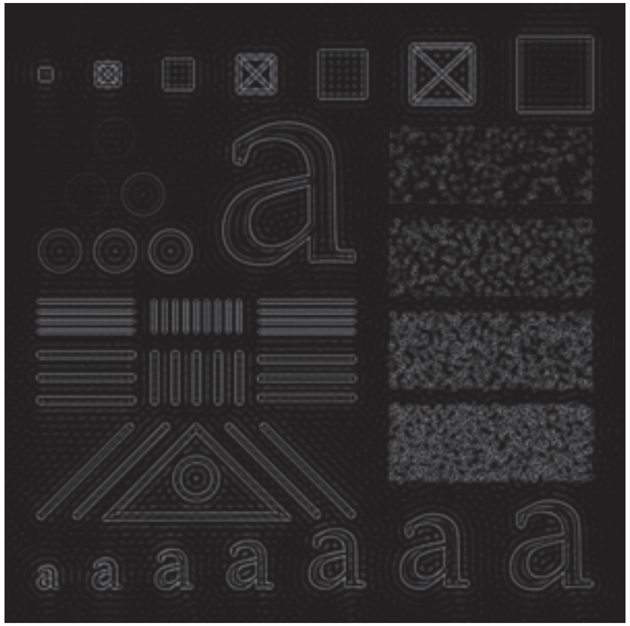
$n = 2$, D_0=60

$n = 2$, D_0=160

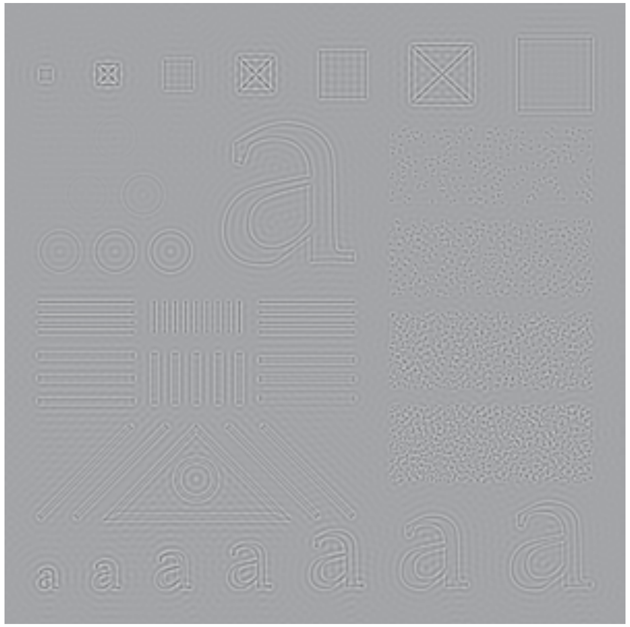
- Comparison of Highpass Filter $D_0 = 60$ (row 1), 160 (row 2), $n = 2$

IHPF GHPF BHPF

- Comparison of Highpass Filter



IHPF GHPF BHPF



Display the full intensity range

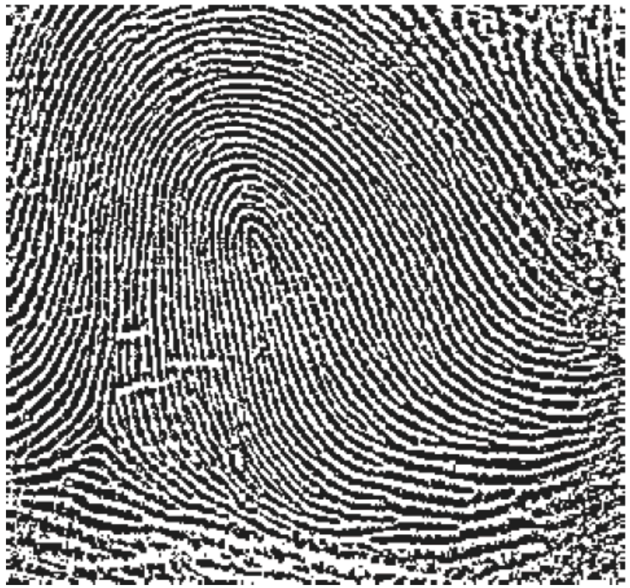
4.9 Image Sharpening Using Highpass Filters

- More Examples of Highpass Filter

Using BHPF and thresholding for image enhancement

Original Image

BHPF



Thresholding

4.9 Image Sharpening Using Highpass Filters - The Laplacian in the Frequency Domain

- **Review - Laplacian in spatial domain (Sec. 3.6)**

$$\nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}$$

Discrete form:

$$\nabla^2 f = [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)] - 4f(x,y)$$

The corresponding spatial filter:

0	1	0
1	−4	1
0	1	0

4.9 Image Sharpening Using Highpass Filters - The Laplacian in the Frequency Domain

$$\nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}$$

Fourier transform pair:

$$\nabla^2 f(t,z) \Leftrightarrow -4\pi^2(\mu^2 + \nu^2)F(\mu,\nu)$$

Table 4.4:

$$\left(\frac{\partial}{\partial t}\right)^m \left(\frac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m (j2\pi\nu)^n F(\mu,\nu)$$

$$\frac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu); \quad \frac{\partial^n f(t,z)}{\partial z^m} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$$

4.9 Image Sharpening Using Highpass Filters - The Laplacian in the Frequency Domain

$$\nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}$$

Fourier transform pair:

$$\nabla^2 f(t,z) \Leftrightarrow -4\pi^2(\mu^2 + \nu^2)F(\mu,\nu)$$

Laplacian in frequency domain:

$$\nabla^2 f(x,y) = \Im^{-1}\left[H(u,v)F(u,v)\right]$$

$$H(u,v) = -4\pi^2(u^2 + v^2)$$

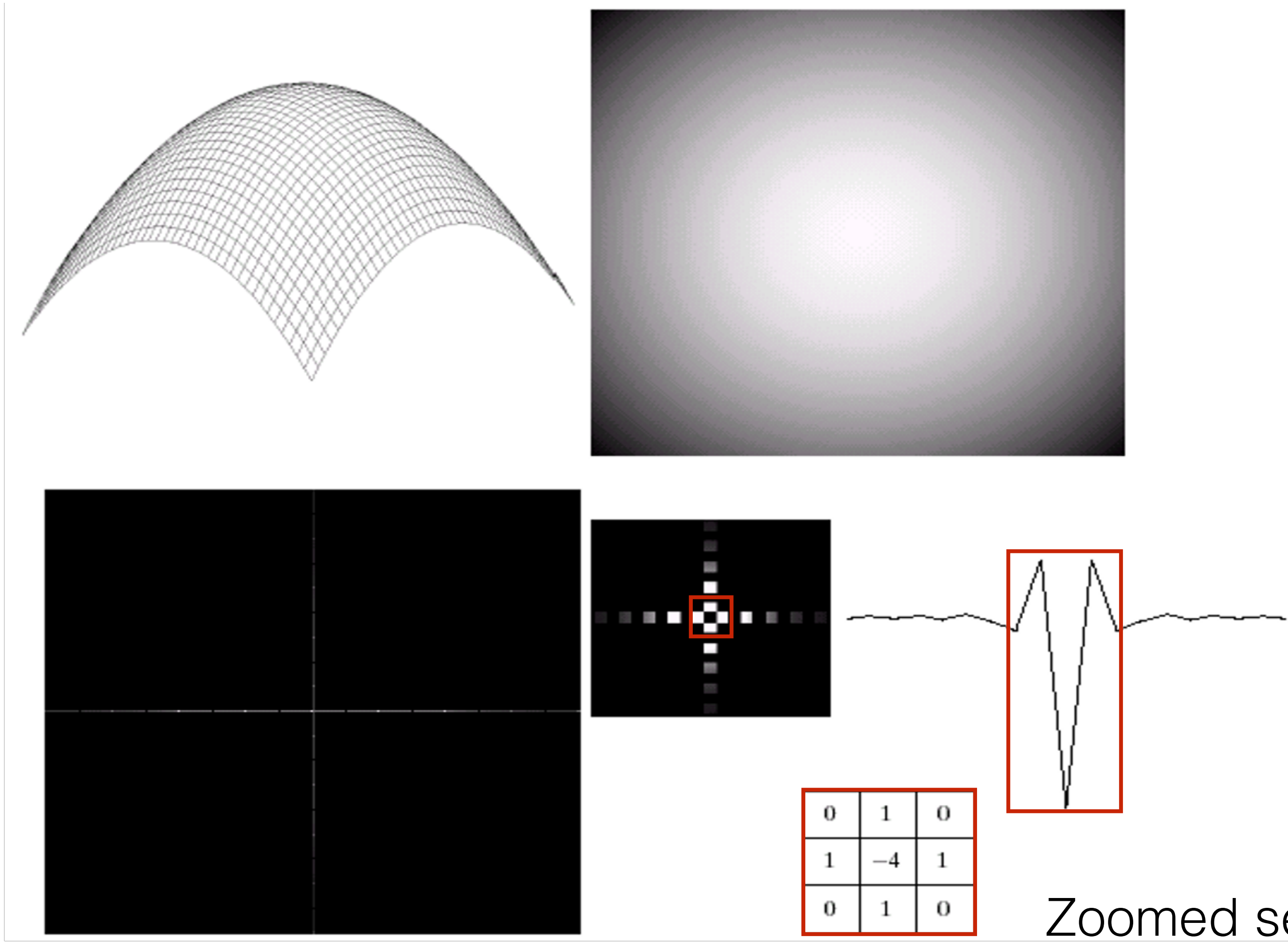
Centered $$H(u,v) = -4\pi^2\left[(u - P/2)^2 + (v - Q/2)^2\right] = -4\pi^2 D^2(u,v)$$

4.9 Image Sharpening Using Highpass Filters - The Laplacian in the Frequency Domain

$$H(u,v) = -4\pi^2\left[(u - P/2)^2 + (v - Q/2)^2\right] = -4\pi^2 D^2(u,v)$$

Laplacian in frequency domain

IDFT of Laplacian in frequency domain



Zoomed section

4.9 Image Sharpening Using Highpass Filters
- The Laplacian in the Frequency Domain

the Laplacian for image sharpening in spatial domain:

$$g(x,y) = f(x,y) - \nabla^2 f$$

the Laplacian for image sharpening in frequency domain:

$$\begin{aligned} g(x,y) &= \Im^{-1}\{F(u,v) - H(u,v)F(u,v)\} \\ &= \Im^{-1}\{[1 - H(u,v)]F(u,v)\} \\ &= \Im^{-1}\{[1 + 4\pi^2 D^2(u,v)]F(u,v)\} \end{aligned}$$

4.9 Image Sharpening Using Highpass Filters - The Laplacian in the Frequency Domain

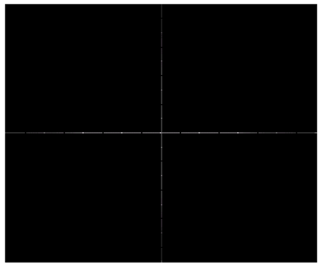
Example of Laplacian image sharpening in the frequency domain

Original Image

Laplacian image sharpening in the frequency domain

Frequency domain vs Spatial domain

Laplacian image sharpening in the frequency domain



Laplacian image sharpening in the spatial domain

1	1	1
1	−8	1
1	1	1

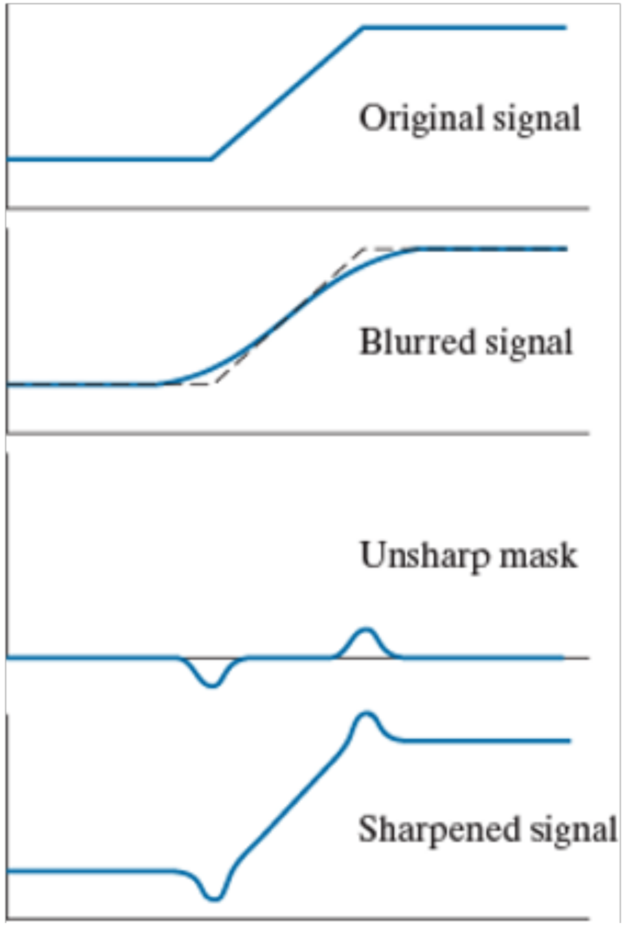
4.9 Image Sharpening Using Highpass Filters - Unsharp Masking and High-boost Filtering

- Review - Unsharp masking and high-boost filtering in spatial domain (Sec. 3.6)

$$b(x,y)=Blur\{f(x,y)\}$$

$$g_{mask}(x,y)=f(x,y)-b(x,y)$$

$$g(x,y)=f(x,y)+k\, g_{mask}(x,y)$$



k = 1: **Unsharp Masking** k > 1: **Highboost Filtering**

4.9 Image Sharpening Using Highpass Filters - Unsharp Masking and High-boost Filtering

Unsharp Mask: $g_{\text{mask}}(x,y) = f(x,y) - f_{\text{LP}}(x,y)$

$$f_{\text{LP}}(x,y) = \Im^{-1}\left[H_{\text{LP}}(u,v)F(u,v)\right]$$

Unsharp Masking ($k = 1$):

High-boost Filtering ($k > 1$):

$$g(x, y) = f(x, y) + k\, g_{mask}(x, y)$$

Frequency domain computations:

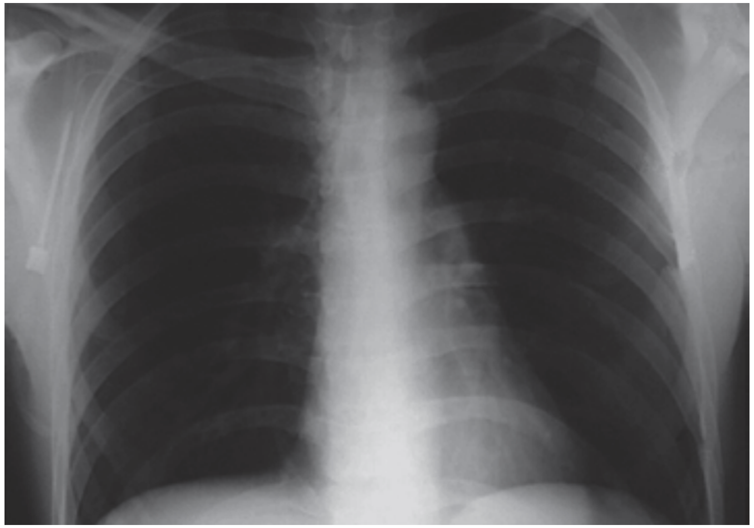
$$g(x,y) = \Im^{-1}\left\{\left(1 + k\left[1 - H_{\text{LP}}(u,v)\right]\right)F(u,v)\right\}$$

or $$g(x,y) = \Im^{-1}\left\{\underline{\left[1 + kH_{\text{HP}}(u,v)\right]}F(u,v)\right\}$$

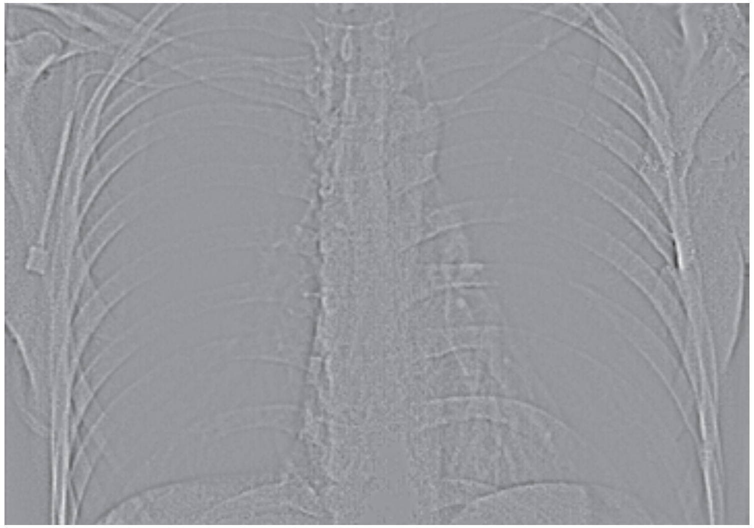
High-frequency Emphasis Filter Transfer Function

general formulation: $$g(x,y) = \Im^{-1}\left\{\left[k_1 + k_2 H_{\text{HP}}(u,v)\right]F(u,v)\right\}$$

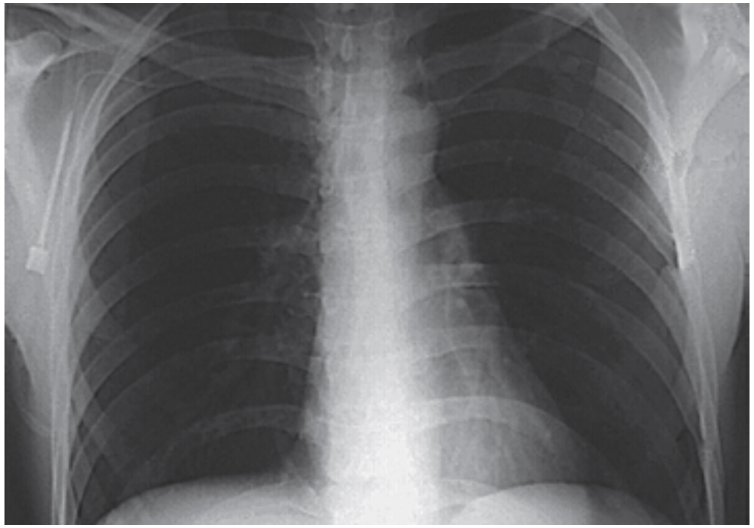
Example of High-Frequency-Emphasis Filtering



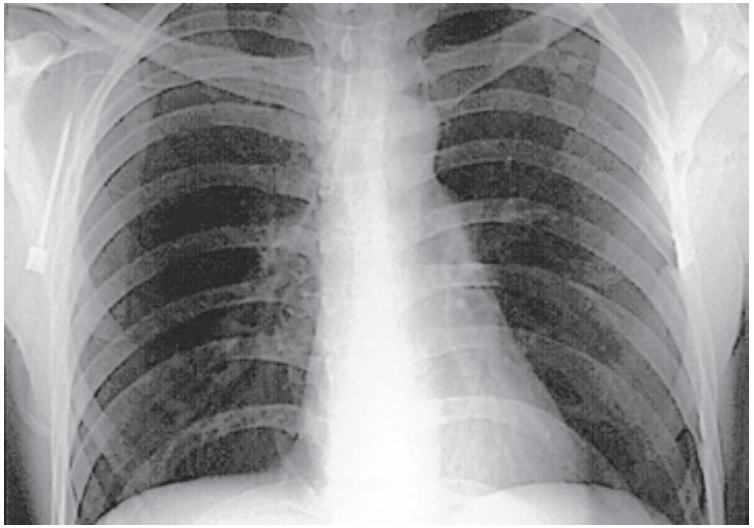
Original Image



GHPF $D_0 = 70$



High-Frequency-Emphasis Filtering $k_1 = 0.5$ and $k_2 = 0.75$



Histogram equalisation of left image

4.9 Image Sharpening Using Highpass Filters - Homomorphic Filtering

- **illumination-reflectance model** (Sec. 2.3)

$$f(x,y) = i(x,y)\, r(x,y)$$

i(x,y): the amount of source illumination incident on the scene

r(x,y): the reflectivity function (or transmissivity function)

illumination: slow spatial variations

reflectivity: vary abruptly

CANNOT be used directly to operate on the frequency components of illumination and reflectance, i.e.

$$\Im[f(x,y)] \neq \Im[i(x,y)]\Im[r(x,y)]$$

Because the Fourier transform of a product is not the product of the transforms

4.9 Image Sharpening Using Highpass Filters - Homomorphic Filtering

Define $$z(x,y) = \ln f(x,y) = \ln i(x,y) + \ln r(x,y)$$

$$\Im[z(x,y)] = \Im[\ln f(x,y)] = \Im[\ln i(x,y)] + \Im[\ln r(x,y)]$$

or $$Z(u,v) = F_i(u,v) + F_r(u,v)$$

Process with a filter transfer function *H*(*u*, *v*):

$$S(u,v) = H(u,v)Z(u,v) = H(u,v)F_i(u,v) + H(u,v)F_r(u,v)$$

$$s(x,y) = \Im^{-1}[S(u,v)] = \Im^{-1}[H(u,v)F_i(u,v)] + \Im^{-1}[H(u,v)F_r(u,v)]$$

4.9 Image Sharpening Using Highpass Filters - Homomorphic Filtering

$$s(x,y) = \Im^{-1}\left[S(u,v)\right] = \Im^{-1}\left[H(u,v)F_i(u,v)\right] + \Im^{-1}\left[H(u,v)F_r(u,v)\right]$$

Define
$$\begin{cases} i'(x,y) = \Im^{-1}\left[H(u,v)F_i(u,v)\right] \\ r'(x,y) = \Im^{-1}\left[H(u,v)F_r(u,v)\right] \end{cases}$$

$$\Rightarrow \quad s(x,y) = i'(x,y) + r'(x,y)$$

$$g(x,y) = e^{s(x,y)} = e^{i'(x,y)}e^{r'(x,y)} = i_0(x,y)r_0(x,y)$$

$$\boxed{z(x,y) = \ln f(x,y)}$$

Where $r_0(x,y) = e^{r'(x,y)}$ and $i_0(x,y) = e^{i'(x,y)}$

4.9 Image Sharpening Using Highpass Filters - Homomorphic Filtering

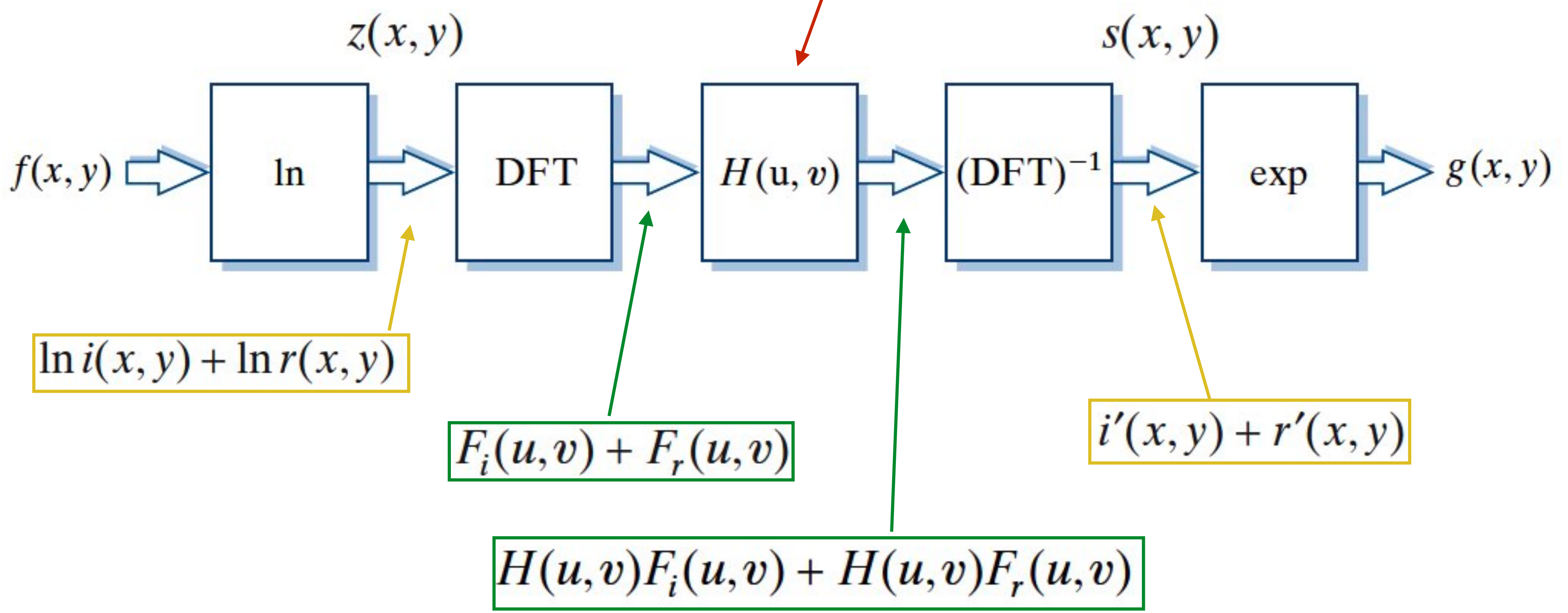
- Homomorphic System

$$g(x,y) = i_0(x,y)\,r_0(x,y)$$

$$\begin{cases} r_0(x,y) = e^{r'(x,y)} \\ i_0(x,y) = e^{i'(x,y)} \end{cases}$$

$$f(x,y) = i(x,y)\,r(x,y)$$

Homomorphic filter

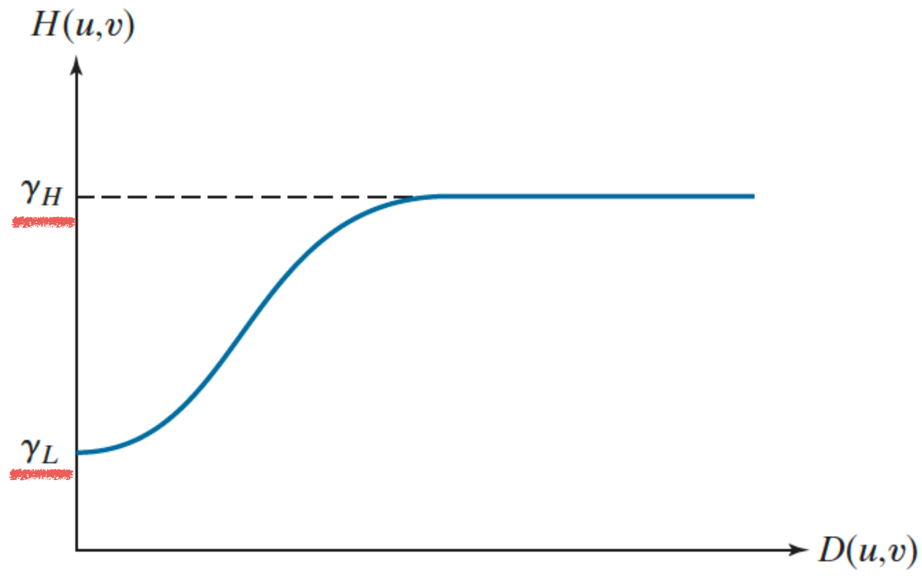


4.9 Image Sharpening Using Highpass Filters - Homomorphic Filtering

- **Homomorphic System**

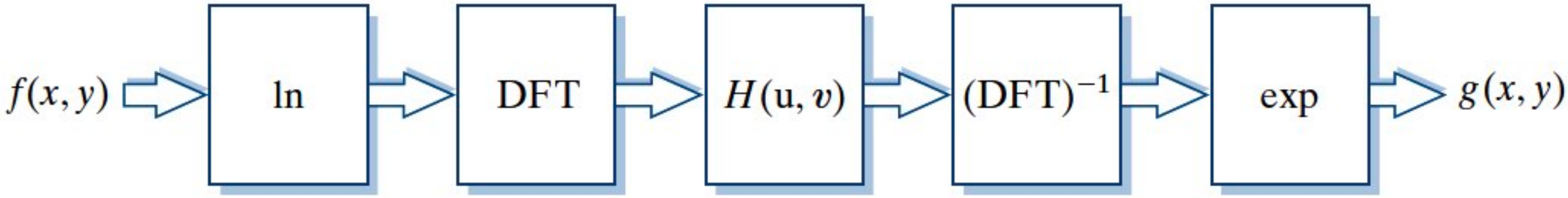
An example of homomorphic filter: using a slightly modified form of the GHPF function

Goal: simultaneous intensity range compression and contrast enhancement



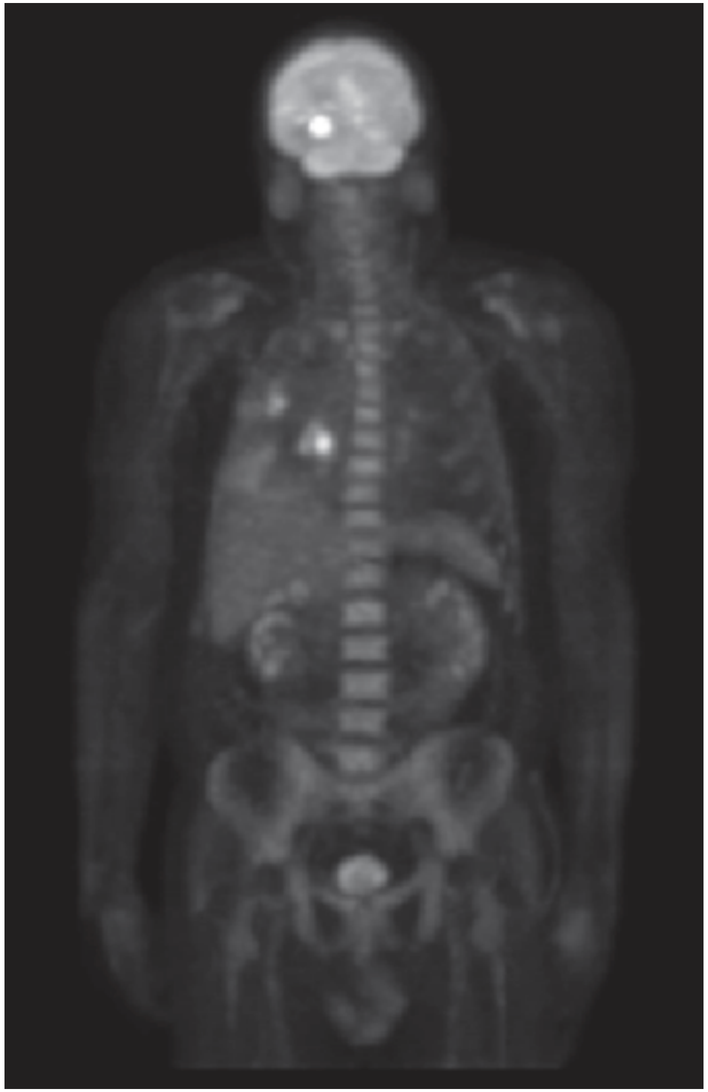
Homomorphic filter

$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-cD^2(u,v)/D_0^2}\right] + \gamma_L$$



Example of Homomorphic Filtering

Reducing the effects of the dominant illumination components (the hot spots)
Sharpening the reflectance components of the image (edge information)



Original Image

$\gamma_L = 0.4$, $\gamma_H = 3.0$, $c = 5$, and $D_0 = 20$

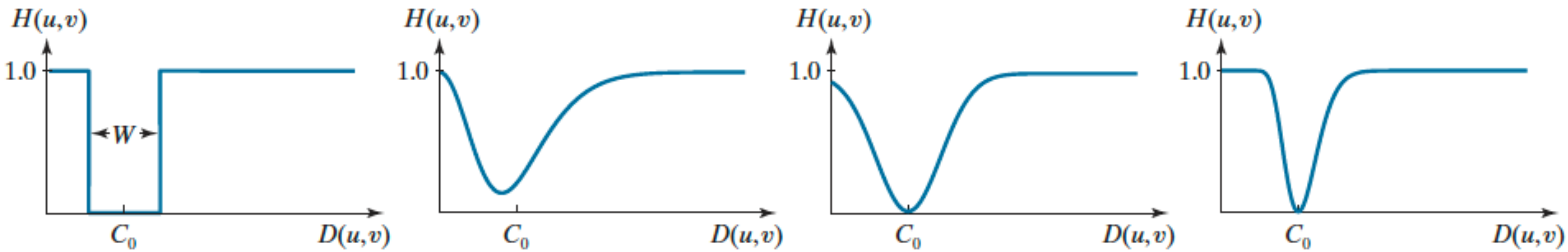
Example of Homomorphic Filtering

Original Image

Homomorphic Filtering

4.10 Selective Filtering
- Bandreject and Bandpass Filters

- Bandreject Filter (notch reject filters)

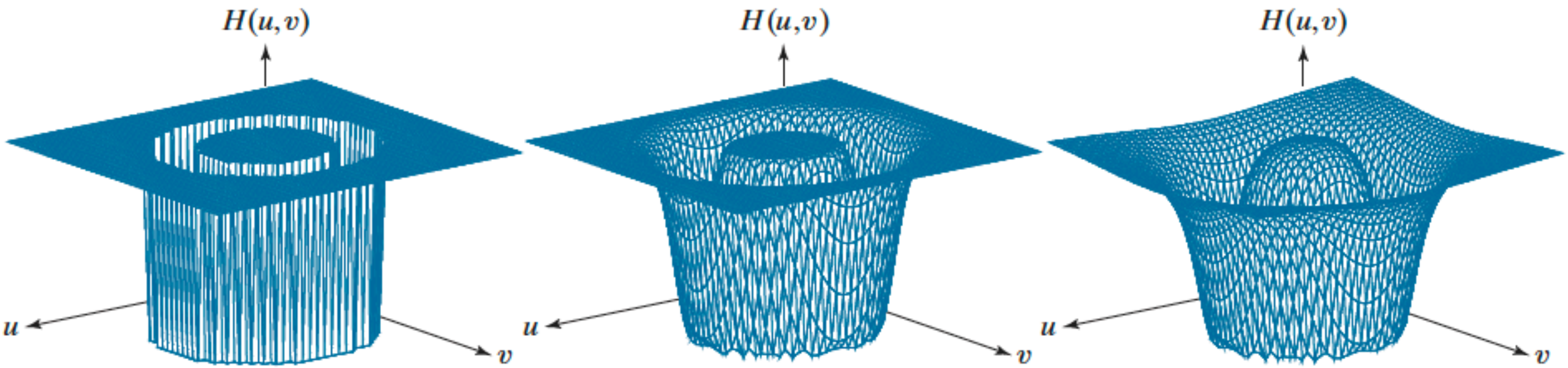


Ideal (IBRF)	Gaussian (GBRF)	Butterworth (BBRF)
$H(u,v) = \begin{cases} 0 & \text{if } C_0 - \frac{W}{2} \le D(u,v) \le C_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = 1 - e^{-\left[\frac{D^2(u,v) - C_0^2}{D(u,v)W}\right]^2}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{D(u,v)W}{D^2(u,v) - C_0^2}\right]^{2n}}$

4.10 Selective Filtering
- Bandreject and Bandpass Filters

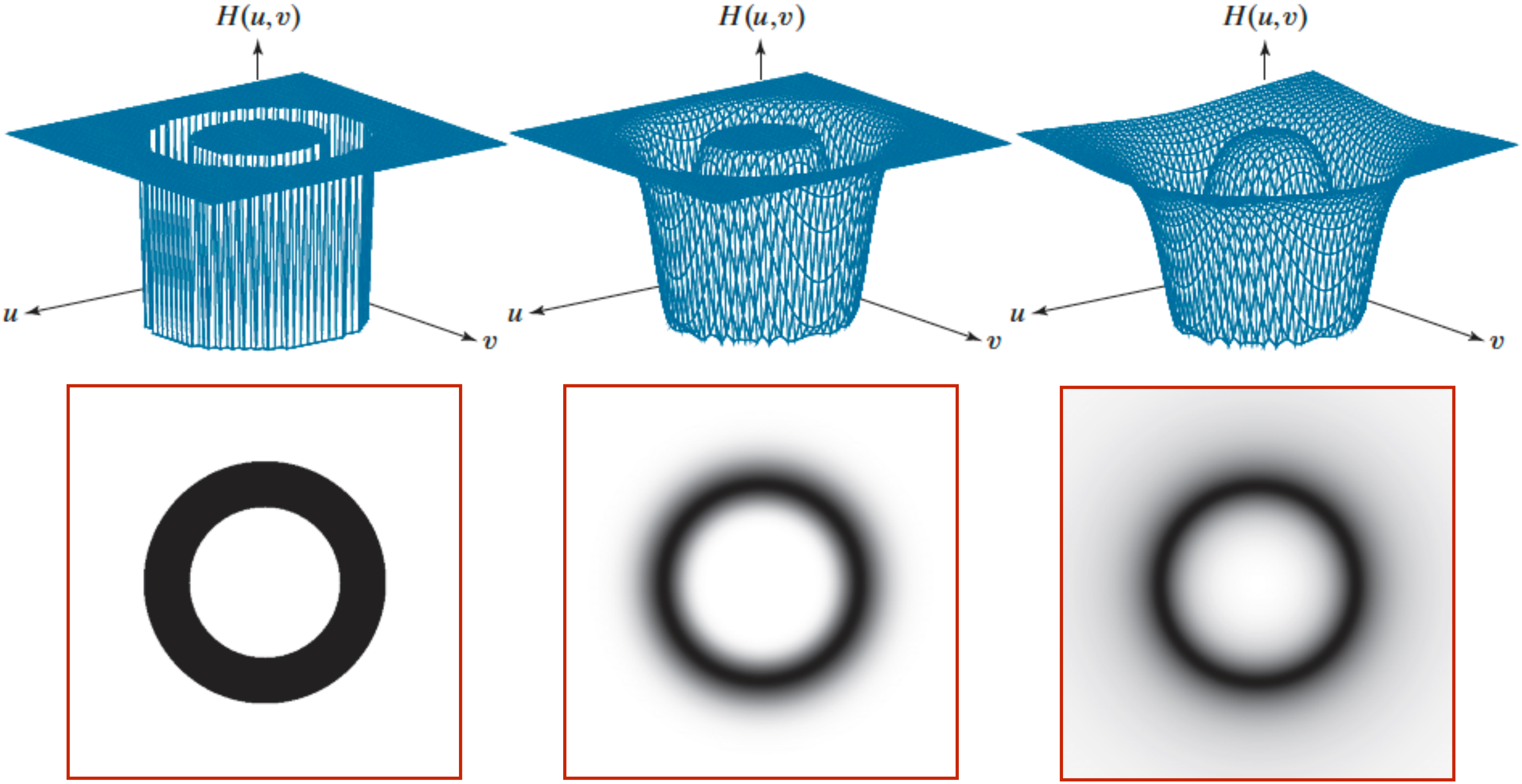
- Bandreject Filter (notch reject filters)

Ideal (IBRF)	Gaussian (GBRF)	Butterworth (BBRF)
$H(u,v) = \begin{cases} 0 & \text{if } C_0 - \frac{W}{2} \le D(u,v) \le C_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = 1 - e^{-\left[\frac{D^2(u,v) - C_0^2}{D(u,v)W}\right]^2}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{D(u,v)W}{D^2(u,v) - C_0^2}\right]^{2n}}$



4.10 Selective Filtering
- Bandreject and Bandpass Filters

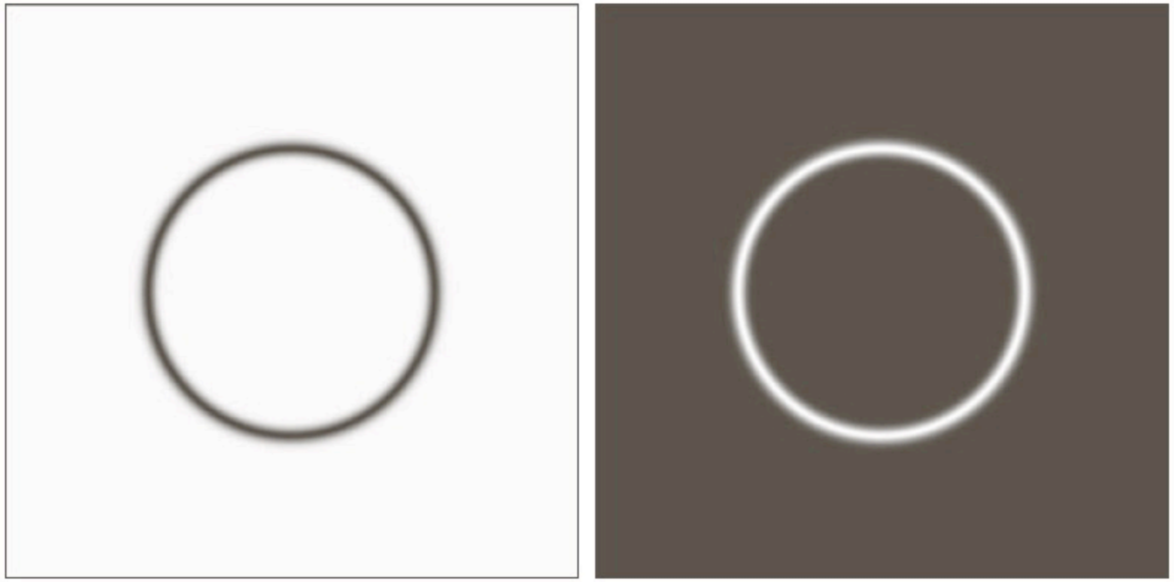
- Bandreject Filter (notch reject filters)



4.10 Selective Filtering
- Bandreject and Bandpass Filters

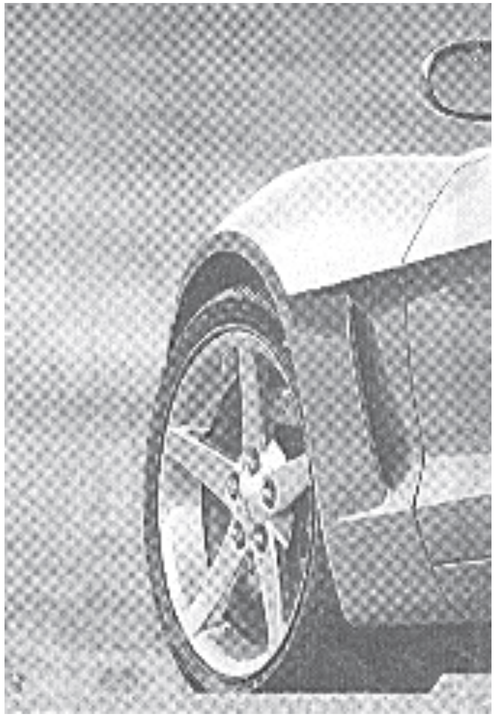
- Bandpass Filter (notch pass filters)

$$H_{\mathrm{BP}}(u,v) = 1 - H_{\mathrm{BR}}(u,v)$$

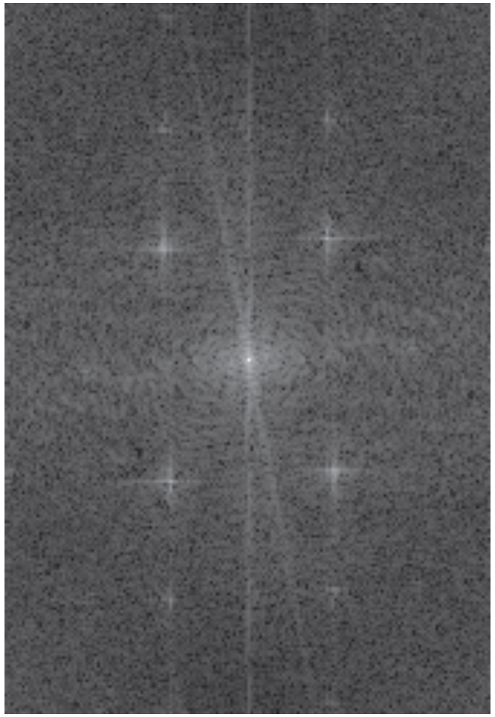


4.10 Selective Filtering
- Bandreject and Bandpass Filters

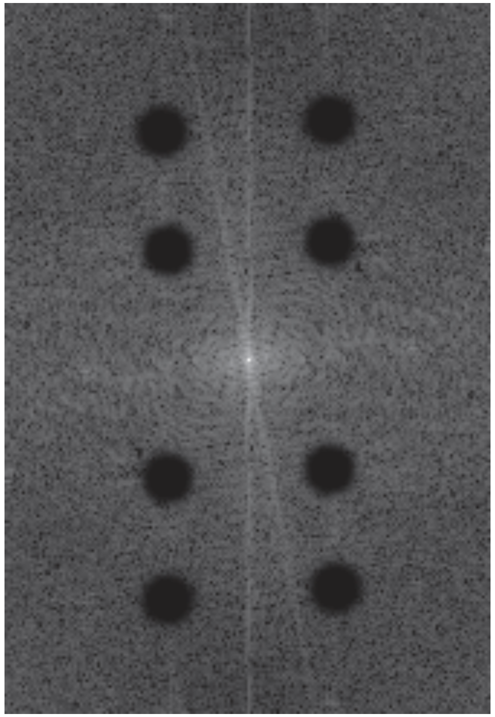
- Example



Sampled newspaper image



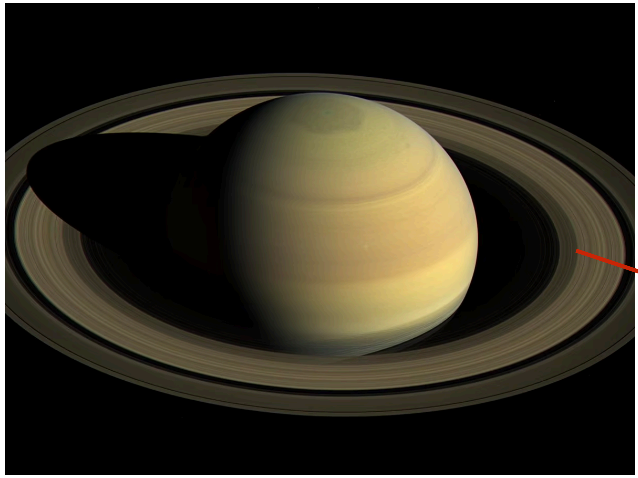
Fourier Spectrum



Butterworth notch reject filter

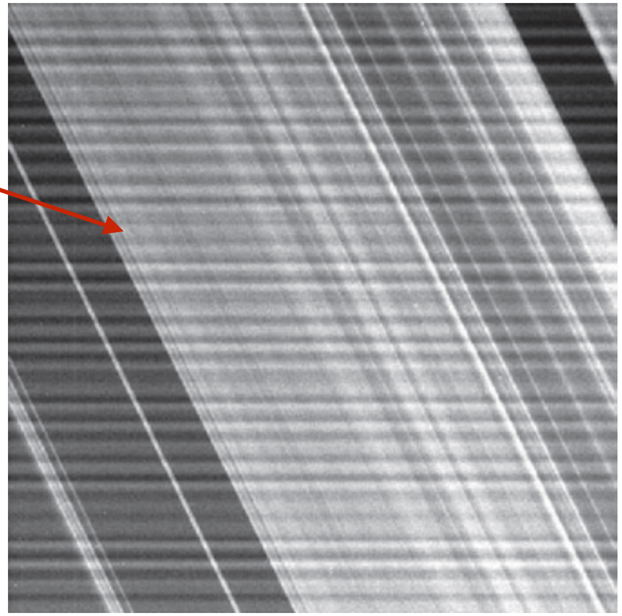
Filtered image

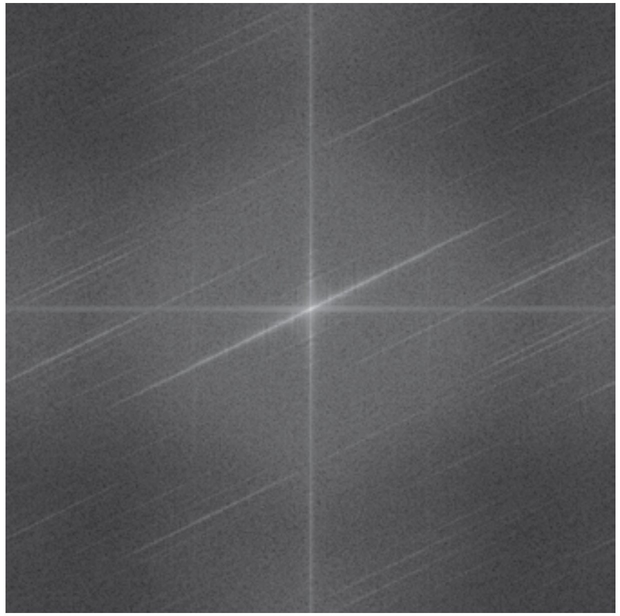
- Example



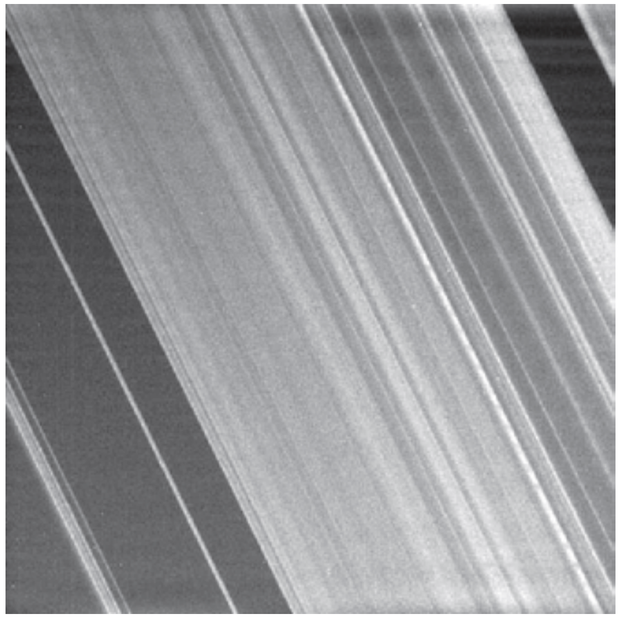
Fourier Spectrum

Original Image



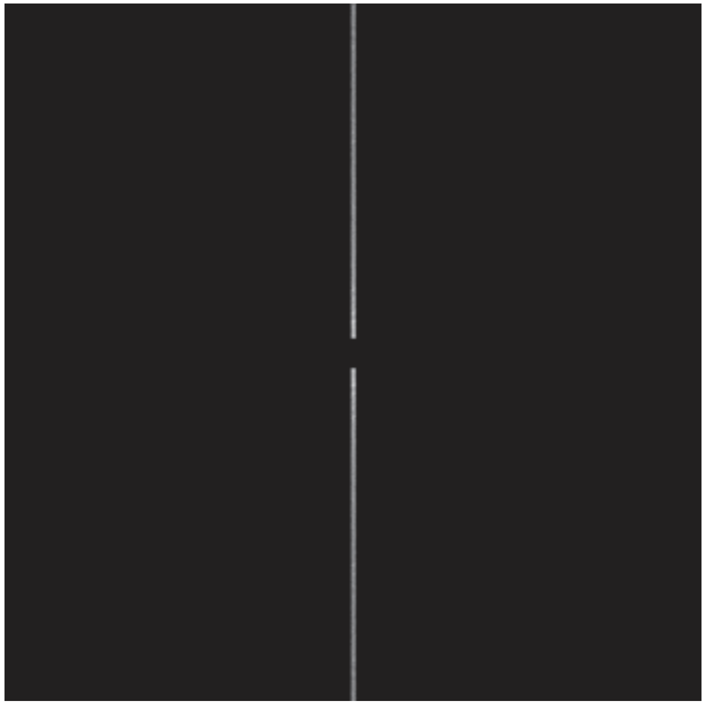


vertical notch
reject filter

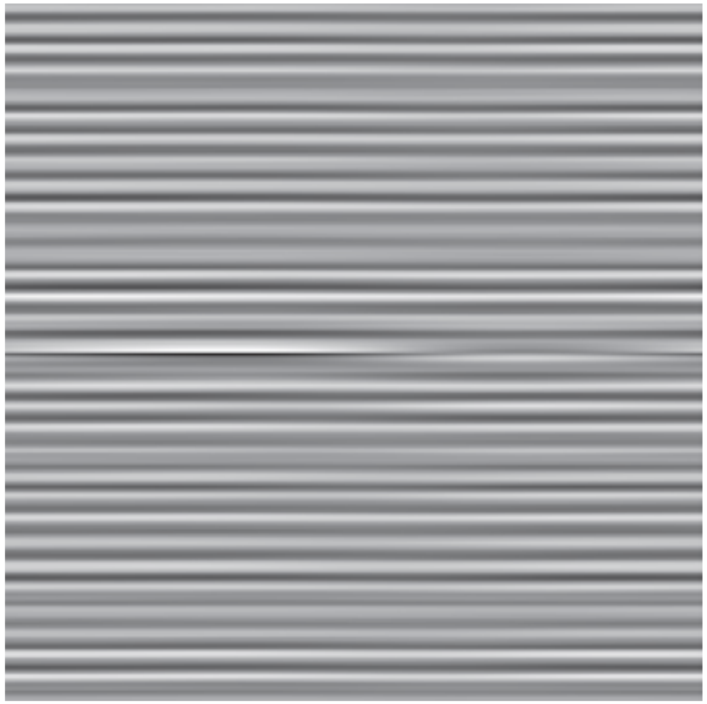


Filtered image

vertical notch
reject filter



vertical notch
pass filter



Its IDFT

4.11 The Fast Fourier Transform

- Fast Fourier Transform (FFT) algorithm reduces the complexity from $\mathbf{O((}MN\mathbf{)^2)}$ to $\mathbf{O(}MN\ \mathbf{log}MN\mathbf{)}$.

Skip the details of this subchapter

Introduction to Image Processing

Ch 4. Filtering in the Frequency Domain

Kuan-Wen Chen