Decidable Term-Modal Logics

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Abstract. The paper considers *term-modal logics* and introduces some decidable fragments thereof. In particular, two fragments will be introduced: one that simulates monotone non-normal logics and another one that simulates normal multi-agent epistemic logics with quantification over groups of agents. These logics are defined semantically. Then, each of them is proof-theoretically characterized by a labelled calculus with good structural properties. Finally, we prove that each fragment considered is decidable, and we characterize the complexity of the validity problem for some of them.

Keywords: Term-modal logics · Monotone modalities
Multi-agent epistemic logics · Decidability · Sequent calculi

1 Introduction

Propositional multi-modal epistemic logics (MELs) have been a key tool for reasoning about knowledge and belief in multi-agent systems (MASs), cf. [9]. Given a set of agents $\{1, \ldots, n\}$, we have formulas $\Box_i \phi$, which may be read as *agent i knows that ϕ*, and given a group of agents G – i.e. a subset of the set of agents – we may have formulas such as $E_G\phi, D_G\phi$, and $C_G\phi$ which may be read, respectively, as *every G knows/it is distributed/common knowledge among the Gs that ϕ*. Most applications are based on the multi-agent logic of belief $\mathbf{KD45^n}$ or on the multi-agent logic of knowledge $\mathbf{S5^n}$ – e.g., interpreted systems [9] are captured by $\mathbf{S5^n}$; but in some cases also other logics, such as $\mathbf{K^n}$ or $\mathbf{S4^n}$, are used. One key aspect of these logics is that they are decidable and have a good computational complexity: if $n \geq 2$, the validity problem is PSPACE-complete if common knowledge is omitted, else it is EXPTIME-complete [16].

Even if MELs allow us to reason about agents' knowledge, they do not allow to reason about agents and groups thereof. The problem, roughly, is that agents are (denoted by) a finite set of indexes that, so to say, live outside of the logic. Therefore, as it is noted in [15], we can only reason about a finite and known set of agents where each name (i.e. index) denotes a different agent and where

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the naming relation is common knowledge. One elegant way of overcoming these limitations is provided by term-modal logics (TMLs) [10]. These are first-order epistemic logics with increasing domains and rigid designators, cf. [2,3], where each epistemic operator is indexed by a term. The term-modal operator $[x]$ – to be read as *the agent (denoted by the possibly bound) x knows that* – is semantically modeled by a three place relation of x-dependent compatibility relation. TMLs enhance our ability to reason about agents and groups in that (i) we do not have to reason about a finite and known set of agents, and (ii) we can express:

– knowledge of a particular agent: $[x]A$ means that *x knows that A*;
– knowledge of a generic agent: $\exists x[x]A$ means that *someone knows that A*;
– knowledge of a first-order definable group: $\forall x(Gx \supset [x]A)$ means that *every member of the group G knows that A*;
– relations between groups of agents: $\forall x(Gx \supset Hx) \wedge \forall y(Hy \supset [y]A)$ means that *each G is an H and each H knows A*, and it entails that $\forall x(Gx \supset [x]A)$.

Despite their great expressive power, the TMLs considered in [10] are not suited for practical applications because: (i) [10] introduces only logics where the negative introspection axiom 5 is not valid, and (ii) being extensions of first-order modal logics, TMLs are undecidable.[1]

We first show that TMLs with any combination of the axioms $D, T, 4$, and 5, as well as their extensions with interaction axioms like the Barcan Formulas, can be proof-theoretically characterized by labelled sequent calculi. This would allow us to introduce the term-modal version of useful quantified MELs such as, e.g., the objectual quantified interpreted systems considered in [2]; see also [5,10,15,19] for scenarios where term-modal-like logics can be applied. Then, we introduce two decidable fragments of TMLs. The first fragment has very limited expressive power, but it is interesting in that it simulates monotone non-normal logics [4,17,25] in a way that is more natural than that in [13,18]. The second fragment – which extends both MELs and the epistemic logic with names $\mathbf{AX}_{\mathcal{N}}$ of [15] – allows us to reason about the propositional knowledge of individual agents and of groups of agents denoted by monadic predicates. Moreover, it allows us to express whether individual agents are members of these groups or not (but not to reason about the relations between groups). The logics defined over this latter fragment are interesting for reasoning about MASs in that they increase the expressive power of MELs without thereby increasing their complexity.

The paper proceeds as follows. Section 2 introduces TMLs and shows that they can be proof-theoretically characterized by labelled sequent calculi with the good structural properties that are typical of **G3**-style calculi. Then, in Sects. 3 and 4, we present two decidable fragments of TMLs. More specifically, in Sect. 3, we consider non-normal monotone logics based on a generalization of multi-relational semantics [4]; we introduce TMLs expressing them; and we characterize their complexity. In Sect. 4, we consider TMLs that simulate MELs with

[1] See [19] for a term-modal like extension of the logic of belief **KD45**; and see [5–7] for extensions of TMLs, called indexed epistemic logics, with non-rigid designators.

explicit quantification over groups of agents; we show that they are decidable; and we characterize the complexity of (most of) them. We conclude, in Sect. 5, by sketching some future direction of research.

2 Term-Modal Logics

2.1 Syntax and Semantics

Language. Let us consider a first-order language $\mathcal{L}$ whose signature contains only predicate symbols of any arity n (each result of the paper can be extended straightforwardly to a signature containing also individual constants). Let Var be an infinite set of variables. The primitive logical symbols are $\neg, \wedge, \forall$, and $[\cdot]$. $\mathcal{L}$-formulas are defined by the following grammar, where P^n is an arbitrary n-ary predicate symbol and $y, x_1, \ldots, x_n \in Var$,

$$A ::= P^n x_1, \ldots, x_n \mid \neg A \mid A \wedge A \mid \forall y A \mid [y]A. \tag{$\mathcal{L}$}$$

We use the following metavariables, all possibly with numerical subscripts: x, y, z for variables; p for atomic formulas; and A, B, C for formulas. The formulas $\bot, \top$, $A \vee B$, $A \supset B$, $\exists x A$ are defined as usual, and $\langle x\rangle A := \neg[x]\neg A$. The notions of *free and bound occurrences of a variable* are defined as expected; in particular, the displayed occurrence of x in $[x]A$ is free, and each occurrence of x in $\forall x A$ is bound. The *height* of a formula, $He(A)$, is the height of the longest branch of its generation tree; its *length*, $Le(A)$, is the number of nodes of its generation tree.

We use $\equiv$ to denote syntactic identity. Without loss of generality, we assume that the variables occurring free in a formula are different from the bound ones, and we identify formulas that differ only in the name of bound variables. By $A(y/x)$ we denote the formula that is obtained from A by substituting each (free) occurrence of x with an occurrence of y. In particular, we have that $([z]A)(y/x) \equiv [z(y/x)](A(y/x))$. Having identified formulas differing only in the name of bound variables, we can assume that y is free for x in A whenever we write $A(y/x)$ – that is to say, no free occurrence of any variable becomes bound after having applied a substitution.

Semantics. We mostly follow [10] in introducing the semantics. The main novelties are (i) that we consider a more general varying domain semantics instead of an increasing domains semantics where the Converse Barcan Formula holds in every frame, cf. [3]; and (ii) that we consider also Euclidean frames.

Definition 1 (Frame). *A* frame *is a tuple* $\mathcal{F} := < \mathcal{W}, \mathcal{U}, \mathcal{D}, \{\overset{a}{\rightarrowtail} : a \in \mathcal{U}\} >$, *where:*

1. *$\mathcal{W}$ is a non-empty set of worlds, denoted by $u, v, w, \ldots$;*
2. *$\mathcal{U}$ is a non-empty set of objects/agents, denoted by $a, b, c, \ldots$. $\mathcal{U}$ is called the* outer domain *of $\mathcal{F}$;*

3. *$\mathcal{D}$ is a set containing, for each $w \in \mathcal{W}$, a possibly empty subset of $\mathcal{U}$ denoted by D_w. D_w is called the* inner domain *of w and it represents the objects existing at w;*
4. *each $\overset{a}{\rightarrowtail}$ is an agent-dependent* compatibility *relation between worlds – $\overset{a}{\rightarrowtail} \subseteq \mathcal{W} \times a \times \mathcal{W}$ – for which we use infix notation. Intuitively, $w \overset{a}{\rightarrowtail} v$ means that world v is compatible with what agent a knows in world w.*

Definition 2 (Models and assignments). *A* model *(based on $\mathcal{F}$) is a pair $\mathcal{M} := < \mathcal{F}, \mathcal{I} >$ where $\mathcal{F}$ is a frame and $\mathcal{I}$ is an* interpretation *function mapping each n-ary $\mathcal{L}$-predicate to a set of $n+1$-tuples made of a world and of n objects from the outer domain $\mathcal{U}$. Formally $\mathcal{I}(P^n) \subseteq \{< w, a_1, \ldots, a_n >: w \in \mathcal{W} \,\&\, a_i \in \mathcal{U}\}$. An* assignment *is a mapping from Var to $\mathcal{U}$. We use σ, τ, ν to denote assignments. By $\sigma^{x \triangleright a}$ we denote the assignment that behaves like σ save for the variable x that is mapped to the object a.*

Definition 3 (Satisfaction). Satisfaction *of a formula A at a world w of a model $\mathcal{M}$ under the assignment σ, to be denoted by $\sigma \vDash_w^{\mathcal{M}} A$, is defined by:*

$\sigma \vDash_w^{\mathcal{M}} P^n x_1, \ldots, x_n$	*iff*	$< w, \sigma(x_1), \ldots, \sigma(x_n) > \in \mathcal{I}(P^n)$
$\sigma \vDash_w^{\mathcal{M}} \neg B$	*iff*	$\sigma \nvDash_w^{\mathcal{M}} B$
$\sigma \vDash_w^{\mathcal{M}} B \wedge C$	*iff*	$\sigma \vDash_w^{\mathcal{M}} B$ *and* $\sigma \vDash_w^{\mathcal{M}} C$
$\sigma \vDash_w^{\mathcal{M}} \forall x B$	*iff*	*for all* $a \in D_w$, $\sigma^{x \triangleright a} \vDash_w^{\mathcal{M}} B$
$\sigma \vDash_w^{\mathcal{M}} [x] B$	*iff*	*for all* $v \in \mathcal{W}$, $w \overset{\sigma(x)}{\rightarrowtail} v$ *implies* $\sigma \vDash_v^{\mathcal{M}} B$

The notions of truth in a world, $\vDash_w^{\mathcal{M}} A$, truth in a model, $\vDash^{\mathcal{M}} A$, validity in a frame, $\mathcal{F} \vDash A$, and validity in a class $\mathcal{C}$ of frames, $\mathcal{C} \vDash A$, are as usual.

Logics. By an $\mathcal{L}$-*logic* we mean the set of all $\mathcal{L}$-formulas that are valid in some class of frames. In this paper we will consider all $\mathcal{L}$-logics that are defined by some combination of the properties in the following correspondence results, whose straightforward proofs can be omitted.

Proposition 4 (Correspondence results). *The following formulas are valid in all and only the frames satisfying the following properties, where $w, v, u \in \mathcal{W}$ and $a, b \in \mathcal{U}$ (the universal closure of T–BF would correspond to the same properties restricted to agents of the inner domains: $\forall a \in D_w$ instead of $\forall a \in \mathcal{U}$),*

- $T := [x]A \supset A$ *iff $\mathcal{F}$ is* reflexive*: $\forall a \forall w (w \overset{a}{\rightarrowtail} w)$*
- $D := \neg [x] \bot$ *iff $\mathcal{F}$ is* serial*: $\forall a \forall w \exists v (w \overset{a}{\rightarrowtail} v)$*
- $4 := [x]A \supset [x][x]A$ *iff $\mathcal{F}$ is* transitive*: $\forall a \forall w, v, u (w \overset{a}{\rightarrowtail} v \& v \overset{a}{\rightarrowtail} u \supset w \overset{a}{\rightarrowtail} u)$*
- $5 := \langle x \rangle A \supset [x] \langle x \rangle A$ *iff $\mathcal{F}$ is* Euclidean*: $\forall a \forall w, v, u (w \overset{a}{\rightarrowtail} v \& w \overset{a}{\rightarrowtail} u \supset v \overset{a}{\rightarrowtail} u)$*
- $NE := \forall x A \supset \exists x A$ *iff $\mathcal{F}$ has* non empty domains*: $\forall w (D_w \neq \varnothing)$*
- $UI := \forall x A \supset A(y/x)$ *iff $\mathcal{F}$ has* single domain*: $\forall w (D_w = \mathcal{U})$*
- $CBF := [x] \forall y A \supset \forall y [x] A$ *iff $\mathcal{F}$ has* increasing domains*:*
  $\forall a, b \forall w, v (w \overset{a}{\rightarrowtail} v \& b \in D_w \supset b \in D_v)$
- $BF := \forall y [x] A \supset [x] \forall y A$ *iff $\mathcal{F}$ has* decreasing domains*:*
  $\forall a, b \forall w, v (w \overset{a}{\rightarrowtail} v \& b \in D_v \supset b \in D_w)$

We use the standard names for the $\mathcal{L}$-extensions of propositional logics in the cube of normal modalities. For example, **K** denotes the set of $\mathcal{L}$-formulas valid in the class of all frames; **T** denotes the set of $\mathcal{L}$-formulas valid in all reflexive frames; and **S5** denotes the set of $\mathcal{L}$-formulas valid in all reflexive, transitive, and Euclidean frames. Moreover, if **X** is the name of one of the $\mathcal{L}$-logics thus defined, **X** $\oplus$ **NE** is the logic of all non empty domains **X**-frames, and analogously for their extensions with UI, CBF, BF, and the combinations thereof. We use **L** for an arbitrary $\mathcal{L}$-logic among the ones we are considering.

2.2 Proof Systems

Labelled Sequent Calculi for TMLs. We are now going to introduce labelled sequent calculi that characterize $\mathcal{L}$-logics. We assume the reader is acquainted with sequent calculi. The calculi for TMLs are like the ones for propositional and quantified modal logics [20, Sects. 11 and 12.1], save that two-place relational atoms, wRv, are replaced by three-places compatibility atoms, $w \overset{x}{\rightarrowtail} v$. More precisely, we introduce an infinite set of fresh variables, called *(world) labels*, for which we use the metavariables w, v, u. A *labelled sequent* is an expression $\Omega; \Gamma \Rightarrow \Delta$, where Ω is a multiset of *domain atoms* $x \in w$ – meaning that x is in the inner domain of world w – and of *compatibility atoms* $w \overset{x}{\rightarrowtail} v$ – meaning that v is compatible with what agent x knows in w; and where Γ and Δ are multisets of *labelled formulas* $w : A$ – meaning that the $\mathcal{L}$-formula A holds at w.

The rules for the calculus **G3tm.K**, which characterizes the $\mathcal{L}$-logic **K**, are given in Table 1; the label u in rule $R\Box$, as well as the variable z in $R\forall$, is an *eigenvariable* – i.e., it cannot occur free in the conclusion of that rule instance.

Table 1. Sequent calculus **G3tm.K**

Initial sequents:

$\Omega; w : p, \Gamma \Rightarrow \Delta, w : p \quad (p \text{ atomic})$

Logical rules:

$$\frac{\Omega; \Gamma \Rightarrow \Delta, w : A}{\Omega; w : \neg A, \Gamma \Rightarrow \Delta}\, L\neg \qquad \frac{\Omega; w : A, \Gamma \Rightarrow \Delta}{\Omega; \Gamma \Rightarrow \Delta, w : \neg A}\, R\neg$$

$$\frac{\Omega; w : A, w : B, \Gamma \Rightarrow \Delta}{\Omega; w : A \wedge B, \Gamma \Rightarrow \Delta}\, L\wedge \qquad \frac{\Omega; \Gamma \Rightarrow \Delta, w : A \qquad \Omega; \Gamma \Rightarrow \Delta, w : B}{\Omega; \Gamma \Rightarrow \Delta, w : A \wedge B}\, R\wedge$$

$$\frac{y \in w, \Omega; w : A(y/x), w : \forall x A, \Gamma \Rightarrow \Delta}{y \in w, \Omega; w : \forall x A, \Gamma \Rightarrow \Delta}\, L\forall \qquad \frac{z \in w, \Omega; \Gamma \Rightarrow \Delta, w : A(z/x)}{\Omega; \Gamma \Rightarrow \Delta, w : \forall x A}\, R\forall, z \text{ eig.}$$

$$\frac{w \overset{x}{\rightarrowtail} v, \Omega; v : A, w : [x]A, \Gamma \Rightarrow \Delta}{w \overset{x}{\rightarrowtail} v, \Omega; w : [x]A, \Gamma \Rightarrow \Delta}\, L\Box \qquad \frac{w \overset{x}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta, u : A}{\Omega; \Gamma \Rightarrow \Delta, w : [x]A}\, R\Box, u \text{ eig.}$$

To obtain calculi for the other $\mathcal{L}$-logics, we use non-logical rules expressing the geometric semantic conditions given in Proposition 4, cf. [20, Sect. 8]. For any $\mathcal{L}$-logic **L**, the calculus **G3tm.L** is obtained by extending **G3tm.K** with the non-logical rules from Table 2 that express the semantic conditions defining **L**. The label u in rule L_D and the variable z in rule L_{NE} are *eigenvariables*. If a calculus contains rule L_5, it also contains its contracted instances L_{5c}. The other rules in Table 2 are such that we do not have do add contracted instances.

Table 2. Non-logical rules

$$\frac{w \overset{x}{\rightarrowtail} w, \Omega; \Gamma \Rightarrow \Delta}{\Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_T} \qquad \frac{v \overset{x}{\rightarrowtail} u, w \overset{x}{\rightarrowtail} v, w \overset{x}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta}{w \overset{x}{\rightarrowtail} v, w \overset{x}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_5} \qquad \frac{v \overset{x}{\rightarrowtail} v, w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}{w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{5c}}$$

$$\frac{w \overset{x}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta}{\Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_D,\, u \text{ eig.}} \qquad \frac{w \overset{x}{\rightarrowtail} u, w \overset{x}{\rightarrowtail} v, v \overset{x}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta}{w \overset{x}{\rightarrowtail} v, v \overset{x}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_4}$$

$$\frac{x \in w, \Omega; \Gamma \Rightarrow \Delta}{\Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{UI}} \qquad \frac{z \in v, z \in w, w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}{z \in w, w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{CBF}}$$

$$\frac{z \in w, \Omega; \Gamma \Rightarrow \Delta}{\Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{NE},\, z \text{ eig.}} \qquad \frac{z \in w, z \in v, w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}{z \in v, w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{BF}}$$

The notion of substitution of world labels is defined as expected. Substitutions are extended to domain and compatibility atoms. Substitutions are extended to sequents by applying them componentwise. A derivation $\mathcal{D}$ of a sequent $\mathcal{S}$ in **G3tm.L** is a tree of sequents that is obtained by applying rules of **G3tm.L**, whose root is $\mathcal{S}$, and whose leaves are initial sequents. The *height* of a derivation $\mathcal{D}$, $He(\mathcal{D})$, is the height of the longest branch of $\mathcal{D}$. We write **G3tm.L** $\vdash^{(n)} \mathcal{S}$ if the sequent $\mathcal{S}$ is **G3tm.L**-derivable (with a derivation of at most height n). We say that a rule is *(height-preserving) admissible* in **G3tm.L** if whenever its premisses are **G3tm.L**-derivable (with height n), its conclusion is **G3tm.L**-derivable (with at most height n). Finally, in the rules in Tables 1 and 2, the multisets Ω, Γ, and Δ are called *contexts*, the formulas displayed in the conclusion are called *principal* and those displayed only in the premiss(es) are called *active*.

Properties of G3tm.L. We are now going to present some properties of the calculi **G3tm.L**. The main results are (i) that these calculi have the good structural properties of **G3**-style calculi – i.e. all rules are invertible, weakening and contraction are height-preserving admissible (hp-admissible), and cut is admissible; and (ii) that each calculus is sound and complete with respect to the corresponding $\mathcal{L}$-logic. Most proofs will be omitted for lack of space. They can be easily obtained by modifying the ones given in [20, Sect. 12.1] for quantified modal logics or the ones given in [6, Sects. 3–4] for indexed epistemic logics.

Lemma 5 (Substitution). *Substitutions are hp-admissible in* **G3tm.L***:*

1. *If* **G3tm.L** $\vdash^n \mathcal{S}$ *then* **G3tm.L** $\vdash^n \mathcal{S}(y/x)$*;*
2. *If* **G3tm.L** $\vdash^n \mathcal{S}$ *then* **G3tm.L** $\vdash^n \mathcal{S}(w/v)$*.*

Lemma 6 (Initial sequents). *Sequents of shapes (i) $\Omega; w : \bot, \Gamma \Rightarrow \Delta$, (ii) $\Omega; \Gamma \Rightarrow \Delta, w : \top$, and (iii) $\Omega; w : A, \Gamma \Rightarrow \Delta, w : A$, with A arbitrary $\mathcal{L}$-formula and w arbitrary world label, are derivable in* **G3tm.L**.

Lemma 7 (Weakening). *The following rules are hp-admissible in* **G3tm.L***:*

$$\frac{\Omega; \Gamma \Rightarrow \Delta}{\Omega', \Omega; \Gamma \Rightarrow \Delta}\ LW_{\Omega} \qquad \frac{\Omega; \Gamma \Rightarrow \Delta}{\Omega; \Pi, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Omega; \Gamma \Rightarrow \Delta}{\Omega; \Gamma \Rightarrow \Delta, \Sigma}\ RW$$

Lemma 8 (Invertibility). *Each rule of* **G3tm.L** *is hp-invertible.*

Lemma 9 (Contraction). *The following rules are hp-admissible in* **G3tm.L***:*

$$\frac{\Omega', \Omega', \Omega; \Gamma \Rightarrow \Delta}{\Omega', \Omega; \Gamma \Rightarrow \Delta}\ LC_{\Omega} \qquad \frac{\Omega; \Pi, \Pi, \Gamma \Rightarrow \Delta}{\Omega; \Pi, \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Omega; \Gamma \Rightarrow \Delta, \Sigma, \Sigma}{\Omega; \Gamma \Rightarrow \Delta, \Sigma}\ RC$$

Theorem 10 (Cut). *The following rule of cut is admissible in* **G3tm.L***:*

$$\frac{\Omega; \Gamma \Rightarrow \Delta, w : A \qquad \Omega'; w : A, \Pi \Rightarrow \Sigma}{\Omega, \Omega'; \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ Cut$$

In order to show that **G3tm.L** is sound and complete with respect to **L**-frames, we extend the notion of validity to sequents. Notice that a semantic proof of the admissibility of the structural rules of inference is an immediate corollary of the completeness theorem.

Definition 11. *Let $\sigma_{\mathcal{F}}$ be a function mapping world labels to worlds of a frame $\mathcal{F}$ and variables to objects of the outer domain of $\mathcal{F}$.*
A sequent $\Omega; \Gamma \Rightarrow \Delta$ is valid *on $\mathcal{F}$ iff for all $\sigma_{\mathcal{F}}$ and all $\mathcal{M}$ based on $\mathcal{F}$,*
if *(i) for all $x \in w$ occurring in Ω we have that $\sigma_{\mathcal{F}}(x) \in D_{\sigma_{\mathcal{F}}(w)}$,*
(ii) for all $w \overset{x}{\rightarrowtail} v$ occurring in Ω we have that $\sigma_{\mathcal{F}}(w) \overset{\sigma_{\mathcal{F}}(x)}{\rightarrowtail} \sigma_{\mathcal{F}}(v)$, and
(iii) for all $w : A$ occurring in Γ we have that $\sigma_{\mathcal{F}} \vDash^{\mathcal{M}}_{\sigma_{\mathcal{F}}(w)} A$,
then *there is some $v : B$ occurring in Δ such that $\sigma_{\mathcal{F}} \vDash^{\mathcal{M}}_{\sigma_{\mathcal{F}}(v)} B$.*

Theorem 12 (Soundness). *If a sequent $\mathcal{S}$ is derivable in* **G3tm.L***, then it is valid in the class of all frames for* **L**.

Proof (Sketch). The proof is by induction on the height of the derivation $\mathcal{D}$ of $\mathcal{S}$. The base case holds trivially. For the inductive step, we have to check that each rule of **G3tm.L** preserves validity over frames for **L**. Each logical rule preserves validity over any frame. Each non-logical rule preserves validity over frames satisfying the corresponding semantic property; cf. [20, Theorem 12.13]. □

Theorem 13 (Completeness). *If a sequent is valid in the class of all frames for* **L***, then it is derivable in* **G3tm.L**.

Proof. The proof is in three steps. First, in Definition 14, we define a notion of **L**-saturated branch of a proof-search for a sequent $\mathcal{S}$. Then, with Definition 15 and Lemma 16, we show that an **L**-saturated branch allows us to define a countermodel for $\mathcal{S}$ that is based on a frame for **L**. Finally, we give a root first **G3tm.L**-proof-search procedure, Proposition 17, that either gives us a **G3tm.L**-derivation of $\mathcal{S}$ – and, by Theorem 12, $\mathcal{S}$ is **L**-valid – or it has an **L**-saturated branch – and, therefore, $\mathcal{S}$ has a countermodel based on an appropriate frame. □

Definition 14 (Saturation). *A branch $\mathcal{B}$ of a* **G3tm.L**-*proof-search tree for a sequent $\mathcal{S}$ is* **L**-saturated *if it satisfies the following conditions, where $\mathbf{\Gamma}$ ($\mathbf{\Delta}$) is the union of the antecedents (succedents) occurring in that branch,*

1. *no $w : p$ occurs in $\mathbf{\Gamma} \cap \mathbf{\Delta}$;*
2. *if $w : \neg A$ is in $\mathbf{\Gamma}$, then $w : A$ is in $\mathbf{\Delta}$;*
3. *if $w : \neg A$ is in $\mathbf{\Delta}$, then $w : A$ is in $\mathbf{\Gamma}$;*
4. *if $w : A \wedge B$ is in $\mathbf{\Gamma}$, then both $w : A$ and $w : B$ are in $\mathbf{\Gamma}$;*
5. *if $w : A \wedge B$ is in $\mathbf{\Delta}$, then at least one of $w : A$ and $w : B$ is in $\mathbf{\Delta}$;*
6. *if both $w : \forall x A$ and $y \in w$ are in $\mathbf{\Gamma}$, then $w : A(y/x)$ is in $\mathbf{\Gamma}$;*
7. *if $w : \forall x A$ is in $\mathbf{\Delta}$, then, for some z, $w : A(z/x)$ is in $\mathbf{\Delta}$ and $z \in w$ is in $\mathbf{\Gamma}$;*
8. *if both $w : [x]A$ and $w \overset{x}{\rightarrowtail} v$ are in $\mathbf{\Gamma}$, then $v : A$ is in $\mathbf{\Gamma}$;*
9. *if $w : [x]A$ is in $\mathbf{\Delta}$, then, for some u, $u : A$ is in $\mathbf{\Delta}$ and $w \overset{x}{\rightarrowtail} u$ is in $\mathbf{\Gamma}$;*
10. *if R is a non-logical rule of* **G3tm.L**, *then for any set of principal formulas of R that are in $\mathbf{\Gamma}$ also the corresponding active formulas are in $\mathbf{\Gamma}$ (for some* eigenvariable *of R, if any).*

Definition 15. *Let $\mathcal{B}$ be* **L**-*saturated. The model $\mathcal{M}^{\mathcal{B}}$ is thus defined: (i) $\mathcal{W}^{\mathcal{B}}$ is the set of world labels occurring in $\mathbf{\Gamma} \cup \mathbf{\Delta}$; (ii) $\mathcal{U}^{\mathcal{B}}$ is the set of all variables occurring free in $\mathbf{\Gamma} \cup \mathbf{\Delta}$; (iii) for each $w \in \mathcal{W}$, $x \in D_w$ iff $x : w$ is in $\mathbf{\Gamma}$; (iv) for each $x \in \mathcal{U}$, $w \overset{x}{\rightarrowtail} v$ iff the formula $w \overset{x}{\rightarrowtail} v$ is in $\mathbf{\Gamma}$; (v) $\mathcal{I}^{\mathcal{B}}(P^n)$ is the set of all $n+1$-tuples $< w, x_1, \dots, x_n >$ such that the formula $w : P^n x_1, \dots, x_n$ is in $\mathbf{\Gamma}$. Given $\mathcal{M}^{\mathcal{B}}$, $\sigma_{\mathcal{B}}$ denotes the assignment given by the identity mapping.*

Lemma 16. *Let $\mathcal{B}$ be an* **L**-*saturated branch. Then (1) for any $\mathcal{L}$-formula A we have that $\sigma_{\mathcal{B}} \vDash_w^{\mathcal{M}^{\mathcal{B}}} A$ iff $w : A$ is in $\mathbf{\Gamma}$; and (2) $\mathcal{M}^{\mathcal{B}}$ is based on a frame for* **L**.

Proof (Sketch). The proof of claim (1) is by induction on $He(A)$. The base case holds by construction of $\mathcal{M}^{\mathcal{B}}$ and of $\sigma_{\mathcal{B}}$, and the inductive cases depend on Definition 14.2–9. Claim (2) follows by Definition 14.10 and by construction of $\mathcal{M}^{\mathcal{B}}$ and of $\sigma_{\mathcal{B}}$. □

Proposition 17. *A* **G3tm.L**-*proof-search tree for a sequent $\mathcal{S}$ is the tree of sequent that has $\mathcal{S}$ as root and whose branches grow according to the following procedure: if the leaf is an initial sequent the branch stops growing, else either no instance of rules of* **G3tm.L** *is applicable root first to it, or k instances are (where rules $L\forall, L\Box, L_T, L_D, L_{UI}, L_{NE}$ are applied w.r.t. all free variables/labels occurring in the leaf). In the first case, the branch stops growing; in this case it is immediate to see that we have a finite* **L**-*saturated branch. In the second case,* *we apply the k rule instances that are applicable in some order (each one will be applied to all end-sequents that are generated at the previous step). If the tree never stops growing then, by König's Lemma, it has an infinite branch which, as the reader can easily check, is* **L***-saturated.*

3 Monotone Non-Normal Modalities

The first decidable fragment of TMLs is obtained by restricting the language to 0-ary predicates – i.e., propositional variables – and by using quantifiers and term-modal operators only to introduce logical operators of shape $\exists x[x]$ and $\forall x\langle x\rangle$. This fragment simulates non-normal monotone epistemic logics [4,17,25] – i.e., logics not closed under deduction nor under necessitation, but only under the weaker rule *RM* – via normal TMLs. Roughly, the monotone formula $\Box\phi$ is expressed by $\exists x[x]\phi$. This simulation is simpler than that via polymodal normal modalities [13,18]. Moreover, as it is done in [14] building on the approach in [13,18], we easily obtain labelled calculi for monotone epistemic logics. Monotone epistemic logics are interesting in at least two respects. First, they model humans' knowledge more appropriately (than normal ones) in that agents need not know every tautology and their knowledge need not be closed under deduction, cf. [25] and [16, p. 377]. Second, and more related to applications, many important logics for reasoning in MASs, such as Parikh's Game Logic [22] and Pauly's Coalition Logic [23], are based on monotone modalities, cf [17].

Monotone Epistemic Logics. We give here a very short introduction to monotone epistemic logics. Given that this will simplify the simulation, we will make use of (a generalization of) multi-relational semantics for monotone epistemic logics [4], and not the usual neighbourhood semantics [17,25]. We generalize the (weak) semantics given in [4] by considering a 'varying domain' version of it where the *necessitation axiom* $\Box\top$ is not valid in all frames. The language $\mathcal{L}^{\Box}$ is generated by the following grammar, where Q^0 is an arbitrary 0-ary predicate,

$$\phi ::= Q^0 \mid \neg\phi \mid \phi\wedge\phi \mid \Box\,\phi \mid \Diamond\,\phi. \qquad (\mathcal{L}^{\Box})$$

A *multi-relational frame* is a tuple $\mathscr{F} =< \mathcal{W}, \{R_1,\dots,R_n\}, \mathcal{R} >$, where (i) $\mathcal{W}$ is a non-empty set of worlds; (ii) $n \geq 1$ and $R_i \subseteq \mathcal{W}\times\mathcal{W}$; and (iii) $\mathcal{R}$ is a mapping from worlds to possibly empty subsets of $\{R_1,\dots,R_n\}$. A *multi-relational model* is a tuple $\mathscr{M} =< \mathscr{F}, \mathcal{V} >$, where $\mathcal{V}$ is a *valuation* mapping 0-ary predicates to subsets of $\mathcal{W}$. *Truth* of an $\mathcal{L}^{\Box}$-formula at a world w of a model $\mathscr{M}$ is defined as in Kripke semantics, save for $\Box\phi$ and $\Diamond\phi$ where we have, respectively

$$\vDash_w^{\mathscr{M}} \Box\phi \text{ iff some } R_i\in\mathcal{R}(w) \text{ is s.t. for all } v\in\mathcal{W}, \text{ if } w\mathcal{R}_i v \text{ then } \vDash_v^{\mathscr{M}} \phi; \qquad (1)$$

$$\vDash_w^{\mathscr{M}} \Diamond\phi \text{ iff for each } R_i\in\mathcal{R}(w) \text{ there is } v\in\mathcal{W} \text{ s.t. } w\mathcal{R}_i v \text{ and } \vDash_v^{\mathscr{M}} \phi. \qquad (2)$$

By a *monotone modal logic* we mean the set of all $\mathcal{L}^{\Box}$-formulas valid in some class of multi-relational frames. In particular, we will consider the classes of multi-relational frames that are defined by some combination of the properties given in Table 3, where $R_j(w) = \{v : wR_jv\}$, and w, v, u are generic worlds. The set **M** of the $\mathcal{L}^{\Box}$-formulas valid in all multi-relational frames is the smallest set containing all $\mathcal{L}^{\Box}$-instances of propositional tautologies, that is closed under modus ponens, and is closed under RM: if $(\phi \supset \psi) \in \mathbf{M}$ then $(\Box\phi \supset \Box\psi) \in \mathbf{M}$. The set **MN** of $\mathcal{L}^{\Box}$-formulas valid in all $\mathscr{F}$ satisfying N^m is the smallest extension of **M** containing $N := \Box\top$. The set **MC** of $\mathcal{L}^{\Box}$-formulas valid in all $\mathscr{F}$ satisfying C^m is the smallest extension of **M** containing $C := \Box\phi \wedge \Box\psi \supset \Box(\phi \wedge \psi)$; and so on for the logics containing all $\mathcal{L}^{\Box}$-instances of the modal axioms $T, D, 4, 5$ (where if $5 \in \mathbf{X}$ then $N \in \mathbf{X}$, and where $D := \neg\Box\bot$ is equivalent to $\Box\phi \supset \Diamond\phi$ only if $C \in \mathbf{X}$), and for the combinations thereof. The proofs are like the ones in [4], save that in [4] it is imposed that $\mathcal{R}(w) \neq \varnothing$ and, therefore, N always holds and the 'existential constituents' of 4^m and 5^m hold trivially.

Table 3. Properties of $\mathscr{F}$

$N^m := \mathcal{R}(w) \neq \varnothing$	$C^m := \forall R_i, R_j \in \mathcal{R}(w) \exists R_k \in \mathcal{R}(w)(R_k(w) \subseteq R_i(w) \cap \mathcal{R}_j(w))$
$T^m := \forall R_i(R_i \in \mathcal{R}(w) \supset wR_iw)$	$D^m := \forall R_i(R_i \in \mathcal{R}(w) \supset R_i(w) \neq \varnothing)$
$4^m := \forall R_i(R_i \in \mathcal{D}(w) \& wR_iv \supset \mathcal{R}(v) \neq \varnothing) \;\&\; \forall R_i, R_k \in \mathcal{D}(w) \forall R_j \in \mathcal{D}(v)(wR_iv \& vR_ju \supset wR_ku)$	
$5^m := \mathcal{R}(w) \neq \varnothing \;\&\; \forall R_i, R_j \in \mathcal{D}(w) \forall R_k \in \mathcal{D}(v)(wR_iv \& wR_ju \supset vR_ku)$	

Term-Modal Logics and Monotone Epistemic Logics. From the perspective of multi-relational semantics, the monotone modalities $\Box$ and $\Diamond$ are similar to normal ones in Kripke semantics. The only novelty is that we have $\exists\forall$ and $\forall\exists$ modalities whose epistemic readings are, respectively, (i) *there is an agent such that in all worlds compatible with his knowledge*...; and (ii) *for each agent some world compatible with his knowledge is such that*.... It should immediately be clear that we can capture these quantifier alternation in the term-modal language.[2] In order to do so, we introduce the following notational conventions: $\boxplus A := \exists x[x]A$ and $\diamondplus A := \forall x\langle x\rangle A$; and we consider TMLs based on the language $\mathcal{L}^{\boxplus}$ defined by the following grammar, where Q^0 is an arbitrary 0-ary predicate,

$$A ::= Q^0 \mid \neg A \mid A \wedge A \mid \boxplus A \mid \diamondplus A. \qquad (\mathcal{L}^{\boxplus})$$

Next, we consider correspondence results between $\mathcal{L}^{\boxplus}$-formulas and properties of term-modal frames.

Proposition 18. *The following $\mathcal{L}^{\boxplus}$-formulas are valid in all and only the term-modal frames satisfying the following conditions (for $w, v, u \in \mathcal{W}$ and $a, b, c \in \mathcal{U}$)*

[2] See [24] for another way of expressing the minimal monotone logic **M** via TMLs.

- $N^{\boxplus} := \boxplus\top$ *iff* $\forall w \exists a(a \in D_w)$
- $C^{\boxplus} := \boxplus A \wedge \boxplus B \supset \boxplus(A \wedge B)$ *iff* $\forall w, v, u\, \forall a, b$
  $(a \in D_w \;\&\; b \in D_w \supset \exists c(c \in D_w \;\; \& \;\; \forall w_1(w \overset{c}{\rightarrowtail} w_1 \supset (w \overset{a}{\rightarrowtail} w_1 \;\&\; w \overset{b}{\rightarrowtail} w_1))))$
- $T^{\boxplus} := \boxplus A \supset A$ *iff* $\forall w\, \forall a(a \in D_w \supset w \overset{a}{\rightarrowtail} w))$
- $D^{\boxplus} := \neg \boxplus \bot$ *iff* $\forall w\, \forall a(a \in D_w \supset \exists v(w \overset{a}{\rightarrowtail} v))$
- $4^{\boxplus} := \boxplus A \supset \boxplus \boxplus A$ *iff* $\forall w, v\, \forall a \in D_w(w \overset{a}{\rightarrowtail} v \supset \exists b(b \in D_v)) \;\; \& \;\; \forall w, v, u$
  $\forall a, c \in D_w \forall b \in D_v(w \overset{a}{\rightarrowtail} v \;\&\; v \overset{b}{\rightarrowtail} u \supset w \overset{c}{\rightarrowtail} u)$
- $5^{\boxplus} := \oplus A \supset \boxplus \oplus A$ *iff* $\forall w \exists a(a \in D_w) \;\; \& \;\; \forall w, v, u \forall a, b \in D_w \forall c \in D_v$
  $(w \overset{a}{\rightarrowtail} v \;\&\; w \overset{b}{\rightarrowtail} u \supset v \overset{c}{\rightarrowtail} u)$

Now, we introduce a 1-1 mapping, TR, between $\mathcal{L}^{\Box}$-formulas and $\mathcal{L}^{\boxplus}$-formulas and another 1-1 mapping, M, between multi-relational models and term-modal models defined over $\mathcal{L}^{\boxplus}$ ($\textsc{tr}^{-1}$and $\textsc{m}^{-1}$ denote the inverse mappings).

Definition 19. *Let ϕ be an $\mathcal{L}^{\Box}$-formula, A an $\mathcal{L}^{\boxplus}$-formula, $\mathscr{M}$ a multi-relational model, and $\mathcal{M}$ a term-modal model over $\mathcal{L}^{\boxplus}$. The mappings* TR *and* M *are:*

- $\textsc{tr}(Q^0) = Q^0$; $\quad \textsc{tr}(\neg\phi) = \neg\textsc{tr}(\phi)$; $\quad \textsc{tr}(\phi \wedge \psi) = \textsc{tr}(\phi) \wedge \textsc{tr}(\psi)$;
  $\textsc{tr}(\Box\phi) = \boxplus\textsc{tr}(\phi)$; $\quad \textsc{tr}(\Diamond\phi) = \oplus\textsc{tr}(\phi)$.
- *For* $\mathscr{M} =< \mathcal{W}, \{R_1, \ldots, R_n\}, \mathcal{R}, \mathcal{V} >$, $\textsc{m}(\mathscr{M})$ *is* $< \mathcal{W}^{\mathscr{M}}, \mathcal{U}^{\mathscr{M}}, \mathcal{D}^{\mathscr{M}}, \rightarrowtail^{\mathscr{M}}, \mathcal{I}^{\mathscr{M}} >$, *where: (i)* $\mathcal{W}^{\mathscr{M}} = \mathcal{W}$; *(ii)* $\mathcal{U}^{\mathscr{M}} = \{R_1, \ldots, R_n\}$; *(iii)* $\mathcal{D}^{\mathscr{M}} = \{D_w : R_i \in D_w$ *iff* $R_i \in \mathcal{R}(w)\}$; *(iv)* $\rightarrowtail^{\mathscr{M}} = \bigcup\{\overset{R_i}{\rightarrowtail}: w \overset{R_i}{\rightarrowtail} v$ *iff* $w\mathcal{R}_i v\}$; *and (v)* $\mathcal{I}^{\mathscr{M}}(Q^0) = \mathcal{V}(Q^0)$.

We can now prove that $\mathcal{L}^{\boxplus}$-logics simulate monotone epistemic logics.

Lemma 20. *If ϕ, A, $\mathscr{M}$, $\mathcal{M}$ are as in Definition 19, then* $(i) \vDash_w^{\mathscr{M}} \phi$ *iff* $\vDash_w^{\textsc{m}(\mathscr{M})}$ $\textsc{tr}(\phi)$, *and, vice versa,* $(ii) \vDash_w^{\mathcal{M}} A$ *iff* $\vDash_w^{\textsc{m}^{-1}(\mathcal{M})} \textsc{tr}^{-1}(A)$.

Proof (Sketch). We can prove that $\vDash_w^{\mathscr{M}} \phi$ iff $\vDash_w^{\textsc{m}(\mathscr{M})} \textsc{tr}(\phi)$ by an easy induction on the height of ϕ. If ϕ is atomic or of shape $\neg\psi$ or $\psi_1 \wedge \psi_2$, the proof is straightforward. If $\phi \equiv \Box\psi$, then: $\vDash_w^{\mathscr{M}} \Box\psi \overset{(1)}{\iff} \exists R_i \in \mathcal{R}(w) \forall v \in w(wR_i v \supset \vDash_w^{\mathscr{M}} \psi) \overset{IH}{\iff}$ $\exists R_i \in \mathcal{R}(w) \forall v \in w(wR_i v \supset \vDash_w^{\textsc{m}(\mathscr{M})} \textsc{tr}(\psi)) \overset{\text{Def. } \textsc{m}(\mathscr{M})}{\iff} \exists R_i \in D_w \forall v \in \mathcal{W}$ $(w \overset{R_i}{\rightarrowtail} v \supset \vDash_w^{m(\mathscr{M})} \textsc{tr}(\psi)) \overset{\text{Def. 3}}{\iff} \vDash_w^{\textsc{m}(\mathscr{M})} \exists x[x]\textsc{tr}(\psi) \overset{\text{Def. } \boxplus + \textsc{tr}}{\iff} \vDash_w^{\textsc{m}(\mathscr{M})} \textsc{tr}(\Box\psi)$. The proofs of case $\phi \equiv \Diamond\psi$ and of claim (ii) are similar and can be omitted. □

Theorem 21. *Validity of $\mathcal{L}^{\boxplus}$-formulas over term-modal frames defined by properties in Proposition 18 and validity of $\mathcal{L}^{\Box}$-formulas over multi-relational frames defined by the corresponding properties (see Table 3) are equivalent problems.*

Proof. An easy corollary of Lemma 20 and of Proposition 18. □

Theorem 22. *The validity problem of $\mathcal{L}^{\boxplus}$-formulas over frames defined by properties in Proposition 18 is decidable. Moreover, for $\mathcal{L}^{\boxplus}$-logics without $C^{\boxplus}$ the validity problem is co-*NP*-complete, and for $\mathcal{L}^{\boxplus}$-logics with $C^{\boxplus}$ it is* PSPACE*-complete.*

Proof. The theorem follows by Theorem 21 and by [25, Theorem 2.3] (assuming that the conjecture [25, p. 251] that $\mathcal{L}^{\Box}$-logics with C are PSPACE-hard is correct). □

Finally, Table 4, together with rule L_{NE} (see Table 2) which expresses condition N^m, gives the non-logical rules that allow us to introduce the labelled calculus **G3tm.X** which characterizes (via TR) the monotone logic **X**. If **G3tm.X** contains rule $L_{5^{\boxplus}}$, then it contains also rule L_{NE}; and if it contains L_C ($L_{5^{\boxplus}}$), then it contains also its contracted instances L_{Cc} ($L_{5c^{\boxplus}}$) and, for $L_{5^{\boxplus}}$ and $L_{4_2^{\boxplus}}$, those where $x \in w \equiv y \in w$. The semantic condition that corresponds to $C^{\boxplus}$ in Proposition 18 is not geometric, but the rule expressing it is made geometric by introducing the three places atomic predicate $\bigcap$ and rule $L_{\bigcap}$, cf. [8]. It can be easily shown that results analogous to Lemma 5–Theorem 13 hold for each calculus **G3tm.X**. It should be possible to define a terminating **G3tm.X**-proof-search procedure, cf. [12,20]. This would solve the open problem that multi-relational models have the finite model property, cf. [4, p. 318].

Table 4. Non-logical rules for monotone epistemic logics

$$\frac{z \in w, \bigcap(z,x,y), x \in w, y \in w, \Omega; \Gamma \Rightarrow \Delta}{x \in w, y \in w, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_C,\, z \text{ eig.}} \qquad \frac{w \overset{x}{\rightarrowtail} v, w \overset{y}{\rightarrowtail} v, w \overset{z}{\rightarrowtail} v, \bigcap(z,x,y), \Omega; \Gamma \Rightarrow \Delta}{w \overset{z}{\rightarrowtail} v, \bigcap(z,x,y), \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{\bigcap}}$$

$$\frac{w \overset{x}{\rightarrowtail} u, x \in w, \Omega; \Gamma \Rightarrow \Delta}{x \in w, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{D^{\boxplus}},\, u \text{ eig.}} \qquad \frac{v \overset{z}{\rightarrowtail} u, x \in w, y \in w, z \in v, w \overset{x}{\rightarrowtail} v, w \overset{y}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta}{x \in w, y \in w, z \in v, w \overset{x}{\rightarrowtail} v, w \overset{y}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{5^{\boxplus}}}$$

$$\frac{z \in v, x \in w, w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}{x \in w, w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{4_1^{\boxplus}},\, z \text{ eig.}} \qquad \frac{w \overset{y}{\rightarrowtail} u, x \in w, y \in w, z \in v, w \overset{x}{\rightarrowtail} v, v \overset{z}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta}{x \in w, y \in w, z \in v, w \overset{x}{\rightarrowtail} v, v \overset{z}{\rightarrowtail} u, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{4_2^{\boxplus}}}$$

$$\frac{w \overset{x}{\rightarrowtail} w, x \in w, \Omega; \Gamma \Rightarrow \Delta}{x \in w, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{T^{\boxplus}}} \quad \frac{z \in w, \bigcap(z,x,x), x \in w, \Omega; \Gamma \Rightarrow \Delta}{x \in w, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{Cc},\, z \text{ eig.}} \quad \frac{v \overset{y}{\rightarrowtail} v, x \in w, y \in v, w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}{x \in w, y \in v, w \overset{x}{\rightarrowtail} v, \Omega; \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{5c^{\boxplus}}}$$

4 Multi-Agent Epistemic Logics with Groups

The second decidable fragment of TMLs that we consider expresses MELs with quantification over groups of agents. This fragment simulates MELs by expressing the multi-agent modalities $\Box_1, \dots, \Box_n$ via the term-modal ones $[x_1], \dots, [x_n]$, and, if P is a monadic predicate, it allows to say that the individual agent x_i is a member of the group P and that each/some member of the group P knows something. Moreover, it simulates the epistemic logic with names $AX_{\mathcal{N}}$ introduced in [15]. Note that groups' knowledge can be expressed also in MELs, but the present formulation is preferable in that it is more succint in the sense of [11].

Formally, we introduce the following conventions: $[\forall P]A := \forall y(Py \supset [y]A)$ and $[\exists P]A := \exists y(Py \wedge [y]A)$. Then, for any $n \in \mathbb{N}$, we consider the language $\mathcal{L}_n^*$ defined by the following grammar, where we restrict the set of variables to $\{y, x_1, \dots, x_n\}$; Q is a 0-ary predicate and P an 1-ary one; and, finally, B is an $\mathcal{L}_n^*$-formula with no subformula of shape Px_i

$$A ::= Q \,|\, Px_i \,|\, \neg A \,|\, A \wedge A \,|\, [x_i]B \,|\, [\forall P]B \,|\, [\exists P]B. \qquad (\mathcal{L}_n^*)$$

In $\mathcal{L}_n^*$-formulas, the variable y is always bound by a quantifier and the x_is are always free; therefore, the x_is can be thought as individual constants and $\mathcal{L}_n^*$ as a monodic fragment of TMLs [21]. The operators $\top, \bot, \vee, \supset$, and $\langle x_i \rangle$ are defined as before. We can also consider the 'group-diamond' operators $\langle \exists P \rangle$ and $\langle \forall P \rangle$ that are the dual of the primitive ones, e.g. the dual of $[\forall P]$ is $\langle \exists P \rangle$ that is defined as $\langle \exists P \rangle A := \neg [\forall P] \neg A$. $[\forall P]$ and $[\exists P]$ are not interdefinable.

$\mathcal{L}$-formulas involving interaction of modalities and quantifiers, like the Barcan Formulas, are not $\mathcal{L}_n^*$-formulas. Thus, the distinction between varying and single domain frames loses much of its interest over the language $\mathcal{L}_n^*$. For the sake of simplicity, we consider here only single domain frames (sd-frames, for short.) – i.e. frames where, for all $w \in \mathcal{W}$, $D_w = \mathcal{U}$. By an $\mathcal{L}_n^*$-*logic* we mean the set of all $\mathcal{L}_n^*$-formulas that are valid in some class of sd-frames obtained by some combination of the semantic properties corresponding to the schemes $T, D, 4$, and 5 (see Proposition 4). We name $\mathcal{L}_n^*$-logics according to the standard propositional conventions. To illustrate, $\mathbf{KD45_n^*}$ is the set of $\mathcal{L}_n^*$-formulas valid over all serial, transitive and Euclidean sd-frames; $\mathbf{S5_n^*}$ is the set of $\mathcal{L}_n^*$-formulas valid over all sd-frames where each $\overset{a}{\rightarrowtail}$ is an equivalence relation. Theorems 12 and 13 imply that $\mathbf{G3tm.X} \oplus \mathbf{UI_n^*}$ is sound and complete w.r.t. theoremhood in $\mathbf{X_n^*}$.

Example 23. The $\mathcal{L}_n^*$-formula $Px_i \wedge [x_i]A \wedge [\forall P](A \supset B) \supset [\exists P]B$ – which expresses the sentence *if x_i is a P that knows that A and if every P knows that A implies B, then some P knows that B* – is valid in every sd-frame, but the $\mathcal{L}_n^*$-formula $[\exists P]A \wedge [\exists P](A \supset B) \supset [\exists P]B$ is not valid in every sd-frame, as it is shown by the following (compressed) $\mathbf{G3tm.K} \oplus \mathbf{UI_n^*}$-proof-searches (where rules $L\exists$, $R\exists$, and $L\supset$ are admissible).

$$
\dfrac{\dfrac{\dfrac{}{w : Px_i \Rightarrow w : Px_i} \qquad \dfrac{\dfrac{}{w : Px_i \Rightarrow w : Px_i} \qquad \dfrac{\dfrac{\dfrac{}{v : A \Rightarrow v : A}\,{\scriptstyle Lem.6} \qquad \dfrac{}{v : B \Rightarrow v : B}\,{\scriptstyle Lem.6}}{w \overset{x_i}{\rightarrowtail} v, \dots, v : A \supset B, v : A \Rightarrow v : B}\,{\scriptstyle L\supset}}{\dots, w : [x_i](A \supset B), w : [x_i]A \Rightarrow w : [x_i]B}\,{\scriptstyle R\Box + L\Box}}{\dots, w : Px_i \supset [x_i](A \supset B) \Rightarrow w : [x_i]B}\,{\scriptstyle L\supset}}{x_i \in w; w : Px_i, w : [x_i]A, w : Px_i \supset [x_i](A \supset B) \Rightarrow w : Px_i \wedge [x_i]B}\,{\scriptstyle R\wedge}}{w : Px_i, w : [x_i]A, w : [\forall P](A \supset B) \Rightarrow w : [\exists P]B}\,{\scriptstyle \text{Defs. } [\forall P] \& [\exists P] + L_{UI} + L\forall + R\exists}
$$

$$
\dfrac{\dfrac{\begin{matrix}\vdots \\ \star\end{matrix} \quad \dfrac{\begin{matrix}\vdots \\ \star\end{matrix} \quad \dfrac{\dfrac{\overset{???}{\dots v : A \Rightarrow v : B, u : B, u : A} \qquad \dfrac{}{\dots u : B \Rightarrow v : B, u : B}\,{\scriptstyle Lem.6}}{w \overset{y}{\rightarrowtail} v, w \overset{z}{\rightarrowtail} u, y, z \in w; v : A, u : A \supset B, \dots \Rightarrow v : B, u : B}\,{\scriptstyle L\supset}}{y, z \in w; w : Py, w : [y]A, w : Pz, w : [z](A \supset B) \Rightarrow w : [y]B, w : [z]B}\,{\scriptstyle R\Box + L\Box}}{y, z \in w; w : Py, w : [y]A, w : Pz, w : [z](A \supset B) \Rightarrow w : [y]B, w : Pz \wedge [z]B}\,{\scriptstyle R\wedge}}{y, z \in w; w : Py, w : [y]A, w : Pz, w : [z](A \supset B) \Rightarrow w : Py \wedge [y]B, w : Pz \wedge [z]B}\,{\scriptstyle R\wedge}}{w : [\exists P]A, w : [\exists P](A \supset B) \Rightarrow w : [\exists P]B}\,{\scriptstyle \text{Def. } [\exists P] + L\exists + L\wedge + R\exists} \quad \Box
$$

We have now all the elements to prove the main result of this section, whose proof is based on that for the epistemic logic with names $AX_{\mathcal{N}}$ [15, Theorem 3.2].

Theorem 24. *Each $\mathcal{L}_n^*$-logic is decidable. Moreover, the validity problem for the $\mathcal{L}_n^*$-logics $\mathbf{K_n^*}$, $\mathbf{S4_n^*}$, $\mathbf{KD45_n^*}$, and $\mathbf{S5_n^*}$ is* PSPACE*-complete.*

Proof (Sketch). We fix an $\mathcal{L}_n^*$-logic $\mathbf{X_n^*}$ and an $\mathcal{L}_n^*$-formula A of length m. First, in **(1)**, we show that the $\mathbf{X_n^*}$-satisfiability problem is decidable by outlining a terminating $\mathbf{G3tm.X} \oplus \mathbf{UI}^*_{\mathbf{n}}$-proof-search procedure for $\mathcal{L}^*_n$-formulas. Then, in **(2)**, we prove that the problem is in PSPACE, and, in **(3)**, we prove that it is PSPACE-hard. Both **(2)** and **(3)** are proved by reduction to a satisfiability problem in propositional MELs which are known to be PSPACE-complete.

(1) First, it is possible to modify the proof-search procedure given in Proposition 17 into a terminating procedure for the sequent $\Rightarrow w : \neg A$. Let us call *agent-creating* any subformula of A that is either of shape $[\exists P]B$ and in the scope of an even number of negations, or of shape $[\forall P]B$ and in the scope of an odd number of negations. It can be shown that, if A is $\mathbf{X}^*_{\mathbf{n}}$-satisfiable, each agent-creating subformula of A can be satisfied by exactly one individual. Thus, in the proof-search procedure, each time we consider an instance of rules $L\exists$ and $R\forall$ whose principal formula has been already analysed (root first) in that branch, we can instantiate it to the same variable (we apply an hp-admissible substitution to rename the *eigenvariable* we introduce). It follows that at most m different term-modal modalities $[z]$ occurs in the proof-search tree, and we can easily adapt the termination procedures for propositional labelled calculi given in [12,20].

(2) Next, we prove PSPACE-completeness. Let, from now on, $\mathbf{X}^*_{\mathbf{n}}$ be one of $\mathbf{K}^*_{\mathbf{n}}$, $\mathbf{S4}^*_{\mathbf{n}}$, $\mathbf{KD45}^*_{\mathbf{n}}$ and $\mathbf{S5}^*_{\mathbf{n}}$. We start by showing that the problem is in PSPACE. Let A be $\mathbf{X}^*_{\mathbf{n}}$-satisfiable, and let $\mathcal{M}^A$ be the model (based on a frame for $\mathbf{X}^*_{\mathbf{n}}$) that is constructed as in Definition 15 from the terminating proof-search procedure for $\Rightarrow w : \neg A$. For each monadic predicate P occurring in A (A-group, for short.), let $|P|$ be the maximum number of agents (i.e. members of $\mathcal{U}^A$) satisfying P in some $w \in \mathcal{W}^A$; where, for each x_i occurring in A, we suppose there is a corresponding singleton A-group X_i. It is clear that for each A-group P, there is a $k \leq m$, with $m = Le(A)$, such that $|P| = k$. We consider a propositional multi-modal logic $\mathbf{X}^{\mathbf{A}}$ where, for each A-group P, we have the modalities $\{\Box_{P_i} : i \leq |P|\}$ and where, if Px_i holds in some world, then $\Box_{X_i} \equiv \Box_{P_j}$ for some $j \leq |P|$.
We map A to the $\mathbf{X}^{\mathbf{A}}$-formula ϕ^A that is obtained by replacing each subformula of shape $[\exists P]B$ with $\bigvee_{i=1}^{|P|} \Box_{P_i} B$; each subformula of shape $[\forall P]B$ with $\bigwedge_{i=1}^{|P|} \Box_{P_i} B$ (if $|P| = 0$, we replace $[\exists P]B$ with $\bot$ and $[\forall P]B$ with $\top$); and each subformula of shape Px_i with a new atomic propositional formula p_{Px_i} having the same semantic value. It is clear that, for all $v \in \mathcal{W}$, $\vDash^{\mathcal{M}^A}_v A$ iff $\vDash^{\mathcal{M}^A}_v \phi^A$.
In [16] it is given a PSPACE algorithm for checking satisfiability of an $\mathbf{X}^{\mathbf{A}}$-formula, which, if adequately implemented, depends on the number of its subformulas, and the number of subformulas of ϕ^A is at most m^2. Thus, the $\mathbf{X}^*_{\mathbf{n}}$-satisfiability problem is in PSPACE.

(3) Finally, we show that the satisfiability problem is PSPACE-hard. Let's assume that the language of $\mathbf{X}^*_{\mathbf{n}}$ is such that at least two agents/groups are expressible in it. To show that the $\mathbf{X}^*_{\mathbf{n}}$-satisfiability problem is PSPACE-hard, it is enough to notice that the logic $\mathbf{X}^{\mathbf{n}}$ is contained in $\mathbf{X}^*_{\mathbf{n}}$, and that, save for $\mathbf{KD45^1}$ and $\mathbf{S5^1}$, the $\mathbf{X}^{\mathbf{n}}$-satisfiability problem is PSPACE-hard. $\Box$

5 Future Work

We have introduced two simple decidable fragments of TMLs, and we have characterized the complexity of the validity problem for logics in these fragments. In the future we plan to introduce terminating **G3tm.L**-proof-search procedures for these logics, and to consider more expressive fragments. One simple extension is the addition of distributed knowledge to $\mathcal{L}_n^*$-logics. Distributed knowledge represents 'what a wise man, who knows what every member of the group knows, would know' [16, p. 321], and, accordingly, it is possible to express the propositional epistemic formula $D_G\phi$ via the $\mathcal{L}$-sentence $\exists x\forall y(Gy \wedge [y]\phi \supset [x]\phi)$. The addition of distributed knowledge to $\mathbf{X^n}$ does not change its complexity [16], and the same should hold for $\mathbf{X_n^*}$.

Notice that the operators $[\exists P]$ and $\langle\forall P\rangle$ can be seen as a monotone epistemic operators over the group P. Since Coalition Logic ($\mathcal{CL}$) can be seen as a logic with monotone modalities, cf. [17], it should be possible to simulate $\mathcal{CL}$ via TMLs (the only interesting step is that of modeling *superadditivity*, see [23, p. 152], in the term-modal framework). Then, it would be natural to consider its PSPACE-complete epistemic extensions $\mathcal{ECL}$ and $\mathcal{CLD}$ [1].

The fragments of TMLs considered thus far are more expressive than MELs, but they cannot express relations between groups – e.g., $\forall x(Px \supset Qx)$ is not expressible – nor agents' knowledge about whether they are member of a group – e.g., $\exists x(Px\wedge[x]Px)$ is not expressible. An important direction for future research is to find 'maximal' decidable fragments of TMLs. It is known, cf. [21], that the monadic fragment of TMLs is not decidable, and that the propositional monodic fragment – which is, roughly, $\mathcal{L}_n^*$ without unary predicates – is decidable. It might well be that the *full* monodic fragment [3, p. 582–86] of TMLs and of indexed epistemic logics [5–7], or some interesting sublanguage thereof, is decidable.

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