Fast Engset computation

Parsiad Azimzadeh, Tommy Carpenter

February 23, 2015

Abstract

The *Engset formula* is used to determine the *blocking probability* P occurring in a queue with a finite number of sources. In particular, P is expressed as a fixed point of some function: $P = f(P)$. We provide a proof of the *existence and uniqueness* of this fixed point, showing that the Engset formula is well-posed. The literature [2, 3] ubiquitously suggests the use of a *fixed point iteration* to compute P, although it is noted that this procedure may not converge [3]. We give sufficient conditions on convergence (along with examples in which convergence does not occur) and demonstrate that it is sometimes outperformed by a simple, *unconditionally convergent bisection method.* We provide a global convergence result for *Newton's method* and demonstrate that this method converges to the fixed point in just a few iterations.

1 Engset formula

The *Engset formula* is used to determine the *blocking probability* P (a.k.a. *probability of congestion*) occurring in a queue with a finite number of sources, such as a telephony circuit group. In teletraffic engineering, this quantity provides the grade of service (GoS): the probability that a new call is rejected because all circuits are busy.

The formula requires that the user knows the *number of servers* M (e.g. circuits), the *number of sources* S (e.g. callers), and the *offered traffic* E (the average number of concurrent calls/requests that would have been carried if there were an unlimited number of servers). The Engset formula is

$$P = \underbrace{\frac{\binom{S-1}{M} A^M}{\sum_{X=0}^{M} \binom{S-1}{X} A^X}}_{f(P)} \text{ where } A = \frac{E}{S - E(1-P)}. \qquad \text{(Engset formula)}$$

Noting that A itself is a function of P, it is not straightforward to find a value of P that satisfies the Engset formula. It is also not particularly obvious that if such a value of P exists, it is unique (the Engset formula would not be very useful if it yielded two or more distinct blocking probabilities, or none at all!). Existence and uniqueness is established in section §2.

Contrast the above with the *Erlang-B formula*:

$$P = \frac{E^M / M!}{\sum_{X=0}^{M} E^X / X!}. \qquad \text{(Erlang-B formula)}$$

Instead of a finite number, the Erlang-B formula assumes an infinite number of sources,[1] yielding a *Poisson arrival process*. Most notably, the quantity appearing in the right-hand side of the Erlang-B formula does not depend on P, and hence computing the blocking probability under the Erlang-B assumption is trivial.

2 Existence and uniqueness

We begin by pruning out uninteresting pathological cases from our line of inquiry. If the number of servers M is zero, any request entering the queue is blocked ($P = 1$). If, on the other hand, there are at least as many servers as there are sources S, any request entering the queue can immediately be serviced ($P = 0$). Finally, we note that the case of zero offered traffic E corresponds to a queue that receives no requests (P is not well-defined). In light of this, we assume the following for the remainder of this work:

Assumption. *M and S are integers with $1 < M < S$. E is a positive number.*

Our first order of business is to verify that the Engset formula yields a unique blocking probability. As mentioned previously, the formula would not be meaningful if it produced two or more distinct blocking probabilities!

Theorem 1. *There exists a unique probability P satisfying the Engset formula.*

A proof of this result is given in appendix §A and also establishes the fact that the map

$$P \mapsto f(P) - P$$

is strictly decreasing on $[0, 1]$, allowing us to approximate the blocking probability via the *bisection method* (algorithm 1). Most notably, the resulting method is *unconditionally convergent*. We will see that the more commonly used *fixed point iteration* is only *conditionally convergent* (section §3) and often outperformed by bisection (section §5).

Starting with the unit interval, the bisection method halves the search interval at each step, so that the maximum possible error at the nth iteration is 2^{-n}. Therefore, given a desired error tolerance *tol*, bisection requires at most $\lceil -\lg(tol) \rceil$ iterations. Bisection gives us an upper bound on the amount of work required to compute the blocking probability.

[1] By taking the limit as $S \to \infty$, the reader can verify that the Erlang-B formula is a special case of the Engset formula.

Algorithm 1 Computing the Engset formula via the bisection method.

```
ENGSET-BISECTION(M, S, E, tol)
lo = 0
hi = 1
while TRUE
    P = (lo + hi) / 2
    if (hi − lo) / 2 ≤ tol
        return P
    if f (P; M, S, E) − P < 0
        hi = P
    else
        lo = P
```

3 Fixed point iteration

A fixed point iteration involves picking an initial guess P_0 for the blocking probability and considers the iterates of f evaluated at P_0. That is to say,

$$\begin{aligned} &P_0 \text{ a probability} \\ &P_n = f\left(P_{n-1}\right) \text{ for } n > 0. \end{aligned} \tag{fixed point iteration}$$

If $P_n \to P$ for some probability P (i.e. the iteration is convergent), we have found a probability that satisfies Engset formula:[2]

$$P = f\left(P\right).$$

Theorem 2. *A fixed point iteration converges to the blocking probability satisfying the Engset formula whenever $S \geq E$ and $|f'(0)| < 1$.*

Evaluating the derivative of f is computationally expensive, and—perhaps more importantly—the form above is not particularly easy to comprehend. Therefore, we present the following relaxation:

Corollary 3. *A fixed point iteration converges to the blocking probability satisfying the Engset formula whenever $S \geq E$ and $S \geq 2M$.*

Proofs of these results are given in appendix §A.

[2]This follows from taking limits on both sides of $P_n = f\left(P_{n-1}\right)$ and using the fact that f defined in the Engset formula is continuous.

4 Newton's method

Newton's method uses first-derivative information to speed up convergence. In particular,

$$
\begin{aligned}
&P_0 \text{ a probability} \\
&P_n = P_{n-1} - \frac{f\left(P_{n-1}\right) - P_{n-1}}{f'\left(P_{n-1}\right) - 1} \text{ for } n > 0. \qquad \text{(Newton's method)}
\end{aligned}
$$

As usual, if $P_n \to P$ for some probability P (i.e. the iteration is convergent), P satisfies the Engset formula.[3]

In general root-finding problems, strong guarantees regarding convergence cannot usually be made about Newton's method (often, convergence results for Newton's method are local in nature; i.e. for an initial guess P_0 sufficiently close to the solution). Therefore, the following nonlocal convergence result—whose proof is given in appendix §A—is pleasantly surprising:

Theorem 4. *Newton's method converges to the blocking probability satisfying the Engset formula whenever $S \geq E$ (independent of the initial guess).*

In practice, fast convergence is observed even when $S < E$, regardless of the initial guess (section §5).

5 Comparison of methods

We compare the performance of bisection (algorithm 1), fixed point iteration, and Newton's method in table 1.

We note that the fixed point iteration does not converge (DNC) or performs poorly precisely when the sufficient conditions in corollary 3 are violated. It is greatly outperformed by bisection in many cases. The erratic performance of the fixed point iteration is due to its oscillatory nature, as demonstrated in figure 1.

Newton's method outperforms both algorithms by a wide margin, often converging in just a few iterations. It is difficult to give a tight bound on the number of iterations, so we must content ourselves with a rough explanation of the underlying phenomenon. Let P denote the fixed point of f and $\epsilon_n = |P - P_n|$ denote the error at the nth step of Newton's method. It is well-known that if the iterates lies in a sufficiently smal neighbourhood N of P,

$$\epsilon_{n+1} \leq \sup_{Q \in N} \frac{1}{2} \left| \frac{f''\left(Q\right)}{f'\left(Q\right) - 1} \right| \epsilon_n^2.$$

[3] This follows from taking limits on both sides of

$$P_n = P_{n-1} - \frac{f\left(P_{n-1}\right) - P_{n-1}}{f'\left(P_{n-1}\right) - 1}$$

and using the facts that f is continuously differentiable and that $f' < 0$ on the nonnegative half-line (see the proof of theorem 1 in appendix A).

Servers	Sources	Offered	Blocking	Number of iterations		
		traffic	probability	Bisection	Fixpoint	Newton
M	S	E		(alg. 1)		
5	10	2	0.016350	24	5	3
5	20	2	0.026584	24	5	3
5	10	5	0.247678	24	14	4
5	10	10	0.552256	24	32	3
5	10	20	0.761709	24	82	3
10	20	20	0.532659	24	45	3
12	20	20	0.442391	24	556	4
15	20	20	0.308416	24	DNC	5
19	20	30	0.384872	24	DNC	8

Table 1: Comparison of algorithms. The stopping criterion and initial guess used in the fixed point iteration and Newton's method are $|P_{n+1} - P_n| \leq tol = 2^{-24}$ (single precision machine-ϵ [6]) and $P_0 = 1/2$, respectively. DNC indicates that the method did not converge.

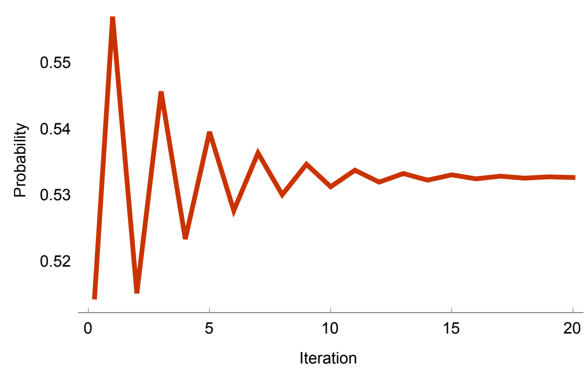

Figure 1: Oscillatory nature of the fixed point iteration demonstrated on $M = 10$, $S = 20$, $E = 20$, and $P_0 = 1/2$.

tol	2^{-24}	2^{-53}	2^{-113}
$\lceil \lg(1 - \lg tol) \rceil - 1$	4	5	6

Table 2: Approximate upper bound on Newton's method iterations required to guarantee results accurate to single, double, and quadruple precision (a.k.a. IEEE 754-2008 [6] binary32, binary64, and binary128) machine-ϵ.

Indeed, if we assume

$$\left| \frac{f''(Q)}{f'(Q) - 1} \right| \leq 1 \tag{1}$$

along with $P_0 = 1/2$ (and hence $\epsilon_0 \leq 1/2$), we arrive at $\epsilon_n \leq 2^{1-2^{n+1}}$. Therefore, the *approximate* number of iterations required for convergence to a desired error tolerance $0 < tol < 1$ is

$$\lceil \lg(1 - \lg tol) \rceil - 1$$

(table 2). This agrees with all but the last two rows of table 1. Presumably, inequality (1) is violated there.

Thanks: We thank Dmitrii B. Karp (Far Eastern Federal University) for enlightening discussions. We thank Srinivasan Keshav and Johnny W. Wong (University of Waterloo) for advice and direction.

References

[1] Milton Abramowitz, Irene A Stegun, et al. *Handbook of mathematical functions*, volume 1. Dover Publications, 1965.

[2] Villy B Iversen. Teletraffic engineering and network planning. ftp://ftp.dei.polimi.it/users/Flaminio.Borgonovo/Teoria/teletraffic_Iversen.pdf, 2010.

[3] Srinivasan Keshav. *An engineering approach to computer networking: ATM networks, the Internet, and the telephone network*, volume 1. Addison-Wesley, 1997.

[4] Richard S Palais. A simple proof of the Banach contraction principle. *Journal of Fixed Point Theory and Applications*, 2(2):221–223, 2007.

[5] Michael Spivak. *Calculus*, volume 3. Publish or Perish, 1994.

[6] Dan Zuras, Mike Cowlishaw, Alex Aiken, Matthew Applegate, David Bailey, Steve Bass, Dileep Bhandarkar, Mahesh Bhat, David Bindel, Sylvie Boldo, et al. IEEE standard for floating-point arithmetic. *IEEE Std 754-2008*, pages 1–70, 2008.

A Proofs

This appendix makes use of the *ordinary hypergeometric function* ${}_2F_1$, for which there exists a wealth of identities [1]. Let $(\cdot)_X$ denote the *Pochhammer symbol*:

$$(c)_X = \begin{cases} c\,(c+1)\cdots(c+X-1)\,, & X \text{ a positive integer} \\ 1 & X = 0. \end{cases} \tag{Pochhammer symbol}$$

Then, the hypergeometric function[4] is given by

$${}_2F_1\,(a,b;c;z) = \sum_{X=0}^{\infty} \frac{(a)_X\,(b)_X}{(c)_X} \frac{z^X}{X!} \text{ whenever } |z| < 1 \text{ or } b \text{ a negative integer.} \tag{hypergeometric function}$$

Note that the Pochhammer symbol can be used to represent *falling factorials* as well. That is,

$$\begin{aligned} c^{(X)} &= c\,(c-1)\cdots(c-X+1) \\ &= (-c)\,(-1)\,(-c+1)\,(-1)\cdots(-c+X-1)\,(-1) = (-c)_X\,(-1)^X\,. \end{aligned}$$

Lemma 5. *$f\,(P)$ defined in the Engset formula can be written*

$$f\,(P) = \frac{1}{{}_2F_1\,(1,-M;S-M;1-P-S/E)}$$

Proof. Taking the reciprocal of A defined in the Engset formula yields

$$\frac{1}{A} = -\,(1-P-S/E)\,. \tag{2}$$

We can use this fact to write the binomial terms appearing in the Engset formula in terms of Pochhammer symbols as follows:

$$\frac{\binom{S-1}{X}}{\binom{S-1}{M}} = \frac{M!}{X!} \frac{(S-1-M)!}{(S-1-X)!} = \frac{M^{(M-X)}}{(S-M)_{M-X}}. \tag{3}$$

Noting that f is nowhere zero, substituting equation (2) and equation (3) into the reciprocal of $f\,(P)$ yields

$$\begin{aligned} \frac{1}{f\,(P)} &= \sum_{X=0}^{M} \frac{\binom{S-1}{X}}{\binom{S-1}{M}} \left(\frac{1}{A}\right)^{M-X} = \sum_{X=0}^{M} \frac{M^{(M-X)}}{(S-M)_{M-X}} \left(-\,(1-P-S/E)\right)^{M-X} \\ &= \sum_{X=0}^{M} \frac{(-M)_X}{(S-M)_X} (1-P-S/E)^X = \sum_{X=0}^{\infty} \frac{(-M)_X}{(S-M)_X} (1-P-S/E)^X\,. \end{aligned} \tag{4}$$

The upper bound of summation was harmlessly relaxed to ∞ in the last equality due to $(-M)_X = 0$ whenever $X > M$. The desired result then follows from multiplying each summand in the series by $(1)_X/X! = 1$. □

[4] We take ${}_2F_1\,(a,-b;-c;z) = \infty$ (complex infinity) whenever c is a positive integer and b is not a positive integer strictly smaller than c. We also employ the convention $0 \cdot \infty = \infty \cdot 0 = 0$ throughout.

Although the queueing theory literature does not make explicit use of the hypergeometric function in its representation, equation (4) is reminiscent of the recursive form of the Engset formula [2] that avoids the unstable computation of the binomial terms appearing therein.

We use the following identity frequently:

Lemma 6. *Suppose b is a negative integer. Then,*

$$ {}_2F_1\left(a,b;c;z+w\right)=\sum_{Y=0}^{\infty}\frac{(a)_Y\,(b)_Y}{(c)_Y}\frac{z^Y}{Y!}{}_2F_1\left(a+Y,b+Y;c+Y;w\right). $$

Proof. This fact follows from an application of the binomial theorem:

$$ \begin{aligned} {}_2F_1\left(a,b,c;z+w\right) &= \sum_{X=0}^{\infty}\frac{(a)_X\,(b)_X}{(c)_X}\frac{(z+w)^X}{X!} \\ = \sum_{X=0}^{\infty}\frac{(a)_X\,(b)_X}{(c)_X}\frac{1}{X!}\sum_{Y=0}^{X}\binom{X}{Y}z^Y w^{X-Y} &= \sum_{Y=0}^{\infty}\frac{z^Y}{Y!}\sum_{X=Y}^{\infty}\frac{(a)_X\,(b)_X}{(c)_X}\frac{w^{X-Y}}{(X-Y)!} \\ &= \sum_{Y=0}^{\infty}\frac{(a)_Y\,(b)_Y}{(c)_Y}\frac{z^Y}{Y!}\sum_{X=Y}^{\infty}\frac{(a+Y)_{X-Y}\,(b+Y)_{X-Y}}{(c+Y)_{X-Y}}\frac{w^{X-Y}}{(X-Y)!}. \end{aligned} $$

The desired result follows by shifting the inner index of summation to $X=0$. □

There is an obvious relaxation of the above lemma to cases where b is not a negative integer, but care must be taken to ensure that both sides of the equality are convergent.

Proof of theorem 1. The statement of theorem 1 is equivalent to the claim that the map

$$ P\mapsto f\left(P\right)-P \tag{5} $$

has a unique root strictly between zero and one.

We first seek to show that $P\mapsto 1/f\left(P\right)$ is a polynomial with positive coefficients. That is to say,

$$ \frac{1}{f\left(P\right)}=\sum_{Y=0}^{M}c_Y P^Y \text{ where } c_Y>0. \tag{6} $$

If $S\geq E$, A is nonnegative, and this fact follows directly from equation (4). The case of $S<E$ is more involved. An application of lemma 6 to the form in lemma 5 reveals that the coefficients are

$$ c_Y=\frac{M^{(Y)}}{(S-M)_Y}{}_2F_1\left(Y+1,-\left(M-Y\right);S-M+Y;1-S/E\right). \tag{7} $$

To arrive at equation (6), it suffices to show that the hypergeometric term appearing above is positive.[5] Another application of lemma 6 along with the identity

$$ {}_2F_1(a, -b; c; 1) = \frac{(c-a)_b}{(c)_b} \text{ whenever } b \text{ is a nonnegative integer.} $$

yields

$$ \begin{aligned} &{}_2F_1(Y+1, -(M-Y); S-M+Y; 1-S/E) \\ &\quad = \sum_{Z=0}^{M-Y} \frac{(S/E)^Z}{Z!} \frac{(Y+1)_Z (M-Y)^{(Z)}}{(S-M+Y)_Z} \frac{(S-M-1)_{M-Y-Z}}{(S-M+Y+Z)_{M-Y-Z}}. \end{aligned} \tag{8} $$

In this form, it is obvious that the hypergeometric term is positive.

We now show that $f(1) < 1$, or equivalently, $1/f(1) > 1$. By the positivity of (8), the map $E \mapsto 1/f(P;E)$ is strictly decreasing. It is possible to pass to the limit and drop higher order terms involving $1/E^Z$ with $Z > 0$ to get

$$ \frac{1}{f(P;E)} > \lim_{E' \to \infty} \frac{1}{f(P;E')} = \sum_{Y=0}^{M} P^Y \frac{(-M)_Y (-1)^Y}{(S-M)_Y} \frac{(S-M-1)_{M-Y}}{(S-M+Y)_{M-Y}}. $$

It is not too hard to verify that if $P = 1$, the above sum is exactly one, yielding $1/f(1) > 1$ (for all $S < E < \infty$), as desired.

By equation (6), the map (5) is strictly decreasing on the nonnegative half-line $[0, \infty)$. Furthermore, since $f(1) < 1$, $f(0) - 0$ and $f(1) - 1$ have opposite signs. Because the map (5) is also continuous, the desired result follows by the intermediate value theorem. $\square$

The proof of theorem 2 requires the following result, whose verification is left to the reader:

Lemma 7. *Let*

$$ A(Q) = \sum_{X=0}^{N_A} a_X Q^X \text{ and } B(Q) = \sum_{X=0}^{N_B} b_X Q^X $$

be distinct polynomials with positive coefficients satisfying $N_B \geq N_A$, $b_0 \leq a_0$, *and* $b_X \geq a_X$ *for* $0 < X \leq N_A$. *Then, the map*

$$ Q \mapsto \frac{A(Q)}{B(Q)} $$

is strictly decreasing on the nonnegative half-line $[0, \infty)$.

[5] A concise proof of this is fact is given by the Euler transform [1]: ${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z)$. We use a more elementary approach instead.

Proof of theorem 2. We employ the *Banach fixed point theorem* [4] to show that $S \geq E$ and $|f'(0)| < 1$ are sufficient conditions for the iterates of f to converge to the blocking probability.

In particular, Let $H = [0, \infty)$ denote the nonnegative half-line. The proof of theorem 1 establishes that $f(H) \subset H$. Therefore, it suffices to show that f is a *contraction* on H. That is, $|f'| \leq L < 1$ on H for some constant $L > 0$.

For notational succinctness, let $N = S - M$. The derivative of the hypergeometric function—with respect to the main argument—is [1]

$$\frac{\partial}{\partial z} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z).$$

This fact combined with the representation in lemma 5 yields

$$|f'(P)| = \frac{M}{N} \frac{A(P + S/E - 1)}{B(P + S/E - 1)}$$

where

$$A(Q) = {}_2F_1(2, -(M-1); N+1; -Q) \text{ and } B(Q) = {}_2F_1(1, -M; N; -Q)^2.$$

We seek to show that the map

$$Q \mapsto \frac{A(Q)}{B(Q)} \tag{9}$$

is strictly decreasing on H. If this is the case, since $P + S/E - 1 \geq 0$ due to $S \geq E$, we have that

$$|f'(P)| = \frac{M}{N} \frac{A(P + S/E - 1)}{B(P + S/E - 1)} \leq \frac{M}{N} \frac{A(S/E - 1)}{B(S/E - 1)} = |f'(0)|,$$

and the desired result follows.

We can write the map (9) as a quotient of polynomials by noting that

$$A(Q) = \sum_{X=0}^{M-1} \frac{(X+1)(M-1)^{(X)}}{(N+1)_X} Q^X$$

and (expanding using the Cauchy product)

$$B(Q) = \left(\sum_{X=0}^{M} \frac{M^{(X)}}{(N)_X} Q^X \right)^2 = \sum_{X=0}^{2M} Q^X \sum_{Y=0}^{X} \frac{M^{(Y)}}{(N)_Y} \frac{M^{(X-Y)}}{(N)_{X-Y}}.$$

We are now in a position to apply lemma 7. To compare the coefficients of the polynomials, denoted a_X and b_X, consider

$$\begin{aligned} B(Q) - A(Q) &= \sum_{X=0}^{M-1} Q^X \left[\left(\sum_{Y=0}^{X} \frac{M^{(Y)}}{(N)_Y} \frac{M^{(X-Y)}}{(N)_{X-Y}} \right) - \frac{(X+1)(M-1)^{(X)}}{(N+1)_X} \right] + R(Q) \\ &= \sum_{X=0}^{M-1} Q^X \sum_{Y=0}^{X} \left(\frac{M^{(Y)}}{(N)_Y} \frac{M^{(X-Y)}}{(N)_{X-Y}} - \frac{(M-1)^{(X)}}{(N+1)_X} \right) + R(Q) \end{aligned} \tag{10}$$

where the remainder term $R(Q)$ is used to gather terms of order M or higher. Because equation (10) yields $a_0 = b_0$ and $b_X > a_X = 0$ for $X \geq M$ (the terms in the remainder receive no contributions from $A(Q)$), it suffices to show that $b_X \geq a_X$ for $0 < X < M$. Indeed, if $0 < X < M$ and $0 \leq Y \leq X$,

$$\frac{M^{(Y)}}{(N)_Y}\frac{M^{(X-Y)}}{(N)_{X-Y}} - \frac{(M-1)^{(X)}}{(N+1)_X} = \frac{M^{(Y)}}{(N)_Y}\left(\frac{M^{(X-Y)}}{(N)_{X-Y}} - \frac{(M-Y-1)^{(X-Y)}}{(N+Y+1)_{X-Y}}\right)$$

is trivially nonnegative. □

Proof of corollary 3. We begin by considering the case of $S > E$; $S = E$ is handled separately. Recall that the proof of theorem 2 shows that the map (9) is strictly decreasing on the nonnegative half-line $[0, \infty)$. Since $S/E - 1 > 0$ and ${}_2F_1(a, b; c; 0) = 1$,

$$|f'(0)| = \frac{M}{S-M}\frac{A(S/E-1)}{B(S/E-1)} < \frac{M}{S-M}\frac{A(0)}{B(0)} = \frac{M}{S-M} \tag{11}$$

so that the desired result follows ($S \geq 2M$ is equivalent to $M/(S-M) \leq 1$).

Suppose now $S = E$. We modify our approach, as the strict inequality appearing in equation (11) no longer holds. The proof of theorem 1 establishes that f is strictly decreasing along with

$$\frac{1}{f(0)} = {}_2F_1(1, -M, S-M; 0) = 1$$

(equations (6) and (7)) and $0 < f(1) < 1$. Therefore, any iterate of f (except possibly the first) resides in $H' = [f(1), 1]$ and hence we can relax the sufficient condition for convergence in theorem 2 to

$$|f'(f(1))| < 1.$$

Then

$$\frac{M}{S-M}\frac{A(f(1))}{B(f(1))} < \frac{M}{S-M}\frac{A(0)}{B(0)} = \frac{M}{S-M}$$

and the desired result follows. □

The following is an extension of [5, Chapter 22, Exercise 14b], and given without proof.

Lemma 8. *Let $g\colon I \to \mathbb{R}$ be a convex differentiable function on an interval I with at least one root there. Then, given $x_0 \in I$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = x_n - g(x_n)/g'(x_n) \text{ for } n > 0$$

converges to a root of g provided that $g'(x_0) \neq 1$ and $x_1 \in I$.

Proof of theorem 4. Suppose $S \geq E$. The proof of theorem 1 establishes that $f' < 0$ on $[0,1]$ and hence $f' \neq 1$ there. Moreover, it establishes the polynomial form with coefficients c_Y given in equation (6). Since

$$c_0 = {}_2F_1\left(1, -M; S-M; 1-S/E\right) \geq {}_2F_1\left(1, -M; S-M; 0\right) = 1,$$

it follows that $0 < f(P) \leq 1$ for all P in I. Finally, the proof of theorem 2 yields that $P \mapsto |f'(P)|$ is decreasing on I. Since $|f'(P)| = -f'(P)$ on I, f is convex there.

Defining $g\colon I \to \mathbb{R}$ by

$$g(P) = f(P) - P,$$

we are now in a position to apply lemma 8 to arrive at the desired result. $\square$