4.4 The Discrete Fourier Transform of One Variable

- Fourier Transform

$$F(\mu)=\int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

(1) $$\boxed{\tilde{f}(t)=f(t)s_{\Delta T}(t)=\sum_{n=-\infty}^{\infty} f(t)\delta(t-n\Delta T)}$$

(2) $$\boxed{f_k=\int_{-\infty}^{\infty} f(t)\delta(t-k\Delta T)dt=f(k\Delta T)}$$

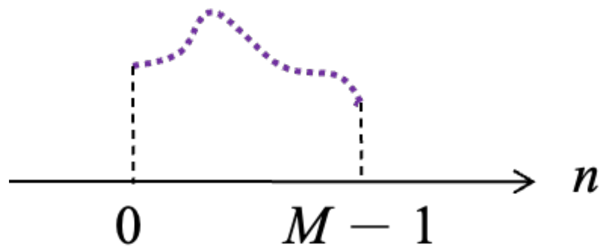
- Discrete Fourier Transform (DFT)

$$\tilde{F}(\mu)=\int_{-\infty}^{\infty} \tilde{f}(t)e^{-j2\pi\mu t}dt$$

$$=\int_{-\infty}^{\infty} \tilde{f}(t)e^{-j2\pi\mu t}dt \overset{(1)}{=} \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(t)\delta(t-n\Delta T)e^{-j2\pi\mu t}dt$$

$$=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\delta(t-n\Delta T)e^{-j2\pi\mu t}dt$$

$$\overset{(2)}{=}\sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}=\sum_{n=0}^{M-1} f_n e^{-j2\pi\mu n\Delta T}$$



4.4 The Discrete Fourier Transform of One Variable

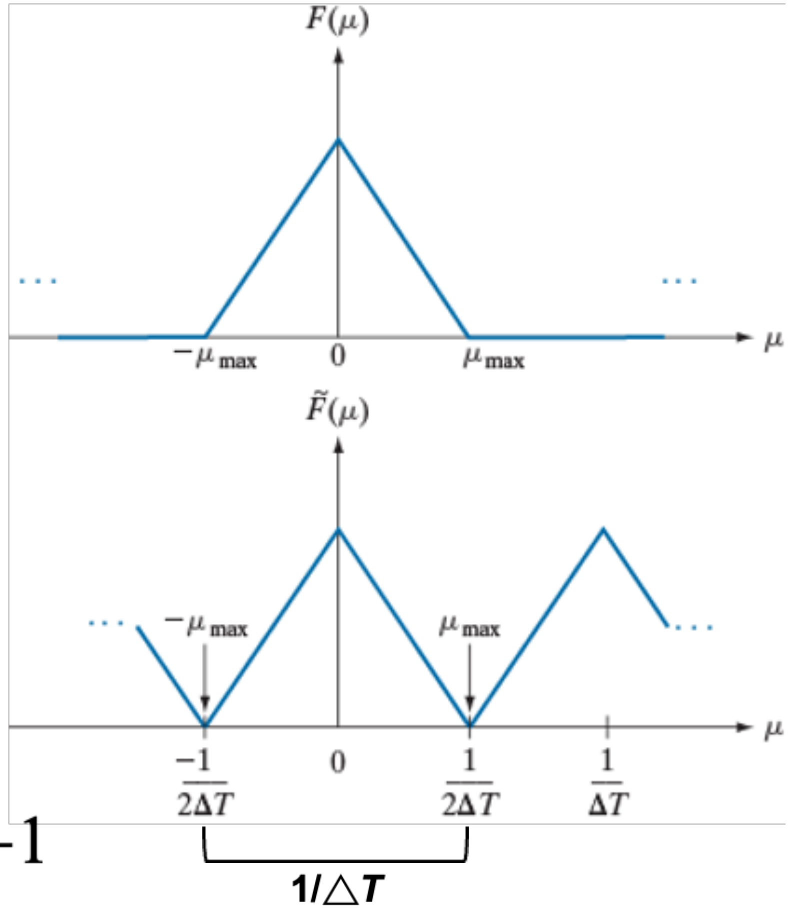
- Discrete Fourier Transform (DFT)

$$\tilde{F}(\mu) = \sum_{n=0}^{M-1} f_n e^{-j2\pi\mu n\Delta T}$$

Obtain *M* equally spaced samples of $\tilde{F}(\mu)$ taken over the period μ = 0 to μ = 1/△*T*

$$\mu = \frac{m}{M\Delta T} \qquad m = 0,1,2,\ldots,M-1$$

$$\tilde{F}(\mu) = \sum_{n=0}^{M-1} f_n e^{-j2\pi\mu n\Delta T}$$



$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0,1,2,\ldots,M-1$$

credit of this slide: Y. P. Hung

4.4 The Discrete Fourier Transform of One Variable

- Inverse Discrete Fourier Transform (IDFT)

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \quad n = 0, 1, 2, \ldots, M-1$$

- 1-D Discrete Fourier Transform (DFT) pair

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad u = 0,1,2,\ldots,M-1$$

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \quad x = 0,1,2,\ldots,M-1$$

more intuitive, especially in two dimensions, to use the notation x and y for image coordinate variables and u and v for frequency variables

4.4 The Discrete Fourier Transform of One Variable

- Inverse Discrete Fourier Transform (IDFT)

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \quad n = 0, 1, 2, \ldots, M-1$$

- 1-D Discrete Fourier Transform (DFT) pair

$$F(u) = \sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \quad u = 0,1,2,\ldots,M-1$$

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \quad x = 0,1,2,\ldots,M-1$$

more intuitive, especially in two dimensions, to use the notation x and y for image coordinate variables and u and v for frequency variables

4.4 The Discrete Fourier Transform of One Variable

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad u = 0,1,2,\ldots,M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \quad x = 0,1,2,\ldots,M-1$$

- **Property**: both transforms are infinitely periodic, with period *M*. (more properties will be discussed in Sec. 4.6)

$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM)$$

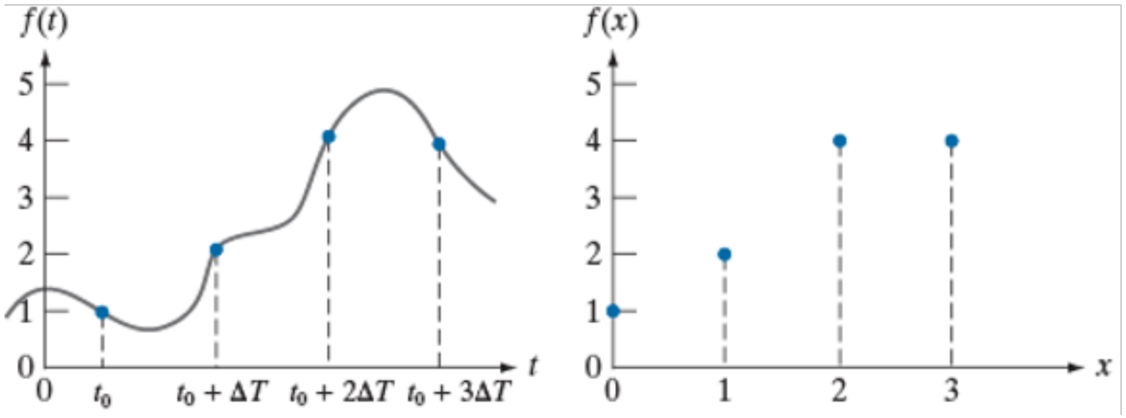
Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

The complex exponential function is periodic

4.4 The Discrete Fourier Transform of One Variable

- The mechanics of computing the DFT



$$F(u)=\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}$$

$$f(x)=\frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M}$$

$$F(0)=\sum_{x=0}^{3} f(x)=f(0)+f(1)+f(2)+f(3)=1+2+4+4=11$$

會考

$$F(1)=\sum_{x=0}^{3} f(x)e^{-j2\pi(1)x/4}=1e^{0}+2e^{-j\pi/2}+4e^{-j\pi}+4e^{-j3\pi/2}=-3+2j$$

Euler's formula

$$e^{j\theta}=\cos\theta+j\sin\theta$$

$$F(2)=-(1+0j) \quad , \quad F(3)=-(3+2j)$$

$$f(0)=\frac{1}{4}\sum_{u=0}^{3} F(u)e^{j2\pi u(0)/4}=\frac{1}{4}\sum_{u=0}^{3} F(u)=\frac{1}{4}[11-3+2j-1-3-2j]=\frac{1}{4}[4]=1$$

4.5 Extensions to Functions of Two Variables - The 2D Impulse and its Sifting Property

- The impulse of two continuous variables

$$\delta(t,z) = \begin{cases} 1 & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)\,dt\,dz = 1$$

- Sifting property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)\,dt\,dz = f(0,0)$$

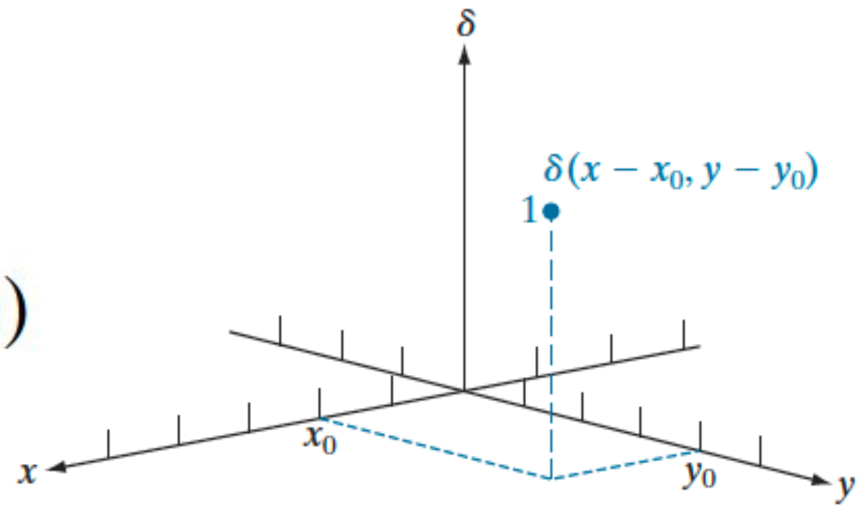
- Sifting property at (t_0, z_0)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t-t_0, z-z_0)\,dt\,dz = f(t_0,z_0)$$

4.5 Extensions to Functions of Two Variables - The 2D Impulse and its Sifting Property

- The impulse of two discrete variables

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$



- Sifting property

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y) = f(0,0)$$

- Sifting property at (x_0, y_0)

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0,y-y_0) = f(x_0,y_0)$$

4.5 Extensions to Functions of Two Variables - The 2-D Continuous Fourier Transform Pairs

- 2D Fourier transform pair:

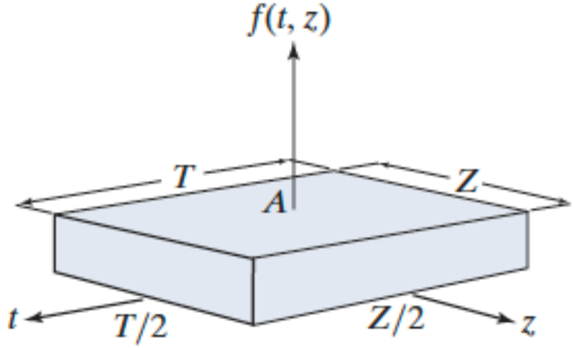
$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t + \nu z)}\,dt\,dz$$

$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu)e^{j2\pi(\mu t + \nu z)}\,d\mu\,d\nu$$

$$\begin{cases} F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt \\ f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu \end{cases}$$

4.5 Extensions to Functions of Two Variables - The 2-D Continuous Fourier Transform Pairs

- Example of the *Fourier transform*



- **Fourier transform:**

f(t, z)

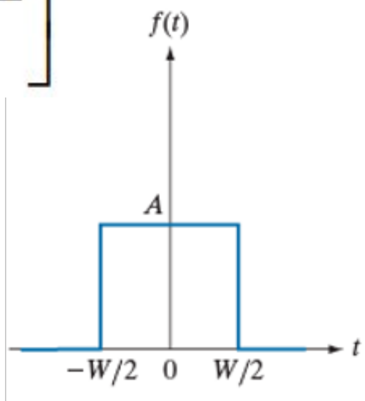
$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}\,dt\,dz$$

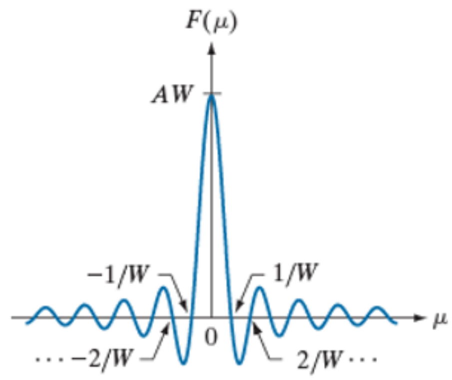
$$= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} Ae^{-j2\pi(\mu t+\nu z)}\,dt\,dz$$

$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$

$$F(\mu) = AW\text{sinc}(\mu W)$$

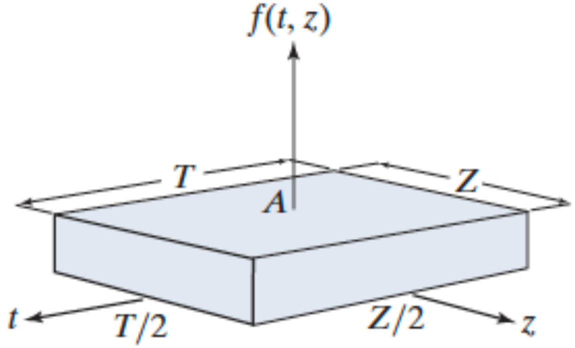
$$\text{sinc}(\text{m}) = \frac{\sin(\pi\text{m})}{(\pi\text{m})}$$





4.5 Extensions to Functions of Two Variables - The 2-D Continuous Fourier Transform Pairs

- Example of the *Fourier transform*



- **Fourier transform:**

$f(t, z)$

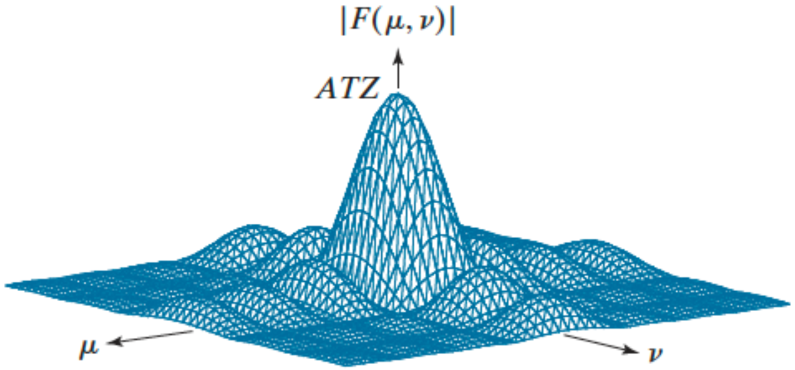
$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t + \nu z)}\,dt\,dz$$

$$= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} A e^{-j2\pi(\mu t + \nu z)}\,dt\,dz$$

$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$

The box is longer along the t-axis, so the spectrum is more contracted along the μ-axis.

- **Fourier spectrum:** $|F(\mu, \nu)|$

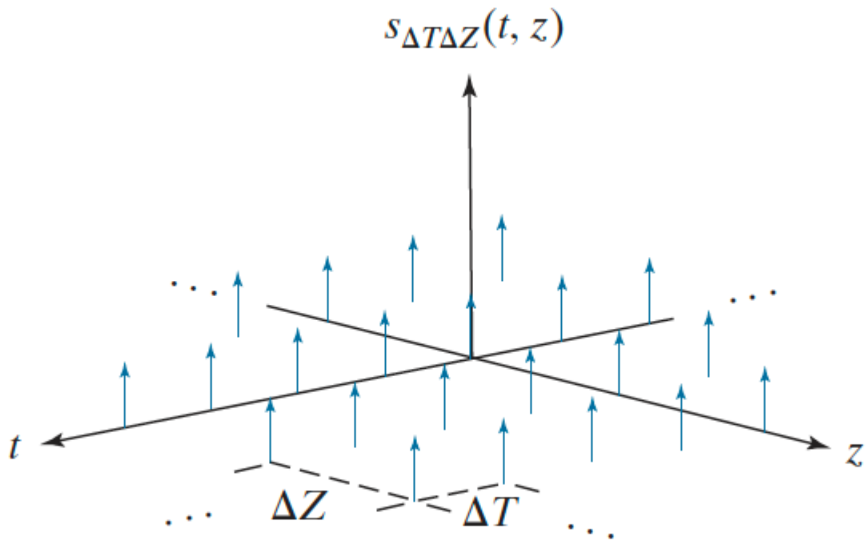


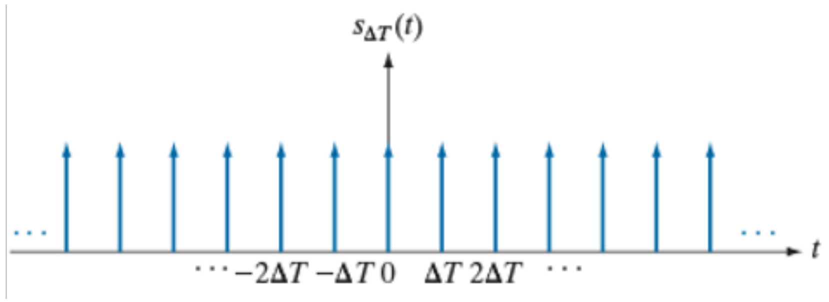
4.5 Extensions to Functions of Two Variables - 2D Sampling and the 2D Sampling Theorem

- 2D Impulse Train

$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$





4.5 Extensions to Functions of Two Variables - 2D Sampling and the 2D Sampling Theorem

- Band-limited function

A function $f(t, z)$ whose Fourier transform is zero for values of frequencies outside a rectangle by the intervals $[-\mu_{max}, \mu_{max}]$ and $[-\nu_{max}, \nu_{max}]$

$$F(\mu,\nu) = 0 \quad \text{for} \quad |\mu| \geq \mu_{\max} \quad \text{and} \quad |\nu| \geq \nu_{\max}$$

- 2D Sampling Theorem

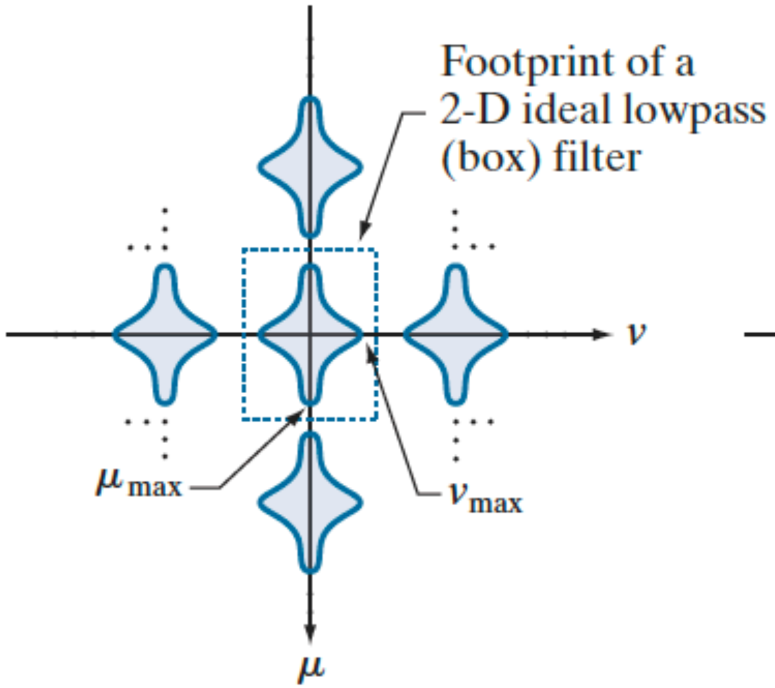
A continuous, band-limited function can be recovered completely from a set of its samples if

$$\frac{1}{\Delta T} > 2\mu_{\max} \quad \text{and} \quad \frac{1}{\Delta Z} > 2\nu_{\max}$$

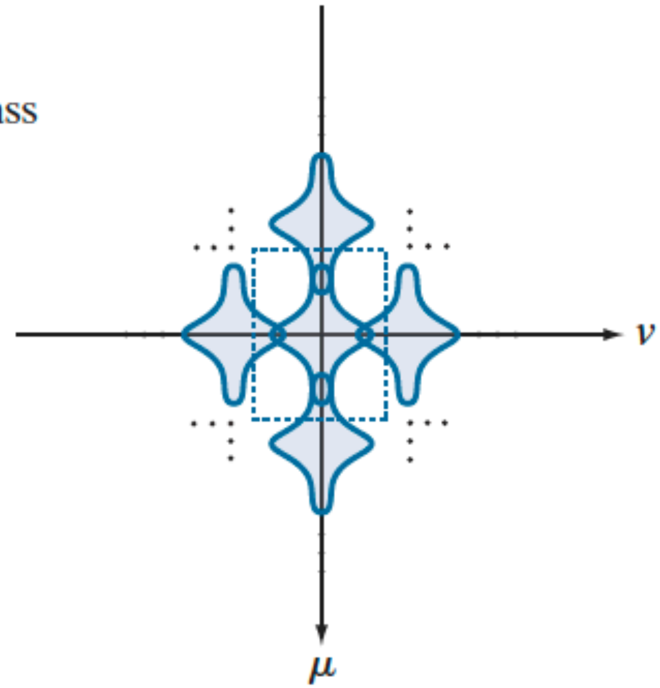
Nyquist rate $\frac{1}{\Delta T} > 2\mu_{\max}$

4.5 Extensions to Functions of Two Variables - 2D Sampling and the 2D Sampling Theorem

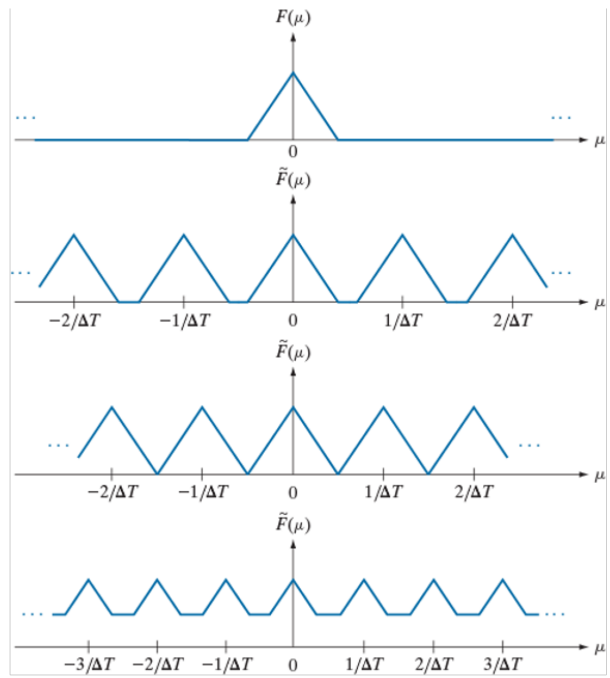
- 2D Sampling Theorem $\dfrac{1}{\Delta T} > 2\mu_{\max}$ and $\dfrac{1}{\Delta Z} > 2\nu_{\max}$



Over-sampled

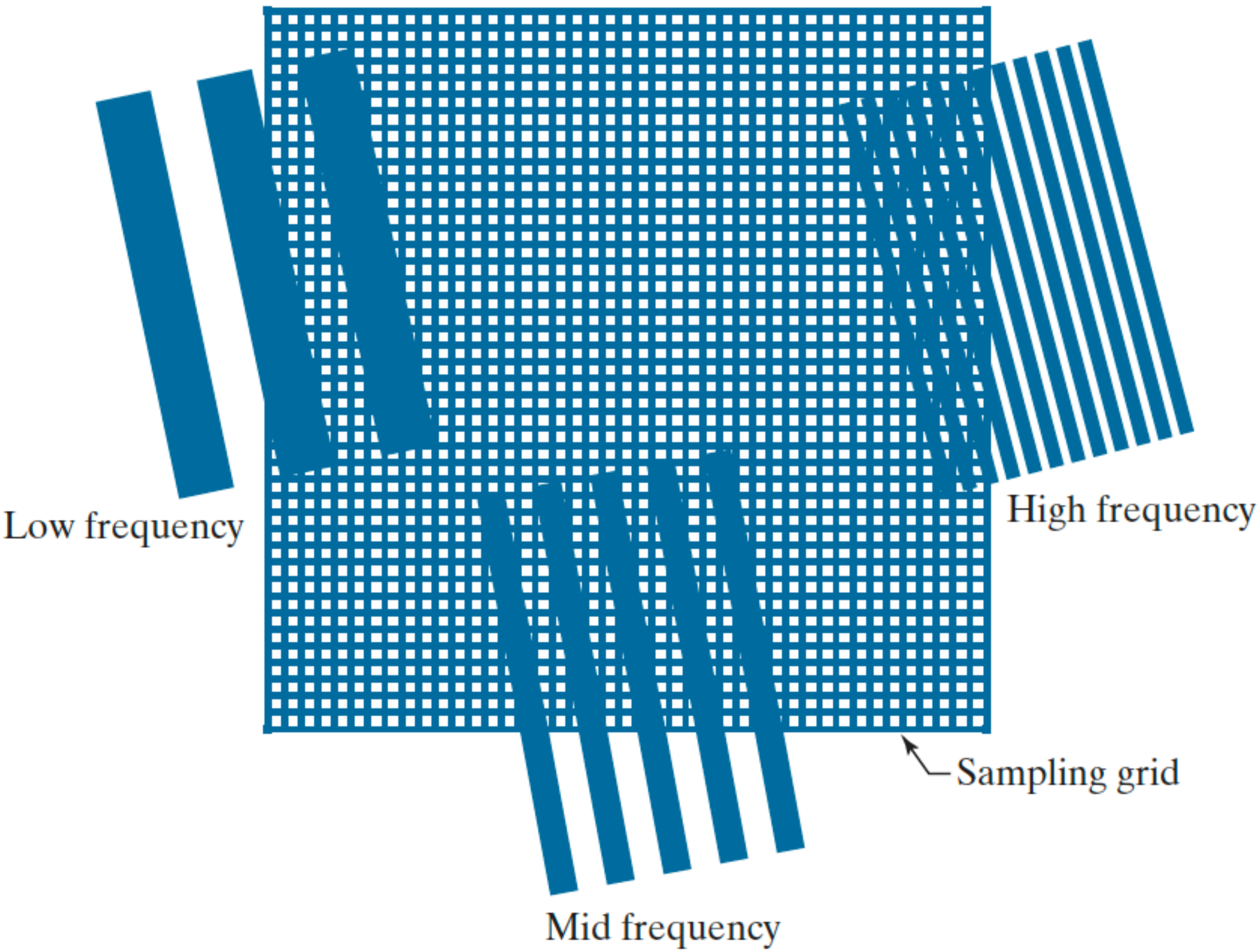


Under-sampled



1D example

4.5 Extensions to Functions of Two Variables - Aliasing in Images



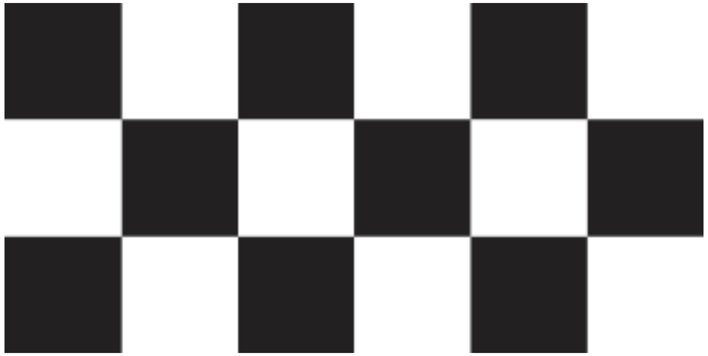
The jaggedness increases as the frequency of the region increases

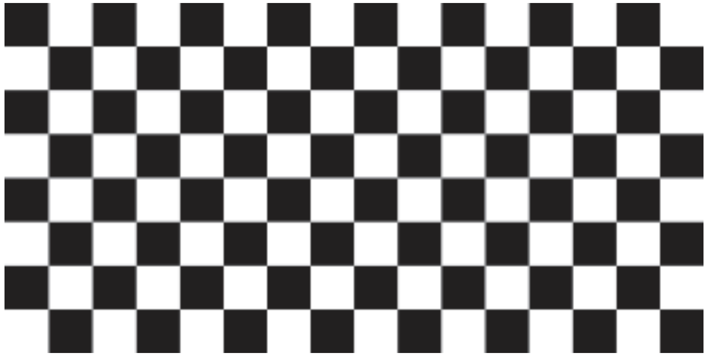
4.5 Extensions to Functions of Two Variables - Aliasing in Images

- **Aliasing due to camera resolution**

Square size

16x16

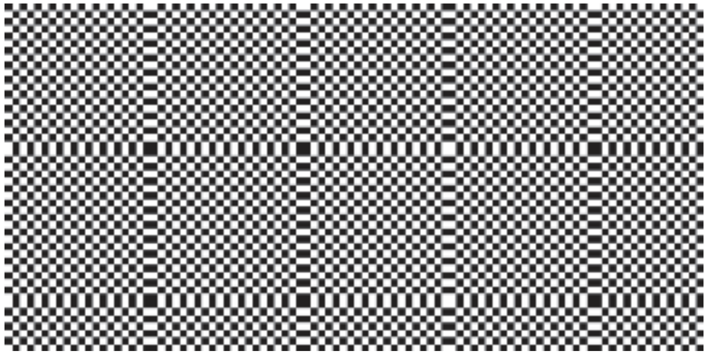


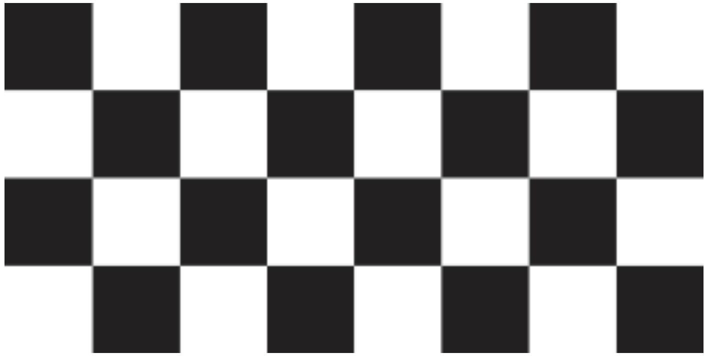


6x6

0.95x0.95

aliased





0.48x0.48

aliased

Anti-aliasing: defocusing the image in digital cameras before the image is sampled

4.5 Extensions to Functions of Two Variables - Aliasing in Images

- **Aliasing due to image re-sampling**
  - Zooming: over-sampling
  - Shrinking: under-sampling

Original Image

Resizing the image to 33% by pixel deletion

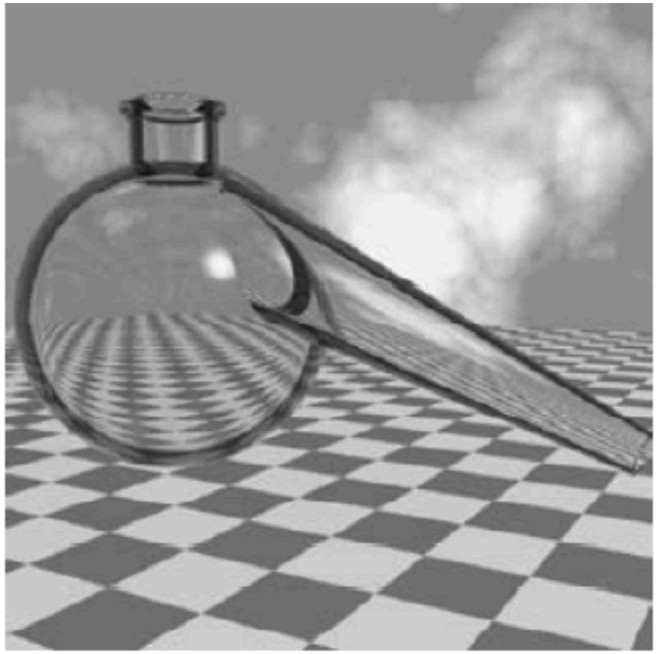
aliased

Same as middle image, but with averaging filter prior to resizing

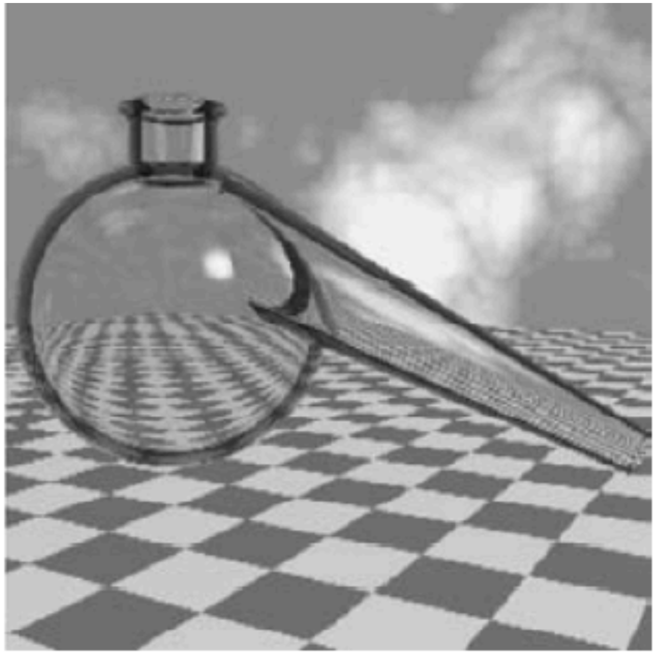
4.5 Extensions to Functions of Two Variables - Aliasing in Images

- **Aliasing due to image interpolation**

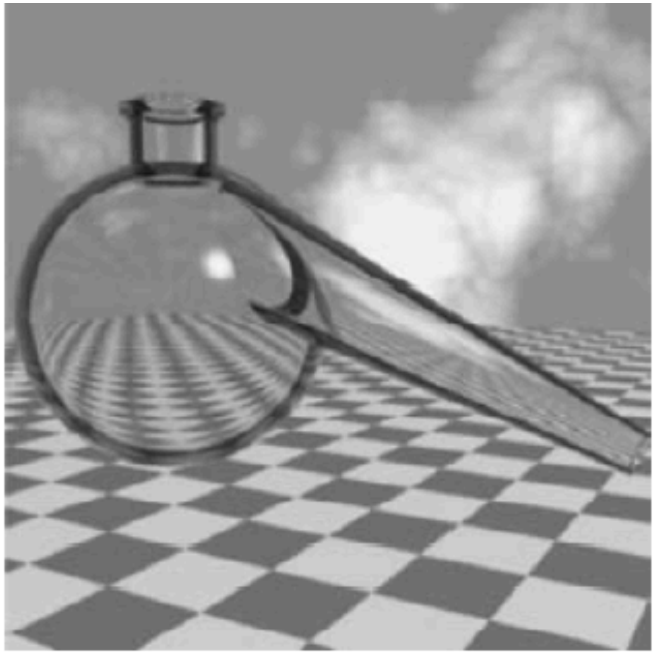
  Jaggies: block-like aliasing in images with strong edges after interpolating images.



(a) 1024x1024 original image



(b) resize to 25% using bilinear interpolation



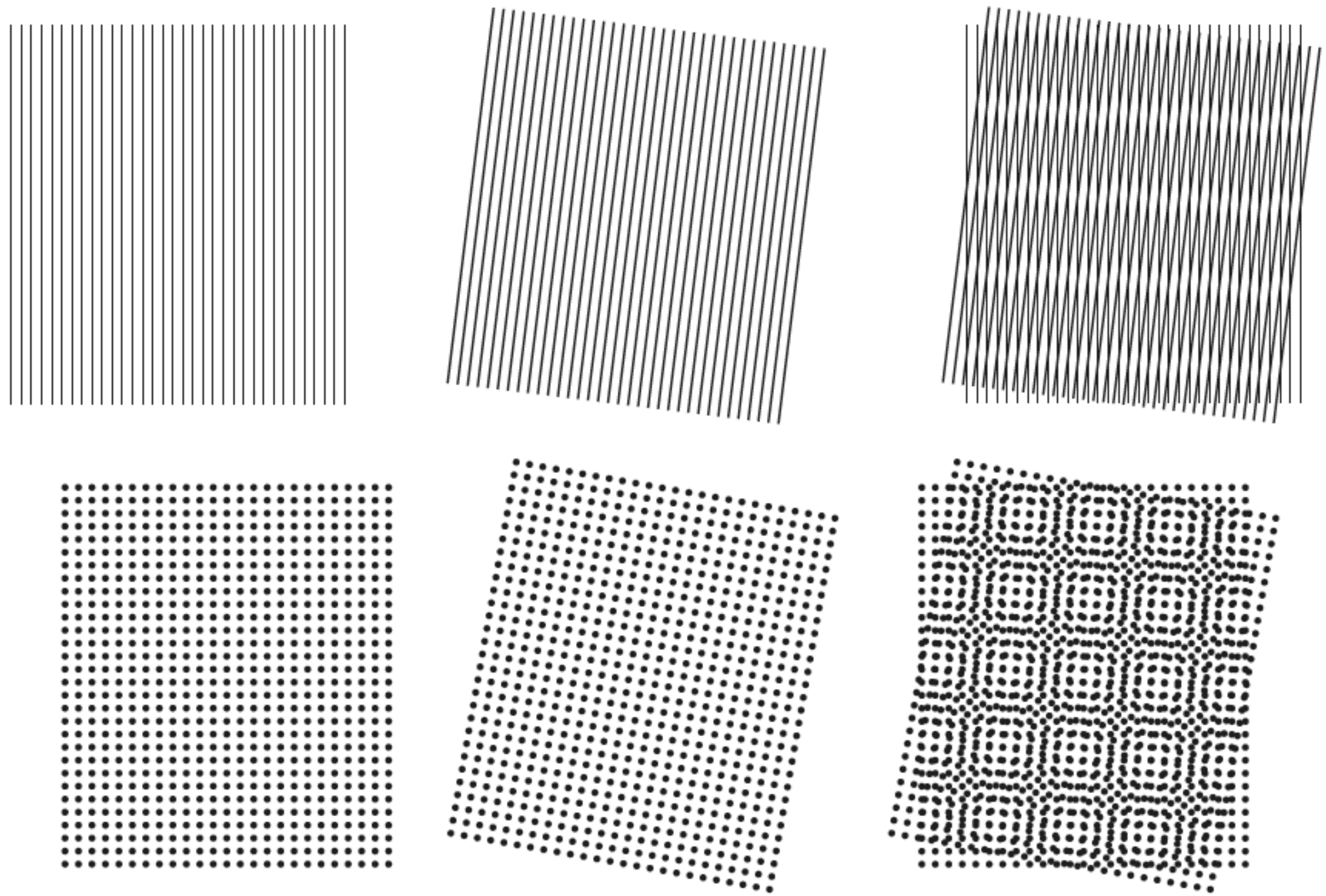
(c) With 5x5 averaging filter prior to resizing it to 25%

credit of this slide: Y. P. Hung

4.5 Extensions to Functions of Two Variables - Aliasing in Images

- **Moiré Patterns**

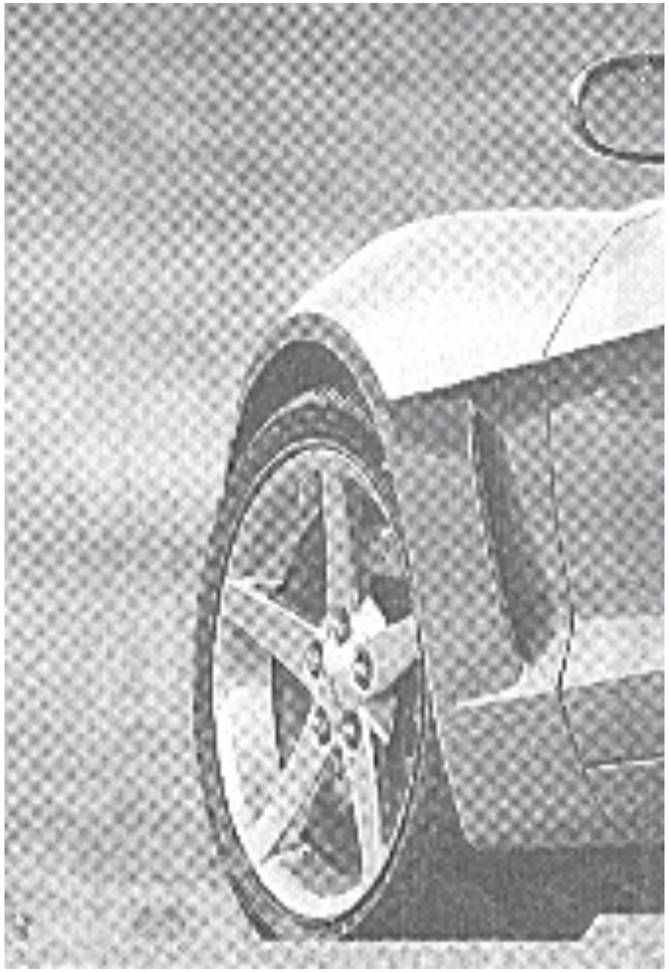
  Produced when a partially opaque ruled pattern with transparent gaps is overlaid on another similar pattern.

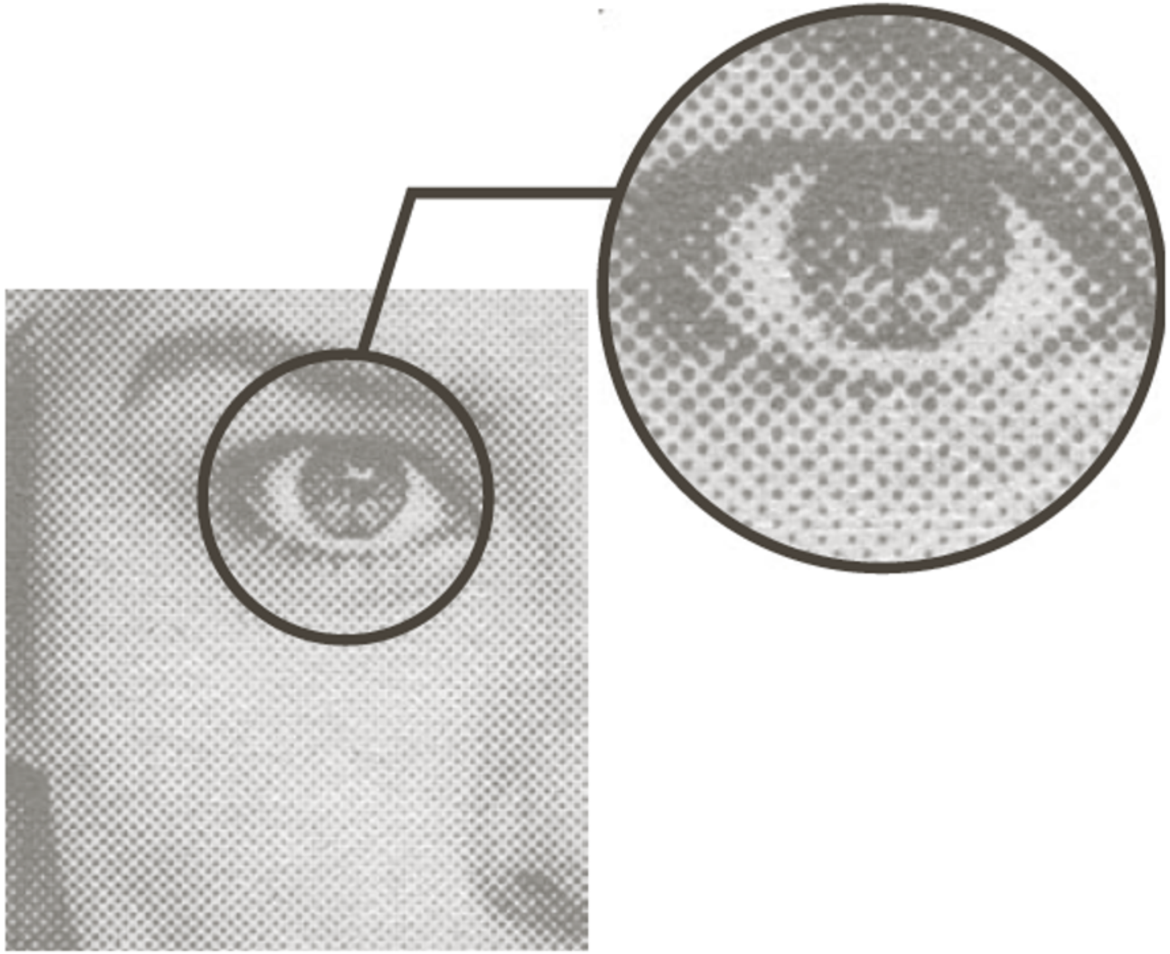


4.5 Extensions to Functions of Two Variables - Aliasing in Images

- **Moiré Patterns**

  Newspapers and other printed materials use so called *halftone dots*.





4.5 Extensions to Functions of Two Variables
- The 2D Discrete Fourier Transform and Its Inverse

- 2-D Discrete Fourier Transform (DFT) pair

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$F(u)=\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \qquad u=0,1,2,\ldots,M-1$$

$$f(x)=\frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \qquad x=0,1,2,\ldots,M-1$$

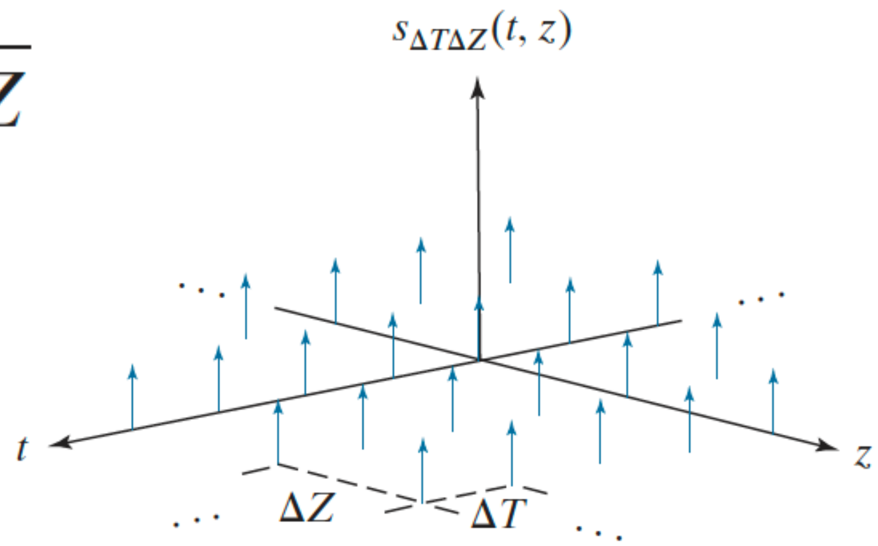
4.6 Some Properties of the 2-D DFT and IDFT

- **Relationships Between Spatial and Frequency Intervals**

Suppose that a continuous function $f(t, z)$ is sampled to form a digital image, $f(x, y)$, consisting of M x N samples. Let ΔT and ΔZ denote the separations between samples. Then

$$\Delta u = \frac{1}{M\Delta T} \qquad \Delta v = \frac{1}{N\Delta Z}$$

Property: separations in both domains are inversely proportional



4.6 Some Properties of the 2-D DFT and IDFT

- Translation

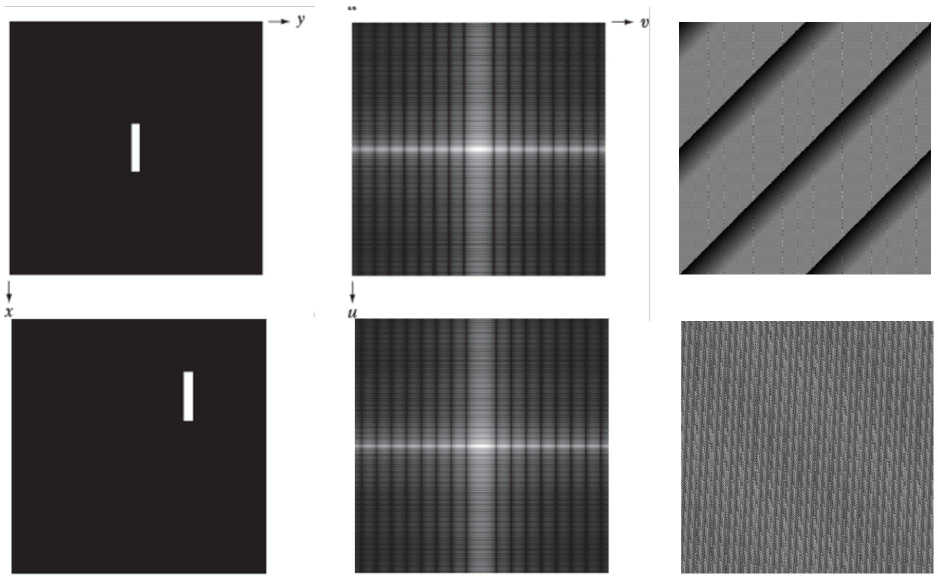
$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

Fourier transform pairs

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0,v-v_0)$$

Similarly, $$f(x-x_0,y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(x_0u/M+y_0v/N)}$$

Property: translation has no effect on the magnitude (spectrum) of $F(u, v)$



4.6 Some Properties of the 2-D DFT and IDFT

- Rotation

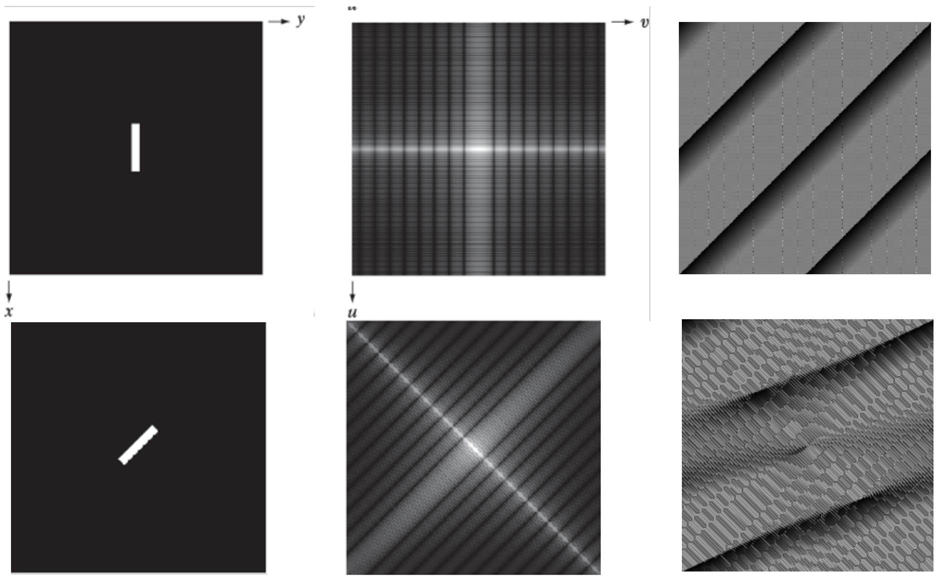
$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

Using the polar coordinate

$$x=r\cos\theta \qquad y=r\sin\theta \qquad u=\omega\cos\varphi \qquad v=\omega\sin\varphi$$

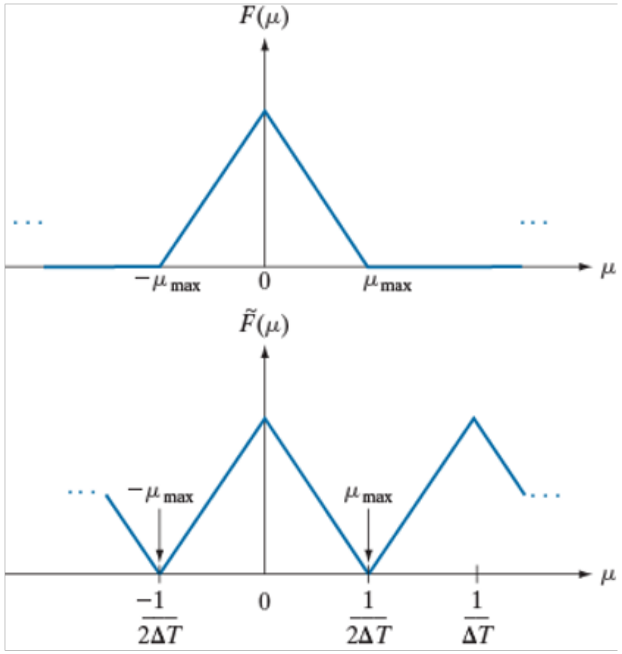
Then, $f(r,\theta+\theta_0)\Leftrightarrow F(\omega,\varphi+\theta_0)$

Property: rotating $f(x, y)$ by an angle θ_0 rotates F(u,v) by the same angle, and vice versa.



4.6 Some Properties of the 2-D DFT and IDFT

- Periodicity



Property: As in the 1-D case, the 2-D Fourier transform and its inverse are infinitely periodic in the *u* and *v* directions, i.e.

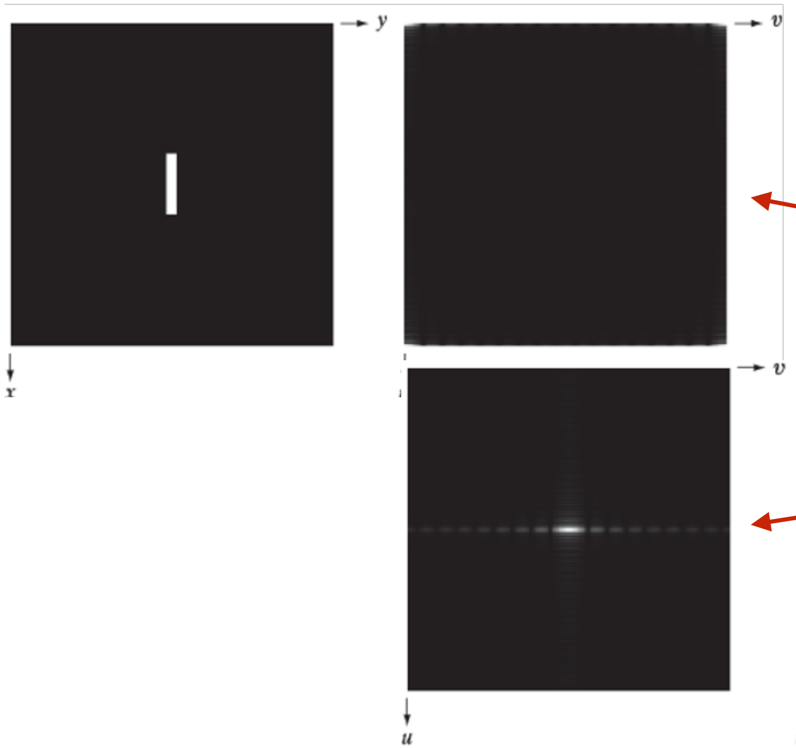
$$F(u,v) = F(u + k_1 M, v) = F(u, v + k_2 N) = F(u + k_1 M, v + k_2 N)$$

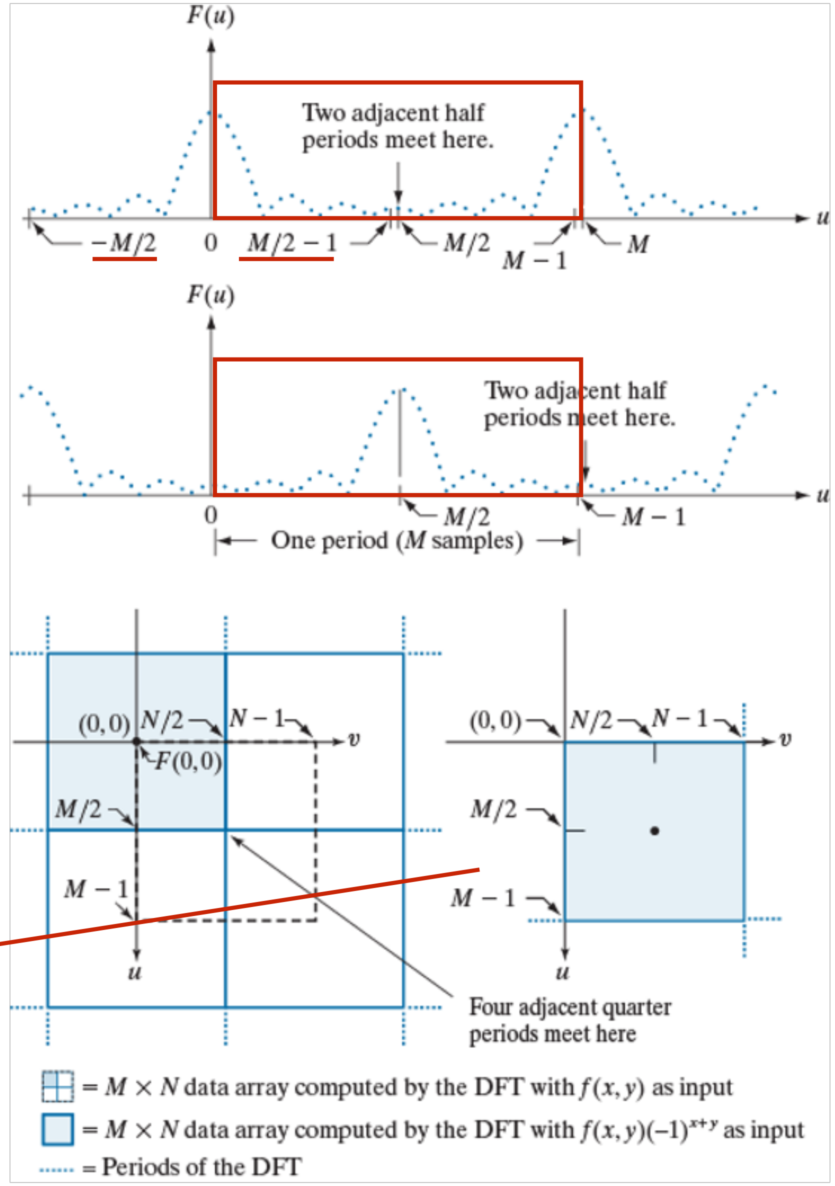
$$f(x,y) = f(x + k_1 M, y) = f(x, y + k_2 N) = f(x + k_1 M, y + k_2 N)$$

4.6 Some Properties of the 2-D DFT and IDFT

- Periodicity

  For display and filtering purposes, it is more convenient to have in this interval a complete period of the transformation.





4.6 Some Properties of the 2-D DFT and IDFT

- Periodicity

For display and filtering purposes, it is more convenient to have in this interval a complete period of the transformation.

$$f(x)e^{j2\pi(u_0 x/M)} \Leftrightarrow F(u-u_0)$$

Let $u_0 = M/2$

$$f(x)(-1)^x \Leftrightarrow F(u - M/2)$$

$$e^{j\pi x} = (-1)^x$$

2-D case:

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$$

4.6 Some Properties of the 2-D DFT and IDFT

- **Symmetry Properties**

DFTs of even and odd functions have some very important characteristics.

From functional analysis is that any real or complex function, $w(x, y)$, can be expressed as the sum of an even and an odd part.

$$w(x,y) = w_e(x,y) + w_o(x,y)$$

even part:

$$w_e(x,y) \triangleq \frac{w(x,y) + w(-x,-y)}{2}$$

odd part:

$$w_o(x,y) \triangleq \frac{w(x,y) - w(-x,-y)}{2}$$

Functional analysis (泛函分析): 泛函分析是現代數學分析的一個分支，隸屬於分析學，其研究的主要物件是函數構成的函數空間。

4.6 Some Properties of the 2-D DFT and IDFT

- **Symmetry Properties**

even part: $$w_e(x,y) \triangleq \frac{w(x,y)+w(-x,-y)}{2}$$

odd part: $$w_o(x,y) \triangleq \frac{w(x,y)-w(-x,-y)}{2}$$

It follows that:

$$w_e(x,y) = w_e(-x,-y)$$ symmetric

$$w_o(x,y) = -w_o(-x,-y)$$ antisymmetric

4.6 Some Properties of the 2-D DFT and IDFT

- **Symmetry Properties**

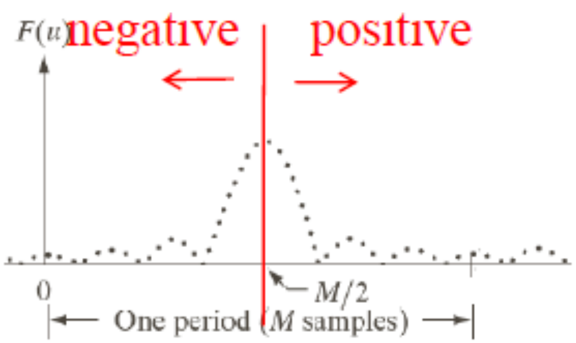
  Because all indices in the DFT and IDFT are positive, when we talk about the symmetry(antisymmetry), we are referring to symmetry (antisymmetry) about the center point of a sequence.

  - Indices to the *right* of the center point => positive
  - Indices to the *left* of the center point => negative

in which case the definitions of even and odd become:

$$w_e(x,y) = w_e(M-x, N-y)$$

$$w_o(x,y) = -w_o(M-x, N-y)$$



credit of this slide: Y. P. Hung

4.6 Some Properties of the 2-D DFT and IDFT

- Symmetry Properties

Example of even and odd function:

Consider the 1-D sequence ($M = 4$):

$$f = \{f(0), f(1), f(2), f(3)\} = \{2, 1, 1, 1\}$$

To test for evenness: $f(x) = f(4-x)$

$$f(0) = \underline{f(4)}, \quad f(1) = f(3), \quad f(2) = f(2), \quad f(3) = f(1)$$

$f(4)$ is outside the range being examined and can be any value

the sequence is even

4.6 Some Properties of the 2-D DFT and IDFT

- Symmetry Properties

Example of even and odd function:

Consider the 1-D sequence (M = 4):

$$f = \{f(0), f(1), f(2), f(3)\} = \{2, 1, 1, 1\}$$

To test for evenness: $f(x) = f(4-x)$

$$f(0) = f(4), \quad f(1) = f(3), \quad f(2) = f(2), \quad f(3) = f(1)$$

In fact, any 4-point even sequence has to have the form

$\{a, b, c, b\}$

4.6 Some Properties of the 2-D DFT and IDFT

- Symmetry Properties

Fourier transform of a real function, $f(x, y)$, is conjugate symmetric:

$$F^*(u,v) = F(-u,-v)$$

Proof:

$$F^*(u,v) = \left[\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)} \right]^*$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \underline{f^*(x,y)} e^{j2\pi(ux/M + vy/N)}$$

$\downarrow$ real

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \underline{f(x,y)} e^{-j2\pi([-u]x/M + [-v]y/N)}$$

$$= F(-u,-v)$$

4.6 Some Properties of the 2-D DFT and IDFT

- Symmetry Properties

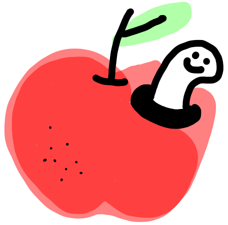
Fourier transform of a real function, $f(x, y)$, is conjugate symmetric:

$$F^*(u,v) = F(-u,-v)$$

Fourier transform of a imaginary, $f(x, y)$, is conjugate antisymmetric:

$$F^*(-u,-v) = -F(u,v)$$

4.6 Some Properties of the 2-D DFT and IDFT



會考證明

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u,v)$ and $I(u,v)$ are the real and imaginary parts of $F(u,v)$, respectively. Use of the word *complex* indicates that a function has nonzero real and imaginary parts.

	Spatial Domain†		Frequency Domain†
1)	$f(x,y)$ real	$\Leftrightarrow$	$F^*(u,v) = F(-u,-v)$
2)	$f(x,y)$ imaginary	$\Leftrightarrow$	$F^*(-u,-v) = -F(u,v)$
3)	$f(x,y)$ real	$\Leftrightarrow$	$R(u,v)$ even; $I(u,v)$ odd
4)	$f(x,y)$ imaginary	$\Leftrightarrow$	$R(u,v)$ odd; $I(u,v)$ even
5)	$f(-x,-y)$ real	$\Leftrightarrow$	$F^*(u,v)$ complex
6)	$f(-x,-y)$ complex	$\Leftrightarrow$	$F(-u,-v)$ complex
7)	$f^*(x,y)$ complex	$\Leftrightarrow$	$F^*(-u,-v)$ complex
8)	$f(x,y)$ real and even	$\Leftrightarrow$	$F(u,v)$ real and even
9)	$f(x,y)$ real and odd	$\Leftrightarrow$	$F(u,v)$ imaginary and odd
10)	$f(x,y)$ imaginary and even	$\Leftrightarrow$	$F(u,v)$ imaginary and even
11)	$f(x,y)$ imaginary and odd	$\Leftrightarrow$	$F(u,v)$ real and odd
12)	$f(x,y)$ complex and even	$\Leftrightarrow$	$F(u,v)$ complex and even
13)	$f(x,y)$ complex and odd	$\Leftrightarrow$	$F(u,v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M-1]$, and y and v in the range $[0, N-1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an *odd* complex function. As before, "$\Leftrightarrow$" indicates a Fourier transform pair.

3)	$f(x,y)$ real	$\Leftrightarrow$	$R(u,v)$ even; $I(u,v)$ odd
4)	$f(x,y)$ imaginary	$\Leftrightarrow$	$R(u,v)$ odd; $I(u,v)$ even
8)	$f(x,y)$ real and even	$\Leftrightarrow$	$F(u,v)$ real and even
9)	$f(x,y)$ real and odd	$\Leftrightarrow$	$F(u,v)$ imaginary and odd
10)	$f(x,y)$ imaginary and even	$\Leftrightarrow$	$F(u,v)$ imaginary and even
11)	$f(x,y)$ imaginary and odd	$\Leftrightarrow$	$F(u,v)$ real and odd
12)	$f(x,y)$ complex and even	$\Leftrightarrow$	$F(u,v)$ complex and even
13)	$f(x,y)$ complex and odd	$\Leftrightarrow$	$F(u,v)$ complex and odd

Property	$f(x)$		$F(u)$
3	$\{1, 2, 3, 4\}$	$\Leftrightarrow$	$\{(10+0j),(-2+2j),(-2+0j),(-2-2j)\}$
4	$\{1j, 2j, 3j, 4j\}$	$\Leftrightarrow$	$\{(0+2.5j),(.5-.5j),(0-.5j),(-.5-.5j)\}$
8	$\{2, 1, 1, 1\}$	$\Leftrightarrow$	$\{5, 1, 1, 1\}$
9	$\{0, -1, 0, 1\}$	$\Leftrightarrow$	$\{(0+0j),(0+2j),(0+0j),(0-2j)\}$
10	$\{2j, 1j, 1j, 1j\}$	$\Leftrightarrow$	$\{5j, j, j, j\}$
11	$\{0j, -1j, 0j, 1j\}$	$\Leftrightarrow$	$\{0, -2, 0, 2\}$
12	$\{(4+4j), (3+2j), (0+2j), (3+2j)\}$	$\Leftrightarrow$	$\{(10+10j),(4+2j),(-2+2j),(4+2j)\}$
13	$\{(0+0j),(1+1j),(0+0j),(-1-j)\}$	$\Leftrightarrow$	$\{(0+0j),(2-2j),(0+0j),(-2+2j)\}$

4.6 Some Properties of the 2-D DFT and IDFT

- **Fourier Spectrum and Phase Angle**

The 2-D DFT is complex in general, it can be expressed in polar form:

$$F(u,v) = R(u,v) + jI(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

The magnitude is called Fourier (or frequency) spectrum

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

Phase angle or phase spectrum:

$$\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$$

Power spectrum:

$$\begin{aligned} P(u,v) &= |F(u,v)|^2 \\ &= R^2(u,v) + I^2(u,v) \end{aligned}$$

4.6 Some Properties of the 2-D DFT and IDFT

- **Fourier Spectrum and Phase Angle**

The Fourier transform of a real function is *conjugate symmetric*, which implies
- the spectrum is even symmetry about the origin

$$|F(u,v)| = |F(-u,-v)|$$

- the phase angle is odd symmetry about the origin

$$\phi(u,v) = -\phi(-u,-v)$$

$$\boxed{F^*(u,v) = F(-u,-v)}$$

$$\boxed{\begin{aligned} F(u,v) &= |F(u,v)|\, e^{j\phi(u,v)} \\ F^*(u,v) &= |F(u,v)|\, e^{-j\phi(u,v)} \\ F(-u,-v) &= |F(-u,-v)|\, e^{j\phi(-u,-v)} \end{aligned}}$$

4.6 Some Properties of the 2-D DFT and IDFT

- Fourier Spectrum and Phase Angle

$$\boxed{F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M + vy/N)}}$$

zero-frequency term of DFT:

$$F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

$$= MN\underline{\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)}$$

$$= MN\underline{\bar{f}}$$

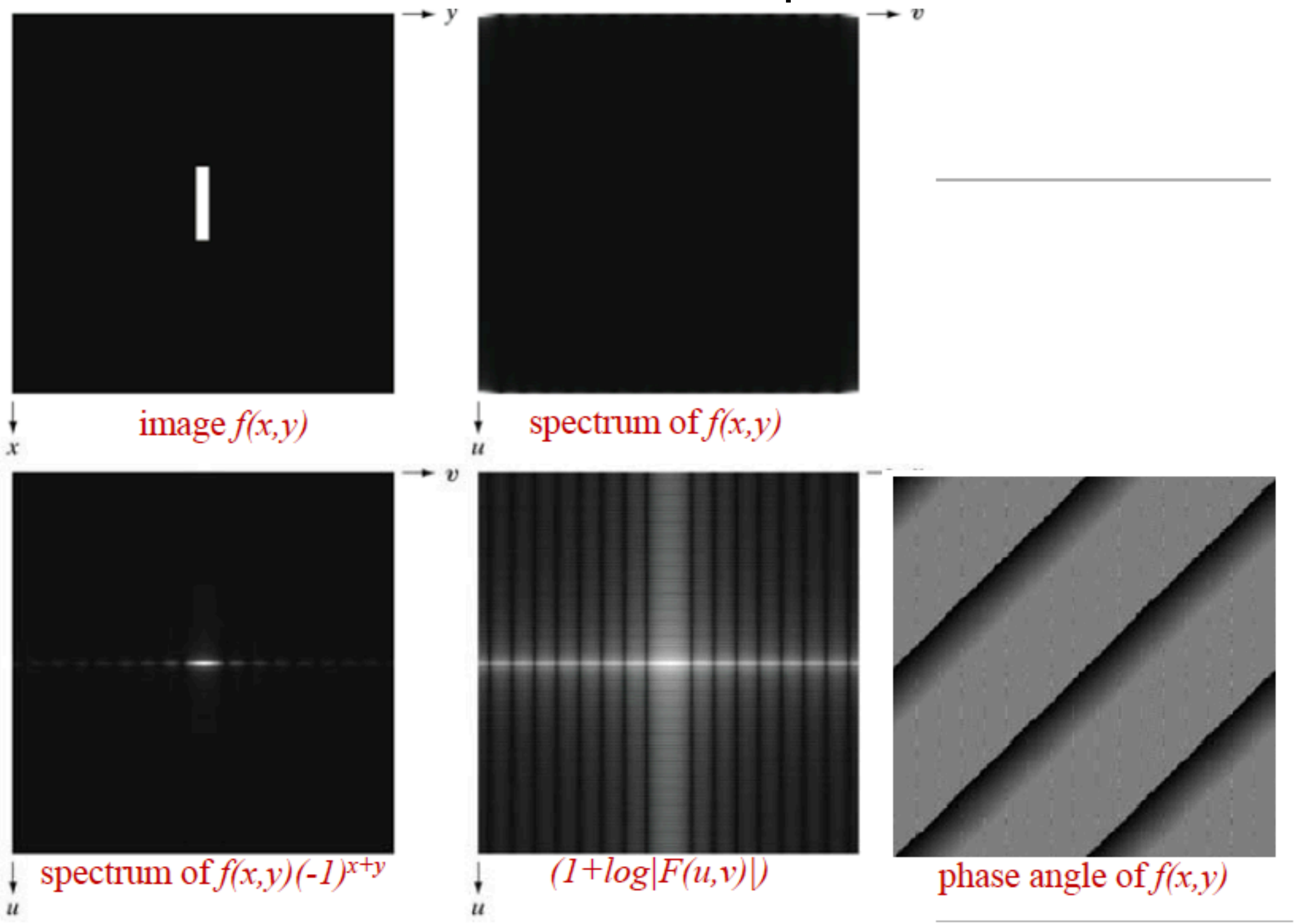
average value of *f*

$$|F(0,0)| = MN|\bar{f}|$$

dc component of the transform

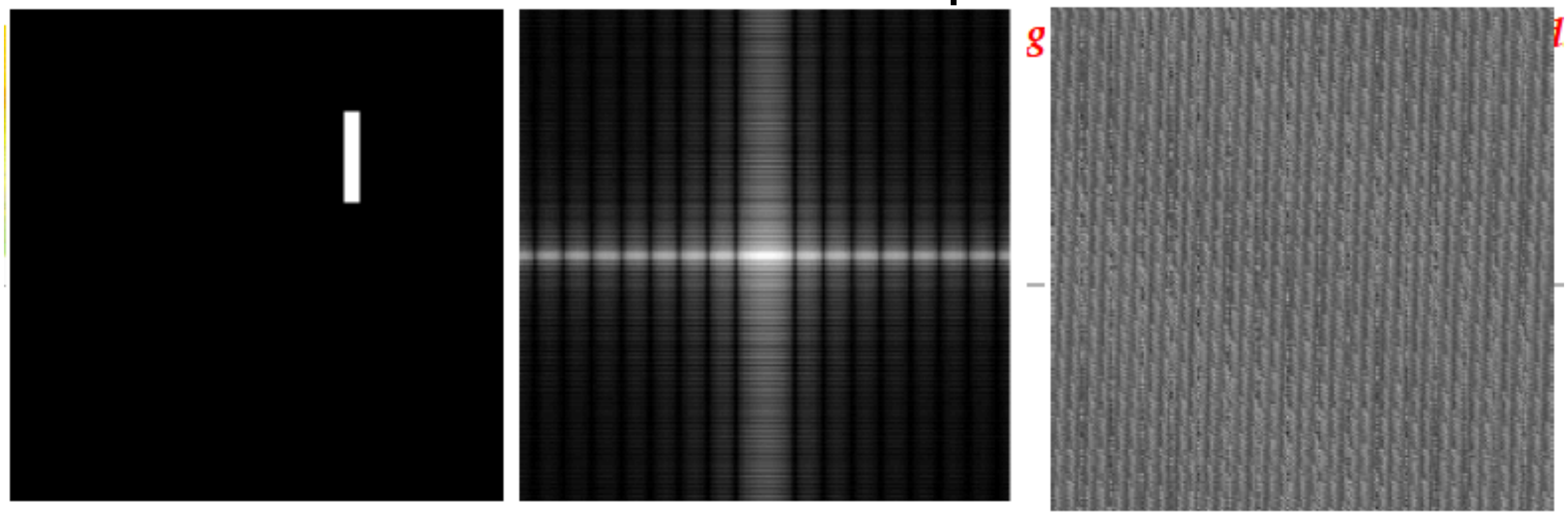
- dc: direct current (i.e., current of zero frequency).

4.6 Some Properties of

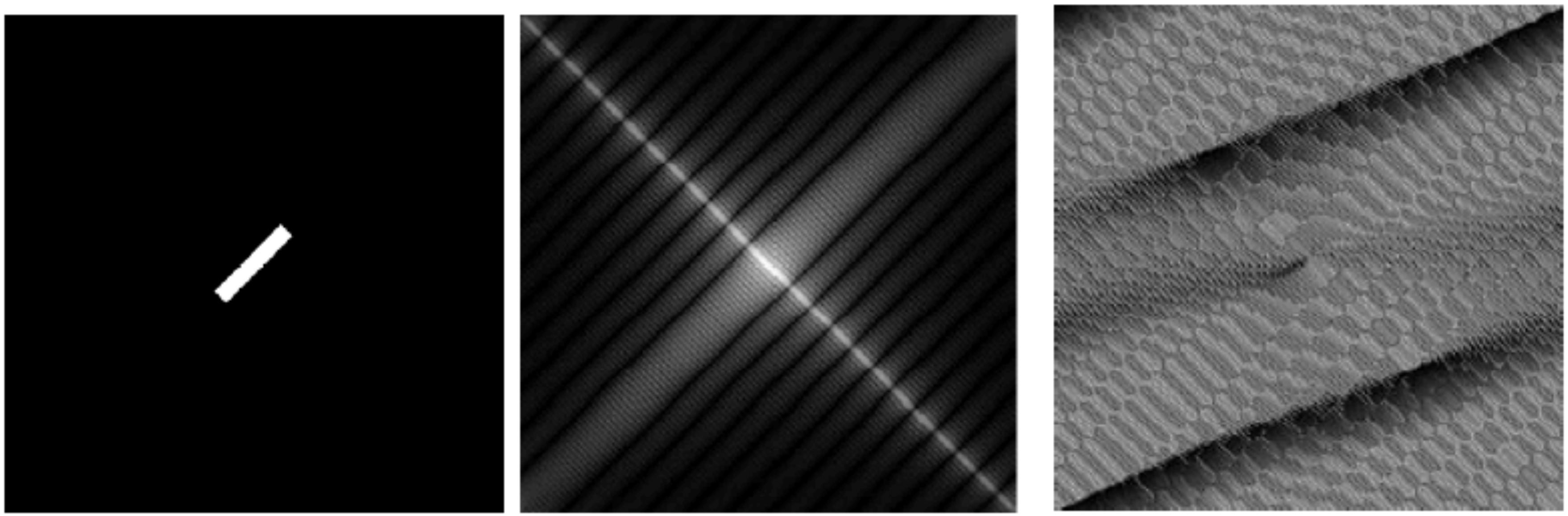


credit of this slide: Y. P. Hung

4.6 Some Properties of



translated image $f(x,y)$ — centered spectrum of $f(x,y)$ — phase angle of $f(x,y)$



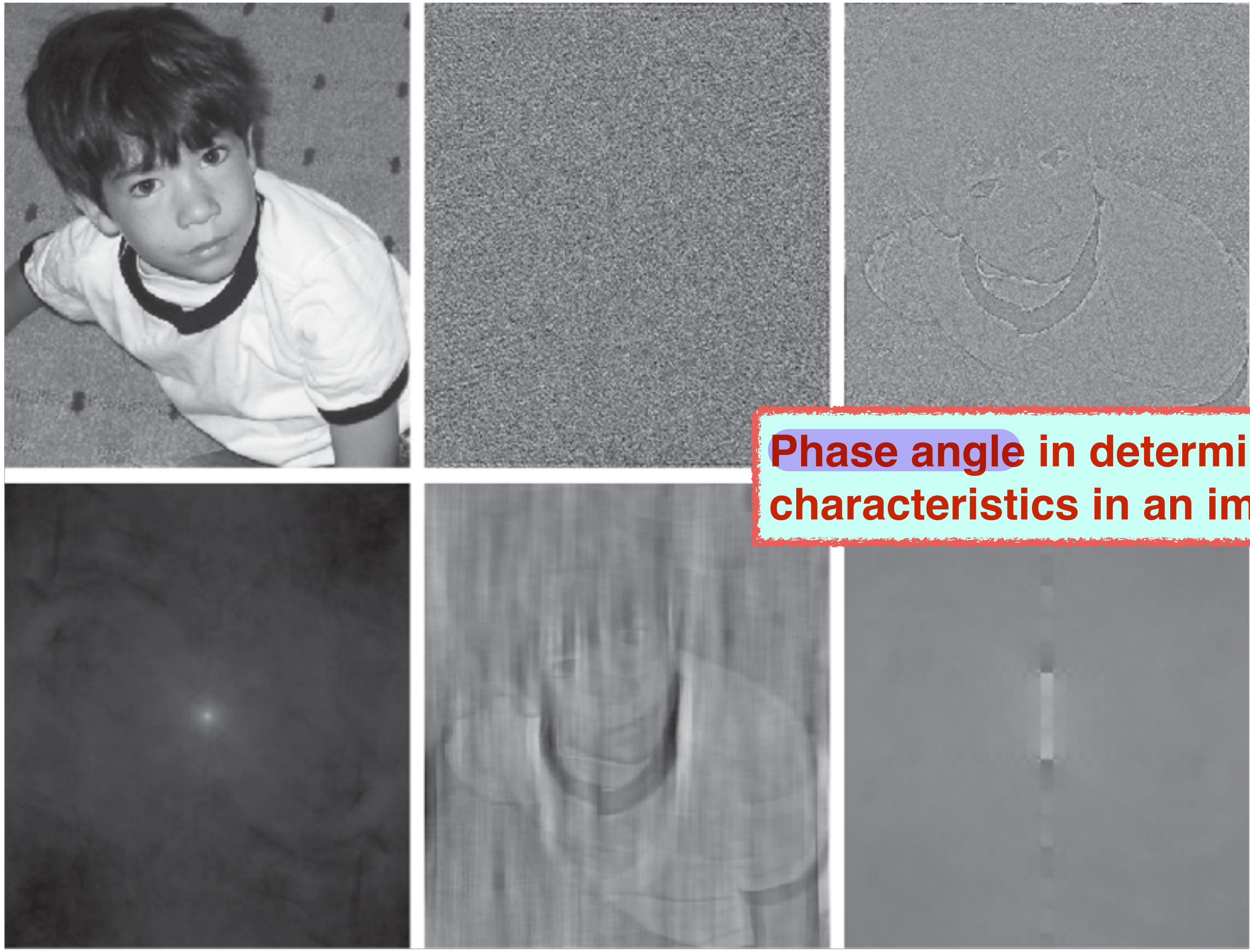
rotated image $g(x,y)$ — centered spectrum of $g(x,y)$ — phase angle of $g(x,y)$

credit of this slide: Y. P. Hung

Boy image

Phase angle

Reconstructed using only its phase angle



Phase angle in determining shape characteristics in an image

Reconstructed using only its spectrum

Reconstructed using its phase angle and the spectrum of the rectangle

Reconstructed using the phase angle of rectangle and the spectrum of the boy image

4.6 Some Properties of the 2-D DFT and IDFT

- **The 2-D Convolution Theorem**

2-D convolution:

$$(f \star h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n)$$

2-D convolution theorem:

$$(f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$$

$$(f \bullet h)(x,y) \Leftrightarrow \frac{1}{MN}(F \star H)(u,v)$$

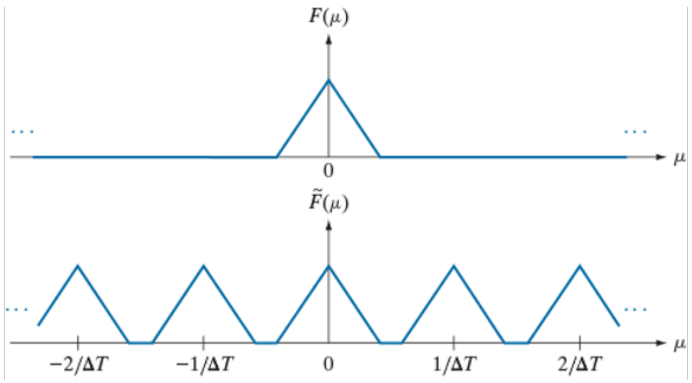
foundation of linear filtering in the frequency domain

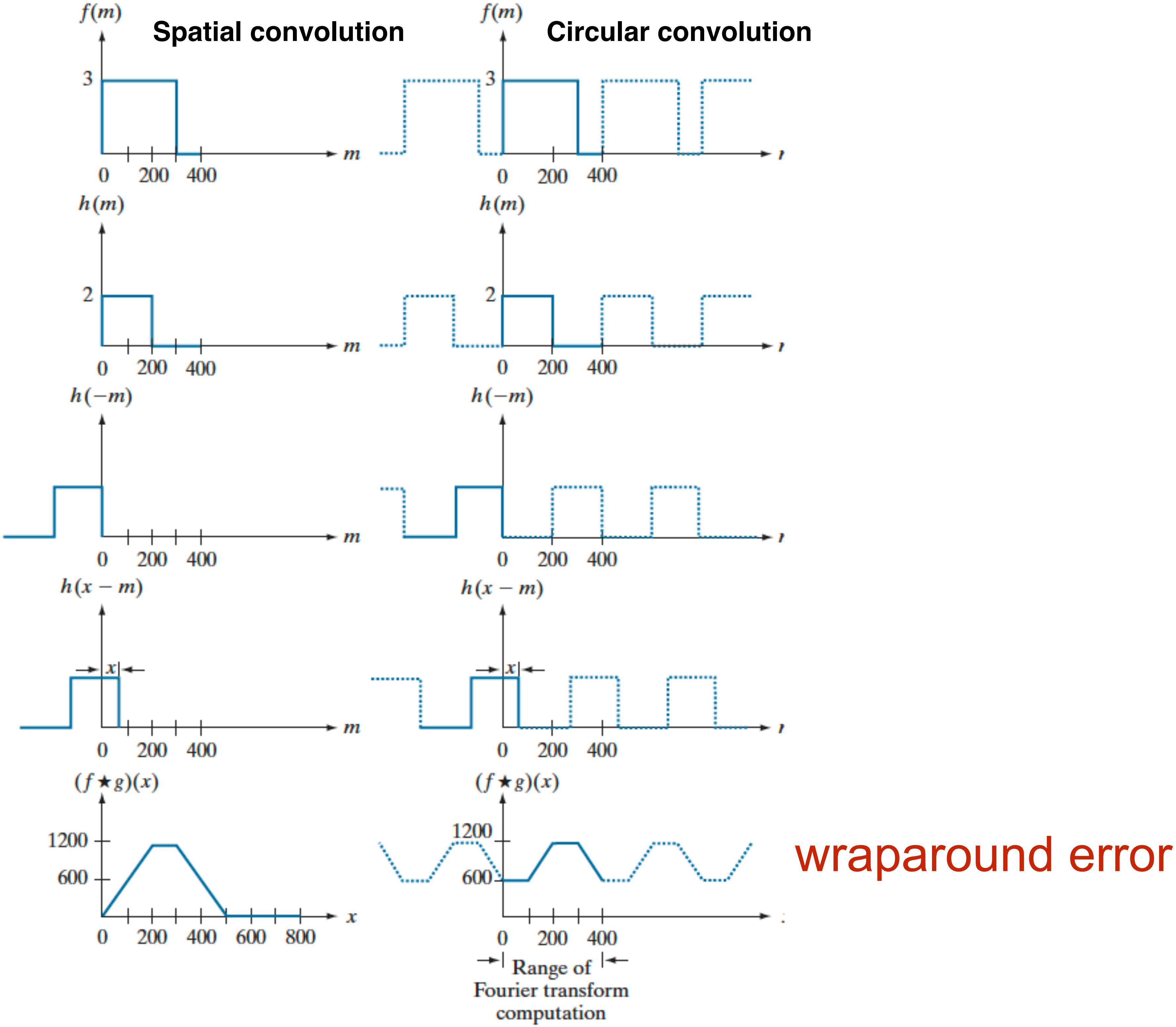
4.6 Some Properties of the 2-D DFT and IDFT

$$(f \star h)(x, y) \Leftrightarrow (F \bullet H)(u, v)$$

- Zero padding for wraparound error

  - DFT algorithm automatically implies periodicity
  - When we take the inverse Fourier transform of the product of the two transforms we would get a circular (i.e., periodic) convolution
  - If we want to compute the spatial convolution using the IDFT of the product of the two transforms, then the *periodicity* issues must be taken into account.



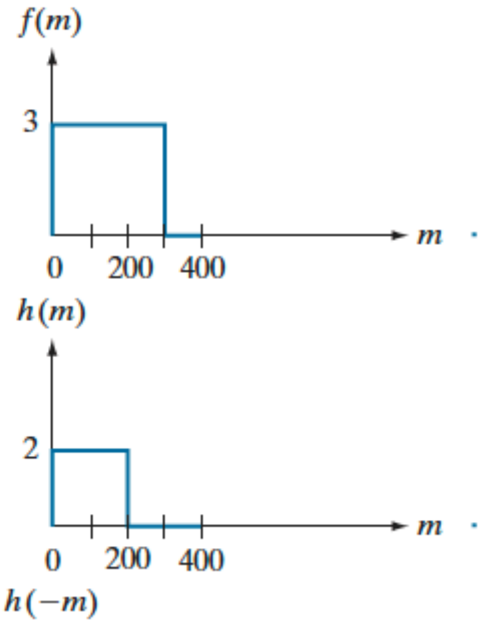


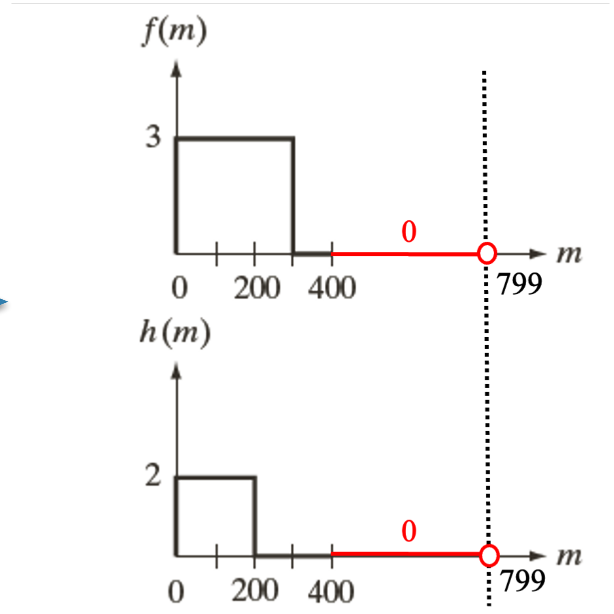
Spatial convolution
Circular convolution
f(m)
3
0
200
400
m
h(m)
2
h(−m)
h(x − m)
x
(f ★ g)(x)
1200
600
600
800
Range of
Fourier transform
computation
wraparound error

4.6 Some Properties of the 2-D DFT and IDFT

- Solution to wraparound error: zero padding
  - Consider *f*(*x*) and *h*(*x*) composed of *A* and *B* samples, respectively
  - Append zeros to both functions so that they have the same length *P*, then wraparound is avoided by choosing

$$P \geq A + B - 1$$





4.6 Some Properties of the 2-D DFT and IDFT

- **Solution to wraparound error: zero padding**
  - Consider $f(x)$ and $h(x)$ composed of A and B samples, respectively
  - Append zeros to both functions so that they have the same length P, then wraparound is avoided by choosing

$$P \geq A + B - 1$$

  - 2-D case: $f(x, y)$: A x B, $h(x, y)$: C x D

$$P \geq A + C - 1 \qquad Q \geq B + D - 1$$

$$f_p(x,y) = \begin{cases} f(x,y) & 0 \leq x \leq A-1 \text{ and } 0 \leq y \leq B-1 \\ 0 & A \leq x \leq P \text{ or } B \leq y \leq Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y) & 0 \leq x \leq C-1 \text{ and } 0 \leq y \leq D-1 \\ 0 & C \leq x \leq P \text{ or } D \leq y \leq Q \end{cases}$$