Intuitionistic Modal Logic: A 15-year retrospective

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Intuitionistic Modal Logic:
A fifteen year retrospective

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The series of workshops on Intuitionistic Modal Logic and Applications (IMLA) owes its existence to the hope that philosophers, mathematical logicians and computer scientists would share information and tools when investigating intuitionistic modal logics and modal type theories, if they knew of each other's work.

More than ten years have passed since the retrospective view of de Paiva, Goré, and Mendler [8], and progress in the area of constructive modal logic has been slow and getting slower. It is our view that differences in the outlook of the various groups of scholars interested in the topic, differences that were once fruitful, now are responsible for a tendency for the new work to be driven by technical issues that have not had wide interest, leading to compartmentalisation and waning interest in the IMLA's big tent. Work on modal type theories seems to have been pursued in narrow tracts. For instance, much work in the symposium on Principles of Programming Languages (POPL), in specific type systems could be considered work in applied constructive modal logic, but this aspect is neglected because it is not considered useful or productive. Generally speaking, topic specialists have stopped expecting outsiders to say anything of interest to them, so they do not make the effort to say anything of interest to outsiders.

This is a shame, since there are big questions in the field that hold broad interest that have not been answered and that might benefit from the breadth that once characterised the field. To this end, we offer a new retrospective of the field, aimed at giving a sense of the dynamics that have driven compartmentalisation, and make the case for reengagement with certain broad issues which offer all parties the hope of systematic theories that are much better suited to all of their needs.

The IMLA interest groups

In the past, IMLA has been a venue for several groups of scholars (call them the interest groups) motivated by quite divergent problems. To pick out the interest groups that might have most visibility outside their subfield, consider:

1. Philosophers exploring aspects of meaning theories and epistemology: modal logic has offered difficult and important problems both for inferentialist theories in the Gentzen–Belnap–Prawitz–Dummett vein [24] and for accounts of belief justification that can handle the many modal epistemological paradoxes;

2. Logicians interested in *good proof calculi* for both intuitionistic and classical modal logics in parallel. There are many conferences and communities dedicated to **classical** modal logic. Advances in Modal Logic (`http://www.aiml.net/`), a collection of bi-annual meetings and books, and Methods for the Modalities (`http://m4m.loria.fr`) are generically about modal logic, while DEON (International Conference on Deontic Logic in Computer Science) and TIME (`http://time.di.unimi.it/TIME_Home.html`) have a yet narrower focus. There are also several conferences on description logics, hybrid logics and the many venues for multi-agent logics, which are related to modal logic. But very little in this modal logic world is about **intuitionistic or constructive** modal logics, although the proof calculus research would benefit from this connection.

   We include in this interest group logicians and computer scientists interested in decision algorithms and complexity bounds for modal logics. Tableau methods have been the most important single technique in this area, and it is well known that they are closely related to sequent calculi. Several groups work on tableau methods for automating proofs in modal logic, see for instance the conference TABLEAUX `http://i12www.ira.uka.de/TABLEAUX/`.

3. Mathematicians interested in categorical logic have been forced to confront the issues of constructive modal logic very early on. The rise of categorical logic has been rich with connections with modal logic, from Grothendieck's topologies [3] to Kripke–Joyal semantics [20], passing through the uses of the double-negation modality in topos theory. These various appearances of an intuitionistic modality that has some characteristics of necessity and some of possibility has been discussed by Goldblatt [16, 15]. Goldblatt explains how there is more to intuitionistic modal logic than will be found by generalising the behaviour of Boolean necessity and possibility. Operators like modalities over an intuitionistic basis include other phenomena that lose their significance when the law of excluded middle is present. In particular, the modality denoted by Goldblatt as ∇ [16] and considered by him as a necessity operator in 1981, was considered by Curry in 1952 a possibility operator in Gentzen-style modal logic [12]. This system reemerged as Propositional Lax Logic in the work of Fairtlough and Mendler [11] in 1997 and as CL-logic in the work of Benton, Bierman and de Paiva [4], motivated by Moggi's Computational Lambda Calculus [22].

4. Computer scientists interested in type theories for both programming languages and compiler optimisation: particularly since Moggi's seminal

work, categorically inspired type theories for the lambda calculus and programming languages have made modal logic fundamental in this area.

Also, intuitionistic and constructive modal logic proved to be useful for computer scientists interested in logic-based security, especially on logics for access control. There is a short survey of the uses of logic in access control due to Abadi [2] as well as a more complete tutorial [1]. One of the basic ideas is to consider the operator `P says A` (where `P` is a principal, i.e. an agent and `A` a proposition) as a modal operator, sometimes as a constructive S4 necessity operator. Garg's doctoral dissertation for modal theories that are classical. This expansion of the area of interest is natural: the "modal" in modal logic has since at least Prior encompassed topics outside the logic of mode such as tense logic, and it has done so for both technical and philosophical reasons, given the similarity of the kind of questions of interest.

Not all of these interest groups discussed above have had fruitful relationships with all of the others, but all have benefitted from the work in classical modal logic that has provided one of the foundations of IMLA. Broadly speaking, what has made classical modal logic coherent and useful has been its provision of a general and highly applicable toolkit, one that has allowed the dizzyingly diverse range of modal theories to be made tractable and manageable. This view of modal logic in terms of model-theoretic, an algebraic and a computational toolkits, applied to issues of completeness, computability, and complexity of modal systems, and related to each other by correspondence theory and algebraic dualities has been advanced by Blackburn, de Rijke and Venema [5], and has been called the 'modern approach to modal logic'. It is clear that we are far from having these kinds of results, and these kinds of toolkits (model-theoretic, algebraic and computational), so far, for constructive modal logic.

One of the challenges facing modal logic in general, and intuitionistic modal logic in particular, is the lack of a 'standard toolkit' similar to the one existing for the proof theory of first-order logic. Some of the pieces that make up the 'standard' toolkit used for first-order logic are missing. In particular we would want, besides the Hilbert-style systems of axioms that do exist, sequent calculus and natural deduction formulations for all the usual systems of modal logic. These, together with the usually associated 'good properties', e.g., cut elimination and the subformula property of sequent calculus, (weak and strong) normalization of Natural Deduction proofs, existence of well-behaved reduction relations, type preservation and Church-Rosserness of these reduction relations between proofs, are missing for either constructive or classical modal logic.

The structural proof theory of modal logic, the part of proof theory concerned with analytic proofs following Gentzen, is much involved than that for first-order logic: to the extent that proof theories exist that can handle most of the modal theories in use, they are more complicated than those most familiar for the first-order case. The proof theories that do exist either treat only a few modal theories or have not been provided with a constructive basis comparable to the formulae-as-types correspondence for constructive type theory.

The Gentzenian proof theory for intuitionistic/constructive modal logic that exists, however, has been quite fruitful. Moggi's work on the computational lambda calculus provides a sense of what can be achieved: the computational model simply is, once the details have been tied up, a proof theory in the formulae-as-types mould for a variant of intuitionistic S4 (where necessity and possibility collapse into a single modality), and this correspondence is quite as pleasing as the paradigmatic cases of the formulae-as-types correspondence often given in propositional and predicate logic [4]. The insights from categorical logic that guided development of this theory enrich it still further, through the relationship to Lawvere's theory of algebras and their application to understanding how to combine the pure lambda calculus with the modelling of imperative computational features, as described for instance in [19].

The schools of intuitionistic modal logic

There are many systems of constructive modal logics, even more so than in classical modal logic. This much is expected, as the constructivization of a notion, usually creates several possibilities that need to be compared. Some relationships are known between systems, e.g. the famous "cube" of modal logics, appearing in [14] for instance, can and has also been considered, recently for intuitionistic and constructive modal logics [27], [21]. Pros and cons are better discussed in terms of specific applications and while we started by mentioning some fields of expertise and applications, many more exist and it seems more productive to discuss them at a higher-level of abstraction.

The proof theory of modal logic is much more subtle and complicated than the one for first-order logic, either classical or intuitionistic. Since one cannot have all the canonical good properties of first-order logic we would like to, choices and compromises have to be made. These choices are the origins of some of the schools below:

- Nerode's temporal modal logics for hybrid systems, especially with Wijesekera (Constructive Concurrent Dynamic Logic [28]) and Davoren (Topological semantics for Intuitionistic modal logics [7]) seems the first to advocate intuitionistic modal logics for computer science;

- Artemov's Justification Logics (including the original Logic of Proofs) is a program unfolding since the early 90s. Justification logic is a refinement of modal logic which studies the concepts of knowledge, belief, and provability. The single modality from the modal language is replaced by a family of justification terms. While a modal formula $\Box A$ can be read as 'A is known/believed', or 'A is provable', a justification counterpart $t : A$ of this formula is read as 'A is known/believed for reason t' or 't is evidence for A', where t is a justification term.

- Judgemental Modal Logic: Pfenning's very influential school of constructive modal logic is based on the design of expressive type systems for practical programming languages which allow for more program errors to be caught at compile-time without sacrificing conciseness or efficiency of programs. Most of the systems are based on an intuitionistic dual version of S4, described in [6, 23].

- Wolter and Zakharyaschev's framework: in a series of papers [31, 29, 30], Wolter and Zakharyaschev extended well-known results on the relationship between non-modal superintuitionistic logics and classical modal logics to modal intuitionistic logics and multi-modal modal logic. They provided some general results on embedding intuitionistic modal logics with n modalities into classical modal logics with $n+1$ modalities. They showed that the embedding reflects decidability, Kripke completeness, the finite model property and tabularity, and used it together with some general results on classical multi-modal modal logics to prove decidability and finite model property for a range of intuitionistic modal logics. Their results provide a useful tool for modal logicians but they are not applicable for certain definitions of $\Box$ and $\Diamond$ in more applied intuitionistic modal logics.

- Separation Logic: Reynolds and others have provided separation logic [25], an extension of Hoare logic, a way of reasoning about programs, using modalities indexed by program constructs. The assertion language of separation logic is a special case of the logic of bunched implications (BI). Separation logic facilitates reasoning about programs that manipulate pointer data structures, "transfer of ownership" (avoidance of semantic frame axioms); and virtual separation (modular reasoning) between concurrent modules. The ideas of separation logic have been very influential and there are many systems based on the original Reynolds conception, adapted to several specific problems of verification of properties of programs.

- Curry–Howard for Constructive Modal Logics. Another less cohesive branch on the work on intuitonistic modal logics is based on the priority of the Curry–Howard correspondence as an organising tool for logic, lambda-calculi and categorical structures. This work had a head start on the others (Mendler and Fairtlough, Benton et al, Bierman and de Paiva, Alechina et al, Mendler and Scheele)

Highly flexible proof calculi

The Gentzenian proof theory for modal logics that exists has been very useful, but there are limits to how far this kind of proof theory can take us. The poster child for the difficulties associated to a Gentzenization of modal logic is the system S5 (as well as system B). Stouppa [26] surveyed the existing attempts to provide proof theories of S5: the theories that could give a clean account for S5 made use of non-traditional proof-theoretic devices. The three most flexible proof approaches were

1. Simpson-style labelled calculi: labelling formulae, allowing fine constraints to be placed on the application of inference rules, constraints that need

not be local to those rules, is a technique that comes naturally if we want to tie our proof calculus closely to tableau systems, and the earliest labelled calculi arose in this way. Using such labelled rules gives enormous (and enormously abusable) flexibility, but highly principled uses of labels exist, such as those in Simpson (1994), where the labels represent points in frames, and the application of labels in inference rules comes directly from frame conditions. The kind of account Simpson gives is quite close in many respects to traditional Getzen proof calculi, but labels can greatly complicate proof analysis, since labels can constrain the applicability of inference rules in ways that are arbitrary from the point of view of traditional proof analysis.

   Unlike the older proof theories for such modal theories as K, T, and S4, the characterisation of the modalities is not done entirely in the introduction and elimination rules (or left/right rules for sequent calculi); the Simpson calculi have additional rules that deal with relationships between labels.

   Theories of labelled natural deduction can give rise to constructive interpretations via the formulae-as-types correspondence, but the naturality and applicability of these theories needs to be shown;

2. Display logic: driven by concerns about what it meant to specify a logical connective by its inference rules, display logic normally gives a very uniform characterisation of logical connectives that is related to natural deduction, but has a quite different kind of structural rule. Since Kracht, display logic has had a powerful, uniform technique for characterising modal proof theories.

3. Calculus of structures: initially formulated to try to give a proof theory for the noncommutative pomset logic of Retore, the calculus of structures rejects many of the inferentialist concerns that have characterised structural proof theory in an attempt to obtain algebraic simplicity. Attempts have been made to provide general schemes for characterising modal theories in the calculus of structures, although the theory remains less well-developed than for display logic.

These represent programmes of investigation into the proof theory of modal logic beyond the older and less general approaches. This diversity has made progress difficult by making the value of results in any particular framework less clear. For several of our groups of research interest, the situation has been frustrating:

- The failure of traditional proof theories to characterise many modal logics is an interesting fact if we are concerned with inferentialism, but the failure to identify a successor framework for proof theory makes identifying valuable positive contributions harder;

- The distance between the modal proof and type theories that computer scientists have found applicable and the theories taken to have broader logical interest has been growing.

- Established application areas are still productive, but the state in IMLA overall may have reduced the attractiveness of decidability results problems relating to modal theories.

What kind of work supports IMLA at present?

If we say the health of IMLA as an interdisciplinary interest area stems from, on the one hand, the investigations into the specific applications that drive the interest of the field and, on the other, the systematic work that builds the toolkit that supports the applied investigations and gives the field coherence, we get a picture of what kind of research can particularly support IMLA. Development of specific applications is always helpful, and the IMLA workshop showed a strong representation motivated by the concerns around the family of epistemic logics, but the problem areas has lain particularly on the systematic side. We suggest four areas, with some continuity with [8]:

Identifying constructive bases for the highly flexible modal calculi

Display logic and the calculus of structures have difficulty supporting a constructive basis in the Curry–Howard "proof terms as constructions" mold, because inference in the calculi make use of many symmetries and permutations. Finding a constructive basis for these theories might best be pursued by finding a family of constructions that can act as realisers for proofs.

The rich spectrum of proof calculi

The diversity of calculi that have been proposed to give an account of analytic proof for modal theories has suggested the fertility of the area as a problem domain, and the emergence of highly flexible proof theories have opened up the question of what a proof theory should look like. The confusion this diversity brings can be greatly ameliorated with yet more diversity, showing how different calculi can follow a pattern, or give a bridge to understand the relationship between particular calculi. There has been some fruitful work in this vein

- Hein and Stewart [18] introduced a notion of *unravelling labelled proofs* that apparently allows many Simpson-style labelled caculi to be turned into unlabelled proofs in the calculus of structures. It is conjectured that this technique can be made to work to create similar proofs in display logic.

- The tree-sequents and nested sequent calculus we see in Galmiche's contribution is like a fugue of CoS and sequent calculus, and brings this kind of work back to intuitionistic logic.

- Goré and Tiu [17] have given preliminary results outlining a relationship between proof theories in the calculus of structures and display logic.

More flexible theories of meaning

The Gentzen–Prawitz–Dummett inferentialist account of proof-theoretic meaning has been a fruitful contact point between structural proof theory and core problems in analytical philosophy. The advent of highly flexible proof theories, however, threatens this relationship, since the account has been closely tied to the mechanics of natural deduction, and open-ended questions about which proof theories work best can be embarassing to narrow assumptions about what a proof theory looks like.

An entirely different kind of problem with this inferentialist account has arisen from the objection that inferentialism provides an account of concepts that is not plausible. Doubts of this kind are quite old, but Dutilh Novaes [10] presented a particularly sharp attack, based on research in psychology, some quite striking, that shows how much actual inference is driven by pragmatic, circumstantial understanding, and how little by formal deduction. Are inferentialists committed to a use-based theory of meaning that isn't grounded in use?

There are at least a few possible responses to this criticism, but generally we think inferentialism needs to be broader and more flexible to accomodate both formalist and non-formal patterns of reasoning. To paraphrase Dummett [9], inferentialism (verificationism or pragmatism) requires "middle axioms" to mediate the higher-level philosophical concerns with the basic mathematical results.

An algebraic perspective

Scholars hailing from less mathematical backgrounds, and sometimes scholars with a very deep background in mathematics, have often criticised the use of algebra, and particularly category theory, for the high level of technicality it brings and the difficulty there is justifying the choices that algebraic logicians make in their formalisms without reference to internal matters of taste.

The adoption, say, of mathematics to a central place in physics was attended by this kind of concern, and occurred because of the fruitfulness of the insights outweighed these costs that came from mathematisation.

Constructive algebraic methods may be made for the particular problems IMLA has at present: it is very good at handling the kinds of variation involved in modal inference, and it (especially categorical logic) provides tools for reconciling different kinds of (suitably presented semantics). It may be that the kind of middle axioms we suggested above that would benefit theories of meaning might most easily come from an algebraic perspective.

Papers in this volume

Hansen, Bolander and Brauner define a family of many-valued hybrid logics where truth values form a finite Heyting algebra. The authors introduce sound and complete tableau systems for the logics and show that their tableau construction terminates and hence the logics are decidable. The authors argue that, since hybrid logic is a way of making modal logic more well-behaved proof-theoretically and since there are interesting applications of multi-valued modal logic such as Fitting's multi-expert systems ([13]), one ought to produce a well-behaved many-valued hybrid logic and they proceed to do so. Their logics are intermediate between intuitionistic hybrid logic and classical hybrid logic, much as Fitting's many-valued modal logic is intermediate between classical modal logic and intuitionistic modal logic. The motivation for these intermediate (modal and hybrid) logics is that these many-valued truth values can be used to model systems where a group of experts have individual opinions on the truth of modal issues, and where there is a relation of dominance between these experts. The truth-values can then be taken to be subsets of experts and the truth-value of a formula is the subset of experts who accepts the formula as true.

Bavera and Bonelli study the computational interpretation of Justification Logic where formulas of the form $[[t]]A$ are interpreted as stating that t is a type derivation justifying the validity of type A. Thus Justification logic refines modal logic by inserting into the logic formulas new terms for justifications. The resulting lambda calculus has history-aware computations, which makes it potentially suitable for security applications. Bavera and Bonelli then describe one such application to audit trails. The main technical challenge encountered in the normalization proof is to conciliate the use of different but "compatible justifications. This is solved by incrementing the logic with a new judgment $Eq(A, s, t)$ denoting that the justifications s and t are compatible in some context.

The paper by Bellin, Carraro & Chiffli provides a continuation of Bellin's project of using Linear Logic, Intuitionistic Logic and Classical Logic systems to produce logical models of pragmatic notions such as assertions, conjectures, justifications and objections. The paper in this volume is an account in this vein, concerned in part with providing a recipe for how to blend these different logics with *co-intuionism*, a system formally dual to Intuitionistic Logic, where instead of implication, we have a notion of co-implicaton or subtraction. Systems are provided with term assignments, Curry-Howard style and cut elimination theorems are proved. This whole project started with Dalla Pozza and Garola [] pragmatic interpretation of intuitionistic logic, where sentences and proofs formalize assertions and their justications and revise it and extended it into very many directions. The new systems seem a bit unwieldy, though.

Finally Galmiche and Salhi, building up on the doctoral work of Salhi, under Galmiche, produce a new version of label-free sequent calculi for intuitionistic modal logics obtained from the combinations of the axioms T, B, 4 and 5. These calculi use multi-contextual sequent structures, called Tree-sequents, based on the ideas of deep inference and the calculus of structures and are shown to work for S5 and for many other combinations of classical axioms. The use of tree-sequents also provides new decision procedures and alternative syntactic proofs of decidability of the intuitonistic modal logics obtained from the combinations of the axioms T, B, 4 and 5. While it is clear that these systems are valuable, it is not clear whether these tree-sequents have other virtues, as far as uniformization and generalization of sequent calculi are concerned or if they are a minor variation of the theme of additions to sequents to make cut-elimination work.

Altogether this collection brings together a very interesting body of work on the subject of intuitonistic modal logics, but one which can still leave the reader with a feeling of looking at a landscape of fragments of a bigger and deeper picture. As we have seen, the landscape of IMLA is littered with half-finished bridges connecting these fragments: the task of finishing these constructions is left as an exercise for the IMLA community.

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