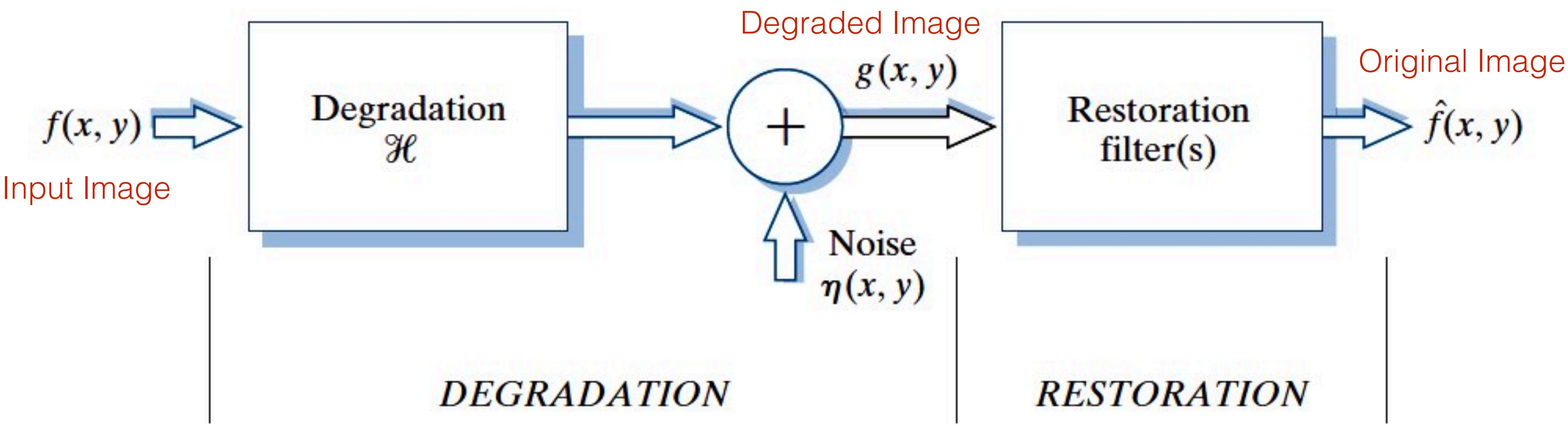
5.5 Linear, Position-Invariant Degradations

- Review - A Model of the Image Degradation/Restoration process (Sec. 5.1)



$$g(x,y) = (h \star f)(x,y) + \eta(x,y)$$

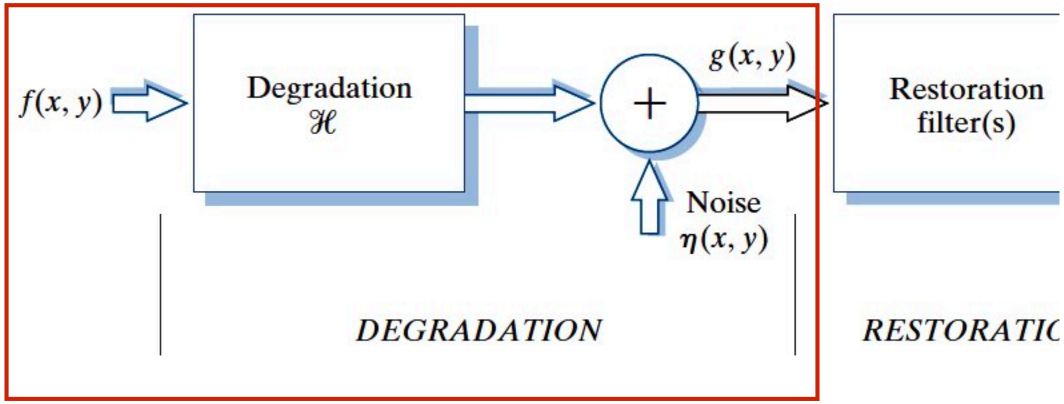
$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

5.5 Linear, Position-Invariant Degradations

- A general degradation stage can be expressed as

$$g(x,y) = \mathcal{H}[f(x,y)] + \eta(x,y)$$

degradation function



- If the degradation function H is *linear* and *position-invariant*, then we can derive (Sec. 5.5, but skip the details in this class) from the above equation to

$$g(x,y) = (h \star f)(x,y) + \eta(x,y)$$

$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

5.5 Linear, Position-Invariant Degradations

- H is *linear*, if

$$\mathcal{H}\left[af_1(x,y) + bf_2(x,y)\right] = a\mathcal{H}\left[f_1(x,y)\right] + b\mathcal{H}\left[f_2(x,y)\right]$$

  - If $a = b = 1$

$$\mathcal{H}\left[f_1(x,y) + f_2(x,y)\right] = \mathcal{H}\left[f_1(x,y)\right] + \mathcal{H}\left[f_2(x,y)\right]$$ **additivity**

  - If $f_2(x,y) = 0$

$$\mathcal{H}\left[af_1(x,y)\right] = a\mathcal{H}\left[f_1(x,y)\right]$$ **homogeneity**

  - Thus, a linear operator possesses both the property of *additivity* and the property of *homogeneity*.

5.5 Linear, Position-Invariant Degradations

- H is *position-invariant*, if

$$\mathcal{H}[f(x-\alpha, y-\beta)] = g(x-\alpha, y-\beta)$$

  - This definition indicates that the response at any point in the image depends only on the value of the input at that point, not on its position.

5.5 Linear, Position-Invariant Degradations

- In summary, a linear, spatially invariant degradation system with additive noise can be modeled as

$$g(x,y) = (h \star f)(x,y) + \eta(x,y)$$

- Because degradations are modeled as being the result of convolution, the restoration are called *image deconvolution* and the filters used are called *deconvolution filters*.

- In the following sections, we will learn how to estimate the degradation function and then apply inverse process to restore the image.

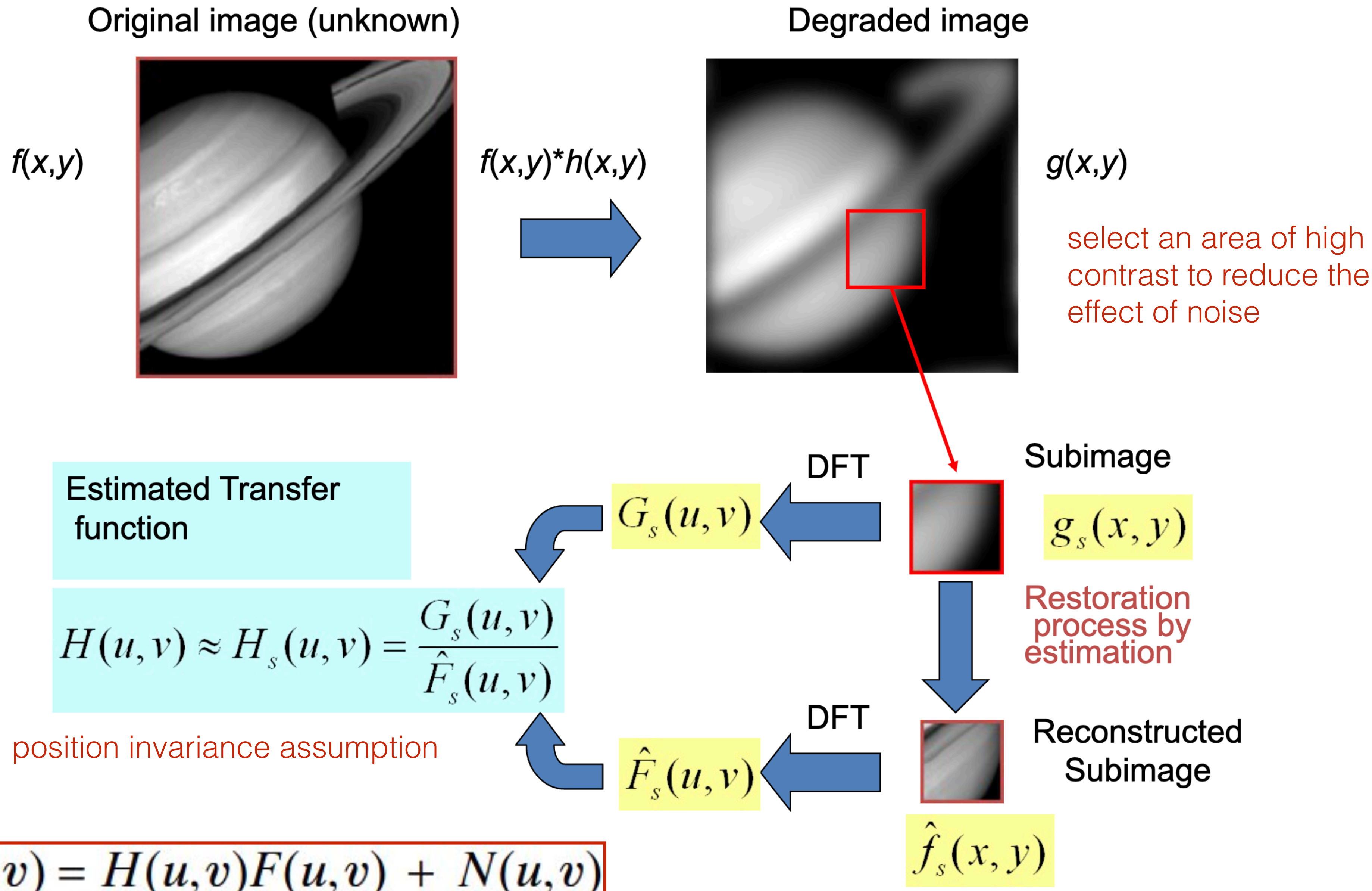
5.6 Estimating the Degradation Function

- There are three principal ways to estimate the degradation function for use in image restoration
    - observation
    - experimentation
    - mathematical modeling

- The process of restoring an image by using a degradation function that has been estimated by any of these approaches sometimes is called *blind deconvolution*

5.6 Estimating the Degradation Function

- **Estimation by Image Observation**



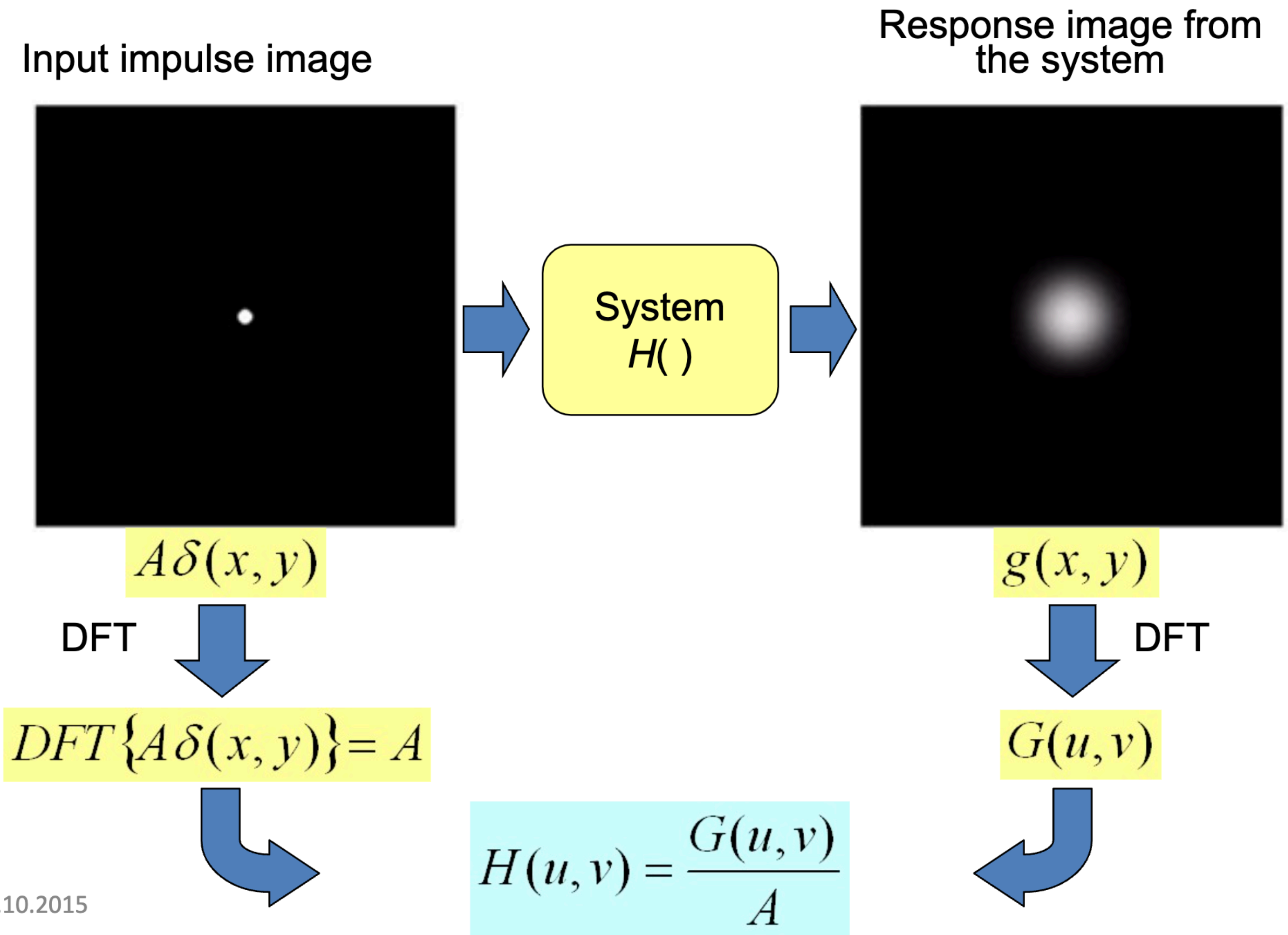
$$\boxed{G(u,v) = H(u,v)F(u,v) + N(u,v)}$$

credit of this slide: C. Paul

5.6 Estimating the Degradation Function

- **Estimation by Image Experimentation**

Used when we have the same equipment set up

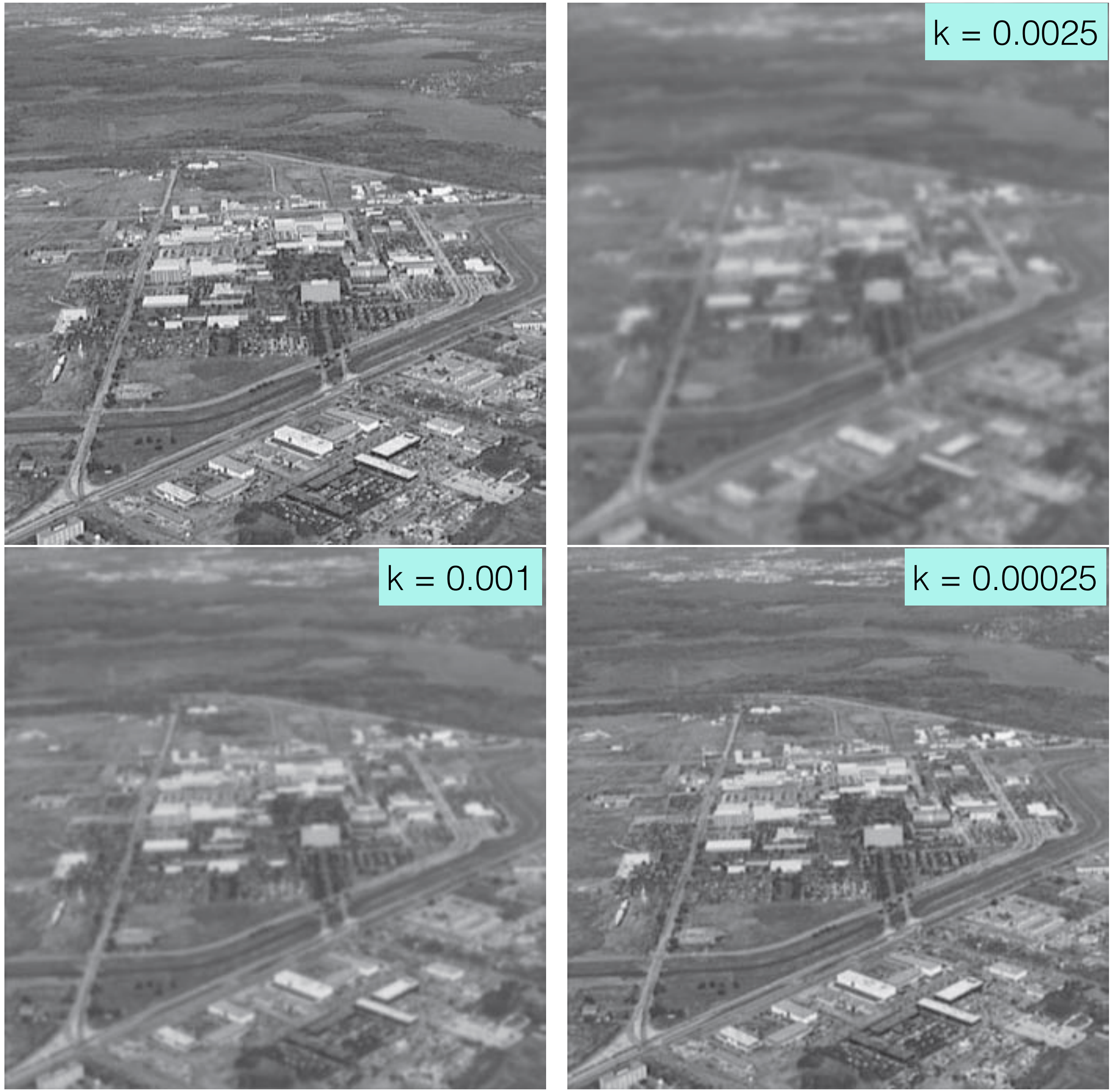


credit of this slide: C. Paul

5.6 Estimating the Degradation Function

- **Estimation by Modeling**

Used when we know physical mechanism underlying the image formation process that can be expressed mathematically



- Example: Atmospheric turbulence model

$$H(u,v) = e^{-k(u^2 + v^2)^{5/6}}$$

credit of this slide: C. Paul

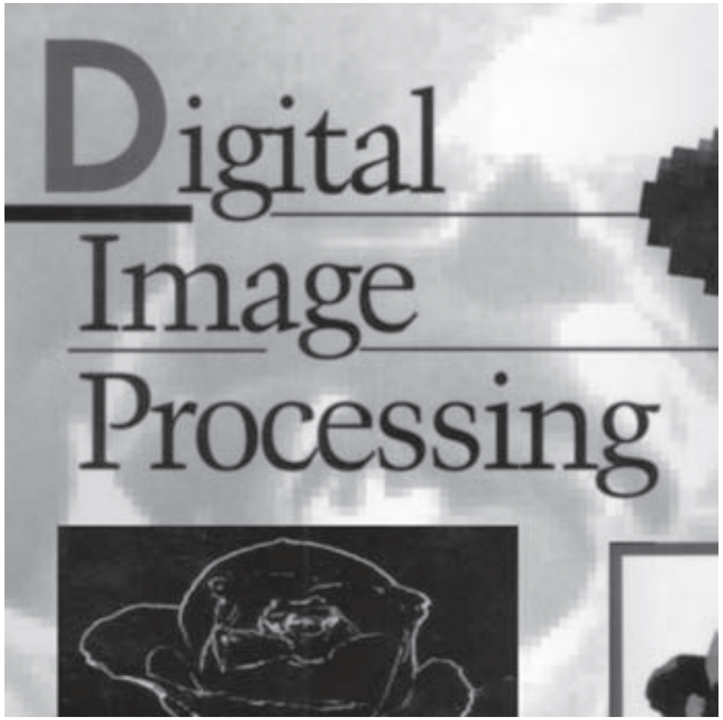
5.6 Estimating the Degradation Function

- **Estimation by Modeling**
  - Example: Image blurred by uniform linear motion

exposure time

$$g(x,y) = \int_0^T f[x - x_0(t),\, y - y_0(t)]\,dt$$

camera velocity



5.6 Estimating the Degradation Function

- **Estimation by Modeling**
  - Example: Image blurred by uniform linear motion

$$g(x,y) = \int_0^T f[x - x_0(t), y - y_0(t)]dt$$

  - Fourier transform

$$G(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(ux+vy)}dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\int_0^T f[x - x_0(t), y - y_0(t)]dt\right]e^{-j2\pi(ux+vy)}dx\,dy$$

$$= \int_0^T \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[x - x_0(t), y - y_0(t)]e^{-j2\pi(ux+vy)}dx\,dy\right]dt$$

5.6 Estimating the Degradation Function

- **Estimation by Modeling**

$$G(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(ux+vy)}dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\int_{0}^{T} f[x - x_0(t), y - y_0(t)]dt\right]e^{-j2\pi(ux+vy)}dx\,dy$$

$$= \int_{0}^{T}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[x - x_0(t), y - y_0(t)]e^{-j2\pi(ux+vy)}dx\,dy\right]dt$$

$$= \int_{0}^{T} F(u,v)\mathrm{e}^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

$$= F(u,v)\int_{0}^{T} \mathrm{e}^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

5.6 Estimating the Degradation Function

- **Estimation by Modeling**

$$G(u,v) = F(u,v)\underline{\int_0^T \mathrm{e}^{-j2\pi[ux_0(t)+vy_0(t)]}\,dt}$$

Let $H(u,v) = \int_0^T \mathrm{e}^{-j2\pi[ux_0(t)+vy_0(t)]}\,dt$

We have $G(u,v) = H(u,v)F(u,v)$

For constant motion: $x_0(t) = at/T$ and $y_0(t) = bt/T$

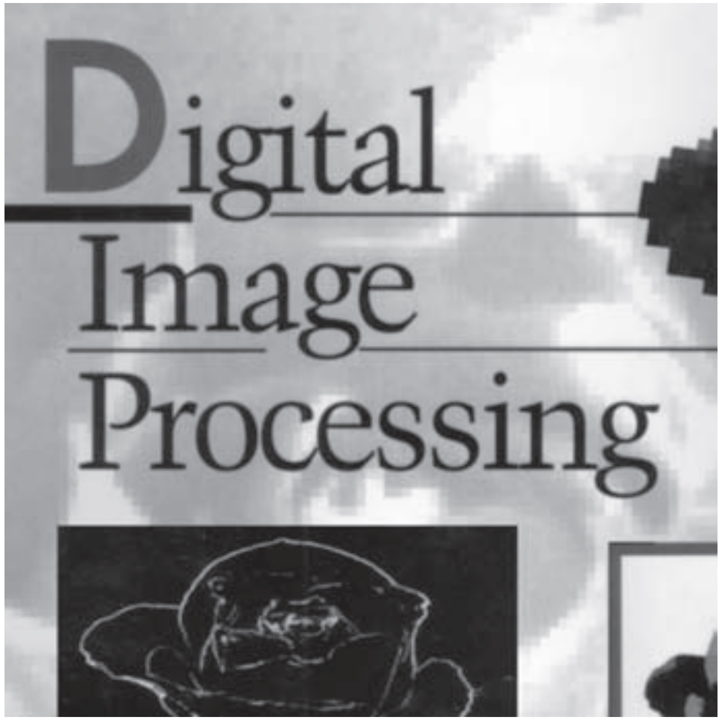
$$H(u,v) = \frac{T}{\pi(ua+vb)}\sin[\pi(ua+vb)]e^{-j\pi(ua+vb)}$$

$$\boxed{H(u,v) = \int_0^T \mathrm{e}^{-j2\pi ux_0(t)}\,dt = \int_0^T \mathrm{e}^{-j2\pi uat/T}\,dt = \frac{T}{\pi ua}\sin(\pi ua)e^{-j\pi ua}}$$

5.6 Estimating the Degradation Function

- **Estimation by Modeling**
  - Example: Image blurred by uniform linear motion

$$H(u,v) = \frac{T}{\pi(ua + vb)} \sin[\pi(ua + vb)] e^{-j\pi(ua + vb)}$$



a = b = 0.1 and T = 1

5.6 Estimating the Degradation Function

- **Estimation by Modeling**
  - Example: Radial distortion

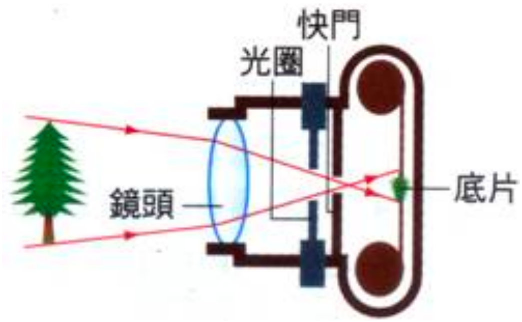
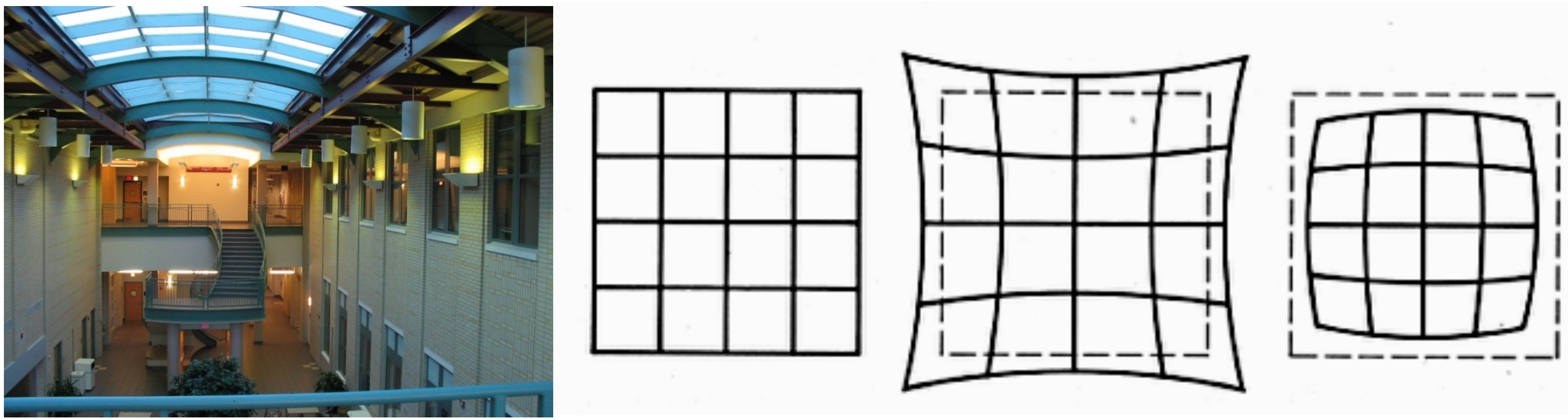


figure from: https://www.phyworld.idv.tw/Nature/Jun_2/htm/B3_4-4_POINT.html



- Radial distortion of the image
  - Caused by imperfect lenses
  - Deviations are most noticeable for rays that pass through the edge of the lens

credit of this slide: Y. Y. Chuang

5.6 Estimating the Degradation Function

- **Estimation by Modeling**
  - Example: Radial distortion

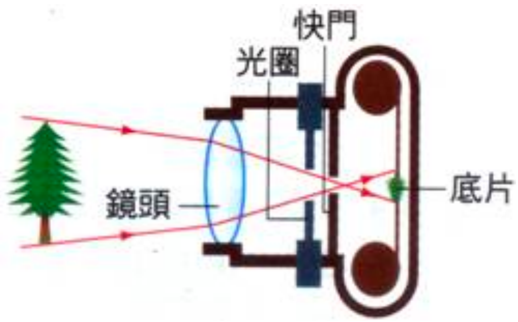


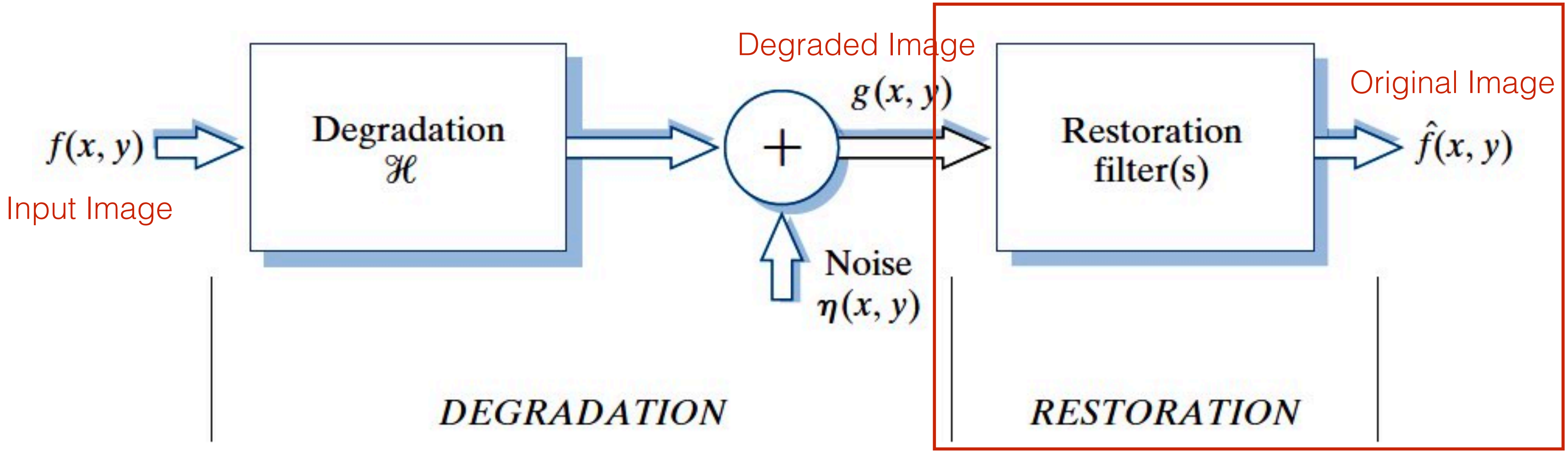
figure from: https://www.phyworld.idv.tw/Nature/Jun_2/htm/B3_4-4_POINT.html

$$x'' = x'*(1 + k_1r^2 + k_2r^4) + 2*p_1x'*y' + p_2(r^2+2*x'^2)$$
$$y'' = y'*(1 + k_1r^2 + k_2r^4) + p_1(r^2+2*y'^2) + 2*p_2*x'*y'$$
$$\text{where } r^2 = x'^2+y'^2$$

dewarp

5.7 Inverse Filtering

- Review - A Model of the Image Degradation/Restoration process (Sec. 5.1)



$$g(x,y) = (h \star f)(x,y) + \eta(x,y)$$

$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

The simplest approach to restoration is direct **inverse filtering**

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)}$$

5.7 Inverse Filtering

The simplest approach to restoration is **direct inverse filtering**

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)}$$

$$\boxed{G(u,v) = H(u,v)F(u,v) + N(u,v)}$$

- But, even if we know ***H***, we cannot recover the undegraded image ***F*** exactly because ***N*** is unknown

$$\hat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

- When there is no noise, then $\hat{F}(u,v) = F(u,v)$
- If ***H*** is 0 or very small, ***N***/***H*** will dominate

Solution: limit the filter frequencies to values near the origin

5.7 Inverse Filtering

Solution: limit the filter frequencies to values near the origin

- We know that *H*(0,0) is usually the highest value of *H*(u, v) in the frequency domain
- Thus, by limiting the analysis to frequencies near the origin, we reduce the likelihood of encountering zero values.

Direct inverse filtering:

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)}$$

Pseudo-inverse filtering:

$$\hat{F}(u,v) = \begin{cases} \dfrac{G(u,v)}{H(u,v)} & |H(u,v)| > \delta \\ 0 & |H(u,v)| \le \delta \end{cases}$$

Radially limited inverse filtering:

$$\hat{F}(u,v) = \begin{cases} \dfrac{G(u,v)}{H(u,v)} & \sqrt{u^2+v^2} \le R \\ 0 & \sqrt{u^2+v^2} > R \end{cases}$$

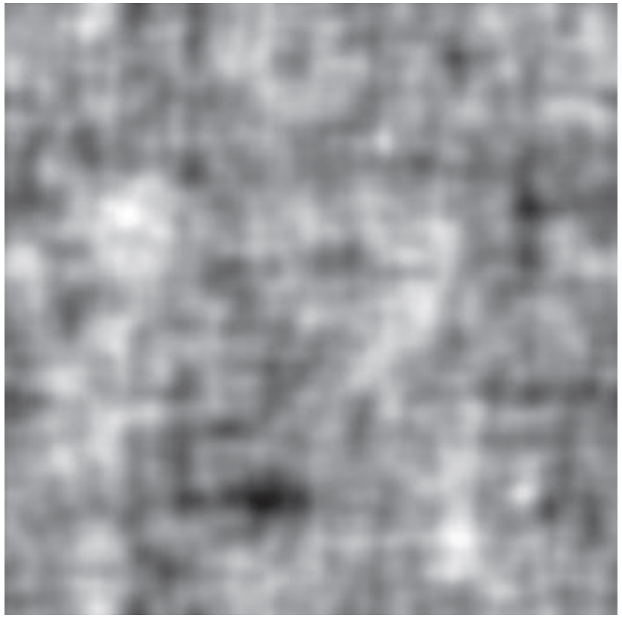
5.7 Inverse Filtering

- Example of radially limited inverse filtering:
    - Atmospheric turbulence model $H(u,v) = e^{-k(u^2 + v^2)^{5/6}}$

- Example of radially limited inverse filtering:

$$H(u,v) = e^{-k(u^2 + v^2)^{5/6}}$$

Direct inverse filter



cut off radius of 40

cut off radius of 70

cut off radius of 85

The basic theme of the following three sections is how to improve on direct inverse filtering

5.8 Minimum Mean Square Error (Wiener) Filtering

考問答題

- So far we assumed nothing about the statistical properties of the image and noise.
- We now consider image and noise as random variables and the objective is to find an estimate of the uncorrupted image f such that the mean square error between the estimate and the image is minimized:

$$\min_{\hat{\mathbf{f}}} \left\{ E\left[(\mathbf{f} - \hat{\mathbf{f}})^2 \right] \right\}$$

where $E[\mathbf{x}]$ is the expected value of vector $\mathbf{x}$.

- The result is known as the *Wiener filter*.

(*minimum mean square error filter* or *the least square error filter*)

credit of this slide: C. Nikou

5.8 Minimum Mean Square Error (Wiener) Filtering

- Assumption:
  - The noise and the image are uncorrelated
  - The intensity levels in the estimate are a linear function of the levels in the degraded image, i.e. $\hat{\mathbf{f}} = \mathbf{P}\mathbf{g}$

the goal is to find the best matrix **P** for $\boxed{\min_{\hat{\mathbf{f}}} \left\{ E\left[(\mathbf{f} - \hat{\mathbf{f}})^2 \right] \right\}}$

$$\hat{F}(u,v) = \left[\frac{H^*(u,v) S_f(u,v)}{S_f(u,v) |H(u,v)|^2 + S_\eta(u,v)} \right] G(u,v)$$

$$= \left[\frac{H^*(u,v)}{|H(u,v)|^2 + S_\eta(u,v) / S_f(u,v)} \right] G(u,v)$$

$$= \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_\eta(u,v) / S_f(u,v)} \right] G(u,v)$$

5.8 Minimum Mean Square Error (Wiener) Filtering

$$\hat{F}(u,v)=\left[\frac{1}{H(u,v)}\frac{|H(u,v)|^2}{|H(u,v)|^2+\underline{S_\eta(u,v)/S_f(u,v)}}\right]G(u,v)$$

unknown

1. $\hat{F}(u,v)$ = Fourier transform of the estimate of the undegraded image.
2. $G(u,v)$ = Fourier transform of the degraded image.
3. $H(u,v)$ = degradation transfer function (Fourier transform of the spatial degradation).
4. $H^*(u,v)$ = complex conjugate of $H(u,v)$.
5. $|H(u,v)|^2 = H^*(u,v)H(u,v)$.
6. $S_\eta(u,v) = |N(u,v)|^2$ = power spectrum of the noise [see Eq. (4-89)]†
7. $S_f(u,v) = |F(u,v)|^2$ = power spectrum of the undegraded image.

assumed white noise, the spectrum is a constant

signal-to-noise ratio: $$\mathrm{SNR}=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F(u,v)|^2\Big/\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|N(u,v)|^2$$

- If the noise is zero, the Wiener filter reduces to the inverse filter.

5.8 Minimum Mean Square Error (Wiener) Filtering

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)} \right] G(u,v)$$

approximation

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + K} \right] G(u,v)$$

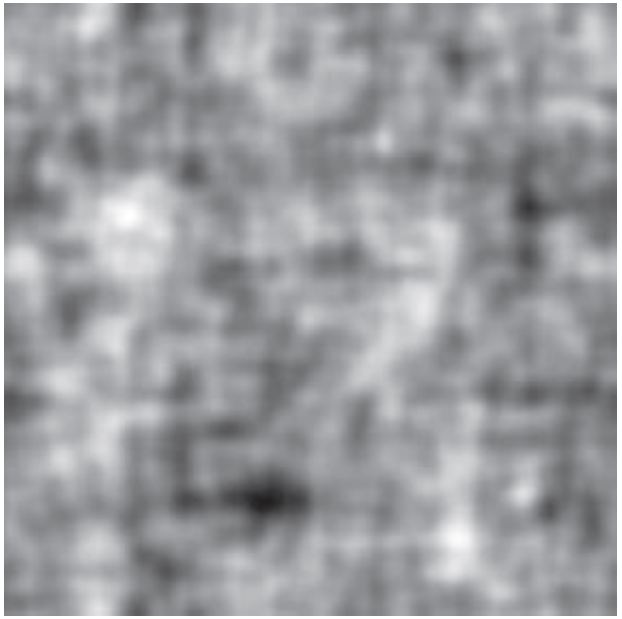
K was chosen interactively to yield the best visual results.

- Example of Wiener filtering:

Original image

Atmospheric turbulence image

Direct inverse filter



Radially limited inverse filter

Wiener filter

- Example of Wiener filtering:

Direct inverse filter | Wiener filter

motion blur and additive noise image

noise variance one order of magnitude less

noise variance five orders of magnitude less

5.9 Constrained Least Squares Filtering

Wiener filter $\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)} \right] G(u,v)$

- When we do not have information on the power spectra the Wiener filter is not optimal.
- Another idea is to introduce a smoothness term in our criterion, such as the second derivative of an image (Laplacian)

Minimize $C = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[\nabla^2 f(x,y) \right]^2$ subject to $\|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 = \|\boldsymbol{\eta}\|^2$

credit of this slide: C. Nikou

5.9 Constrained Least Squares Filtering

Minimize $C = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[\nabla^2 f(x,y) \right]^2$ subject to $\left\| \mathbf{g} - \mathbf{H}\hat{\mathbf{f}} \right\|^2 = \left\| \boldsymbol{\eta} \right\|^2$

- The optimal frequency domain solution:

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + \gamma \left|P(u,v)\right|^2} \right] G(u,v)$$

- γ: control the degree of smoothness.
  - If $\gamma = 0$, inverse filter
  - If $\gamma = \infty$, f = 0, i.e. ultra smooth solution
- $P(u, v)$: Fourier transform of the function

$$p(x,y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

5.9 Constrained Least Squares Filtering

- Comparison between Wiener filter and constrained least squares filter

Wiener filter:

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + \underline{S_\eta(u,v)/S_f(u,v)}} \right] G(u,v)$$

can be approximated as *K and K* was chosen interactively

Constrained least squares filter:

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \underline{\gamma |P(u,v)|^2}} \right] G(u,v)$$

γ can be chosen interactively or iterative estimation

- Comparison between Wiener filter and constrained least squares filter

Both K *and* γ were selected manually to yield the best visual results

motion blur and additive noise image

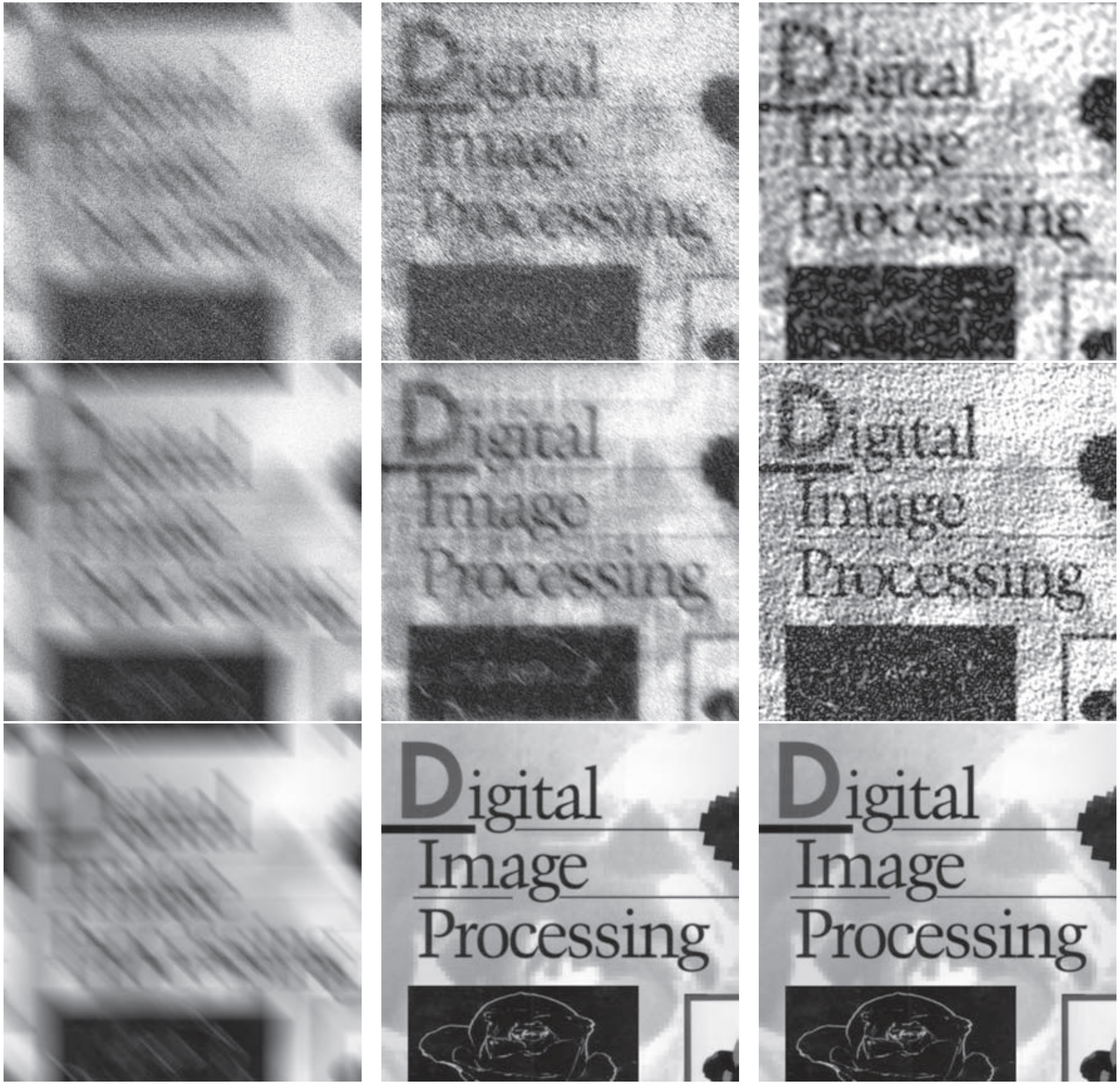
constrained least squares filter yielded better results, because γ is a true scalar but *K* is a scalar approximation to the ratio of two unknown

noise variance one order of magnitude less

noise variance five orders of magnitude less

Wiener filter

constrained least squares filter



5.10 Geometric Mean Filter

This filter represents a family of filters combined into a single expression

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2}\right]^{\alpha} \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \beta\left[\dfrac{S_\eta(u,v)}{S_f(u,v)}\right]}\right]^{1-\alpha} G(u,v)$$

$\alpha = 1$ → the inverse filter
$\alpha = 0$ → the Parametric Wiener filter
$\alpha = 0,\ \beta = 1$ → the standard Wiener filter
$\beta = 1,\ \alpha < 0.5$ → More like the inverse filter
$\beta = 1,\ \alpha > 0.5$ → More like the Wiener filter

Another name: the spectrum equalization filter

credit of this slide: K. Vant

5.11 Image Reconstruction from Projections

- This section deals with the problem of reconstructing an image from a series of projections, with a focus on X-ray computed tomography (CT).

Skip the details of this subchapter

Introduction to Image Processing

Ch 5. Image Restoration and Reconstruction

Kuan-Wen Chen