Introduction to Image Processing

Ch 4. Filtering in the Frequency Domain

Kuan-Wen Chen

Ch 4. Filtering in the Frequency Domain

4.1 Background
- Jean Baptiste Joseph Fourier

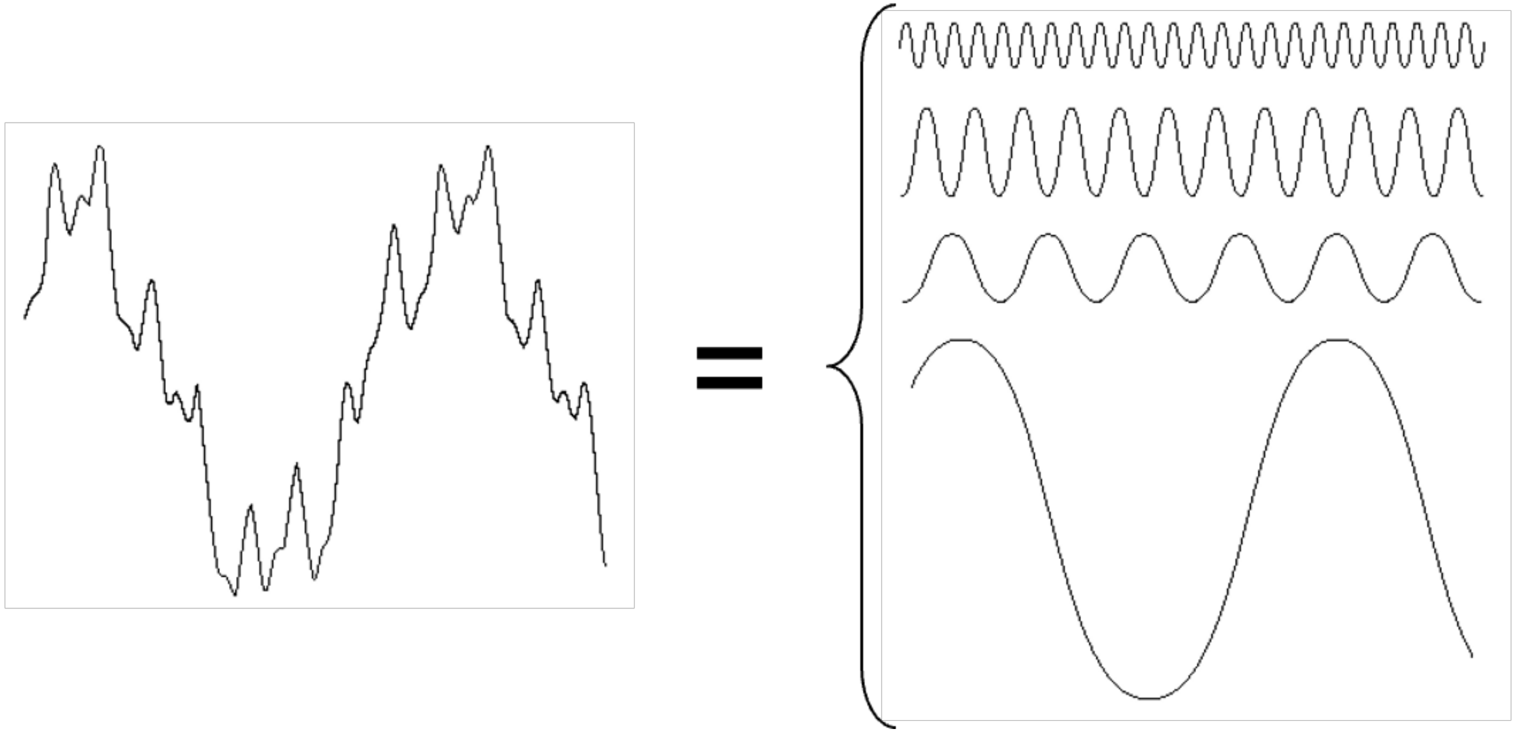
Fourier was born in Auxerre, France in 1768.

- Most famous for his work "*La Théorie Analitique de la Chaleur*" published in 1822.
- *Translated into English in 1878: "The Analytic Theory of Heat".*《熱的解析理論》

- Nobody paid much attention when the work was first published.
- ***Fourier series*** - one of the most important mathematical theories in modern engineering.

credit of this slide: C. Nikou

4.1 Background
- Fourier Series



Any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient. – ***Fourier series***

4.2 Preliminary Concepts

- Complex Numbers

$$C = R + jI \qquad C^* = R - jI$$

$$C = |C|(\cos\theta + j\sin\theta) \quad \text{where } |C| = \sqrt{R^2 + I^2}$$

- Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$C = |C|\, e^{j\theta}$$

- Fourier Series

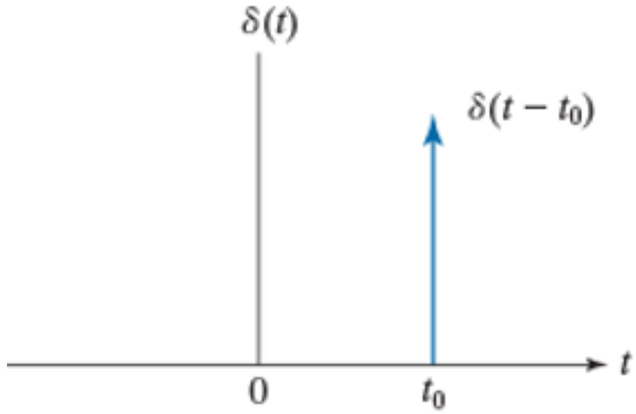
a function $f(t)$ of a continuous variable, t, that is periodic with a period, T.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} \qquad \text{where} \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t}\, dt$$

4.2 Preliminary Concepts
- Impulses and Their Sifting Property

- **Impulse**
  - It may be considered both as continuous and discrete.
  - Useful for the representation of discrete signals through sampling of continuous signals.

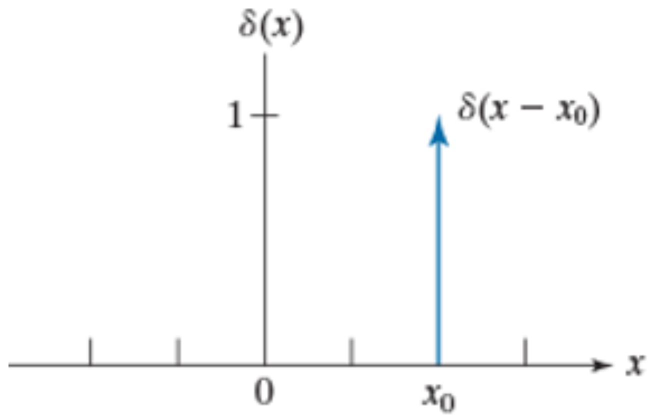
Continuous impulse



$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

Impulse at $t = 0$

Unit discrete impulse



$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Impulse at $x = 0$

4.2 Preliminary Concepts - Impulses and Their Sifting Property

- Impulse at $t = 0$ or $x = 0$

Continuous impulse

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Unit discrete impulse

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

- **Sifting Property**: Sifting simply yields the value of the function at the location of the impulse

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

Impulse at $t = t_0$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x-x_0) = f(x_0)$$

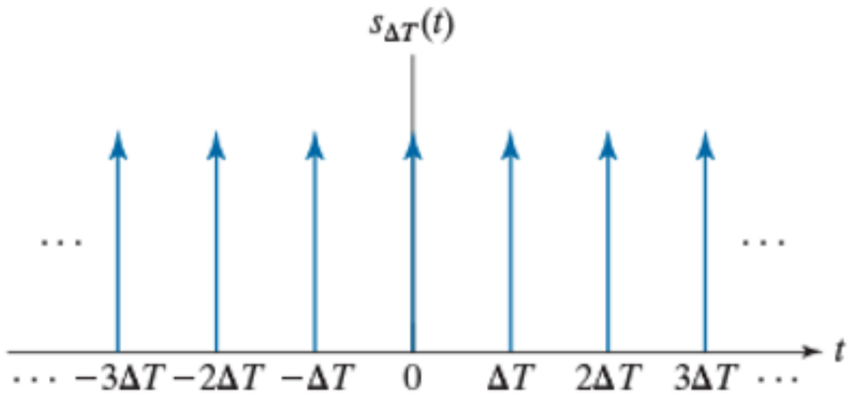
Impulse at $x = x_0$

4.2 Preliminary Concepts
- Impulses and Their Sifting Property

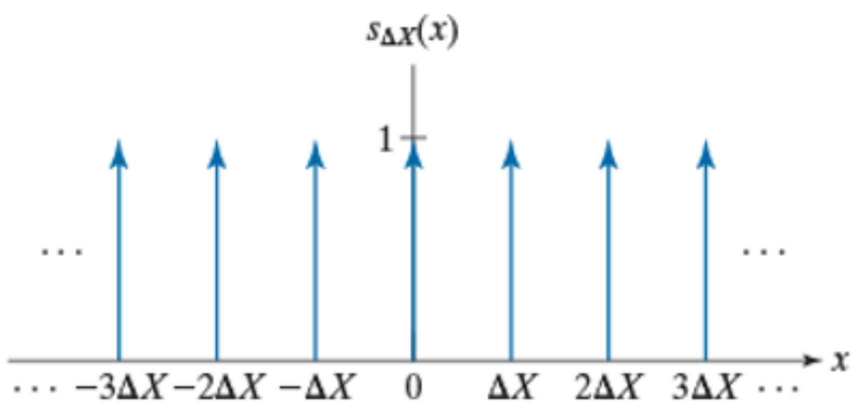
- **Impulse train**

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

Continuous



Discrete



4.2 Preliminary Concepts - Impulses and Their Sifting Property

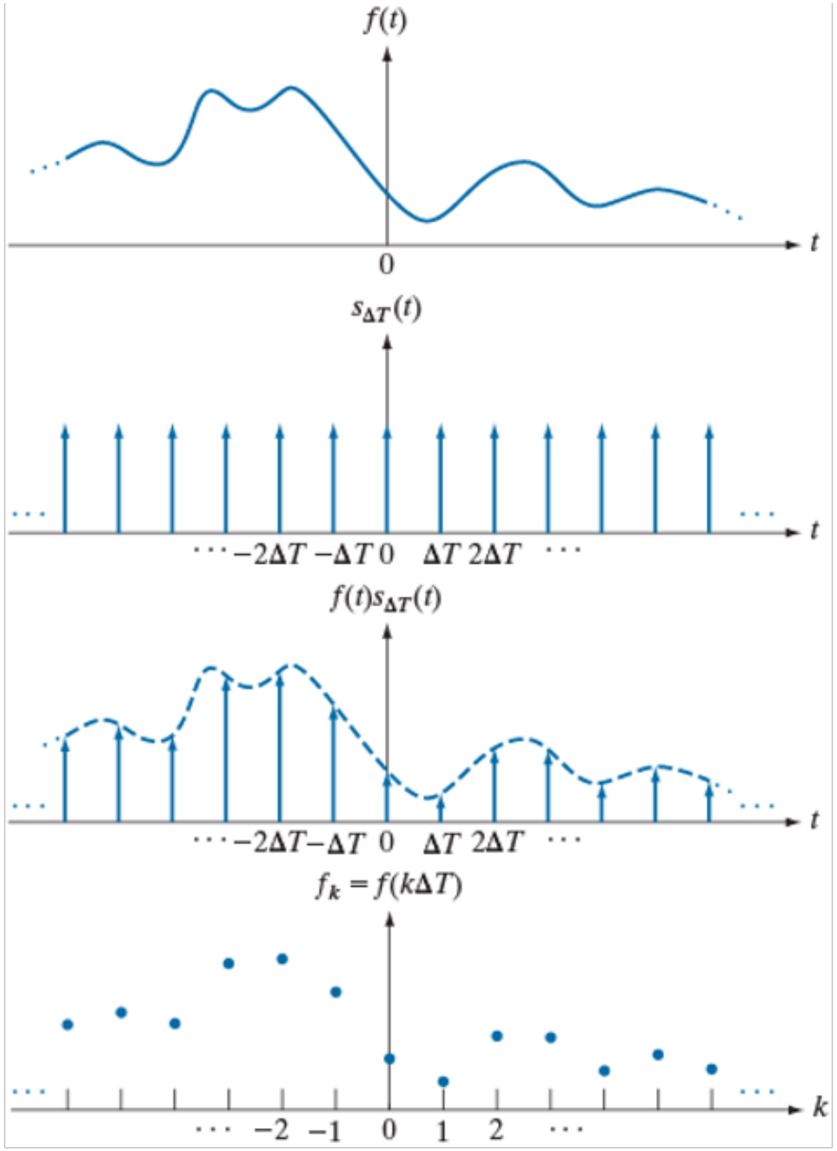
- **Impulse train**

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

Applied for sampling:

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt = f(k\Delta T)$$



4.2 Preliminary Concepts - Fourier Transform

- **Definition**: the *Fourier transform* of a continuous function $f(t)$ of a continuous variable t:

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$= \int_{-\infty}^{\infty} f(t)\big(\cos(2\pi\mu t) - j\sin(2\pi\mu t)\big)dt$$

Fourier transform pair

- inverse *Fourier transform*

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

4.2 Preliminary Concepts - Fourier Transform

- **Definition**: the *Fourier transform* of a continuous function $f(t)$ of a continuous variable t:

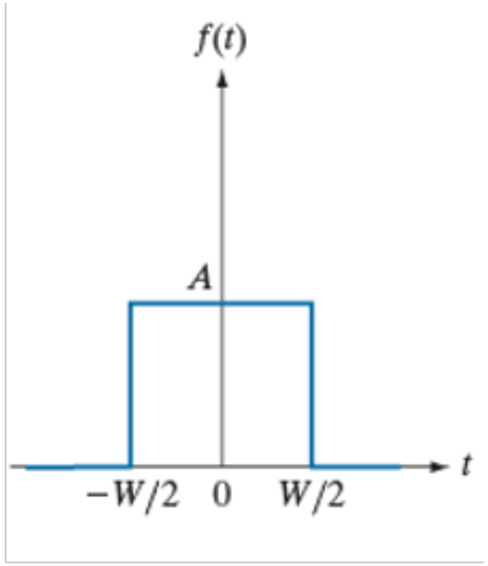
$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

$$= \int_{-\infty}^{\infty} f(t)\left(\cos(2\pi\mu t) - j\sin(2\pi\mu t)\right)dt$$

- Note: Fourier transform is an expansion of $f(t)$ multiplied by sinusoidal terms whose frequencies are determined by the values of μ. Thus, because the only variable left after integration is frequency, we say that the domain of the Fourier transform is the *frequency domain*.

4.2 Preliminary Concepts - Fourier Transform

- Example of the *Fourier transform*



f(t)

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt$$

$$= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right] = AW \frac{\sin(\pi\mu W)}{(\pi\mu W)} = AW\text{sinc}(\mu W)$$

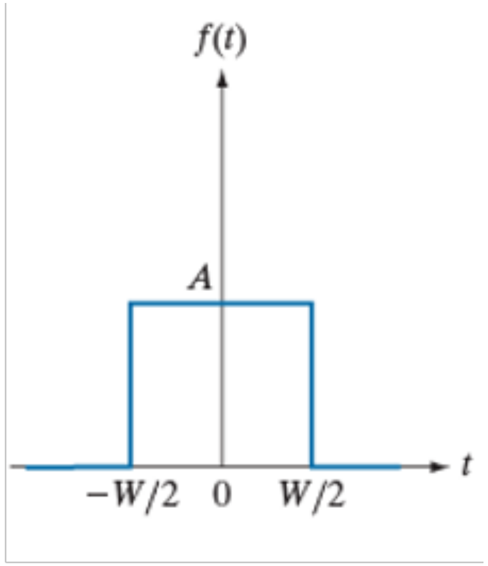
$$\boxed{\sin\theta = (e^{j\theta} - e^{-j\theta})/2j}$$

sinc function: $\text{sinc}(\text{m}) = \dfrac{\sin(\pi \text{m})}{(\pi \text{m})}$

where $\begin{cases} \text{sinc}(0) = 1 \\ \text{sinc}(\text{m}) = 0 \end{cases}$ For all other integer value of m

4.2 Preliminary Concepts - Fourier Transform

- Example of the *Fourier transform*



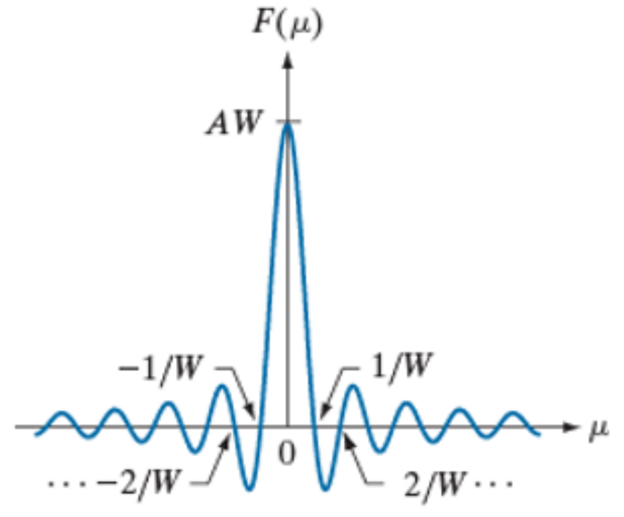
f(t)

$$F(\mu)=\int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt=\int_{-W/2}^{W/2} Ae^{-j2\pi\mu t}dt$$

$$=\frac{A}{j2\pi\mu}\left[e^{j\pi\mu W}-e^{-j\pi\mu W}\right]=AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}=AW\mathrm{sinc}(\mu W)$$

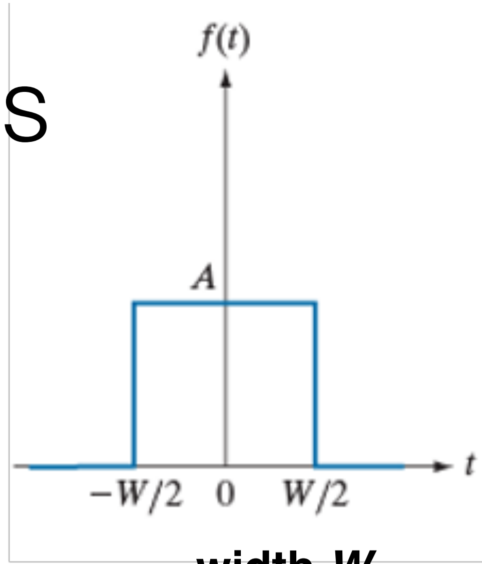
$$\boxed{\sin\theta=(e^{j\theta}-e^{-j\theta})/2j}$$

sinc function:



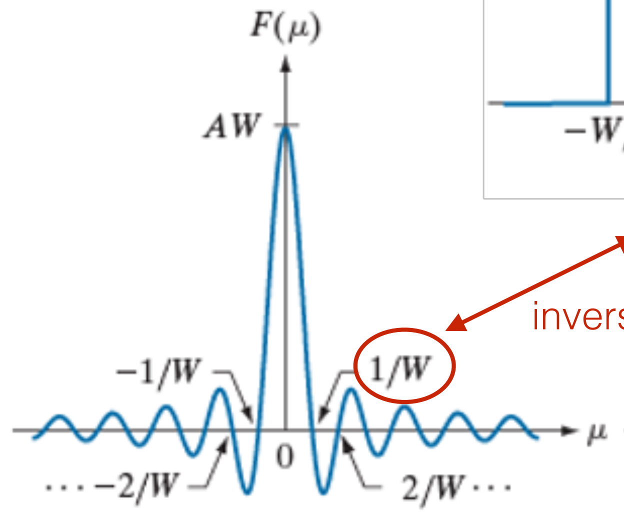
4.2 Preliminary Concepts - Fourier Transform

- Example of the *Fourier transform*



width *W*

inversely proportional



  - **Fourier transform:**

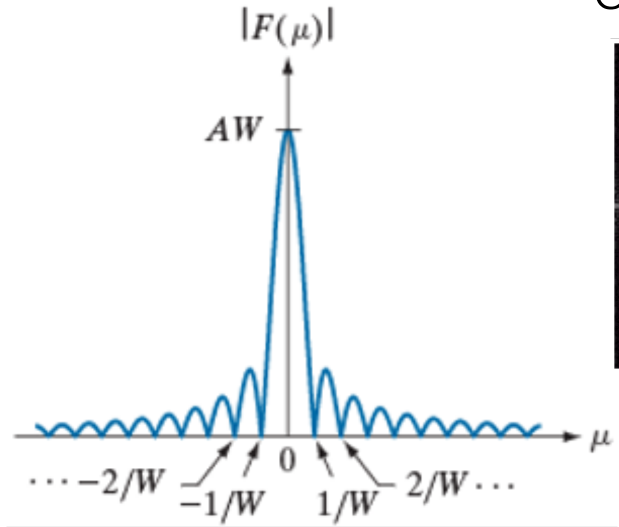
$$F(\mu) = AW\mathrm{sinc}(\mu W)$$

contains complex terms

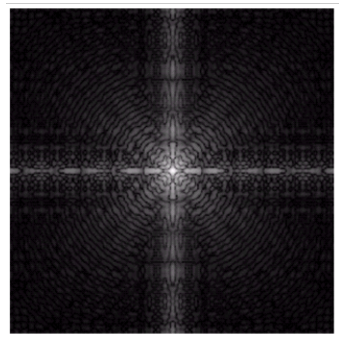
  - **Fourier spectrum:**

$$|F(\mu)|$$

for display purposes to
work with the magnitude



Ch 3.2



4.2 Preliminary Concepts - Fourier Transform

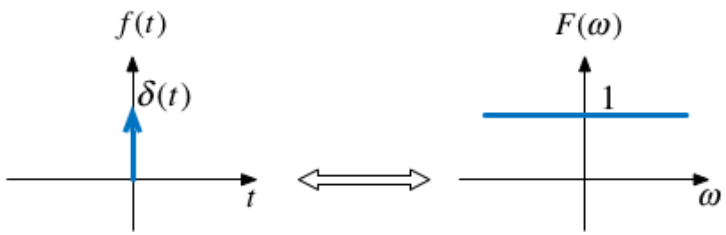
- Example of the *Fourier transform of an impulse*

f(*t*)

$$\delta(t)=\begin{cases}\infty & \text{if } t=0\\ 0 & \text{if } t\neq 0\end{cases}$$

$$F(\mu)=\int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

$$=\int_{-\infty}^{\infty} \delta(t)e^{-j2\pi\mu t}dt=e^{-j2\pi\mu 0}=e^{0}=1$$

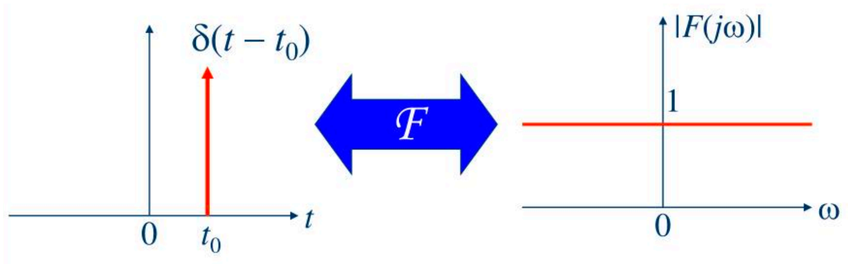


For impulse at *t* = *t*₀

$$F(\mu)=\int_{-\infty}^{\infty} \delta(t-t_0)e^{-j2\pi\mu t}dt=e^{-j2\pi\mu t_0}$$

$$=\cos(2\pi\mu t_0)-j\sin(2\pi\mu t_0)$$

a unit circle centered on the origin of the complex plane



4.2 Preliminary Concepts - Fourier Transform

- Example of the *Fourier transform of an* *impulse train*

  - **Fourier transform pair:**

$$\begin{cases} F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt \\ f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu \end{cases}$$

$$\Im\{F(t)\} = \int_{-\infty}^{\infty} F(t) e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} F(t) e^{j2\pi(-\mu)t} dt = f(-u)$$

$$\Im\{\delta(t - t_0)\} = e^{-j2\pi\mu t_0}$$

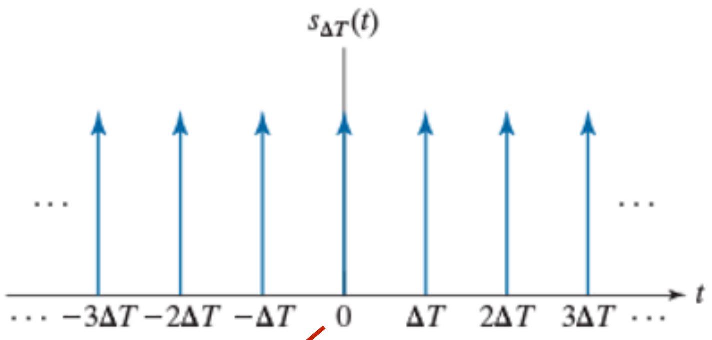
$$\Rightarrow \quad \Im\{e^{-j2\pi\mu t_0}\} = \delta(-\mu - t_0) = \delta((-t_0) - \mu) = \delta(\mu - (-t_0))$$

credit of this slide: Y. P. Hung

4.2 Preliminary Concepts - Fourier Transform

- Example of the *Fourier transform of an impulse train*
- **Impulse train**

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



expressed as a ***Fourier series***

$$= \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

where

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T}$$

$$\Rightarrow \quad s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

credit of this slide: Y. P. Hung

4.2 Preliminary Concepts - Fourier Transform

- Example of the *Fourier transform of an impulse train*

$$S(\mu) = \Im\{s_{\Delta T}(t)\} = \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

$$\boxed{\Im\{e^{-j2\pi\mu t_0}\} = \delta(\mu - (-t_0))}$$

Fourier transform of an impulse train is also an impulse train, but with inverse proportional period

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty}\delta(t - n\Delta T)$$

4.2 Preliminary Concepts - Convolution

Convolution of two continuous function

$$(f \star h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

Fourier transform

$$\Im\{f(t) * h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau\right]e^{-j2\pi\mu t}\,dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\underbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}\,dt\right]}_{\Im\{h(t-\tau)\}}d\tau$$

credit of this slide: Y. P. Hung

4.2 Preliminary Concepts - Convolution

$$\Im\{h(t-\tau)\}$$

$$\Im\{h(t-\tau)\} = \int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}dt$$

$$= \int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu(t-\tau)}e^{-j2\pi\mu\tau}d(t-\tau)$$

$$= e^{-2\pi\mu\tau}\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu(t-\tau)}d(t-\tau)$$

Let $t' = t - \tau$

$$= e^{-2\pi\mu\tau}\int_{-\infty}^{\infty} h(t')e^{-j2\pi\mu t'}dt'$$

$$= e^{-2\pi\mu\tau}H(\mu)$$

$$\Im\{h(t-\tau)\} = H(\mu)e^{-j2\pi\mu\tau}$$

credit of this slide: Y. P. Hung

4.2 Preliminary Concepts - Convolution

$$\Im\{f(t)*h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right]e^{-j2\pi\mu t}dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}dt\right]d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu\tau}\right]d\tau$$

$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}d\tau$$

$$= H(\mu)F(\mu)$$

$$\Im\{h(t-\tau)\} = H(\mu)e^{-j2\pi\mu\tau}$$

credit of this slide: Y. P. Hung

4.2 Preliminary Concepts - Convolution

- **Convolution Theorem**

$$\begin{cases} f(t) * h(t) \Leftrightarrow H(\mu)F(\mu) \\ f(t)h(t) \Leftrightarrow H(\mu) * F(\mu) \end{cases}$$

Denote $*$ as *convolution* here

4.3 Sampling and the Fourier Transform of Sampled Functions

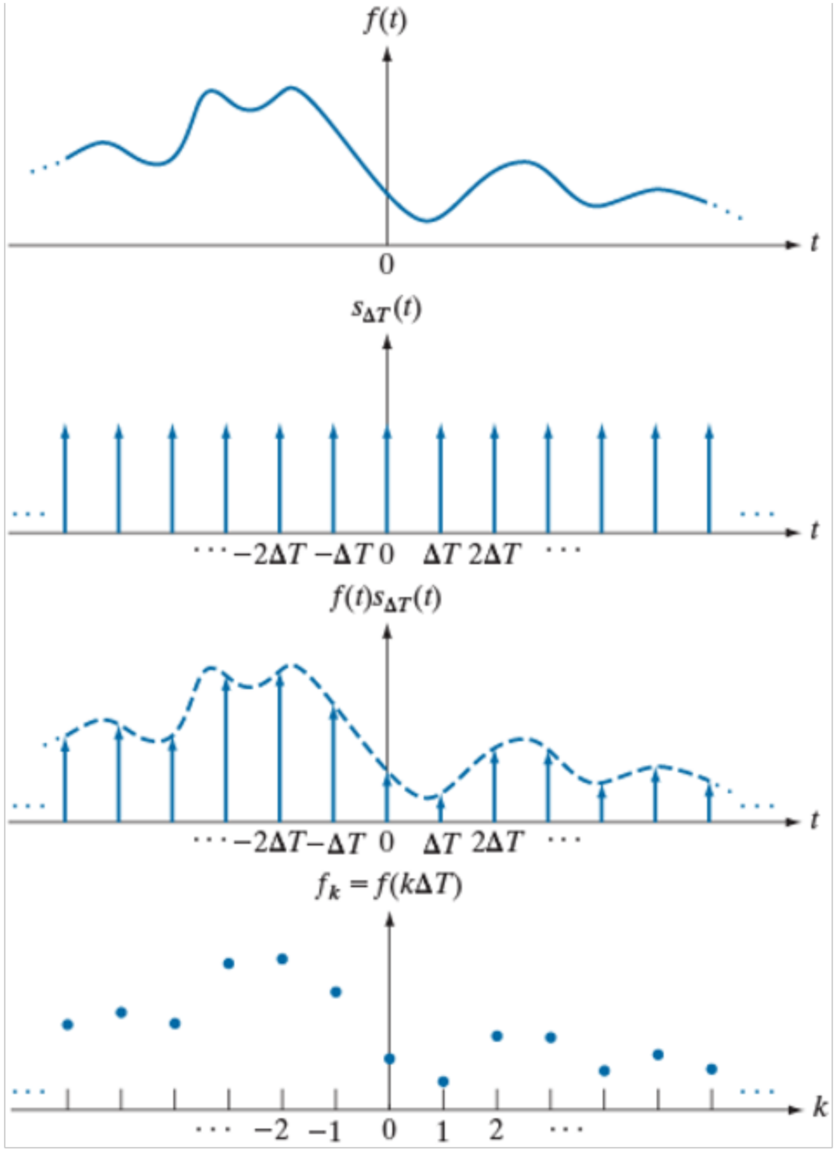
- Sampling

Sampling rate: $1/\Delta T$

Sampled function

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt = f(k\Delta T)$$



4.3 Sampling and the Fourier Transform of Sampled Functions

- the Fourier Transform of Sampled Functions

$$\tilde{F}(\mu)=\Im\left\{\tilde{f}(t)\right\}=\underline{\Im\left\{f(t)s_{\Delta T}(t)\right\}=F(\mu)*S(\mu)}$$

Convolution Theorem

$$=\int_{-\infty}^{\infty}F(\tau)S(\mu-\tau)d\tau$$

$$S(\mu)=\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu-\frac{n}{\Delta T}\right)$$

$$=\frac{1}{\Delta T}\int_{-\infty}^{\infty}F(\tau)\sum_{n=-\infty}^{\infty}\delta\left(\mu-\tau-\frac{n}{\Delta T}\right)d\tau$$

$$=\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}F(\tau)\delta\left(\mu-\tau-\frac{n}{\Delta T}\right)d\tau$$

$$=\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}F(\tau)\delta\left((\mu-\frac{n}{\Delta T})-\tau\right)d\tau$$

Sifting property of impulse

$$=\boxed{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}F(\mu-\frac{n}{\Delta T})}$$

credit of this slide: Y. P. Hung

4.3 Sampling and the Fourier Transform of Sampled Functions

- the Fourier Transform of Sampled Functions

$$\widetilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$$

$$n = 0 \Rightarrow F(\mu)$$

$$n = 1 \Rightarrow F(\mu - \frac{1}{\Delta T})$$

$$n = 2 \Rightarrow F(\mu - \frac{2}{\Delta T})$$

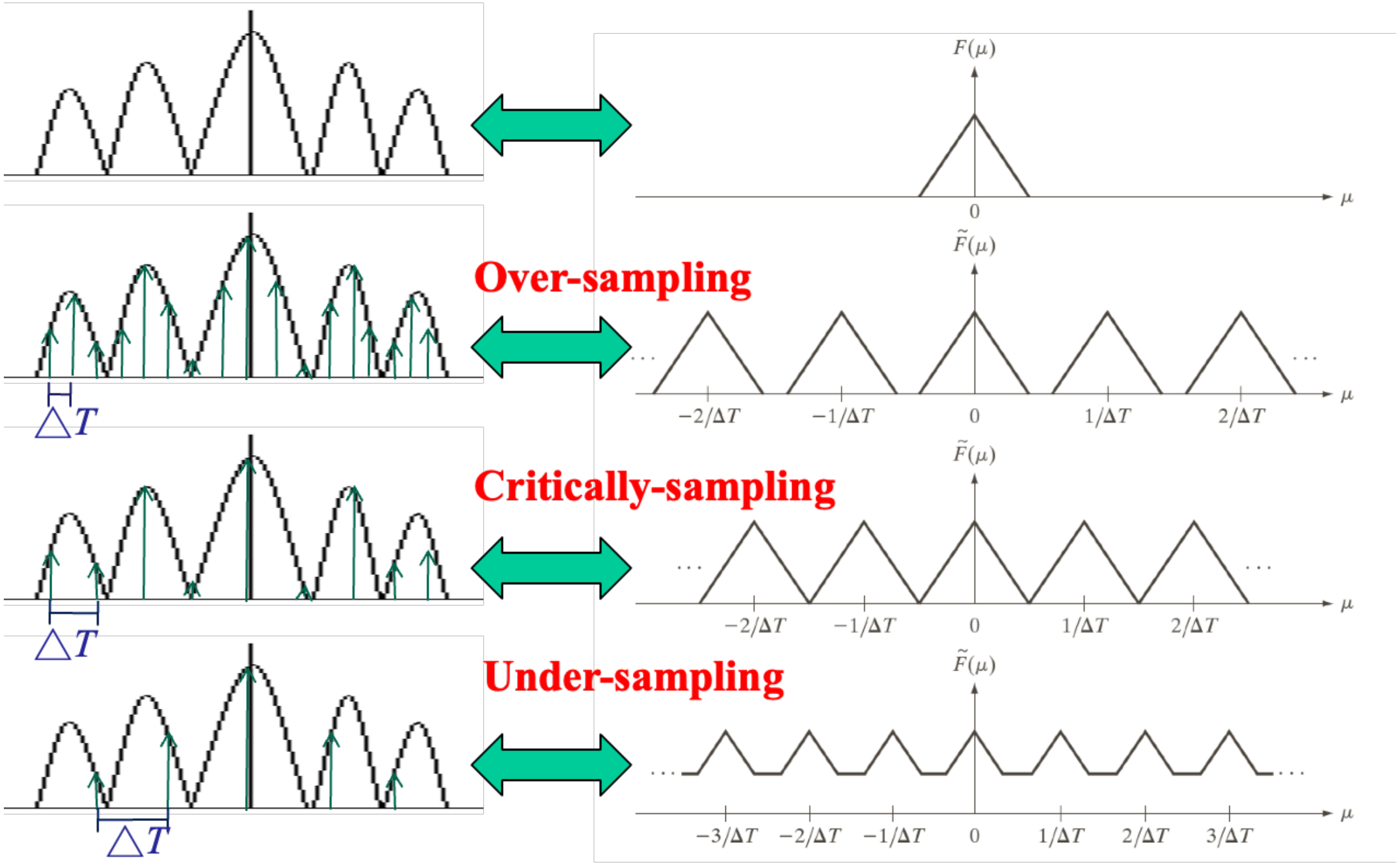
An infinite, periodic sequence of copies of the transform of the original, continuous function $F(\mu)$

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$$

$$\Im\{tri(t)\} = \text{sinc}^2(\mu)$$

4.3 Sampling and the Fourier Transform of Sampled Functions

- Fourier transform of a band-limited function



credit of this slide: Y. P. Hung

4.3 Sampling and the Fourier Transform of Sampled Functions -The Sampling Theorem

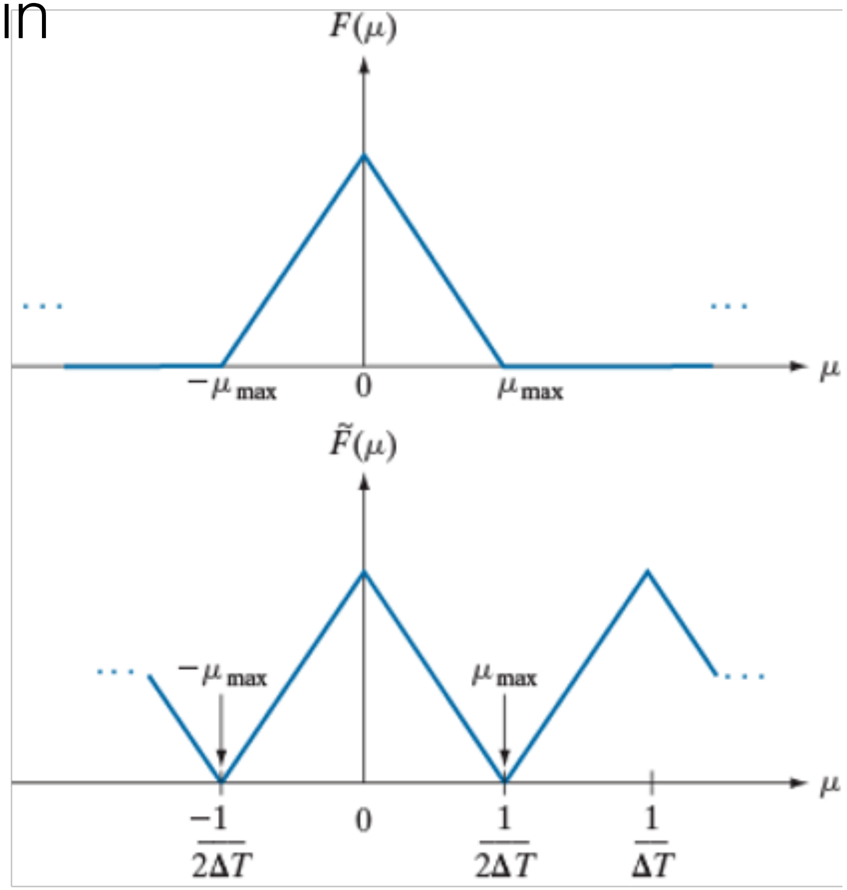
- **Band-limited function**

  A function $f(t)$ whose Fourier transform is zero for values of frequencies outside a finite interval (band) $[-\mu_{max}, \mu_{max}]$ about the origin

- **Sampling Theorem**

  A continuous, band-limited function can be recovered completely from a set of its samples if

$$\frac{1}{\Delta T} > 2\mu_{\max}$$ *Nyquist rate*



4.3 Sampling and the Fourier Transform of Sampled Functions -The Sampling Theorem

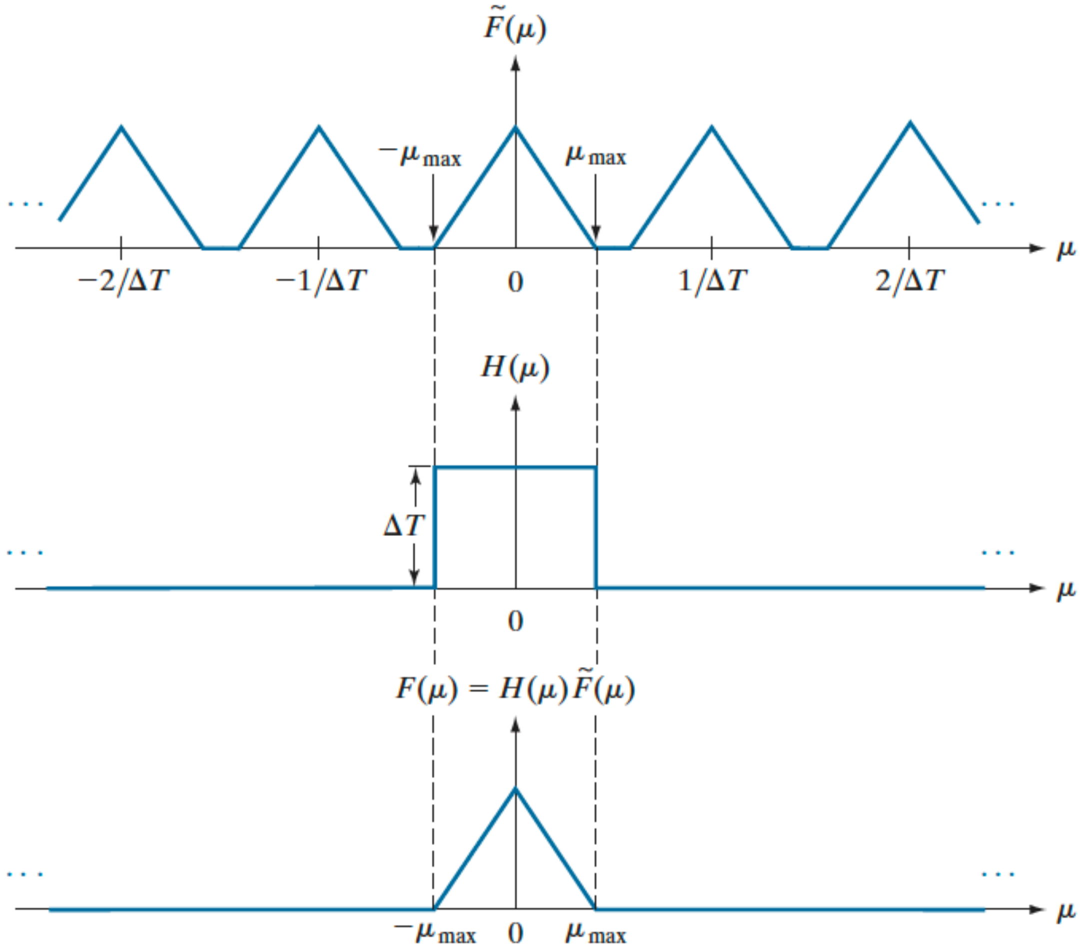
- **Reconstruction filters**
  - – Provided a correct sampling, the continuous signal **may** be perfectly reconstructed by its samples.

$$F(\mu) = H(\mu)\widetilde{F}(\mu)$$

Ideal lowpass filter

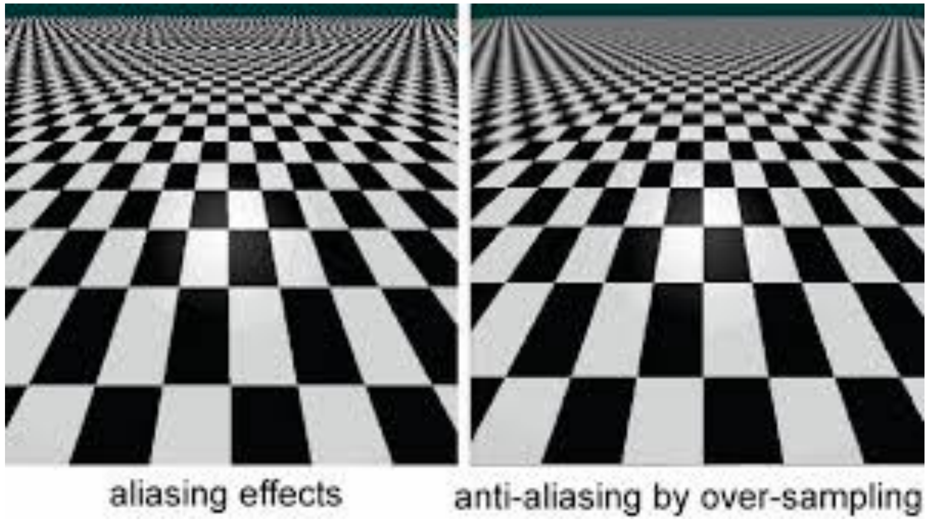
$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \leqq \mu \leqq \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

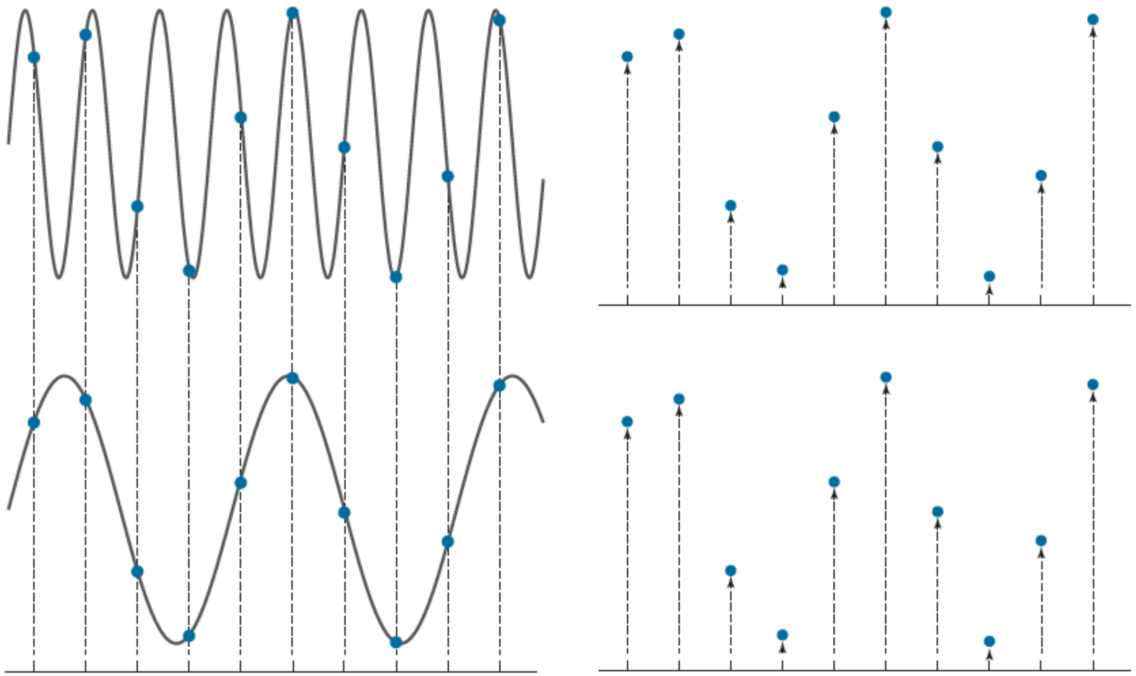


4.3 Sampling and the Fourier Transform of Sampled Functions - Aliasing

Aliasing refers to sampling phenomena that cause different signals to become indistinguishable from one another after sampling (under sampling).

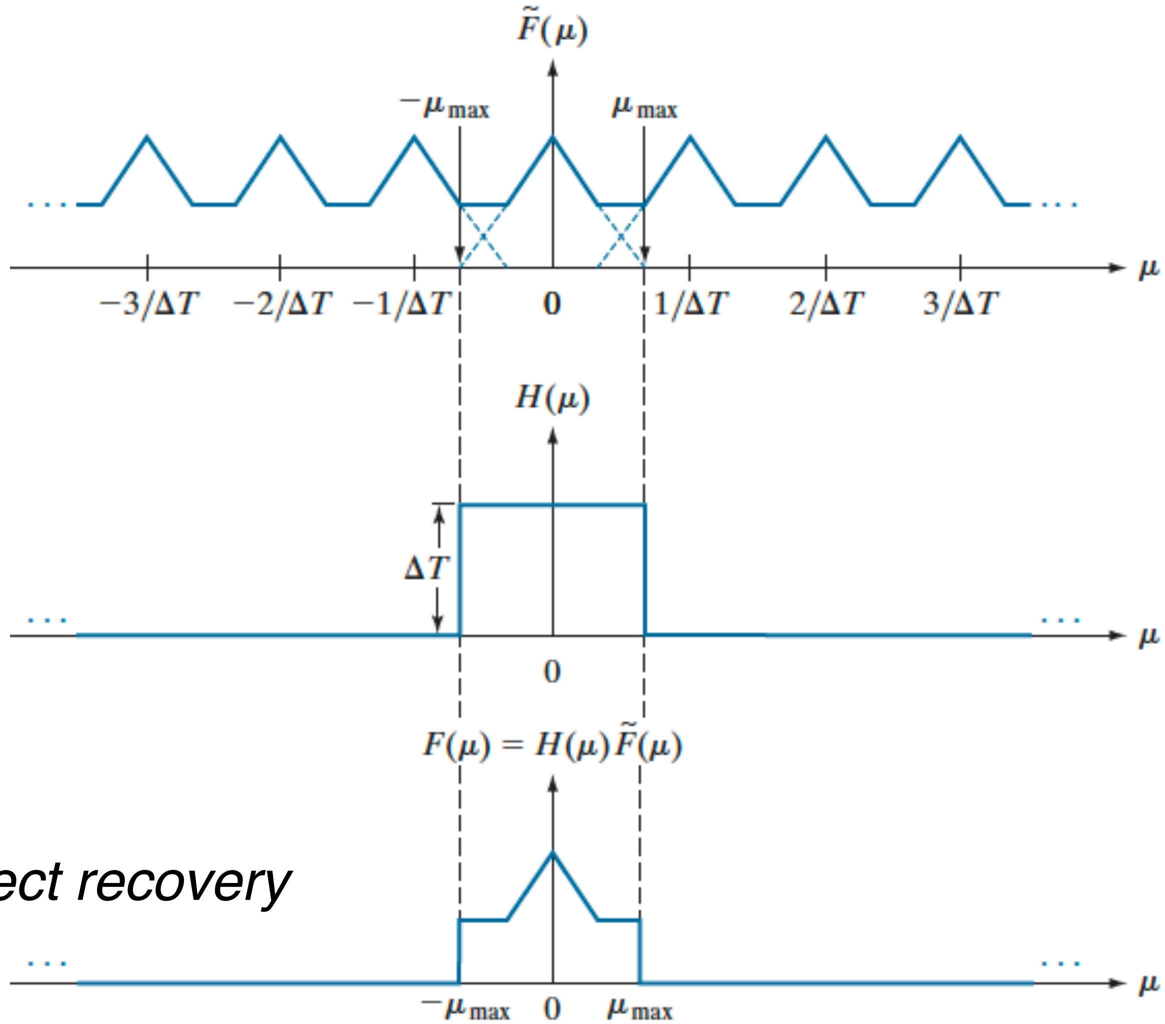


Example of ***aliased pair***



4.3 Sampling and the Fourier Transform of Sampled Functions - Aliasing

Aliasing refers to sampling phenomena that cause different signals to become indistinguishable from one another after sampling (under sampling).

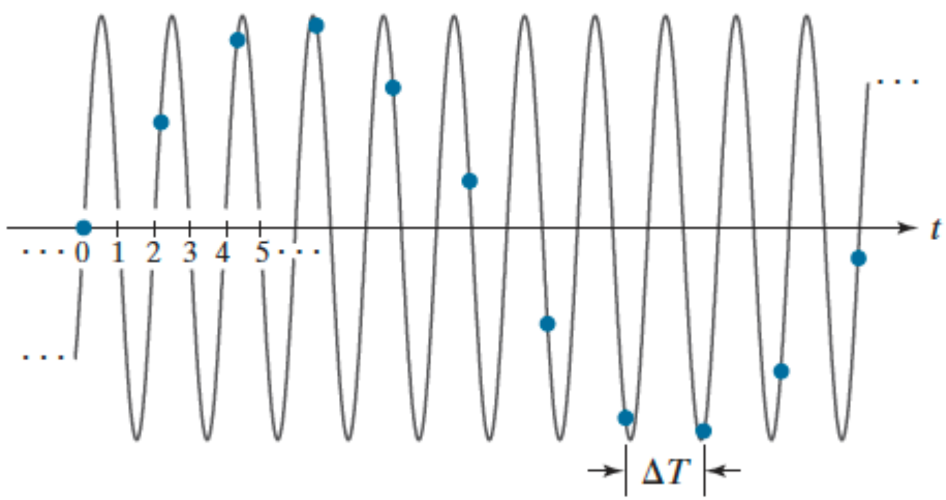


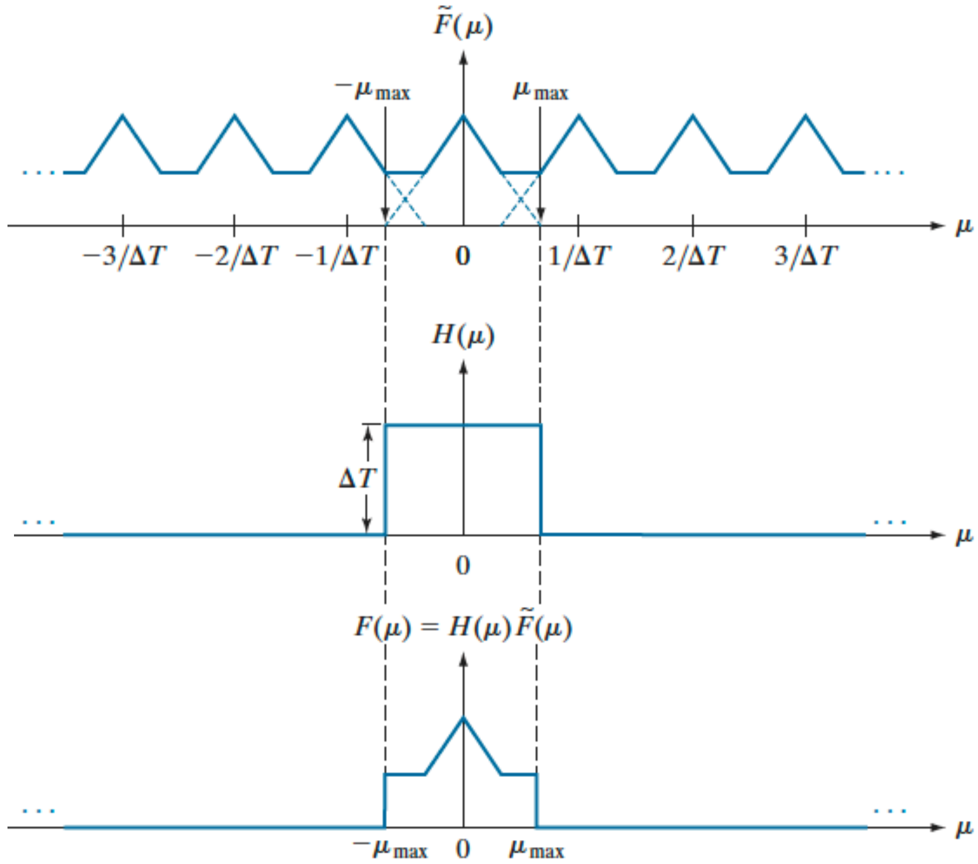
Smooth

prevents perfect recovery

4.3 Sampling and the Fourier Transform of Sampled Functions - Aliasing

Anti-aliasing:
the effects of aliasing can be reduced by smoothing (lowpass filtering) the input function to attenuate its higher frequencies.





4.3 Sampling and the Fourier Transform of Sampled Functions
- Function Reconstruction (Recovery) from Sampled Data

$$f(t) = \mathfrak{I}^{-1}\{F(\mu)\} = \mathfrak{I}^{-1}\{H(\mu)\tilde{F}(\mu)\} = h(t) * \tilde{f}(t)$$

Convolution Theorem

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\text{sinc}\left[(t - n\Delta T)/\Delta T\right]$$

- It shows the perfectly reconstructed function, $f(t)$, is an infinite sum of sinc functions weighted by the sample values.
- The reconstructed function is identically equal to the sample values at multiple integer increments of ΔT.
- Between sample points, values of $f(t)$ are interpolations formed by the sum of the sinc functions.

$$\begin{cases} \text{sinc}(0) = 1 \\ \text{sinc}(m) = 0 \quad \text{For all other integer value of } m \end{cases}$$

credit of this slide: Y. P. Hung

$$f(t) = \Im^{-1}\{F(\mu)\} = \Im^{-1}\{H(\mu)\tilde{F}(\mu)\} = h(t) * \tilde{f}(t)$$

$$\Rightarrow \quad f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\underline{\mathrm{sinc}\left[(t - n\Delta T)/\Delta T\right]}$$

$$\tilde{f}(t) * h(t) = \int_{-\infty}^{\infty} \tilde{f}(\tau)h(t-\tau)d\tau$$

$$\boxed{\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)}$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(\tau)\delta(\tau - n\Delta T)h(t-\tau)d\tau$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)\delta(\tau - n\Delta T)h(t-\tau)d\tau$$

Sifting property of impulse

$$= \sum_{n=-\infty}^{\infty} f(n\Delta T)\underline{h(t - n\Delta T)}$$

we have to show that $h(t - n\Delta T) = \mathrm{sinc}\left[(t - n\Delta T)/\Delta T\right]$

credit of this slide: Y. P. Hung

$$h(t-n\Delta T)=\mathrm{sinc}\left[(t-n\Delta T)/\Delta T\right]$$

$$H(\mu)=\begin{cases}\Delta T & -1/(2\Delta T)\leqq\mu\leqq 1/(2\Delta T)\\ 0 & \text{otherwise}\end{cases}$$

$$h(t)=\int_{-\infty}^{\infty}H(\mu)e^{j2\pi\mu t}d\mu=\int_{-1/2\Delta T}^{1/2\Delta T}\Delta Te^{j2\pi\mu t}d\mu$$

$$=\frac{\Delta T}{j2\pi t}\left(e^{j\pi t/\Delta T}-e^{-j\pi t/\Delta T}\right)$$

$$\boxed{\sin\theta=(e^{j\theta}-e^{-j\theta})/2j}$$

$$=\frac{\Delta T}{\pi t}\sin(\pi t/\Delta T)=\frac{\sin(\pi t/\Delta T)}{\pi t/\Delta T}$$

$$\boxed{\mathrm{sinc(m)}=\frac{\sin(\pi\mathrm{m})}{(\pi\mathrm{m})}}$$

$$=\mathrm{sinc}(t/\Delta T)$$

$$\therefore h(t-n\Delta T)=\mathrm{sinc}\left[(t-n\Delta T)/\Delta T\right]$$

credit of this slide: Y. P. Hung