# Propositional Game Logic

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## Abstract

We define a propositional logic of games which lies in expressive power between the Propositional Dynamic Logic of Fischer and Ladner [FL] and the μ-calculus of Kozen [K]. We show that the logic is decidable and give a very simple, complete set of axioms, one of the rules being Brouwer's bar induction. Even though decidable, this logic is powerful enough to define well orderings. We state some other results, open questions and indicate directions for further research.

## Introduction

Two person games have traditionally played an important part in Mathematical Economics and in Logic. In Economics the work of von Neumann and Morgenstern [vNM] has played a fundamental role. In Logic Ehrenfeucht [E] has given a characterisation of First Order Logic in terms of games. In set theory the axiom of determinacy (AD), a game theoretic axiom made famous by Mycielski and Steinhaus [MS], has played an important role in recent research.

The appearance of games in Computer Science is still more recent, but has been equally fruitful. The alternating Turing machines of Chandra, Kozen and Stockmeyer [CKS] give important connections between time and space complexities. But ATMs are clearly game theoretic objects with disjunctive states and conjunctive states corresponding, respectively, to the moves of player I and player II. Gurevich and Harrington [GH] have recently used games to give a new proof of Rabin's famous result [Ra] of the decidability of SnS. Proving the correctness of a concurrent algorithm can also be seen as a game theoretic situation with the scheduler in the role of the adversary player (player II from now on).

Probably the usefulness of games comes from the fact that games are logically very complex and powerful objects - Solovay has shown for instance that ZF with AD is much stronger than ZF with a measurable cardinal - and yet we have strong intuitive feelings about games. (Possibly because of having played them as children?)

In this paper we develop a Propositional Logic of Games which lies in expressive power between the PDL of Fischer and Ladner [FL] and the μ-calculus of Kozen [K]. It is stronger than the first, but might possibly be equal in expressive power to the second. We give a complete axiomatisation of one version of Game Logic but leave open the question of axiomatising an apparently stronger logic with the dual operator. The dual operator is one which converts a game into its dual where player I is permitted the moves of player II and



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vice versa. In certain contexts the dual operator can be replaced by two negations. (More or less the way the two quantifiers, existential and universal, can be defined from each other. Indeed the quantifiers themselves are simple, one move games which are duals of each other.) However, this device fails when the dual operator is allowed to fall within the scope of a * and then duality appears to make the logic much stronger.

A normal (human) game , like chess, consists of three parts, the initial position, the legal moves, and the positions that are winning for player I. In our version of games, we shall have no draws, and even infinite games will have a winner and a loser, the loser being that player whose responsibility it was (at that point) to see that the game ended. Thus the winning positions for player II are automatically defined from those for player I. Moreover, we shall take the game *itself* to consist *only* of the middle part, i.e. the legal moves, and that *will* include a determination of who loses if the game does not terminate. However, the initial position and the winning states for player I will be, as it were, supplied at the last moment when we convert a game into a statement.

Thus a typical statement of our logic will look like: $s \models (\alpha)A$ where s is a state of some model, $\alpha$ is a game, and A is some formula. Here the state s plays the role of the initial position, and the set of states satisfying A plays the role of the winning states for player I. The whole thing means: player I has a winning strategy, starting in state s, to play the game $\alpha$ in such a way, that either the game ends with A true, or else it fails to terminate in such a way, that player II will get the blame.

However, notice that $(\alpha)$ is also a predicate transform, converting the predicate A into the new predicate $(\alpha)A$. This predicate transform is monotonic in that if A implies B then $(\alpha)A$ implies $(\alpha)B$. In this it resembles the modalities $\langle\alpha\rangle$ and $[\alpha]$ of PDL. However, these modalities of PDL are disjunctive and conjunctive respectively. $\langle\alpha\rangle$ commutes with disjunction and $[\alpha]$ with conjunction. This can be explained in game theoretic terms by saying that $\langle\alpha\rangle$, like $\vee$, is a game where player I is the only one who moves, and player II is the only one who moves in $[\alpha]$, and in $\wedge$. More general games where both players move, share the monotonicity of $\langle\alpha\rangle$ and $[\alpha]$, but cannot be either disjunctive or conjunctive.

## Syntax and Semantics

We assume that the reader is familiar with the syntax and semantics of PDL and the $\mu$-calculus. (See [FL], [KP1], [P] for the first and [K] for the second.) However, those for Game Logic *can* be defined independently. We have a finite supply $g_1,\ldots,g_n$ of atomic games and a finite supply $P_1,\ldots,P_m$ of atomic formulae. Then we define games $\alpha$ and formulae A by induction.

1. Each $P_i$ is a formula.
2. If A and B are formulae, then so are $A \vee B$, $\neg A$.
3. If A is a formula and $\alpha$ is a game, then $(\alpha)A$ is a formula.
4. Each $g_i$ is a game.
5. If $\alpha$ and $\beta$ are games, then so are $\alpha;\beta$ (or simply $\alpha\beta$), $\alpha\vee\beta$, $\langle\alpha^*\rangle$, and $\alpha^d$. Here $\alpha^d$ is the dual of $\alpha$.
6. If A is a formula then $\langle A\rangle$ is a game.

We shall write $\alpha\wedge\beta$, $[\alpha^*]$ and $[A]$ respectively for the duals of $\alpha\vee\beta$, $\langle\alpha^*\rangle$ and $\langle A\rangle$.

Intuitively, the games can be explained as follows. $\alpha;\beta$ is the game: play $\alpha$ and then $\beta$. The game $\alpha\vee\beta$ is: player I has the first move and in it he decides whether $\alpha$ or $\beta$ is to be played, and then the chosen game is played. The game $\alpha\wedge\beta$ is similar except that player II makes the decision. In $\langle\alpha*\rangle$, the game $\alpha$ is played repeatedly (perhaps zero times) until player I decides to stop. If he never says "stop" then he loses. He may not stop in the middle of some play of $\alpha$. Similarly with $[\alpha*]$ and player II. In $\alpha^d$, the two players interchange roles. Finally, with $\langle A\rangle$, the formula A is evaluated. If A is false, then I loses, otherwise we go on. (Thus $\langle A\rangle B$ is equivalent to $A\wedge B$.) Similarly with $[A]$ and II.

Formally, a model for game logic consists of a set W of worlds, for each atomic P a subset $\pi(P)$ of W and for each primitive game g a subset $\rho(g)$ of $W\times P(W)$, where P(W) is the power set of W. $\rho(g)$ must satisfy the monotonicity condition: if $(s,X) \in \rho(g)$ and $X \subseteq Y$, then $(s,Y) \in \rho(g)$. We shall find it convenient to think of $\rho(g)$ as an operator from P(W) to itself, given by the formula $\rho(g)(X) = \{s \mid (s,X)\in\rho(g)\}$. It is then monotonic in X. We define $\pi(A)$ and $\rho(\alpha)$ for more complex formulae and games as follows:

2'. $\pi(A\vee B) = \pi(A)\cup\pi(B)$

$\pi(\neg A) = W-\pi(A)$

3'. $\pi(\langle\alpha\rangle A) = \{s \mid (s,\pi(A))\in\rho(\alpha)\}$

$= \rho(\alpha)(\pi(A))$

and

5'. $\rho(\alpha;\beta)(X) = \rho(\alpha)(\rho(\beta)(X))$

$\rho(\alpha\vee\beta)(X) = \rho(\alpha)(X)\cup\rho(\beta)(X)$

$\rho(\langle\alpha*\rangle)(X) = \mu Y(X\subseteq Y\wedge\rho(Y)\subseteq Y)$

$\rho(\alpha^d)(X) = W - \rho(\alpha)(W-X)$

6'. $\rho(\langle A\rangle)(X) = \pi(A)\cap X$

It is easily checked that $\rho(\alpha\wedge\beta)(X) = \rho(\alpha)(X)\cap\rho(\beta)(X)$, $\rho([A])(X) = (W-\pi(A))\cup X$, and $\rho([\alpha*])(X) = \nu Y(Y\subseteq X\wedge\rho(\alpha)(Y)\subseteq Y)$ where $\nu Y$ means "the largest Y such that".

We shall have occasion to use both ways of thinking of $\rho$, as a map from P(W) to itself, and also as a subset of WXP(W). In particular we shall need the (easily checked) fact that $(s,X)\in\rho(\beta;\gamma)$ iff there is a Y such that $(s,Y)\in\rho(\beta)$ and for all $t\in Y$, $(t,X)\in\rho(\gamma)$. Similarly, $(s,X)\in\rho(\beta\vee\gamma)$ iff $(s,X)\in\rho(\beta)$ or $(s,X)\in\rho(\gamma)$.

So far we have made no connection with PDL. However, given a language of PDL we can associate with it a Game Logic where to each program $a_i$ of PDL we associate two games $\langle a_i\rangle$ and $[a_i]$. We take

$\rho(\langle a\rangle)(X) = \{s \mid \exists t\ (s,t)\in R_a \text{ and } t\in X\}$

and

$\rho([a])(X) = \{s \mid \forall t\ (s,t)\in R_a \text{ implies } t\in X\}$

and the formulae of PDL can be translated easily into those of game logic. Note that if the program a is to be run and player I wants to have A true after, then if *he* runs a, only $\langle a\rangle A$ needs to be true. However, if player II is going to run the program a then $[a]A$ needs to be true for I to win in any case. Note that if there are no a-computations beginning at the state s, then player II is unable to move, $[a]A$ is true and player I wins. In other words, unlike chess, a situation where a player is unable to move is regarded as a loss for that player in both PDL and Game Logic. However, Game Logic is more expressive than PDL. The formula $\langle[b]*\rangle$false of game logic says that there is no infinite computation of the program b, a notion that can be expressed in Streett's $PDL^{\Delta}$ but not in PDL. We suspect that the formula $\langle(\langle a\rangle;[b])*\rangle$false, which says that player I can make "a" moves to player II's "b" moves in such a way that eventually player II will be deadlocked, cannot be expressed in $PDL^{\Delta}$ either. (NB. Moshe Vardi has shown recently that this is indeed the case.) Finally, let us show how well-foundedness can be defined in Game Logic. Given a linear ordering R over a set W, consider the model of Game Logic where $\rho(g)$ denotes $[a]$ and $R_a$ is the inverse relation of R. Then R is well-founded over W iff the formula $\langle g*\rangle$false is true. Since player I cannot terminate the game without losing, and he also loses if he never terminates, the only way he can win is to keep saying to player II, keep playing, and hope that player II will sooner or later be deadlocked. (the subgame $[a]$ of $\langle[a]*\rangle$ is a game where player II moves, and in the main game $\langle[a]^*\rangle$, player I is only responsible for deciding how many times is [a] played). Thus I wins iff there are no infinite descending sequences of R on W.

However, game logic can be translated into the $\mu$-calculus of [K] and by the decision procedure of [KP2], is decidable. This translation is not as obvious as might appear at first sight. The $\mu$-calculus, like PDL, is based on binary relations and while every binary relation R can be regarded as a game in two ways, as $\langle R\rangle$ and $[R]$, not every game can be written in one of these two forms. However, it *is* true that every game g over W can be written as $\langle a\rangle[b]$ over W' where W' is a superset of W and its size is exponential in that of W. This allows us to give a translation as we claimed above. We do not know if there is an elementary decision procedure for the full Game Logic with the dual operator. An elementary decision procedure for dual-free Game Logic will follow as the consequence of the completeness result, given below.

## Completeness

The following axioms and rules are complete for the "dual-free" part of game logic.

*The axioms of game logic*

1) All tautologies
2) $(\alpha)(A \Rightarrow B) \Rightarrow ((\alpha)A \Rightarrow (\alpha)B)$
3) $(\alpha;\beta)A \Leftrightarrow (\alpha)(\beta)A$
4) $(\alpha\vee\beta)A \Leftrightarrow (\alpha)A \vee (\beta)A$
5) $(\langle\alpha^*\rangle)A \Leftrightarrow A \vee (\alpha)(\langle\alpha^*\rangle)A$
6) $(\langle A?\rangle)B \Leftrightarrow A\wedge B$

*Rules of Inference*

1) Modus Ponens

$$\frac{A \qquad A \Rightarrow B}{B}$$

2) Monotonicity

$$\frac{A \Rightarrow B}{(\alpha)A \Rightarrow (\alpha)B}$$

3) Bar Induction

$$\frac{(\alpha)A \Rightarrow A}{(\langle\alpha^*\rangle)A \Rightarrow A}$$

The soundness of these axioms and rules is quite straightforward. The completeness proof is similar to that in [KP1] and proceeds through four lemmas.

*Lemma 1:* (Substitutivity of Equivalents) If $\vdash A \Leftrightarrow B$ and D is obtained from C by replacing some occurrences of A by B, then $\vdash C \Leftrightarrow D$.
Proof: Quite straightforward, using tautologies and rule 2).

To show that every consistent formula is satisfiable, we construct a model for a given consistent $A_0$ as follows. Let FL be the Fischer-Ladner closure of $A_0$. This is quite similar to the Fischer-Ladner closure as in PDL. Let W be the set of all consistent conjunctions of elements of FL and their negations (as in [KP1]). We let $\pi(P_i)$ be as before, the set of those atoms $A \lesssim P_i$. Finally we take $\rho(g_i)$ as $\{(A,X) \mid A\wedge X' \text{ is consistent}\}$ where $X \subseteq W$ and X' is the disjunction of all the elements of X. We give first the proof for the case where all tests in all games are atomic. We remind the reader of our convention from [KP1] that for *formulas* X and Y, $X \lesssim Y$ means $\vdash X \Rightarrow Y$.

*Lemma 2:* For all $X \subseteq W$, $A\in W$, all atomic test games $\alpha$, if $A\wedge(\alpha)X'$ is consistent, then $(A,X)\in\rho(\alpha)$.

*Proof:* By induction on the complexity of $\alpha$.
1) If $\alpha$ is some $g_i$ then the lemma holds by definition of $\rho(g_i)$.
2) If $\alpha$ is $\langle P\rangle$ for some P, then by hypothesis and axiom 6, $A\wedge X'\wedge P$ is consistent, so $A\in X$ and $A \models P$ by definition of the model. So $(A,X)\in\rho(\alpha)$.
3) $\alpha$ is $\beta\vee\gamma$. Then, by hypothesis and axiom 4, $(A\wedge(\beta)X')\vee(A\wedge(\gamma)X')$ is consistent. Hence once of the disjuncts, say $(A\wedge(\beta)X')$ is consistent. Hence $(A,X)\in\rho(\beta)$ so that $(A,X)\in\rho(\alpha)$.

4) $\alpha$ is $\beta;\gamma$. Then by axiom 3 $A\wedge(\beta)(\gamma)X'$ is consistent. Hence $A\wedge(\beta)(T\wedge(\gamma)X')$ is consistent, by lemma 1 where T is *true*. So $A\wedge(\beta)(\quad(B\wedge(\gamma)X':B\in W)$ is consistent. Let $Y=\{B\mid B\wedge(\gamma)X'$ is consistent$\}$. Then we can disregard B not in Y above. So $A\wedge(\beta)(\quad(B\wedge(\gamma)X':B\in Y)$ is consistent. Hence $A\wedge(\beta)Y'$ is consistent. It is easily seen, using induction hypothesis that $(A,Y)\in\rho(\beta)$ and for all $B\in Y$, $(B,X)\in\rho(\gamma)$. Hence $(A,X)\in\rho(\alpha)$.

5) $\alpha$ is $\langle\beta*\rangle$. Let Z be the smallest set such that $X\subseteq Z$ and for all B, if $B\wedge(\beta)Z'$ is consistent, then $B\in Z$. Then note that for all $B\in Z$, $(B,X)\in\rho(\langle\beta*\rangle)=\rho(\alpha)$. Then it is immediate that $\vdash X'\Rightarrow Z'$ and also, by induction hypothesis $\vdash(\beta)Z'\Rightarrow Z'$. Hence, by bar induction, $\vdash(\langle\beta*\rangle)Z'\Rightarrow Z'$. Now, $A\wedge(\alpha)X'$ is consistent, so $A\wedge(\alpha)Z'$ is also consistent, so $A\wedge Z'$ is consistent, by above, and so A must be in Z. Thus $(A,X)\in\rho(\alpha)$. QED

Given X in FL let $X^+$ denote the set of all atoms B such that $B\lesssim X$. Then $X^{+'}$ will be a formula equivalent to X.

*Lemma 3:* If $(\alpha)X\in FL$ then (i) $(A,X^+)\in\rho(\alpha)$ iff (ii) $A\wedge(\alpha)X$ is consistent iff (iii) $A\lesssim(\alpha)X$

*Proof:* Clearly the last two cases are equivalent since A is an atom and either $(\alpha)X'$ or its negation occurs in A. Also (ii)=>(i) by lemma 2. So assume (i) and use induction on the complexity of $\alpha$. The cases where $\alpha$ is $P_i$ or $g_j$ or $\beta\vee\gamma$ are all easy.

2) Suppose $\alpha$ is $\beta;\gamma$. Since $(A,X)\rho(\alpha)$ there must exist Y such that $(A,Y)\in\rho(\beta)$ and for all $B\in Y$, $(B,X)\in\rho(\gamma)$. By induction hypothesis, for all $B\in Y$, $B<(\gamma)X'$. Hence $Y'\lesssim(\gamma)X'$. Also $A\lesssim(\beta)Y'$. Hence $A\lesssim(\beta)(\gamma)X'$. Thus $A\lesssim(\alpha)X'$.

3) Suppose $\alpha$ is $\langle\beta*\rangle$. If $A\in X$ then $A\lesssim X'$ so $A\lesssim(\alpha)X'$ by axiom 5. Otherwise $(A,X)\in\rho(\beta;\beta*)$. Define $Z_0=X$ and $Z_{n+1}=Z_n\cup\{B\mid(B,Z_n)\in\rho(\beta)\}$. Since A is in the union of the $Z_n$, take the least m such that $A\in Z_m$. By induction hypothesis (on n), for all $B\in Z_{m-1}$, $B\lesssim\langle\alpha\rangle X'$. Also $(A,Z_{m-1})\in\rho(\beta)$. By an argument like that of case 2 above, we get $A\lesssim(\beta)(\langle\beta*\rangle)X'$ so $A\lesssim(\langle\beta*\rangle)X'$. QED.

*Lemma 4:* For all A in W, all X in FL, $A\lesssim X$ iff $A\models X$.

*Proof:* By induction on the complexity of X. The atomic case and the cases for negation and disjunction are easy.

Suppose X is $(\alpha)Y$.

Then $A\models X$ iff $(A,\pi(Y))\in\rho(\alpha)$

iff $(A,Y^+)\in\rho(\alpha)$ (IH)

iff $A\lesssim(\alpha)Y^{+'}$

iff $A\lesssim(\alpha)Y$ iff $A\lesssim X$. QED

The completenes result follows at once. For if a formula X is consistent then there is an atom A such that $A<X$ and then X is satisfiable.

To see the case for the "rich-test" version of Game Logic, suppose we define, by induction, forumlae of level n to be formulae whose programs have tests all of level n-1. Then notice that the argument will go through for level n assuming that all lemmas were true at level n-1. For the only fact we used in the proof above for *atomic* P is that $A\lesssim P$ iff $A\models P$. This fact will now be the induction hypothesis for formulae of level n-1 rather than part of the definition of the model, and the proof goes through.

It follows also that Game Logic is decidable in nondeterministic exponential time. For given a formula X, if X has a model, then it has one where W is at most exponential in the size of X. We can guess such a model and a state where X is to hold and proceed to verify this guess in time which is polynomial in the size of W. The only novel feature here that does not appear in PDL is the presence of the $\mu X$ operator used in defining the extension of $\langle\alpha*\rangle$. However, if W has k elements then $\mu X\phi(X)$ is merely $X_k$ where $X_0$ is the empty set and $X_{i+1}$ is $\phi(X_i)$. Thus if $\phi$ can be calculated in time polynomial in the size of W, so can $\mu X\phi(X)$. We suspect that the decision procedure can be made to run in *deterministic* exponential time using the familiar techniques due to Pratt.

We conjecture that the addition of the following axiom scheme results in a system complete for the dual operator.

$$7)\ (\alpha^d)A \Leftrightarrow \neg(\alpha)\neg A$$

*Other work and open problems:* We have already mentioned two open problems, one is to determine the exact complexity of Game Logic, and the other is to give a completeness proof for Game Logic with the dual operator. An application of many-person Game Logic to prove the correctness of a cake cutting algorithm, has appeared in [Pa]. Other applications, e.g. to concurrent computation, are under progress.

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