Annals of Pure and Applied Logic 50 (1990) 271–301
North-Holland

CONSTRUCTIVE MODAL LOGICS I

Duminda WIJESEKERA*
Mathematical Sciences Institute, Cornell University, Ithaca, NY 14853-2602, USA

Communicated by A. Nerode
Received 9 November 1989; revised 11 July 1990

We often have to draw conclusions about states of machines in computer science and about states of knowledge and belief in artificial intelligence (AI) based on partial information. Nerode (1990) suggested using constructive (equivalently, intuitionistic) logic as the language to express such deductions and also suggested designing appropriate intuitionistic Kripke frames to express the partial information. Following this program, Nerode and Wijesekera (1990) developed syntax, semantics and completeness for a system of intuitionistic dynamic logic for proving properties of concurrent programs. Like all dynamics logics, this was a logic of many modalities, each expressing a program, but in intuitionistic rather than in classical logic. In that logic, both box and diamond are needed, but these two are not intuitionistically interdefinable and, worse, diamond does not distribute over 'or', except for sequential programs. This also happens in other contemplated computer science and AI applications, and leads outside the class of constructive logics investigated in the literature. The present paper fills this gap. We provide intuitionistic logics with independent box and diamond without assuming distribution of diamond over 'or'. The completeness theorem is based on intuitionistic Kripke frames (partially ordered sets of increasing worlds), but equipped with an additional, quite separate accessibility relation between worlds. In the interpretation of Nerode and Wijesekera (1990), worlds under the partial order represent states of partial knowledge, the accessibility represents change in state of partial knowledge resulting from executing a specific program. But there are many other computer science interpretations. This formalism covers all computer science applications of which we are aware. We also give a cut elimination theorem and algebraic and topological formulations, since these present some new difficulties. Finally, these results were obtained prior to those in Nerode and Wijesekera (1990).

1. Syntax, semantics and completeness

1.1. Introduction

We develop propositional and predicate modal intuitionistic logic based on minimal axioms for the modal connectives box (necessity) and diamond (possibility). We give a correspondingly general modal intuitionistic Kripke semantics with completeness theorems. The minimality of the axioms is indicated by the observation that, with these axioms, box and diamond are not interdefinable. In fact, they should not be interdefinable if an intuitionistic meaning is given to their 'possible worlds' interpretations. (See also [2] for a treatment of independent box and diamond. Their system was based on purely philosophical considerations, not

* Supported by NSF contract DMS-8902797 and ARO grant DAA29-85-C018.

computer science. Our systems are motivated instead by intended computer science applications to areas such as dynamic logic and the logic of belief.) Our minimal set of axioms and semantics are general enough to cover the natural axioms and semantics of all intended computer science areas of application of which we know, while still meeting stringent requirements of constructivity. Our systems are fully capable of being implemented naturally in such constructive programming environments as Constable's NUPRL (an implementation of extensions of Martin-Löf's predicative intuitionistic type theory) or Huet–Coquand's CONSTRUCTIONS (an implementation of extensions of Girard's impredicative intuitionistic type theory). For other work on intuitionistic modal logics and their applications, see [3, 6–10, 13, 28, 29, 33, 37].

The intuition behind our semantics is that a possible world for the intuitionistic Kripke model should represent a *partial* state of knowledge about a full classical relational system. Further, the modality should be based on an accessibility relation between these *partial* states of knowledge. So for worlds (partial states of knowledge) w, v, $w \leqslant v$ has the intuitive interpretation that partial state of knowledge v has at least as much knowledge as partial state of knowledge w. Let us use the computer science special case of constructive concurrent dynamic logic (CCDL) of [27] to motivate our minimal system of intuitionistic modal logic more completely. This is historically inaccurate, since our minimal system was developed first, at Nerode's suggestion, but CCDL is too good a motivation to omit. Our minimal system abstracts the intuitionistic modal logic of a single imperative program P executed on a single nondeterministic machine, sequential or concurrent. This minimal system, unlike dynamic logic, wholly omits all apparatus for building up complex programs from simple programs and concentrates instead on the intrinsic modal intuitionistic logic of the transition, or accessibility, relation R from the state S_1 to state S_2, where $S_1 \, R \, S_2$ say P, starting on the machine in state S_1, has at least one execution sequence completing execution in states S_2. In CCDL [27], a state S_1 on which R acts, represents partial information, rather than total information, about a complete machine state S. (In classical dynamic logic, only complete machine states are considered.) In the intuitionistic Kripke frames used in CCDL, each world represents partial information about a single complete machine state. If worlds S_1, S_2 represent partial information about the same machine state S, then in the Kripke frame partial ordering $\leqslant$ of worlds, $S_1 \leqslant S_2$ iff S_2 contains at least as much information about state S as S_1 does. If T_1 is a world representing partial information about a complete machine state T other than the complete machine state S referred to above, then T_1 is incomparable under $\leqslant$ with the $S_1, S_2, \ldots$ representing partial information about S. For classical dynamic logic, the abstract features of a single program accessibility are reflected algebraically by a modal Boolean algebra. There should be a corresponding notion of an intuitionistic modal algebra, corresponding to the intuitionistic modal logic of the accessibility relation of a single program acting on partial information about machine states. Our minimal modal intuitionistic logic has this role. Its algebraic form is modal Heyting algebra, developed in a later section. Finally, our minimal system is broad enough to be used in other computer science and artificial intelligence applications under development jointly with Nerode.

Axioms, rules of inference, Kripke semantics, algebraic semantics, and topological semantics are developed in Sections 1 and 2. Constructive content is brought out by a cut elimination theorem for an LJ-style calculus in Section 1.2. Soundness of axioms and rules of inference with respect to the intended semantics is proved in Section 1.3. Completeness is proved in Section 1.4. The cut rule is shown to be eliminable in Section 1.5; cut elimination is useful for suggesting term rewriting implementations of modal intuitionistic logics in high level functional programming languages such as Constable's NUPRL, or Huet–Coquand's CONSTRUCTIONS

Language

The language is that of classical predicate modal logic with $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\exists$, $\forall$, $\Box$ and $\Diamond$ as logical connectives. The deductive theory will be constructive (intuitionistic). The semantics will be Kripke intuitionistic frames, equipped with an additional accessibility relation between worlds.

Notational conventions. Lower-case letters from the beginning of the alphabet, possibly with integer subscripts, denote individual constants. C_0 is the set of individual constants. Upper-case letters from the beginning of the alphabet, possibly with integer subscripts, denote atomic predicates. Lower-case letters from the end of the alphabet, possibly with integer subscripts, denote individual variables. X is the set of all variables x_i. Lower-case Greek letters stand for formulas. Upper-case Greek letters stand for sets of formulas.

Definition 1.1.1. *Terms* are individual variables or individual constants.

Definition 1.1.2. Suppose that $A(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$ is an atomic formula, and that $t_i, \ldots, t_j$ are terms. Then $A(x_1, \ldots, t_i/x_i, \ldots, t_j/x_j, \ldots, x_n)$ is called the *instantiation* of $x_i, \ldots, x_j$ to $t_i, \ldots, t_j$ in the atomic formula $A(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$.

Definition 1.1.3. The *inductive definition of formula* is as follows.

(1) Every instantiation of every atomic formula is a formula.

(2) If ψ, θ are formulas, then so are $(\neg\psi)$, $(\psi \rightarrow \theta)$, $(\psi \wedge \theta)$, $(\psi \vee \theta)$, $(\Box\psi)$ and $(\Diamond\psi)$.

(3) If $\psi(x)$ is a formula, then so are $((\exists x)\psi(x))$ and $((\forall x)\psi(x))$.

We omit parentheses for readibility. We follow standard usage for the notions of free and bound variables.

Definition 1.1.3 is easily extended to an arbitrary instantiation of terms to free variables in an arbitrary formula. We omit the details. Let $\psi(x_1, \ldots, t_i/x_i, \ldots, t_j/x_j, \ldots, x_n)$ denote the instantiation of the terms $t_i, \ldots, t_j$ for the free instances of the variables $x_i, \ldots, x_j$. Finally as usual, a *sentence* is a formula with no free variables.

Semantics

Definition 1.1.4. A quintuple $\langle K, \leqslant, D, R, \Vdash\rangle$ is an *intuitionistic modal frame* if

(1)(i) $(K, \leqslant)$ is a partially ordered set and R is any binary relation on K;

(ii) D is a function assigning nonempty sets to the elements of K satisfying: for all $k, k' \in K$, $k \leqslant k'$ implies $D(k) \subseteq D(k')$ and for all $k, k' \in K$, $k\,R\,k'$ implies $D(k) \subseteq D(k')$.

(2) With additional constant symbols added to the language for each element of $D = \bigcup\{D(k): k \in K\}$, we assume given a relation $\Vdash$ is a relation between elements of K and atomic statements in this extended language such that

$$k \Vdash A(\boldsymbol{d}_n) \quad \text{implies} \quad d_i \in D(k) \quad \text{for } 1 \leqslant i \leqslant n,$$

$$k \Vdash A(\boldsymbol{d}_n) \quad \text{implies} \quad k' \Vdash A(\boldsymbol{d}_n) \quad \text{for all } k' \geqslant k.$$

(3) Then $\Vdash$ is extended to all sentences φ in the extended language allowing constants in $D(k)$ by the inductive definition below.

(i) $k \Vdash \varphi \vee \psi$ if $k \Vdash \varphi$ or $k \Vdash \psi$.

(ii) $k \Vdash \varphi \wedge \psi$ if $k \Vdash \varphi$ and $k \Vdash \psi$.

(iii) $k \Vdash \varphi \rightarrow \psi$, if, for all $k' \geqslant k$, $k' \Vdash \varphi$ implies $k' \Vdash \psi$.

(iv) $k \Vdash \neg\varphi$ if, for all $k' \geqslant k$, it is not the case that $k' \Vdash \varphi$.

(v) $k \Vdash \Box\varphi$ if, for all $k' \geqslant k$ and for all k'' such that $k'\,R\,k''$, we have that $k'' \Vdash \varphi$.

(vi) $k \Vdash \Diamond\varphi$ if, for all $k' \geqslant k$, there is a k'' such that $k'\,R\,k''$ and $k'' \Vdash \varphi$.

(vii) $k \Vdash \exists x\, \varphi(x)$ if there exists a $d \in D(k)$ such that $k \Vdash \varphi(d)$.

(viii) $k \Vdash \forall x\, \varphi(x)$, if, for all $k' \geqslant k$ and for all $d \in D(k')$, $k' \Vdash \varphi(d)$.

Satisfaction

Definition 1.1.5. Let $M = \langle K, D, \leqslant, R, \Vdash\rangle$ be an intuitionistic modal frame.

(1) A sentence φ is *satisfied* at $k \in K$ if $k \Vdash \varphi$. A set Γ of sentences is satisfied at k if every member of Γ is satisfied at k.

(2) φ is *satisfiable* in $M = \langle K, D, \leqslant, R, \Vdash\rangle$ if there is $k \in K$ with φ satisfied at k.

(3) φ is *satisfiable* if there is an intuitionistic modal frame M in which φ is satisfiable.

(4) If Γ is a set of statements, $\Gamma \Vdash_M \varphi$ if for each $k \in K$, $k \Vdash \Gamma$ implies $k \Vdash \varphi$.

(5) $\Gamma \Vdash \varphi$ if for all intuitionistic modal frames M, $\Gamma \Vdash_M \varphi$.

Notice that our definition of $\Gamma \Vdash \varphi$ is 'local' to one k at a time. It does *not* say that if M is any intuitionistic modal frame and for all k in K all ψ in Γ are satisfied at k, then for all k in K, φ is satisfied at k. This is a necessary change from most modal logic literature, better suited for such applications as dynamics logic in computer science. Now we prove the fundamental elementary property of any intuitionistic forcing relation.

Lemma 1.1.6 (Monotonicity lemma). *Let $M = \langle K, D, \leqslant, R, \Vdash\rangle$ be an intuitionistic modal frame. Let $k, k' \in K$ and assume that $k \leqslant k'$. Let φ be any sentence. If $k \Vdash \varphi$, then also $k' \Vdash \varphi$.*

Proof. The proof is as usual by induction on the length of sentences. We give only the inductive cases involving modal connectives.

For $\Box$. Suppose that $k \Vdash \Box\varphi$ and that $k' \geqslant k$. Let k'' and k''' be such that $k'' \geqslant k'$ and $k'' R\, k'''$. Then $k \leqslant k''$. Hence $k \Vdash \Box\varphi$ implies that $k''' \Vdash \varphi$.

For $\Diamond$. Suppose that $k \Vdash \Diamond\varphi$ and that $k'' \geqslant k' \geqslant k$. Then $k'' \geqslant k$ implies that there is a k''' such that $k'' R\, k'''$ and $k''' \Vdash \varphi$. This gives $k' \Vdash \Diamond\varphi$. $\Box$

1.2. Sequent calculus

Here we formulate a modal intuitionistic Gentzen sequent calculus, which we prove is sound and complete with respect to the proposed semantics. All the sequent calculus notation of Takeuti [38] is used. Any questions of notation can be resolved by consulting that source. For this section, upper-case Greek letters represent finite sequences of sentences. Then φ, Γ represents the sequent obtained by appending φ to the head of Γ. Similarly Γ, φ represents the sequent obtained by appending φ to the tail of Γ. If $\Gamma = \langle \gamma_1, \ldots, \gamma_n\rangle$, then let $\Box\Gamma$ denote $\langle \Box\gamma_1, \ldots, \Box\gamma_n\rangle$.

Initial sequents (Axioms)

$$A \gg A$$

Structural rules

$$\frac{\Gamma \gg \Delta}{\varphi, \Gamma \gg \Delta}, \qquad \frac{\Gamma \gg \Delta}{\Gamma \gg \Delta, \varphi} \qquad \text{Weakening}$$

$$\frac{\varphi, \varphi, \Gamma \gg \Delta}{\varphi, \Gamma \gg \Delta}, \qquad \frac{\Gamma \gg \varphi, \varphi, \Delta}{\Gamma \gg \varphi, \Delta}, \qquad \text{Contraction}$$

$$\frac{\Gamma, \varphi, \theta, \Pi \gg \Delta}{\Gamma, \theta, \varphi, \Pi \gg \Delta}, \qquad \frac{\Gamma \gg \Delta, \varphi, \theta, \Lambda}{\Gamma \gg \Delta, \theta, \varphi, \Lambda}, \qquad \text{Exchange}$$

$$\frac{\Gamma \gg \Delta, \varphi \quad \varphi, \Pi \gg \Lambda}{\Gamma, \Pi \gg \Delta, \Lambda}, \qquad \text{Cut}$$

Logical rules

$$\frac{\Gamma \gg \Delta, \varphi}{\neg\varphi, \Gamma \gg \Delta}, \qquad \frac{\varphi, \Gamma \gg}{\Gamma \gg \neg\varphi},$$

$$\frac{\varphi, \Gamma \gg \Delta}{\varphi \wedge \theta, \Gamma \gg \Delta}, \qquad \frac{\theta, \Gamma \gg \Delta}{\varphi \wedge \theta, \Gamma \gg \Delta}, \qquad \frac{\Gamma \gg \Delta, \varphi \quad \Gamma \gg \Delta, \theta}{\Gamma \gg \Delta, \varphi \wedge \theta},$$

$$\frac{\varphi, \Gamma \gg \Delta \quad \theta, \Gamma \gg \Delta}{\varphi \vee \theta, \Gamma \gg \Delta}, \qquad \frac{\Gamma \gg \Delta, \varphi}{\Gamma \gg \Delta, \varphi \vee \theta}, \qquad \frac{\Gamma \gg \Delta, \theta}{\Gamma \gg \Delta, \varphi \vee \theta},$$

$$\frac{\Gamma \gg \Delta, \varphi \quad \psi, \theta \gg \Lambda}{\varphi \to \psi, \Gamma, \theta \gg \Delta, \Lambda}, \qquad \frac{\varphi, \Gamma \gg \psi}{\Gamma \gg \varphi \to \psi},$$

$$\frac{\psi(t), \Gamma \gg \Delta}{\forall x\, \psi(x), \Gamma \gg \Delta}, \qquad \frac{\Gamma \gg \psi(a)}{\Gamma \gg \forall x\, \psi(x)},$$

provided the lower sequent is free of a.

$$\frac{\varphi(a), \Gamma \gg \Delta}{\exists x\, \varphi(x), \Gamma \gg \Delta}, \qquad \frac{\Gamma \gg \Delta, \varphi(t)}{\Gamma \gg \Delta, \exists x\, \varphi(x)},$$

where the lower sequent is free of a.

Modal rules

$$\frac{\Gamma, \varphi \gg \psi}{\Box\Gamma, \Diamond\varphi \gg \Diamond\psi}, \qquad \frac{\Gamma, \varphi \gg}{\Box\Gamma, \Diamond\varphi \gg}, \qquad \frac{\Gamma \gg \psi}{\Box\Gamma \gg \Box\psi}.$$

Definition 1.2.1. A sequent of the form $\Gamma > \Delta$ is *LJ-like* if Δ contains at most one sentence.

Adopt the convention that $\Gamma \vdash \Delta$ means that we use the same proof rules as above, but with the consequent Δ always restricted to contain at most one sentence. This is called the *LJ-like form of the calculus*, following Takeuti [38].

Lemma 1.2.2. *The sequent calculus presented above and its LJ-like forms are equivalent, i.e.,* $\Gamma \gg B_1, \ldots, B_n$ *iff* $\Gamma \vdash B_1 \vee \cdots \vee B_n$.

Proof. This is by induction on the proof tree following the lines of [38]. The modal rules can be applied only for *LJ*-like sequents. Therefore, in the inductive proof, the modal rules do not present extra cases. $\Box$

1.3. Soundness of the sequent calculus

By abuse of notation, for a sequent of the form $\Gamma \vdash \varphi$, $\Gamma \Vdash \varphi$ will mean that $\{\gamma_1, \ldots, \gamma_n\} \Vdash \varphi$, where $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$.

Theorem 1.3.1 (Soundness). *If* $\Gamma \vdash \varphi$, *then* $\Gamma \Vdash \varphi$.

Proof. The method of proof is by induction on the proof tree as usual. First, one shows that for each proof rule of the form

$$\frac{\Delta \vdash \theta}{\Delta' \vdash \theta'},$$

if $\Delta \Vdash \theta$, then so is $\Delta' \Vdash \theta'$. Similarly, for rules of the form

$$\frac{\Delta_1 \vdash \theta_1 \quad \Delta_2 \vdash \theta_2}{\Delta \vdash \theta},$$

one shows that if $\Delta_1 \Vdash \theta_1$ and $\Delta_2 \Vdash \theta_2$, then $\Delta \Vdash \theta$. Finally, for the only axiom, $\varphi \vdash \varphi$, $\varphi \Vdash \varphi$. By induction on the proof tree of $\Gamma \vdash \varphi$, it follows $\Gamma \Vdash \varphi$.

For the nonmodal rules the proof is the same as that for showing the soundness of the LJ-rules with respect to nonmodal Kripke frames, so these cases are omitted. For the modal rules, assume that $\langle K, D, \leqslant, R, \Vdash \rangle$ is any intuitionistic modal frame. Let $k \in K$. For the rule

$$\frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi},$$

suppose that $k \Vdash \Box\Gamma$. To show that $k \Vdash \Box\varphi$, let k', k'' be such that $k \leqslant k'$ and $k' R k''$. Since $k \Vdash \Box\Gamma$, $k'' \Vdash \Gamma$. Therefore $k'' \Vdash \varphi$. Hence $k \Vdash \Box\varphi$.

For the rule

$$\frac{\Gamma, \varphi \vdash \psi}{\Box\Gamma, \Diamond\varphi \vdash \Diamond\psi},$$

suppose that $k \Vdash \Box\Gamma$ and that $k \Vdash \Diamond\varphi$. The latter implies that for all $k' \geqslant k$, there is a $k'' \in K$ such that $k' R k''$ and $k'' \Vdash \varphi$. Since $k \Vdash \Box\Gamma$, we get $k'' \Vdash \Gamma$. Hence $k'' \Vdash \Gamma, \varphi$. By the inductive assumption, $k'' \Vdash \psi$. Hence $k \Vdash \Diamond\psi$.

For the rule

$$\frac{\Gamma, \varphi \vdash}{\Box\Gamma, \Diamond\varphi \vdash},$$

suppose that $k \Vdash \Box\Gamma, \Diamond\varphi$. Then for all $k' \geqslant k$, there is a $k'' \in K$ such that $k' R k''$ and $k'' \Vdash \varphi$. Since $k \Vdash \Box\Gamma$, we get $k'' \Vdash \Gamma, \varphi$. By the inductive assumption this is impossible.

This completes the proof of soundness. $\Box$

1.4. Completeness of the sequent calculus

This section proves that the sequent calculus of Section 1.2 is complete with respect to the semantics given in Section 1.1. The usual completeness proof for classical modal logic uses as worlds maximal consistent sets closed under modalities and uses an accessibility relation between these worlds. There is no partial ordering of increasing information present. The usual completeness proof for intuitionistic nonmodal logic uses prime filters as possible worlds, partially ordered by inclusion. There are no modalities in the calculus, and no separate accessbility in the frame. The first thought is that for intuitionistic modal logic we should combine these two methodologies, and use certain prime filters and an appropriate accessibility between prime filters to reflect the modality. If we had assumed the distributivity of diamond over 'or', we can easily carry this program out. But to cover all intended computer science applications, we have to avoid using this axiom of distributivity. In the completeness proof for our systems, but not in the direct definition of intuitionistic modal frames, we need a more subtle 'set accessibility' in which *single* prime filters access *sets* of prime filters instead of individual prime filters. This device is embodied in the notion of a 'segment' given below. Intuitionistic modal logics needing such 'segments' arise in interpreting partial information in computer science, see [27]. There a 'set' accessibility arises in dynamic logic whenever we interpret a concurrent program in Peleg concurrent transition systems which are not sequential. Not allowing such 'set' accessibility would restrict attention to sequential programs, and indeed corresponds roughly to assuming diamond distributes over 'or'.

Notation 1.4.1. Let Γ be a set of sentences. By abuse of notation, write $\Gamma \vdash \varphi$ if there is a finite subset of sentences Γ' of Γ such that $\Gamma' \vdash \varphi$. Then φ is called a *consequence* of Γ. Call Γ *deductively closed* if Γ contains all its consequences.

Definition 1.4.2. Let Γ be a set of sentences in language L. Then Γ is *saturated* in L if the following conditions hold:

(i) Γ is consistent and deductively closed.

(ii) If $\varphi \vee \psi \in \Gamma$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

(iii) If $\exists x\, \psi(x) \in \Gamma$, then $\psi(c) \in \Gamma$ for some c in L.

Notation 1.4.3. Let L be a countable language, let C be a countable set of individual constants outside L. Let $L(C)$ denotes the language formed from L by including C.

Lemma 1.4.4. *In a countable intuitionistic modal logic language L, suppose that $(\Gamma \vdash \varphi)$ is not true. Then there is a $\Gamma' \supseteq \Gamma$ such that Γ' is saturated in $L(C)$ and $(\Gamma' \vdash \varphi)$ is not true.*

Proof. See [40]. □

Theorem 1.4.5 (Completeness). *Suppose that $(\Gamma \vdash \varphi)$ is not true. Then there exists a modal frame $\langle K, D, \leqslant, R, \Vdash \rangle$ and a $k_0 \in K$ such that $k_0 \Vdash \Gamma$, but not $(k_0 \Vdash \varphi)$.*

Proof. Work entirely with sets C of constants which are subsets of a countable master set of constants $\boldsymbol{C}$ outside L. Call (Γ, C) a *base* if Γ is saturated with respect to $L(C)$. For (Γ_0, C_{Γ_0}) a base and U a set of bases, call $((\Gamma_0, C_{\Gamma_0}), U)$ a *segment* if:

(i) $C_{\Gamma_0} \subset C_\Gamma$ for each $(\Gamma, C_\Gamma) \in U$.
(ii) If $\Box\varphi \in \Gamma_0$, then $\varphi \in \Gamma$ for each $(\Gamma, C_\Gamma) \in U$.
(iii) If $\Diamond\varphi \in \Gamma$, then there is a $(\Delta, C_\Delta) \in U$ such that $\varphi \in \Delta$.

For the segment $S = ((\Gamma_0, C_{\Gamma_0}), U)$, call (Γ_0, C_{Γ_0}) the head of the segment and denote it by $H(S)$, call U the tail of the segment and denote it by $T(S)$. So segment $S = (H(S), T(S))$. Let $\boldsymbol{S}$ denote the set of all segments constructed out of the sentences of $L(\boldsymbol{C})$. Define $\lesssim$ on segments $L(\boldsymbol{C})$ as follows:

$$((\Gamma, C_\Gamma), U) \lesssim ((\Gamma', C_{\Gamma'}), U') \quad \text{iff} \quad \Gamma \subseteq \Gamma' \text{ and } C_\Gamma \subseteq C_{\Gamma'}.$$

Let Γ be a set of sentences and let φ be a sentence such that $(\Gamma \vdash \varphi)$ does not hold. By Lemma 1.4.4, such a Γ can be extended to a saturated Γ_0 in an $L(C_{\Gamma_0})$ where $(\Gamma_0 \vdash \varphi)$ is not true. Let $\bar{\Gamma}_0 = \{\varphi : \Box\varphi \in \Gamma_0\}$. For each sentence of the form $\Diamond\theta$ with $\Diamond\theta \in \Gamma_0$, let $\Gamma_\theta = \bar{\Gamma}_0 \cup \{\theta\}$. Now Γ_θ is consistent, since $\Gamma_\theta \vdash$ implies that there exist sentences $\gamma_1, \ldots, \gamma_n \in \bar{\Gamma}_0$ for which $\gamma_1, \gamma_2, \ldots, \gamma_n, \theta \vdash$. The $\Diamond$ rule implies then that $\Box\gamma_1, \Box\gamma_2, \ldots, \Box\gamma_n, \Diamond\theta \vdash$, which implies $\Gamma_0 \vdash$. This contradicts Γ_0 being saturated. Since each Γ_θ is consistent we can use Lemma 1.4.4 to extend this set to a set Γ'_θ saturated with respect to $L(C_{\Gamma'_\theta})$, with $C_{\Gamma'_\theta} \supseteq C_{\Gamma_0}$. Call $(\Gamma'_\theta, C_{\Gamma'_\theta})$, B_θ. Then $((\Gamma_0, C_{\Gamma_0}), \{B_\theta : \Diamond\theta \in \Gamma_0\})$ is a segment. Let $S_0 = ((\Gamma_0, C_{\Gamma_0}), \{B_\theta : \Diamond\theta \in \Gamma_0\})$, let $\mathrm{SEG}_0 = \{S_0\}$. For each i, define T_i and SEG_i by induction as follows.

$$\text{Let } T_i = \{S \in \boldsymbol{S} : H(S) \in T(S') \text{ for some } S' \in \mathrm{SEG}_i\}$$
$$\text{and } \mathrm{SEG}_{i+1} = \{S \in \boldsymbol{S} : S \gtrsim S' \text{ for some } S' \in T_i \cup \mathrm{SEG}_i\}.$$

Let $\mathrm{SEG} = \bigcup_{i=0}^{\infty} \mathrm{SEG}_i$. The modal intuitionistic frame $\langle K, D, \leqslant, R, \Vdash \rangle$ is defined as follows. Let $K = \mathrm{SEG}$. Define $\leqslant$ on K as $\lesssim$. Define accessibility R on K as $S\,R\,S'$ iff $H(S') \in T(S)$. For each $k \in K$ define $D(k) = C_\Gamma \cup C_0$, where $H(k) = (\Gamma, C_\Gamma)$. The domain function satisfies the required monotonicity conditions. Define $\Vdash$ on the atomic statements by

$$k \Vdash A(\boldsymbol{d}_n) \quad \text{iff} \quad d_i \in D(k) \text{ for } 1 \leqslant i \leqslant n \text{ and } A(\boldsymbol{d}_n) \in \Gamma, \text{ where head}(k) = (\Gamma, C_\Gamma).$$

Extend $\Vdash$ to all sentences as it was done in Definition 1.1.4. Forcing is of course monotone.

Lemma 1.4.6. *Let $H(k) = (\Gamma, C_\Gamma) \in K$. Then for any sentence φ in the language of $L(C_\Gamma)$, $k \Vdash \varphi$ iff $\varphi \in \Gamma$.*

Proof. This is proved by induction on the complexity of φ. When φ is an instance of an atomic formula, the result follows from the definition of $\Vdash$. When φ is a nonatomic formula, our induction assumption is that the result is true for all subformulas of φ and for all $k \in K$.

(1) Suppose that φ is $\varphi_1 \wedge \varphi_2$. If $k \Vdash \varphi_1 \wedge \varphi_2$, then $k \Vdash \varphi_1$ and $k \Vdash \varphi_2$, therefore by the induction assumption $\varphi_1 \in \Gamma$ and $\varphi_2 \in \Gamma$. Since Γ is deductively closed, $\varphi_1 \wedge \varphi_2 \in \Gamma$. Conversely, if $\varphi_1 \wedge \varphi_2 \in \Gamma$, then, since Γ is deductively closed, we get $\varphi_1 \in \Gamma$ and $\varphi_2 \in \Gamma$. Hence $k \Vdash \varphi_1$ and $k \Vdash \varphi_2$, and therefore $k \Vdash \varphi_1 \wedge \varphi_2$.

(2) Suppose that φ is $\varphi_1 \vee \varphi_2$. If $k \Vdash \varphi_1 \vee \varphi_2$, then either $k \Vdash \varphi_1$ or $k \Vdash \varphi_2$. Without loss of generality, if $k \Vdash \varphi_1$, then by the inductive assumption $\varphi_1 \in \Gamma$. Since Γ is deductively closed, $\varphi_1 \vee \varphi_2 \in \Gamma$. Conversely, if $\varphi_1 \vee \varphi_2 \in \Gamma$, then Γ is saturated implies that either $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$. Without loss of generality, if $\varphi_1 \in \Gamma$, then by the inductive assumption, $k \Vdash \varphi_1$. Hence $k \Vdash \varphi_1 \vee \varphi_2$.

(3) Suppose that φ is $\varphi_1 \rightarrow \varphi_2$. If $k \Vdash \varphi_1 \rightarrow \varphi_2$ and $\varphi_1 \rightarrow \varphi_2 \notin \Gamma$, then $(\Gamma \vdash \varphi_1 \rightarrow \varphi_2)$ is not true. The proof rules for $\rightarrow$ imply that $(\Gamma, \varphi_1 \vdash \varphi_2)$ is not true either. By Lemma 1.4.4 choose a Γ' saturated with respect to some $L(C_{\Gamma'})$, where $C_{\Gamma'} \supseteq C_\Gamma$ and $\Gamma' \supseteq \Gamma \cup \{\varphi_1\}$, but $\varphi_2 \notin \Gamma'$. Carry out the same proof that was used to construct S_0 in the proof of the main theorem and construct a segment $k' = ((\Gamma', C_{\Gamma'}), \Delta_{\Gamma'})$, for some set of bases $\Delta_{\Gamma'}$. By the inductive assumption for Γ' there is a $k' \geqslant k$ with $k' \Vdash \varphi_1$ but not $k' \Vdash \varphi_2$. But this contradicts $k \Vdash \varphi_1 \rightarrow \varphi_2$. Conversely, suppose that $\varphi_1 \rightarrow \varphi_2 \in \Gamma$ and not $k \Vdash \varphi_1 \rightarrow \varphi_2$. Then there exists a $k' \in K$ with $k' \geqslant k$ and $k' \Vdash \varphi_1$ and not $k' \Vdash \varphi_2$. Therefore, by the inductive assumption, we get that $\varphi_1 \in \Gamma'$ and $\varphi_2 \notin \Gamma'$, where $H(k') = (\Gamma', C_{\Gamma'})$. Since $k' \geqslant k$, we get that $\Gamma' \supseteq \Gamma$. So $\varphi_1 \rightarrow \varphi_2 \in \Gamma'$. Since Γ' is deductively close, we conclude that $\varphi_2 \in \Gamma'$, a contradiction.

(4) Suppose that φ is $\neg\varphi_1$. The proof is similar to that for case 3.

(5) Suppose that φ is $\exists x\, \varphi_1(x)$. If $\Gamma \Vdash \exists x\, \varphi(x)$, then $\Gamma \Vdash \varphi_1(c)$ for some $c \in C_\Gamma \cup C_0$. Hence, by the inductive assumption, $\varphi_1(c) \in \Gamma$. Since Γ is deductively closed, $\exists x\, \varphi_1(x) \in \Gamma$. Conversely, if $\exists x\, \varphi_1(x) \in \Gamma$, then by the saturation of Γ, $\varphi_1(c) \in \Gamma$ for some $c \in C_\Gamma$. By inductive assumption, $k \Vdash \varphi_1(c)$. Hence $k \Vdash \exists x\, \varphi_1(x)$.

(6) Suppose that φ is $\forall x\, \varphi_1(x)$. Suppose that $k \Vdash \forall x\, \varphi_1(x)$ and $\forall x\, \varphi_1(x) \notin \Gamma$. Then for a new d not appearing in $C_\Gamma \cup C_0$, it is not the case that $\Gamma \vdash \varphi_1(d)$. Hence by Lemma 1.4.4, we can extend (Γ, C_Γ) to a $(\Gamma', C_{\Gamma'})$ such that $\varphi_1(d) \notin \Gamma'$ and $d \in C_{\Gamma'}$. By using the same construction as for S_0 in Theorem 1.4.5, we can construct a segment $((\Gamma', C_{\Gamma'}), \Delta_{\Gamma'})$. So it is not the case that $k' \Vdash \varphi_1(d)$, where k' corresponds to the segment $((\Gamma', C_{\Gamma'}), \Delta_{\Gamma'})$ with $d \in D(k')$. Conversely, suppose that $\forall x\, \varphi_1(x) \in \Gamma$. Then for all $K' \geqslant K$, where $H(K') = (\Gamma', C_{\Gamma'})$, we have $\forall x\, \varphi_1(x) \in \Gamma'$. Since Γ' is deductively closed, $\varphi_1(d) \in \Gamma'$ for any $d \in C_{\Gamma'} \cup C_0$. Hence $k \Vdash \forall x\, \varphi_1(x)$.

(7) Suppose that $\varphi = \Diamond\varphi_1$. If $k \Vdash \Diamond\varphi_1$ and $\Diamond\varphi_1 \notin \Gamma$, then for each $\Diamond\theta \in \Gamma$, let $\Gamma_\theta = \{\theta\} \cup \bar{\Gamma}$, with $\bar{\Gamma} = \{\lambda : \Box\lambda \in \Gamma\}$. Then it is not the case that $(\Gamma_\theta \vdash \varphi_1)$; for otherwise, θ, $\bar{\Gamma} \vdash \varphi_1$, hence by the diamond rule $\Diamond\theta$, $\Gamma \vdash \Diamond\varphi_1$, a contradiction. So by Lemma 1.4.4 we can extend Γ_θ to a Γ'_θ saturated with respect to some $C_{\Gamma_\theta} \supseteq C_\Gamma$ and $\varphi_1 \notin \Gamma'_\theta$. Now we see that $k' = ((\Gamma', C_{\Gamma'}), \{(\Gamma_\theta, C_{\Gamma_\theta}) : \Diamond\theta \in \Gamma\})$ is a segment, where $(\Gamma', C_{\Gamma'}) = (\Gamma, C_\Gamma)$. Note that $k' R k''$ implies $H(k'') = (\Gamma_\theta, C_{\Gamma_\theta})$

for some Γ_θ. Because $\varphi_1 \notin \Gamma_\theta$, by the inductive hypothesis it is not true that $((\Gamma_\theta, C_{\Gamma_\theta}) \Vdash \varphi_1)$. Hence there exists a $k' \in K$ such that no k'' with $k' R k''$ satisfies $(k'' \Vdash \varphi_1)$. This contradicts $k \Vdash \Diamond\varphi_1$. Conversely, if $\Diamond\varphi \in \Gamma$, then for all $k' \geqslant k$, for $H(k') = (\Gamma', C_{\Gamma'})$, we get that $\Diamond\varphi \in \Gamma'$. Hence $\varphi \in \Gamma''$ for some k'' with $k' R k''$. If $H(k'') = (\Gamma'', C_{\Gamma''})$, by the inductive assumption we get $k'' \Vdash \varphi$.

(8) Suppose that φ is $\Box\varphi_1$. If $k \Vdash \Box\varphi_1$ and $\Box\varphi_1 \notin \Gamma$, then for each $\Diamond\theta \in \Gamma$, let Γ_θ be a saturated extension of $\{\theta\} \cup \bar{\Gamma}$ as before. It is false that $(\bar{\Gamma} \vdash \varphi_1)$, hence we can extend $\bar{\Gamma}$ to a saturated Γ' so that $\varphi_1 \notin \Gamma'$. Now

$$((\Gamma, C_\Gamma), \{(\Gamma', C_{\Gamma'})\} \cup \{(\Gamma_\theta, C_{\Gamma_\theta}) \colon \Diamond\theta \in \Gamma\}) = k'$$

is a segment. Using the same method as at the beginning of the proof of Theorem 1.4.5, we can create segments corresponding to $T(k')$. Let k'' be the segment with $H(k'') = (\Gamma', C_{\Gamma'})$. Then we have k', k'' with $k \leqslant k'$ and $k' R k''$, but not $(k'' \Vdash \varphi)$. This contradicts $(k \Vdash \Box\varphi)$. Conversely, suppose that $\Box\varphi_1 \in \Gamma$. Then $\Box\varphi_1 \in \Gamma'$ for each $k' \geqslant k$ with $H(k') = \Gamma'$. Therefore for any k'' with $k' R k''$ and $(\Gamma'', C_{\Gamma''}) = H(k'')$, we have that $\varphi_1 \in \Gamma''$. By the inductive hypotheses we conclude that $k'' \Vdash \varphi_1$.

This ends the proof of Lemma 1.4.6, and Lemma 1.4.6 completes the proof of the completeness Theorem 1.4.5. $\Box$

1.5. Hilbert style axioms

It is possible to develop Hilbert style axioms from the sequent formulation of Section 1.2. We devote this section to that task.

Axioms

(i) Axioms for intuitionistic predicate logic.
(ii) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
(iii) $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$.
(iv) $\Box\varphi \wedge \Diamond(\varphi \to \psi) \to \Diamond\psi$.
(v) $\Diamond(\varphi \wedge \neg\varphi) \to \varphi \wedge \neg\varphi$.

Rules of inference

(i) (Modus ponens) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$.

(ii) If $\vdash\varphi$, then $\vdash\Box\varphi$.

(iii) Quantifier rules

$$\frac{\varphi \to \psi(x)}{\varphi \to \forall x\, \psi(x)} \qquad \frac{\psi(x) \to \varphi}{\exists x\, \psi(x) \to \varphi},$$

with the usual restrictions.

The equivalence of this Hilbert style system with the sequent formulation previously given is the content of the following lemma.

Lemma 1.5.1. *Let $\Lambda \mathrel{|\!\!-} \psi$ mean that ψ is provable from Δ in the Hilbert style system. Then $\Lambda \vdash \psi$ if and only if $\Lambda \mathrel{|\!\!-} \psi$.*

Proof. To show that 'if $\Lambda \mathrel{|\!\!-} \psi$, then $\Lambda \vdash \psi$', it is enough to show that the axioms of the Hilbert style formulation are theorems of the sequent calculus and to notice that the proof rules of the Hilbert system are valid instances of rules in the sequent calculus.

(1) The axioms of intuitionistic predicate calculus are proved as usual for sequent calculus.

(2) For (ii),

$$\begin{array}{l} \varphi \to \psi, \varphi \vdash \psi \\ \hline \Box(\varphi \to \psi), \Box\varphi \vdash \Box\psi \\ \hline \Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi \\ \hline \vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \end{array}.$$

For (iii),

$$\begin{array}{l} \varphi \to \psi, \varphi \vdash \psi \\ \hline \Box(\varphi \to \psi), \Diamond\varphi \vdash \Diamond\psi \\ \hline \Box(\varphi \to \psi) \vdash \Diamond\varphi \to \Diamond\psi \\ \hline \vdash \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi) \end{array}.$$

For (iv),

$$\begin{array}{l} \varphi \to \psi, \varphi \vdash \psi \\ \hline \Diamond(\varphi \to \psi), \Box\varphi \vdash \Diamond\psi \\ \hline \vdash \Box\varphi \wedge \Diamond(\varphi \to \psi) \to \Diamond\psi \end{array}.$$

For (v)

$$\begin{array}{l} \varphi \vdash \varphi \\ \hline \varphi, \neg\varphi \vdash \\ \hline \varphi \wedge \neg\varphi \vdash \\ \hline \Diamond(\varphi \wedge \neg\varphi) \vdash \\ \hline \Diamond(\varphi \wedge \neg\varphi) \vdash \varphi \wedge \neg\varphi \\ \hline \vdash \Diamond(\varphi \wedge \neg\varphi) \to (\varphi \wedge \neg\varphi) \end{array}.$$

The proof rules are instances of *LJ*-like rules.

Conversely, to show that 'if $\Lambda \vdash \psi$, then $\Lambda \vdash \varphi$', it is enough to justify the steps that use the modal rules in a sequent-style proof by means of the Hilbert system. So we show that the modal rules are derived rules of the Hilbert system.

(1) $$\frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi}.$$

By inductive hypothesis, assume that $\Gamma \vdash \varphi$. By repeated application of the deduction theorem, $\vdash \gamma_1 \to (\gamma_2 \to (\gamma_3 \to \cdots (\gamma_n \to \varphi)))$, where $\Gamma = \{\gamma_i : i = 1, n\}$. By generalization, $\vdash \Box(\gamma_1 \to (\gamma_2 \to (\gamma_3 \to \cdots (\gamma_n \to \varphi))))$. By applying $\Box(\varphi \to \theta) \to (\Box\varphi \to \Box\theta)$, we get $\vdash (\Box\gamma_1 \to (\Box\gamma_2 \to (\Box\gamma_3 \to \cdots (\Box\gamma_n \to \Box\varphi))))$. By modus ponens, we get $\Box\Gamma \vdash \Box\varphi$.

(2) $$\frac{\Gamma, \varphi \vdash \theta}{\Box\Gamma, \Diamond\varphi \vdash \Diamond\theta}.$$

By inductive hypothesis, suppose that Γ, $\varphi \vdash \theta$. Then $\Gamma \vdash \varphi \to \theta$. Hence $\Box\Gamma \vdash \Box(\varphi \to \theta)$ by the argument used above. By modus ponens and axiom (iii), $\Box\Gamma \vdash \Diamond\varphi \to \Diamond\theta$. By the deduction theorem, $\Box\Gamma$, $\Diamond\varphi \vdash \Diamond\theta$.

(3) For

$$\frac{\Gamma, \varphi \vdash}{\Box\Gamma, \Diamond\varphi \vdash},$$

the argument follows from (2) and the axiom $\Diamond(\theta \wedge \neg\theta) \to (\theta \wedge \neg\theta)$. $\Box$

2. Cut elimination, algebraic and topological semantics

2.1. Freedom of cut and its consequences

This section gives a proof of the 'freedom of cut' in the *LJ*-like sequent calculus of Section 1.2. The notation and general outline of Takeuti's proof of cut elimination [38, p. 21–28] are used, the reader is referred there for all unexplained notations. Since Takeuti's calculus has no modalities, we have to define exactly all notions where modalities play an essential role. The cases that are necessary to establish the freedom of cut with the modal connectives are the only ones presented here. Notions from sequent calculus such as 'deduction tree' and a 'thread' in a deduction tree are used without definition. The notion of a 'principal formula' of an inference figure is assumed known and is indicated beside the figure—e.g.

$$\frac{s_1 \quad s_2}{s}(\varphi)$$

is an inference figure with principal formula φ. When the sequent Γ occurs in the above inference figure, Γ^* denotes the sequent Γ with occurrences of φ deleted.

The following rule, known as Mix rule, is substituted for cut.

$$\text{Mix rule} \quad \frac{\Gamma \vdash \varphi \quad \Pi \vdash \theta}{\Gamma, \Pi^* \vdash \theta}(\varphi).$$

The following lemma is now a triviality.

Lemma 2.1.1. *The Cut and Mix rules are equivalent.*

Proof. See [38]. □

Definition 2.1.2. Let P be a proof which contains a mix rule only as the last inference

$$\frac{s_1 \quad s_2}{s}(\varphi),$$

where φ is the mix formula.

(1) The *rank of a thread above* s_1 is the number of consecutive sequents that contain the formula φ, starting from s_1 and counting upwards.

(2) *Left rank of* $P = \mathrm{rank}_{\mathrm{L}}(P) = \max\{\mathrm{rank}(F, P) : F$ is a left thread$\}$,
Right rank of $P = \mathrm{rank}_{\mathrm{R}}(P) = \max\{\mathrm{rank}(F, P) : F$ is a right thread$\}$.

(3) $\mathrm{Rank}(P) = \mathrm{rank}_{\mathrm{R}}(P) + \mathrm{rank}_{\mathrm{L}}(P)$.

(4) $\mathrm{Grade}(\varphi) =$ number of logical connectives occurring in φ.

(5) *Grade of a mix rule* = grade of the principal formula of the mix rule.

Lemma 2.1.3. *If P is a proof of $\Gamma \vdash \varphi$ in which only one mix occurs, and that occurring as the last inference, then $\Gamma \vdash \varphi$ is provable without a mix.*

Proof. The proof is by double induction on the rank and the grade. The inductive hypothesis is that the statement of the lemma is valid for all formulas of lower grade, and we prove it for proofs of current grade.
Case 1. rank = 2. Let P be

$$\frac{s_1 \quad s_2}{s}(\psi).$$

Case 1.1. Either s_1 or s_2 is an initial sequent. The proof is the same as Takeuti's.

Case 1.2. Either s_1 or s_2 is a structural rule. The proof is the same as Takeuti's.

Case 1.3. Both s_1 and s_2 are logical rules. In this case, rank = 2 implies $\mathrm{rank}_{\mathrm{L}} = 1 = \mathrm{rank}_{\mathrm{R}}$. Hence the principal formula must be the mix formula. Only where a modal rule is involved will it be presented. The other cases are the same as in [38].

1.3.1. $$\frac{s_1 \quad s_2}{s} = \frac{\dfrac{\Lambda \vdash \theta}{\Box\Lambda \vdash \Box\theta} \quad \dfrac{\Pi \vdash \theta_1}{\Box\Pi \vdash \Box\theta_1}}{\Box\Lambda, \Box\Pi^* \vdash \Box\theta_1}(\Box\theta).$$

By the inductive assumption there is a mix-free proof of

$$\frac{\Lambda \vdash \theta \quad \Pi \vdash \theta_1}{\Lambda, \Pi^* \vdash \theta_1} (\theta).$$

Now apply the modal rule $\Box$ to get

$$\frac{\dfrac{\Lambda \vdash \theta \quad \Pi \vdash \theta_1}{\Lambda, \Pi^* \vdash \theta_1}}{\Box\Lambda, \Box\Pi^* \vdash \Box\theta_1}.$$

1.3.2. Let P be

$$\frac{\dfrac{\Lambda \vdash \theta}{\Box\Lambda \vdash \Box\theta} \quad \dfrac{\Pi, \theta_1 \vdash \theta_2}{\Box\Pi, \Diamond\theta_1 \vdash \Diamond\theta_2}}{\Box\Lambda, \Box\Pi^*, \Diamond\theta_1 \vdash \Diamond\theta_2} (\Box\theta).$$

Replace the antecedent by the mix-free proof of

$$\frac{\Lambda \vdash \theta \quad \Pi, \theta_1 \vdash \theta_2}{\Lambda, \Pi^* \theta_1 \vdash \theta_2} (\theta),$$

and apply the $\Diamond$ rule to get a

$$\frac{\dfrac{\Lambda \vdash \theta \quad \Pi, \theta_1 \vdash \theta_2}{\Lambda, \Pi^*, \theta_1 \vdash \theta_2} (\theta)}{\Box\Lambda, \Box\Pi^*, \Diamond\theta_1 \vdash \Diamond\theta_2}.$$

1.3.3. Let P be

$$\frac{\dfrac{\Lambda \vdash \theta}{\Box\Lambda \vdash \Box\theta} \quad \dfrac{\Pi, \theta_1 \vdash}{\Box\Pi, \Diamond\theta_1 \vdash}}{\Box\Lambda, \Box\Pi^*, \Diamond\theta_1 \vdash} (\Box\theta).$$

By the same argument in 1.3.2 we get

$$\frac{\dfrac{\Lambda \vdash \theta \quad \Pi, \theta_1 \vdash}{\Lambda, \Pi^* \theta_1 \vdash} (\theta)}{\Box\Lambda, \Box\Pi^*, \Diamond\theta_1 \vdash}.$$

1.3.4.

$$P = \frac{\dfrac{\Lambda, \theta \vdash \theta_1}{\Box\Lambda, \Diamond\theta \vdash \Diamond\theta_1} \quad \dfrac{\Pi, \theta_1 \vdash \theta_2}{\Box\Pi, \Diamond\theta_1 \vdash \Diamond\theta_2}}{\Box\Lambda, \Box\Pi, \Diamond\theta \vdash \Diamond\theta_2} (\Diamond\theta_1).$$

This can be replaced by the following $\Diamond\theta_1$ mix-free proof

$$\frac{\dfrac{\Lambda, \theta \vdash \theta_1 \quad \Pi, \theta_1 \vdash \theta_2}{\Lambda, \Pi, \theta \vdash \theta_2} (\theta)}{\Box\Lambda, \Box\Pi, \Diamond\theta \vdash \Diamond\theta_2}.$$

1.3.5. Let P be

$$\frac{\dfrac{\Lambda, \theta \vdash \theta_1}{\Box\Lambda, \Diamond\theta \vdash \Diamond\theta_1} \quad \dfrac{\Pi, \theta_1 \vdash}{\Box\Pi, \Diamond\theta_1 \vdash}}{\Box\Lambda, \Box\Pi, \Diamond\theta \vdash}(\Diamond\theta_1).$$

This can be replaced by the $\Diamond\theta_1$ mix-free proof

$$\dfrac{\dfrac{\Lambda, \theta \vdash \theta_1 \quad \Pi, \theta_1 \vdash}{\Lambda, \Pi, \theta \vdash}(\theta_1).}{\Box\Lambda, \Box\Pi, \Diamond\theta \vdash}$$

The proof, when the logical rules involved are nonmodal, are as they appear in [38].

Case 2. $\text{rank}(P) > 2$.

Case 2.1. $\text{Rank}_{\text{R}}(P) > 1$.

2.1.1. Let the inference be of the form

$$\frac{\Lambda \vdash \theta_2 \quad \dfrac{\Pi_0 \vdash \theta_0}{\Pi_1 \vdash \theta_1}}{\Lambda, \Pi_1^* \vdash \theta_1},$$

where the mix formula does not occur in Λ and s_2 is a logical rule. The thing to notice is that the logical rule involved cannot be a modal rule for, otherwise $\text{rank}_{\text{R}} = 1$. Hence the proof follows as in Takeuti [38].

2.1.2. All other forms of inference with $\text{rank}(P) > 2$ and $\text{rank}_{\text{R}}(P) > 1$ are as in Takeuti [38].

Case 2.2. $\text{rank}_{\text{R}}(P) = 1$. Then $\text{rank}_{\text{L}}(P) > 1$. In this case the only way a modal rule can be used in the left-hand antecedent is when it is of the form

$$\frac{\Lambda, \theta \vdash}{\Box\Lambda, \Diamond\theta \vdash},$$

but this cannot be involved in a mix as the consequent is empty. All other instances of the present case are a repetition of the standard proof and are therefore omitted. $\Box$

Theorem 2.1.4. *If $\Gamma \vdash \varphi$ is proved in the LJ-like calculus as originally proposed (i.e., with cut instead of mix), then $\Gamma \vdash \varphi$ is provable without a cut.*

Proof. The usual proof works with the following two observations.

(i) Cut and mix rules are equivalent.

(ii) We can inductively traverse down a proof tree, and replace its first cut by a cut-free subtree, and then proceed down to the root. $\Box$

Consequences of freedom of cut

The standard proofs of interpolation theorems follow from modifications, rather than the addition of cases, to usual proofs.

Lemma 2.1.5. *Let $\Gamma \vdash \varphi$ be provable and let Γ_1, Γ_2 be a partition of Γ. Then there is a sentence ψ such that*

(i) *$\Gamma_1 \vdash \psi$ and $\psi, \Gamma_2 \vdash \varphi$;*

(ii) *the nonlogical symbols of $\psi \subseteq L_1 \cup L_2$, where $L_i = \{$nonlogical symbols of $\Gamma_i\}$ for $i = 1, 2$.*

Proof. The proof is by induction on the number k of inferences in the cut-free proof of $\Gamma \vdash \varphi$.

Case (i). $k = 0$. Then $\Gamma \vdash \varphi$ must be $\varphi \vdash \varphi$. Take ψ to be φ.

Case (ii). $k > 0$. Then we need a case analysis on the last inference of the proof. Once again we give only the cases where the last inference is a modal rule and refer the rest to [38].

(1) Suppose the last inference is

$$\frac{\Lambda \vdash \theta}{\Box\Lambda \vdash \Box\theta}$$

and $\Lambda = \Lambda_1 \cup \Lambda_2$ is the partition of Λ. Then by inductive assumption we get a θ_1 satisfying $\Lambda_1 \vdash \theta_1$ and $\Lambda_2, \theta_1 \vdash \theta$. By applying $\Box$ we get

$$\frac{\Lambda_1 \vdash \theta_1}{\Box\Lambda_1 \vdash \Box\theta_1} \quad \text{and} \quad \frac{\Lambda_2, \theta_1 \vdash \theta}{\Box\Lambda_2, \Box\theta_1 \vdash \Box\theta}.$$

Now $\Box\Lambda_1 \cup \Box\Lambda_2$ is a partition of $\Box\Lambda$, and $\Box\theta_1$ is the new sentence.

(2) Suppose the last inference is

$$\frac{\Lambda, \theta_1 \vdash \theta_2}{\Box\Lambda, \Diamond\theta_1 \vdash \Diamond\theta_2}.$$

Let $\Box\Lambda_1$, $\Diamond\theta_1$ and $\Box\Lambda_2$ be a partition of the lower antecedent. Then Λ_1, θ_1 and Λ_2 are partitions of the upper antecedent. Hence by induction we can find ψ satisfying conditions (i) and (ii) of the lemma such that $\Lambda_1, \theta_1 \vdash \psi$ and $\Lambda_2, \psi \vdash \theta_2$. Then the following are admissible:

$$\frac{\Lambda_1, \theta_1 \vdash \psi}{\Box\Lambda_1 \Diamond\theta_1 \vdash \Diamond\psi}, \qquad \frac{\Lambda_2, \psi \vdash \theta_2}{\Box\Lambda_2 \Diamond\psi \vdash \Diamond\theta_2}.$$

Here the required sentence is $\Diamond\psi$.

(3) Suppose the last inference is

$$\frac{\Lambda, \psi \vdash}{\Box\Lambda, \Diamond\psi \vdash}$$

and $\Box\Lambda_1$, $\Diamond\psi$ and $\Box\Lambda_2$ are a partition. Hence Λ_1, ψ and Λ_2 are partitions of the upper antecedent. Therefore we get θ satisfying $\Lambda_2 \vdash \theta$ and $\theta, \Lambda_1, \psi \vdash$, from which we get

$$\frac{\theta, \Lambda_1, \psi \vdash}{\Box\theta, \Box\Lambda_1, \Diamond\psi \vdash} \quad \text{and} \quad \frac{\Lambda_2 \vdash \theta}{\Box\Lambda_2 \vdash \Box\theta}.$$

Suppose we want a λ such that $\Box\Lambda_1$, $\Diamond\psi \vdash \lambda$ and λ, $\Box\Lambda \vdash$. Then by the inductive assumption we can get θ such that Λ_1, $\psi \vdash$ and θ, $\Lambda_2 \vdash$. Hence

$$\frac{\Lambda_1,\ \psi \vdash \theta}{\Box\Lambda_1,\ \Diamond\psi \vdash \Diamond\theta} \quad \text{and} \quad \frac{\theta,\ \Lambda_2 \vdash}{\Diamond\theta,\ \Box\Lambda_2 \vdash}.$$

So λ can be taken as $\Diamond\theta$. $\Box$

Theorem 2.1.6 (Craig's interpolation theorem). *Suppose φ and ψ are two formulas such that $\varphi \vdash \psi$. Then there is a formula θ such that*

(i) *$\varphi \vdash \theta$ and $\theta \vdash \psi$;*

(ii) *{nonlogical symbols of θ} $\subseteq$ {nonlogical symbols of φ} $\cap$ {nonlogical symbols of ψ}.*

Proof. Apply Lemma 2.1.5 with $\{\varphi\}$, $\{\cdot\}$ as the partition to get the required θ. $\Box$

2.2. Heyting modal algebras

This section provides an algebraic semantics by defining Heyting modal algebras extending Heyting algebras, analogous to classical modal algebras extending Boolean algebras. The remaining sections are restricted to the algebraic version of propositional intuitionistic modal logic only. The extension to modal intuitionistic predicate logic can be developed following Rasiowa–Sikorski and Henkin–Tarski–Monk, and is not developed here. An additional symbol $\perp$ is added to the syntax already given for falsehood.

Definition 2.2.1. A structure $\langle H, \leqslant, \wedge, \vee, \Rightarrow, \perp, T, L, M\rangle$ is a *Heyting modal algebra* if

(1) $\langle H, \leqslant, \wedge, \vee, \Rightarrow, \perp, T\rangle$ is a Heyting algebra,

(2)(i) $L: H \to H$ satisfies $L(T) = T$, $L(x \cap y) = L(x) \cap L(y)$,

(ii) $M: H \to H$ satisfies M is monotone, $M(\perp) = \perp$, $L(x) \cap M(y) \leqslant M(x \cap y)$.

Definition 2.2.2. An *assignment* of intuitionistic propositional modal logic in a modal algebra H is a function h with domain the set of atomic propositions, range a subset of H. Each assignment h has a unique extension to an 'algebraic interpretation' or homomorphism h mapping the set of all modal propositions to H as follows:

$$\begin{aligned}
&h(\perp) = \perp,\\
&h(\varphi \wedge \phi) = h(\varphi) \wedge h(\phi),\\
&h(\varphi \vee \phi) = h(\varphi) \vee h(\phi),\\
&h(\varphi \to \phi) = h(\theta) \Rightarrow h(\phi),\\
&h(\Box\varphi) = L(h(\varphi)),\\
&h(\Diamond\varphi) = M(h(\varphi)).
\end{aligned}$$

Then φ is called *true*, or *valid*, in the algebraic interpretation if $h(\varphi) = T$.

Theorem 2.2.3 (Soundness theorem). *If* $\vdash \varphi$, *then* φ *is true in all algebraic interpretations.*

Proof. Let h be an algebraic interpretation in a Heyting modal algebra. It is shown below that, if φ is an axiom, then $h(\varphi) = T$. Also, if θ is derived from Γ by rules of proof and if $h(\Gamma) = T$, then $h(\theta) = T$.

Axioms

(1) The axioms of intuitionistic logic are valid in a Heyting algebra.

(2) If φ is $\Box(\theta_1 \rightarrow \theta_2) \rightarrow (\Box\theta_1 \rightarrow \Box\theta_2)$, then

$$\begin{aligned} h(\Box(\theta_1 \rightarrow \theta_2) \rightarrow (\Box\theta_1 \rightarrow \Box\theta_2)) &= h[\Box(\theta_1 \rightarrow \theta_2)] \Rightarrow [h(\Box\theta_1) \Rightarrow h(\Box\theta_2)] \\ &= L(h(\theta_1 \rightarrow \theta_2) \Rightarrow [L(h(\theta_1)) \Rightarrow L(h(\theta_2))] = T, \end{aligned}$$

as it can be seen that $L(P \Rightarrow Q) \leqslant L(P) \Rightarrow L(Q)$ because of the identity $L(x \cap y) = L(x) \cap L(y)$ and the monotonicity of L.

(3) If φ is $\Box(\theta_1 \rightarrow \theta_2) \rightarrow (\Diamond\theta_1 \rightarrow \Diamond\theta_2)$, then

$$\begin{aligned} h(\varphi) &= h(\Box(\theta_1 \rightarrow \theta_2)) \Rightarrow [h(\Diamond\theta_1) \Rightarrow h(\Diamond\theta_2)] \\ &= L[h(\theta_1) \Rightarrow h(\theta_2)] \Rightarrow [M(h(\theta_1)) \Rightarrow M(h(\theta_2))] = T, \end{aligned}$$

because from the argument below $L(P \Rightarrow Q) \wedge M(P) \leqslant M(Q)$. (Since $L(P \Rightarrow Q) \wedge M(P) \leqslant M((P \Rightarrow Q) \wedge P) \leqslant M(Q)$, it follows that $L(P \Rightarrow Q) \leqslant (M(P) \Rightarrow M(Q)$.)

(4) If φ is $\Box\theta_1 \wedge \Diamond(\theta_1 \rightarrow \theta_2) \rightarrow \Diamond\theta_2$, observe that

$$L(h(\theta_1)) \wedge M(h(\theta_1) \Rightarrow h(\theta_2)) \leqslant M[h(\theta_1) \wedge (h(\theta_1) \Rightarrow h(\theta_2))] \leqslant M[h(\theta_2)].$$

Hence $L(h(\theta_1)) \wedge M(h(\theta_1)) \Rightarrow M(h(\theta_2)) = T$. Therefore

$$\begin{aligned} &h[(\Box\theta_1 \wedge \Diamond(\theta_1 \rightarrow \theta_2)) \rightarrow \Diamond\theta_2] \\ &\qquad = L(h(\theta_1)) \wedge M(h(\theta_1)) \Rightarrow M(h(\theta_2)) \Rightarrow M(h(\theta_2)) = T. \end{aligned}$$

(5) The axiom $\Diamond(\varphi \wedge \neg\varphi) \rightarrow \varphi \wedge \neg\varphi$ translates to $\Diamond\bot \rightarrow \bot$, and

$$h(\Diamond\bot \rightarrow \bot) = M(h(\bot)) \Rightarrow \bot = M(\bot) \Rightarrow \bot = \bot \Rightarrow \bot = T.$$

Rules of inference

Modus ponens follows from the usual argument. As for necessitation—i.e., if $\vdash \varphi$, then $\vdash \Box\varphi$, assume that $h(\varphi) = T$. Then $h(\Box\varphi) = L(h(\varphi)) = L(T) = T$. The soundness of the axiom system now follows. $\Box$

Theorem 2.2.4 (Completeness theorem). *If* φ *is valid in all algebraic interpretations, then* $\vdash \varphi$.

Proof. The contrapositive is easily shown by assuming that $\vdash \varphi$ is not true and using the appropriate Lindenbaum algebra construction sketched below. Define the equivalence relation $\sim$ on propositions as $\varphi \sim \theta$ iff $\varphi \vdash \theta$ and $\theta \vdash \varphi$. Let H be the set of equivalence classes $[\varphi]$ under the relation $\sim$. Define $\wedge$, $\vee$, $\rightarrow$ on $H \times H$ as usual and $L: H \rightarrow H$ and $M: H \rightarrow H$ as $L([\varphi]) = [\Box\varphi)$ and $M([\varphi]) = [\Diamond\varphi]$. Then L and M are well-defined because $\varphi \vdash \theta$ implies $\Box\varphi \vdash \Box\theta$ and also that $\Diamond\varphi \vdash \Diamond\theta$. The algebraic interpretation h: propositions $\rightarrow H$ is defined by $h(\varphi) = [\varphi]$. This implies that $h(\varphi) = T$ if $\vdash \varphi$.

- $L(T) = T$ follows since $T \vdash \Box T$,
- $L(x) \cap L(x) = L(x \cap y)$ follows since $\vdash \Box\varphi_1 \wedge \Box\varphi_2 \leftrightarrow \Box(\varphi_1 \wedge \varphi_2)$,
- $M(\perp) = \perp$ follows from $\vdash \Diamond\perp \rightarrow \perp$,
- that M is monotonic follows since $\varphi_1 \vdash \varphi_2$ implies $\Diamond\varphi_1 \vdash \Diamond\varphi_2$,
- $L(x) \cap M(y) \leqslant M(x \cap y)$ follows since $\Box\varphi \wedge \Diamond\theta \vdash \Diamond(\varphi \wedge \theta)$.

It is now seen that there is an algebraic interpretation h such that $h(\varphi) \neq T$, contradicting the assumption. $\Box$

2.3. *Topological models of constructive modal logic.*

In this section the standard interpretation of intuitionistic logic by open sets in a topological space is extended to accommodate the modal operators. The basic structure used is a topological space with a distinguished relation on its points.

Definition 2.3.1. Let $\langle T, \tau \rangle$ be a topological space and let τ be the collection of all open subsets of T. Let $R \subseteq T \times T$ be a relation. For each $v \in \tau$ define

$$\Box v = \bigcup \{u \in \tau : R(u) \subseteq v\}, \qquad \Diamond v = \bigcup \{u \in \tau : u \in R^{-1}(v)\},$$

where

$$R(u) = \{y \in T : R(x, y) \text{ for some } x \in u\},$$

and

$$R^{-1}(u) = \{x \in T : R(x, y) \text{ for some } y \in u\}.$$

Definition 2.3.2. A *topological interpretation of intuitionistic propositional modal logic in a topological model* is a structure $\langle T, \tau, R, h \rangle$ such that

(i) $\langle T, \tau \rangle$ is a topological space and R is a relation on T.

(ii) h: the set of propositions $\rightarrow \tau$ satisfying the following properties:

$$h(\perp) = \emptyset \quad \text{and} \quad h(T) = T, \tag{1}$$

$$h(\varphi_1 \vee \varphi_2) = h(\varphi_1) \cup h(\varphi_2), \tag{2}$$

$$h(\varphi_1 \wedge \varphi_2) = h(\varphi_1) \cap h(\varphi_2), \tag{3}$$

$$h(\varphi \rightarrow \varphi_2) = \text{int}(h(\varphi_1)^c \cup h(\varphi_2)), \tag{4}$$

$$h(\Box\varphi) = \Box(h(\varphi)), \tag{5}$$

$$h(\Diamond\varphi) = \Diamond(h(\varphi)). \tag{6}$$

Definition 2.3.3. A proposition φ is *true* (or *valid*) in a topological interpretation $\langle T, \tau, R, h \rangle$ if and only if $h(\varphi) = T$. A proposition φ is *true* (*valid*) if φ is true in all topological interpretations.

Lemma 2.3.4 (Soundness). *If φ is provable, then φ is true in all topological interpretations.*

Proof. In order to prove this, it is only necessary to see that the axioms are true and the rules of inference preserve truth. Only the modal axioms will be shown to be true. For the others, see a standard reference such as [40]. Pick any topological interpretation $\langle T, \tau, R, h\rangle$.

(1) Suppose that φ is $\Box(\varphi_1 \to \varphi_2) \to (\Box\varphi_1 \to \Box\varphi_2)$. To show that $h(\varphi) = T$, it is enough to show that

$$\left(\bigcup \{u \in \tau : R(u) \subseteq \operatorname{int}(h(\varphi_1)^c \cup h(\varphi_2))\}\right) \cap \left(\bigcup \{u \in \tau : R(u) \subseteq h(\varphi_1)\}\right)$$

$$\subseteq \bigcup \{u \in \tau : R(u) \subseteq h(\varphi_2)\}.$$

Let $A = h(\varphi_1)$ and $B = h(\varphi_2)$. Pick any u_1 and u_2 for which $R(u_1) \subseteq \operatorname{int}(A^c \cup B)$ and $R(u_2) \subseteq A$. Then

$$R(u_1 \cap u_2) \subseteq R(u_1) \cap R(u_2) \subseteq \operatorname{int}(A^c \cup B) \cap A \subseteq B.$$

Then $u_1 \cap u_2 \in (\bigcup \{u \in \tau : R(u) \subseteq h(\varphi_2)\})$, justifying (1).

(2) Suppose that φ is $\Box(\varphi_1 \to \varphi_2) \to (\Diamond\varphi_1 \to \Diamond\varphi_2)$. To show that $h(\varphi) = T$, it is enough to show that $(\bigcup \{u \in \tau : R(u) \subseteq \operatorname{int}(A^c \cup B)\}) \cap (\bigcup \{u : u \subseteq R^{-1}(A)\}) \subseteq \bigcup(\{v : v \subseteq R^{-1}(B)\})$, where $A = h(\varphi_1)$ and $B = h(\varphi_2)$. Pick any $x \in u_1 \cap u_2$, where $R(u_1) \subseteq \operatorname{int}(A^c \cup B)$ and $u_2 \subseteq R^{-1}(A)$ for $u_1, u_2 \in \tau$. Then $x \in u_2$ implies that there is a $y \in A$ with $R(x, y)$. Then $x \in u_1$ and $R(u_1) \subseteq \operatorname{int}(A^c \cup B)$ imply $y \in \operatorname{int}(A^c \cup B)$ but $y \notin A^c$. Hence $y \in B$. Hence $x \in R^{-1}(B)$. Hence $u_1 \cap u_2 \subseteq R^{-1}(B)$, justifying (2).

(3) Suppose that φ is $\Box\varphi_1 \wedge \Diamond(\varphi_1 \to \varphi_2) \to \Diamond\varphi_2$. To show that $h(\varphi) = T$ it is enough to show that

$$\left(\bigcup \{u \in \tau : R(u) \subseteq A\}\right) \cap \left(\bigcup \{u \in \tau : u \subseteq R^{-1}(\operatorname{int}(A^c \cup B))\}\right)$$

$$\subseteq \left(\bigcup \{u \in \tau : u \subseteq R^{-1}(B)\}\right), \quad \text{where } A = h(\varphi_1) \text{ and } B = h(\varphi_2).$$

Pick any $x \in u_1 \cap u_2$, $R(u_1) \subseteq A$ and $u_2 \subseteq R^{-1}(\operatorname{int}(A^c \cup B))$ with $u_1, u_2 \in \tau$. Then there exists a $y \in \operatorname{int}(A^c \cup B)$ such that $R(x, y)$. Then $x \in u_1$ and $R(u_1) \subseteq A$ imply $y \in A$, because $y \in \operatorname{Int}(A^c \cup B)$. Hence $y \in B$. Thus (3) follows.

(4) Suppose that φ is $\Diamond(\bot) \to \bot$. Note that $h(\Diamond\bot) = \Diamond(h(\bot)) = \Diamond\emptyset = \emptyset$. Hence $h(\Diamond\bot \to \bot) = T$.

For the modal rules, we must show that $h(\phi) = T$ implies $h(\Box\phi) = T$. This is true because $h(\Box\phi) = \Box(h\phi) = \Box T = \bigcup \{u : R(u) \subseteq T\} = T$.

This, combined with the proof of soundness of the topological interpretation for intuitionistic propositional logic, gives the soundness result that is sought. □

Lemma 2.3.5 (Completeness of topological semantics). *If ψ is valid in all topological models, then ψ is provable.*

Proof. Suppose that ψ is not provable. Then by the completeness with respect to Kripke semantics, there is a Kripke frame $\langle W, \leqslant, R, \Vdash \rangle$ and a $\omega_0 \in W$ where $\omega_0 \Vdash \psi$ is false.

Define the topological model as follows. Let $T = W$. Let its set τ of all open sets be the set of all upward closed subsets of W. Let $\langle T, \tau, R \rangle$ be the topological space over which we define the topological interpretation. For atomic propositions φ, define $h(\varphi) = \{\omega \in W : \omega \Vdash \varphi\}$. Note that $h(\varphi) \in \tau$. Extend h to all formulas by

$$h(\varphi_1 \vee \varphi_2) = h(\varphi_1) \cup h(\varphi_2),$$

$$h(\varphi_1 \wedge \varphi_2) = h(\varphi_1) \cap h(\varphi_2),$$

$$h(\varphi_1 \rightarrow \varphi_2) = \operatorname{int}(h(\varphi_1)^c \cup h(\varphi_2)),$$

$$h(\Diamond\varphi_1) = \Diamond h(\varphi_1),$$

$$h(\Box\varphi_1) = \Box h(\varphi_1),$$

where for any open u the definition of $\Diamond u$ and $\Box u$ are given by Definition 2.3.1. Note also that $h(\varphi) \in \tau$ for all propositions φ.

Claim 2.3.6.

$$h(\varphi) = \{\omega \in W : \omega \Vdash \varphi\}.$$

Proof. The proof is by induction on the structure of φ. Only the modal cases are verified below. The other cases are as in [40].

(1) Suppose that $h(\varphi) = \{\omega \in W : \omega \Vdash \varphi\}$. We show that $h(\Box\varphi) = \Box h(\varphi) = \{\omega \in W : \omega \Vdash \Box\varphi\}$, i.e., that $\{\omega \in W : \omega \Vdash \Box\varphi\} = \bigcup \{u \in \tau : R(u) \subseteq h(\varphi)\}$, note that $\{\omega \in W : \omega \Vdash \Box\varphi\} \in \tau$ and that if for any $\omega, \omega' \in W$ $\omega \Vdash \Box\varphi$ and $R(\omega, \omega')$, then $\omega' \Vdash \varphi$. This gives $\omega' \in h(\varphi)$ by the inductive hypothesis. Hence $R(\{\omega \in W : \omega \Vdash \Box\varphi\}) \subseteq h(\varphi)$, and therefore $\{\omega \in W : \omega \Vdash \Box\varphi\} \subseteq \bigcup \{u : R(u) \subseteq h(\varphi)\}$.

Conversely, suppose that $\omega \in u \in \tau$ for $R(u) \subseteq h(\varphi)$. Now for any ω', ω'' such that $\omega \leqslant \omega'$ and $R(\omega', \omega'')$ and u upward closed implies that $\omega' \in u$ and $R(\omega', \omega'')$. Since $R(u) \subseteq h(\varphi)$, by the inductive hypothesis $\omega'' \Vdash \varphi$. Hence $\omega \Vdash \Box\varphi$.

(2) Suppose that $h(\varphi) = \{\omega \in W : \omega \Vdash \varphi\}$. We show that $h(\Diamond\varphi) = \Diamond h(\varphi) = \{\omega \in W : \omega \Vdash \Diamond\varphi\}$, i.e., $\{\omega \in W : \omega \Vdash \Diamond\varphi\} = \bigcup \{u \in \tau : u \subseteq R^{-1}(h(\varphi))\}$. First note that $\{\omega \in W : \omega \Vdash \Diamond\varphi\} \in \tau$. If we pick ω from this set, then $\omega \Vdash \Diamond\varphi$, and therefore $R(\omega, \omega')$ and $\omega' \Vdash \varphi$ for some $\omega' \in W$. By the inductive assumption we get $\omega' \in h(\varphi)$. Hence $\{\omega \in W : \omega \Vdash \Diamond\varphi\} \subseteq R^{-1}(h(\varphi))$.

Conversely, if $\omega \in u \in \tau$ and $u \subseteq R^{-1}(h(\varphi))$, to show that $\omega \Vdash \Diamond\varphi$, choose any $\omega' \geqslant \omega$. By the upward closure of u, $\omega' \in u$. Because $u \subseteq R^{-1}(h(\varphi))$, there is a

$\omega'' \in h(\varphi)$ satisfying $R(\omega', \omega'')$. Now, again by the inductive assumption, $\omega'' \Vdash \varphi$. Hence $\omega \Vdash \Diamond\varphi$. $\square$ Claim 2.3.6.

Now, back to the proof of Lemma 2.3.5. Note that $h(\varphi) = \{\omega \in W : \omega \Vdash \varphi\}$ by Claim 2.3.6. But $\omega_0 \nVdash \psi$, hence $\langle T, \tau, R, h\rangle$, as given above, is a topological interpretation in which $h(\psi) \neq T$, so ψ fails. $\square$

2.4. A homomorphism

Every Heyting algebra is embeddable in a topological Heyting algebra. We generalize this, but get only a homomorphism.

Definition 2.4.1. Let $(H, \leqslant, \wedge, \vee, \Rightarrow, \perp, T, L, M)$, $(H_1, \leqslant_1, \wedge_1, \vee_1, \Rightarrow_1, \perp_1, T_1, L_1, M_1)$ be Heyting modal algebras. Then $f: H \to H'$ is a *homomorphism* if

(i) f preserves the structure of the Heyting algebra operations,

(ii) $f(L(\varphi)) = L_1(f(\varphi))$ and $f(M(\varphi)) = M_1(f(\varphi))$.

Definition 2.4.2. An *intuitionistic topological modal algebra* is a topological space $\langle T, \tau\rangle$ with an arbitrary relation R in which $\leqslant$, $\wedge$, $\vee$, $\Rightarrow$, $\perp$, T, L, and M are interpreted as $\subseteq$, $\cap$, $\cup$, $\Rightarrow$, $\emptyset$, T, $\square$ and $\Diamond$ over the topology τ, where

(i) $\square u$ and $\Diamond u$ for $u \in \tau$ are defined as in Definition 2.3.1,

(ii) $u \Rightarrow v$ is defined as $\mathrm{int}(u^c \cup v)$ for $u, v \in \tau$.

Theorem 2.4.3. *Every Heyting modal algebras is homomorphically embeddable in an intuitionistic topological modal algebra by the f defined below.*

Proof. Let $(H, \leqslant, \wedge, \vee, \Rightarrow, \perp, T, \square, \Diamond)$ be a given Heyting modal algebra. A topological modal algebra will be produced so that H is embeddable in it. Let H^* be defined as $\{(u, \Diamond\varphi): u$ is a prime filter of H and $\Diamond\varphi \notin u\}$. Define R, a binary relation on H^*, by $(u, \Diamond p)\, R\, (v, \Diamond q)$ if (i) if $\square y \in u$, then $y \in u$; (ii) $p \notin v$.

Let $f: H \to P(H^*)$ be defined as $f(h) = \{(u, \Diamond p) \in H^* : h \in u\}$, and let τ be the topology generated by taking $\{f(h): h \in H\} = S$ as a basis. S is closed under finite intersections. Also,

$$f(h_1 \vee h_2) = f(h_1) \cup f(h_2),$$
$$f(h_1 \wedge h_2) = f(h_1) \cap f(h_2),$$
$$f(h_1 \Rightarrow h_2) = \mathrm{int}(f(h_1)^c \cup f(h_2)).$$

We verify

(1) $\quad f(\square a) = \square f(a) = \bigcup \{v \in \tau : R(v) \subseteq f(a)\}$,

(2) $\quad f(\Diamond a) = \Diamond f(a) = \bigcup \{v \in \tau : v \subseteq R^{-1}(f(a))\}$.

Verification of (1). We show that $\{(u, \Diamond p) \in H^* : \square a \in u\} = \bigcup \{v \in \tau : R(v) \subseteq f(a)\}$. If $\square a \in u$ and $(u, \Diamond p)\, R\, (u_1, \Diamond p_1)$, then by definition of R, $a \in u_1$. Hence $(u_1, \Diamond p_1) \in f(a)$. Note that $\{(u, \Diamond p) \in H^* : \square a \in u\} \in \tau$. Hence $R(\{(u, \Diamond p) \in H^* : \square a \in u\}) \subseteq f(a)$.

Conversely, if $(u, \Diamond x) \in \bigcup \{v \in \tau; R(v) \subseteq f(a)\}$, then there is a basic open set $v \in \tau$ such that $(u, \Diamond x) \in v$. Since v is basic open, $(u, \Diamond\bot) \in v$. Let $w' = \{b : \Box b \in u\}$. Note that w is upward closed and $\wedge$-closed. If $\bot \in w$, then $\Box\bot \in u$, and hence $\Box a \in u$ since $\Box\bot \leqslant \Box a$. Then there is nothing more to prove.

If $\bot \notin w'$ and $a \notin w'$, then there exists a $w \supseteq w'$ such that w is a prime filter and $a \notin w$. Hence $(u, \Diamond\bot)\, R\, (w, \Diamond\bot)$, but $a \notin w$. This contradicts $R(v) \subseteq f(a)$. Hence $\Box a \in u$, hence $(u, \Diamond x) \in \{(u, \Diamond p) \in H^* : \Box a \in u\}$.

Verification of (2). We show that $f(\Diamond a) = \Diamond f(a))$, i.e., that

$$\{(u, \Diamond p) \in H^* : \Diamond a \in u\} = \bigcup \{v \in \tau : v \subseteq R^{-1}(f(a))\}.$$

Notice that $(u, \Diamond p) \in H^* : \Diamond a \in u\} \in \tau$. To show that

$$\{(u, \Diamond p) \in H^* : \Diamond a \in u\} \subseteq \bigcup \{v \in \tau : v \subseteq R^{-1}(f(a))\},$$

pick any $(u, \Diamond p) \in H^*$ with $\Diamond a \in u$. Let $w' = \{a\} \cup \{x : \Box x \in u\}$.

Claim 2.4.4. *There is a nontrivial prime filter $w \supseteq w'$ with $p \notin w$ such that $(u, \Diamond p)\, R\, (w, \Diamond\bot)$.*

Proof. The filter v generated by w' is nontrivial and avoids p. Hence w exists by the usual argument. So $(u, \Diamond p)\, R\, (w, \Diamond\bot)$ and $(w, \Diamond\bot) \in f(a)$ since $a \in w$.

Conversely, suppose that $(u, \Diamond x) \in \bigcup \{v \in \tau : v \subseteq R^{-1}(f(a))\}$, and that $\Diamond a \notin u$. Then there exists a basic open set v_0 with $(u, \Diamond x) \in v_0$ and $v_0 \subseteq R^{-1}(f(a))$. If $\Diamond a \notin u$, then $(u, \Diamond a) \in v_0$. This contradicts $(u, \Diamond a) \in R^{-1}(f(a))$. Hence $\Diamond a \in u$. So f is a homomorphism. □

3. Further axioms

3.1. Two further axioms

We introduce two further axioms and their corresponding semantics. They are useful for concurrent dynamic logic [27]. Soundness and completeness will be proven. Section 3 ends with a brief description of the additional axioms and corresponding semantics for a system of Ewalds [7], also discovered by Plotkin and Stirling [33]. These are systems of intuitionistic modal logics that are in between our weakest possible system of Section 1 and the systems of Ewalds, Plotkin and Stirling, which already exist in the literature. The language used will be the same as before but will also include $\bot$ and T in addition to $\Diamond$ and $\Box$ as new symbols.

Additional axioms

(i) $\neg(\Diamond T) \rightarrow \Box\bot$,

(ii) $(\Diamond T \rightarrow \Box\varphi) \rightarrow \Box\varphi$.

Definition 3.1.1 (Semantics). $\langle W, \leqslant, R, \Vdash \rangle$ is here called a *model* if
- W is a collection of classical models,
- $\leqslant$ is a partial ordering on W,
- $R \subseteq W \times P(W)$,
- for all ω, ω', U with $\omega \leqslant \omega'$, if $R(\omega, U)$, then there is a $V \in P(W)$ such that
  (i) $R(\omega', V)$,
  (ii) for all $v \in V$ there is $u \in U$ such that $v \geqslant u$.

Forcing is defined as $\Vdash$ in Definition 1.1.4 except for the following cases:

(i) $\omega \Vdash \Box\varphi$ if for any $\omega' \in W$, and for any $U \subseteq W$ satisfying $\omega \leqslant \omega'$ and $R(\omega', U)$, we have that for all $w'' \in U$, $\omega'' \Vdash \varphi$.

(ii) $\omega \Vdash \Diamond\varphi$ if there exists a $U \in P(W)$ with $R(\omega, W)$ such that for all $\omega' \in U$, we have $\omega' \Vdash \varphi$.

Lemma 3.1.2. *If $\omega \Vdash \varphi$ and $\omega \leqslant \omega'$, then $\omega' \Vdash \varphi$.*

Theorem 3.1.3 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \Vdash \varphi$.*

Proof. Only the two new axioms will be verified.

(i) $\neg(\Diamond T) \to (\Box\bot)$. If not $\omega \Vdash \Box\bot$, then there exist ω', ω'', U such that $\omega, \omega' \in W$ and $U \subseteq W$ and $\omega \leqslant \omega'$ and $R(\omega', U)$ and $\omega'' \in U$ and not $(\omega'' \Vdash \bot)$. This implies not $(\omega \Vdash \neg(\Diamond T))$.

(ii) $(T \to \Box\varphi) \to \Box\varphi$. Suppose that $(\omega \Vdash \Box\varphi)$ is false. Then there exist $\omega, \omega', \omega'' \in W$ and $U \subseteq W$ with $\omega'' \in U$ satisfying $\omega \leqslant \omega'$ and $R(\omega', U)$ but not $(\omega'' \Vdash \varphi)$. Hence not $(\omega \Vdash \Diamond T \to \Box\varphi)$. $\Box$

Definition 3.1.4. Let F be the set of nontrivial prime filters of formulas that are deductively closed with respect to all the proof rules and axioms. Then $(\omega, U) \subseteq F \times P(F)$ is said to be an *accessibility pair* if,

(i) $\phi \in \omega'$ for all $\omega' \in U$ implies $\Diamond\phi \in \omega$,

(ii) $\Box\phi \in \omega$ implies $\phi \in \omega'$ for all $\omega' \in U$.

Lemma 3.1.5. *Suppose that $\omega \in F$ and $\Diamond\phi \in \omega$. Then there exists a $U \subseteq F$, with (ω, U) an accessibility pair, such that $\phi \in u$ for all $u \in U$.*

Proof. Let $V = \{\phi\} \cup \{\theta : \Box\theta \in \omega\}$; let U be the set of all deductively closed minimal prime filters extending V, constructed in stages as given below. Let $\{\psi_i \vee \theta_i\}$ be an enumeration of all sentences of the form $\psi \vee \theta$. Construct the set U_n at stage n; U is the union of all U_n. Index the elements of U_n by $\sigma \in \{0, 1\}^{<\omega}$.

Stage 0. Let $U_0 = V_0$, with V_0 the deductive closure of V.

Stage n. For each $j \leqslant n$ do the following. For each $V_\sigma \in U_n$, if $\psi_j \vee \theta_j \in V_\sigma$, but $\psi_j, \theta_j \notin V_\sigma$, and if $\{\psi_j\} \cup V_\sigma$ is consistent, define $V_{\sigma\frown 0}$ as the deductive closure of $V_\sigma \cup \{\psi_j\}$. Otherwise leave $V_{\sigma\frown 0}$ undefined. If $\{\theta_j\} \cup V_\sigma$ is consistent, then define

$V_{\sigma^\frown 1}$ as the deductive closure of $V_\sigma \cup \{\theta_j\}$). Otherwise leave $V_{\sigma^\frown 1}$ undefined. If $\psi_j \vee \theta_j \notin V_\sigma$, define $V_{\sigma^\frown 0}$, $V_{\sigma^\frown 1}$ as V_σ. Include $V_{\sigma^\frown 0}$ and $V_{\sigma^\frown 1}$ in U_{n+1}. If U_n is finite, then so is U_{n+1}. Let

$$U = \{\bigcup_{n \in \omega} V_{\sigma^{<n}} : \sigma \in \omega^\omega \text{ and } V_{\sigma^{<n}} \text{ is defined for all } n\},$$

where $\sigma^{<n}$ denotes the finite sequence that consist of the first n entries of σ. Condition 3.1.4(ii) is satisfied by construction. For 3.1.4(i), suppose that $\Diamond\psi \notin \omega$. Then $\psi \notin V_0$. Hence there exists a $V_\sigma \in U$ with $\psi \notin V_\sigma$. Hence (ω, U) is an accessibility pair. $\square$

Lemma 3.1.6. *Let $\omega, \omega' \in F$ with $\omega \leqslant \omega'$ and (ω, U) an accessibility pair constructed as in Lemma 3.1.5. Then there exists a $U' \subseteq F$ such that*
(1) *(ω', U') is an accessibility pair,*
(2) *for each $u' \in U'$, there exists a $u \in U$ such that $u \leqslant u'$.*

Proof. Let ω, ω' and U be as given. Let V' be the deductive closure of $V_1 \cup V_2$, where $V_1 = \{\phi : \Box\phi \in \omega'\}$ and $V_2 = \{\phi : \phi \in u \text{ for all } u \in U\}$. Then V' is consistent. For otherwise, $\phi_1, \phi_2 \vdash \bot$ for some $\Box\phi_1 \in \omega'$ and $\Diamond\phi_2 \in \omega$. Hence $\Box\phi_1$, $\Diamond\phi_2 \vdash \Diamond\bot \vdash \bot$, giving $\bot \in \omega'$. Carry out the same construction as we did in the previous lemma using the V' given in the present proof to obtain deductively closed prime filters which we name V'_σ. We verify that if V'_σ is defined, then so is V_σ, and $V_\sigma \subset V'_\sigma$. At the nth stage of the construction, for $\psi_i \vee \varphi_i$, if ψ_i or φ_i is consistent with $V'_{\sigma^{<n}}$, then so is ψ_i or φ_i consistent with $V_{\sigma^{<n}}$. Hence if either of $V'_{\sigma^{<n+1}}$ is defined, then so is $V_{\sigma^{<n+1}}$. Further, if this is the case, then $V_{\sigma^\frown k} \subset V'_{\sigma^\frown k}$ for $k = 0, 1$. The only thing left to verify is that for all $p' \in U'$, there exists a $p' \in U$ with $p \leqslant p'$. This follows from the way that p' is obtained in the usual proof extending a filter to a prime filter. Namely, let $\{\alpha_i \vee \beta_i\}$ be a list of formulas such that $\alpha_i \vee \beta_i \in U$ but neither $\alpha_i, \beta_i \in U$. Since p' is a prime filter, for each i, either α_i or β_i belongs to p'. Let that one be γ_i. Now take p to be the minimal prime filter created by choosing γ_i for each step in extending U to a prime filter. Then $\bot \notin p$, otherwise $\bot \in p'$. $\square$

Theorem 3.1.7 (Completeness with the additional axioms). *If $\Gamma \vdash \varphi$, then $\Gamma \Vdash \varphi$.*

Proof. Suppose φ is not provable from Γ. Extend Γ to Γ_0 with $\Gamma_0 \in F$ and $\varphi \notin \Gamma_0$ by Lemma 1.4.4. Construct the model as follows. Let $W = F$. Define $\leqslant$ on W by $\omega_1 \leqslant \omega_2$ iff $\omega_1 \subseteq \omega_2$. Define R on $W \times P(W)$ as $R(\omega, U)$ if there is an $\Diamond\varphi \in \omega$, with U defined from $\{\varphi\} \cup \{\theta : \Box\theta \in \omega\}$ as given in Lemma 3.1.5. For each $\omega \in W$ and for an atomic proposition θ, define $\omega \Vdash \theta$ if $\theta \in \omega$. Extend $\Vdash$ to all formulas as given in Definition 3.1.1. It can be verified that $\omega \Vdash \theta$ if and only if $\theta \in \omega$ by induction on the complexity of θ. Only the inductive cases for the modal connectives are provided below.

For $\Diamond$. By Lemma 3.1.5, if $\Diamond\theta \in \omega \in F$, then there is a U such that $R(\omega, U)$ and $\theta \in \omega'$ for all $\omega' \in U$, and hence $\omega' \Vdash \theta$ by the inductive assumption. Therefore $\omega \Vdash \Diamond\theta$. Conversely, if $\Diamond\theta \notin \omega$ and if $R(\omega, U)$, then by the proof of Lemma 3.1.5, there exists a $u \in U$ with $\theta \notin u$. Hence $(\omega \Vdash \Diamond\theta)$ is false.

For $\Box$. Suppose $\Box\theta \in \omega$. Then for any $\omega' \in F$ and any $U \subseteq F$, if $\omega \leqslant \omega'$ and $R(\omega', U)$, then $\Box\theta \in \omega'$. Hence, by the definition of R, $\theta \in u$ for all $u \in U$, thus by the inductive assumption, $u \Vdash \theta$. Hence $\omega \Vdash \Box\theta$. Conversely, if $\Box\theta \notin \omega$, then $\Box(\bot) \notin \omega$. Hence $\neg(\Diamond T) \notin \omega$ by Axiom (i). Therefore $\omega \cup \{\Diamond T\}$ is consistent, but $\Diamond T$, $\omega \nvdash \Box\theta$. For otherwise, $\Diamond T$, $\omega \vdash \Box\theta$. Hence $\omega \vdash \Diamond T \rightarrow \Box\theta$, and by Axiom (ii), $\omega \vdash \Box\theta$. So $\Box\theta \in \omega$, a contradiction. Let $\omega' = \omega \cup \{\Diamond T\}$, and extend ω' to $\omega'' \in F$ so that $\Box\theta \notin \omega''$. Let $U = \{\psi : \Box\psi \in \omega''\}$. Notice that U is a proper filter, and $\theta \notin u$. Hence if we carry out the same procedure as in Lemma 3.1.5, we get $U' \subset W$ satisfying $R(\omega'', U')$ with $\theta \notin u'$ for some $u' \in U$. Then by the inductive assumption, $u' \nVdash \theta$. Hence $\omega \nVdash \Box\theta$. $\Box$

3.2. *Algebraic and topological semantics*

Topological and algebraic semantics that correspond to the system of Section 3.1 are presented in this section. The Heyting modal algebra satisfies $\neg\Diamond(T) \rightarrow \Box(\bot)$ and $(\Box(T) \rightarrow \Diamond(x)) \rightarrow \Diamond(x)$, corresponding to the new axioms added in Section 3.1. Interpretation in an algebraic model and the standard results of soundness and completeness follow by means of the method employed in Section 2.

Definition 3.2.1 (Topological semantics for the additional axioms). Let $\langle T, \tau\rangle$ be a topological space with τ the topology on T. Let $R \subseteq T \times (P(T) - \{\emptyset\})$ satisfy $V \in \tau$ implies $R^{-1}(V) \in \tau$. Then $\langle T, \tau, R\rangle$ is called a *topological model* for the system under consideration.

For $U \in \tau$, let $\Diamond U = \{x \in T : V \subseteq U$ and $R(x, V)$ for some subset $V\} = R^*(U)$ (say) and $\Box U = \bigcup\{V : R(V) \subseteq U\}$, where $R(V) = \{x \in T :$ there is a subset $U \subseteq V$ with $R(x, U)\}$. The interpretation in a topological model is the same as in Definition 2.3.2.

Theorem 3.2.2 (Soundness for the additional axioms). *If φ is provable, then φ is valid.*

Proof. Suppose $\langle T, \tau, R, h\rangle$ is an interpretation into a topological model. Then it is necessary to verify that if $\vdash \varphi$, then $h(\varphi) = T$. What is presented below is only a verification of the modal axioms and rules of inference.

(1) When φ is the axiom $\Box(\varphi_1 \rightarrow \varphi_2) \rightarrow (\Box\varphi_1 \rightarrow \Box\varphi_2)$, it is enough to show that

$$\left(\bigcup\{U : R(U) \subseteq \operatorname{int}(h(\varphi_1)^c \cup h(\varphi_2))\}\right) \cap \left(\bigcup\{U : R(U) \subseteq h(\varphi_1)\}\right) \subseteq \bigcup\{U : R(U) \subseteq h(\varphi_2)\}.$$

Pick any U_1, U_2 with $R(U_1) \subseteq \mathrm{int}(h(\varphi_1)^c \cup h(\varphi_2))$ and $R(U_2) \subseteq h(\varphi_1)$. Since $R(U_1 \cap U_2) \subseteq R(U_1) \cap R(U_2)$, we get $R(U_1 \cap U_2) \subseteq h(\varphi_2)$.

(2) When φ is the axiom $\Box(\varphi_1 \to \varphi_2) \to (\Diamond\varphi_1 \to \Diamond\varphi_2)$, it is enough to show that

$$\left(\bigcup \{U \in \tau : R(U) \subseteq \mathrm{int}(h(\varphi_1)^c \cup h(\varphi_2))\}\right) \cap R^*(h(\varphi_1)) \subseteq R^*(h(\varphi_2)).$$

Pick an $x \in U_1 \cap R^*(h(\varphi_1))$, where $R(U_1) \subseteq \mathrm{int}(h(\varphi_1)^c \cup h(\varphi_2))$. Then there exists a $V \subseteq h(\varphi_1)$ satisfying $R(x, V)$. By definition of $R(U_1)$ we get $V \subseteq \mathrm{int}(h(\varphi_1)^c \cup h(\varphi_2))$. Hence $x \in \mathrm{int}(h(\varphi_1)^c \cup h(\varphi_2)) \cap h(\varphi_1) \subset h(\varphi_2)$. Hence $x \in R^*(h(\varphi_2))$.

(3) When φ is $\Diamond\bot \to \bot$, this is true since $R^*(\emptyset) = \emptyset$.

(4) When φ is $\neg\Diamond T \to \Box(\bot)$, we need to show that

$$\mathrm{int}[R^*(T)^c] \subseteq \bigcup \{U : R(U) \subseteq \emptyset\}.$$

This is true since $x \in \mathrm{int}[R^*(T)^c]$ implies that there is no $V \neq \emptyset$ with $R(x, V)$.

(5) When φ is $(\Diamond T \to \Box\varphi) \to \Box\varphi$, we need to show that

$$\mathrm{int}\left[R^*(T)^c \cup \left(\bigcup \{U \in \tau : R(U) \subseteq h(\varphi)\}\right)\right] \subseteq \bigcup \{U : R(U) \subseteq h(\varphi)\}.$$

This is true because if $x \in R^*(T)^c$, then $R(x, V)$ is not true for any V. □

Theorem 3.2.3 (Completeness for the additional axioms). *If $\Gamma \Vdash \varphi$, then $\Gamma \vdash \varphi$.*

Proof. Assume not $\Gamma \vdash \varphi$ and produce a Kripke model for not $\Gamma \Vdash \varphi$. Use the same methods as in Lemma 2.3.5 to turn this into a topological model. □

3.3. Ewald's axioms

The system of modal logic developed by Ewalds [7] uses more axioms than presented here, and is too restrictive to cover our intended application to constructive concurrent dynamic logic. Plotkin and Stirling [33] presented an equivalent set of axioms. We sketch the Ewalds–Stirling–Plotkin system here, because it has the right conditions for describing constructive *sequential* dynamic logic.

Semantics

Definition 3.3.1. $\langle W, \leqslant, R\rangle$ is said to be an *(Ewald–Plotkin–Stirling) Kripke model* if

(i) W is a collection of classical models,

(ii) $\leqslant$ is a partial order on W and $R \subseteq W \times W$ is a relation satisfying the following conditions:

(a) $u_1 \geqslant \omega_1$ and $R(\omega_1, \omega_2)$ implies that there exists a $u_2 \geqslant \omega_2$ with $R(u_1, u_2)$.

(b) $u_2 \geqslant \omega_2$ and $R(\omega_1, \omega_2)$ implies that there exists a $u_1 \geqslant \omega_1$ with $R(u_1, u_2)$.

Definition 3.3.2. The definition of *forcing* is the same as before except for the modal connectives, for which they are defined as

(i) $\omega \Vdash \Diamond\varphi$ if there exists a ω' satisfying $R(\omega, \omega')$ and $\omega' \Vdash \varphi$,

(ii) $\omega \Vdash \Box\varphi$ if for all ω', ω'' satisfying $\omega' \geqslant \omega$ and $R(\omega', \omega'')$, we have $\omega'' \Vdash \varphi$.

Axioms

(1) Axioms of intuitionistic propositional logic.

(3.1) $\neg(\Diamond\varphi) \to \Box\neg\varphi$,

(3.2) $(\perp) \to \perp$,

(3.3) $(\Diamond\varphi \to \Box\theta_1 \vee \Box\theta_2) \to \Box(\varphi \to \theta_1 \vee \theta_2)$,

(3.4) $\Diamond(\varphi \vee \theta) \to \Diamond\varphi \vee \Diamond\theta$,

(3.5) $\Box(\varphi \to \theta) \to (\Box\varphi \to \Box\theta)$,

(3.6) $\Diamond\varphi \wedge \Box(\varphi \to \theta) \to \Diamond\theta$,

(3.7) $\Box\varphi \wedge \Diamond(\varphi \to \theta) \to \Diamond\theta$.

Rules of inference

(1) $\varphi, \varphi \to \theta \vdash \theta$,

(2) $$\frac{\Gamma, \varphi \vdash \theta}{\Box\Gamma, \Diamond\varphi \vdash \Diamond\theta},$$

(3) $$\frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi}.$$

Note. Provability now means with respect to all new axioms and proof rules.

Theorem 3.3.3. $\Gamma \vdash \varphi$ *if and only if* $\Gamma \Vdash \varphi$.

Proof. In Ewalds [7]. □

Theorem 3.3.4. *Adding* $\Diamond T \vee \neg(\Diamond T)$ *and* $\Box(\varphi \vee \theta) \to \Box\varphi \vee \Box\theta$ *to the axioms and imposing further conditions* (c) *and* (d) *below on the Kripke models of Definition* 3.3.1 *gives a sound and complete system.*

(c) $w_1 \leqslant u_1$ and $R(u_1, u_2)$ *implies there is a* $w_2 \leqslant u_2$ *with* $R(w_1, w_2)$;

(d) $w_2 \leqslant u_2$ *and* $R(u_1, u_2)$ *implies there is a* $w_1 \leqslant u_1$ *with* $R(w_1, w_2)$.

Proof. See Ewalds [7]. □

Plotkin and Stirling also presented an equivalent system in their extended abstract [33].

Acknowledgements

The author thanks Anil Nerode, Richard A. Shore, Robert L. Constable and James B. Lipton for many fruitful discussions.

References

[1] S. Abramsky, Domain theory in logical form, Ann. Pure Appl. Logic, to appear.
[2] M. Bozic and K. Dosen, Models for nomal intuitionistic logics, Studia Logica 43 (1984) 217–245.
[3] R. Bull, MIPC as the formalization of an intuitionistic concept of modality, J. Symbolic Logic 31 (1966) 609–616.
[4] R. Burstall, Program proving as hand simulation with a little induction, Inform. Process. 74 (1974) 308–312.
[5] A. Chandra, D. Kozen and L. Stockmeyer, Alternation, J. Assoc. Comput. Mach. 28 (1981) 114–133.
[6] K. Dosen, Models for strongly normal intuitionistic modal logics, Studia Logica 44 (1985) 39–70.
[7] W.B. Ewalds, Intuitionistic tense and modal logic, J. Symbolic Logic 51 (1986) 166–179.
[8] G. Fisher-Servi, On modal logic with an intuitionistic base, Studia Logica 36 (1977) 2–36.
[9] G. Fisher-Servi, Semantics for a class of intuitionistic modal calculi, Italian Studies in the Philosophy of Science (1981) 59–72.
[10] F.B. Fitch, Intuitionistic modal logic with quantifiers, Portugal. Math. 7 (1949) 113–118.
[11] M.C. Fitting, Proof Methods for Modal and Intuitionistic Logic (Reidel, Dordrecht, 1983).
[12] R. Floyd, Assigning meaning to programs, Proc. Amer. Math. Soc. 19 (1967) 19–32.
[13] M. Font, Modality and possibility in some intuitionistic modal Logics, Notre Dame. Formal Logic 27 (1986) 533–546.
[14] D.M. Gabbay, Investigations in Modal and Tense Logics with Applications to Problems in Philosophy and Linguistics (Reidel, Dordrecht, 1977).
[15] D.M. Gabbay, Semantical Investigations in Heyting's Intuitionistic Logic (Reidel, Dordrecht, 1980).
[16] D.M. Gabbay and F. Guenthner, Handbook of Philosophical Logic, Volumes I, II, III (Reidel, Dordrecht, 1986).
[17] R.I. Goldblatt, Logic of Time and Computation, CSLI Lecture Notes 7 (1987).
[18] R.I. Goldblatt, Topoi. The Categorial Analysis of Logic (North-Holland, Amsterdam, 1981).
[19] D. Harel, Dynamic Logic in: [16].
[20] D. Harel, Dynamic Logic, Lecture Notes in Comput. Sci. 68 (1976).
[21] C. C. Hoare, An axiomatic basis for computer programming, Comm. ACM 12 (1969) 576–583.
[22] D. Kozen and R. Parikh, An elementary proof of the completeness of PDL (Theoret. Comput. Sci. 14 (1981) 113–118.
[23] D. Kozen and J. Tiuryn. Logics of programs, in: J. Van Leeuwen, ed., Handbook of Theoretical Computer Science Vol. B (Elsevier, Amsterdam, 1990) Chapter 14, 789–840.
[24] D. Leivant, Proof theoretic methodology for propositional dynamic logic, Lecture Notes in Comput. Sci. 107 (1981) 356–373.
[25] A. Nerode, Some lecture notes in modal logic, in: CIME Montecatini Volume (Springer, Berlin, 1990), to appear.
[26] A. Nerode, Some lecture notes in intuitionistic logic, NATO Adv. Sci. Ser., to appear.
[27] A. Nerode and D. Wijesekera, Constructive Concurrent Dynamic Logic, Mathematics and Artificial Intelligence 1 (2) (1990), to appear.
[28] H. Nishimura, Semantical analysis of constructive PDL, Publ. Res. Inst. Math. Sci. 18(2) (1982) 427–438.
[29] H. Ono, On some intuitionistic modal logics, Publ. Res. Inst. Math. Sci. 13 (1977) 687–722.
[30] R. Parikh, The logic of games and its applications, Ann. Discrete Math. 24 (1985) 195–200.
[31] D. Peleg, Concurrent dynamic logic, J. Assoc. Comput. Mach. 34(2) (1987) 450–479.

[32] D. Peleg, Communication in concurrent dynamic logic, J. Comput. System Sci. 35 (1987) 23–58.
[33] G. Plotkin and C. Stirling, A framework for intuitionistic modal logic, in: J.Y. Halpern, ed., Theoretical Aspects of Reasoning About Knowledge (1986) 399–406.
[34] V. Pratt, Semantical considerations on Floyd–Hoare logic, in: Proc. 17th Annual IEEE Symp. on Found. Comput. Sci., New York, 1976, 109–121.
[35] V. Pratt, Applications of modal logic to programming, Studia Logica 39 (1980) 257–274.
[36] D. Scott, Identity and existence in intuitionistic logic, Lecture Notes in Math. 753 (1981) 580–695.
[37] C. Stirling, Modal logics for communicating systems, Technical Report CSR-193-85, Dept. Comput. Sci., Univ. Edinburgh, 1985.
[38] G. Takeuti, Proof Theory, 2nd rev. ed. (North-Holland, Amsterdam, 1987).
[39] A.S. Troelstra and D. Van Dalen, Constructivism in Mathematics: An Introduction, I, II (North-Holland, Amsterdam, 1988).
[40] D. Van Dalen, Intuitionistic logic, in: D.M. Gabbay and F. Guenther, eds., Handbook of Philosophical Logic (Reidel, Dordrecht, 1986) 255–339.