A Stochastic Interpretation of Propositional Dynamic Logic: Expressivity

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Abstract. We propose a probabilistic interpretation of Propositional Dynamic Logic (PDL). We show that logical and behavioral equivalence are equivalent over general measurable spaces. Bisimilarity is also discussed and shown to be equivalent to logical and behavioral equivalence, provided the base spaces are Polish spaces. We adapt techniques from coalgebraic stochastic logic and point out some connections to Souslin's operation $\mathcal{A}$ from descriptive set theory.

1 Introduction

Propositional Dynamic Logic (PDL) is a modal logic originally proposed for modelling program behavior. Its basic operators are of the form $\langle\pi\rangle$, where π is a non-deterministic program; a formula $\langle\pi\rangle\varphi$ holds in a state s iff some terminating execution of π in s may lead to a state in which φ holds. Programs are composed from basic programs by sequential composition, iteration, and by non-deterministic choice. Usually a test operator is available as well: if φ is a formula, then program $\varphi?$ tests whether φ is true; if it is, the program continues, if not, it fails. This dynamic logic is interesting from an application point of view, see [1] for an overview from a semantic perspective. It has attracted attention as a possible model for two-person games, where the programs are thought of as games for modelling the behavior of the players, see [9].

We are proposing an interpretation of PDL through stochastic Kripke models. This appears to be new, it is motivated by two observations. First, games and economic behavior are successfully modelled through probabilistic models, and PDL permits capturing games and their semantics, it might be interesting to know what probabilistic properties are reflected by the logic. For example, the notion of bisimilarity is of some interest in modelling the equivalence of games [9, Section 4], it has also been extensively studied in the area of coalgebras, modal logics and their probabilistic interpretations [3,4], so it is worthwhile to dwell on this common interest. The second observation addresses the dynamic nature of PDL. When interpreting modal or coalgebraic logics, each modal operator is assigned a relation or a predicate lifting which is associated with its interpretation. This property has to be addressed for a probabilistic interpretation of PDL. Closely connected with the interpretation of logics is the question of expressivity

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of their models. This problem is addressed in the rest of the paper, performing some transformations from one model class to others, for which the characterization of expressivity has been undertaken already.

After proposing a probabilistic interpretation – what do we do with it? First, we have a look at the structure of the sets of states for which a formula holds. These set are usually measurable in probabilistic interpretations of modal logics, but this is not automatically the case in the present interpretation. We require the Souslin operation from descriptive set theory in order to make sure that these sets are well-behaved. We then show how different models for the logic do compare to each other. Logically equivalent models are behaviorally equivalent, thus having the same theory is for two models equivalent to finding a model onto which both can be mapped by a morphism. This result is well-known in "static" modal logics. Bisimulations are considered next, and here we need to restrict the generality of the models under consideration to those working over Polish spaces, i.e., over complete and separable metric spaces. A fair amount about models for logics of the type FRAG over Polish spaces, so we transform the problem again, solve it in the sphere of FRAG, and transform the solution back to PDL. We show that bisimilarity is equivalent to logical and behavioral equivalence, provided the models are defined over Polish spaces.

Organization. Section 2 collects some notions from probability. Section 3 defines the set of programs we are working with, and defines some logics of interest. The interpretation proposed in Section 4 is analyzed in terms of infinite well-founded trees, and some results on measurability are derived. Morphisms for comparing models are introduced in Section 5. Section 6 defines issues of expressivity formally and derives the main result in two steps, isolating topological requirements from general questions of measurability. Section 7 wraps it all up and proposes further research.

2 Preliminaries

The interpretation of logics through stochastic Kripke models requires some tools from measure theory. For more information on these topics the reader is referred to Srivastava's treatise [12] on Borel sets, or to the tutorial Chapter 1 in [4].

Measurable Spaces. Let $(X, \mathfrak{C})$ be a measurable space, i.e., a set X with a σ-algebra $\mathfrak{C}$ of subsets; the elements of $\mathfrak{C}$ are called $\mathfrak{C}$-*measurable sets* (or just *measurable sets*, if no confusion arises). Denote by $\sigma(\mathfrak{C}_0)$ the smallest σ-algebra which contains the family $\mathfrak{C}_0$ of sets. $\mathfrak{S}(X, \mathfrak{C})$ denotes the set of all subprobabilities on $(X, \mathfrak{C})$. Let $(Y, \mathfrak{D})$ be another measurable space, a map $f : X \to Y$ is called $\mathfrak{C}$-$\mathfrak{D}$-*measurable* iff $f^{-1}[D] \in \mathfrak{C}$ for all $D \in \mathfrak{D}$. This implies that a real-valued map f on X is $\mathfrak{C}$-*measurable* iff the set $\{x \in X \mid f(x) < r\}$ is a member of $\mathfrak{C}$ for each $r \in \mathbb{R}$.

Let $\mu \in \mathfrak{S}(X, \mathfrak{C})$ be a subprobability and $f : X \to Y$ be $\mathfrak{C}$-$\mathfrak{D}$-measurable. Put $\mu^f(D) := \mu(f^{-1}[D])$ whenever $D \in \mathfrak{D}$, then μ^f is the *image of measure* μ *under* f; apparently $\mu^f \in \mathfrak{S}(Y, D)$. The integral with respect to an image measure can be computed through the original measure, as the *change of variables formula* shows.

Lemma 1. *Let* $(X, \mathfrak{C})$ *and* $(Y, \mathfrak{D})$ *be measurable spaces,* $f : X \to Y$ *be a* $\mathfrak{C}$-$\mathfrak{D}$-*measurable map, and* $\mu \in \mathfrak{S}(X, \mathfrak{C})$. *Then* $\int_Y g(y)\, \mu^f(dy) = \int_X (g \circ f)(x)\, \mu(dx)$ *for each* $\mathfrak{D}$-*measurable and bounded* $g : Y \to \mathbb{R}$. $\dashv$

Stochastic Relations. A *stochastic relation* $K : (X, \mathfrak{C}) \rightsquigarrow (Y, \mathfrak{D})$ between the measurable spaces $(X, \mathfrak{C})$ and $(Y, \mathfrak{D})$ is a map $K : X \to \mathfrak{D} \to [0,1]$ with these properties: $K(x)$ is for each $x \in X$ a subprobability on $(Y, \mathfrak{D})$, and the map $x \mapsto K(x)(D)$ is $\mathfrak{C}$-measurable for each $D \in \mathfrak{D}$. In the parlance of probability theory, stochastic relations are called transition probabilities. We note in particular that $K(x)(Y) \leq 1$ for $x \in X$, hence $K(x)(Y) < 1$ may occur, so that mass may vanish. This caters for the observation that, e.g., programs sometimes do not terminate.

If $L : (Y, \mathfrak{D}) \rightsquigarrow (Z, \mathfrak{E})$ is another stochastic relation, then the *convolution* $L{*}K$ *of* L *and* K is defined through $\bigl(L{*}K\bigr)(x)(E) := \int_Y L(y)(E)\ K(x)(dy)$ $(x \in X, E \in \mathfrak{E})$. Standard arguments show that $L{*}K : (X, \mathfrak{C}) \rightsquigarrow (Z, \mathfrak{E})$ is a stochastic relation between $(X, \mathfrak{C})$ and $(Z, \mathfrak{E})$. Note that the convolution has identities I_X, I_Y with $K{*}I_X = I_Y{*}K = K$: put $I_X(x)(C) := (x \in C ? 1 : 0)$ for $x \in X, C \in \mathfrak{C}$; this is the *indicator function for* $C \subseteq X$.

Completion And Operation $\mathcal{A}$. Given $\mu \in \mathfrak{S}(X, \mathfrak{C})$, a set $N \subseteq X$ is called a *μ-null set* iff there exists $N \subseteq N_0 \in \mathfrak{C}$ with $\mu(N_0) = 0$; $\mathfrak{N}_\mu$ is the set of all μ-null sets. Define the *μ-completion* $\overline{\mathfrak{C}}^\mu$ of $\mathfrak{C}$ as $\sigma(\mathfrak{C} \cup \mathfrak{N}_\mu)$, thus $M \in \overline{\mathfrak{C}}^\mu$ iff there exists $M_1 \subseteq M \subseteq M_2$ with $M_1, M_2 \in \mathfrak{C}$ and $\mu(M_2 \setminus M_1) = 0$. The *universal completion* $\overline{\mathfrak{C}}$ is defined as $\overline{\mathfrak{C}} := \bigcap\{\overline{\mathfrak{C}}^\mu \mid \mu \in \mathfrak{S}(X, \mathfrak{C})\}$. A measurable space is called *complete* iff it coincides with its completion.

V^w denotes for a set V the set of all finite words with letters from V including the empty string ϵ. Let $\{A_s \mid s \in \mathbb{N}^w\}$ be a collection of subsets of a set X indexed by all finite sequences of natural numbers, then the *Souslin operation* $\mathcal{A}$ on this collection is defined as $\mathcal{A}\bigl(\{A_s \mid s \in \mathbb{N}^w\}\bigr) := \bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \bigcap_{n \in \mathbb{N}} A_{\alpha|n}$, where $\alpha|n$ are just the first n letters of the sequence α. This operation is intimately connected with the theory of analytic sets [12].

Proposition 1. *The completion $\overline{\mathfrak{B}}^\mu$ of σ-algebra $\mathfrak{B}$ is closed under the Souslin operation $\mathcal{A}$ whenever $\mu \in \mathfrak{S}(X, \mathfrak{B})$. Thus a complete measurable space is closed under this operation.* $\dashv$

3 The Logic

The logic under consideration is a modal logic, the modal operators of which are given through a set of programs. The programs in turn are composed from a set of basic statements, which cannot be decomposed further. This section defines programs and the logic. We also give here a fragment of the logic which models straight line programs, and we define for comparison, reference and motivation a classical variant of the logic which is to be interpreted by set theoretic relations.

Programs. Let $\mathbb{U}$ be a set of ur-programs, i.e., of programs that cannot be decomposed further. This set is fixed. Given $\mathbb{U}$, we define the set $\mathbb{P}$ of programs through this grammar $\pi ::= \upsilon \mid \pi_1; \pi_2 \mid \pi_1 \cup \pi_2 \mid \pi_1^*$ with $\upsilon \in \mathbb{U}$ an ur-program. Thus a program is an ur-program, the *sequential composition* $\pi_1; \pi_2$ which executes first π_1 and then π_2, the *nondeterministic choice* $\pi_1 \cup \pi_2$, which selects nondeterministically among π_1 and π_2 which one to execute, or the (indefinite) *iteration* π^* of program π_1 which executes π_1 a finite number of times, possibly not at all.

sPDL *and the Fragment* FRAG. A formula in sPDL is given through this grammar

$$\varphi ::= \top \mid p \mid \varphi_1 \wedge \varphi_2 \mid [\pi]_q\varphi.$$

Here p is an atomic proposition, taken from a fixed set $\mathbb{A}$ of atomic propositions, $\pi \in \mathbb{P}$ is a program, and $q \in \mathbb{Q} \cap [0,1]$ is a rational number from the interval $[0,1]$. The informal meaning of formula $[\pi]_q\varphi$ being true in state s is that after executing program π in state s, the probability of reaching a state in which formula φ holds is not greater than q. Denote by $\mathbf{F}_{\mathsf{sPDL}}$ the set of all formulas of sPDL.

We will also consider the fragment FRAG of sPDL in which the set of programs is restricted to members of $\mathbb{U}^w$, hence to sequential compositions of ur-programs. Denote the set of formulas of FRAG by $\mathbf{F}_{\mathsf{FRAG}}$.

These logics do have only the bare minimum of logical operators, negation and disjunction are missing. It will turn out that for discussing expressivity of these logics, negation or disjunction need not be present, but — strange enough — conjunction must be there. The technical reasons for preferring conjunction over disjunction will be discussed after stating Proposition 9.

The second remark addresses the intended meaning of the modal operator $[\pi]_q$ which specifies a probability **at most** q. Usually a modal operator of the form $\langle a\rangle_q$ is defined with the intended meaning that $\langle a\rangle_q\varphi$ holds in state s iff after executing action a in state s the system will be brought into a state in which φ holds with probability **at least** q, see [3,4]. Since we will recursively collect probabilities along different paths, it is intuitively more satisfying to argue with an upper bound than with a lower bound.

Vanilla PDL. Propositional dynamic logic PDL is defined in modal logic through this grammar

$$\varphi ::= \top \mid p \mid \neg\varphi_1 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \langle\pi\rangle\varphi.$$

with $p \in \mathbb{A}$ and $\pi \in \mathbb{P}$. The set $\mathbb{P}$ of programs is unchanged [1, Example 1.15]. Note that we admit as operations disjunction as well as negation to PDL, and that the modal operator $\langle\pi\rangle$ associated with program π is not decorated with a numeric argument. The intuitive interpretation of $\langle\pi\rangle\varphi$ holding in a state s is that upon executing program π in state s a state can be reached in which formula φ holds.

We remind the reader of this logic, because we will draw some motivation from its interpretation when defining the interpretation of sPDL.

4 Interpretation of sPDL

sPDL will be interpreted through Kripke pre-models. Whereas a Kripke model assigns to each modal operator a relation through which the operator is to be interpreted, or, in coalgebraic logics, a predicate lifting, we cannot do that in the present context. This is due to the fact that the programs, i.e., the modal operators, have a specific structure which is not matched by the stochastic relations. This is possible, however, for Kripke models of the fragment FRAG, and these models will be used for generating the interpretation of the full logic. The correspondence of Kripke pre-models for sPDL and Kripke models for FRAG will be observed further when investigating expressivity in Section 6.

Interpreting PDL. We recall the interpretation of Vanilla PDL first (see [1, Example 1.26]). Let $\mathcal{R} := \bigl(S, \{R_\pi \mid \pi \in \mathbb{P}\}, (V_p)_{p\in\mathbb{A}}\bigr)$ be a Kripke model, i.e., S is a set of possible worlds, R_π is for each program $\pi \in \mathbb{P}$ a relation on S, and V_p is for each atomic proposition $p \in \mathbb{A}$ a subset of S. The family of relations satisfies these conditions: $R_{\pi_1;\pi_2} = R_{\pi_1} \circ R_{\pi_2}, R_{\pi_1\cup\pi_2} = R_{\pi_1} \cup R_{\pi_2}, R_{\pi^*} = \bigcup_{n\in\mathbb{N}_0} R_{\pi^n}$ with $A \circ B$ as the relational composition of relations A and B, and $R_{\pi^0} = \varepsilon$, the identity relation. Thus the relational structures on R match the algebraic structure on the set of all programs: sequential composition of programs corresponds to relational composition, and nondeterministic choice of programs corresponds to the union of the corresponding relations.

The interpretation of $\top$ and the Boolean operations in PDL is straightforward, and we put $\mathcal{R}, s \models p \Leftrightarrow s \in V_p; \mathcal{R}, s \models \langle\pi\rangle\varphi \Leftrightarrow \exists t \in S : \langle s, t\rangle \in R_\pi$ and finally $\mathcal{R}, t \models \varphi$. Hence we find, e.g., $\mathcal{R}, s \models \langle\pi^*\rangle\varphi \Leftrightarrow \exists n \in \mathbb{N}_0 : \mathcal{R}, s \models \langle\pi^n\rangle\varphi$.

Stochastic Kripke Pre-Models. A *stochastic Kripke pre-model* $\mathcal{K} = \bigl((S, \mathfrak{B}), (K_\pi)_{\pi\in\mathbb{U}}, (V_p)_{p\in\mathbb{A}}\bigr)$ for the logic sPDL is a measurable space $(S, \mathfrak{B})$, the space of states, a family of stochastic relations $K_\pi : (S, \mathfrak{B}) \rightsquigarrow (S, \mathfrak{B})$, the transition law, indexed by the ur-programs, and a family of measurable sets $V_p \in \mathfrak{B}$, indexed by the atomic propositions. $K_\pi(s)(D)$ is the probability for the new state to be an element of D after executing program π in state s.

We assume that we have transition laws for the programs' building blocks only, and not, as in the case of relational Kripke models, for each program, thus we use the term "pre-model" rather than "model". Whereas the algebraic structure of $\mathbb{P}$ can be modelled in the set of relations, this is not the case for stochastic relations: there is a natural composition operator, given by the convolution, but there does not seem to be an intuitively satisfying way of modelling the non-deterministic choice or the indefinite iteration.

All the same, the relational approach will be used as a source for guidelines for the probabilistic approach. Fix a stochastic Kripke pre-model $\mathcal{K}$. Let $A \in \mathfrak{B}$ be a measurable set; we define recursively a set-valued map $\mathcal{I}_q^A$ from the set of programs to the subsets of S, indexed by the rationals on the interval $[0, 1]$. For the time being, the informal interpretation of $\mathcal{I}_q^A(\pi)$ is the characterization of all those states which through executing program π bring the system into a state in A with probability less than q.

As auxiliary sets we define for $q \in \mathbb{Q} \cap [0, 1]$ and $n \in \mathbb{N}$

$$Q^{(n)}(q) := \{a \in (\mathbb{Q} \cap [0, 1])^n \mid a_1 + \cdots + a_n \le q\},$$
$$Q^{(\infty)}(q) := \{a \in (\mathbb{Q} \cap [0, 1])^\infty \mid a_0 + a_1 + \cdots \le q\}.$$

Put $K_\varepsilon := I_S$ for simplicity and for uniformity: the empty program does not do anything.

a. If $\pi \in \mathbb{U} \cup \{\varepsilon\}$, then $\mathcal{I}_q^A(\pi) := \{s \in S \mid K_\pi(s)(A) < q\}$.This yields all states for which the execution of program π leads to A with probability less that q.
b. Let $s \in \mathcal{I}_q^A(\pi_1; \ldots; \pi_n)$ iff $\bigl(K_{\pi_n} * K_{\pi_{n-1}} * \ldots * K_{\pi_1}\bigr)(s)(A) < q$, for $\pi_1, \pi_2, \ldots, \pi_n \in \mathbb{U}$, thus executing programs $\pi_1, \ldots, \pi_n$ sequentially is modelled through the convolution of the corresponding transition probabilities, and we determine all states for which the combined programs yield a probability less that q to be in set A.

Consider the case $n = 2$. By the definition of the convolution,

$$(K_{\pi_2} * K_{\pi_1})(s)(A) < q \text{ iff } \int_S K_{\pi_2}(t)(A)\ K_{\pi_1}(s)(dt) < q,$$

thus upon executing program π_1 in state s the system goes into an intermediate state t, and executing π_2 in this intermediate state the probability of entering A is determined. Since the transitions happen at random, averaging through the corresponding transition probability yields the desired probability, which is then tested against q.

c. Let $\pi_1, \pi_2 \in \mathbb{P}$, then $\mathcal{I}_q^A(\pi_1 \cup \pi_2) := \bigcup\{\left(\mathcal{I}_{a_1}^A(\pi_1) \cap \mathcal{I}_{a_2}^A(\pi_2)\right) \mid a \in Q^{(2)}(q)\}$. Selecting nondeterministically one of the programs π_1 or π_2, $\mathcal{I}_{a_1}^A(\pi_1)$ accounts for all states which are lead by executing π_1 to a state in the set A with probability at most a_1, similarly, $\mathcal{I}_{a_2}^A(\pi_2)$ for π_2. Since we want to bound the probability from above by q, we require $a_1 + a_2 \leq q$.

d. Let $\pi_1, \pi_2, \pi_3 \in \mathbb{P}$, then $\mathcal{I}_q^A\big(\pi_1; (\pi_2 \cup \pi_3)\big) := \mathcal{I}_q^A(\pi_1; \pi_2 \cup \pi_1; \pi_3)$, similarly, $\mathcal{I}_q^A\big((\pi_1 \cup \pi_2); \pi_3\big)$ is defined. This corresponds to the distributive laws.

e. If $\pi_1, \pi_2 \in \mathbb{P}$, define $\mathcal{I}_q^A(\pi_1^*) := \bigcup\{\bigcap_{n\in\mathbb{N}_0} \mathcal{I}_{a_n}^A(\pi_1^n) \mid a \in Q^{(\infty)}(q)\}$, similarly, $\mathcal{I}_q^A(\pi_1; \pi_2^*)$ and $\mathcal{I}_q^A(\pi_1^*; \pi_2)$ are defined. If executing program π_1 exactly n times results in a member of A with probability not exceeding a_n, then executing π_1 a finite number of times (including not executing it at all) results in a member of A with probability at most $a_0 + a_1 + \ldots$, which should be bounded above by q for the resulting state to be a member of A with probability at least q.

The definition of $\mathcal{I}_q^A(\pi)$ shows that the elementary building blocks from which these sets are computed are the sets $\mathcal{I}_q^A(\pi_1; \ldots; \pi_n)$ for ur-programs $\pi_1, \ldots, \pi_n$. These building blocks are combined through elementary set operations, they in turn are determined by the stochastic relations which come with the Kripke pre-model, either directly ($n = 1$) or through convolutions ($n > 1$).

The intuition says that executing π^* is somewhat akin either doing nothing at all or to execute π followed by executing π^*, hence to executing $\varepsilon \cup \pi; \pi^*$.

Example 1. Let $\pi \in \mathbb{P}$, then

$$\begin{aligned}
\mathcal{I}_q^A(\varepsilon \cup \pi; \pi^*) &= \bigcup_{a\in Q^{(2)}(q)} \left(\mathcal{I}_{a_1}^A(\varepsilon) \cap \mathcal{I}_{a_2}^A(\pi; \pi^*)\right) \\
&= \bigcup_{a\in Q^{(2)}(q)} \Big(\mathcal{I}_{a_1}^A(\pi^0) \cap \bigcup_{b\in Q^{(\infty)}(a_2)} \bigcap_{n\in\mathbb{N}_0} \mathcal{I}_{b_n}^A(\pi^{n+1})\Big) \\
&= \bigcup_{c\in Q^{(\infty)}(q)} \bigcap_{n\in\mathbb{N}_0} \mathcal{I}_{c_n}^A(\pi^n) = \mathcal{I}_q^A(\pi^*).
\end{aligned}$$

$\dashv$

Define the *interpretation order* on the set of programs through $\pi_1 \sqsubseteq \pi_2$ iff $\mathcal{I}_q^A(\pi_1) \subseteq \mathcal{I}_q^A(\pi_2)$ for all q and all A, then $\sqsubseteq$ is reflexive and transitive. Put $\equiv := \sqsubseteq \cap \sqsupseteq$ as the associated equivalence relation, then $\pi_1 \equiv \pi_2$ iff π_1 and π_2 have the same interpretation. We will consider programs rather than classes for convenience.

Proposition 2. *The interpretation order has the following properties:*

a. $\pi_1 \sqsubseteq \pi_2$ *iff* $\pi_1 \cup \pi_2 \equiv \pi_2$*, and* $\pi_1 \sqsubseteq \pi_2$ *implies* $\pi; \pi_1 \sqsubseteq \pi; \pi_2$ *for all* π.
b. $\pi^* = \sup_{n \in \mathbb{N}_0} \pi^n$.
c. π^* *is the smallest fixed point of the monotone map* $\tau \mapsto \varepsilon \cup \pi; \tau$.

Proof. Property a. is trivial. For establishing property b. one proves first by induction that $\pi^n \sqsubseteq \pi^*$ for all $n \in \mathbb{N}_0$. Moreover, if $\pi^n \sqsubseteq \pi'$ for all $n \in \mathbb{N}_0$, then it is not difficult to see that $\pi^* \sqsubseteq \pi'$. Thus π^* is the smallest upper bound to $\{\pi^n \mid n \in \mathbb{N}_0\}$. Finally, the map $\tau \mapsto \varepsilon \cup \pi; \tau$ is monotone, and π^* is a fixed point of $\tau \mapsto \varepsilon \cup \pi; \tau$ by Example 1. If $\widetilde{\tau}$ is another fixed point of this map, then $\widetilde{\tau} \equiv \varepsilon \cup \pi^1 \cup \cdots \cup \pi^n \cup \pi^{n+1}; \widetilde{\tau}$, so that by part a. $\pi^n \sqsubseteq \widetilde{\tau}$ for all $n \in \mathbb{N}_0$. Thus π^* is in fact the smallest fixed point. $\dashv$

Consequently, the semantics for the iteration construct defined through $\mathcal{I}$ uses actually a fixed point with the order adapted to the programs' effects. This appears to be a sensible alternative to associating the set-theoretically smallest fixed point (in the fashion of the μ-calculus) to this construct, see the approach proposed in, e.g., [9].

Associate with each program $\pi \in \mathbb{P}$ a tree $\mathcal{T}(\pi)$. We will be using the following format for writing down a tree with root node labelled ν and at most countable offsprings $\sigma_0, \sigma_1, \ldots$ from left to right: $\lceil\, \nu \;\|\; \sigma_0 \mid \sigma_1 \mid \ldots \rfloor$.

A. If $\pi \in \mathbb{U}$, then $\mathcal{T}(\pi) := \pi$, thus ur-programs constitute the leaves of the tree.
B. If $\pi_1, \ldots, \pi_n \in \mathbb{U}$ $(n > 1)$, then $\mathcal{T}(\pi_1; \ldots; \pi_n) := \lceil\, \mathfrak{comp} \;\|\; \pi_1 \mid \cdots \mid \pi_n \rfloor$,thus the tree associated with a finite sequence of ur-programs has the composition symbol $\mathfrak{comp}$ as the label of the root, and the ur-programs as offsprings.
C. Define recursively $\mathcal{T}(\pi_1 \cup \pi_2) := \lceil\, \mathfrak{union} \;\|\; \mathcal{T}(\pi_1) \mid \mathcal{T}(\pi_2) \rfloor$, for the programs $\pi_1, \pi_2, \pi_3 \in \mathbb{P}$, similarly, $\mathcal{T}(\pi_1; (\pi_2 \cup \pi_2))$ and $\mathcal{T}((\pi_1 \cup \pi_2); \pi_3)$ are defined. The root has the label $\mathfrak{union}$, the left and right offspring correspond to the operands.
D. Given the programs $\pi_1, \pi_2 \in \mathbb{P}$, define recursively $\mathcal{T}(\pi_1^*) := \lceil\, \mathfrak{star} \;\|\; \varepsilon \mid \mathcal{T}(\pi_1) \mid \cdots \mid \mathcal{T}(\pi_1^n) \mid \ldots \rfloor$, the trees $\mathcal{T}(\pi_1; \pi_2^*)$ and $\mathcal{T}(\pi_1^*; \pi_2)$ are defined similarly. Thus the root node is decorated with the annotation $\mathfrak{star}$, and it has countably many offsprings, each corresponding to executing the program exactly n times. Note that ε serves to indicate that the program is not executed at all.

Fig. 1 gives an impression what the partially expanded tree $\mathcal{T}((\pi_0 \cup \pi_1; \pi_2^*)^*)$ looks like.

Lemma 2. *The tree* $\mathcal{T}(\pi)$ *for each program* $\pi \in \mathbb{P}$ *is well-founded.* $\dashv$

The set operations for determining $\mathcal{I}_q^A$ are not always countable, so one might suspect that they go beyond what can be represented through a σ-algebra. If the underlying state space is a measurable space which is closed under the Souslin operation, however, we can establish that we stay within the realm of measurable sets. The proof of the following Proposition proceeds by induction on the program tree $\mathcal{T}(\pi)$.

Proposition 3. *Let* $(S, \mathfrak{B})$ *be a measurable space which is closed under the Souslin operation* $\mathcal{A}$*, then* $\mathcal{I}_q^A(\pi) \in \mathfrak{B}$ *for each program* $\pi \in \mathbb{P}$*, provided* $A \in \mathfrak{B}$. $\dashv$

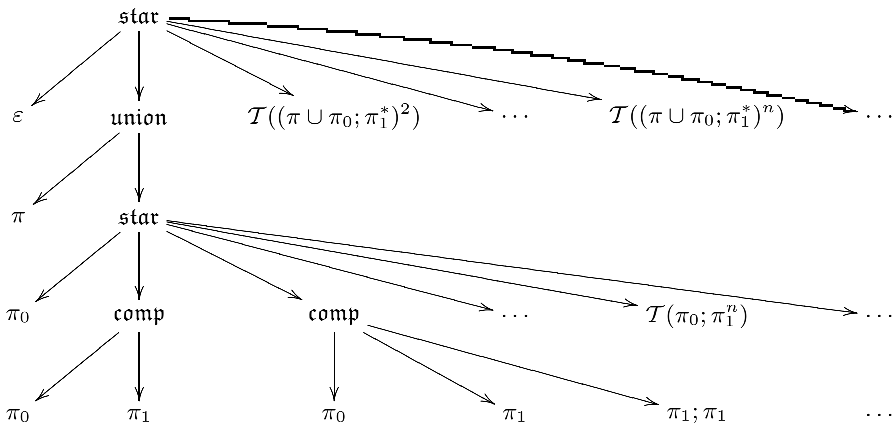

Fig. 1. Partially expanded tree for $(\pi_0 \cup \pi_1; \pi_2^*)^*$

The Semantics for sPDL. The semantics for the formulas is defined now. For this, define the *extent* $[\![\varphi]\!]_{\mathcal{K}}$ *of formula* φ as the set of all worlds $\{s \in S \mid \mathcal{K}, s \models \varphi\}$ of the Kripke pre-model $\mathcal{K}$ in which formula φ is true. Define the semantics of formula φ through $\mathcal{K}, s \models \varphi \Leftrightarrow s \in [\![\varphi]\!]_{\mathcal{K}}$,where $[\![\varphi]\!]_{\mathcal{K}}$ is defined recursively in this way.

$$
\begin{aligned}
[\![\top]\!]_{\mathcal{K}} &:= S, \\
[\![p]\!]_{\mathcal{K}} &:= V_p, \ \text{if } p \in \mathbb{A}, \\
[\![\varphi_1 \wedge \varphi_2]\!]_{\mathcal{K}} &:= [\![\varphi_1]\!]_{\mathcal{K}} \cap [\![\varphi_2]\!]_{\mathcal{K}}, \\
[\![[\pi]_q \varphi]\!]_{\mathcal{K}} &:= \mathcal{I}_q^{[\![\varphi]\!]_{\mathcal{K}}}(\pi), \ \text{if } \pi \in \mathbb{P}, q \in \mathbb{Q} \cap [0,1].
\end{aligned}
$$

Concerning the structure of these sets, we state

Proposition 4. *If the state space* $(S, \mathfrak{B})$ *of the Kripke pre-model* $\mathcal{K}$ *is closed under operation* $\mathcal{A}$*, then* $[\![\varphi]\!]_{\mathcal{K}} \in \mathfrak{B}$ *for all formulas* φ*.* $\dashv$

Thus if the state space $(S, \mathfrak{B})$ of the Kripke pre-model $\mathcal{K}$ equals the completion $\overline{\mathfrak{C}}^{\mu}$ for some $\mu \in \mathfrak{S}(S, \mathfrak{C})$, or if it equals the universal completion $\overline{\mathfrak{C}}$ for some σ-algebra $\mathfrak{C}$, then $[\![\varphi]\!]_{\mathcal{K}} \in \mathfrak{B}$ for all formulas φ, see Proposition 1. If, however, the state space is not closed under operation $\mathcal{A}$, then for some formulas φ the set $[\![\varphi]\!]_{\mathcal{K}}$ may not be representable as an event.

The Semantics of Fragment FRAG. We cannot associate with each program a stochastic relation for sPDL, but we are able to do this for fragment FRAG. A *Kripke model* $\mathcal{M} = \big((S, \mathfrak{B}), (M_\pi)_{\pi \in \mathbb{U}^w}, (V_p)_{p \in \mathbb{A}}\big)$ for the fragment FRAG of sPDL is defined just as a pre-models for sPDL, with the exception that $M_\pi : (S, \mathfrak{B}) \rightsquigarrow (S, \mathfrak{B})$ is a stochastic relation for each $\pi \in \mathbb{U}^w$, hence for each finite sequence of ur-programs.

We define the semantics of formulas in FRAG in an obvious way, the interesting case is the treatment of the modal operator$[\pi]_q$: $[\![[\pi]_q \varphi]\!]_{\mathcal{M}} := \{s \in S \mid M_\pi(s)\big([\![\varphi]\!]_{\mathcal{M}}\big) < q\}$. Structural induction establishes measurability of the formulas' extension.

Proposition 5. *Let $\mathcal{M} = \big((S, \mathfrak{B}), (M_\pi)_{\pi\in\mathbb{U}^w}, (V_p)_{p\in\mathbb{A}}\big)$ be a Kripke model for* FRAG*, then $[\![\varphi]\!]_{\mathcal{M}} \in \mathfrak{B}$ for all formulas φ of* FRAG*.* $\dashv$

Measurability of the sets $[\![\varphi]\!]_{\mathcal{M}}$ is easier to obtain for models than for pre-models, because the structure of the formulas is considerably simpler, in particular we may restrict our attention to programs that have a predetermined number of components, in contrast to iterations which are finite but of indefinite length.

Given a stochastic Kripke pre-model $\mathcal{K}$ over the state space $(S, \mathfrak{B})$ with stochastic relations $(K_\pi)_{\pi\in\mathbb{U}}$, we construct a stochastic Kripke model $\mathcal{K}^\dagger$ for the fragment FRAG with stochastic relations $(M_\pi)_{\pi\in\mathbb{U}^w}$ such that $M_{\pi_1;\ldots;\pi_n} := K_{\pi_n} * K_{\pi_{n-1}} * \ldots * K_{\pi_1}$, all other components are inherited from $\mathcal{K}$. Conversely, a stochastic Kripke model $\mathcal{M}$ for FRAG with relations $(M_\pi)_{\pi\in\mathbb{U}^w}$ yields a pre-model $\mathcal{M}_\ddagger$ for sPDL with stochastic relations $(K_\pi)_{\pi\in\mathbb{U}}$ upon setting $K_\pi := M_\pi$ for the ur-program $\pi \in \mathbb{U}$, again all other components remaining the same. Then $\big(\mathcal{K}^\dagger\big)_\ddagger = \mathcal{K}$ for each pre-model $\mathcal{K}$. Structural induction on the formula shows that

Lemma 3. *Let $\mathcal{K}$ be a stochastic Kripke pre-model, then $[\![\varphi]\!]_{\mathcal{K}} = [\![\varphi]\!]_{\mathcal{K}^\dagger}$ holds for each formula $\varphi \in \mathbf{F}_{\mathsf{FRAG}}$.* $\dashv$

The interplay of pre-model $\mathcal{K}$ with model $\mathcal{K}^\dagger$, and of model $\mathcal{M}$ with the pre-model $\mathcal{M}_\ddagger$ will be of considerable interest when investigating expressivity in Section 6.

5 Morphisms

Morphisms are used to relate pre-models to each other. A morphism preserves the structure of an interpretation: the states in which atomic propositions are true are related to each other, and the probabilistic structure, i. e., the transition laws, are compared against each other.

Let $\mathcal{K} = \big((S, \mathfrak{B}), (K_\pi)_{\pi\in\mathbb{U}}, (V_p)_{p\in\mathbb{A}}\big)$ and $\mathcal{L} = \big((T, \mathcal{C}), (L_\pi)_{\pi\in\mathbb{U}}, (W_p)_{p\in\mathbb{A}}\big)$ be stochastic Kripke pre-models. A $\mathfrak{C}$-$\mathfrak{D}$-measurable map $f : S \to T$ is called a *pre-model morphism* $f : \mathcal{K} \to \mathcal{L}$ iff $f^{-1}[W_p] = V_p$ for each atomic proposition $p \in \mathbb{A}$, and $K_\pi^f = L_\pi \circ f$ for each ur-program $\pi \in \mathbb{U}$. The first condition states that $s \in V_p$ iff $f(s) \in W_p$ for each atomic proposition. The second condition states that $L_\pi\big(f(s)\big)(D) = K_\pi(s)\big(f^{-1}[D]\big)$ for each $s \in S$, each measurable set $D \in \mathfrak{D}$, and each ur-program $\pi \in \mathbb{U}$. Thus the probability in $\mathcal{L}$ of bringing state $f(s)$ into D through executing ur-program π is the same as executing π in state s and ending up in $f^{-1}[D]$ in $\mathcal{K}$.

This property is preserved through sequential program composition.

Proposition 6. *Let $\mathcal{K}$ and $\mathcal{L}$ be stochastic Kripke pre-models as above, and $f : \mathcal{K} \to \mathcal{L}$ be a morphism. Then we have for each $s \in S, D \in \mathfrak{D}$ and ur-programs $\pi_1, \ldots, \pi_n \in \mathbb{U}$ $\big(L_{\pi_1} * \ldots * L_{\pi_n}\big)(f(s))(D) = (K_{\pi_1} * \ldots * K_{\pi_n})(s)\big(f^{-1}[D]\big)$.* $\dashv$

Comparing Morphisms. Dealing with Kripke models $\mathcal{M}$ and $\mathcal{N}$, the definition of a *model morphism* $f : \mathcal{M} \to \mathcal{N}$ remains essentially the same. To be specific, if the stochastic relations $(M_\pi)_{\pi\in\mathbb{U}^w}$ and $(N_\pi)_{\pi\in\mathbb{U}^w}$ govern the probabilistic behavior, then the first condition on atomic propositions remains as it is, and the second one is replaced by $M_\pi^f = N_\pi \circ f$ for each finite sequence $\pi \in \mathbb{U}^w$ of ur-programs. We obtain as an immediate consequence.

Corollary 1. *Consider for a measurable map $f : S \to T$ these statements, where $\mathcal{K}$ and $\mathcal{L}$ are pre-models, and $\mathcal{M}$ and $\mathcal{N}$ are models with state spaces S resp. T. Then $f : \mathcal{K} \to \mathcal{L}$ is a pre-model morphism iff $f : \mathcal{K}^\dagger \to \mathcal{L}^\dagger$ is a model morphism, and if $f : \mathcal{M} \to \mathcal{N}$ is a model morphism, then $f : \mathcal{M}_\ddagger \to \mathcal{N}_\ddagger$ is a pre-model morphism.* $\dashv$

Morphisms preserve and reflect the validity of formulas both in sPDL and in the fragment FRAG.

Proposition 7. *Let $f : \mathcal{K} \to \mathcal{L}$ be a pre-model morphism, then $\mathcal{K}, s \models \varphi \Leftrightarrow \mathcal{L}, f(s) \models \varphi$ for each state s in $\mathcal{K}$ and each formula $\varphi \in \mathbf{F}_{\mathsf{sPDL}}$. If $g : \mathcal{M} \to \mathcal{N}$ a model morphism, then $\mathcal{M}, s \models \varphi \Leftrightarrow \mathcal{N}, g(s) \models \varphi$ for each state s in $\mathcal{M}$ and each formula $\varphi \in \mathbf{F}_{\mathsf{FRAG}}$.* $\dashv$

The Test Operator. The usual definition of PDL includes a test operator. With each formula φ a program $\varphi?$ is associated which tests whether φ holds. Intuitively, if the test succeeds, then the program continues. Given a Kripke model $\mathcal{R}$ with S as the set of possible worlds, the set of relations for the interpretation of the logic is extended by the relation $R_{\varphi?} := \{\langle s, s\rangle \mid s \in S, \mathcal{R}, s \models \varphi\}$, see [1, p. 23].

The test operator is integrated into sPDL in this way. Given a Kripke pre-model $\mathcal{K}$ over state space $(S, \mathfrak{B})$ and a formula φ, we define $K_{\varphi?}(s)(B) := I_S(s)(B \cap [\![\varphi]\!]_{\mathcal{K}})$, and $K_{\overline{\varphi}?}(s)(B) := I_S(s)\big(B \cap (S \setminus [\![\varphi]\!]_{\mathcal{K}})\big)$ for $B \in \mathfrak{B}$. $K_{\varphi?}$ is the relation associated with testing for φ, and $K_{\overline{\varphi}?}$ tests whether φ does not hold — recall that the logic is negation free.

Lemma 4. *Let $\mathcal{K}$ be a Kripke pre-model over state space $(S, \mathfrak{B})$ which is closed under the Souslin operation $\mathcal{A}$. Then both $K_{\varphi?}$ and $K_{\overline{\varphi}?}$ are a stochastic relations for each formula φ.* $\dashv$

Thus, if the set $[\![\varphi]\!]_{\mathcal{K}}$ is not measurable, the test operator associated with φ is not representable within the framework of the Kripke model; consequently, we may not be able to test *within this model*, e.g., whether an iteration has terminated. This emphasizes again the importance of operation $\mathcal{A}$.

Example 2. Let $p \in \mathbb{A}$ be an atomic proposition, $\pi \in \mathbb{U}$ an ur-program and φ a formula. Then $[\![[p?;\pi]_q\varphi]\!]_{\mathcal{K}} = \{s \in S \mid K_\pi(s)(V_p \cap [\![\varphi]\!]_{\mathcal{K}}) < q\}$, and $[\![[\pi;\overline{p}?]_q\varphi]\!]_{\mathcal{K}} = \big(S \setminus V_p \cap [\![[\pi]_q\varphi]\!]_{\mathcal{K}}\big) \cup V_p$. $\dashv$

Tests are compatible with morphisms.

Proposition 8. *Let $f : \mathcal{K} \to \mathcal{L}$ be a morphism between pre-models. Then $K^f_{\varphi?} = L_{\varphi?} \circ f$ for every formula $\varphi \in \mathbf{F}_{\mathsf{sPDL}}$.* $\dashv$

This is the reason why we are not interested in this operator when looking into the expressivity of Kripke pre-models is that these operators do not contribute to the relevant properties of this problem.

6 Expressivity

We are able to compare the behavior of two (pre-) models once we know that there exists a morphism between then. This leads to the notion of behavioral equivalence: there exists a reference system onto which the pre-models may be mapped. On the other hand, we can compare pre-models through their theories: two pre-models are logically equivalent iff for each state in one model there exists a state in the other model in which renders exactly the same formulas are true. It will be shown that these notions of expressivity are equivalent, and it is easy to see that behaviorally equivalent models are logically equivalent. The proof for the other direction proceeds technically as follows: we show that logically equivalent pre-models give rise to logically equivalent models, and we know for models that logical and behavioral equivalence coincide. This result is then carried over to the world of pre-models. There is a small catch, though: the equivalence mentioned holds for models of another, albeit closely related, logic. This has to be taken care of.

Define for a state s in pre-model $\mathcal{K}$ the sPDL-*theory* $Th_{\mathcal{K}}(s)$ *of* s as the set of all formulas in $\mathbf{F}_{\mathsf{sPDL}}$ which are true in s, formally $Th_{\mathcal{K}}(s) := \{\varphi \in \mathbf{F}_{\mathsf{sPDL}} \mid \mathcal{K}, s \models \varphi\}$. Similarly, the FRAG-theory $Th_{\mathcal{M}}(s)$ of state s in model $\mathcal{M}$ is defined as all formulas in the fragment FRAG which are true in s; we use the same notation for both logics, trusting that no confusion arises. These sets are closely related. Just for the record:

Lemma 5. *Let $\mathcal{K}$ be a pre-model, then $Th_{\mathcal{K}^\dagger}(s) = Th_{\mathcal{K}}(s) \cap \mathbf{F}_{\mathsf{FRAG}}$ for each state s in $\mathcal{K}$.* $\dashv$

Logical vs. Behavioral Equivalence. The pre-models $\mathcal{K}$ and $\mathcal{L}$ are called *logically equivalent* iff given a state in $\mathcal{K}$, there exists a state in $\mathcal{L}$ which has the same theory, and vice versa. This translates to $\{Th_{\mathcal{K}}(s) \mid s \text{ is a state in } \mathcal{K}\} = \{Th_{\mathcal{L}}(t) \mid t \text{ is a state in } \mathcal{L}\}$. Logical equivalence for models is defined in the same way. We infer from Lemma 5.

Corollary 2. *If the pre-models $\mathcal{K}$ and $\mathcal{L}$ are logically equivalent, so are the models $\mathcal{K}^\dagger$ and $\mathcal{L}^\dagger$.* $\dashv$

The pre-models $\mathcal{K}$ and $\mathcal{L}$ are called *behaviorally equivalent* iff there exists a model $\mathcal{Q}$ and surjective pre-model morphisms $\mathcal{K} \xrightarrow{f} \mathcal{Q} \xleftarrow{g} \mathcal{L}$. In fact, let s be a state in $\mathcal{K}$ and φ be a formula, then there exists a state t in $\mathcal{L}$ such that $f(s) = g(t)$, consequently by Proposition 7 $\mathcal{K}, s \models \varphi \Leftrightarrow \mathcal{Q}, f(s) \models \varphi \Leftrightarrow \mathcal{Q}, g(t) \models \varphi \Leftrightarrow \mathcal{L}, t \models \varphi$.

This accounts for the name, and the argumentation shows

Lemma 6. *Behaviorally equivalent pre-models are logically equivalent.* $\dashv$

Behavioral equivalence is defined for Kripke models in the same way through the existence of a pair of morphisms with the same target.

Lemma 7. *Let $\mathcal{K}$ and $\mathcal{L}$ be behaviorally equivalent pre-models, then $\mathcal{K}^\dagger$ and $\mathcal{L}^\dagger$ are behaviorally equivalent models.* $\dashv$

Logical equivalence and behavioral equivalence are closely related for Kripke models, as the following proposition shows. We have to show that we can find a pre-model for logically equivalent models which serves as a target for morphisms which are defined on the given models. This construction is technically somewhat involved, but we can fortunately draw on the analogous results for another, closely related logic; this result is massaged into being suitable for the present scenario.

Proposition 9. *If the Kripke models $\mathcal{M}$ and $\mathcal{N}$ are logically equivalent, then they are behaviorally equivalent as well.*

Proof. 1. $\mathcal{M} = \big((S, \mathfrak{B}), (M_\pi)_{\pi \in \mathbb{U}^w}, (V_p)_{p \in \mathbb{A}}\big)$ and $\mathcal{N} = \big((S, \mathfrak{C}), (N_\pi)_{\pi \in \mathbb{U}^w}, (W_p)_{p \in \mathbb{A}}\big)$ are the Kripke models under consideration. Construct as in [4, Section 2.6] a measurable space $(H, \mathfrak{E})$ and surjective maps $f : S \to H$ and $g : T \to H$ together with stochastic relations $H_\pi : (H, \mathfrak{E}) \rightsquigarrow (H, \mathfrak{E})$ for $\pi \in \mathbb{U}^w$ with these properties:

i. f is $\mathfrak{B}$-$\mathfrak{E}$-measurable, g is $\mathfrak{C}$-$\mathfrak{E}$-measurable,
ii. $f\big[[\![\varphi]\!]_\mathcal{M}\big] = g\big[[\![\varphi]\!]_\mathcal{N}\big]$ for all formulas $\varphi \in \mathbf{F}_{\mathsf{sPDL}}$,
iii. $[\![\varphi]\!]_\mathcal{M}$ is f-invariant, $[\![\varphi]\!]_\mathcal{N}$ is g-invariant for all formulas $\varphi \in \mathbf{F}_{\mathsf{sPDL}}$ (thus, e.g., $s \in [\![\varphi]\!]_\mathcal{M}$ and $f(s) = f(s')$ implies $s' \in [\![\varphi]\!]_\mathcal{M}$),
iv. $H_\pi^f = M_\pi \circ f$ and $H_\pi^g = N_\pi \circ g$ for all $\pi \in \mathbb{U}^w$,
v. $\mathfrak{E} = \sigma\big(\{f\big[[\![\varphi]\!]_\mathcal{M}\big] \mid \varphi \in \mathbf{F}_{\mathsf{sPDL}}\}\big) = \sigma\big(\{g\big[[\![\varphi]\!]_\mathcal{N}\big] \mid \varphi \in \mathbf{F}_{\mathsf{sPDL}}\}\big)$.

The actual construction in [4, Section 2.6] is carried out, however, for a logic given by the grammar

$$\psi ::= \top \mid \psi_1 \wedge \psi_2 \mid \langle a \rangle_q \psi,$$

where a is taken from an arbitrary non-empty set of actions, $q \in \mathbb{Q} \cap [0, 1]$, and the interpretation of formula $\langle a \rangle_q \psi$ is that $\langle a \rangle_q \psi$ is true in state s iff action a in s leads the model to a state in which ψ holds **with probability at least** q. A careful analysis of the proofs in [4, Section 2.6] (in particular of, resp., Proposition 2.6.8 and Lemma 2.6.15), however, shows that the latter condition may be replaced everywhere by the definition of validity for $[\pi]_q \varphi$ proposed in the present paper, without changing the proofs' substance.

2. Define $X_p := f\big[V_p\big]\big(= g\big[W_p\big]\big)$ for the atomic proposition $p \in \mathbb{A}$, then $X_p \in \mathfrak{E}$. Due to the invariance property of f resp., g, we conclude that both $f^{-1}\left[X_p\right] = V_p, g^{-1}\left[X_p\right] = W_p$. hold. In fact, if $f(s) \in X_p = f\big[V_p\big]$, there exists $s' \in V_p$ with $f(s) = f(s')$. Since V_p is f-invariant, we conclude $s \in V_p$, so that $f^{-1}\left[X_p\right] \subseteq V_p$. The inclusion $f^{-1}\left[X_p\right] \supseteq V_p$ is obvious. 3. Consequently, $\mathcal{H} := \big((H, \mathfrak{E}), (H_\pi)_{\pi \in \mathbb{U}^w}, (X_p)_{p \in \mathbb{A}}\big)$ is a Kripke model with morphisms $\mathcal{M} \xrightarrow{f} \mathcal{H} \xleftarrow{g} \mathcal{N}$. Thus the logically equivalent Kripke models $\mathcal{M}$ and $\mathcal{N}$ are behaviorally equivalent. $\dashv$

Some remarks are in order. ❶ The proofs from [4, Section 2.6] are adapted, they require the logical equivalence of the underlying Kripke models. They factor the Kripke models according to the logic, saying that two states are equivalent iff they satisfy exactly the same formulas in the respective models. The factor spaces are related to each other by glueing the equivalence classes for s in $\mathcal{M}$ and for t in $\mathcal{N}$ iff $Th_\mathcal{M}(s) = Th_\mathcal{N}(t)$; logical equivalence renders this construction permissible. The combined space is essentially the state space for model $\mathcal{H}$. A suitable σ-algebra on $\mathcal{H}$ is constructed, and suitable stochastic relations are defined. For these constructions the underlying logic is required to be closed under conjunctions for measure theoretic reasons. ❷ The observation just made is the technical reason for requiring the logic being closed under conjunctions; the alternative of closing under disjunctions, however, is for measure theoretic purposes not equally attractive. ❸ Incidentally, since we work in a σ-algebra, we do not need negation, which tells us when a formula is *not* true. On the level of models we can state that formula φ does not hold in state s iff $s \in S \setminus [\![\varphi]\!]_{\mathcal{M}}$, which is a member of the σ-algebra over which we are working, hence a member to our universe of discourse, provided $[\![\varphi]\!]_{\mathcal{M}}$ is. ❹ An alternative to the proof presented would have been through a coalgebraic approach by defining for each modal operator $[\pi]_q$ a suitable predicate lifting, and by investigating the corresponding logic over a coalgebra, see [4, Section 4.4] or [5]. Balancing the — considerable — effort of doing so against modifying a construction which works with conventional methods resulted in the proof presented above.

We are ready to prove the first installment of our main result

Proposition 10. *Let $\mathcal{K}$ and $\mathcal{L}$ be Kripke pre-models, then $\mathcal{K}$ and $\mathcal{L}$ are behaviorally equivalent if and only if $\mathcal{K}$ and $\mathcal{L}$ are logically equivalent.* $\dashv$

It may be noted that the result above holds for any measurable space, independent of whether or not the validity sets of the formulas are measurable (which require, by Proposition 3, closedness under the Souslin operation $\mathcal{A}$).

In contrast, relating bisimilarity to the logical and behavioral equivalence makes fairly strong assumptions on the base space.

Bisimilarity. Call the stochastic Kripke pre-models $\mathcal{K}$ and $\mathcal{L}$ *bisimilar* iff there exists a pre-model $\mathcal{M}$ such that $\mathcal{K} \xleftarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{L}$ for suitable pre-model morphisms f and g. Pre-model $\mathcal{M}$ is sometimes called *mediating*. Bisimilarity is a key concept in modal logics as well as in the theory of coalgebras. The reader is referred to [1] and to [10] for comprehensive discussions, where also the relationship to the coalgebraic aspects of Milner's original concept of bisimilar concurrent systems [7] is discussed. An immediate observation is (cp. Proposition 7 and Lemma 6).

Lemma 8. *Bisimilar pre-models are logically equivalent.* $\dashv$

We cannot show in general measurable spaces that logically equivalent models are bisimilar; for this, we have to specialize the base spaces in which we are working to Polish spaces. A topological space (X, τ) is called *Polish* iff it is second countable, and there exists a metric for τ which is complete, see [12]. The σ-algebra $\mathfrak{B}(X, \tau)$ of *Borel sets* for a topological space (X, τ) is the smallest σ-algebra on X which contains the open sets, hence $\mathfrak{B}(X) = \sigma(\tau)$; as usual, we omit the Borel sets, when we talk about a topological space. A map from a topological space (X, τ) to a measurable space $(S, \mathfrak{C})$ is called *Borel* iff it is $\mathfrak{B}(X)$-$\mathfrak{C}$-measurable.

For Polish spaces we can establish the equivalence of all three notions of expressivity.

Theorem 1. *Let $\mathcal{K}$ and $\mathcal{L}$ be Kripke pre-models, and consider these statements.*

a. $\mathcal{K}$ and $\mathcal{L}$ are logically equivalent.
b. $\mathcal{K}$ and $\mathcal{L}$ are behaviorally equivalent.
c. $\mathcal{K}$ and $\mathcal{L}$ are bisimilar.

Then $(c) \Rightarrow (a) \Leftrightarrow (b)$*, and if the state spaces of* $\mathcal{K}$ *and* $\mathcal{L}$ *are Polish spaces, then all three statements are equivalent.*

Proof. 1. The implications (c) $\Rightarrow$ (a) $\Leftrightarrow$ (b) are Proposition 10 together with Lemma 8. Hence the implication b $\Rightarrow$ c remains to be established.

2. Let $\mathcal{K} = (S, (K_\pi)_{\pi \in \mathbb{U}}, (V_p)_{p \in \mathbb{A}})$ and $\mathcal{L} = (T, (L_\pi)_{\pi \in \mathbb{U}}, (W_p)_{p \in \mathbb{A}})$ be behaviorally equivalent Kripke pre-models over the Polish spaces S and T. Then the Kripke models $\mathcal{K}^\dagger$ and $\mathcal{L}^\dagger$ are behaviorally equivalent models by Lemma 7, so there exists a model $\mathcal{M} = \big((Q, \mathcal{H}), (M_\pi)_{\pi \in \mathbb{U}^w}, (X_p)_{p \in \mathbb{A}}\big)$ and model morphisms $\mathcal{K}^\dagger \xleftarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{L}^\dagger$. We infer from [5, Proposition 6.18] that we may assume Q to be a separable metric space with $\mathcal{H} = \mathfrak{B}(Q)$. Put $Y := \{\langle s, t\rangle \mid s \in S, t \in T, f(s) = g(t)\}$, and let $\beta : Y \to S, \gamma : Y \to T$ be the projections. Since f and g are surjective and Borel, β and γ are surjective and Borel. We infer from [6, Theorem 3.8] that we can find for each $\pi \in \mathbb{U}^w$ a stochastic relation $N_\pi : Y \rightsquigarrow Y$ such that $K_\pi^\dagger \circ \beta = N_\pi^\beta, L_\pi^\dagger \circ \beta = N_\pi^\gamma$. Define $Z_p := Y \cap V_p \times W_p$ for the atomic proposition $p \in \mathbb{A}$, then $\mathcal{N} := \big(Y, (N_\pi)_{\pi \in \mathbb{U}^w}, (Z_p)_{p \in \mathbb{A}}\big)$ is a Kripke model with $\mathcal{K}^\dagger \xleftarrow{\beta} \mathcal{N} \xrightarrow{\gamma} \mathcal{L}^\dagger$. Consequently, the pre-model $\mathcal{N}_\ddagger$ is a mediating pre-model for $\mathcal{K}$ and $\mathcal{L}$. $\dashv$

The crucial step in the proof is the existence of the stochastic relations N_π for each $\pi \in \mathbb{U}^w$. This actually requires some heavy machinery from measurable selection theory which is available in Polish spaces, but not in general measurable spaces.

Thus we have carried over the result of the equivalence of logical equivalence, bisimilarity and behavioral equivalence. It has been established for the logics mentioned in the proof of Proposition 9 (they are sometimes called *Hennessy-Milner logics*) for the case that the underlying space is Polish, and that there is a countable set of actions, see [4, Section 2.3]. The latter assumption is made in order to make sure that the factor spaces which are needed for the constructions are well-behaved. Note that by separating concerns we did not need an assumption on countability, and that bisimilarity only required the assumption on a Polish base space.

7 Conclusion

We propose a probabilistic interpretation for sPDL, the modal operators of which are given by $[\pi]_q$ for a program π with the intended meaning that $[\pi]_q\varphi$ holds in a state s if a terminating execution of program π in state s will reach a state in which formula φ holds has a probability not greater than q. This deviates slightly from the usual probabilistic interpretation of modal logics, see [3,4], because the proposed interpretation is more adequate for the present logic. It is shown that behavioral and logical equivalence are the same, they are equivalent to bisimilarity in the case of models based on Polish spaces.

The models we investigate do not show an interpretation for each modal operator. We have to generate the interpretation from the model for a fragment. The technique seems to be interesting in itself. The question arises whether it can be applied to dynamic coalgebraic logics, i.e., to dynamic logics in which the modal operators are given through predicate liftings. Coalgebraic logic has recently attracted some interest [11,4,5] as a unifying and powerful generalization of modal logics. Another promising line is the exploration of ideas pertaining to the probabilistic interpretations of games.

The interpretation proposed here shows some rough edges: First, the semantics of formulas $[\pi^*]_q\varphi$ is defined through an iterative fixed point rather than a smallest one, cp. Example 1 and Proposition 2. This keeps it in line with the interpretation of sPDL in modal logics [2, Sections 6.6, 6.7], but in contrast to a very similar approach in game logics [9], see also the discussion of belief structures in [8]. This topic will have to be investigated further. Second, we establish bisimulations only for Kripke models on Polish spaces; on the other hand, the extent of formula $[\pi^*]\varphi$ can only be shown to be measurable if the underlying measurable space is closed under the Souslin operation $\mathcal{A}$. But non-discrete Polish spaces are *never* closed under this operation. This follows from the observation that in such spaces there exist analytic sets which are not Borel measurable; the collection of analytic sets is closed under operation $\mathcal{A}$, however [12, Theorems 4.1.5, 4.1.13]. Third, we propose an interpretation of FRAG through a stochastic Kripke model, but then we continue in a nondeterministic fashion. We will investigate how this nondeterminism can be replaced by a purely stochastic approach.

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