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Models for Stronger Normal Intuitionistic Modal Logics*

Abstract. This paper, a sequel to "Models for normal intuitionistic modal logics" by M. Božić and the author, which dealt with intuitionistic analogues of the modal system K, deals similarly with intuitionistic analogues of systems stronger than K, and, in particular, analogues of $S4$ and $S5$. For these propositional logics Kripke-style models with two accessibility relations, one intuitionistic and the other modal, are given, and soundness and completeness are proved with respect to these models. It is shown how the holding of formulae characteristic for particular logics is equivalent to conditions for the relations of the models. Modalities in these logics are also investigated.

§ 1. Introduction. As in [3], first we shall deal with systems with $\Box$ primitive, and next with systems with $\Diamond$ primitive. Unlike what we have in modal logic with a classical basis, these two kinds of systems are not reducible to each other. This means that with the usual semantical definitions of the holding of $\Box A$ and $\Diamond A$ we cannot use the same models for both kinds of systems.

Some of the systems we shall consider are already known in the literature. For example, systems equivalent to our $HS4\Box$ (see § 3) were considered in [11] and [8] (see [3] for a survey of works on intuitionistic modal logic). However, we do not presuppose for this paper an acquaintance with these earlier works. What we presuppose for it is a certain acquaintance with Kripke models for intuitionistic propositional logic and Kripke models for normal modal logics based on classical logic (for the former the reader may consult [7] or [1], Ch. 9, and for the latter [6]). We also presuppose an acquaintance with [3]. Nevertheless, in order to make this paper more self-contained, we shall review briefly in the next section some of the terminology and results of [3]. (A reader reading this paper in conjunction with [3] may skip this section.)

§ 2. Review of terminology and results of [3]. The language $L\Box$ is the language of propositional modal logic with denumerably many propositional variables, for which we use the schemata $p, q, r, p_1, \ldots$, and the connectives $\rightarrow$, $\wedge$, $\vee$, $\neg$ and $\Box$ ($\leftrightarrow$ is defined as usual in terms of $\rightarrow$ and $\wedge$, and in formulae $\wedge$ and $\vee$ bind more strongly than $\rightarrow$ and $\leftrightarrow$).

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As schemata for formulae we use $A, B, \ldots, A_1, \ldots$, and as schemata for sets of formulae we use capital Greek letters.

The system $\boldsymbol{HK}\Box$ is an extension of the Heyting propositional calculus in $L\Box$ with

$\Box 1.\quad \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$

$\Box 2.\quad \Box(A \rightarrow A)$

$R\Box.\quad \dfrac{A \rightarrow B}{\Box A \rightarrow \Box B}.$

The relation $\vdash$ in $\Phi \vdash A$ is defined as the usual relation of deducibility from hypotheses *using only modus ponens*. The expression $Cl(\Phi)$ stands for $\{A \mid \Phi \vdash A\}$.

An $\boldsymbol{H}\Box$ frame is $\langle X, R_I, R_M\rangle$ where (i) $X \neq \emptyset$, (ii) $R_I \subseteq X^2$ is reflexive and transitive, (iii) $R_M \subseteq X^2$ and (iv) $R_I R_M \subseteq R_M R_I$. The variables $x, y, z, t, u, v, x_1, \ldots$ range over X. An $\boldsymbol{H}\Box$ model is $\langle X, R_I, R_M, V\rangle$ where $\langle X, R_I, R_M\rangle$ is an $\boldsymbol{H}\Box$ frame and V, called a valuation, is a mapping from the set of propositional variables of $L\Box$ to the power set of X such that for every p, $\forall x, y\big(xR_I y \Rightarrow (x \in V(p) \Rightarrow y \in V(p))\big)$. The relation $\models$ in $x \models A$ is defined as usual, except that for $\rightarrow$ and $\neg$ it involves R_I, whereas for $\Box$ it involves R_M. A formula A holds in a model $\mathscr{M}$ ($\mathscr{M} \models A$) iff $\forall x . x \models A$; A holds in a frame Fr ($Fr \models A$) iff A holds in every model with this frame; and A is valid ($\models A$) iff A holds in every frame. If an $\boldsymbol{H}\Box$ frame (model) is condensed, then $R_I R_M = R_M$, and if it is strictly condensed, then $R_I R_M = R_M R_I = R_M$. The system $\boldsymbol{HK}\Box$ is sound and complete with respect to $\boldsymbol{H}\Box$ models (condensed $\boldsymbol{H}\Box$ models, strictly condensed $\boldsymbol{H}\Box$ models).

The language $L\Diamond$ differs from $L\Box$ in having $\Diamond$ instead of $\Box$, and the system $\boldsymbol{HK}\Diamond$ is an extension of the Heyting propositional calculus in $L\Diamond$ with:

$\Diamond 1.\quad \Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$

$\Diamond 2.\quad \neg\Diamond\neg(A \rightarrow A)$

$R\Diamond.\quad \dfrac{A \rightarrow B}{\Diamond A \rightarrow \Diamond B}.$

An $\boldsymbol{H}\Diamond$ frame (model) differs from an $\boldsymbol{H}\Box$ frame (model) by having $R_I^{-1} R_M \subseteq R_M R_I^{-1}$ for clause (iv) (also V maps the propositional variables of $L\Diamond$ rather than $L\Box$). The relation $\models$ involves R_M for $\Diamond$. If an $\boldsymbol{H}\Diamond$ frame (model) is condensed, then $R_I^{-1} R_M = R_M$, and if it is strictly condensed, then $R_I^{-1} R_M = R_M R_I^{-1} = R_M$. The system $\boldsymbol{HK}\Diamond$ is sound and complete with respect to $\boldsymbol{H}\Diamond$ models (condensed $\boldsymbol{H}\Diamond$ models, strictly condensed $\boldsymbol{H}\Diamond$ models).

Any extension of $\boldsymbol{HK}\Box$ in $L\Box$, or of $\boldsymbol{HK}\Diamond$ in $L\Diamond$, which is closed under substitution for propositional variables is called *normal*. (An extension of a system S is closed under the primitive rules of S.)

SYSTEMS IN $L\Box$

§ 3. Some normal extensions of $\boldsymbol{HK}\Box$. In this section we shall present some normal extensions of $\boldsymbol{HK}\Box$ which later (in § 5) we shall prove sound and complete with respect to specific classes of $\boldsymbol{H}\Box$ models. Consider the following schemata:

$\Box D$. $\quad \neg\Box\neg(A\to A)$
$\Box T$. $\quad \Box A\to A$
$\Box 4$. $\quad \Box A\to\Box\Box A$
$\Box 5$. $\quad (\Box A\to\Box B)\to\Box(\Box A\to\Box B)$
$\Box 5.1$. $\quad \neg\Box A\to\Box\neg\Box A$.

The schema $\Box D$ can easily be shown equivalent with $\Box A\to\neg\Box\neg A$ in any extension of $\boldsymbol{HK}\Box$, since in $\boldsymbol{HK}\Box$ we can prove:

$$\begin{aligned}(\Box A\to\neg\Box\neg A) &\leftrightarrow \neg(\Box A\wedge\Box\neg A)\\ &\leftrightarrow \neg\Box(A\wedge\neg A)\\ &\leftrightarrow \neg\Box\neg(A\to A).\end{aligned}$$

It is well known that we can extend the modal system $\boldsymbol{K}$ in $L\Box$, based on classical propositional logic, with $\Box D$ in order to get the system $\boldsymbol{D}$, with $\Box T$ in order to get the system $\boldsymbol{T}$, with $\Box T$ and $\Box 4$ to get $\boldsymbol{S4}$, and with $\Box T$ and $\Box 5.1$ to get $\boldsymbol{S5}$ (cf. [6], p. 131). It can be proved that $\Box 5$ can replace $\Box 5.1$ in $\boldsymbol{S5}$ yielding the same set of theorems. First, to show that $\Box 5$ is a theorem of $\boldsymbol{S5}$ we have

$$\dfrac{\dfrac{\dfrac{\Box\neg\Box B\to\neg\Box B}{\Box B\to\neg\Box\neg\Box B}\qquad \neg\Box\neg\Box B\to\Box\neg\Box\neg\Box B}{\Box B\to\Box\neg\Box\neg\Box B}\qquad \dfrac{\dfrac{(*)\ \dfrac{\neg\Box B\to\Box\neg\Box B}{\neg\Box\neg\Box B\to\neg\neg\Box B}}{\neg\Box\neg\Box B\to\Box B}}{\Box\neg\Box\neg\Box B\to\Box\Box B}}{\Box B\to\Box\Box B;}$$

$$(*)\ \dfrac{\dfrac{\neg\Box A\to\Box\neg\Box A}{\neg\Box A\to\Box(\Box A\to\Box B)}\qquad \dfrac{\Box B\to\Box\Box B}{\Box B\to\Box(\Box A\to\Box B)}}{(\Box A\to\Box B)\to\Box(\Box A\to\Box B)}$$

(the steps marked (*) are not intuitionistically valid). Next, as an instance of $\Box 5$ we have $\big(\Box A\to\Box\neg(B\to B)\big)\to\Box\big(\Box A\to\Box\neg(B\to B)\big)$, which together with $\Box\neg(B\to B)\leftrightarrow\neg(B\to B)$ and $\big(C\to\neg(B\to B)\big)\leftrightarrow\neg C$ yields $\Box 5.1$.

We shall consider the normal systems named on the left, obtained by adding to $\boldsymbol{HK}\Box$ the schemata on the right:

$\boldsymbol{HD}\Box$	:	$\Box D$
$\boldsymbol{HT}\Box$	:	$\Box T$
$\boldsymbol{HS4}\Box$	:	$\Box T$ and $\Box 4$
$\boldsymbol{HS5.1}\Box$	:	$\Box T$, $\Box 4$ and $\Box 5.1$
$\boldsymbol{HS5}\Box$	:	$\Box T$ and $\Box 5$.

Later we shall see that $\boldsymbol{HS5.1}\Box$ is a proper subsystem of $\boldsymbol{HS5}\Box$, and hence that $\Box 5$ and $\Box 5.1$ are not interreplaceable anymore. In general, we shall see that

$$\boldsymbol{HD}\Box \subset \boldsymbol{HT}\Box \subset \boldsymbol{HS4}\Box \subset \boldsymbol{HS5.1}\Box \subset \boldsymbol{HS5}\Box,$$

where $S_1 \subset S_2$ means that S_1 is a proper subsystem of S_2. We shall also prove that each of these systems constitutes a proper subsystem of the corresponding system based on classical propositional logic.

Let $Triv\,\Box$ be the extension of the classical propositional calculus in $L\Box$ with $\Box A \leftrightarrow A$. It can be shown that an extension of $\boldsymbol{D}$ closed under substitution for propositional variables is consistent iff it is a subsystem of $Triv\,\Box$. To show this for $\boldsymbol{HD}\Box$ essentially the same methods can be used.

LEMMA 1. *A normal extension of* $\boldsymbol{HD}\Box$ *is consistent iff it is a subsystem of* $Triv\,\Box$.

PROOF. Suppose $\boldsymbol{S}$ is a consistent normal extension of $\boldsymbol{HD}\Box$ which is not a subsystem of $Triv\,\Box$. So, there must be a theorem of $\boldsymbol{S}$ such that when we delete all the squares from A, the resulting formula $d(A)$ is not a tautology, i.e., for a valuation v ascribing $\top$ or $\bot$ to the propositional variables of A, $v(d(A)) = \bot$. Let A' be obtained from A in the following way:

if $v(p) = \top$, then replace p in A by $p \to p$
if $v(p) = \bot$, then replace p in A by $\neg(p \to p)$.

Since $\boldsymbol{S}$ is closed under substitution for propositional variables, A' is also a theorem of $\boldsymbol{S}$. But in $\boldsymbol{S}$ we can prove:

$$\Box(B \to B) \leftrightarrow (B \to B)$$
$$\Box\neg(B \to B) \leftrightarrow \neg(B \to B)$$

and then, using these two theorems and theorems of the Heyting propositional calculus, we can show by an easy induction on the complexity of A' that $A' \leftrightarrow \neg(B \to B)$ is also a theorem of $\boldsymbol{S}$. From that it follows that $\boldsymbol{S}$ is inconsistent, which contradicts our assumption, and proves one direction of the Lemma. The other direction is trivial. q.e.d.

Note that $\boldsymbol{HK}\Box$ does not satisfy Lemma 1 as $\boldsymbol{HD}\Box$ does. For example, the extension of $\boldsymbol{HK}\Box$ with $\Box(\Box A \to A) \to \Box A$ is consistent (it is a subsystem of the logic G of [2]: a mapping replacing $\Box A$ in formulae by $A \to A$ can be used to show the consistency of this extension), but this extension is not a subsystem of $\mathit{Triv}\,\Box$.

We can prove the following lemma for $\boldsymbol{HD}5\Box$, analogous to a lemma demonstrable for $\boldsymbol{S}5$.

LEMMA 2. (i) *Let A be a formula of $L\Box$ in which every propositional variable is in the scope of a square. Then* $\vdash_{\boldsymbol{HS5}\Box} A \leftrightarrow \Box A$.

(ii) *Let A be a formula in which every propositional variable p occurs in subformulae of the form $\Box p$, and let A' be obtained from A by deleting all squares except those in subformulae of the form $\Box p$. Then* $\vdash_{\boldsymbol{HS5}\Box} A \leftrightarrow A'$.

PROOF. It is easily shown that in $\boldsymbol{HS5}\Box$ we can prove:

$(\Box B \alpha \Box C) \;\leftrightarrow\; \Box(\Box B \alpha \Box C)$, where α is $\to$, $\wedge$, or $\vee$ (for $\wedge$ and $\vee$ this holds in $\boldsymbol{HS4}\Box$)

$\neg\Box B \;\leftrightarrow\; \Box\neg\Box B$

$\Box B \;\leftrightarrow\; \Box\Box B.$

Then using these theorems (i) is provable by an easy induction on the complexity of A. For (ii) we first use these theorems to transform A into an equivalent formula in which every subformula is of the form $\Box B$, for some B, and then we show by induction that this formula is equivalent to A'. q.e.d.

An equivalent formulation of $\boldsymbol{HS5}\Box$ would be given by extending the Heyting propositional calculus in $L\Box$ with the following rules:

$\boldsymbol{R}\Box 1.$ $\quad \dfrac{A \to B}{\Box A \to B}$

$\boldsymbol{R}\Box 2.$ $\quad \dfrac{A \to B}{A \to \Box B}$, provided every propositional variable in A is in the scope of a square.

The rule $\boldsymbol{R}\Box 1$ is derivable in $\boldsymbol{HS5}\Box$ because of $\Box T$, and the derivability of $\boldsymbol{R}\Box 2$ follows from $\boldsymbol{R}\Box$ and Lemma 2 (i). On the other hand, it is not difficult to prove $\Box T$ using $\boldsymbol{R}\Box 1$, and $\Box 5$ using $\boldsymbol{R}\Box 2$. Also $\Box 1$, $\Box 2$ and $\boldsymbol{R}\Box$ are easily derivable with the help of $\boldsymbol{R}\Box 1$ and $\boldsymbol{R}\Box 2$. This formulation shows that $\boldsymbol{HS5}\Box$ coincides with the theorems in $L\Box$ proved by using only the rules with $\Box$ of the intuitionistic modal calculus *MIPC* of Prior and Bull (see [4] and [5]). The calculus *MIPC* is formulated with both $\Box$ and $\Diamond$, and the proviso for $\boldsymbol{R}\Box 2$ in *MIPC* is that every propositional variable in A is in the scope of a *modal operator*, and not only a *square*. In § 7 the remaining rules of *MIPC* will be given. In [5] models are built for *MIPC*, which exploit the analogy between this system and Heyting's predicate logic.

Let $d(A)$ be the result of deleting all squares from a formula A of $L\Box$. Then we can easily show that if A is a theorem of any of the normal extensions we have introduced, $d(A)$ is a theorem of the Heyting propositional calculus in $L\Box$ without $\Box$. This immediately establishes that all our extensions are conservative with respect to this calculus, and also shows that these extensions are consistent. (This consistency also follows from Lemma 1.)

A set of formulae Φ has the disjunction property iff $A \vee B \in \Phi$ implies $A \in \Phi$ *or* $B \in \Phi$. A system has this property iff the set of its theorems has this property. In order to prove that the normal extensions we have introduced possess the disjunction property we use a variant of Kleene's slash defined in [3] (§ 2) (this definition is fairly standard, save for the case of $|\Box A$, which is defined as $\Vdash A$, i.e., $\vdash A$ and $|A$).

Lemma 3. *Any extension of* $\boldsymbol{HK}\Box$ *with some of the schemata* $\Box D$, $\Box T$, $\Box 4$, $\Box 5$ *and* $\Box 5.1$ *has the disjunction property.*

Proof. We show by induction on the length of proof of A that $\vdash A \Rightarrow |A$, where $\vdash$ stands for $\vdash_S$ and S is a system from the Lemma. This enlarges the proof of Lemma 1 of [3], where S was $\boldsymbol{HK}\Box$. All the additional cases in the basis of the induction are quite straightforward (for $\Box D$ it is enough to note its analogy with $\Diamond 2$ and refer to the proof of Lemma 15 of [3]; for $\Box T$ we have $\vdash \Box A$ *and* $\Vdash A \Rightarrow |A$; for $\Box 4$ we have $\Vdash \Box A \Rightarrow \Vdash \Box A$; for $\Box 5$ we have $\Vdash \Box A \to \Box B \Rightarrow \Vdash \Box A \to \Box B$; and for $\Box 5.1$ we have $\Vdash \neg\Box A \Rightarrow \Vdash \neg\Box A$).

Then, if $\vdash A \vee B$, it follows that $|A \vee B$, which by the definition of the slash means $\Vdash A$ *or* $\Vdash B$, and this implies $\vdash A$ *or* $\vdash B$. q.e.d.

As a particular case of $\vdash A \Rightarrow |A$ of the proof above we have $\vdash \Box A \Rightarrow \Vdash A$, which shows that any of the systems of Lemma 3, as well as $\boldsymbol{HK}\Box$, is closed under the rule

$$\frac{\Box A}{A}.$$

This is, of course, trivial for systems with $\Box T$, but for some of those without $\Box T$ it shows that, in this respect, they differ from their classical counterparts (cf. [6], p. 168).

§ 4. Equivalence of $\Box D$, $\Box T$, $\Box 4$, $\Box 5$ and $\Box 5.1$ with conditions on $\boldsymbol{H}\Box$ frames. In this section we shall present a series of lemmata about the formulae characteristic of the normal extensions of $\boldsymbol{HK}\Box$ we have introduced. We shall show that the holding of these formulae in $\boldsymbol{H}\Box$

frames is equivalent to specific conditions concerning the relations of the frames. These lemmata will also help us to obtain soundness and completeness proofs in the next section.

In the proofs of our lemmata we shall make use of the fact that in $\boldsymbol{H}\Box$ or $\boldsymbol{H}\Diamond$ frames $\forall x, y\big(xR_I y \Rightarrow \varphi(y)\big) \Leftrightarrow \forall y \varphi(y)$, provided x does not occur free in $\varphi(y)$, which is an easy consequence of the reflexivity of R_I and first-order logic. Next we show the following preliminary lemma.

LEMMA 4. *Let Q be $R_1R_2\ldots R_n$, $n \geqslant 0$, where R_i, $1 \leqslant i \leqslant n$, is either R_I, or R_M, or R_I^{-1}, or R_M^{-1}, and let $y \in X$, where $\langle X, R_I, R_M\rangle$ is an $\boldsymbol{H}\Box$ or an $\boldsymbol{H}\Diamond$ frame. Then:*

(i)	$\forall x(x \vDash p$	$\Leftrightarrow$	$yQR_Ix)$	*or*
(ii)	$\forall x(x \vDash p$	$\Leftrightarrow$	*not* $xR_IQy)$	*or*
(iii)	$\forall x(x \vDash p$	$\Leftrightarrow$	*not* $yQR_I^{-1}x)$	*or*
(iv)	$\forall x(x \vDash p$	$\Leftrightarrow$	$xR_I^{-1}Qy)$	

implies that $\forall x_1, x_2(x_1R_Ix_2 \quad \Rightarrow \quad (x_1 \vDash p \Rightarrow x_2 \vDash p))$.

PROOF. Suppose (i) *and* $x_1R_Ix_2$ *and* $x_1 \vDash p$. It follows that yQR_Ix_1. By the transitivity of R_I we obtain yQR_Ix_2, from which $x_2 \vDash p$ follows.

Suppose (ii) *and* $x_1R_Ix_2$ *and not* $x_2 \vDash p$. It follows that x_2R_IQy. By the transitivity of R_I we obtain x_1R_IQy, from which *not* $x_1 \vDash p$ follows.

Suppose (iii) *and* $x_1R_Ix_2$ *and not* $x_2 \vDash p$. It follows that $yQR_I^{-1}x_2$. By the transitivity of R_I we obtain $yQR_I^{-1}x_1$, from which *not* $x_1 \vDash p$ follows.

Suppose (iv) *and* $x_1R_Ix_2$ *and* $x_1 \vDash p$. It follows that $x_1R_I^{-1}Qy$. By the transitivity of R_I we obtain $x_2\ R_I^{-1}Qy$, from which $x_2 \vDash p$ follows. q.e.d.

Throughout this section $\boldsymbol{Fr}$ will be an $\boldsymbol{H}\Box$ frame $\langle X, R_I, R_M\rangle$ and the symbol $R_\Box$ an abbreviation for R_MR_I. Then we start our series of lemmata.

LEMMA 5. $\boldsymbol{Fr} \vDash \neg\Box\neg(A \to A) \quad \Leftrightarrow \quad \forall x \exists y.\ xR_\Box y$
$\Leftrightarrow \quad \forall x \exists y.\ xR_M y$

(i.e., $R_\Box$ and R_M are serial).

PROOF. It follows immediately from the meaning of $R_\Box$ and the reflexivity of R_I that the right-hand sides are equivalent. Next we have:

$$\begin{aligned}\forall z.\ z \vDash \neg\Box\neg(A \to A) \quad &\Leftrightarrow \quad \forall z, x(zR_Ix \Rightarrow \exists y(xR_\Box y \text{ and } y \vDash A \to A)\\ &\Leftrightarrow \quad \forall x\, \exists y(xR_\Box y \text{ and } y \vDash A \to A)\\ &\Leftrightarrow \quad \forall x\, \exists y.\ xR_\Box y. \qquad \text{q.e.d.}\end{aligned}$$

In the proofs of the following lemmata we shall use the fact that in $\boldsymbol{H}\Box$ models

$$x \vDash \Box A \quad \Leftrightarrow \quad \forall y(xR_\Box y \Rightarrow y \vDash A)$$

which was proved in Lemma 4 of [3].

LEMMA 6. $Fr \vDash \Box A \to A \Leftrightarrow \forall x.\ xR_\Box x$ (*i.e.*, $R_\Box$ *is reflexive*).

PROOF. ($\Rightarrow$) Suppose for some x *not* $xR_\Box x$. Let $\forall y (y \vDash p \Leftrightarrow xR_\Box y)$. By Lemma 4 (i) there is a valuation such that this is satisfied. With this valuation, $\forall y (xR_M y \Rightarrow xR_\Box y)$, which holds by the reflexivity of R_I, entails $x \vDash \Box p$. On the other hand, *not* $x \vDash p$, and hence *not* $x \vDash \Box p \to p$.

$$\begin{aligned}(\Leftarrow)\quad \forall x.\ xR_\Box x\ &\Rightarrow \forall x \big(\forall y (xR_\Box y \Rightarrow y \vDash A) \Rightarrow x \vDash A\big)\\ &\Rightarrow \forall z, x \big(zR_I x \Rightarrow (x \vDash \Box A \Rightarrow x \vDash A)\big)\\ &\Rightarrow \forall z.\ z \vDash \Box A \to A. \qquad \text{q.e.d.}\end{aligned}$$

LEMMA 7. $Fr \vDash \Box A \to \Box\Box A \Leftrightarrow R_\Box^2 \subseteq R_\Box$ (*i.e.*, $R_\Box$ *is transitive*).

PROOF. ($\Rightarrow$) Suppose for some x, y *and* z, $xR_\Box y$ and $yR_\Box z$ *and not* $xR_\Box z$. Let $\forall t (t \vDash p \Rightarrow xR_\Box t)$. By Lemma 4 (i) there exists a valuation such that this is satisfied. With this valuation we obtain $x \vDash \Box p$ and *not* $x \vDash \Box\Box p$, and hence *not* $x \vDash \Box p \to \Box\Box p$.

$$\begin{aligned}(\Leftarrow)\ R_\Box^2 \subseteq R_\Box &\Rightarrow \forall y, z, t (\forall u (yR_\Box u \Rightarrow u \vDash A) \text{ and } yR_\Box z \text{ and } zR_\Box t \Rightarrow t \vDash A)\\ &\Rightarrow \forall x.\ x \vDash \Box A \to \Box\Box A. \qquad \text{q.e.d.}\end{aligned}$$

LEMMA 8. *Let*

$$\varphi \Leftrightarrow_{df} \forall x, y, z \big(xR_\Box y \text{ and } yR_\Box z \Rightarrow \exists t \big(xR_I t \text{ and } tR_\Box z \text{ and } \forall u (tR_\Box u \Rightarrow yR_\Box u)\big)\big).$$

Then $Fr \vDash (\Box A \to \Box B) \to \Box(\Box A \to \Box B) \Leftrightarrow \varphi$.

PROOF. ($\Rightarrow$) Suppose *not* φ, i.e., for some x, y and z

(1) $\quad xR_\Box y$ *and* $yR_\Box z$ *and* $\forall t (xR_I t$ *and* $\forall u (tR_\Box u \Rightarrow yR_\Box u) \Rightarrow$ *not* $tR_\Box z)$

Let $\forall v (v \vDash p \Leftrightarrow yR_\Box v)$ and $\forall v (v \vDash q \Leftrightarrow \text{not}\, vR_I z)$. By Lemma 4 (i) and (ii), there is a valuation such that this is satisfied. We show first that with this valuation $x \vDash \Box p \to \Box q$.

Suppose $v_1 R_M v_2$ *and* $v_2 R_I z$. Hence, $v_1 R_\Box z$, but then from (1) we obtain *not* $xR_I v_1$ *or not* $\forall u (v_1 R_\Box u \Rightarrow yR_\Box u)$. Next, suppose $xR_I v_1$; hence, for some u, $v_1 R_\Box u$ *and not* $yR_\Box u$. It follows that for some v_4, $v_1 R_M v_4$ *and* $v_4 R_I u$. Supposing now $\forall v_3 (v_1 R_M v_3 \Rightarrow yR_\Box v_3)$, we obtain $yR_\Box v_4$, which together with $v_4 R_I u$ and the transitivity of R_I gives us $yR_\Box u$. But we already have *not* $yR_\Box u$; so we can conclude that:

$$\forall v_1, v_2 (xR_I v_1 \text{ and } \forall v_3 (v_1 R_M v_3 \Rightarrow yR_\Box v_3) \text{ and } v_1 R_M v_2 \Rightarrow \text{not } v_2 R_I z)$$

which together with our valuation yields $x \vDash \Box p \to \Box q$.

On the other hand, with this valuation from $xR_\Box y$ and $yR_\Box z$ of (1), and $\forall v_3 (yR_M v_3 \Rightarrow yR_\Box v_3)$, which holds by the reflexivity of R_I, we can show that *not* $x \vDash \Box(\Box p \to \Box q)$. So, the left-hand side of the Lemma is false.

($\Leftarrow$) Suppose for some x, $x \models \Box A \to \Box B$ *and not* $x \models \Box(\Box A \to \Box B)$. Hence, for some y, $xR_\Box y$ *and* $y \models \Box A$ *and not* $y \models \Box B$. Hence, for some z, $yR_\Box z$ *and not* $z \models B$. But then, from φ, it follows that for some t, $xR_I t$ *and* $tR_\Box z$ *and* $\forall u(tR_\Box u \Rightarrow yR_\Box u)$. Since $y \models \Box A$, we have $\forall u(tR_\Box u \Rightarrow u \models A)$, i.e. $t \models \Box A$. On the other hand, if $t \models \Box B$, from $tR_\Box z$, it follows that $z \models B$, which is a contradiction. Hence, *not* $t \models \Box B$ and this contradicts $xR_I t$ *and* $x \models \Box A \to \Box B$. q.e.d.

LEMMA 9. *Let*
$\varphi' \Leftrightarrow_{df} \forall x, y, z(xR_\Box y \text{ and } yR_\Box z \Rightarrow \exists t(xR_I t \text{ and } tR_\Box z \text{ and } yR_\Box t))$.
Then, if $R_\Box$ *is reflexive in* Fr, $Fr \models (\Box A \to \Box B) \to \Box(\Box A \to \Box B) \Leftrightarrow \varphi'$.

PROOF. We shall prove that if $R_\Box$ is reflexive, $\varphi \Leftrightarrow \varphi'$, where φ is as in Lemma 8. For that it is enough to show that $\forall u(tR_\Box u \Rightarrow yR_\Box u) \Leftrightarrow yR_\Box t$. From left to right we use $tR_\Box t$. For the other direction it is enough to show that φ' (or φ) implies the transitivity of $R_\Box$. We have:

$$
\begin{aligned}
xR_\Box y \text{ and } yR_\Box z &\Rightarrow \exists t(xR_I t \text{ and } tR_\Box z)\\
&\Rightarrow xR_I R_M R_I z\\
&\Rightarrow xR_M R_I R_I z, \text{ since } R_I R_M \subseteq R_M R_I\\
&\Rightarrow xR_\Box z, \text{ since } R_I \text{ is transitive.} \quad \text{q.e.d.}
\end{aligned}
$$

LEMMA 10. *Let*
$\psi \Leftrightarrow_{df} \forall x, y\big(xR_\Box y \Rightarrow \exists t\big(xR_I t \text{ and } \forall u(tR_\Box u \Rightarrow yR_\Box u)\big)\big)$.
Then $Fr \models \neg\Box A \to \Box\neg\Box A \Leftrightarrow \psi$.

PROOF. ($\Rightarrow$) Suppose *not* ψ, i.e., for some x and y

(1) $\quad xR_\Box y \text{ and } \forall t\big(xR_I t \Rightarrow \exists u(tR_\Box u \text{ and not } yR_\Box u)\big)$.

Let $\forall z(z \models p \Leftrightarrow yR_\Box z)$. By Lemma 4 (i) there is a valuation such that this is satisfied. It follows from the second conjunct of (1) that with this valuation $x \models \neg\Box p$. On the other hand, we have from (1) and the reflexivity of R_I, $xR_\Box y$ *and* $\forall v_1(yR_M v_1 \Rightarrow yR_\Box v_1)$, which with our valuation entails *not* $x \models \Box\neg\Box p$. So, the left-hand side is false.

($\Leftarrow$) Suppose for some x, $x \models \neg\Box A$ *and not* $x \models \Box\neg\Box A$. Hence, for some x_1, $xR_M x_1$ *and not* $x_1 \models \neg\Box A$, and for some y, $xR_\Box y$ *and* $y \models \Box A$. But then from ψ it follows that for some t, $xR_I t$ *and* $\forall u(tR_\Box u \Rightarrow yR_\Box u)$. Since $y \models \Box A$, we have $\forall u(tR_\Box u \Rightarrow u \models A)$, i.e., $t \models \Box A$, which contradicts $xR_I t$ *and* $x \models \neg\Box A$. q.e.d.

LEMMA 11. *Let*
$\psi' \Leftrightarrow_{df} \forall x, y\,(xR_\Box y \Rightarrow \exists t(xR_I t \text{ and } yR_\Box t))$ (*i.e.*, $R_\Box \subseteq R_I R_\Box^{-1}$).
Then, if $R_\Box$ *is reflexive and transitive in* Fr, $Fr \models \neg\Box A \to \Box\neg\Box A \Leftrightarrow \psi'$.

The proof of this lemma is analogous to the proof of Lemma 9.

The conditions φ, φ', ψ and ψ' of Lemmata 8–11 are rather entangled, and we shall try to explain them a little bit. Let $Fr_c = \langle X, R\rangle$ be an ordinary Kripke frame for modal logic, where X is the set of worlds and R the accessibility relation, and let:

$$\begin{array}{lll}
\varphi_c & \Leftrightarrow_{\mathrm{df}} & \forall x, y, z(xRy \text{ and } yRz \Rightarrow xRz \text{ and } \forall u(xRu \Rightarrow yRu)) \\
\varphi_c' & \Leftrightarrow_{\mathrm{df}} & \forall x, y, z(xRy \text{ and } yRz \Rightarrow xRz \text{ and } yRx) \\
\psi_c & \Leftrightarrow_{\mathrm{df}} & R^{-1}R \subseteq R \text{ (i.e., } R \text{ is } \textit{euclidean}) \\
\psi_c' & \Leftrightarrow_{\mathrm{df}} & R \subseteq R^{-1} \text{ (i.e., } R \text{ is } \textit{symmetric}).
\end{array}$$

By fairly standard methods from modal logic it can be shown that:

$$\begin{array}{l}
Fr_c \models (\Box A \to \Box B) \to \Box(\Box A \to \Box B) \Leftrightarrow \varphi_c \\
R \text{ is reflexive } \Rightarrow (Fr_c \models (\Box A \to \Box B) \to \Box(\Box A \to \Box B) \Leftrightarrow \varphi_c') \\
Fr_c \models \neg\Box A \to \Box\neg\Box A \Leftrightarrow \psi_c \\
R \text{ is reflexive } \Rightarrow (Fr_c \models \neg\Box A \to \Box\neg\Box A \Leftrightarrow \psi_c' \text{ and } R \text{ is transitive}).
\end{array}$$

Now, φ_c is a conflation of transitivity and "euclideanity", and φ_c' a conflation of transitivity and symmetry. Diagrams from which we can read out the structure of φ_c, φ_c', ψ_c and ψ_c' are given in Fig. 1.

The conditions φ, φ', ψ and ψ' insert into these diagrams an extra R_I relation. Diagrams from which we can read out the structure of these conditions are given in Fig. 2 (dotted arrows stand for R_I and continuous arrows for $R_\Box$).

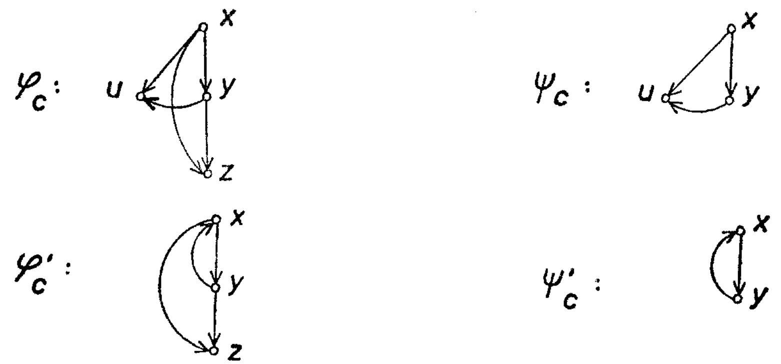


Fig. 1

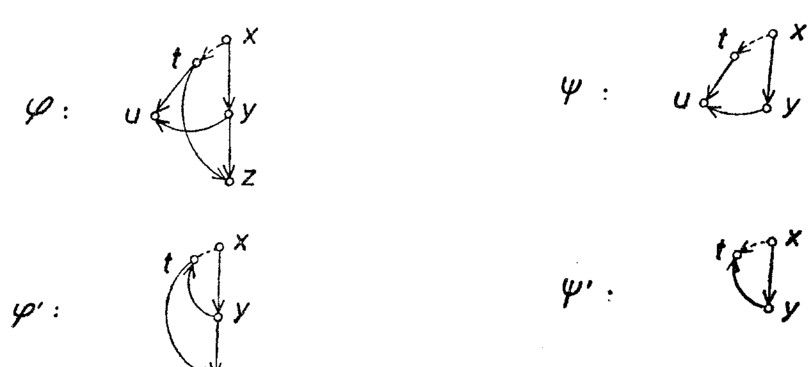


Fig. 2

Now it should be clear that when $R_\Box$ is serial (and *a fortiori*, when it is reflexive), $\varphi \Rightarrow \psi$ and $\varphi' \Rightarrow \psi'$ — but not conversely, since, as it is shown in the proof of Lemma 9, φ, or φ', implies the transitivity of $R_\Box$, and a simple counterexample can show that ψ, or ψ', does not. None of the conditions φ, φ', ψ and ψ' entails that $R_\Box$ is euclidean, or symmetric. This follows from the following lemma, which we shall state without proof.

LEMMA 12.

(i) $\quad Fr \models \Box A \vee \Box\neg\Box A \Leftrightarrow R_\Box^{-1} R_\Box \subseteq R_\Box \quad$ (*i.e.*, $R_\Box$ *is euclidean*).

(ii) $\quad Fr \models A \vee \Box\neg\Box A \Leftrightarrow R_\Box \subseteq R_\Box^{-1} \quad$ (*i.e.*, $R_\Box$ *is symmetric*).

Adding to $\boldsymbol{HS5}\Box$ either of the schemata mentioned on the left-hand sides of this lemma gives us a proper extension in which the disjunction property fails.

The schemata $\Box 5$ and $\Box 5.1$ give a kind of pseudo-euclideanity. We shall also mention a schema which somewhat similarly gives a kind of pseudo-transitivity. Let

$$\pi \quad \Leftrightarrow_{\mathrm{df}} \quad \forall x, y\big(xR_\Box y \Rightarrow \exists t\big(yR_I t \ \textit{and} \ \forall u(tR_\Box u \Rightarrow xR_\Box u)\big)\big).$$

Diagrams corresponding to transitivity $\tau \Leftrightarrow_{\mathrm{df}} \forall x, y\,\big(xR_\Box y \Rightarrow \forall u(yR_\Box u \Rightarrow \Rightarrow xR_\Box u)\big)$ and to π are given in Fig. 3.

Fig. 3

Then it is possible to show that $Fr \models \Box A \to \Box\neg\neg\Box A \Leftrightarrow \pi$. The schema $\Box A \to \Box\neg\neg\Box A$ is provable in $\boldsymbol{HS5.1}\Box$ without $\Box 4$ (see the proof of $\Box B \to \Box\Box B$ in $\boldsymbol{S5}$, in §3).

We note incidentally that it can be proved that $Fr \models \Box A \vee \neg\Box A \Leftrightarrow R_I^{-1} R_\Box \subseteq R_\Box (\Leftrightarrow R_I^{-1} R_M \subseteq R_M R_I)$. (This should be compared with the condition for condensed $\boldsymbol{H}\Diamond$ frames, and with what we had in connection with $\Box 3$ in § 11 of [3].) We also note that if we interpret the square as "it is intuitionistically provable that", since our disjunction is intuitionistic, $\Box(A \vee B) \to \Box A \vee \Box B$ might seem plausible. For this formula, which is not a theorem of $\boldsymbol{HS5}\Box$, we can show $Fr \models \Box(A \vee B) \to \Box A \vee \Box B \Leftrightarrow \forall x, y, z\,\big(xR_M y$ *and* $xR_M z \Rightarrow \exists t(xR_M t$ *and* $tR_I y$ *and* $tR_I z)\big)$. Extending with this formula any normal extension of $\boldsymbol{HK}\Box$ having the disjunction property would not spoil this property.

§ 5. Soundness and completeness of normal extensions of $\boldsymbol{HK}\Box$. In this section we shall prove that the normal extensions of $\boldsymbol{HK}\Box$ introduced in § 3 are sound and complete with respect to specific classes of $\boldsymbol{H}\Box$ models which are defined as follows:

DEFINITION 1. An $\boldsymbol{H}\Box$ frame $\langle X, R_I, R_M\rangle$ (model $\langle X, R_I, R_M, V\rangle$) is an:

(i) $\boldsymbol{HD}\Box$ frame (model) iff $R_\Box$ is serial
(ii) $\boldsymbol{HT}\Box$ frame (model) iff $R_\Box$ is reflexive
(iii) $\boldsymbol{HS4}\Box$ frame (model) iff $R_\Box$ is reflexive and transitive
(iv) $\boldsymbol{HS5.1}\Box$ frame (model) iff $R_\Box$ is reflexive and transitive, and ψ' of Lemma 11 holds
(v) $\boldsymbol{HS5}\Box$ frame (model) iff $R_\Box$ is reflexive and φ' of Lemma 9 holds.

A set of formulae Γ is *nice* iff Γ is consistent, deductively closed (i.e., $Cl(\Gamma) \subseteq \Gamma$) and it has the disjunction property. In [3] (§ 4) we introduced the canonical $\boldsymbol{S}$ frame $\langle X^c, R^c_I, R^c_M\rangle$ (canonical $\boldsymbol{S}$ model $\langle X^c, R^c_I, R^c_M, V^c\rangle$) by taking for X^c the set of all sets which are nice with respect to the system $\boldsymbol{S}$, whereas $\Gamma R^c_I \Delta$ was defined as $\Gamma \subseteq \Delta$ and $\Gamma R^c_M \Delta$ as $\Gamma_\Box \subseteq \Delta$, where $\Gamma_\Box = \{A \mid \Box A \in \Gamma\}$ ($V^c(p)$ was defined as $\{\Gamma \mid p \in \Gamma\}$). Then the following lemma can be proved.

LEMMA 13. *Let* $\boldsymbol{S}$ *be* (i) $\boldsymbol{HD}\Box$, (ii) $\boldsymbol{HT}\Box$, (iii) $\boldsymbol{HS4}\Box$, (iv) $\boldsymbol{HS5.1}\Box$, *or* (v) $\boldsymbol{HS5}\Box$. *Then the canonical* $\boldsymbol{S}$ *frame* (*model*) *is an* $\boldsymbol{S}$ *frame* (*model*).

PROOF. (i) In the canonical $\boldsymbol{HD}\Box$ frame $R^c_M R^c_I$ is serial. Otherwise, for some nice set Γ, $\forall\Delta$ *not* $\Gamma R^c_M R^c_I \Delta$. Since R^c_I is reflexive, $\forall \Delta$ *not* $\Gamma R^c_M \Delta$, i.e., $\forall\Delta$ *not* $\Gamma_\Box \subseteq \Delta$. It follows from Lemma 6 of [3] that every consistent set can be extended to a nice set — hence, $\Gamma_\Box$ is inconsistent, i.e., $\neg(A \to A) \in \Gamma_\Box$. But then $\Box\neg(A\to A) \in \Gamma$, and this yields a contradiction, since $\neg\Box\neg(A\to A) \in \Gamma$ and Γ is nice.

(ii) In the canonical $\boldsymbol{HT}\Box$ frame $R^c_M R^c_I$ is reflexive. This is shown as follows. Since $\Box A \to A$ belongs to every nice set Γ, we have $\Gamma_\Box \subseteq \Gamma$, i.e., $\Gamma R^c_M \Gamma$, and hence $\Gamma R^c_M R^c_I \Gamma$, by the reflexivity of R^c_I.

(iii) In the canonical $\boldsymbol{HS4}\Box$ frame $R^c_M R^c_I$ is reflexive and transitive. For reflexivity we proceed as in (ii). For transitivity we have:

$$
\begin{aligned}
\Gamma R^c_M R^c_I \Delta \textit{ and } \Delta R^c_M R^c_I \Theta \;&\Rightarrow\; \Gamma R^c_M \Delta \textit{ and } \Delta R^c_M \Theta \text{ since } R^c_M R^c_I = R^c_M\\
&\Rightarrow\; \Gamma_\Box \subseteq \Delta \textit{ and } \Delta_\Box \subseteq \Theta\\
&\Rightarrow\; (\Gamma_\Box)_\Box \subseteq \Delta_\Box \textit{ and } \Delta_\Box \subseteq \Theta\\
&\Rightarrow\; (\Gamma_\Box)_\Box \subseteq \Theta\\
&\Rightarrow\; \Gamma_\Box \subseteq \Theta, \text{ since by } \Box 4,\ \Gamma_\Box \subseteq (\Gamma_\Box)_\Box\\
&\Rightarrow\; \Gamma R^c_M R^c_I \Theta.
\end{aligned}
$$

(iv) In the canonical $\boldsymbol{HS5.1}\Box$ frame $R^c_M R^c_I$ is reflexive and transitive, and ψ' of Lemma 11 holds, where R_I is R^c_I and $R_\Box$ is $R^c_M R^c_I$. For the reflexivity and transitivity of this last relation we proceed as in (ii) and (iii). For ψ', suppose $\Gamma R^c_M R^c_I \Delta$. It follows that $\Gamma_\Box \subseteq \Delta$. Then we have that $\Gamma \cup \Delta_\Box$ is consistent. (Otherwise, for some $A \in \Gamma$ and some $B_1, \ldots, B_n \in \Delta_\Box$, $\{A, B_1, \ldots, B_n\} \vdash \neg(C \to C)$, and we get

$$\begin{array}{rll} & \vdash & B_1 \wedge \ldots \wedge B_n \;\to\; \neg A \\ & \vdash & \Box B_1 \wedge \ldots \wedge \Box B_n \;\to\; \Box\neg A \\ \Delta & \vdash & \Box\neg A. \end{array}$$

Since Δ is nice, $\Box\neg A \in \Delta$, and $\neg\Box\neg A \notin \Delta$, and since $\Gamma_\Box \subseteq \Delta$, $\Box\neg\Box\neg A \notin \Gamma$. Then by $\Box 5.1$, $\neg\Box\neg A \notin \Gamma$, and $A \notin \Gamma$, since $A \to \neg\Box\neg A$, which follows from $\Box\neg A \to \neg A$, belongs to Γ.) Since $\Gamma \cup \Delta_\Box$ is consistent, by Lemma 6 of [3], there is a nice set Θ such that $\Gamma \subseteq \Theta$ and $\Delta_\Box \subseteq \Theta$. From this $\Gamma R^c_I \Theta$ *and* $\Delta R^c_M R^c_I \Theta$ follows.

(v) In the canonical $\boldsymbol{HS5}\Box$ frame $R^c_M R^c_I$ is reflexive and φ' of Lemma 9 holds, where R_I is R^c_I and $R_\Box$ is $R^c_M R^c_I$. For the reflexivity of $R^c_M R^c_I$ we proceed as in (ii) and for φ' we proceed in the following way.

Suppose $\Gamma R^c_M R^c_I \Delta$ *and* $\Delta R^c_M R^c_I \Theta$. It follows that $\Gamma_\Box \subseteq \Delta$ *and* $\Delta_\Box \subseteq \Theta$. Let $Z =_{\mathrm{df}} \{\Psi \mid \Gamma \cup \Delta_\Box \subseteq \Psi$ *and* $\Psi_\Box \subseteq \Theta$ *and* $Cl(\Psi) \subseteq \Psi\}$. First, we prove that $Cl(\Gamma \cup \Delta_\Box) \in Z$. The only difficult part of this is to show that $\big(Cl(\Gamma \cup \Delta_\Box)\big)_\Box \subseteq \Theta$. We have:

$$\begin{array}{lll} A \in \big(Cl(\Gamma\cup\Delta_\Box)\big)_\Box & \Rightarrow & \Box A \in Cl(\Gamma \cup \Delta_\Box) \\ & \Rightarrow & \Gamma \cup \{B_1, \ldots, B_n\} \vdash \Box A, \text{ where } B_1, \ldots, B_n \in \Delta_\Box \\ & \Rightarrow & \Gamma \vdash \Box(B_1 \wedge \ldots \wedge B_n) \to \Box A, \text{ by the Deduction Theorem and } \Box T \\ & \Rightarrow & \Gamma \vdash \Box(\Box(B_1 \wedge \ldots \wedge B_n) \to \Box A), \text{ by } \Box 5 \\ & \Rightarrow & \Box(B_1 \wedge \ldots \wedge B_n) \to \Box A \in \Delta, \text{ since } \Gamma \text{ is nice and } \Gamma_\Box \subseteq \Delta \\ & \Rightarrow & \Box B_1 \wedge \ldots \wedge \Box B_n \to \Box A \in \Delta, \text{ using } \boldsymbol{HK}\Box \\ & \Rightarrow & A \in \Theta, \text{ since } \Box B_1, \ldots, \Box B_n \in \Delta \text{ and } \Delta_\Box \subseteq \Theta. \end{array}$$

Hence, Z is nonempty, and it is easy to show that Z is closed under unions of nonempty chains. So by Zorn's Lemma, Z has a maximal element Φ with respect to $\subseteq$. We show first that

(v.i) Φ is consistent.

Otherwise, $\Phi \vdash \Box\neg(A \to A)$, and since $Cl(\Phi) \subseteq \Phi$ we have $\Box\neg(A \to A) \in \Phi$. Hence, $\neg(A \to A) \in \Theta$, which contradicts the supposition that Θ is nice. Next, we infer immediately from $\Theta \in Z$ that

(v.ii) Φ is deductively closed.

Now, suppose that Φ does not have the disjunction property i.e., for some B and C, $B \vee C \in \Phi$ *and* $B \notin \Phi$ *and* $C \notin \Phi$. Since $\Phi \cup \{B\}$ and $\Phi \cup \{C\}$ are proper supersets of Φ, they cannot be in Z. *A fortiori*, $Cl(\Phi \cup \{B\})$

and $Cl(\Phi\cup\{C\})$ are not in Z. This is possible only if for some $\Box B_1$ from the first and some $\Box C_1$ from the second of these sets, $B_1 \notin \Theta$ *and* $C_1 \notin \Theta$. On the other hand, we have:

$$
\begin{aligned}
\Phi\cup\{B\}\vdash \Box B_1 \textit{ and } \Phi\cup\{C\}\vdash\Box C_1 \quad &\Rightarrow\quad \Phi\vdash B\vee C\rightarrow\Box B_1\vee\Box C_1\\
&\Rightarrow\quad \Phi\vdash\Box B_1\vee\Box C_1,\ \text{since } B\vee C\in\Phi\\
&\Rightarrow\quad \Phi\vdash\Box(\Box B_1\vee\Box C_1),\ \text{since in } \boldsymbol{HS5}\Box \text{ we can prove } \Box B_1\vee\Box C_1 \rightarrow \Box(\Box B_1\vee\Box C_1)\\
&\Rightarrow\quad \Box B_1\vee\Box C_1\in\Theta,\ \text{since (v.ii) and } \Phi_\Box\subseteq\Theta\\
&\Rightarrow\quad B_1\in\Theta \textit{ or } C_1\in\Theta,\ \text{since } \Theta \text{ is nice, and using } \Box T
\end{aligned}
$$

which gives a contradiction. So,

(v.iii) Φ has the disjunction property

and we can conclude that Φ is nice. We have that $\Gamma\subseteq\Phi$ *and* $\Delta_\Box\subseteq\Phi$ *and* $\Phi_\Box\subseteq\Theta$, and from this φ' follows.

Using (i)-(v) and Lemma 7 of [3], asserting that the canonical $\boldsymbol{S}$ frame (model) is an $\boldsymbol{H}\Box$ frame (model), we obtain the Lemma. q.e.d.

Next we prove the following soundness and completeness theorem.

THEOREM 1. *Let* $\boldsymbol{S}$ *be* $\boldsymbol{HD}\Box$, $\boldsymbol{HT}\Box$, $\boldsymbol{HS4}\Box$, $\boldsymbol{HS5.1}\Box$, *or* $\boldsymbol{HS5}\Box$. *Then*

$$\vdash_{\boldsymbol{S}}^{\!\!-} A \quad\Leftrightarrow\quad \textit{for every } \boldsymbol{S} \textit{ frame } Fr,\ Fr\vDash A.$$

PROOF. ($\Rightarrow$) Soundness follows from the ($\Leftarrow$) parts of Lemmata 5, 6, 7, 9 and 11, and from the first part of the proof of Theorem 1 of [3], which treats of the soundness of $\boldsymbol{HK}\Box$ with respect to $\boldsymbol{H}\Box$ frames.

($\Leftarrow$) For completeness we proceed as follows. Suppose that for every $\boldsymbol{S}$ frame Fr, $Fr\vDash A$. By Lemma 13, A holds in the canonical $\boldsymbol{S}$ frame, and consequently A holds in the canonical $\boldsymbol{S}$ model, i.e., $\forall\Gamma\in X^c.\ \Gamma\vDash A$. By Lemma 8 of [3], which asserts that in the canonical $\boldsymbol{S}$ model $\forall\Gamma\in X^c$ $(\Gamma\vDash A\Leftrightarrow A\in\Gamma)$, we have $\forall\Gamma\in X^c.\ A\in\Gamma$. Since the set of theorems of $\boldsymbol{S}$ is nice (it is consistent, deductively closed and, according to Lemma 3, it has the disjunction property), it follows that $\vdash_{\boldsymbol{S}} A$. q.e.d.

(For systems without the disjunction property, the completeness proof would use the possibility of extending the set of theorems to a nice set to which a non-theorem does not belong, which is guaranteed by Lemma 6 of [3].)

With Theorem 1 we can easily check the proper inclusions among our systems, mentioned in § 3. We can also prove the following soundness and completeness theorem, which follows from the soundness part of

Theorem 1 and from $R^c_M R^c_I = R^c_M$ (cf. Theorem 2 of [3], which covers the case when S is $\boldsymbol{HK}\Box$).

THEOREM 2. *Let* S *be as in Theorem* 1. *Then*:

$\vdash_S A \;\Leftrightarrow\;$ *for every condensed* $\boldsymbol{S}$ *frame* Fr, $Fr \models A$

$ \;\Leftrightarrow\;$ *for every strictly condensed* $\boldsymbol{S}$ *frame* Fr, $Fr \models A$.

It is interesting to note that in strictly condensed $\boldsymbol{H}\Box$ frames where $R_M R_I$, and hence also R_M, are reflexive and transitive, $R_I R_M = R_M R_I = R_M$ is interreplaceable with $R_I \subseteq R_M$. That the former condition implies the latter is an easy consequence of the reflexivity of R_M, and for the converse we use the transitivity of R_M and the reflexivity of R_I.

The Heyting propositional calculus can be embedded in the modal system $\boldsymbol{S4}$ by a translation which prefixes a square to every (proper) subformula, save (possibly) subformulae with $\wedge$ or $\vee$ as main connectives. It is possible to embed in a similar way some of the extensions $\boldsymbol{HX}\Box$ of $\boldsymbol{HK}\Box$ we have studied, in modal systems based on classical propositional logic with two squares: one ($\Box_I$) is an $\boldsymbol{S4}$ square which is used for the translation of intuitionistic connectives, and the other ($\Box_M$) is an X square. Axioms connecting these two squares would correspond to the conditions connecting the R_I and R_M relations in $\boldsymbol{H}\Box$ models. We have studied this problem in some detail for $\boldsymbol{HK}\Box$ in § 6 of [3], and from results presented there it is easy to infer that $\boldsymbol{HD}\Box$, $\boldsymbol{HT}\Box$ and $\boldsymbol{HS4}\Box$ could be embedded in systems where the axioms connecting the squares correspond to the conditions for strictly condensed $\boldsymbol{H}\Box$ models. In the system for $\boldsymbol{HS4}\Box$ we would then have an $\boldsymbol{S4}$ square $\Box_M$ and the schema $\Box_M A \to \Box_I A$, which is equivalent to $R_I \subseteq R_M$ in the sense of Lemma 10 of [3].

This is of some interest for the *topological interpretation* of $\boldsymbol{HS4}\Box$. That this interpretation is possible is a consequence of the well known interpretation of Heyting's propositional logic in algebras of open subsets of a topological space, and of the interpretation of $\boldsymbol{S4}$ in topological Boolean algebras (see [12], Chs. 3, 4, 9 and 11). Now, the system with two squares in which we can embed $\boldsymbol{HS4}\Box$, which we have mentioned above, could be interpreted in topological Boolean algebras with two *nested* interior operators, i.e., we would have $Int_M a \subseteq Int_I a$ (some results concerning nested topologies applicable to the topological interpretation of $\boldsymbol{HS4}\Box$ are given in [9]).

We note incidentally that a straightforward embedding into a system with $\Box_M$ an $\boldsymbol{S5}$ square is provided for systems with the schemata from Lemma 12 rather than for $\boldsymbol{HS5}\Box$ or $\boldsymbol{HS5.1}\Box$.

Consider the translation mentioned above with which we can embed the Heyting propositional calculus in $\boldsymbol{S4}$. What non-modal system can be embedded with this translation in $\boldsymbol{HS4}\Box$? The answer is again: the Heyting propositional calculus. The reasons for the fact that if A is a translation of a non-modal formula, then $\vdash_{S4} A \Leftrightarrow \vdash_{HS4\Box} A$, are essentially the same as those adduced for the coincidence of classical and intuitionistic strict implication in § 5 of [3].

§ 6. Modalities in *HS4*□ and *HS5*□. By a *modality* we shall understand, as usual, a sequence, possibly empty, made of □, ◇ and ¬. If M_1 and M_2 are two modalities, M_1 *implies* M_2 in the system S iff for every formula A, $\vdash_S M_1A \to M_2A$, and M_1 is *equivalent* to M_2 in S iff for every A, $\vdash_S M_1A \leftrightarrow M_2A$. When we say that the number of modalities in S is k, we mean that the set of all modalities can be divided into k equivalence classes using the equivalence of modalities in S. And when we display these k modalities, we display the shortest representatives of these equivalence classes. A modality is *positive* iff the number of ¬ in it is zero or even — otherwise, it is *negative*. In this section we shall consider modalities in $L\Box$, i.e., made only of □ and ¬, whereas in § 10 we shall deal with modalities in $L\Diamond$, i.e., made only of ◇ and ¬.

Using our models we can show that the number of modalities in ***HT***□, and subsystems of it, is infinite. Let $\Box^n$ stand for a sequence of n squares, where $n \geqslant 0$. Then for every $n \geqslant 0$, $\Box^n$ and $\Box^{n-1}$ are not equivalent in ***HT***□.

On the other hand, the number of modalities in ***HS4***□ is 31, of which 17 are positive and 14 negative. (It is well known that this number is 14 in ***S4*** — there we have 7 positive and 7 negative modalities.) In Fig. 4 we display the positive and in Fig. 5 the negative modalities of ***HS4***□. Arrows indicate that the modality from which the arrow starts implies in ***HS4***□ the modality at the other end, but not conversely. Dotted lines indicate that the modalities connected are equivalent in ***S4***, continuous lines that they are not.

In order to reduce modalities to those in Fig. 4 and Fig. 5 we use essentially the following theorems of ***HS4***□:

(1) $\Box\Box A \leftrightarrow \Box A$

(2) $\neg\neg\neg A \leftrightarrow \neg A$

(3) $\Box\neg\neg\Box\neg A \leftrightarrow \Box\neg A$

(4) $\neg\Box\neg\neg\Box A \leftrightarrow \neg\Box A$

(5) $\Box\neg\Box\neg\Box\neg\Box A \leftrightarrow \Box\neg\Box A$.

It is not hard to show either syntactically or model-theoretically that these formulae are theorems and that the implications in Fig. 4 and Fig. 5 do hold. It is also not difficult to produce ***HS4***□ models falsifying the converse implications, or would-be implications where we have drawn no arrows in Fig. 4 and Fig. 5. (Charts similar to these are given in [10], but the chart corresponding to Fig. 4 is slightly defective: the implication

$\neg\neg\Box\neg\Box\neg\Box\neg\neg A \rightarrow \neg\neg\Box\neg\Box\neg A$ is missing. Also the equivalence (i) of Lemma 1 of this paper does not obtain and is probably a misprint for our (5).)

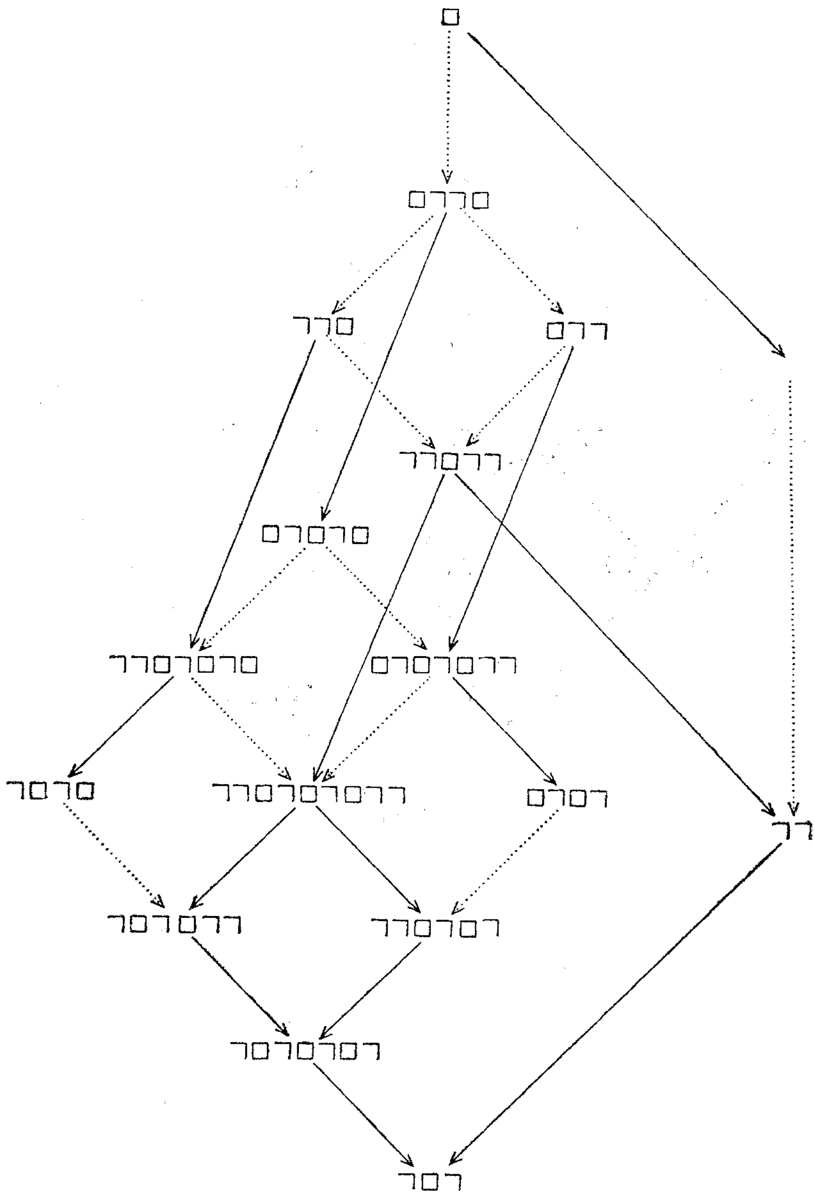


Fig. 4

The number of modalities in ***HS*5**$\Box$ and ***HS*5.1**$\Box$ is identical and it equals 10. Of these modalities 6 are positive and 4 negative. (It is well known that in **S5** there are 6 modalities, of which 3 are positive and 3 negative.) In Fig. 6 we display the positive and in Fig. 7 the negative modalities of $\boldsymbol{HS5}\Box$, or $\boldsymbol{HS5.1}\Box$. These charts are interpreted as those of Fig. 4 and Fig. 5, modulo the change from $\boldsymbol{HS4}\Box$ and **S4** to $\boldsymbol{HS5}\Box$ ($\boldsymbol{HS5.1}\Box$) and **S5**.

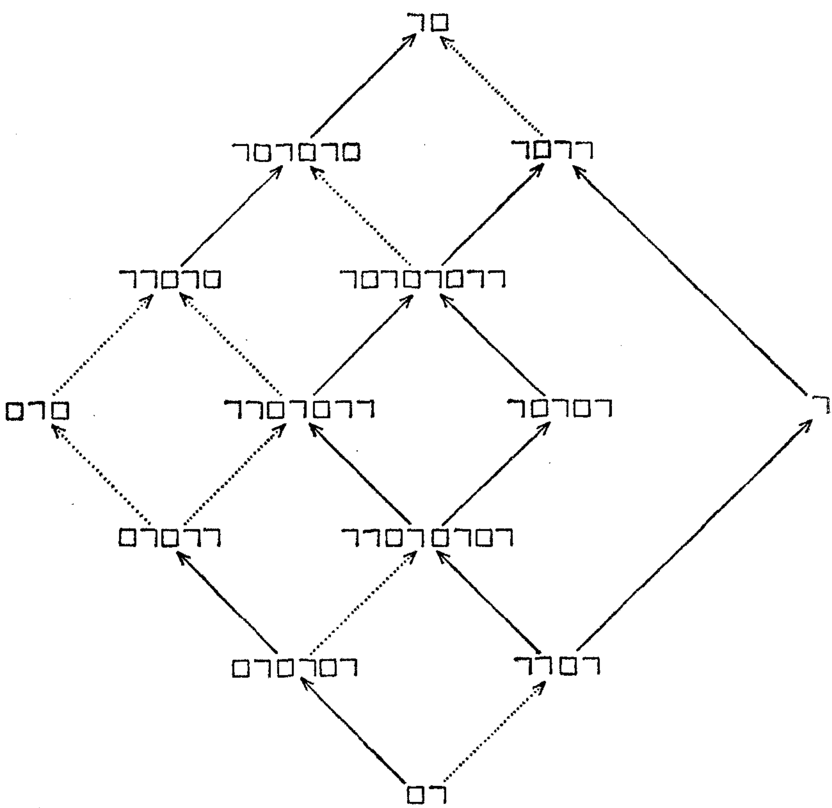
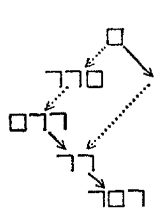


Fig. 5



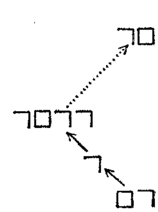


Fig. 6 Fig. 7

In order to reduce modalities to those in Fig. 6 and Fig. 7 we use essentially in addition to (1)-(5) the following theorems of $\boldsymbol{HS5.1}\Box$:

(6) $\Box\neg\Box A \leftrightarrow \neg\Box A$

(7) $\Box\neg\neg\Box A \leftrightarrow \neg\neg\Box A$

(8) $\neg\neg\Box\neg A \leftrightarrow \Box\neg A.$

As before, there is no special difficulty in proving these theorems and the implications of Fig. 6 and Fig. 7, and in falsifying with $\boldsymbol{HS5}\Box$ models the converse implications, or would-be implications where we have drawn no arrows.

In the extension of $\boldsymbol{HS5}\Box$ with the schemata of Lemma 12 the charts of modalities of Fig. 6 and Fig. 7 are changed only by having $\neg\neg\Box A \leftrightarrow \Box A$ — the rest remains unchanged.

SYSTEMS IN $L\Diamond$

§ 7. Some normal extensions of $\boldsymbol{HK}\Diamond$. In this section we shall present some normal extensions of $\boldsymbol{HK}\Diamond$ which later (in § 9) we shall prove sound and complete with respect to specific classes of $\boldsymbol{H}\Diamond$ frames. Consider the following schemata:

$\Diamond D.$ $\quad \Diamond(A \to A)$

$\Diamond T.$ $\quad A \to \Diamond A$

$\Diamond 4.$ $\quad \Diamond\Diamond A \to \Diamond A$

$\Diamond 5.$ $\quad \Diamond(\Diamond A \to \Diamond B) \to (\Diamond A \to \Diamond B)$

$\Diamond 5.1$ $\quad \Diamond\neg\Diamond A \to \neg\Diamond A.$

It is well known that we can extend the modal system $\boldsymbol{K}$ in $L\Diamond$, based on classical propositional logic, with $\Diamond D$ in order to get the system $\boldsymbol{D}$, with $\Diamond T$ in order to get the system $\boldsymbol{T}$, with $\Diamond T$ and $\Diamond 4$ in order to get $\boldsymbol{S4}$, and with $\Diamond T$ and $\Diamond 5.1$ in order to get $\boldsymbol{S5}$ (cf. [6], pp. 131–133). Extending any extension of $\boldsymbol{K}$ with $\Diamond 5$ yields the same set of theorems as extending this extension with $\Diamond 4$ and $\Diamond 5.1$. First, suppose we have $\Diamond 5$; then we obtain $\Diamond 4$ as follows:

$$\begin{array}{lll}
\Diamond A & \to & (\Diamond\Diamond A \to \Diamond A) \\
\Diamond\Diamond A & \to & \Diamond(\Diamond\Diamond A \to \Diamond A) \\
\Diamond\Diamond A & \to & (\Diamond\Diamond A \to \Diamond A) \\
\Diamond\Diamond A & \to & \Diamond A.
\end{array}$$

Next we have $\Diamond(\Diamond A \to \Diamond\neg(B \to B) \to (\Diamond A \to \Diamond\neg(B \to B))$, which together with $\Diamond\neg(B \to B) \leftrightarrow \neg(B \to B)$ (obtained from $\Diamond 2$ of § 2) yields $\Diamond 5.1$. Now, suppose we have $\Diamond 4$ and $\Diamond 5.1$; then

$$\begin{array}{lll}
\Diamond(\neg\Diamond A \vee \Diamond B) & \to & \Diamond\neg\Diamond A \vee \Diamond\Diamond B, \quad \text{by } \Diamond 1 \text{ of } \S 2 \\
 & \to & \neg\Diamond A \vee \Diamond B.
\end{array}$$

Since it is known that $\Diamond 4$ is a theorem of $\boldsymbol{S5}$ (cf. with the proof of $\Box 4$ in $\boldsymbol{S5}$, in § 3), it follows that $\Diamond 5$ can be used for $\Diamond 5.1$ in $\boldsymbol{S5}$ yielding the same set of theorems.

(Even though $\Box 5$ can replace $\Box 5.1$ for $\boldsymbol{S5}$ in $L\Box$, it is *not* the case that $\Box 5$ can be replaced by $\Box 4$ and $\Box 5.1$ in *any* extension of $\boldsymbol{K}$ in $L\Box$, but only in those which are extensions of $\boldsymbol{D}$ in $L\Box$. Note that in § 3 we have used $\Box\neg(B\to B)\leftrightarrow\neg(B\to B)$, which follows from $\Box D$, in order to derive $\Box 5.1$ from $\Box 5$. This derivation is impossible without $\Box D$. This can be shown with the help of an ordinary Kripke model for modal logic with two worlds x_1 and x_2 such that the accessibility relation connects only x_1 with x_2, and p is falsified in x_2. Then we can check that this model satisfies φ_c from § 4, and that $\neg\Box p\to\Box\neg\Box p$ is falsified in x_1. So, there is a slight lack of duality between $\Box 5$ and $\Diamond 5$.)

We shall consider the normal systems named on the left which are obtained by adding to $\boldsymbol{HK}\Diamond$ the schemata on the right:

$\boldsymbol{HD}\Diamond$	:	$\Diamond D$
$\boldsymbol{HT}\Diamond$	:	$\Diamond T$
$\boldsymbol{HS4}\Diamond$	:	$\Diamond T$ and $\Diamond 4$
$\boldsymbol{HS5.1}\Diamond$	:	$\Diamond T$, $\Diamond 4$ and $\Diamond 5.1$
$\boldsymbol{HS5}\Diamond$	:	$\Diamond T$ and $\Diamond 5$.

Later we shall see that $\boldsymbol{HS5.1}\Diamond$ is a proper subsystem of $\boldsymbol{HS5}\Diamond$, and hence $\Diamond 5$ and $\Diamond 5.1$ are not interreplaceable anymore. In general, we shall see that

$$\boldsymbol{HD}\Diamond \subset \boldsymbol{HT}\Diamond \subset \boldsymbol{HS4}\Diamond \subset \boldsymbol{HS5.1}\Diamond \subset \boldsymbol{HS5}\Diamond .$$

We shall also see that each of these systems is a proper subsystem of the corresponding system based on classical propositional logic.

A lemma analogous to Lemma 1 can be shown for $\boldsymbol{HD}\Diamond$ and the system $\boldsymbol{Triv}\Diamond$, which is the extension of the classical propositional calculus in $L\Diamond$ with $\Diamond A\leftrightarrow A$. To prove a lemma analogous to Lemma 2 for $\boldsymbol{HS5}\Diamond$, replacing squares by diamonds, we only need to show that in $\boldsymbol{HS5}\Diamond$ we can prove

$(\Diamond A\alpha\Diamond B) \leftrightarrow \Diamond(\Diamond A\alpha\Diamond B)$, where α is $\to$, $\wedge$, or $\vee$ (for $\wedge$ and $\vee$ this holds in HS4$\Diamond$)

$\neg\Diamond A \leftrightarrow \Diamond\neg\Diamond A$

$\Diamond A \leftrightarrow \Diamond\Diamond A.$

An equivalent formulation of $\boldsymbol{HS5}\Diamond$ would be given by extending the Heyting propositional calculus in $L\Diamond$ with the following rules:

$\boldsymbol{R}\Diamond 1.$ $\dfrac{A\to B}{\Diamond A\to B}$, provided every propositional variable in A is in the scope of a diamond

$\boldsymbol{R}\Diamond 2.$ $\dfrac{A\to B}{A\to\Diamond B}$.

This is parallel to what we had for $\boldsymbol{HS5}\Box$ in §3, and the proof proceeds in an analogous way. This formulation shows that $\boldsymbol{HS5}\Diamond$ coincides with the theorems in $L\Diamond$ proved by using only the rules with $\Diamond$ of the calculus *MIPC* mentioned in § 3. Note that the proviso for $R\Diamond 1$ in *MIPC* is that every propositional variable in A is in the scope of a *modal operator*, and not only a *diamond*. The calculus *MIPC* is obtained by extending $L\Box$ with $\Diamond$, and then extending the Heyting propositional calculus with the rules $R\Box 1$, $R\Box 2$ (see § 3), $R\Diamond 1$ and $R\Diamond 2$, with modified provisos for $R\Box 2$ and $R\Diamond 1$.

To show that any of the extensions of $\boldsymbol{HK}\Diamond$ we have introduced is conservative with respect to the Heyting propositional calculus in $L\Diamond$ without $\Diamond$, we proceed as for the corresponding systems in $L\Box$ using a mapping which deletes diamonds. In this way we also establish the consistency of our extensions. We can also show the following lemma with the help of the slash mentioned in § 3 extended by the clause $|\Diamond A \Leftrightarrow_{df} \Vdash A$.

LEMMA 14. *Any extension of* $\boldsymbol{HK}\Diamond$ *with some of the schemata* $\Diamond D$, $\Diamond T$, $\Diamond 4$, $\Diamond 5$ *and* $\Diamond 5.1$ *has the disjunction property.*

PROOF. This proof is analogous to the proof of Lemma 3. It enlarges the proof of Lemma 15 of [3] (which asserts that $\boldsymbol{HK}\Diamond$ has the disjunction property), save that for $\Diamond 1$ we can appeal to Lemma 14 of [3] (asserting that no formula of the form $\Diamond A$ is a theorem of $\boldsymbol{HK}\Diamond$) only with systems without $\Diamond D$ and $\Diamond T$. With other systems we appeal to the rule

$$\frac{A}{\Diamond A}$$

which is derivable in the presence of $\Diamond D$ or $\Diamond T$, in order to obtain $\vdash\Diamond(A\vee B)$ *and* $\vdash A\vee B$ *and* $(\Vdash A$ *or* $\Vdash B) \Rightarrow (\vdash\Diamond A$ *and* $\Vdash A)$ *or* $(\vdash\Diamond B$ *and* $\Vdash B)$ from which $|\Diamond(A\vee B)\to\Diamond A\vee\Diamond B$ follows. All the additional cases in the basis of the induction are quite straightforward (for $\Diamond D$ it is enough to note its analogy with $\Box 2$ and refer to the proof of Lemma 1 of [3]; for $\Diamond T$ we have $\Vdash A \Rightarrow \Vdash A$; for $\Diamond 4$ we have $\vdash\Diamond\Diamond A$ *and* $\Vdash\Diamond A \Rightarrow |\Diamond A$; for $\Diamond 5$ we have $\vdash\Diamond(\Diamond A\to\Diamond B)$ *and* $\Vdash\Diamond A\to\Diamond B \Rightarrow |\Diamond A\to\Diamond B$; and for $\Diamond 5.1$ we have $\vdash\Diamond\neg\Diamond A$ *and* $\Vdash\neg\Diamond A \Rightarrow |\neg\Diamond A$). When we have established that $\vdash A \Rightarrow |A$, we proceed as before. q.e.d.

Analogously to what we had after the proof of Lemma 3, the proof of Lemma 14 establishes that any of the systems of this lemma is closed under the rule

$$\frac{\Diamond A}{A}.$$

(This is trivial for systems like $\boldsymbol{HK}\Diamond$ in which there are no theorems of the form $\Diamond A$.) In this respect some of our systems differ from their classical counterparts (cf. [6], p. 168). This might be of interest if we note that closure under the rule corresponding to the rule above, where the place of the diamond is taken by the existential quantifier (which is up to a point analogous to the diamond), is usually considered a desirable property for intuitionistic systems, on a pair with the disjunction property.

§ 8. Equivalence of $\Diamond D$, $\Diamond T$, $\Diamond 4$, $\Diamond 5$ and $\Diamond 5.1$ with conditions on $H\Diamond$ frames. This section, which is analogous to § 4, will contain a series of lemmata about the formulae characteristic of the normal extensions of $HK\Diamond$ we have introduced. It is our aim to show that the holding of these formulae in $H\Diamond$ frames is equivalent to specific conditions concerning the relations of the frames. These lemmata will also help us to obtain soundness and completeness proofs in the next section. Throughout this section Fr will be an $H\Diamond$ frame $\langle X, R_I, R_M\rangle$, and the symbol $R\Diamond$ is an abbreviation for $R_M R_I^{-1}$.

LEMMA 15. $Fr \vDash \Diamond(A \to A) \Leftrightarrow \forall x \exists y.\ xR_\Diamond y \Leftrightarrow \forall x \exists y.\ xR_M y$ *(i.e., $R_\Diamond$ and R_M are serial).*

PROOF. It follows immediately from the meaning of $R_\Diamond$ and the reflexivity of R_I that the right-hand sides are equivalent. Next we have:

$$\begin{aligned}\forall x.x \vDash \Diamond(A \to A) &\Leftrightarrow \forall x \exists y (xR_M y \text{ and } y \vDash A \to A)\\ &\Leftrightarrow \forall x \exists y.\ xR_M y. \quad \text{q.e.d.}\end{aligned}$$

In the proofs of the following lemmata we shall use the fact that in $H\Diamond$ models

$$x \vDash \Diamond A \Leftrightarrow \exists y (xR_\Diamond y \text{ and } y \vDash A),$$

which was shown in Lemma 18 of [3].

LEMMA 16. $Fr \vDash A \to \Diamond A \Leftrightarrow \forall x.\ xR_\Diamond x$ *(i.e., $R_\Diamond$ is reflexive).*

PROOF. ($\Rightarrow$) Suppose for some x *not* $xR_\Diamond x$. Let $\forall y (y \vDash p \Leftrightarrow \text{not } xR_\Diamond y)$. By Lemma 4 (iii) there is a valuation such that this is satisfied. It is easy to show that with this valuation *not* $x \vDash p \to \Diamond p$.

($\Leftarrow$) Suppose $xR_I y$ *and* $y \vDash A$. Using $yR_\Diamond y$ we obtain $\exists z (yR_\Diamond z \text{ and } z \vDash A)$, i.e., $y \vDash \Diamond A$. Hence, $x \vDash A \to \Diamond A$. q.e.d.

LEMMA 17. $Fr \vDash \Diamond\Diamond A \to \Diamond A \Leftrightarrow R_\Diamond^2 \subseteq R_\Diamond$ *(i.e., $R_\Diamond$ is transitive).*

PROOF. ($\Rightarrow$) Suppose for some x, y and z, $xR_\Diamond y$ *and* $yR_\Diamond z$ *and not* $xR_\Diamond z$. Let $\forall t (t \vDash p \Leftrightarrow \text{not } xR_\Diamond t)$. By Lemma 4 (iii) there is a valuation such that this is satisfied. It is easy to show that with this valuation *not* $x \vDash \Diamond\Diamond p \to \Diamond p$.

($\Leftarrow$) Suppose $xR_I y$ *and* $y \vDash \Diamond\Diamond A$. So we have $\exists z, t(yR_\Diamond z$ *and* $zR_\Diamond t$ *and* $t \vDash A)$. Then by using $R_\Diamond^2 \subseteq R_\Diamond$, we obtain $\exists t(yR_\Diamond t$ *and* $t \vDash A)$, i.e., $y \vDash \Diamond A$. Therefore, $x \vDash \Diamond\Diamond A \to \Diamond A$. q.e.d.

LEMMA 18. *Let*

$\varrho \Leftrightarrow_{\mathrm{df}} \forall x, y, z\big(xR_\Diamond y$ *and* $xR_\Diamond z \Rightarrow \exists t\big(yR_I t$ *and* $tR_\Diamond z$ *and* $\forall u(tR_\Diamond u \Rightarrow xR_\Diamond u)\big)\big)$.

Then $Fr \vDash \Diamond(\Diamond A \to \Diamond B) \to (\Diamond A \to \Diamond B) \Leftrightarrow \varrho$.

PROOF. Suppose *not* ϱ, i.e., for some x, y and z

(1) $xR_\Diamond y$ *and* $xR_\Diamond z$ *and* $\forall t\big(yR_I t$ *and* $tR_\Diamond z \Rightarrow \exists u(tR_\Diamond u$ *and not* $xR_\Diamond u)\big)$.

Let $\forall v(v \vDash p \Leftrightarrow zR_I v)$ and $\forall v(v \vDash q \Leftrightarrow not\, xR_\Diamond v)$. By Lemma 4 (i) and (iii), there is a valuation such that this is satisfied. We establish first that with this valuation $x \vDash \Diamond(\Diamond p \to \Diamond q)$. We have:

$$
\begin{aligned}
(1) \quad &\Rightarrow \quad \forall t\big(yR_I t \text{ and } \exists v(tR_M v \text{ and } zR_I v) \Rightarrow \exists u(tR_\Diamond u \text{ and } u \vDash q)\big)\\
&\Rightarrow \quad \forall t(yR_I t \text{ and } t \vDash \Diamond p \Rightarrow t \vDash \Diamond q)\\
&\Rightarrow \quad y \vDash \Diamond p \to \Diamond q\\
&\Rightarrow \quad x \vDash \Diamond(\Diamond p \to \Diamond q).
\end{aligned}
$$

On the other hand, with this valuation we can easily show that *not* $x \vDash \Diamond p \to \Diamond q$, and so, the left-hand side of the Lemma is false.

($\Leftarrow$) First note that in the Heyting propositional calculus we can prove

$$\big(\Diamond(\Diamond A \to \Diamond B) \to (\Diamond A \to \Diamond B)\big) \quad \leftrightarrow \quad \big(\Diamond(\Diamond A \to \Diamond B) \wedge \Diamond A \to \Diamond B\big).$$

Now, suppose for some x, $x \vDash \Diamond(\Diamond A \to \Diamond B)$ *and* $x \vDash \Diamond A$ *and not* $x \vDash \Diamond B$. Hence, for some y, $xR_\Diamond y$ *and* $y \vDash \Diamond A \to \Diamond B$, and for some z, $xR_\Diamond z$ *and* $z \vDash A$. But then from ϱ it follows that for some t, $yR_I t$ *and* $tR_\Diamond z$ *and* $\forall u(tR_\Diamond u \Rightarrow xR_\Diamond u)$. From $tR_\Diamond z$ and $z \vDash A$ we have $t \vDash \Diamond A$, and since $y \vDash \Diamond A \to \Diamond B$ and $yR_I t$, we obtain $t \vDash \Diamond B$. So for some v, $tR_\Diamond v$ and $v \vDash B$. From $\forall u(tR_\Diamond u \Rightarrow xR_\Diamond u)$ we obtain $xR_\Diamond v$. Hence, $xR_\Diamond v$ *and* $v \vDash B$, which contradicts *not* $x \vDash \Diamond B$. q.e.d.

LEMMA 19. *Let*

$\varrho' \Leftrightarrow_{\mathrm{df}} \forall x, y\big(xR_\Diamond y \Rightarrow \exists t(yR_I t$ *and* $tR_\Diamond x$ *and* $xR_\Diamond t)\big)$.

Then, if $R_\Diamond$ *is reflexive and transitive in* Fr, $Fr \vDash \Diamond(\Diamond A \to \Diamond B) \to (\Diamond A \to \Diamond B) \Leftrightarrow \varrho'$.

PROOF. Let

$\varrho'' \Leftrightarrow_{\mathrm{df}} \forall x, y, z\big(xR_\Diamond y$ *and* $xR_\Diamond z \Rightarrow \exists t(yR_I t$ *and* $tR_\Diamond z$ *and* $xR_\Diamond t)\big)$.

If $R_\Diamond$ is reflexive and transitive, $\varrho \Leftrightarrow \varrho''$, where ϱ is as in Lemma 18. For that it is enough to show that $\forall u(tR_\Diamond u \Rightarrow xR_\Diamond u) \Leftrightarrow xR_\Diamond t$. From left to right we use $tR_\Diamond t$, and the other direction is an immediate consequence of the transitivity of $R_\Diamond$.

Now suppose ϱ. It follows that ϱ'', and from that we obtain as an instance $xR_\Diamond y$ *and* $xR_\Diamond x \Rightarrow \exists t(yR_I t$ *and* $tR_\Diamond x$ *and* $xR_\Diamond t)$. Then using $xR_\Diamond x$ we obtain ϱ'.

For the converse, suppose ϱ' and $xR_\Diamond y$ *and* $xR_\Diamond z$. By ϱ' there is a t such that $yR_I t$ *and* $tR_\Diamond x$ *and* $xR_\Diamond t$. From $tR_\Diamond x$ and $xR_\Diamond z$ by the transitivity of $R_\Diamond$ we obtain $tR_\Diamond z$. From that ϱ'' follows, and hence also ϱ. q.e.d.

LEMMA 20. *Let*
$\sigma \Leftrightarrow_{\mathrm{df}} \forall x, y, z(xR_\Diamond y$ *and* $xR_\Diamond z \Rightarrow yR_I R_\Diamond z)$
where $yR_I R_\Diamond z \Leftrightarrow \exists t(yR_I t$ *and* $tR_\Diamond z)\big)$. *Then* $Fr \models \Diamond\neg\Diamond A \rightarrow \neg\Diamond A \Leftrightarrow \sigma$.

PROOF. First note that in the Heyting propositional calculus we can prove

$$(\Diamond\neg\Diamond A \rightarrow \neg\Diamond A) \quad\leftrightarrow\quad \neg(\Diamond\neg\Diamond A \wedge \Diamond A).$$

($\Rightarrow$) Suppose *not* σ, i.e., for some x, y and z

(1) $xR_\Diamond y$ *and* $xR_\Diamond z$ *and not* $yR_I R_\Diamond z$.

Let $\forall u(u \models p \Leftrightarrow$ *not* $yR_I R_\Diamond u)$. By Lemma 4 (iii) there is a valuation such that this is satisfied. We shall show that with this valuation *not* $x \models \neg(\Diamond\neg\Diamond p \wedge \Diamond p)$. By the reflexivity of R_I we have $\forall u(\exists v(yR_I v$ *and* $vR_M u) \Rightarrow yR_I R_\Diamond u)$, which entails $\forall v\big(yR_I v \Rightarrow \forall u(vR_M u \Rightarrow$ *not* $u \models p)\big)$. From this we obtain $y \models \neg\Diamond p$, which together with (1) gives us $x \models \Diamond\neg\Diamond p$. It also follows from (1) that $z \models p$, which implies $x \models \Diamond p$. Thus, $x \models \Diamond\neg\Diamond p \wedge \Diamond p$, and then it remains only to appeal to $xR_I x$.

($\Leftarrow$) Suppose for some x, $x \models \Diamond\neg\Diamond A$ *and* $x \models \Diamond A$. Hence, for some y and z, $xR_\Diamond y$ *and* $y \models \neg\Diamond A$ *and* $xR_\Diamond z$ *and* $z \models A$. Then, from σ it follows that for some t, $yR_I t$ *and* $tR_\Diamond z$. From $y \models \neg\Diamond A$ and $yR_I t$ it follows that *not* $t \models \Diamond A$. But this contradicts $tR_\Diamond z$ and $z \models A$. q.e.d.

LEMMA 21. *Let*
$\sigma' \Leftrightarrow_{\mathrm{df}} \forall x, y(xR_\Diamond y \Rightarrow yR_I R_\Diamond x)$ (*i.e.*, $R_\Diamond \subseteq R_\Diamond^{-1} R_I^{-1}$).
Then, if $R_\Diamond$ *is reflexive and transitive in* Fr, $Fr \models \Diamond\neg\Diamond A \rightarrow \neg\Diamond A \Leftrightarrow \sigma'$.

PROOF. As an instance of σ from Lemma 20 we have $xR_\Diamond y$ *and* $xR_\Diamond x \Rightarrow yR_I R_\Diamond x$, and then using $xR_\Diamond x$, we obtain $\sigma \Rightarrow \sigma'$. For the converse suppose σ' and $xR_\Diamond y$ *and* $xR_\Diamond z$. By σ' there is a t such that $yR_I t$ *and* $tR_\Diamond x$. From $tR_\Diamond x$ and $xR_\Diamond z$ by the transitivity of $R_\Diamond$ we get $tR_\Diamond z$. From that σ follows. q.e.d.

It is not hard to prove that in $\boldsymbol{H}\Diamond$ frames where $R_\Diamond$ is reflexive σ' entails $R_\Diamond \subseteq R_\Diamond^{-1} R_\Diamond$.

Since the conditions ϱ, ϱ', σ and σ' of Lemmata 18–21 are rather entangled, we shall try to explain them a little bit. We have seen in § 4 how the conditions φ, φ', ψ and $\psi'$of Fig.2 insert an extra R_I relation into their classical counterparts of Fig.1. The conditions ϱ, ϱ', σ and σ' will also insert an extra R_I relation into their classical counterparts, but not in the same way as φ, φ', ψ and ψ'. It follows from the replaceability of $\Diamond 5$ by $\Diamond 5.1$ and $\Diamond 4$, mentioned in § 7, that the condition corresponding classically to $\Diamond 5$ is the transitivity and euclideanity of the accessibility relation; and if this relation is also reflexive, we obtain transitivity and symmetry. As we know, the condition corresponding classically to $\Diamond 5.1$ is the euclideanity of the accessibility relation; if this relation is also reflexive and transitive, we obtain symmetry. In order to represent how ϱ, ϱ', σ and σ' modify these conditions, diagrams are produced in Fig. 8 from which we can read out the structure of ϱ, ϱ', σ and σ' (dotted arrows stand for R_I and continuous arrows for $R_\Diamond$).

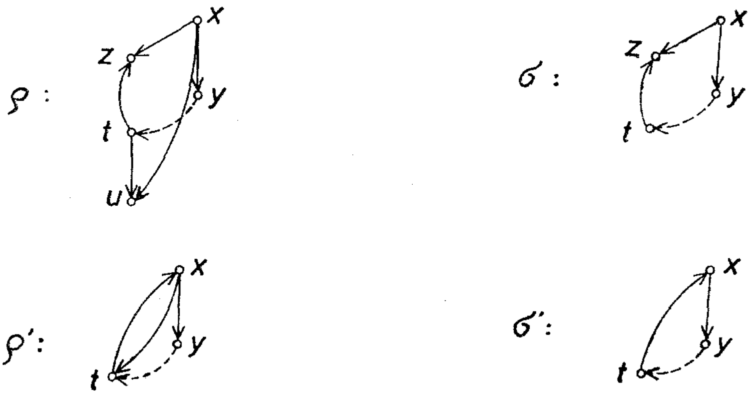


Fig. 8

Now it should be clear that $\varrho \Rightarrow \sigma$ and $\varrho' \Rightarrow \sigma'$, but not conversely, as easy counterexamples can show. We can also prove the following lemma.

LEMMA 22. *$\varrho \Rightarrow R_\Diamond$ is transitive.*

PROOF. Suppose $xR_\Diamond y$ *and* $yR_\Diamond z$. This implies for some v

(1) $xR_\Diamond y$ *and* $yB_M v$ *and* $vR_I^{-1}z$.

From ϱ we obtain $xR_\Diamond y \Rightarrow \exists t\big(yR_I t$ *and* $tR_\Diamond y$ *and* $\forall u(tR_\Diamond u \Rightarrow xR_\Diamond u)\big)$. Hence, by (1), for some t

(2) $yR_I t$ *and* $tR_\Diamond y$ *and* $\forall u(tR_\Diamond u \Rightarrow xR_\Diamond u)$.

Then

$$
\begin{aligned}
yR_M v \text{ and } yR_I t \quad &\Rightarrow \quad tR_I^{-1}R_M v \\
&\Rightarrow \quad tR_M R_I^{-1} v \\
&\Rightarrow \quad xR_\Diamond z, \text{ using the last conjuncts of (2)} \\
&\qquad \text{and (1), and the transitivity of} \\
&\qquad R_I. \quad \text{q.e.d.}
\end{aligned}
$$

We note incidentally it can be proved that $Fr \models \Diamond A \vee \neg\Diamond A \Leftrightarrow R_I R_\Diamond \subseteq R_\Diamond$ ($\Leftrightarrow R_I R_M \subseteq R_M R_I^{-1}$). (This should be compared with the condition for condensed $\boldsymbol{H}\Box$ frames.)

§ 9. Soundness and completeness of normal extensions of $\boldsymbol{HK}\Diamond$. In this section we intend to prove that the normal extensions of $\boldsymbol{HK}\Diamond$ introduced in § 7 are sound and complete with respect to specific classes of $\boldsymbol{H}\Diamond$ models which are defined as follows.

DEFINITION 2. An $\boldsymbol{H}\Diamond$ frame $\langle X, R_I, R_M\rangle$ (model $\langle X, R_I, R_M, V\rangle$) is an

(i)	$\boldsymbol{HD}\Diamond$	frame (model)	iff $R_\Diamond$ is serial
(ii)	$\boldsymbol{HT}\Diamond$	frame (model)	iff $R_\Diamond$ is reflexive
(iii)	$\boldsymbol{HS4}\Diamond$	frame (model)	iff $R_\Diamond$ is reflexive and transitive
(iv)	$\boldsymbol{HS5.1}\Diamond$	frame (model)	iff $R_\Diamond$ is reflexive and transitive, and σ' of Lemma 21 holds
(v)	$\boldsymbol{HS5}\Diamond$	frame (model)	iff $R_\Diamond$ is reflexive and transitive, and ϱ' of Lemma 19 holds.

In [3] (§ 9) we have introduced the canonical frame (model) for a system in $L\Diamond$ by defining $\Gamma R_M^c \Delta$ as $\Delta^\Diamond \subseteq \Gamma$, where $\Delta^\Diamond = \{\Diamond A \mid A \in \Delta\}$. The rest is as for canonical models mentioned in § 5. Then, using this notion of canonical models, we can show the following lemma.

LEMMA 23. *Let* $\boldsymbol{S}$ *be* (i) $\boldsymbol{HD}\Diamond$, (ii) $\boldsymbol{HT}\Diamond$, (iii) $\boldsymbol{HS4}\Diamond$, (iv) $\boldsymbol{HS5.1}\Diamond$ *or* (v) $\boldsymbol{HS5}\Diamond$. *Then the canonical* $\boldsymbol{S}$ *frame* (*model*) *is an* $\boldsymbol{S}$ *frame* (*model*).

PROOF. (i) In the canonical $\boldsymbol{HD}\Diamond$ frame $R_M^c(R_I^c)^{-1}$ is serial. This is shown as follows. Let Θ be the set of theorems of $\boldsymbol{HD}\Diamond$. From Lemma 14, and from the consistency and deductive closure of Θ, it follows that Θ is nice. Since $\boldsymbol{HD}\Diamond$ is closed under the rule

$$\frac{A}{\Diamond A}$$

(as we have already mentioned in the proof of Lemma 14), we have $\Theta^\Diamond \subseteq \Theta$. For every nice set Γ, $\Theta \subseteq \Gamma$; hence, $\Theta^\Diamond \subseteq \Gamma$, and $\forall \Gamma \exists \Delta.\ \Gamma R_M^c \Delta$, which by the transitivity of R_I^c implies $\forall \Gamma \exists \Delta.\ \Gamma R_M^c (R_I^c)^{-1} \Delta$.

(ii) In the canonical $\boldsymbol{HT}\Diamond$ frame $R_M^c(R_I^c)^{-1}$ is reflexive. This is shown as follows. For every nice set Γ, since $A \to \Diamond A \in \Gamma$, we have $\Gamma^\Diamond \subseteq \Gamma$. Hence $\Gamma R_M^c \Gamma$, and the reflexivity of R_I^c yields $\Gamma R_M^c (R_I^c)^{-1} \Gamma$.

(iii) In the canonical $\boldsymbol{HS4}\Diamond$ frame $R_M^c(R_I^c)^{-1}$ is reflexive and transitive.

For reflexivity we proceed as in (ii). For transitivity we have

$$
\begin{aligned}
\Gamma R^c_M(R^c_I)^{-1}\Delta \textit{ and } \Delta R^c_M(R^c_M)^{-1}\Theta \;&\Rightarrow\; \Gamma R^c_M\Delta \textit{ and } \Delta R^c_M\Theta, \text{ since } R^c_M(R^c_I)^{-1} \\
&\qquad = R^c_M \text{ (see the proof of Theorem 5} \\
&\qquad \text{of [3])} \\
&\Rightarrow\; \Delta^\Diamond \subseteq \Gamma \textit{ and } \Theta^\Diamond \subseteq \Delta \\
&\Rightarrow\; \Delta^\Diamond \subseteq \Gamma \textit{ and } (\Theta^\Diamond)^\Diamond \subseteq \Delta^\Diamond \\
&\Rightarrow\; (\Theta^\Diamond)^\Diamond \subseteq \Gamma \\
&\Rightarrow\; \Theta^\Diamond \subseteq \Gamma, \text{ by } \Diamond 4 \\
&\Rightarrow\; \Gamma R^c_M(R^c_I)^{-1}\Theta.
\end{aligned}
$$

(iv) In the canonical $\boldsymbol{HS5.1}\Diamond$ frame $R^c_M(R^c_I)^{-1}$ is reflexive and transitive, and σ' of Lemma 21 holds, where R_I is R^c_I and $R_\Diamond$ is $R^c_M(R^c_I)^{-1}$. To prove the reflexivity and transitivity of this last relation we proceed as in (ii) and (iii). For σ' suppose $\Gamma R^c_M(R^c_I)^{-1}\Delta$. Since $R^c_M(R^c_I)^{-1}_{\ } = R^c_M$, it follows that $\Delta^\Diamond \subseteq \Gamma$. Next, $\Gamma^\Diamond \cup \Delta$ is consistent. (Otherwise, for some $B_1, \ldots, B_n \in \Gamma$ and $A \in \Delta$, $\{B, A\} \vdash \neg(C \to C)$, where B is $\Diamond B_1 \wedge \ldots \wedge \Diamond B_n$. Since $\vdash_{\boldsymbol{HS5.1}\Diamond} \Diamond(\Diamond B_1 \wedge \ldots \wedge \Diamond B_n) \to \Diamond B_1 \wedge \ldots \wedge \Diamond B_n$, we have $\{\Diamond B, A\} \vdash \neg(C \to C)$, and hence

$$
\begin{aligned}
\vdash A &\to \neg\Diamond B\\
\vdash \Diamond A &\to \Diamond\neg\Diamond B\\
\vdash \Diamond A &\to \neg\Diamond B, \text{ by } \Diamond 5.1\\
\vdash \Diamond B &\to \neg\Diamond A.
\end{aligned}
$$

Since $\Diamond B \in \Gamma$ and Γ is nice, $\neg\Diamond A \in \Gamma$, and hence $\Diamond A \notin \Gamma$. But as $\Delta^\Diamond \subseteq \Gamma$, $A \notin \Delta$, which is a contradiction.) Since $\Gamma^\Diamond \cup \Delta$ is consistent, by Lemma 6 of [3], which holds in the present context too, there is a nice set Θ such that $\Delta \subseteq \Theta$ and $\Gamma^\Diamond \subseteq \Theta$. From this $\Delta R^c_I R^c_M(R^c_I)^{-1}\Gamma$ follows.

(v) In the canonical $\boldsymbol{HS5}\Diamond$ frame $R^c_M(R^c_I)^{-1}$ is reflexive and transitive, and ϱ' of Lemma 19 holds, where R_I is R^c_I and $R_\Diamond$ is $R^c_M(R^c_I)^{-1}$. For the reflexivity and transitivity of this last relation we proceed as in (ii) and (iii) ($\Diamond 4$ is a theorem of $\boldsymbol{HS5}\Diamond$; see the derivation of $\Diamond 4$ from $\Diamond 5$ in § 7). For ϱ' we proceed as follows.

Suppose $\Gamma R^c_M(R^c_I)^{-1}\Delta$. Since $R^c_M(R^c_I)^{-1} = R^c_M$, it follows that $\Delta^\Diamond \subseteq \Gamma$. Let $Z =_{df} \{\Psi \mid \Gamma^\Diamond \cup \Delta \subseteq \Psi \textit{ and } \Psi^\Diamond \subseteq \Gamma \textit{ and } Cl(\Psi) \subseteq \Psi\}$. First, we prove that $Cl(\Gamma^\Diamond \cup \Delta) \in Z$. The only difficult part of this is to show that $(Cl(\Gamma^\Diamond \cup \Delta))^\Diamond \subseteq \Gamma$. We have:

$$
\begin{aligned}
C \in \big(Cl(\Gamma^\Diamond \cup \Delta)\big)^\Diamond \;&\Rightarrow\; A \in Cl(\Gamma^\Diamond \cup \Delta), \text{ where } \Diamond A \text{ is } C\\
&\Rightarrow\; \{\Diamond B_1, \ldots, \Diamond B_n\} \cup \Delta \vdash A, \text{ where } B_1, \ldots, B_n \in \Gamma\\
&\Rightarrow\; \Delta \vdash B \to A, \text{ where } B \text{ is } \Diamond B_1 \wedge \ldots \wedge \Diamond B_n\\
&\Rightarrow\; \Delta \vdash \Diamond B \to \Diamond A, \text{ using } \Diamond T, \text{ and since in } \boldsymbol{HS5}\Diamond \text{ we}\\
&\qquad \text{can prove } \Diamond(\Diamond B_1 \wedge \ldots \wedge \Diamond B_n) \to \Diamond B_1 \wedge \ldots \wedge \Diamond B_n\\
&\Rightarrow\; \Diamond(\Diamond B \to \Diamond A) \in \Gamma, \text{ since } \Delta \text{ is nice and } \Delta^\Diamond \subseteq \Gamma\\
&\Rightarrow\; \Diamond B \to \Diamond A \in \Gamma, \text{ using } \Diamond 5, \text{ and since } \Gamma \text{ is nice}\\
&\Rightarrow\; C \in \Gamma, \text{ since } \Diamond B \in \Gamma.
\end{aligned}
$$

Hence, Z is nonempty, and it is easy to show that it is closed under unions of nonempty chains. So by Zorn's Lemma, Z has a maximal element Θ with respect to $\subseteq$. We show first that

(v.i) Θ is consistent.

Otherwise, $\Theta \vdash \neg(A \to A)$, and since $Cl(\Theta) \subseteq \Theta$ we have $\neg(A \to A) \in \Theta$. Hence, $\Diamond\neg(A \to A) \in \Gamma$, but as $\neg\Diamond\neg(A \to A) \in \Gamma$, this contradicts the supposition that Γ is nice. Next, we infer immediately from $\Theta \in Z$ that

(v.ii) Θ is deductively closed.

Now suppose that Θ does not have the disjunction property, i.e., for some B and C, $B \vee C \in \Theta$ *and* $B \notin \Theta$ *and* $C \notin \Theta$. Since $\Theta \cup \{B\}$ and $\Theta \cup \{C\}$ are proper supersets of Θ, they cannot be in Z. *A fortiori*, $Cl(\Theta \cup \{B\})$ and $Cl(\Theta \cup \{C\})$ are not in Z. This is possible only if for some B_1 from the first and some C_1 from the second of these sets, $\Diamond B_1 \notin \Gamma$ *and* $\Diamond C_1 \notin \Gamma$. On the other hand,

$$\begin{aligned}
\Theta \cup \{B\} \vdash B_1 \textit{ and } \Theta \cup \{C\} \vdash C_1 \quad &\Rightarrow \quad \Theta \vdash B \vee C \to B_1 \vee C_1 \\
&\Rightarrow \quad \Theta \vdash B_1 \vee C_1, \text{ since } B \vee C \in \Theta \\
&\Rightarrow \quad \Diamond B_1 \vee \Diamond C_1 \in \Theta, \text{ using } \Diamond T, \Diamond 1 \text{ and (v.ii)} \\
&\Rightarrow \quad \Diamond(\Diamond B_1 \vee \Diamond C_1) \in \Gamma \text{ since } \Theta^{\Diamond} \subseteq \Gamma \\
&\Rightarrow \quad \Diamond B_1 \in \Gamma \textit{ or } \Diamond C_1 \in \Gamma, \text{ since in } \boldsymbol{HS5}\Diamond \text{ we} \\
&\qquad \text{can prove } \Diamond(\Diamond B_1 \vee \Diamond C_1) \to \Diamond B_1 \vee \Diamond C_1 \text{ and} \\
&\qquad \Gamma \text{ is nice}
\end{aligned}$$

which gives a contradiction. So,

(v.iii) Θ has the disjunction property

and we can conclude that Θ is nice. Since, $\Delta \subseteq \Theta$ *and* $\Gamma^{\Diamond} \subseteq \Theta$ *and* $\Theta^{\Diamond} \subseteq \Gamma$, we obtain ϱ'.

Using (i)–(v) and Lemma 20 of [3], which asserts that the canonical $\boldsymbol{S}$ frame (model) is an $\boldsymbol{H}\Diamond$ frame (model), we obtain the Lemma. q.e.d.

Next we prove the following soundness and completeness theorem.

THEOREM 3. *Let* $\boldsymbol{S}$ *be* $\boldsymbol{HD}\Diamond$, $\boldsymbol{HT}\Diamond$, $\boldsymbol{HS4}\Diamond$, $\boldsymbol{HS5.1}\Diamond$ *or* $\boldsymbol{HS5}\Diamond$. *Then*

$$\vdash_S A \quad \Leftrightarrow \quad \textit{for every } \boldsymbol{S} \textit{ frame } Fr,\ Fr \models A.$$

PROOF. ($\Rightarrow$) Soundness follows from the ($\Leftarrow$) parts of Lemmata 15, 16, 17, 19 and 21, and from the first part of the proof of Theorem 4 of [3], which treats of the soundness of $\boldsymbol{HK}\Diamond$ with respect to $\boldsymbol{H}\Diamond$ frames.

($\Leftarrow$) For completeness we proceed as follows. Suppose that for every $\boldsymbol{S}$ frame Fr, $Fr \models A$. By Lemma 23, A holds in the canonical $\boldsymbol{S}$ frame, and consequently A holds in the canonical $\boldsymbol{S}$ model, i.e., $\forall \Gamma \in X^c.\ \Gamma \models A$. By Lemma 22 of [3], which asserts that in the canonical $\boldsymbol{S}$ model $\forall \Gamma \in X^c$ $(\Gamma \models A \Leftrightarrow A \in \Gamma)$, we have $\forall \Gamma \in X^c.\ A \in \Gamma$. Since the set of theorems of $\boldsymbol{S}$ is nice (it is consistent, deductively closed and, according to Lemma 14, it has the disjunction property), it follows that $\vdash_S A$. q.e.d.

(For systems without the disjunction property we could appeal to the parenthetical remark after the proof of Theorem 1.)

With Theorem 3 we can easily check the proper inclusions among our systems mentioned in § 7. We are also in the position to prove the following soundness and completeness theorem, which follows from the soundness part of Theorem 3 and from $R_M^c(R_I^c)^{-1} = R_M^c$ (cf. Theorem 5 of [3], covering the case when $\boldsymbol{S}$ is $\boldsymbol{HK}\Diamond$).

THEOREM 4. *Let $\boldsymbol{S}$ be as in Theorem 3. Then*

$$\begin{aligned}\vdash_S A \quad &\Leftrightarrow \quad \text{for every condensed } \boldsymbol{S} \text{ frame } Fr,\ Fr \models A\\ &\Leftrightarrow \quad \text{for every strictly condensed } \boldsymbol{S} \text{ frame } Fr,\ Fr \models A.\end{aligned}$$

It is interesting to note that in strictly condensed $\boldsymbol{H}\Diamond$ frame where $R_M R_I^{-1}$, and hence also R_M, are reflexive and transitive, $R_I^{-1}R_M = R_M R_I^{-1} = R_M$ is interreplaceable with $R_I^{-1} \subseteq R_M$, analogously to what we remarked after Theorem 2. Remarks analogous to those in § 5 could also be made about the embedding of $\boldsymbol{HS4}\Diamond$ in a modal system based on classical propositional logic with an "intuitionistic" **S4** square ($\Box_I$) and an independent **S4** diamond ($\Diamond_M$). The schema $A \to \Box_I \Diamond_M A$, which is equivalent to $R_I^{-1} \subseteq R_M$ in the sense of Lemma 23 of [3], could be used for this system (note that it corresponds in a certain sense to the schema characteristic of the *Brouwersche* system $\boldsymbol{B}$; see [6], p. 131). In this light it is possible to consider a topological interpretation of $\boldsymbol{HS4}\Diamond$ where $a \subseteq Int_I Cl_M a$.

§ 10. Modalities in $\boldsymbol{HS4}\Diamond$ and $\boldsymbol{HS5}\Diamond$. Analogously to what we had in § 6 the number of modalities in $\boldsymbol{HT}\Diamond$ and its subsystems is infinite. But now, in contradistinction to what we had for $\boldsymbol{HS4}\Box$, the number of modalities in $\boldsymbol{HS4}\Diamond$ is also infinite. First, we establish the following lemma.

LEMMA 24. *For every $m \geqslant 0$ and every $n \geqslant 1$*

$$(1) \qquad \underbrace{\neg\Diamond\neg \ldots \neg\Diamond\neg}_{m} p \;\to\; \underbrace{\neg\Diamond\neg \ldots \neg\Diamond\neg}_{m+n} p$$

(*where m and $m+n$ indicate the number of blocks of $\neg\Diamond\neg$*) *is not a theorem of $\boldsymbol{HS4}\Diamond$.*

PROOF. We shall show that for every $m \geqslant 0$ and every $n \geqslant 1$ we can construct an $\boldsymbol{HS4}\Diamond$ model $\langle X, R_I, R_M, V\rangle$ which falsifies (1). The set X has $2(m+n)+1$ members connected as follows:

$$x_1 R_I x_2 R_M x_3 \ldots x_{2j+1} R_I x_{2j+2} R_M x_{2j+3} \ldots x_{2(m+n)} R_M x_{2(m+n)+1}$$

where $j \geqslant 0$. For every i, $1 \leqslant i \leqslant 2(m+n)$, $x_i \vDash p$, and *not* $x_{2(m+n)+1} \vDash p$. Then it can be proved that the antecedent of (1) holds in x_1, but the consequent does not; hence, (1) is falsified in x_1. q.e.d.

An immediate corollary of this lemma is that the number of modalities in ***HS*4**$\Diamond$ is infinite.

Note that if $\neg\Diamond\neg$ is replaced by $\Box$, and if $m = n = 1$, (1) becomes the characteristic S4 schema in $L\Box$. The converse of (1) for $m \geqslant 1$ is provable in ***HS*4**$\Diamond$, since we have $\vdash_{\boldsymbol{HS4}} \neg\Diamond\neg\neg A \rightarrow \neg A$; but $\neg\Diamond\neg A \rightarrow A$ is not provable – i.e., replacing $\neg\Diamond\neg$ by $\Box$, the characteristic $\boldsymbol{T}$ schema in $L\Box$ is not provable. On the other hand, we have both $\vdash_{\boldsymbol{HS4}\Box} \neg\Box\neg\neg\Box\neg A \rightarrow \neg\Box\neg A$ (see (4) in § 6) and $\vdash_{\boldsymbol{HS4}\Box} A \rightarrow \neg\Box\neg A$. Still, we have *not* $\vdash_{\boldsymbol{HS4}\Box} \neg\Box\neg(A \vee B) \rightarrow \neg\Box\neg A \vee \neg\Box\neg B$ as well as *not* $\vdash_{\boldsymbol{HS4}\Box} \neg\Diamond\neg A \wedge \neg\Diamond\neg B \rightarrow \neg\Diamond\neg(A \wedge B)$. So, roughly speaking, ***HS*4**$\Box$ and ***HS*4**$\Diamond$ cannot be interpreted naturally in each other.

Duality with the corresponding systems in $L\Box$ is restorted with ***HS*5**$\Diamond$ and ***HS*5.1**$\Diamond$: the number of modalities in ***HS*5**$\Diamond$ and ***HS*5.1**$\Diamond$ is identical and it equals 10. Of these 6 are positive and 4 negative. In Fig. 9 we display the positive and in Fig. 10 the negative modalities of ***HS*5**$\Diamond$ (or ***HS*5.1**$\Diamond$). These charts are interpreted as those in Figs. 4–7, modulo the change of systems to ***HS*5**$\Diamond$ (***HS*5.1**$\Diamond$) and ***S*5**.

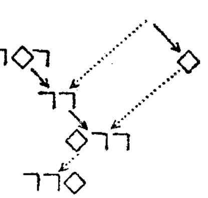


Fig. 9

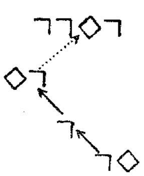


Fig. 10

In order to reduce modalities to those in Fig. 9 and Fig. 10 we use essentially the following theorems of ***HS*5.1**$\Diamond$

(1) $\Diamond\Diamond A \leftrightarrow \Diamond A$
(2) $\neg\neg\neg A \leftrightarrow \neg A$
(3) $\Diamond\neg\Diamond A \leftrightarrow \neg\Diamond A$
(4) $\Diamond\neg\neg\Diamond A \leftrightarrow \neg\neg\Diamond A$
(5) $\neg\Diamond\neg\neg A \leftrightarrow \neg\Diamond A$.

There is no special difficulty in showing either syntactically or model-theoretically that these formulae are theorems and that the implications in Figs. 9 and 10 do hold. We can also easily produce ***HS*5**$\Diamond$ models falsifying the converse implications, or would-be implications where we have drawn no arrows in Fig. 9 and Fig. 10.

It is of some interest to compare the modalities of ***HS*5**$\Box$ and ***HS*5**$\Diamond$ with the modalities of the system *MIPC*, which is formulated with both $\Box$

and $\Diamond$, and which was mentioned in § 3 and § 7. We display the positive and negative modalities of *MIPC* in respectively, Fig. 11 and Fig. 12, which are to be interpreted analogously to Figs. 4–7, 9, 10. The chart of Fig. 11 (12) is obtained by conflating the charts of Fig. 6 (7) and Fig. 9 (10).

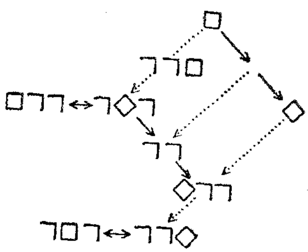


Fig. 11

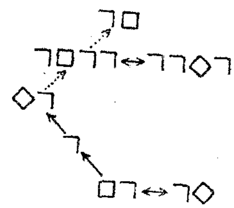


Fig. 12

It is not hard to show with our models that we still have *not* $\vdash_{HS5_\Box} \neg\Box\neg(A\vee B) \to \neg\Box\neg A \vee \neg\Box\neg B$ and *not* $\vdash_{HS5_\Diamond} \neg\Diamond\neg A \wedge \neg\Diamond\neg B \to \neg\Diamond\neg(A\wedge B)$, so that we cannot interpret naturally $\boldsymbol{HS5}_\Box$ and $\boldsymbol{HS5}_\Diamond$ in each other.

It is possible to envisage stronger normal intuitionistic modal logics with both $\Box$ and $\Diamond$, and models for them, on the lines of the system $\boldsymbol{HK}\Box\Diamond$ of [3], where $\Box$ had a certain primacy. But we shall leave this topic for another occasion.

Added in proof. After this paper went into print I have learned of "Modal theories with intuitionistic logic" by Vladimir H. Sotirov (in *Mathematical Logic*, Proceedings of the Conference on Mathematical Logic dedicated to Markov, Bulgarian Academy of Sciences, Sofia 1984, pp. 139–172). This paper (which was announced in the abstract [30] of [3]), covers some of the topics treated here, though Sotirov's approach is somewhat different, as explained in [3]. Appart from that, Sotirov deals with neighbourhood semantics and decidability.

I am indebted to a referee for pointing out that H. Ono in his paper "On some intuitionistic modal logics" (*Publ. Res. Inst. Math. Sci.* (*Kyoto*) 13 (1977), pp. 687–722) covers some of the topics treated in this paper. In particular, he gives completeness proofs for analogues of $\boldsymbol{S}5$ with respect to specific classes of $\boldsymbol{H}\Box$ models.

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