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AN OVERVIEW OF FIBRED SEMANTICS AND THE COMBINATION OF LOGICS

Abstract. This paper presents an overview of the authors methodology of Fibred Semantics and The Combinations of Logics presented in a series of papers under the same title. We explain the ideas behind fibring and illustrate them in several case studies. We include fibring modal and intuitionistic logics, fibring with fuzzy logics as well as self fibring of predicate logics.

1. Introduction

The purpose of this paper is to present an intuitive explanation of the author's fibred semantics methodology for combining logics and systems, and to give a brief overview of the main results, ideas and open problems in the area.

The problem of combining logics and systems is central to modern logic, both pure and applied.

The need to combine logics arises both from applications and from within logic itself as a discipline. As logic is being used more and more to formalise field problems in philosophy, language, artificial intelligence, logic programming and computer science, the kind of logics required become more and more complex. An increasing number of features from the application area need to be formally represented. These features are highly mutually interactive and a formal study of their combined nature becomes necessary. A methodology for combining systems has now become essential for most logic applications.

A simple example will illustrate the complexity of the problem we are facing. There exist well known and well studied logical systems and theories of belief and knowledge. Such systems use notation such as $B\varphi$ (φ is believed) and $K\varphi$ (φ is known). The formal theory of such notions has been studied in the past 30 years. Similarly there are well known systems

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of temporal logics, with operators such as $G\varphi$ (φ will always be true) and $H\varphi$ (φ has always been true). Such systems were developed in connection with the study of language and some problems in the philosophy of time. With the rise of artificial intelligence as a consumer of logic, various theories and models of reactive agents were put forward. Such theories involve a cycle of temporally evolving and changing repositories of knowledge and beliefs, as well as theories of actions for agents making use of their changing knowledge and beliefs.

Clearly any logical system modelling reactive agents should be a combined system of logics of knowledge, belief, time and modal logics (of actions).

It is therefore clear that a good understanding and practical methodology for combining logics can provide the consumer of logic (e.g. a robotics designer) with tools which make him more competent in designing his own system. Such a methodology can help decompose the problem of designing a complex system into developing components (logics) and combining them.

Furthermore, such a methodology might allow the designer to use existing components (which may be very well understood and/or efficiently implemented) and put them together into a desired useful combined system, at a relatively acceptable intellectual and practical cost.

In fact, a good combining methodology allows the study of the complex application area in terms of some of its pure components, yielding both theoretical and practical benefits.

A good methodology of combining systems should study *transfer theorems*. Such theorems prove that if the components satisfy property $\mathcal{P}$, then we are assured that the combined system satisfies (a possibly slightly different) property $\mathcal{P}'$. Moreover, transfer theorems can also help validate existing complex systems by verifying properties of their components.

It turns out that our fibring methodology does not only help realise many of the above expectations, but also seems to bring out unexpected theoretical properties of pure logic itself, showing new connections between existing themes of logic.

The following is a list of the main areas where the fibring methodology makes a serious contribution.

- Straightforward combination of logics such as multimodal logics or modal intuitionistic logics.
- Combining a metalevel language with its object level language. For example the problem of bringing a consequence relation into the object language as a conditional.
- Allowing for a single language to be syntactically ambivalent, and regarding the various ambivalent 'aspects' as a combined system. For example allowing classical logic formulas to apply to themselves (e.g. $\varphi(\varphi)$).

– Modelling the temporal behaviour of a system by fibring it with a temporal logic.
– Making a system fuzzy by fibring it with a fuzzy logic.

The power of the concept of fibring is also emphasised by some other recent research activities that arose independently from our own work. In (Pfalzgraf, 1991; Pfalzgraf & Stokkermans, 1994) the introduction of *logical fibrings* has been strongly motivated by the classical notion of fibre bundles and sheaves. Concrete applications aim at modelling logical control of cooperating robots (agents) scenarios. In (Baader & Schulz, 1995a; Baader & Schulz, 1995; Baader & Schulz, 1995b; Kepser & Schulz, 1996), another variant of fibring is used for combining solution domains and constraint solvers for symbolic constraints.

Our plan for this paper is first to give quick details of the above areas in subsections 1.1–1.5 and then explain the idea of the fibring methodology in Section 2. Further sections present sample case studies and the final section concludes with a discussion.

1.1. COMBINING PURE LOGICAL SYSTEMS

Problems of combining two logics arise both in pure logical theory and in practical applications. We give several typical examples

1. *Modal Intuitionistic Logics*
   The logic of modality $\Box$ was originally philosophically motivated. The same is true for intuitionistic logic. Intuitionistic logic arose as the logic for constructive mathematics. It was natural therefore, for philosophers and pure logicians, to try and study the nature of modal necessity $\Box$ and possibility $\Diamond$ from a constructive point of view. Such studies gave rise to systems involving both intuitionistic implication $\Rightarrow$ and the modal operators $\Box$ and $\Diamond$. In the past 30 years many modal intuitionistic systems were put forward, by philosophically motivated considerations. These systems were formulated directly for the mixed language and were not presented as combined systems. Various methods were used in the presentation, some semantical, some theoretical, some straightforward and some rather roundabout and highly individual. Among the early philosophically motivated systems by famous logicians were Fitch (Fitch, 1969), Bull (Bull, 1965) and Ono (Ono, 1977; Ono, 1987).
   Later, as modal logic has had serious applications in computer science, a new wave of interest in modal intuitionistic logic arose, by virtue of its applicability. The modality could stand for possible algorithmic options and the intuitionistic proof theory for the constructive nature of the

execution. Sample papers in this category are Wijesekera (Wijesekera, 1990) and A. K. Simpson (Simpson, 1994).
After a while logicians got fascinated by modal intuitionistic logics for formal technical reasons. Many papers were written without any special philosophical or field applications motivation. Among them are (Amati & Pirri, to appear; Font, 1986; Fischer-Servi, 1977; Fischer-Servi, 1984; Fischer-Servi, 1980; Došen, 1985; Suzuki, 1990; Božić & Došen, 1984; Ewald, 1986; Ono, 1987; Suzuki, 1988).
Among the technical problems involved in the modal intuitionistic logic area are:

- proof theory for the system. Hilbert formulation, Gentzen formulation, tableaux formulation, etc.;
- Kripke type semantics for the system;
- decision procedures;
- finite model property;
- comparison with other systems of modal intuitionistic logics, in terms of expressive power and relative interpretability.

We shall see that our fibring methodology can yield many of the above systems and results automatically and in bulk. Our main paper in this area is (Gabbay, 1993), which includes a detailed discussion of the main systems in the literature. In this survey in Section 3, we quote the main theorems of our paper.

2. *syntactical Modality*
Here we mean that a modal operator may arise naturally in another logic by a syntactic definition. For example in Girard linear logic, which is independently and very deeply motivated, some modalities (the exclamation !) arise naturally by definition. Studying their properties is of course the study of a combined system. Another example is the modality $\Box A =$ def$(A \Rightarrow A) \Rightarrow A$ in Anderson and Belnap's *Entailment and Relevant Logics* (Anderson & Belnap, 1975). The above does turn out to be some sort of modal-relevant system. See also the paper of Meyer and Mares (Meyer & Mares, 1993).
Modalities can also be obtained as by-products of translations and interpretations. We mention two well-known examples.
   (a) the strict implication of modal **S4** for boxed formulas is intuitionistic implication, thus giving rise to a combined modal intuitionistic logic.
   (b) the semantics translation of modal logic into classical predicate logic gives rise to a modal intuitionistic logic when classical predicate logic is weakened into intuitionistic predicate logic.

3. *Multimodal Logics*
   These arise from applications where several modal operators interact. The application may be a philosophical study of some notions and its paradoxes, such as systems of obligation and permission, see (Jones & Pörn, 1985; Jones & Pörn, 1986) or a direct formalisation of some practical application area, e.g. logical omniscience, see (Fagin *et al.*, 1990).
   A large very useful area of multimodal logic is dynamic logic. In fact, some applied logicians believe that multimodal logic is *the logic* for computer science applications.

The bulk of the literature on combined systems is mainly concerned with the presentation of individual combined systems. There are a few papers on methodological questions and transfer theorems. Besides my own papers on fibring methodology, there are some theoretical studies of multiple modalities as well as some algebraic studies of transfer properties. See (Fischer-Servi, 1977; Fitting, 1969; Fine & Schurz, to appear; Kracht & Wolter, 1991; Goranko & Passy, 1992). Again, I am happy to say, that the fibred semantics methodology gives bulk results in multimodal logics. See (Gabbay, 1993).

1.2. COMBINING META-LEVEL WITH THE OBJECT LEVEL

This is a new application of the fibred methodology.

Consider a consequence relation of the form $\Delta \mid\!\sim B$ between wffs of a logic. Assume it is a non-monotonic consequence relation on classical language satisfying Reflexivity ($\Delta \mid\!\sim A$ if $A \in \Delta$); Restricted monotonicity ($\Delta \mid\!\sim A$ and $\Delta \mid\!\sim B$ imply $\Delta, A \mid\!\sim B$) and Cut ($\Delta \mid\!\sim A$ and $\Delta, A \mid\!\sim B$ imply $\Delta \mid\!\sim B$). See (Gabbay, 1985; Kraus *et al.*, 1990). We would like to bring $\mid\!\sim$ into the language as a connective $A > B$ and find a conservative extension $\mid\!\sim^*$ of $\mid\!\sim$ such that the following holds

- $\varnothing \mid\!\sim^* A > B$ iff $A \mid\!\sim B$.

The above means that $>$ represents $\mid\!\sim$ in the language itself.

The fibred semantics methodology allows us to do this in a methodological way.

Delgrande has already observed (Delgrande, 1988) the formal similarities between non-monotonic consequence relations $\mid\!\sim$ and the conditional $>$. Of course $>$ is an object level connective and $\mid\!\sim$ is a metalevel consequence. Our fibred methodology turns $\mid\!\sim$ into a conditional. In our paper (Gabbay, 1995d) we indeed get the traditional semantics for the conditional out of the fibred semantics of the construction that brings $\mid\!\sim$ into the object level.

1.3. SELF-FIBRING OF PREDICATE LOGICS

This is another surprising application of the fibred methodology. It allows us to give solid and meaningful semantics to expressions of the form $A(t, \varphi(f(\alpha))$ where $A(x, y)$ is a formula, $\alpha, \varphi(x)$ are formulas and $f(x)$ a function symbol. As you can see, we have substituted formulas within other formulas and other function symbols.

Such expressions and languages are widely used in logic, philosophy and linguistics and have extensive and important applications. We mention a few typical examples:

- Logics of non-denoting singular terms
- In the logical analysis of language, we try and analyse statements like 'John said that Mary believed that he did not love her'
  Here we use the predicates, *say* (t, φ) and *believe* (t, φ).
- In logic programming there is a widespread use of metapredicates such as
  $Demo(A, B)$ and
  Y if $Not(X)$

  Ordinary clauses for the transitive closure of a relation R are written as

  $$(cl(Z))(X, Y) \text{ if } Z(X, Y)$$
  $$(cl(Z))(X, Y) \text{ if } Z(X, V) \wedge ((cl(Z))(V, X))$$

  or expressions like

  $$\forall x.\ \text{bel}(\text{John, friend (John, } x)) \rightarrow \text{ exists } (y,\ \text{loves } (x, y))),$$

- Tarski truth predicate

  $$\mathbf{T}(\varphi) \leftrightarrow \varphi$$

  and the diagonalisation function symbol

  $$d(\varphi) = \sim \varphi(\varphi)$$

We make sense of the above expressions using fibred semantics.

1.4. TEMPORALISING A SYSTEM

This example is very practical. Given a system $\mathcal{S}$, we can describe it in some logic $\mathbf{L}$. We now let the system vary in time. We can get snapshots $\mathcal{S}_t$ of it at different times t. What would be a suitable logic to describe the evolving and changing system?

The answer is the combination of $\mathbf{L}$ with a suitable temporal logic of our choice. The fibred semantics methodology will tell us what is the semantics of the new combined logic and the availability of transfer theorems will tell us what properties to expect of the combined system, given what we know of the components. This is done in, e.g. (Finger & Gabbay, to appear) and (Finger & Gabbay, 1992).

1.5. MAKING YOUR LOGIC FUZZY

This is the most satisfactory application of fibred semantics. Fuzzy logic has been controversial among pure logicians for a long time. It lacked respectability. People had the impression that we can take any system and make it fuzzy in an ad hoc way, by taking any $\{0, 1\}$ function we can find and turning it into a fuzzy $[0, 1]$ function. There is no methodology and no logic.

Take modal logic for example. Its models have the form (S, R, a, h) where S is the set of possible worlds, $a \in S$, R is a crisp $\{0, 1\}$ relation and h is the assignment, giving for each $t \in S$ and atomic q a crisp $\{0, 1\}$ value $h(t, q) \in \{0, 1\}$.

How can we make modal logic fuzzy?

We can turn h into a $[0, 1]$ function. We can turn R into a $[0, 1]$ function. We can even make fuzzy which world is the actual world. This seems arbitrary. See (Fitting, 1991; Fitting, 1992; Fitting, 1994) for examples of where it is done. Can we make sense of it?

The answer is yes. The very same fibring methodology used to combine intuitionistic and modal logics and used to combine two modalities, can be used here to combine modal logic with Łukasiewicz infinite valued logic.

Depending on how we fibre we can get the above (or all) different options of making modal logic fuzzy.

2. The Idea of Fibring

The basic problem of combining systems can be formally understood as follows:

COMBINING SYSTEMS PROBLEM

We are given two systems $\mathcal{S}_1$ and $\mathcal{S}_2$ in languages $\mathbf{L}_1$ and $\mathbf{L}_2$. We combine the languages to form $\mathbf{L}$. We ask the following:

1. How can we define a system $\mathcal{S}$ for the combined language, which is a conservative extension of each $\mathcal{S}_1$ and $\mathcal{S}_2$? This is not an easy problem, because $\mathcal{S}_1$ and $\mathcal{S}_2$ may be presented to us in two completely different and incompatible ways. For example even though $\mathcal{S}_1$ and $\mathcal{S}_2$ may be two modal logics with modalities $\Box_1$ and $\Box_2$, the system for $\Box_1$ may

be defined via some algebraic semantics while the system for $\Box_2$ may be a tableaux system.

2. How many options for $\mathcal{S}$ are there and how do they relate to each other?
3. If we want $\mathcal{S}_1$ and $\mathcal{S}_2$ to interact, how do we do that? Can we develop a methodology for interaction and have a set of prearranged and well understood 'interactive axioms'?
4. Suppose we know that both $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy some property $\mathcal{P}$. Can we prove transfer theorems which ensure that the combined $\mathcal{S}$ satisfies property $\mathcal{P}'$ (a variation of $\mathcal{P}$)?
5. Suppose we are given a system $\mathcal{S}$ directly defined in a language $\mathbf{L}$. Assume $\mathbf{L}$ is a syntactic combination of $\mathbf{L}_1$ and $\mathbf{L}_2$. Can we decompose $\mathcal{S}$ into $\mathcal{S}_1$ and $\mathcal{S}_2$, where $\mathcal{S}_i$ are some projections onto the sublanguages $\mathbf{L}_i$, and then can we reconstruct $\mathcal{S}$ back as some combination of $\mathcal{S}_1$ and $\mathcal{S}_2$ with possibly additional interaction axioms? If $\mathcal{S}_i$ happens to be well understood systems, we would have certainly 'simplified' $\mathcal{S}$.

The fibring methodology we are about to describe allows one to combine systems through their semantics and is a very successful framework for answering the above questions.

2.1. APPRECIATION OF THE DIFFICULTIES INVOLVED IN COMBINING SYSTEMS

The problem of combining systems can be pretty difficult in practice, not only because the two systems to be combined may be presented in two completely different ways, but also because that even when they are presented in the same way, it is not clear how to combine them. The next examples will illustrate.

Example 2.1 (Two modal logics) *Let $\Box_1$ and $\Box_2$ be two modalities. We define two systems as follows:*

Let $\mathcal{S}_1$ be the system obtained by adding to classical propositional logic the modality $\Box_1$ with the following axioms and rules:

1. $\Box_1 A$, *where A is substitution instances of truth functional tautology;*
2. $\Box_1(A \to B) \to (\Box_1 A \to \Box_1 B)$
3. $\Box_1(\Box_1(A \to B) \to (\Box_1 A \to \Box_1 B))$
4. $\Box_1 A \to \Box_1\Box_1 A$
5. $\Box_1(\Box_1 A \to \Box_1\Box_1 A)$
6. $\Box_1 A \to A$
7. *Modus ponens.*

$$\vdash A; \vdash A \to B \textit{ imply } \vdash B.$$

Note that we do not require necessitation. So although $\Box_1 A \to A$ is a theorem $\Box_1(\Box_1 A \to A)$ is not provable.

Let $\mathcal{S}_2$ be the system obtained by adding $\Box_2$ to classical propositional logic. The theorems of $\mathcal{S}_2$ are defined semantically as follows.

Let E^2 be Euclidean plane with the usual topology. An assignment h is a function giving to each atom q, a set of points $h(q) \subseteq E^2$.

h can be extended to all wffs of $\mathcal{S}_2$ as follows

$$\begin{array}{rcl} h(\neg A) & = & E^2 - h(a) \\ h(A \wedge B) & = & h(A) \cap h(B) \\ h(A \vee B) & = & h(A) \cup h(B) \\ h(A \to B) & = & (E^2 - h(A)) \cup h(B) \\ h(\Box_2 A) & = & \textit{Topological interior of } h(A). \end{array}$$

Let $\vDash_2 A$ mean that for all h, $h(A) = E^2$.

Problem: *Combine $\mathcal{S}_1$ and $\mathcal{S}_2$ into a system $\mathcal{S}$ which is the smallest logical system for the combined language which is a conservative extension of both $\mathcal{S}_1$ and $\mathcal{S}_2$.*

The two systems are presented in totally different ways. How are we going to combine them?

We shall fibre (combine) them in the next subsection by observing that the $\Box_2$ modality is really **S4** *modality, complete for the class of Kripke models (S, R, a, h) where R is reflexive and transitive and that the $\Box_1$ modality is complete for all Kripke models (S, R, a, h) such that R is transitive and aRa holds. Without this extra knowledge we cannot combine them. With this extra knowledge we can apply our methodology and combine them, as we shall see in the next subsection.*

Even when the two systems are presented in the same way, it is not clear what to do and how to combine them. In the next example, we try to combine classical implication with intuitionistic implication. The two systems will be presented in the same way, with classical implication having one more axiom than intuitionistic implication. We cannot be more 'compatible' than that, can we?

We shall combine them in a straightforward way by taking the union of the axioms, taking each axiom for its own language. The next example will show that we still get into trouble!

See F. del Cerro and A. Herzig (del Cerro & Herzig, 1996) on how to do it correctly.

Example 2.2 (Fibring counterexample) *Let $\mathbf{L}_1$ be the language with $\wedge_1$ and $\twoheadrightarrow_1$ and $\mathbf{L}_2$ with $\wedge_2$ and $\twoheadrightarrow_2$. $\wedge_i$ is intended to be conjunction and $\twoheadrightarrow_i$ is intended to be implication. Let $\mid\!\sim_1$ and $\mid\!\sim_2$ be the respective consequence relations for $\mathbf{L}_1$ and $\mathbf{L}_2$, with $\mid\!\sim_1$ intending to define the $\wedge, \to$ fragment of intuitionistic logic and $\mid\!\sim_2$ intended to define the $\wedge, \to$ fragment of classical* *logic. Both $\mid\!\sim_i$ can be defined as the smallest consequence relations satisfying Reflexivity, Monotonicity and Cut and closed under the additional rules below. $\mid\!\sim_2$, which supposed to define classical implication can be assumed to be closed also under Peirce's rule $(A \twoheadrightarrow_2 B) \twoheadrightarrow_2 A \mid\!\sim_2 A$.*

- $A \wedge_i B \mid\!\sim_i C$ *iff* $A \mid\!\sim_i B \twoheadrightarrow_i C$
- $A \wedge_i B \vdash_i A$
- $A \wedge_i B \vdash_i B$
- $\dfrac{A \mid\!\sim_i B; A \mid\!\sim_i C, B \wedge_i C \mid\!\sim_i D}{A \mid\!\sim_i D}$.

Let us now combine the languages and let $\mid\!\sim$ be the smallest consequence relation in the combined language closed under both rules, for the respective languages. We show that $\twoheadrightarrow_1$ and $\twoheadrightarrow_2$ become equal.

Since $A \twoheadrightarrow_i B \mid\!\sim_i A \twoheadrightarrow_i B$ we get $(A \twoheadrightarrow_i B) \wedge_i A \mid\!\sim_i B$.

We also have:

1. $(A \twoheadrightarrow_1 B) \wedge_2 A \mid\!\sim A$
2. $(A \twoheadrightarrow_1 B) \wedge_2 A \mid\!\sim A \twoheadrightarrow_1 B$
3. $(A \twoheadrightarrow_1 B) \wedge_1 A \mid\!\sim B$

Hence from (1)–(3)

$$(A \twoheadrightarrow_1 B) \wedge_2 A \mid\!\sim B$$

and hence

$$(A \twoheadrightarrow_1 B) \mid\!\sim (A \twoheadrightarrow_2 B).$$

The above is impossible since $\twoheadrightarrow_1$ can be classical implication and $\twoheadrightarrow_2$ intuitionistic implication, and we know from **S4** *Kripke model semantics that they do not collapse when combined..*

2.2. THE BASIC IDEA OF FIBRING

This subsection explains how fibring works. We do it in two stages. First we take a well known logic, modal logic, and show how it works in that case, and then we give a general schematic definition.

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two modal languages with $\Box_1$ and $\Box_2$. We assume that $\mathbf{L}_1$ and $\mathbf{L}_2$ share (are built up from) the same set Q of atomic propositions. Consider two logics $\mid\!\sim_1$ in $\mathbf{L}_1$ and $\mid\!\sim_2$ in $\mathbf{L}_2$. These are our *systems* $\mathcal{S}_1$ of $\mathbf{L}_1$ and $\mathcal{S}_2$ of $\mathbf{L}_2$. To be specific let $\mathcal{S}_1$ be modal logic $\mathbf{K}_1$ for $\Box_1$ and let $\mathcal{S}_2$ be modal logic **S4** for $\Box_2$, as described in Example 2.1. We have to say how these logics are *presented* to us. To simplify matters let us assume that the logics are presented to us via classes $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, of Kripke models for which the logics are sound and complete. Indeed for our fibring methodology to work there is no need to assume the classes are frame classes or normal or anything special. Let $\mathcal{K}_1$ be the class of all Kripke models of the form (S, R, a, h), where R is transitive and aRa holds. Let $\mathcal{K}_2$ be the class of all models where R is reflexive and transitive. To distinguish between the classes, we write $\mathbf{m}^1 = (S^1, R^1, a^1, h^1)$ for a model in $\mathcal{K}_1$ and $\mathbf{m}^2 = (S^2, R^2, a^2, h^2)$ for a model in $\mathcal{K}_2$.

To explain in principle how fibring works, consider a mixed wff of the form $\alpha = \Diamond_1\Box_2 q$.

We say this wff α is in the language $\mathbf{L}_{(1,2)}$, namely we have outer connectives of $\mathbf{L}_1$ and inside there are connectives of $\mathbf{L}_2$. We shall define later the language $\mathbf{L}_{(x_1,\ldots,x_n)}$ in general, where $(x_1, \ldots, x_n)$ is an alternating sequence of numbers in $\{1, 2\}$.

We now motivate the definition of fibred models for formulas of $\mathbf{L}_{(1,2)}$, by looking at our example α.

Let us consider α as a formula of $\mathcal{S}_1$ (since its outer connective is $\Diamond_1$). From the point of view of $\mathbf{L}_1$ this formula has the form $\Diamond_1 p$, where $p = \Box_2 q$ is *atomic*, since $\mathcal{S}_1$ does not recognise $\Box_2$.

To give a model for $\Diamond_1 p$ we take any $\mathcal{S}_1$ model. Take for example $\mathbf{m}^1 = (S^1, R^1, a^1, h^1)$ and check whether r$a^1 \vDash_1 \Diamond_1 p$.

For this to hold we need to check whether for some $t \in S^1$ such that $a^1 R^1 t$, we have $t \vDash_1 p$, i.e. whether $t \vDash_1 \Box_2 q$. Since $\Box_2$ is not in the language of $\mathcal{S}_1$, we do not know how to evaluate it! We observe, however, that we need no more than a truth value for $t \vDash_1 \Box_2 q$. We need an answer, yes or no.

The basic idea of fibring is to associate, with each $t \in S^1$, a model $\mathbf{m}_t^2 = (S_t^2, R_t^2, a_t^2, h_t^2)$ of $\mathcal{S}_2$, and get our answer by evaluating at the associated model i.e. we have

(*) $t \vDash_1 \Box_2 q$ iff $a_t^2 \vDash_2 \Box_2 q$.

Of course $\Box_2 q$ can be evaluated at $\mathbf{m}_t^2$ because it is in the right language.

Let $\mathbf{F}^1$ be the fibring function, i.e. let $\mathbf{f}^1(t) = \mathbf{m}_t^2$.

We can now say that our fibred semantics for the combined language $\mathbf{L}_{(1,2)}$, has models of the form $(S^1, R^1, a^1, h^1, \mathbf{F}^1)$, where $\mathbf{F}^1$ is as above and we use (*) above in our evaluation. Figure 1 below describes the model schematically:

In principle fibred models for formulas with nestings of modalities (e.g. $\Diamond_1\Box_2\Diamond_1\Diamond_1 q$) can be defined inductively by iterating the process of fibring.

We shall see later that these models can be (just for the case of modal logic) greatly simplified, using various technical devices, which will lead us to the notion of *SFM*-models.

Before we do that let us discuss the general case. The above idea of fibring can be schematically generalised, and the rest of this section is going to give an intuitive description of general fibring.

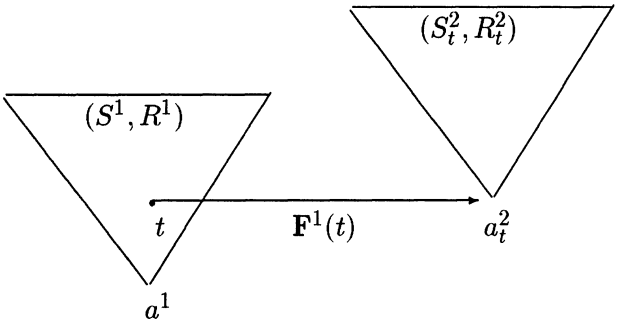

Figure 1.

2.3. GENERAL FIBRING

Since the fibring methodology is applicable in diverse and unconnected areas of logic, the best approach is to give a very general definition of fibring and then show how the fibred semantics can be simplified for each case (area) it is used. The general definition will highlight the basic general assumptions needed to execute the fibring methodology.

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two languages, and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two systems (usually logics, but they could be any systems, for example theorem provers or programming environments). To be specific we advise the reader to think of three typical examples:

- combining two modal logics $\mathcal{S}_1$ and $\mathcal{S}_2$
- $\mathcal{S}_1$ a logic programming language
  $\mathcal{S}_2$ a C-language
  For example, we want to allow C-code to be put into PROLOG code[1]
- Combining two partially ordered systems, with equality $=$ and ordering $\leq$.

We assume the systems satisfy the following assumptions:

Basic assumption for general fibring

1. SYNTACTICAL ASSUMPTION
   The expressions E^i of the language $\mathbf{L}_i$ is built up using $\mathbf{L}_i$-constructors (connectives) from a set of atomic units Q_i. It is sometimes convenient to assume that $Q_1 = Q_2 = Q$, i.e. the languages share the atoms. We

[1]Our fibring methodology has been widely applied in the logic context. I believe it can also be applied for general systems as well. The details for programming systems have not yet been worked out. I am asking the reader to think of the PROLOG-C example because it can help understand the logic case studies.

schematically write $E^i = E^i(q_1^i, \ldots, q_n^i)$ to indicate that E^i is built up from the atoms $Q_1^i, \ldots, q_n^i \in Q_i$. Let $\mathbb{C}^i$ be the set of constructors of $\mathcal{S}_i$. We assume $\mathbb{C}^1 \cap \mathbb{C}^2 = \varnothing$.

2. SEMANTICAL ASSUMPTION
   The system $\mathcal{S}_i$ is 'characterised' (in case of logics $\mathcal{S}_i$ this means completeness) by a class $\mathcal{K}_i$ of models $\{\mathbf{m}_1^i, \mathbf{m}_2^i, \ldots\}$. Each model $\mathbf{m}_n^i$ is built up from a set S_n^i of basic semantic components $t \in S_n^i$. In many cases the model $\mathbf{m}_n^i$ comes with a distinguished point $a_n^i \in S_n^i$ (the actual world in Kripke models). This distinguished point plays a role in the semantic evaluation in the model.
   For technical reasons it is convenient to assume that for $m \neq n, S_n^i \cap S_m^i = \varnothing$. Let $S^i = \bigcup_n S_n^i$. We also assume that $S^1 \cap S^2 = \varnothing$. This can always be achieved by renaming.[2]
3. THE EVALUATION FUNCTION
   An evaluation function val^i is available which can give values ($\top$ or $\bot$ or solutions or any other output) to the following:
   (a) $val^i(\mathbf{m}_n^i, E^i)$.
   This is the value of a general expression E^i of $\mathcal{S}_i$, evaluated at a model $\mathbf{m}_n^i$. In many cases (e.g. modal logic) $val^i(\mathbf{m}_n^i, E^i)$ is defined to be $val^i(a_n^i, E^i)$ where a_n^i is the distinguished point of $\mathbf{m}_n^i$.
   (b) $val^i(t, E^i), t \in S_n^i$.
   The value of a general expression at a unit $t \in S_n^i$.
   (c) The value $val^i(\mathbf{m}_n^i, E^i(q_1, \ldots, q_k))$ and $val^i(t, E^i)$ is functionally reducible through some possibly algorithmic, inductive process to the values $\{val^i(t, q_j)\}$ where $t \in S^i$ and q_j are the atoms appearing in E. These latter values are arbitrary within an acceptable family of assignments. Let us write this as $\{val^i(t, q)\} \in Acceptable^i$.
   Note that there may be some metapredicates involved in the formulation of *Acceptable*, relating $val^i(t, q)$ with $val^i(s, q)$. The most common (and in fact most general) is to impose an ordering $\prec$ on S^i and some relation on *val* (also denoted by $\prec$) and require something like
   – $t \prec s$ implies $val^i(t, q) \prec val^i(s, q)$.
4. DEFINITION OF THE FIBRED LANGUAGE
   We now define the fibred languages $\mathbf{L}_{(x_1, \ldots, x_n)}$, as follows:
   – Let $\mathbf{L}_{(1)}$ be $\mathbf{L}_1$ and $\mathbf{L}_{(2)}$ be $\mathbf{L}_2$.

[2]Note that the nature of this assumption is not clear for the case of programming languages. See previous footnote.

- Let $\bar{y} = (1, y_1, \ldots, y_k)$. Let $\mathbf{L}_{(2)*\bar{y}}$ be defined as the family of all expressions of the form $\alpha \in \mathbf{L}_{(y_1,\ldots,y_k)}$ or $\alpha = E^2(q_1/A_1, \ldots, q_n/A_n)$ where $E^2(q_1, \ldots, q_n) \in \mathbf{L}_2$ and $A_1, \ldots, A_n$ are in $\mathbf{L}_{\bar{y}}$, and q_j/A_j indicate the substitution of A_j for q_j in E^2.

  In other words, $\mathbf{L}_{(2,1,y_1,\ldots,y_k)}$ is the set of all expressions with outer constructor from $\mathbf{L}_2$ and with no more than $k+2$ nested alternation of constructors from $\mathbf{L}_1$ and $\mathbf{L}_2$.
- We similarly define $\mathbf{L}_{(1,2,y_1,\ldots,y_k)}$.
- Let $\mathbf{L}_\infty = \bigcup_{\bar{y}} \mathbf{L}_{\bar{y}}$.

5. THE FIBRING FUNCTION $\mathbf{F}$

   We are now ready for our last assumption for fibring. We require that the values $val^2(\mathbf{m}_n^2, E^2)$, for models $\mathbf{m}_n^2$ of $\mathcal{S}_2$ and E^2 of $\mathbf{L}_2$ are acceptable values for the function $val^1(t, q)$, for $t \in S^1$ and q atomic. In symbols:

   - For all $\mathbf{m}_n^2$ and E^2, $\{val^2(\mathbf{m}_n^2, E^2)\} \in \mathit{Acceptable}^1$.

   Let us see what this means. Take for example $\mathbf{L}_1$ as intuitionistic logic and take a Kripke model for intuitionistic logic, $(S, \leq, a, h)$. the assignments satisfy persistence:

   $$t \leq s \text{ and } h(t,q) = 1 \text{ imply } h(s,q) = 1.$$

   So if we fibre models $\mathbf{m}_t$ to t and $\mathbf{m}_s$ to s then for any formula α of $\mathbf{L}_2$ (the other language, which can be for example, modal logic) we must have $\mathbf{m}_t \vDash_2 \alpha$ implies $\mathbf{m}_s \vDash_2 \alpha$. This has to be ensured, so we cannot fibre arbitrarily in the intuitionistic case. In general the nature of the allowed fibring function $\mathbf{F}$ has to be worked out for each application with possibly some correctness theorems involved.
   Consider the case of PROLOG and C-languages.
   If we encounter a C-code α in the middle of a PROLOG program, then $val^C(\alpha)$ will probably be a numerical value. This value must be usable to the *Prolog* program, which only recognised predicates. So α may be an equation or constraints verified in C, etc.
   Under the above assumptions we can define the family of fibred models for $\mathbf{L}_{(1,2)}$. These have the form $(\mathbf{m}_n^1, \mathbf{F}_n^{(1,2)})$, where $\mathbf{F}_n^{(1,2)}$ is a fibring function on S_n^1, assigning for each $t \in S_n^1$ a model $\mathbf{m}_t^2 \in \mathcal{K}_2$.
   The basic evaluation clause for $t \in S^1$ is

   - $val^{(1,2)}(t, E^2) = val^2(\mathbf{m}_t^2, E^2)$

6. SIMPLIFYING THE FIBRING FUNCTION.

   The fibring function can be simplified, for the following reasons:

(a) Since $S^1_n, n = 1, 2, \ldots$ are all disjoint, the domain of $\mathbf{F}^{(1,2)}_n$ can be $S^1 = \bigcup_n S^1_n$.

(b) Since for each $t, \mathbf{F}^{(1,2)}_n(t)$ is a model $\mathbf{m}^2_t$, this model must have an index n_t, i.e. $\mathbf{m}^2_t = \mathbf{m}^2_{n_t}$.

It is sufficient to regard $\mathbf{F}^{(1,2)}_n$ as giving either the numerical value n_t or some $a \in S^2_{n_t}$. Since the model $S^2_1, S^2_2, \ldots$ are pairwise disjoint, any $a_t \in S^2$ will characterise a model $\mathbf{m}^2_{n_t}$ uniquely. In fact we mentioned in item 2 above that the models $\mathbf{m}^i_n$ can come with a distinguished element $a^i_n \in S^i_n$. We can let $\mathbf{F}^{(1,2)}(t)$ be $a^2_{n_t} \in S^2$.

(c) Also because we can always duplicate names of models, we can always assume that

(*) $t \neq s, t, s, \in S^1$ imply $n_t \neq n_s$.

Thus we can view the fibring function $\mathbf{F}^{(1,2)}$ as a one-to-one function from S^1 into S^2 satisfying (*) above.

(d) Since the models $\mathbf{m}^1_n$, $n = 1, 2, \ldots$ can themselves be duplicated, we can assume that if the $t \in S^1_k$ and $s \in S^1_m$ and $k < m$ then the fibred models $\mathbf{F}^{(1,2)}(t) = a^2_{n_t}$ and $\mathbf{F}^{(1,2)}(s) = a^2_{n_s}$ satisfy $n_t < n_s$.

(e) Furthermore a function $\mathbf{F}^{(1,2)}$ can be similarly defined from S^2 to S^1 for the fibred language $\mathbf{L}_{(2,1)}$. The two functions can be joined as one function

$$\mathbf{F} : S^1 \cup S^2 \mapsto S^1 \cup S^2$$

satisfying the following

- If $t \in S^1$ then $\mathbf{F}(t) = a^2_{n_t} \in S^2$.
- If $t \in S^2$ then $\mathbf{F}(t) = a^1_{n_t} \in S^1$.
- $\mathbf{F}$ is one-to-one and satisfies the condition in (d).

We shall see that for each particular fibring application, astonishing simplifications are possible, showing connections between fibring and other known concepts in logic.

7. GENERAL DEFINITION OF FIBRING.
We can now define the fibred model for the mixed language $\mathbf{L}_\infty = \bigcup_{\bar{x}} \mathbf{L}_{\bar{x}}$ as follows:

(a) A fibred model for $\mathbf{L}$ has the form $\mathbf{n} = (\mathbf{m}^i_n, \mathbf{F})$ where $\mathbf{F}$ is the simplified fibring function as in (6e) above. Note that although $\mathbf{F}$ was defined in (6e) for the case of $\mathbf{L}_{(1,2)}$ and $\mathbf{L}_{(2,1)}$, the form it was given in (6e) allows it to be used for any $\mathbf{L}_{\bar{x}}$.

(b) We now define $val(\mathbf{n}, E)$ as follows.
We can assume that $E \in L_{(x_1,\ldots,x_n)}$. E has the form $E =$

$E^i(q_j/A_j)$ where $A_j \in \mathbf{L}_{(x_1,\ldots,x_n)}$. We know that the $val^i(\mathbf{m}_n^i, E^i)$ depends directly on

$$V = \{val^i(t, q_j) \mid t \in S^1, i = 1, 2, \ldots\}.$$

Let $f_{E^i}(V)$ symbolise this dependency. Therefore if we replace $val^i(t, q_j)$ by the new $V' = \{val(t, A_j)\}$, which we can assume we know how to get, then the same dependency will give us

$$val(\mathbf{n}, E) = f_{E^1}(V').$$

8. THE NOTION OF DOVETAILING.
   We saw in item 6 above that the function $\mathbf{F}$ can be viewed as a function giving for each $t \in S^1 \cup S^2$, an element $\mathbf{F}(t) \in S^1 \cup S^2$ such that if $t \in S^i$ then $\mathbf{F}(t) \in S^j, i \neq j$.
   Consider the case where the language $\mathbf{L}_1$ and $\mathbf{L}_2$ share the same set of atoms Q.
   We can compare the values $val^i(t, q)$ and $val^j(\mathbf{F}(t), q)$ for atom q. These two values need *not* be identical. If we require from our fibring function $\mathbf{F}$ that for each $t \in S^i, \mathbf{F}(t) \in S^j$ and each $q \in Q$ we have

   † $val^i(t, q) = val^j(\mathbf{F}(t), q)$

   Then this fibring case is referred to as *dovetailing.* We shall see later that dovetailing allows for serious simplifications in the combined system.

2.4. CASE STUDY: MODAL LOGIC FIBRING AND DOVETAILING

We take this opportunity to show how the simplifications can be done in the case of fibring modal logics.

In the modal logic case each model involved has the form $\mathbf{m}_n^i = (S_n^i, R_n^i, a_n^i, h_n^i)$.

Since we assumed that all S_n^i are pairwise disjoint, we can put both semantics together in one big set of possible worlds W.

$$W = \bigcup_{(i,n)} S_n^i.$$

The relations R_n^i can be all unified under a single relation $R \subseteq W^2$,

$$R = \bigcup_{(i,n)} R_n^i.$$

This can be done without loss of generality. For any two elements $t, s \in W, tRs$ can hold only if they are both in the same S_n^i and only if $tR_n^i s$

holds. Thus $R^i_n = R \restriction (S^i_n)^2$. Can we retrieve S^i_n out of W? The answer is yes if we record the actual worlds a^i_n. We can assume, from modal logic considerations, that the model (S^i_n, R^i_n, a^i_n) has the property that each $t \in S^i_n$ is accessible from a^i_n via a chain of elements i.e.

$$\forall t \exists k \exists t_1, \ldots, t_k (a^i_n R^i_n t_1 \wedge \ldots t_{k-1} R^i_n t_k \wedge t_k R^i_n t).$$

With this assumption S^i_n is retrievable from a^i_n using R^i_n. So if we let $W_i = \{a^i_n \mid n = 1, 2, \ldots\}$ and $W_a = W_1 \cup W_2$ (a for 'actual') be the subsets of W for all actual worlds, then knowing W_a, we can get back our models. For each $x \in W_a$ let $S^x = \{t \mid \exists k \exists t_1, \ldots, t_k (xRt_1 \wedge \ldots \wedge t_k R_t\}$. So for $x = a^i_n, S^x = S^i_n$.

We can now simplify the fibring function $\mathbf{F}$. For each $t, \mathbf{F}(t)$ is a model $\mathbf{m}^i_n$. Since this model is characterised by a^i_n, we can let $\mathbf{F}(t)$ be a^i_n. Thus

$$\mathbf{F} : W \mapsto W_a.$$

$\mathbf{F}$ satisfies the same properties as before, namely it is one to one and always switching semantics, (i.e. for $t \in S^1, \mathbf{F}(t) \in W_2$ and for $t \in S^2, \mathbf{F}(t) \in W_1$).

The above considerations make $\mathbf{F}$ a jump function which corresponds to a unary operator $\mathbb{J}A$ of the form:

- $t \vDash \mathbb{J}A$ iff $\mathbf{F}(t) \vDash A$

We thus conclude that fibring two modalities is reduced to one modality with a jump connective $\mathbb{J}$.

We can now define the notion of *SFM* model (Simplified Fibred Models) for mult-modal logic, with modalities $\{\Box_i \mid i \in I\}$.

Definition 2.3 (SFM-model) *An SFM-model has the form* $(W, W_i, W_a, R, w_0, h, \mathbf{F}_i), i \in I$, *where*

1. W *is a set of worlds for some* $i_0, w_0 \in W_{i_0}$.
2. $W_i \subseteq W, i \in I$ *are pairwise disjoint and nonempty, with* $W_a = \bigcup_i W_i$.
3. *For* $t, s \in W_i$ *let* $S^t = \{x \mid \exists n \;\; tR^n x\}$.
   Then

   - $t \neq s \to S^t \cap S^s = \varnothing$.
   - $W = \bigcup_{t \in W_a} S^t$.

4. $\mathbf{F}_i$ *is a function satisfying*

   - $x \in S^t$ *and* $t \in W_i \to \mathbf{F}_i(x) = x$
   - $x \in S^t$ *and* $t \notin W_i \to \mathbf{F}_i(x) \in W_i$
   - $x \neq y \to \mathbf{F}_i(x) \neq \mathbf{F}_i(y)$.

5. *For each $t \in W_i$ the model* $\mathbf{m}_t = (S^t, R \restriction S^t \times S^t, t, h \restriction S^t)$ *is in the semantics $\mathcal{K}_i$ of* $\mathbf{L}_i$.
   In case $\mathbf{L}_i$ is complete for the models $\{\mathbf{m}_t \mid t \in W_i\}$ we say the model is a universal model.
6. *We can also assume that* $\mathbf{F}$ generates W *as follows.*
   Let $W^0 = S^{w_0}$
   $W^{n+1} = W^n \cup \bigcup_{y \in W^n, i \ arbitrary} S^{\mathbf{F}_i(y)}$
   Then $W = \bigcup_n W^n$.

Condition (f) can be assumed because evaluation at a model is evaluation at w_0. Any point not in $\bigcup_n W^n$ is not reachable from w_0 by embedded modalities so cannot affect truth values. See Figure 2.

The figure shows a model which has possible worlds $W = S^t \cup S^s \cup \ldots$. $t \in W_j$ means t is an actual world of a model of $\Box_j$. $\mathbf{F}_i(x) = s$ means the model with actual world s (of $\Box_i$) is fibred at the point x.

When dovetailing, the point s is identified with the point x. To distinguish between xRz in S^t and xRy in S^s, we write xR_jz and xR_iy.

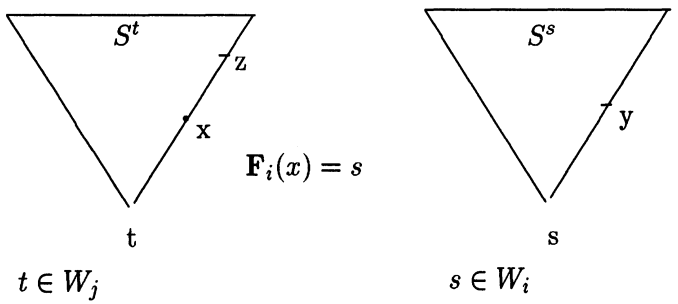

Figure 2.

We now take a new look at the fibring function $\mathbf{F}$. We can simplify the fibred semantics even further. Let us introduce new unary connectives of the form $\mathbb{J}_iA$ ($\mathbb{J}_i$ for a 'jump' operator)and modal operators $\Box$ and $\Diamond$ with the table:

- $t \vDash \mathbb{J}_iA$ iff $\mathbf{F}_i(t) \vDash A$.
- $t \vDash \Diamond A$ iff for some $s(tRs$ and $s \vDash A)$.
- $t \vDash \Box A$ iff for all $s(tRs \to s \vDash A)$.

According to this definition we can let $\Box_i = \mathbb{J}_i\Box$, $\Diamond_i = \mathbb{J}_i\Diamond$. We are referring to $\mathbb{J}_i$ as 'jump' operators because their truth table 'jumps' the evaluation from a world t to the world $\mathbf{F}_i(t)$.

Thus fibring several modalities together is like adding several unary jump operators to the modalities of the fibred semantics.

Conversely, assume we are given a modal logic with modalities $\Box$ and $\Diamond$ and several $\mathbb{J}_i$ and a class $\mathcal{K}$ of *SFM*-models of the form $(W, W_i, R, w_0, h, \mathbf{F}_i)$, as defined above.

We can define modal semantics classes $\mathcal{K}_i$ by letting $\mathcal{K}_i$ to be the set of all models of the form $(S^t, R \restriction S^t \times S^t, t, h \restriction S^t), t \in W_i$.

In the case of a finite number of modalities, one fibring function $\mathbf{F}$ and one jump operator $\mathbb{J}$ are sufficient. $\Box_1$ can be interpreted as $\Box$ and $\Box_k$ as $\mathbb{J}^k\Box$. $\mathbf{F}$ can be chosen to take any point x into $\mathbf{F}(x)$, an actual world of the next semantics in cyclical order.

Note that the jump operators are slightly more expressive. For q atomic, $\mathbb{J}q$ changes the index of evaluation of q from t to $\mathbf{F}(t)$. This cannot be done using modalities.

Example 2.4 (Dovetailing modal Kripke semantics) *Continuing the previous example, the case of dovetailing is when we require that for all atomic q*

$$h(t, q) = h(\mathbf{F}_i(t), q)$$

for all i. This means the actual world of the model fibred at t can be identified with t. We must be careful however. For each t, we must first introduce the notation tR_iy iff (definition) $\mathbf{F}_i(t)Ry$. Then we can identify the point $\mathbf{F}_i(t)$ with t and the dovetailed model can be equivalently represented as (S, R_i, a, h) with

$$\begin{array}{lll} t \vDash \Box_i A & \textit{iff} & \forall y(tR_iy \to y \vDash A) \\ t \vDash \Diamond_i A & \textit{iff} & \exists y(tR_iy \wedge y \vDash A) \end{array}$$

The function $\mathbf{F}$ is no longer needed, since we identified t with $\mathbf{F}_i(t)$, so the dovetailed model reduces to (W, R_i, w_o, h).

It is interesting to see the role of the jump operators in the simplified dovetailed semantics.

We have in the fibred model $(W, R, w_0, h, \mathbf{F}_i)$

$$\begin{array}{lll} t \vDash \mathbb{J}_i\Box A & \textit{iff} & \mathbf{F}_i(t) \vDash \Box A \\ & \textit{iff} & \forall x(\mathbf{F}_i(t)Rx \textit{ implies } x \vDash A) \end{array}$$

This becomes

$$t \vDash \mathbb{J}_i\Box A \textit{ iff } \forall x(tR_ix \textit{ implies } x \vDash A).$$

$\mathbb{J}_i$ becomes a mode shifting operator, changing the mode of evaluation, namely of which R_i to use.

We can consider the two clauses for a satisfaction $\vDash_i$ dependent on mode i.

– $t \vDash_i \Box A$ *iff* $\forall y(tR_iy$ *implies* $y \vDash_i A)$.

– $t \vDash_i \mathbb{J}_j A$ *iff* $t \vDash_j A$.

2.5. STEP BY STEP SCENARIO FOR FIBRING TWO LOGICS

In view of the previous discussion, we need to describe the steps we take in fibring two arbitrary logics $\mathcal{S}_1$ and $\mathcal{S}_2$.

Step 1: Check the form in which these two logics are presented and see whether they can be recast into the format of subsection 2.2.

Step 2: If worst come to worst, extract the consequence relations $\mid\!\sim_1$ and $\mid\!\sim_2$ of these two logics.

Step 3: Using theorems of the next section, give $\mid\!\sim_1, \mid\!\sim_2$ basic relational semantics $\mathcal{K}_1$ and $\mathcal{K}_2$.

Step 4: Using definitions of next section, fibre the two logics.

Step 5: We can assume, one way or another, that we fibred the two systems, using some semantics.

Depending on what the logics are (e.g. modal logics, intuitionistic logics, fuzzy logics, etc.), use general theorems available in the following respective sections to assert transfer of decidability, finite model property, recursive axiomatisation, etc.

2.6. PLAN OF THE REST OF THIS OVERVIEW

So far we have described the fibring methodology in general. From now on we have to do case by case study. In each case special features of the logics involved will allow us to obtain:

1. A more simplified form of the fibring and additional properties of it and connections with other parts of logic.
2. Specific completeness and other transfer theorems.
3. Specific open problems for the logics involved.
4. Comparison with existing results in the literature.

For more details, see my papers (Gabbay, 1995a; Gabbay, 1993; Gabbay, 1995b; Gabbay, 1995d; Gabbay, 1995c; D'Agostino *et al.*, 1995; D'Agostino & Gabbay, 1996; Dörre *et al.*, to appear; Finger & Gabbay, 1992; Finger & Gabbay, to appear).

3. Logics and their Semantics

Our considerations in the general discussion of combining logics in the previous sections indicates that we may be presented with two logics that are completely incompatible in their style and formulation.

In such a case we cannot combine them directly. Our strategy in this case is to do the following:

Step 1: Extract from the given presentation of the logics two respective consequence relations $\mid\!\sim_1$ and $\mid\!\sim_2$ for them.

Step 2: Use the method of this section (as presented below) to give general basic point relational semantics for $\mid\!\sim_1$ and $\mid\!\sim_2$.

Step 3: Fibre the basic point relational semantics to obtain the combined system $\mid\!\sim$.

Note that we offer here a technical solution. The basic point relational semantics may not mean much for the applications or motivations of these logics. We merely need it to bring the logic into some common background so that they can be combined. We can then use the semantics to prove transfer theorems and formulate an axiom system for the combined logic and possibly some confirmed properties of this system (e.g. decidability).

The purpose of this section is then to explain the following points.

- What is a logic and what is a basic relational semantics.
- Under what conditions on $\mid\!\sim$ can we give it basic point relational semantics.
- How to fibre two such semantics.
- Transfer properties for the above.

Let us begin.

Definition 3.1 (Basic logic) *1. A propositional language is a tuple* $\mathbf{L} = (Q, \mathbb{C})$ *where* Q *is a set of atomic propositions and* $\mathbb{C}$ *is a set of connectives. It may be convenient to include the connectives* $\bot$ *and* $\top$.

2. The set of well formed formulas of $\mathbf{L}$, $WFF_{\mathbf{L}}$ *is defined inductively in the usual manner.*

3. Let $\mathbf{L}_1 \cup \mathbf{L}_2$ *denote the language* $\mathbf{L} = (Q_1 \cup Q_2, \mathbb{C}_1 \cup \mathbb{C}_2)$.

4. A basic logic *on a language* $\mathbf{L}$ *is a binary relation between wffs of* $\mathbf{L}$ *of the form* $A\mid\!\sim B$ *satisfying the following conditions:*[3]

[3]The general notion of a *logic* allows for relations between sets of wffs $\Delta\mid\!\sim\Gamma$, and even lists or other structures. Most common is the form $\Delta\mid\!\sim A$, where Δ is a set of wffs including the empty set $\varnothing$. In such a case we require $\mid\!\sim$ to satisfy:

- *Reflexivity*
  $\Delta\mid\!\sim A$, if $A \in \Delta$
- *Restricted monotonicity*

$$\frac{\Delta\mid\!\sim A;\ \Delta\mid\!\sim B}{\Delta, A\mid\!\sim B}$$

- Identity
  $A \mid\!\sim A$
- Logical equivalence
  Let $A \equiv B$ abbreviate '$A \mid\!\sim B$ and $B \mid\!\sim A$' then for all A, B, C

$$\frac{A \equiv B}{(C \mid\!\sim A \text{ iff } C \mid\!\sim B)}$$

and

$$\frac{A \equiv B}{A \mid\!\sim C \text{ iff } B \mid\!\sim C}.$$

- Transitivity

$$\frac{A \mid\!\sim B; B \mid\!\sim C}{A \mid\!\sim C}.$$

- *If $\top$ and $\bot$ are present we have*

$$\bot \mid\!\sim A \mid\!\sim \top.$$

Definition 3.2 (Strengthening) *Let **L** be a language, let $\sharp(A_1, \ldots, A_n)$ be a connective in the language and $\mid\!\sim$ be a basic logic for **L**.*

1. *We say that $\sharp$ satisfies upward (respectively, downward) strengthening in coordinate i, relative to $\mid\!\sim$, iff the following holds:*
   -
$$\frac{X \mid\!\sim Y \,(\textit{respectively } Y \mid\!\sim X)}{\sharp(A_1, \ldots, A_{i-1}, X, A_{i+1}, \ldots, A_n) \mid\!\sim \sharp(A_1, \ldots, A_{i-1}, Y, A_{i+1}, \ldots, A_n)}$$
2. *We say $\mid\!\sim$ satisfies strengthening iff each connective $\sharp$ satisfies either upward or downward (or both) strengthening in each coordinate.*
3. *A logic is said to be* isotonic *(I-Logic) iff it satisfies strengthening.*

Lemma 3.3 *Let **L** be a language and let $\mid\!\sim$ be a I-logic in **L**. Let $C(q)$ be a wff of **L** with atomic q and let $C(q/X)$ denote the substitution of the wff X for q in C. Then the following holds*

$$\frac{A \equiv B}{C(q/A) \equiv C(q/B)}$$

- *Cut*

$$\frac{\Delta, A \mid\!\sim B; \Delta \mid\!\sim A}{\Delta \mid\!\sim B}$$

Proof. By structural induction on C, using strengthening. ■

Definition 3.4 (Rules) *Let* $\mathbf{L}$ *be a language and let* $\mid\!\sim$ *be a logic.*

1. *A logic rule* ρ *has the form*

$$\frac{A_i \mid\!\sim B_i, i = 1, \ldots, m}{A \mid\!\sim B}.$$

2. $\mid\!\sim$ *is said to satisfy the rule* ρ *iff whenever for all* i, $A_i(q_j/C_j)\mid\!\sim B(q_j/C_j)$ *holds then* $A(q_j/C_j)\mid\!\sim B(q_j/C_j)$ *holds.*
3. *Let* $\mathbb{R}$ *be a set of rules. Denote by* $\mid\!\sim_{\mathbb{R}}$ *the smallest consequence relation satisfying all the rules in* $\mathbb{R}$.

Lemma 3.5 *Let* $\mathbf{L}$ *be a language and* $\mid\!\sim$ *be an* I*-logic on* $\mathbf{L}$. *Let* $\sharp$ *be a connective such that* $\sharp$ *satisfies both upward and downward strengthening in its first coordinate. Then* $\mid\!\sim$ *is independent of its first coordinate.*

Proof. We will prove the lemma for the case either $\top$ or $\bot$ are present in the language. Since $\bot\mid\!\sim A\mid\!\sim\top$ for arbitrary A, we get $\sharp(\top) \equiv \sharp(\bot) \equiv \sharp(A)$ and hence $\sharp(A)$ is independent of A. If neither $\bot$ not $\top$ are present the proof is more complex, and we are not going to address it here. See Gabbay (Gabbay, 1994; Gabbay, 1995). ■

Definition 3.6
1. *Let* $\mathbf{L}$ *be a language and let* $\mid\!\sim$ *be an* I*-logic. Let* $\sharp$ *be a connective. We can assume that each coordinate in* $\sharp$ *satisfies uniquely either downward or upward strengthening. We can therefore present* $\sharp$ *in the form* $(A_1, \ldots, A_r) \twoheadrightarrow_\sharp (B_1, \ldots, B_s)$ *where* A_i *are the wffs in the downward coordinates and* B_j *are the wffs in the upward coordinates.*
2. *From now on we present all* I*-logics using connectives of the form* $\{\twoheadrightarrow_1, \ldots, \twoheadrightarrow_k\}$, *where for each* i, *the arities of the connective* $\twoheadrightarrow_i$ *are written as* (r_i, s_i).
3. *Let* $\mathbf{L}$ *be a language with connectives* $\mathbb{C} = \{\twoheadrightarrow_i \mid i = 1, \ldots, k\}$ *with arities* (r_i, s_i), *respectively. We denote by* $\mid\!\sim_{I(\mathbb{C})}$ *the smallest* I*-logic (i.e. smallest consequence relation) with the connectives* $\mathbb{C}$.

Definition 3.7 (Basic point relational structures) *Let* $\mathbf{L}$ *be a language and* $\mid\!\sim$ *an* I*-logic for* $\mathbf{L}$. *Assume its connectives are* $\{\twoheadrightarrow_1, \ldots, \twoheadrightarrow_k\}$ *with arities* $m_i = r_i + s_i, i = 1, \ldots, k$. *We present the connectives in the form* $(A_1, \ldots, A_{r_i}) \twoheadrightarrow_i (B_1, \ldots, B_{s_i})$.

1. *A basic relational model for this language has the form*

$$\mathbf{m} = (S, \prec, a, R_1, \ldots, R_k, h)$$

where S *is a non-empty set,* $a \in S$, $\prec$ *is a reflexive and antisymmetric and transitive relation on* S. *We assume that for some* $t_S \in S$ *we have* $\forall x \in S(t_S \prec x)$. *Further, for* $i = 1, \ldots, k, R_i$ *is an* $m_i + 1$ *place relation on* S *satisfying the following conditions.*

1.1. $\forall x_1, \ldots, x_{r_i}, y_1, \ldots, y_{s_i} \exists! t$
$[R_i(t, x_1, \ldots, y_1, \ldots) \wedge \forall u(R_i(u, x_1, \ldots, y_1, \ldots) \Leftrightarrow t \prec u)]$.
We can denote this unique t *by* $t = f_i(x_1, \ldots, x_{r_i}, y_1, \ldots, y_{s_i})$, *and call the function* f_i *the function associated with* R_i. *By 1.2 below, we have that* f_i *is monotonic upwards in* $y_1, \ldots, y_{s_i}$ *and downwards in* $x_1, \ldots, x_{r_i}$ *(relative to* $\prec$*).*

1.2. $\forall x_1, \ldots, x_{r_i}, y_1, \ldots, y_{s_i}, x'_1, \ldots, x'_{r_i}, y_1, \ldots, y'_{s_i}$
$[R_i(t, x_1, \ldots, y_1, \ldots) \wedge \bigwedge_j x_j \prec x'_j \wedge \bigwedge_j y'_j \prec y_j \Rightarrow R_i(t, x'_1, \ldots, y'_1, \ldots s)]$.

h is an assignment to the atoms. For atomic q, *there exists a* $t_q \in S$ *such that* $h(q) = \{x \mid t_q \prec x\}$.

2. *Satisfaction is defined as follows:*
   - $t \vDash q$ *iff* $t \in h(q)$, q *atomic.*
   - $t \vDash (A_1, \ldots, A_{r_i}) \twoheadrightarrow_i (B_1, \ldots, B_{s_i})$ *iff for all* $x_j \vDash A_j, j = 1, \ldots, r_i$ *there exist* $y_j \vDash B_j, j = 1, \ldots, s_i$ *such that* $R_i(t, x_1, \ldots, x_{r_i}, y_1, \ldots, y_{s_i})$.
   - $\mathbf{m} \vDash B$ *iff* $a \vDash B$.
   - *Let* $\mathcal{K}$ *be a class of models. We define* $A \vDash_{\mathcal{K}} B$ *iff for all* $\mathbf{m} \in \mathcal{K}, \mathbf{m} \vDash A$ *implies* $\mathbf{m} \vDash B$.

   We let $\vDash_{\mathcal{K}} B$ *iff for all* $\mathbf{m} \in \mathcal{K}, \vDash_{\mathbf{m}} B$.
3. *Note that there is another way of defining satisfaction in a model, which we denote by* $\vDash^*$. *Let* $\hat{A} = \{t \mid t \vDash A\}$. *Define* $A \vDash^*_{\mathbf{m}} B$ *iff for all* $\prec$ *minimal points* $x \in \hat{A}$, *we have* $x \vDash_{\mathbf{m}} B$.
   Similarly $A \vDash^*_{\mathcal{K}} B$ *is defined. It is easy to see that if* $\mathcal{K}$ *is a class of models such that if* $(S, \prec, a, R_i, h) \in \mathcal{K}$ *then for all* $x \in S$ *we have also* $(S, \prec, x, R_i, h)$ *is in* $\mathcal{K}$ *then* $A \vDash^*_{\mathcal{K}} B$ *iff* $A \vDash_{\mathcal{K}} B$.
4. *Consider the first-order language of classical logic with binary relation* $\prec$ *and relations* $R_1, \ldots, R_k$ *and possibly function symbols* $f_1, \ldots, f_k$. *Then the class of basic relational structures can be taken as the class of all models of some first-order theory* β, *comprised of all the conditions in (1) above.*

Lemma 3.8 *Let* $\mathbf{m}$ *be a basic relational structure. Let* $\hat{B}$ *be defined as* $\hat{B} = \{t \mid t \vDash B\}$. *Then there exists a* $t_B \in S$ *such that*

$$\hat{B} = \{t \mid t_B \prec t\}.$$

Since $\prec$ *is transitive, we have* $A \vDash_{\mathbf{m}} B$ *iff* $t_B \prec t_A$.

Proof. By definition, for q atomic we have:

$$h(q) = \{t \mid t_q \prec t\} = \hat{q},$$

where t_q is in S.

We show by induction that for any B there exists a t_B such that $\hat{B} = \{t \mid t_B \prec t\}$. We examine the sample inductive case of a binary $B \twoheadrightarrow C$. Let $R(t, x, y)$ be its relation.

Using the induction hypothesis we assume that t_B and t_C exist for $\hat{B}$ and $\hat{C}$ and that:

$$\widehat{B \twoheadrightarrow C} = \{x \mid \forall y(t_B \prec y \to \exists z(t_C \prec z \wedge R(x, y, z))\}.$$

We are looking for $t_{B \twoheadrightarrow C}$. By condition (1.2) on R, there exists a t_0 such that $R(t_0, t_B, t_C)$ holds and for every $u, R(u, t_B, t_C)$ holds if $t_0 \prec u$. (I.e. $t_0 = f(t_B, t_C)$.). We claim t_0 is $t_{B \twoheadrightarrow C}$.

Let $t_0 \prec u$. Show $u \in h(B \twoheadrightarrow C)$. We know that $R(u, t_B, t_C)$ holds. Let $t_B \prec y$, and choose $z = t_C$. We need to show $R(u, y, t_C)$.

By condition (1.1) since $t_B \prec y$ and $t_C \prec t_C$ we get $R(u, y, t_C)$.

Assume $u \in \widehat{B \twoheadrightarrow C}$ and show $t_0 \prec u$. Choose $y = t_B$. Then for some z such that $t_C \prec z$ we have $R(u, t_B, z)$ and by (1) we have $R(u, t_B, t_C)$ and we have $t_0 \prec u$.

This concludes the proof of the lemma. ∎

Remark 3.9 *The proof of the previous theorem shows the basic point relational structures can be equivalently presented in the form*

$$\mathbf{m} = (S, \prec, a, f_1, \ldots, f_k, h)$$

where each $f_i(x_1, \ldots, x_{r_i}, y_1, \ldots, y_{s_i})$ is monotonic upwards in the y coordinates and downward in the x coordinates. The part without the h is called the frame *of the model and h, of course, is the assignment.*

The inductive truth definition is the following:

- $t \vDash (A_1, \ldots, A_{r_i}) \twoheadrightarrow_i (B_1, \ldots, B_{s_i})$ *iff for all* $x_j \vDash A_j, j = 1, \ldots, r_i$, *there exist* $y_j \vDash B_j, j = 1, \ldots, s_i$ *such that* $f_i(x_1, \ldots, x_{r_i}, y_1, \ldots, y_{s_i}) \prec t$.

The next definition and lemma shows how to calculate t_B, for any wff B in such a model.

Definition 3.10 *Let* $\mathbf{m}$ *be a basic point relational model for a language* $\mathbf{L}$.

With each formula $C(q_1, \ldots, q_n)$ with atoms $q_1, \ldots, q_n$ we can define a function $f_C(x_1, \ldots, x_n)$ on the domain of $\mathbf{m}$ *as follows*

- $f_{q_i}(x_i) = x_i$
- $f_C(x_1, \ldots, x_n) = f_i(f_{A_1}, \ldots, f_{A_{r_i}}, f_{B_1}, \ldots, f_{B_{S_i}})$ *where* $C = (A_1, \ldots, A_{r_i}) \twoheadrightarrow_i (B_1, \ldots, B_{s_i})$.

Lemma 3.11 *Let* $\mathbf{m}$ *be a basic point relational model and let* $C(q_1, \ldots, q_n)$ *be a wff built up from the atoms* $q_1, \ldots, q_n$. *Assume that* $h(q_i) = \{t \mid x_i \prec t\}$ *then*

$$\hat{C} = \{t \mid f_C(x_1, \ldots, x_n) \prec t\}.$$

Note that $\hat{\top}$ *is the smallest element of* $(S, \prec)$ *and* $\hat{\bot}$ *is the largest.*

Proof. Follows from the proof of a previous lemma. ∎

Theorem 3.12 (Strong completeness) *Let* $\mathbf{L}$ *be a language and* $\mid\!\sim$ *be an I-logic. Let* $\mathbb{R}$ *be a set of rules such that* $\mid\!\sim = \mid\!\sim_{\mathbb{R}}$.

Then there exists a class $\mathcal{K}$ *of basic point relational structures such that* $\mid\!\sim = \models_{\mathcal{K}}$, *and such that for each rule* ρ *of* $\mathbb{R}$ *of the form:*

$$\frac{A_i(q_1, \ldots, q_m) \mid\!\sim B_i(q_1, \ldots, q_m)}{A(q_1, \ldots, q_m) \mid\!\sim B(q_1, \ldots, q_m)}$$

the models satisfy in classical logic the following formula β_ρ:

$$\begin{array}{l}\forall x_1, \ldots x_m [\bigwedge_i f_{B_i}(x_1, \ldots, x_m) \prec f_{A_i}(x_1, \ldots, x_n) \rightarrow \\ \quad f_B(x_1, \ldots, x_n) \prec f_A(x_1, \ldots, x_m))].\end{array}$$

Proof.

1. *Soundness*
   This can be verified.
2. *Completeness*
   Let S be the set of equivalence classes of $\mathrm{WFF}_{\mathbf{L}}$ over $\equiv$.
   Let $A/\equiv \prec B/\equiv$ hold iff
   $$B \mid\!\sim A.$$
   For each connective $\twoheadrightarrow_i$ of $\mathbf{L}$, let
   $$\begin{array}{l}f_i(A_1/\equiv, \ldots, A_{r_i}/\equiv, B_1/\equiv, \ldots, B_{s_i}/\equiv) = \\ \quad ((A_1, \ldots, A_{r_i}) \twoheadrightarrow_i (B_1, \ldots, B_{s_i}))/\equiv .\end{array}$$
   Let $R_i(t, x_1, \ldots, x_{r_i}, y_1, \ldots, y_{s_i})$ holds iff $f_i(x_1, \ldots, x_{r_i}, y_1, \ldots, y_{s_i}) \prec t$.
   Let $h(q) = \{t \mid q/\equiv \prec t\}$.
   It is easy to show by induction that
   - $\hat{A} = \{B/\equiv \mid B \mid\!\sim A\} = \{t \mid A/\equiv \prec t\}$.
   - $f_C(q_1/\equiv, \ldots, q_n/\equiv) = C(q_1, \ldots, q_n)/\equiv$.
   - $A \models B$ iff $A \mid\!\sim B$ iff $f_B \prec f_A$
   - Each formula β_ρ, for $\rho \in \mathbb{R}$ holds in the model because $\mid\!\sim$ satisfies ρ

The above shows that the canonical model we have defined is the desired model. Assume $A \not\mathrel{|\!\sim} B$, then a countermodel would be $(S, \prec, A/\equiv, R_i, h)$.

■

We shall fibre the basic relational semantics for the logics.

Remark 3.13 (Fibring of canonical models) *We are now going to be more specific about the classes of models $\mathcal{K}_1$ and $\mathcal{K}_2$ which we use. The completeness proof in Theorem 3.12 shows that the semantics of the logic $\mathrel{|\!\sim}_i$ is obtained by taking one canonical frame, namely $\mathbf{m}^i_\alpha = (S^i, \prec^i, \alpha/\equiv, f^i_1, \ldots, f^i_{k_i}, h^i)$, (where $S^i = WFF_{\mathbf{L}_i}/\equiv$ and $A/\equiv \prec^i B/\equiv$ is $B \mathrel{|\!\sim}_i A$ etc.) and letting α run over all wffs. Whenever $A \not\mathrel{|\!\sim} B$ we let $\alpha = A$ and the model $\mathbf{m}^i_\alpha \vDash A$ but $\mathbf{m}^i_\alpha \not\vDash B$. Thus we can let $\mathcal{K}_i = \{\mathbf{m}^i_\alpha \mid \alpha \in WFF_{\mathbf{L}_i}\}$. These models, however, are not pairwise disjoint because $S^i_\alpha = WFF_{\mathbf{L}_i}/\equiv$. To obtain pairwise disjoint models we can best base our WFF on* new atoms.

Let

$$\mathbf{L}_1 = (Q, \mathbb{C}^1)$$
$$\mathbf{L}_2 = (Q, \mathbb{C}^2)$$

Assume $Q = \{q_1, q_2, q_3, \ldots\}$. Let us make more copies of the atoms. Let

$$Q^{(i,n)} = \{q_1^{i,n}, q_2^{i,n}, \ldots, \}.$$

Consider the language

$$\mathbf{L}_{i,n} = (q^{(i,n)}, \mathbb{C}^i).$$

Each formula $\alpha(q^*_1, \ldots, q^*_m) \in WFF_{\mathbf{L}_i}$ built up from $Q^*_1, \ldots, q^*_m \in Q$ has its 'isomorphic' counterpart $\alpha^n \in WFF_{\mathbf{L}_{i,n}}$ obtained by the simultaneous substitution of any atom $q_r \in Q$ by its corresponding $q^{i,n}_r \in Q^{(i,n)}$. Let $\{\alpha^i_1, \alpha^i_2, \alpha^i_3, \ldots\}$ be an enumeration of $WFF_{\mathbf{L}_i}$.

We can now regard the semantics $\mathcal{K}_i$ as having the models

$$\mathbf{M}^i_{\alpha^i_n} = (WFF_{\mathbf{L}_{i,n}}/\equiv, \prec^i_n, \alpha^{i,n}_n, f^i_{r,n}, h^i_n).$$

Now all models in $\mathcal{K}_i$ are pairwise disjoint.

A fibring function $\mathbf{F}$ should associate with each $\beta^{i,n}_n/\equiv \in WFF_{\mathbf{L}_{i,n}}$ an actual world of a model of the other language, i.e. a formula $\alpha^{j,m}_m, j \neq i$. The persistence requirement can be observed by the following trick:

Let $\mathbf{f}$ and $\mathbf{g}$ be two functions $\mathbf{f} : WFF_{\mathbf{L}_1} \mapsto WFF_{\mathbf{L}_2}$ and $\mathbf{g} : WFF_{\mathbf{L}_2} \mapsto WFF_{\mathbf{L}_1}$ such that

$\alpha \mathrel{|\!\sim}_1 \beta$ implies $\mathbf{f}(\alpha) \mathrel{|\!\sim}_2 \mathbf{f}(\beta)$

$\gamma \mathrel{|\!\sim}_2 \delta$ implies $\mathbf{g}(\gamma) \mathrel{|\!\sim}_1 \mathbf{g}(\delta)$.

Now define a fibring function $\mathbf{F}$ as follows:

$$\mathbf{F}(\beta_n^{i,n}) = \alpha_m^{j,m}, m > n$$

satisfying the following

– *if* $i = 1$ *then* $\alpha_m^{2,m} = \mathbf{f}(\beta_n^{1,n})$

– *if* $i = 2$ *then* $\alpha_m^{1,n} = \mathbf{f}(\beta_n^{2,n})$.

The properties of $\mathbf{f}$ *and* $\mathbf{g}$ *will ensure that* $\mathbf{F}$ *satisfies persistence.*

Theorem 3.14 *Let* $\mid\!\sim_1$ *and* $\mid\!\sim_2$ *be two I-logics in* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *respectively. Let* $\models$ *be the fibred consequence relation arising from the fibred semantics for the combined language as defined in Remark 3.13. Then* $\models$ *is the smallest I-logic in the combined language which is a conservative extension of each pure component.*

Proof. Let $\mid\!\sim$ be the smallest I-logic in the combined language conservatively containing both $\mid\!\sim_1$ and $\mid\!\sim_2$. Such a logic exists as the intersection of all conservative extensions, $\models$ being one of them. Note that being an extension does not mean that $\mid\!\sim$ is closed under the rules defining $\mid\!\sim_1$ or $\mid\!\sim_2$, but only that $A \mid\!\sim_i B$ implies $A' \mid\!\sim B'$ where A', B' are substitution instances (in the combined language) of A and B respectively. We show $\mid\!\sim\, = \models$. We know that $\mid\!\sim\, \subseteq \models$, because $\mid\!\sim$ is the smallest conservative extension. We need to show $A_0 \not\mid\!\sim B_0$ implies $A_0 \not\models B_0$. Let $\mathbf{m}_{A_0}$ be a canonical model of $\mid\!\sim$. We know that the canonical models as constructed in Theorem 3.12 have the form

$$\mathbf{m}_\alpha = (\mathrm{WFF}/\equiv, \prec, \alpha/\equiv, f_i, g_j, h)$$

where $A/\equiv\, \prec B/\equiv$ iff $B \mid\!\sim A$, and f_i, g_j are the functions associated with the connectives f_i for $\mathbf{L}_1$ and g_j for $\mathbf{L}_2$.

We will turn $\mathbf{m}_\alpha$ into a fibred model and this will show that $A_0 \not\models B_0$.

Each node $\beta/\equiv$ in the model is either atomic or has the form $(c_1, \ldots, c_k) \twoheadrightarrow (D_1, \ldots, D_m)$ where $\twoheadrightarrow$ is a connective in the language $\mathbf{L}_i, i = 1$ or $i = 2$.

$\mathbf{m}_\alpha$ can be viewed as a model of $\mathbf{L}_i$, where any β with main connective $\twoheadrightarrow$ not from $\mathbf{L}_i$ is regarded as atomic. We display this point of view by writing $\mathbf{m}_\alpha^i$. Thus $\mathbf{m}_\alpha^i$ is the same model $\mathbf{m}_\alpha$ but where any wff with main connective from $\mathbf{L}_j$ $j \neq i$ is regarded as atomic. We can now define a fibring function $\mathbf{F}$ on $\mathbf{m}_\alpha^i$. Take any $\beta/\equiv$ in the model $\mathbf{m}_\alpha^i$.

We can let $\mathbf{F}(\beta) = \mathbf{m}_\beta^j$ now regarded as a model of the $\mathbf{L}_j$ language, $j \neq i$. Thus $\mathbf{F}$ fibres with each node $\beta/\equiv$ a model of the other language. The assignment h_α^i in $\mathbf{m}_\alpha^i$ is defined as $h_\alpha^i(q) = \{t \mid q \prec t\}$.

We know that satisfaction in the canonical model is equal to $\mid\!\sim$ i.e. B holds at $A/\equiv$ iff $A \mid\!\sim B$. We now show that satisfaction in the fibred model is also the same i.e. $A/\equiv \models_{i,\beta} B$ iff $A \mid\!\sim B$.

The proof is by induction.

1. *Case q atomic*
   $A/\equiv\models_{i,\alpha} q$ iff $A/\equiv\in h^i_\alpha(q)$ iff $A\mid\!\sim q$.
2. *Case the main connective of B is in the language $\mathbf{L}_i$*
   $A/\equiv\models_{i,\alpha} (C_m) \twoheadrightarrow (D_n)$ iff $\forall x_m \models_{i,\alpha} C_m\ \exists y_n \models_{i,\alpha} D_n$ s.t. $f_{\twoheadrightarrow}(x_m, y_n) \prec t$. By the induction hypothesis $x_m \models_{i,\alpha} C_m$ is the same $x_m\mid\!\sim C_m$ and $y_n \models_{i,\alpha} D_n$ is the same as $y_n\mid\!\sim D_n$ and hence we continue, iff $A\mid\!\sim B$.
3. *Case B has main connective in the language $\mathbf{L}_j \neq \mathbf{L}_i$*
   $A/\equiv\models_{i,\alpha} B$ iff $\mathbf{F}(A/\equiv) \models_{j,A/\equiv} B$ iff, by the previous case, $A\mid\!\sim B$.
   Thus the model $\mathbf{m}_{A_0} \not\models B_0$.

This completes the proof of the theorem. ■

4. Combining Modal and Intuitionistic Logics

This section gives an overview of how the fibring methodology can be applied to combining traditional modal logics as well as modal and intermediate logics. The modal logics need not be normal. We show transfer of recursive axiomatizability, decidability and finite model property. Detailed proofs are given in (Gabbay, 1993).

Some results on combining logics (normal modal extensions of $\mathbf{K}$) have recently been introduced by Kracht and Wolter, Goranko and Passy and by Fine and Schurz as well as a multitude of special combined systems existing in the literature of the past 20-30 years. We hope our methodology will help organise the field systematically.

We already saw in section 2.2 item 7 what the fibred and dovetailed modal models look like. So proceeding directly from that point, we just quote here the theorems we get. For full details, see (Gabbay, 1993).

Theorem 4.1 (Completeness theorem for the fibred logic $\mathbf{L}^F_I$) *Let $\mathbf{L}_i, i \in I$ be modal logics with classes of structures $\mathcal{K}_i$ and set of theorems $\mathbf{T}_i$. (i.e. $\mathbf{T}_i = \{A$ of $\mathbf{L}_i \mid A$ is valid in all $\mathcal{K}_i$ models $\}$). Let $\mathbf{T}^F_I$ be the following set of wffs of $\mathbf{L}^F_I$.*

1. $\mathbf{T}_i \subseteq \mathbf{T}^F_I$

2. **Modal Fibring Rule:**[4]
   If $\Box_i$ is the modality of $\mathbf{L}_i$ and $\Box_j$ of $\mathbf{L}_j, i, j$ arbitrary $i \neq j$ and $C = \bigwedge^n_{k=1} \Box_i A_k \to \bigvee^m_{k=1} \Box_i B_k \in \mathbf{T}^F_I$ then for all d, $\Box^d_j C \in \mathbf{T}^F_I$.

[4]The meaning of the modal fibring rule will be apparent from the proof below. Intuitively, it has to do with substitutions of wffs of one language into a formula of the other language. If the substituted wffs are related (proof theoretically) we want to propagate this relation into the other language.

There are formal similarities between the modal fibring rule and necessitation. Consider a formula C built up only from 'atoms' of the form $\Box_i B_k$. Then our special rule of necessitation says that from $\vdash C$ we can deduce $\vdash \Box^d_j C$ for any modality $\Box_j$ *other than* $\Box_i$.

3. $\mathbf{T}_I^F$ *is the smallest set closed under (1), (2), modus ponens and substitution.*

Then $\mathbf{T}_I^F$ *is the set of all wffs of* $\mathbf{L}_I^F$ *valid in all the fibred structures of* $\mathbf{L}_I^F$.

Example 4.2 *Consider two logics* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *with two* $\mathbf{K}$ *modalities* $\Box_1$ *and* $\Box_2$. *Consider the fibred combination* $\mathbf{K}_{1,2}^F$. *By the previous theorem it can be axiomatised by taking all theorems of* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *together with the following modal fibring rule:*

$$\frac{\vdash \bigwedge_k \Box_i A_k \to \bigvee_k \Box_i B_k}{\vdash \Box_j [\bigwedge_k \Box_i A_k \to \bigvee_k \Box_i B_k]}$$

where $i, j \in \{1, 2\}$, $i \neq j$.

Note that necessitation is not available. This example investigates under what conditions necessitation is admissible.

The logic $\mathbf{K}$ *has the following special properties:*

1. $\vdash \Box \top$
2. $\vdash \Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$
3. *The disjunction property*

$$\frac{\vdash \Box A \to \Box B \vee \Box C}{\vdash (\Box A \to \Box B) \text{ or } \vdash (\Box A \to \Box C)}$$

The above three properties can be used to prove that the modal fibring rule is equivalent to necessitation.

First the rule can be reduced to

$$\frac{\vdash \Box_i A \to \Box_i B}{\vdash \Box_j (\Box_i A \to \Box_i B)}$$

using (2) and (3) above.

Second, since $(\top \to A) \leftrightarrow A$ *we get from (1) that* $(\Box \top \to \Box A) \leftrightarrow \Box A$.

This can be used to derive

$$\frac{\vdash \Box_i A}{\vdash \Box_j \Box_i A}$$

Since necessitation is admissible in each component we get that

$$\frac{\vdash A}{\vdash \Box_i A}$$

is admissible in the combination $\mathbf{K}_{1,2}$.

The key property which we have used is the disjunction property. This holds for $\mathbf{K}$, $\mathbf{K4}$, *and some other system and for all of these systems the above reduction of the modal fibring rule to necessitation stands.*

Theorem 4.3 *Assume $\mathbf{L}_i, i \in I$ has the finite model property then so does $\mathbf{L}_I^F$.*

Assume $\mathbf{L}_i, i \in I$ are recursively axiomatisable, then so is $\mathbf{L}_I^F$.

Dovetailing arises in many applications where the fibred model at t has the world t itself as its actual world. A major example is multimodal logics for action. In this example, assume we are at a given state t of a system. There are several possible actions α one can take. With each α we associate an accessibility relation R_α, where $tR_\alpha s$ means that s can nondeterministically arise after action α is applied to state s. We associate the modality $\Box_\alpha A$ to read: A holds in all states arising from the application of α. Such a multimodal logic, with modalities $\{\Box_\alpha\}$ is dovetailed, not fibred.

Theorem 4.4 (Completeness theorem for the dovetailed logic $\mathbf{L}_I^D$) *Let $\mathbf{L}_i, i \in I$ be modal logics with semantical classes of structures $\mathcal{K}_i$ and set of theorems $\mathbf{T}_i$. Let $\mathbf{T}_I^D$ be the following set of wffs of $\mathbf{L}_I^D$.*

1. $\mathbf{T}_i \subseteq \mathbf{T}_I^D$
2. **Modal Dovetailing Rule**:[5]
   If $\Box_i$ is the modality of $\mathbf{L}_i$ and $\Box_j$ of $\mathbf{L}_j, i, j$ arbitrary $i \neq j$ and

$$C = \bigwedge_{k=1}^{n} \Box_i A_k \wedge \bigwedge_{k=1}^{m} \Diamond_i \sim B_k \to \bigvee_{k=1}^{r} q_k \in \mathbf{T}_I^D$$

   then for all d $\Box_j^d C \in \mathbf{T}_I^D$. Where q_k are atoms or their negations, and $q_1, \ldots, q_r$ list all the atoms or their negations appearing in any A_k or $B_k, k = 1, 2, \ldots$.
3. *$\mathbf{T}_I^D$ is the smallest set closed under (1), (2) modus ponens and substitution.*

Then $\mathbf{T}_I^D$ is the set of all wffs of $\mathbf{L}_I^D$ valid in all the dovetailed structures of $\mathbf{L}_I^D$.

Theorem 4.5 *Assume $\mathbf{L}_i, i \in I$ all are extensions of $\mathbf{K}$ formulated using traditional Hilbert axioms and the rule of necessitation, then $\mathbf{L}_I^D$ (the dovetailing of $\mathbf{L}_i$) can be axiomatised by taking the union of the axioms and the rules of necessitation for each modality $\Box_i$ of each $\mathbf{L}_i$*

Theorem 4.6 *If $\mathbf{L}_i, i \in I$ admit necessitation and satisfy the disjunction property, then $\mathbf{L}_I^F = \mathbf{L}_I^D$.*

[5]The modal dovetailing rule is really a necessitation rule. It says that if C is a wff built up from 'atomic' units of the form $\Box_i A_s$ and ordinary atoms q_k and $\Box_j$ is a modality *different* from $\Box_i$, then a limited necessitation rule holds: $\vdash C$ implies $\vdash \Box_j^d C$ for any natural number d.

Theorem 4.7 *If $\mathbf{L}_i, i \in I$ all have the finite model property (are finitely axiomatisable), so is $\mathbf{L}_I^D$.*

Theorem 4.8 (Completeness theorem for fibring and dovetailing pure modalities) *Let $\mathbf{L}_i, i \in I$ be modal logics in a language with connectives $\Box_i$ and/or $\Diamond_i$ together with atomic propositional variables. We assume for convenience that $\top, \bot$ are available.*

Let $\mathcal{K}_i$ be semantics (class of Kripke models) for which $\mathbf{L}_i$ is complete. Let $\mathbf{L}_I^F$ be the fibring (resp. Let $\mathbf{L}_I^D$ be the dovetailing) of $\mathbf{L}_i, i \in I$. Let $\mid\!\sim_i$ be the consequence relation of $\mathbf{L}_i$.
Let $\mid\!\sim$ be the consequence relation of $\mathbf{L}_I^F$ (resp. $\mathbf{L}_I^D$) defined by the following rules.

1. *Any substitution instance of $\Delta_i \mid\!\sim_i \Gamma_i$, where $\Delta_i \mid\!\sim \Gamma_i$ is in the language of $\mathbf{L}_i$.*
2. Pure Modalities fibring (resp. dovetailing) rules

$$\frac{A_1, \ldots, A_n \mid\!\sim C_1, \ldots, C_m}{\Box A_1, \ldots, \Box A_n, \Diamond B \mid\!\sim \Diamond C_1, \ldots, \Diamond C_m}$$

In case of fibring, A_i and C_j must begin with a modality.

$$\frac{A_1, \ldots, A_n \mid\!\sim C, B_1, \ldots, B_m}{\Box A_1, \ldots, \Box A_n, \mid\!\sim \Box C, \Diamond B_1, \ldots, \Diamond B_m}$$

In case of fibring, A_i, B_j and C must begin with a modality.

3. *$\mid\!\sim$ is the smallest consequence relation (monotonic, reflexive and transitive) closed under the above rules.*

The following holds.
Let Δ, Γ be two finite sets of wffs of $\mathbf{L}_I^F$ (resp. $\mathbf{L}_I^D$). If $\Delta \not\mid\!\sim \Gamma$, then there exists a fibred model $\mathbf{m}$ of $\mathbf{L}_I^F$ (resp $\mathbf{L}_I^D$) such that

- *$A \in \Delta$ implies $\mathbf{m} \vDash A$*
- *$A \in \Gamma$ implies $\mathbf{m} \nvDash A$*

Theorem 4.9 *Assume $\mathbf{L}_i, i \in I$ are all decidable then so are $\mathbf{L}_I^F$ and $\mathbf{L}_I^D$.*

The next theorem studies the combination of pure modality $\Box$ and or $\Diamond$ into intuitionistic logic. This is the most general combination. The completeness theorem is rather involved, mainly because classical negation is not available, and we need to deal with negative information in a roundabout way, see (Gabbay, 1993) for details.

A lot of effort needs to be spent to ensure that the persistency condition

$$(*) \quad t \vDash A \text{ and } t \leq s \text{ implies } s \vDash A$$

holds in the fibred model. The above condition is the main difference between fibring of modalities alone and fibring of modalities with intuitionistic and intermediate logics.

We want a general fibring of an intuitionistic or intermediate logic connective $\Rightarrow$ with one or several modalities $\Box_1$ and/or $\Diamond_2$. Thus for example we would like to fibre the $\Rightarrow$ of Dummett's **LC** with the **K** modalities $\Box$ and $\Diamond$. The result of the fibring, as well as the axiomatisation, is very much dependent on the semantics we take for the modalities and on the additional connectives which happen to be present, such as $\wedge, \vee$ and $\neg$. Thus no general theorem can be easily given. We will therefore choose one type of modality, **K** modality and only $\Rightarrow$ and do the fibring and dovetailing for that.

We now fibre several intermediate logics $\Rightarrow_i$ with several **K** modalities $\Box_j, \Diamond_j$. Since we might not have $\neg, \wedge$ and $\vee$, we need to formulate the logics to be fibred using the notion of consequence relation. We choose the notion $\Delta \mid\!\sim \Gamma$ where both Δ and Γ are sets of wffs. Our proof works for the notion $\Delta \mid\!\sim B$, i.e. where Γ is a single formula. We therefore have to assume that the case of $\Delta \mid\!\sim \Gamma$ can be reduced to that of $\Delta \mid\!\sim B$. If the logic has disjunction then we can take B to be $\bigvee \Gamma$. Otherwise we an assume that $\mid\!\sim$ is *disjunctive*, namely that $\Delta \mid\!\sim \Gamma$ iff for some $B \in \Gamma, \Delta \mid\!\sim B$.

This condition is not too restrictive. The implicational fragment of intuitionistic logic is disjunctive i.e. if Δ, Γ are in the pure implicational fragment then $\Delta \mid\!\sim \bigvee \Gamma$ iff for some $B \in \Gamma, \Delta \mid\!\sim B$.

Dummett's **LC** does not satisfy the above. Its implicational fragment can be axiomatised by the additional axiom

$$((A \Rightarrow B) \Rightarrow C) \Rightarrow (((B \Rightarrow A) \Rightarrow C) \Rightarrow C)$$

Thus the conditions of Theorem 4.10 below do not apply to it. This does not mean, however, that we cannot prove the result for this case by some other means.

Theorem 4.10 (Completeness theorem for fibred (resp. dovetailed) intuitionistic modal logics).

1. *Let $\mathbf{L}_i$ be a logic with either an intermediate intuitionistic implication $\Rightarrow_i$ (and possibly $\wedge, \vee$ and $\bot$) or with one or both of the $\mathbf{L}_i$ modalities $\Box_i, \Diamond_i$. Let $\mathcal{K}_i$ be the semantics for $\mathbf{L}_i$. Let $\mid\!\sim_i$ be the consequence relation of $\mathbf{L}_i$. Then $\{A_1, \ldots, A_n\} \mid\!\sim_i \{B_1, \ldots, B_m\}$ means that in every Kripke model of $\mathbf{L}_i$ (modal or intuitionistic) of the form (S, R, a, h) whenever for all $j, a \vDash A_j$ then for some $k, a \vDash B_k$. We regard the consequence relation as a relation between finite sets of formulas. The*

completeness theorem characterises the consequence relation of $\mathbf{L}_I^F$ (denoted by $\mid\!\sim$) in terms of the consequence relations of $\mathbf{L}_i$ (denoted by $\mid\!\sim_i$). We make the following assumptions.

(*) *For intuitionistic $\mathbf{L}_i$, we assume that either disjunction is available or that it is disjunctive.*

2. *Let $\mid\!\sim$ be the smallest consequence relation containing $\mid\!\sim_i$ for $i \in I$, closed under substitution (and of course under reflexivity, monotonicity and cut) and closed under the following rules, where $\Rightarrow$ is a connective of an arbitrary $\mathbf{L}_{i_1}$ and $\Box, \Diamond$ of an arbitrary $\mathbf{L}_{i_2}$.*

   (a)

$$\frac{A_1, \ldots, A_n \mid\!\sim C_1, \ldots, C_m}{\Box A_1, \ldots, \Box A_n, \Diamond B \mid\!\sim \Diamond C_1, \ldots, \Diamond C_m}$$

   ]for the case of fibring A_i and C_j must begin with $\Box, \Diamond$ or $\Rightarrow$.

$$\frac{A_1, \ldots, A_n \mid\!\sim C, B_1, \ldots, B_m}{\Box A_1, \ldots, \Box A_n \mid\!\sim \Box C, \Diamond B_1, \ldots, \Diamond B_m}$$

   for the case of fibring A_i, B_j and C must begin with $\Box, \Diamond$ or $\Rightarrow$.

   (b) *Deduction theorem for each $\Rightarrow_i$.*

Then $\mid\!\sim$ is the consequence relation of all the fibred models of $\mathbf{L}_I^F$ (resp. dovetailed models of $\mathbf{L}_I^D$).

Example 4.11 (Some dovetailed modal intuitionistic systems)
Note that Theorem 4.10 allows us to fibre or dovetail any intermediate logic with any modal logic provided the intermediate logic has disjunction in it or is disjunctive. The theorem gives us the axiom system for the combined logic. Especially for the case of dovetailing, the method is very powerful. Consider any modal logic where axioms are reduction of modalities, for example

- $\Box A \vdash A$
- $\Box A \vdash \Box\Box A$
- $\Diamond A \vdash \Box\Diamond A$
- $\Diamond\Box A \vdash \Box\Diamond A$
- $\Diamond^5\Box^6 A \vdash \Diamond^{18}\Box^{11}\Diamond^{12} A$
- $\Box\Diamond A \wedge \Diamond\Box B \vdash \Diamond\Diamond(A \wedge B)$

Some of these axioms you would recognize and some I have just invented. It does not matter. In the combined logic any $\bigwedge_i M_i A_i \vdash MA$ becomes a theorem $\bigwedge_i M_i A_i \Rightarrow MA$, where M_i, M are strings of modalities. We can thus immediately have semantics and axioms for the dovetailed system of *any intermediate logic with disjunction and any major modal logic (e.g.* **K**, **T**, **B**, **S4**, **S5**, *etc.).*

To be more specific, the above theorem allows us to fibre or dovetail the following:

1. *The intermediate logic* **KC** *obtained by adding the schema* $\neg A \vee \neg\neg A$ *to intuitionistic logic with* **S4** *modality* $\Box$ *and* $\Diamond$.
2. *Dummetts* **LC** *(obtained by adding the schema* $(A \Rightarrow B) \vee (B \Rightarrow A)$ *with the* **K** *modality* $\Diamond$.

The axioms of the combined system are obtained from the completeness theorem. For example, dovetailing the systems in (1) yields:

1. **KC** *axioms with Modus Ponens.*
2. $\Box A \wedge \Box(A \Rightarrow B) \Rightarrow \Box B$
   $\Box(A \Rightarrow B) \wedge \Diamond A \Rightarrow \Diamond B$
   $\Diamond\Diamond A \Rightarrow \Diamond A$
   $\Box A \vee \Box\Box A$
   $\Box A \Rightarrow \Diamond A \vee \Box B$
   $\Diamond\neg A \vee \Box\neg\neg A$
   $\Diamond\neg\neg A \vee \Box\neg A$
   $\Box A \Rightarrow A$
   $A \Rightarrow \Diamond A$
3. $\dfrac{\vdash A}{\vdash \Box A}$

Further note that the dovetailing method of adding modality to any intermediate logic is systematic. We check all modalities reductions of the form $\bigwedge_i M_i A_i \vdash MA$ *and turn them into an axiom*

$$M_1 A_1 \Rightarrow (\ldots \Rightarrow (M_n A_n \Rightarrow MA)\ldots).$$

There is nothing arbitrary here.

Another way to obtain intuitionistic modal logic is to take the modal logic formulated axiomatically and change the underlying logic from classical to intuitionistic. Thus modal **K**, *when axiomatised by*

- $\Box(A \to B) \to (\Box A \to \Box B)$
- $\dfrac{\vdash A}{\vdash \Box A}$

yields by chance, when '$\to$' is changed to '$\Rightarrow$', the same system as dovetailing. but if we use an equivalent axiom

- $\Box(\neg A \vee B) \vee \neg\Box A \vee \Box B$

we get a different modal intuitionistic system.

Theorem 4.12 *Let* $\mathbf{L}_1, 1 \in I$, *be an intermediate logic and let* $\mathbf{L}_i, i \neq 1, i \in I$ *be modal logics . Let* $\mathbf{L}_I^D$ *and* $\mathbf{L}_I^F$ *be the dovetailing, resp. fibring of the logics. Then if* $\mathbf{L}_i, i \in I$ *are all decidable, so are* $\mathbf{L}_I^D$ *and* $\mathbf{L}_I^F$.

COMPARISON WITH SOME OTHER WORK

The reader should compare our work with (Eiben *et al.*, 1992; Kracht & Wolter, 1991; Fine & Schurz, to appear) and (Goranko & Passy, 1992). (Eiben *et al.*, 1992) combines first-order theories and can combine logics by translation into classical logic. (Kracht & Wolter, 1991; Fine & Schurz, to appear; Goranko & Passy, 1992) combine Hilbert systems of normal modal logics. Our method is more general, is applicable to a wide variety of systems and yields transfer theorems on bulk. Their ideas may be most effective and transportable perhaps to combining theorem provers?

All in all, I think that given all the available papers in the literature, the subject is ready to take off. See also (Blackburn & de Rijke, to appear) for a discussion.

5. How to Make Your Logic Fuzzy

This section overviews results that show that fibring an arbitrary logic with a known fuzzy logic (e.g. Łukasiewicz infinite valued logic) is a general methodology for making ones logic fuzzy.

To appreciate the need for such a methodology, consider for example the modal propositional logic **K**, with one modality $\Box$, and let us examine our options for turning it into a fuzzy system. This logic is complete for the crisp (i.e. $\{0,1\}$ valued), Kripke semantics. Kripke models have the form $\mathbf{m} = (S, R, a, h)$, where $S \neq \varnothing$ is a set of possible worlds, $R \subseteq S \times S$ is a crisp binary relation, (of the form $R : S \times S \mapsto \{0,1\}$), $a \in S$ is the actual world, and h is a binary function assigning to each $t \in S$ and each atomic q a crisp value $h(t,q) \in \{0,1\}$.

h can be extended to all wffs in the usual way with the inductive evaluation of $h(t, \Box A)$ being

$h(t, \Box A) = 1$ iff for all y such that tRy we have $h(y, A) = 1$.

or

$h(t, \Box A) = \text{ Inf } \{h(y, A) \mid tRy\}$.

We say $\mathbf{m} \vDash A$ if $h(a, A) = 1$.

Let us try and turn this logic fuzzy!

Working intuitively, one may turn modal logic into a fuzzy modal logic in several ways (Fitting, 1991)):

1. changing the function $h(t,q)$ into a fuzzy function $h^\sharp(t,q) \in [0,1]$ (obtaining real number values);
2. changing the crisp relation R into a fuzzy one $R^\sharp : S^2 \mapsto [0,1]$;
3. making $a \in S$ fuzzy.
4. any combination of the above.

Is there a methodology involved to the above or do we just go from logic to logic and make fuzzy whatever semantical component we find?

What if we use a different semantics for $\mathbf{K}$ and make fuzzy the functions involved in that semantics? Do we get yet another batch of fuzzy modal logics?

In our particular example, let $Ł_\infty$ be Łukasiewicz infinite valued logic (with values in $[0,1]$) and let us apply our fibring machinery to $\mathbf{K}$ and $Ł_\infty$. We get the following:

1. The fibred semantics for $\mathbf{K}(Ł_\infty)$ is the fuzzy semantics with h fuzzy, R crisp.
2. The fibred semantics for $Ł_\infty(\mathbf{K})$ is the semantics with R fuzzy and h crisp.
3. The semantics for $Ł_\infty(\mathbf{K}(Ł_\infty))$ is the semantics where both R and h are fuzzy.

So in short, when you ask me how to make your logic $\mathbf{L}_1$ fuzzy, I would answer—take a pure fuzzy logic $\mathbf{L}_2$ (e.g. $Ł_\infty$ or any other) and fibre it to $\mathbf{L}_1$ in different ways.

Example 5.1 (Motivating fuzzy values) *We now give a concrete example of a fibred model of level 1. Figure 3 shows a* $\Box_1$ *Kripke model.*

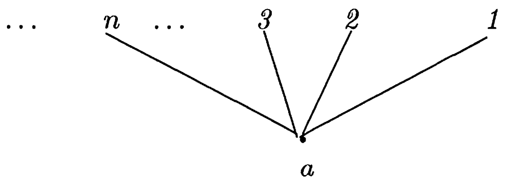

Figure 3.

Here $S = \{a\} \cup \{1,2,3,\ldots\}$ *with* aRn *holding, for* $n = 1,2,\ldots$.

Assume $h(a,q) = 0$ *and* $h(n,q) = 1$ *for* $n = 1,2,\ldots$. *Try to evaluate* $\Diamond_1\Box_2 q$.

$a \vDash \Diamond_1\Box_2 q$ *iff for some* n, $n \vDash \Box_2 q$. *Since* $\Box_2$ *is in the* $\mathbf{L}_2$ *language, we cannot continue to evaluate. We need an* $\mathbf{L}_2$ *model to get a value at* n. *The fibring function* $\mathbf{F}(n)$ *gives an* $\mathbf{L}_2$ *model* (S_n, R_n, a_n, h_n). *Let* $S_m = \{a_m\} \cup \{(m,n) \mid n = 1,2,3,\ldots\}$. *Let* R_m *be defined by*

$$xR_my \textit{ iff } \begin{cases} x = a_m \\ \textit{or} \\ x = (m,n_1) \textit{ and } y = (m,n_2) \textit{ and } n_2 \leq n_1 \end{cases}$$

and let

$$h_m(a_m, q) = 0$$

and

$$h_m((m,n), q) = 1 \text{ iff } m \leq n.$$

To complete the picture, let $\mathbf{F}(a) = \mathbf{F}(1)$. *Thus* $\Box_2 q$ *is false at* a_n *in all the models* $\mathbf{F}(n)$, *but we have*

$$(m,n) \vDash \Box_2 q \text{ iff } m \leq n.$$

This particular fibred model has a special feature which is important. All the models $\mathbf{F}(n)$, *have isomorphic* frames; *they are isomorphic to* $(T = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}, \leq, 0)$, *through* π_m, *where* $\pi_m(a_m) = 0$, *and* $\pi_m(m, -n) = \frac{1}{n}$, *and they differ only in the assignment* $h_m(q)$. *The image of the truth set* $h_m(\Box_2 q) = \{y \mid y \vDash \Box_2 q\}$ *is projected on* $\{0, 1, \ldots, \frac{1}{n} \mid n = 1, 2, 3, \ldots\}$ *gets larger and larger as* m *increases. In the limit we have*

$$\bigcup_m \pi_m h_m(\Box_2 q) = \{\frac{1}{n} \mid n = 1, 2, \ldots\}.$$

Since we are interested in $a \vDash (\Diamond_1 \Box_2 q)$, *where the table for* $\Diamond_1$ *is existential, we can say that* $\Diamond_1$ almost *holds; it* approaches the *'fuzzy' (or 'modal-*$\mathbf{L}_2$*') truth set* $\{\frac{1}{m} \mid m = 1, 2, \ldots\}$.

This is quite a conceptual jump. The model (S, R, a, h) *is a model of* $\mathbf{L}_1$ *and has no business getting set values from the set* T *via the mappings* π_m *of the models of* $\mathbf{L}_2$. *However, since all the fibred models* $\mathbf{F}(t), t \in S$ *are based on isomorphic frames, we can extend the evaluation from the fibred models back into the* $\mathbf{L}_1$ *language.*

It is important to note that the way we extended the evaluation from the fibred model to $\Diamond_1$ *of* $\mathbf{L}_1$ *was arbitrary. We chose a way of doing it which was reasonable, but nevertheless it was a choice. We could have said let us take as value for* $a \vDash \Diamond_1 \Box_2 q$, *not the union of* $\pi_m h_m(\Box_2 q)$ *but the maximum or some other reasonable definition.*

Having adopted a good definition, we now consider the expanded model $(S, R, a, h, \mathbf{F}, T, \pi)$.

We can define an $\mathbf{L}_2$*-fuzzy value* $\mu_t(A)$, *for* $t \in S$ *and any* A *as follows:*

- $\mu_t(A) = \pi_t h_t(A) = \{\pi_t(s) \mid s \in S_t \text{ and } s \vDash A\}$ *for* A *in* $\mathbf{L}_2$ *or* A *atomic.*
- $\mu_t(A \wedge B) = \mu_t(A) \cap \mu_t(B)$
- $\mu_t(\sim A) = T - \mu_t(A)$
- $\mu_t(\Diamond_1 A) = \bigcup_{\{s \mid tRs\}} \mu_s(A)$
- $\mu_t(\Box_1 A) = \bigcap_{\{s \mid tRs\}} \mu_s(A)$

What we have done can be best understood in algebraic terms. Let $\mathbb{B}$ be the Boolean algebra of the set $T = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with the interior operation $Q^{\Box_2}$, for $Q \subseteq T$ begin

$$Q^{\Box_2} = \{x \in T \mid \textit{ for all } y \geqq x, y \in Q\}.$$

Assign to each atom q and $t \in S$ the 'fuzzy' algebraic subset $\mu_t(q) \subseteq T$. In our particular model we assign

$$\mu_n(q) = \{1, \tfrac{1}{2}, \ldots, \tfrac{1}{n}\}$$
$$\mu_a(q) = \mu_1(q)$$

We extend the assignment by

- $\mu_t(\Box_2 A) = (\mu_t(A))^{\Box_2}$
- $\mu_t(\Box_1 A) = \bigcap_{\{s|tRs\}} \mu_s(A)$

The next example brings the idea forward even more clearly.

Example 5.2 (Many valued modal logic) *This is an example of fibring semantical models (modal logic) with algebraic models (Lukasiewicz many-valued logic). We consider the modal language $\mathbf{L}_1$ with $\Box$ and the many valued language $\mathbf{L}_2$, with $\{\wedge, \vee, \rightarrow, \neg\}$ and with truth values at the real interval $[0, 1]$. We study $\mathbf{L}_1(\mathbf{L}_2)$. The algebraic models of $\mathbf{L}_2$ are linearly ordered Abelian groups which are embeddable in $[0, 1]$. So it is sufficient to consider assignments μ of values and truth table for values in $[0, 1]$. The following are the algebraic functions:*

- *The domain is* $[0, 1]$
- $\leq$ *is numerical* $\leq$.
- $\top = \{0\}$ *(0 is truth).*
- $\bot$ *is 1 (1 is falsity).*
- $f_\wedge(x, y) = max\ (x, y)$.
- $f_\vee(x, y) = min(x, y)$
- $f_\neg(x) = 1 - x$
- $f_\rightarrow(x, y) = max(0, y - x)$.

We now turn to fibring.

Let $\mathbf{m} = (S, R, a, h)$ be a Kripke model for $\Box$. The fibring function $\mathbf{F}$ associates with each $t \in S$ an algebraic model $\mathbf{a}_t = (A_t, \leq, f_\wedge, f_\vee, f_\rightarrow, f_\neg, \{0\}, \mu_t)$. Since $A_t = [0, 1]$, fibring algebras $\mathbf{a}_t$ to t is nothing more than associating with each t an arbitrary many-valued assignment μ_t to the atoms of the modal language.

Let us now evaluate $\Box(q \rightarrow p), q, p$ atomic, at the model $\mathbf{m}$.

- $a \vDash \Box(q \rightarrow p)$ *iff for all $t \in S$ such that $aRt, t \vDash q \rightarrow p$.*

- *Since the main connective of* $q \to p$ *is many-valued, we have* $t \vDash q \to p$ *iff* $\mathbf{a}_t \vDash q \to p$ *iff* $\mu_t(q \to p) = 0$ *iff* $max(0, \mu_t(p) - \mu_t(q)) = 0$ *iff* $\mu_t(p) \leq \mu_t(q)$.

We would like to highlight a point which will be of importance later. Consider the above fibring. We start with $\mathbf{m} = (S, R, a, h)$. *Then with each* $t \in S$, *we fibre an algebra* $\mathbf{a}_t$. *Since all the algebras have the same domain, the fibring reduces to* μ_t, *the assignment. Let us pause at this stage and consider the entity* (S, R, a, h, μ) *and let us try to evaluate* $t \vDash \Box q$. *Since* $\Box q$ *contains no many-valued connectives, we get* $t \vDash \Box q$ *holds iff* $\forall s$ *(tRs implies* $s \vDash q$*) iff* $\forall s(tRs$ *implies* $h(t, q) = 1)$. *Consider the wff* $Iq = def\ (q \to q) \to q$. *Really* Iq *is* q *but it is formally a many-valued wff. So we have to evaluate it at the algebra* $\mathbf{a}_t$. *We have* $\mathbf{a}_t \vDash I(q)$ *iff* $\mu_t(q) = 0$. $t \vDash \Box I(q)$ *iff for all* $s(tRs$ *implies* $\mu_t(q) = 0)$.

To summarise, consider $t \vDash \Box q$*; we have two ways of looking at it.*

1. *Regard 'q' as an atom of the modal language, in which case*

$$t \vDash \Box q \text{ iff for all } S, tRs \text{ implies } h(s, q) = 1$$

2. *Regard 'q' as an atom of the many-valued language, in which case*

$$t \vDash \Box q \text{ iff for all } s, tRs \text{ implies } \mu_s(q) = 0.$$

The two evaluations need not give the same result.

We now have the opportunity to make $t \vDash \Box q$ *fuzzy (i.e. 'fuzzle' the satisfaction* $\vDash$, *or in other words, 'fuzzle' the modal logic) by extending* μ_t *to* $\Box q$*:*

($\sharp$) $\mu_t(\Box q) = Sup_{\{s|tRs\}}\mu_s(q)$.

The reader should note that this definition is a chosen one and we could have chosen some other 'averaging' function.

Using ($\sharp$) *we can now fuzzle any wff of the modal logic and extend* μ_t *to all wffs, by taking the many- valued table for* $\wedge, \vee, \neg$ *and* $\to$. *We have thus by understandable intuitive definition, through* ($\sharp$), *turned* (S, R, a, μ) *into a sort of modal many-valued logic by changing the crisp* $\{0,1\}$ *assignment* h *into a fuzzy* μ. *Note that what we are getting is not fibring, it is something new.*

Example 5.3 (Persistence) *This example will fibre modal logic to the intermediate logic Dummett's* **LC**. *It will serve to prepare the ground for fibring in the presence of persistence. Let* $\Rightarrow$ *be intuitionistic implication.* **LC** *is the extension of intuitionistic logic with the axiom schema*

$$(p \Rightarrow q) \vee (q \Rightarrow p)$$

or if disjunction is not available, we can write an implicational axiom schema

$$(p \Rightarrow q) \Rightarrow (((q \Rightarrow p) \Rightarrow r) \Rightarrow r).$$

Let $\mathbf{L}_1$ *be the language with* $\{\Rightarrow, \wedge, \vee, \bot\}$ *and let* $\mathbf{L}_2$ *be modal logic with* $\Box$. *Consider the intuitionistic* **LC** *model with* $U = [0,1]$ *(unit real numbers interval) of the form* $(U, \leq, 0, h)$. *Since we are dealing with intuitionistic model, we must have persistence, i.e. for all atomic* q *and any* $t, s \in U$.

(*) $t \leq s$ *and* $h(t,q) = 1$ *imply* $h(s,q) = 1$.

(**) We also require, for technical reasons, that for all $q, h(1,q) = 1$.

Satisfaction is defined as follows:

- $t \vDash A \wedge B$ *iff* $t \vDash A$ *and* $t \vDash B$
- $t \vDash A \vee B$ *iff* $t \vDash A$ *or* $t \vDash B$
- $t \vDash A \Rightarrow B$ *iff* $\forall s(t \leq s \wedge s \vDash A$ *imply* $s \vDash B)$.
- $t \vDash \bot$ *iff* $t = 1$

The reader familiar with **t***-conorms can view the above as follows:*
For each atomic q *let*

$$\mu(q) = \ Inf\, \{t \mid h(t,q) = 1\}$$

We have (because of persistence) that μ *can be extended to all wffs as follows:*

- $\mu(A \wedge B) = \ max(\mu(A), \mu(B))$
- $\mu(A \vee B) = \ min(\mu(A), \mu(B))$
- $\mu(A \to B) = \ Inf\, \{t \mid \ max\ (t, \mu(A)) \geq \mu(B)\}$

For each $t \in U$, *let* $\mathbf{F}(t) = (S_t, R_t, a_t, h_t)$ *be a modal model of* $\Box$. *Note that* $\Diamond$ *is not intuitionistically definable from* $\Box$ *and so we have to explicitly include* $\Diamond$ *if we want. Here we assume we have* $\Box$ *only.*

By general fibring principles, we must have persistence for modal formulas as well, for example, for $\Box^k A$.

[*] $\mathbf{F}(t) \vDash \Box^k A$ *and* $t \leq s$ *imply* $\mathbf{F}(s) \vDash \Box^k A$

This means that

$$t \leq s \to [\forall y[a_t R_t^k y \to y \vDash A] \to \forall y[a_s R_s^k y \to y \vDash A]]$$

It is possible to show that we can assume without loss of generality (i.e. without changing the semantic consequence relation) that:

(†) $t \leq s \wedge x R_s y \to x R_t y$.[6]

[6]This condition is for $\Box$. For $\Diamond$ we need $t \leq s \wedge x R_t y \to x R_s y$.

In fact if we let $S = \bigcup_t S_t$ we can assume that the fibred models are

$$\mathbf{F}(t) = (S, R_t, a_t, h_t).$$

We are going to assume the following additional properties: $a_t = a$ for some fixed a and $t \leq s$ and $h_t(x,q) = 1$ imply $h_s(x,q) = 1$ for all $x \in S$ and atomic q. We believe one can show that such assumptions can be made without loss of generality.

So the models differ only in their accessibility relation R_t which satisfies (†) above, and the assignment h_t.

Define functions $h^\sharp(x,q) \in U, q$ atomic, $x \in S$ and $R^\sharp : S^2 \mapsto U$ by letting

$$\begin{aligned} h^\sharp(x,q) = & \ \mathit{Inf}\,\{t \mid h_t(x,q) = 1\}. \\ R^\sharp(x,y) = & \ \mathit{Sup}\,\{t \mid xR_t y\}. \end{aligned}$$

(Let us assume the Sup is attained.)

Consider the system $(U, \leq, 0, \mu, S, R^\sharp, a, h^\sharp)$. We can view this system in two ways:

1. *An **LC** model $(U, \leq, 0, \mu)$ with a fibring of modal models (S, R_t, a, h_t), where $xR_t y$ holds iff $R^\sharp(x,y) \geq t$, and $h_t(x,q) = 1$ iff $t \geq h^\sharp(x,q)$.*
2. *A fuzzy model $(S, R^\sharp, a, h^\sharp)$ where the accessibility relation $R^\sharp$ and the assignment $h^\sharp$ are fuzzy and where the fuzzy truth set is $(U, \leq, 0, \mu)$ and evaluation is done using the **t**-conorm* max, *as indicated above.*[7]

Let us explore further the fuzzy model $(S, R^\sharp, a, h^\sharp)$. Consider, for $x \in S$, the statement $x \vDash_t A$, i.e. $x \vDash A$ in the model $\mathbf{F}(t)$. Because of persistence, we can define

$$\mu^\sharp(x,A) = \ \mathit{Inf}\,\{t \mid x \vDash_t A\}.$$

Consider $\mu^\sharp(x, \Box A)$

$$\begin{aligned} \mu^\sharp(x, \Box A) &= \mathit{Inf}\{t \mid x \vDash_t \Box A\} \\ &= \mathit{Inf}\,\{t \mid \forall y (xR_t y \text{ implies } y \vDash_t A)\} \end{aligned}$$

but $xR_t y$ holds iff $t \leq R^\sharp(x,y)$ and $y \vDash_t A$ holds iff $\mu^\sharp(y,A) \leq t$.

Hence

$$\mu^\sharp(x, \Box A) = \ \mathit{Inf}\,\{t \mid \forall y (t \leq R^\sharp(x,y) \text{ implies } \mu^\sharp(y,A) \leq t)\}$$

[7] If we choose a different **t**-conorm, say

$$\begin{aligned} \mu(A \wedge B) = & \ \min\,(1, \mu(A) + \mu(B)) \\ \mu(A \to B) = & \ \max\,(0, \mu(B) - \mu(A)) \\ \mu(A \vee B) = & \ \max\,(0, \mu(A) + \mu(B) - 1) \end{aligned}$$

we get evaluation which makes the accessibility relation Łukasiewicz fuzzy.

The previous two examples show that modal and many valued logic can be put together in two different ways. If we start with a modal model (S, R, a, h) then we can fuzzle (make fuzzy) h by changing it into a many valued assignment μ and extend to the entire modal language. If we start with a many valued model μ then we can fuzzle μ by changing it into a function into elements of a modal algebra. This turned out to be equivalent to looking at modal models where the possible world relation is fuzzy but the assignment is crisp. I.e. models of the form $(S, R^\sharp, a, \mu)$ where $R^\sharp(x, y) \in [0, 1]$, while μ is a $\{0, 1\}$ assignment. μ can be extended to all wffs, in which case it becomes a $[0, 1]$ valued function.

The obvious combination of the two approaches is to make both $R^\sharp$ and $\mu^\sharp$ fuzzy. This leads us to the following definition.

Definition 5.4 *An algebraic fuzzled many valued modal model has the form $(S, R^\sharp, a, \mu^\sharp)$, where $R^\sharp; S^2 \mapsto [0, 1]$ is a fuzzy possible world relation and for each $s \in S$ and atomic $q, \mu^\sharp(q) \in [0, 1]$.*

$\mu^\sharp_s$ can be extended to arbitrary formulas as follows:

$$\mu^\sharp_s(A * B) = f_*(\mu^\sharp_s(A), \mu^\sharp_s(B))$$

where $* \in \{\wedge, \vee, \rightarrow, \neg\}$ and f_* is the many valued truth table for $*$.

$$\mu^\sharp_x(\Box A) = \mathrm{Inf}_t[\textit{for all } y, R^\sharp(x, y) \geq x \textit{ implies } \mu^\sharp_y(A) \leq t].$$

Summary 5.5 *We summarise the ideas of this section.*

- *Making fuzzy is identical with fibring in a special way.*
- *Any logic $\mathbf{L}_1$ can be 'made fuzzy' by fibring it with $\mathbf{L}_2$ as $\mathbf{L}_1(\mathbf{L}_2)$.*
- *If $\mathbf{L}_1$ is the Lukasiewicz infinite valued logic and $\mathbf{L}_2$ is modal logic then $\mathbf{L}_1(\mathbf{L}_2)$ can be understood as modal logic with fuzzy accessibility but crisp assignment to atoms while $\mathbf{L}_2(\mathbf{L}_1)$ is modal logic with fuzzy assignment to atoms (but crisp accessibility).*

  In case of either $\mathbf{L}_1(\mathbf{L}_2(\mathbf{L}_1))$ or $\mathbf{L}_2(\mathbf{L}_1(\mathbf{L}_2))$ we get fuzzy accessibility and fuzzy assignment. Furthermore, any further iterations of the form $\mathbf{L}_1(\mathbf{L}_2(\mathbf{L}_2(\mathbf{L}_2 \ldots) \ldots)$ can be given semantics which is comprised of fuzzy SFM *models, (where both the accessibility and assignment are meny valued) of Definition 2.3 for the case of one modality and one jump operator.*

*More details are in the full paper (*Gabbay, 1995b*).*

6. Self Fibring for Predicate Logics

Fibred semantics can be used to give meaning to expressions of the form $\varphi(t, A)$ or $P(A)$, where $\varphi(x, y)$ is a formula with x, y free and $P(x)$ is a unary predicate and A is a formula and t is a term. Such logical expressions arise extensively in logic programming, logical models of natural language and metalevel considerations. For example, 'John believes A', 'John said that A', '*Demo* (A_1, A_2)', '*Provable* (x, A)', '*Hold* (t, A)' and the like. See (Gabbay, 1994).

The basic idea of our approach for classical logic can easily be explained.

Suppose we start with an ordinary model $\mathbf{m}$ classical logic. This has a domain D. Let $P(x)$ be a unary predicate of the language. Then P is assigned a subset of D. See Figure 4.

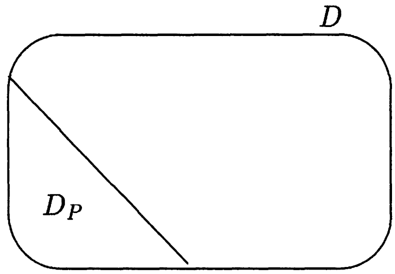

Figure 4.

For any element $a \in D$ we can evaluate the 'expression' $P(a)$. The evaluation gives us a truth value $\top$ or $\bot$. If $a \in D_P$ the value is $\top$. If $a \notin D_P$ the value is $\bot$. Now consider the expression $P(\sim P(a))$ or $P(c)$ where c is a non-denoting singular term.

We do not know how to evaluate it because the expression $y = \sim P(a)$, or $y = c$ inside the predicate is not assigned, (in our model theory), an element of the domain. However, we do notice that all we need to have is an answer to the question: is the truth value of $P(y)$ $\top$, or is it $\bot$? To get an answer we use t he 'fibring' methodology. We need a way of associating a value to the two 'parameters', P and y. We associate with D_P another model $\mathbf{F}(D_P)$, of the appropriate kind, which can help us get a value. The function $\mathbf{F}$ giving us the model is called the *fibring function.*

- For the case of a constant $y = c$, it is sufficient that the fibring function BF assigns a new model $\mathbf{n} = \mathbf{F}(D_P)$, for the language with c based on the do main $D' \supseteq D$. $\mathbf{n}$ of course assigns a value for c in D'. Thus $\mathbf{n} \vDash P(c)$ can be evaluated and so we can let
  $\mathbf{m} \vDash P(c)$ iff $\mathbf{F}(D_P) \vDash P(c)$.
- In the case of $y = \sim P(a)$, the model $\mathbf{n} = \mathbf{F}(D_P)$ is taken as another classical model, in which the expression y can be evaluated, because y

is a formula. So we can let
$\mathbf{m} \vDash P(\sim P(a))$ iff $\mathbf{F}(D_P) \vDash \sim P(a)$.

The reader should note the strong connection between the two cases, $y = c$ and y a formula. Indeed the connection is with Free logic. Free logic systems allow for non-denoting terms such as Pegasus, thus there are constants y in the language where $P(y)$ gets a truth value but y does not get assigned an element of the domain. For this kind of logic the rule $\forall x P(x) \to P(y)$ does hold but $P(y) \to \exists x P(x)$ does not hold. Depending on the system of free logic we might even not have $\forall x \forall y$ and $\forall y \forall x$ not being the same. See Bencivenga (Bencivenga, 1986), and also see (Lambert & Fraassen, 1972).

Our fibred semantics can also be of service to free logic, because we have the domain D of existing elements and also the non-existing 'elements' y, being the formulas of the language, or some other non-denoting constants.

The reader should note and observe that our method is not restricted to classical logic. We can equally study formulas of the $P(c), c$ non-denoting, or of the form $A(t, \varphi(s), f(B))$ in intuitionistic logic. The same approach applies. See my paper (Gabbay, 1995c) for full details.

We start with fibred semantics for self reference in monadic classical logic without function symbols and then gradually make the language more complicated. First add unary function symbols, then binary function symbols and then binary relations. In each case we develop a suitable fibred semantics and explore its properties. Further results can be found in a forthcoming paper, (Gabbay, 1995c).

We begin with our definitions and results.

Definition 6.1 (Fibred monadic syntax) *Let* $\mathbf{L}$ *be a language with monadic predicates, the classical connectives and quantifiers. We allow for two kinds of universal quantifiers,* $(\forall x)$ *and* (x)*.* $(\forall x)$ *is intended as quantifying over terms and formulas and* (x) *quantifiers only over terms (i.e. it is a sort of 'free logic' quantifier). We can define* $(\exists x) =\sim (\forall x) \sim$ *and* $(Ex) =\sim (x) \sim$.

We have no function symbols, but we do allow individual constants. We define the notion of a fibred wff (f-wff) of the language as follows

1. *A is a f-wff of level 0 if A is an ordinary free classical wff of* $\mathbf{L}$ *(built up with $\forall x$ and (x)).*
2. *A is a f-wff of level $n+1$ if there exists a free classical wff $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ of* $\mathbf{L}$*, with free variables $x_1, \ldots, x_n, y_1, \ldots, y_m$ and f-wffs $A_1, \ldots, A_n$ of level $\leq n$ such that $A = \varphi(x_1/A_1, \ldots, x_n/A_n, y_1, \ldots, y_m)$, where x_i/A_i denotes the simultaneous substitution of A_i for x_i in φ. The free variables of A are $y_1, \ldots, y_m$ together with all the*

free variables of $A_1, \ldots, A_n$. We assume the substitution is carefully made and does not result in incorrect binding of variables.

The reader will know from context whether we assume $\mathbf{L}$ *to contain only* $(\forall x)$ *or only* (x) *or both.*

Definition 6.2 (Fibred monadic models) *1. A classical (free logic) model for the language* $\mathbf{L}$ *has the form* $\mathbf{m} = (D, D', h, g)$ *where* $D \neq \varnothing$ *and* $D \cap D' = \varnothing$. *D is the domain of existing elements ((x) ranges over D) and D′ is the domain of non-existing elements* (($\forall x$) *ranges over* $D^* = D \cup D'$). *h is an assignment function assigning to each predicate P a subset* $h(P) \subseteq D^*$. *g is a function assigning to each variable x a value* $g(x) \in D$ *and to each constant c value* $g(c) \in D^*$ *(thus* $g(x)$ *always exists in D but* $g(c)$ *may not exist and be in* D'.
In our fibred semantics D′ is intended to be the set of all wffs.

2. *Traditional classical models are obtained by letting* $D' = \varnothing$. *In such a case we present the model as* (D, h, g).
   A fibred model is obtained by letting D′ = set of all closed f-wffs of the language. In fact D′ can be formally taken as all f-wffs—with free variables. The assignment g assigns elements of D to the variables which will fix the domain D′ to be all wffs of the language based on the ' constants' $\{g(x_1), g(x_2), \ldots\}$. *With this understanding we can define satisfaction for self fibred model later on.*
3. *Satisfaction for classical models is defined using the mixed notation* $\varphi(x)[x/\alpha]$ *to mean the formula* $\varphi(x)$ *with the intended value* α *for x, where* α *is any value element of* D^* *(or as we shall see in the case of fibred models indeed a formula of the language).*
   With the above notation we let
   - $\mathbf{m} \vDash P(x)[x/d]$, *for* $d \in D^*$ *iff* $d \in g(P)$
   - $\mathbf{m} \vDash P(t)$ *iff* $g(t) \in h(P)$.
     The above can also be written as
     $$\mathbf{m} \vDash P(t)[t/g(t)].$$
   - $\mathbf{m} \vDash (x)\varphi(x)$ *iff* $\mathbf{m} \vDash \varphi(x)[x/d]$ *for all* $d \in D$
   - $\mathbf{m} \vDash \forall x \varphi(x)$ *iff* $\mathbf{m} \vDash \varphi(x)[x/d]$, *for all* $d \in D^*$.
     In the self fibred model, D′ is the set of all f-wffs and so $\forall x$ *quantifies over real elements and formulas while (x) quantifies over real elements only.*
   - $\mathbf{m} \vDash t_1 = t_2$ *iff* $g(t_1) = g(t_2)$.
4. *A fibred model has the form* $\mathbf{n}_0 = (D, h_0, g, \mathbf{F})$, *where* (D, h_0, g) *is a classical model and* $\mathbf{F}$ *is a function giving for each assignment h and each subset* $X \subseteq D$ *a new assignment* $h' = \mathbf{F}(X, h)$.

5. *A relation* $\vDash$ *is said to be a* satisfaction *relation between* f*-wffs* A *and fibred models* $\mathbf{n} = (D, h, g, \mathbf{F})$ *if the following holds:*
   - $\mathbf{n} \vDash P(t)$ *iff* $g(t) \in h(P)$, *for* t *variable or constant and* P *atomic.*
   - $\mathbf{n} \vDash P(x)[x/d], d \in D$ *iff* $d \in h(P)$.
   - $\mathbf{n} \vDash \forall x A(x)$ *iff for all* $\alpha \in D$ *and all wffs* $\alpha, \mathbf{n} \vDash A(x)[x/\alpha]$.
   - $\mathbf{n} \vDash (x)A(x)$ *iff for all* $\alpha \in D, \mathbf{n} \vDash A(x)[x/\alpha]$.
     Note that we obtain D' *as the set of all* f*-wffs of the language.*
   - $\mathbf{n} \vDash P(\varphi)$ *iff* $(D, h', g, \mathbf{F}) \vDash \varphi$ *for* $h' = \mathbf{F}(h(P), h)$.
   - $\mathbf{n} \vDash (A \wedge B)[x/\alpha]$ *iff* $\mathbf{n} \vDash A[x/\alpha]$ *and* $\mathbf{n} \vDash B[x/\alpha]$
   - $\mathbf{n} \vDash (\sim A)[x/\alpha]$ *iff* $\mathbf{n} \nvDash A[x/\alpha]$.
6. *The function* $\mathbf{F}$ *need not be defined for all subsets* X *of* D. *It is sufficient to have it defined for all subsets of the form* $h(P)$, *for atomic* P *of the language. This would make the model more effectively computable. Also note that we can relinquish the requirement that* $\mathbf{F}$ *is a function and let* $\mathbf{F}$ *be a relation. Thus* $\mathbf{F}(X, h)$ *would be a set of assignments and the truth condition for* $P(\varphi)$ *would be:*
   - $\mathbf{n} \vDash P(\varphi)$ *iff* $(D, h', g, \mathbf{F}) \vDash \varphi$ *for all* $h' \in \mathbf{F}(h(P), h)$.

   This version of $\mathbf{F}$ *is of particular interest for the interpretation of* $P(\varphi)$ *as a modality on* φ. *We shall discuss this later in the section.*

Note that the above definition makes $\forall x P(x)$ range over all $x \in D$ and over all wffs φ as well.

Also note that for the case of $\forall$ the above definition of $\vDash$ is an implicit and not a recursive definition. Consider $\mathbf{m} \vDash \forall x P(x)$. We have to evaluate $\mathbf{m} \vDash P(\forall x P(x))$ and move to a new model $\mathbf{n} \vDash \forall x P(x)$. The variable x is instantiated also to the value $\forall x P(x)$. This causes an infinite recursive process. If we only have the quantifier (x) which ranges over $x \in D$, then $\vDash$ is recursively defined.

Further note that a more sophisticated satisfaction clause for $\mathbf{n} \vDash P(\varphi)$ can be the following

•• $\mathbf{n} \vDash P(\varphi)$ iff $(D, h', g, \mathbf{F}) \vDash \psi_P(\varphi)$ where $h' = \mathbf{F}(h(P), h)$ and $\psi_P(Q)$ is a classical wff involving a predicate Q and $\psi_P(\varphi)$ is the result of substituting φ for Q.

The previous satisfaction clause is obtained by letting $\psi_P(Q) = Q$.

The definition of the fibring function $\mathbf{F}$ makes it dependent on both assignments h and subsets of D. The dependence on h makes a difference. If $\mathbf{F}$ were independent of h and dependent only on subsets the following f-wff would be valid.

- $(x)[P_1(x) \leftrightarrow Q(P_2(x))] \to [P_1(\varphi) \leftrightarrow Q(P_2(\varphi))]$

The f-wff says that if P_1 in $\mathbf{m}$ has the same extension as P_2 in $\mathbf{m}_Q$ then $\mathbf{m} \vDash P_1(\varphi)$ iff $\mathbf{m}_Q \vDash P_2(\varphi)$, for arbitrary φ.

Theorem 6.3 (Axiomatisation of the system with $\forall$ only) *Consider the logic with all classical axioms and rules together with the following additional axioms and rules:*

- $P(\forall x A(x)) \leftrightarrow \forall x P(A(x))$
- $P(A \wedge B) \leftrightarrow P(A) \wedge P(B)$
- $\vdash A$ *implies* $\vdash P(A)$
- $\sim P(A) \leftrightarrow P(\sim A)$.

Then the logic is complete for satisfaction in fibred models where the universal quantifier $\forall$ ranges over all elements in the domain and all wffs as well.[8]

If we drop the last axiom ($\sim P(A) \leftrightarrow P(\sim A)$) we get completeness for the fibring of set function $\mathbf{F}$.

Proof. See (Gabbay, 1995c). ■

Remark 6.4 (Connection with modal logic) *We already say that $P(\varphi)$ can be viewed as a modal operator $\Box_P\varphi$. This is reminiscent of dynamic logic where there are modalities associated with regular expressions σ of the form $\Box_\sigma\varphi$ and there are operations on the expressions such as*

$$(\sigma_1, \sigma_2) \mapsto \sigma_1 * \sigma_2$$

and various connected axioms such as

$$\Box_{\sigma_1}\Box_{\sigma_2}\varphi \leftrightarrow \Box_{\sigma_1 * \sigma_2}\varphi.$$

In our case the connections between $\Box_{P_1}, \Box_{P_2}$ etc. are done through a full quantified theory of monadic predicate logic. Furthermore the domain of the modal model is the same as the domain used to connect the modal operators. This is like doing dynamic logic with expressions of the form $\Box_{\sigma(x)}\varphi(x,y)$ with x appearing both in σ and in φ. In fact in our case we can get expressions like $\Box_{B(x,y)}(\varphi_1(x,y), \varphi_2(x,y))$; any complex formula $B(x,y)$ can be a sort of modality $B(\varphi_1, \varphi_2)$.

Let us see what we can express in this language: consider the well known irreflexivity rule in modal logic.

$$\frac{\vdash \sim q \wedge \Box q \rightarrow \varphi}{\vdash \varphi}$$

[8]Notice the similarity of the axioms with that of a modal operator. In this case the modal operator is a *next* operator. The axioms without $\sim P(A) \leftrightarrow P(\sim A)$ define a $\mathbf{K}$ modality and correspond to a fibring $\mathbf{F}$ which is a set function.

provided q is not in φ,

This translates into

$$\forall\alpha[\sim (Ey(y=\alpha)]\wedge \sim \alpha \wedge P(\alpha) \to \varphi)] \to \varphi$$

$Ey(y=\alpha)$ is $\sim (y) \sim (y=\alpha)$ and it means α is a formula variable.

Remark 6.5 (Fibred binary relations) *Let us indicate how we might deal with the language of binary relations. We try to give fibred semantics for f-wffs of the form $R(A,B)$, where $R(x,y)$ is atomic binary predicate and A,B wffs. There are two main ways of dealing with $R(A,B)$.*

1. *Following the previous discussion, where we had a unary predicate $P(A)$ and a formula $\psi_P(Q)$, with $(D,h,\mathbf{F}) \vDash P(A)$ iff $(D,h',\mathbf{F}) \vDash \psi_P(A)$, we can define*

   $$(D,h,\mathbf{F}) \vDash R(A,B) \text{ iff } (D,h',\mathbf{F}) \vDash \psi_R(A,B)$$

   where here $\mathbf{F}$ fibred a new model with each binary relation, i.e.

   $$h' = \mathbf{F}(h(R),h), \text{ and } \psi_R(Q_1,Q_2)$$

   is a formula with two unary predicates Q_1,Q_2.
2. *We can adopt a more clever approach. Consider $R(x,t)$. For fixed t, we get a unary predicate in x. Let us use the notation $\lambda x R_t(x)$ for this predicate. Given a wff A, we can semantically evaluate $R_t(A)$ in the unary fibred models of the previous discussion.*

   $$\mathbf{m} \vDash R_t(A) \text{ iff } \mathbf{F}(\lambda x R_t(x),h) \vDash A.$$

We can now consider $R_t(A)$, for A fixed, as a predicate in t, i.e. consider $\{t \mid \mathbf{m} \vDash R_t(A)\}$ for this predicate. Hence, for a wff B, we can evaluate $R_B(A)$

$$\mathbf{m} \vDash_1 R_B(A \mid \text{ iff } \mathbf{F}(\{t \mid \mathbf{m} \vDash R_t(A),h) \vDash B.$$

Summarising, we get

$$\mathbf{m} \vDash_1 R(A,B) \text{ iff } \mathbf{F}(\{t \mid \mathbf{F}(\lambda x R(x,t),h) \vDash A\},h) \vDash B.$$

We obtained $\vDash_1$ by looking at $\lambda x R(x,t)$ first. If we were to start with $\lambda t R(x,t)$ as our initial parameterised unary predicate, we get

$$\mathbf{m} \vDash_2 R(A,B) \text{ iff } \mathbf{F}(\{x \mid \mathbf{F}(\lambda t R(x,t),h) \vDash B\},h) \vDash A.$$

The two definitions are not the same. $R(A,B)$ is syntactically symmetrical and we do not know whether A was substituted first and then B or the other way round, or perhaps both were substituted simultaneously.

*We leave the investigation of these options to the full paper (*Gabbay, 1995c*).*

We now turn to the case where function symbols are available in the language. We assume *unary* function symbols $e_i(x), i = 1, 2, 3, \ldots$ as well as unary predicates. Let $e(x)$ be a term. It has the form $e_1^{k_1}(e_2^{k_2} \ldots (e_n^{k_n}(x)) \ldots)$. The atomic expressions have the form $P(e(x))$. We thus have to give semantics to expressions of the form $e(\varphi)$ and $P(e(\varphi))$.

The idea is to treat $P(e(\varphi))$ as $P_1(\varphi)$ where $P_1(x)$ is defined as $P(e(x))$. We therefore have to let $e(\varphi)$ to be the formula $\varphi_1 = e(\varphi)$ where for any P

$$P(\varphi_1) = P(e(\varphi)) = P_1(\varphi).$$

Semantically we need to have

$$\mathbf{m} \vDash P(\varphi_1) \text{ iff } \mathbf{m}_P \vDash \varphi_1$$

which equals

$$\mathbf{m}_P \vDash e(\varphi) \text{ iff } \mathbf{m}_{e(P)} \vDash \varphi,$$

where e is now regarded as a function from predicates to predicates

$$e(P)(x) = P(e(x)).$$

Equivalently, for $X \subseteq D$, let:

$$e^{-1}(X) = \{y \mid e(y) \in X\}.$$

Definition 6.6 (Fibred syntax with unary functions) *Let* $\mathbf{L}$ *be the classical language of Definition 6.1. With additional unary function symbols* $\{e_1, e_2, e_3, \ldots\}$*. The notion of a well formed formula of level 0 is modified accordingly.*

The notion of f-wff of level $n+1$ *is defined similarly with the added clause that if* A *is of level* n *and* $e(x)$ *is a term then* $e(A)$ *is a wff of level* $n+1$.

Definition 6.7 (Semantics for fibred unary functions) *We modify Definition 6.2. Since our language contains function symbols, the assignment* g *must assign to each function symbol* e *a function* $g(e) : D^* \mapsto D^*$*. We require that* $g(e)$ *takes* D *into* D *and* D' *into* D'*. With each set* $X^* \subseteq D^*$ *and each* $g(e)$*, we let* $g(e)^{-1}(X^*) = \{y \mid g(e)(y) \in X^*\}$.

Notice that all terms are rigid designators!

Satisfaction is defined inductively as before, with the following additional inductive clause for function symbol e.

- $(D, h, g, \mathbf{F}) \vDash e(\varphi)$ *iff for all* $Y \subseteq D$ *and all* h_1 *such that* $h = \mathbf{F}(Y, h_1)$ *we have that* $(D, h', g, \mathbf{F}) \vDash \varphi$ *where* $h' = \mathbf{F}(g(e)^{-1}Y, h_1)$.

The meaning of the clause is as follows:

We want to evaluate $h \vDash e(\varphi)$*. We essentially look at all* h_1 *and* Q *such that* $h = \mathbf{F}(h(Q), h_1)$ *and evaluate* $h_1 \vDash Q(e(\varphi))$.

We stop our discussions here. For more see the full paper (Gabbay, 1995c).

7. Conclusion and Discussion

We saw that the idea of fibred semantics is basically very simple. If we encounter a symbol of a foreign language while evaluating an expression in our own language, then we regard this symbol as atomic (in our language) and turn to the semantics of the foreign language to get a value for it.

The above methodology, when iterated, gives fibred semantics for the combined language. Once we have this fibred semantics, there are some natural general questions to ask about the combined system:

1. How do the properties of the combined system depend on the properties of the components (transfer theorems)?
2. What is the landscape of possible interaction axioms and their corresponding conditions on the fibred semantics?
3. Can the fibring process be mechanised?
4. Can we reformulate existing known systems as fibred systems?
5. Extend our methodology to quantified systems.
6. Check interpolation theorems for the combined language.

There are also specific problems to ask for each of the application areas. Here we list some currently active research problems.

Case of modal and intuitionistic logics

We saw that for this case decidability and finite model property transfer to the combined system. We can also axiomatise the combined system in terms of the axiomatisations of the components. What remains to be done is to investigate the landscape of natural interaction axioms and their corresponding semantic conditions. For example, when dovetailing two modalities $\Box_1$ and $\Box_2$, where both are extensions of modal **K**, the basic models have the form (S, R_1, R_2, a, h). The interaction axiom $\Box_1 A \to \Box_2 A$ corresponds to the condition $R_2 \subseteq R_1$. This is easy to show. It is more difficult to find what condition corresponds to the commutativity axiom: $\Box_1\Box_2 = \Box_2\Box_1$. It is not clear whether we have completeness for models of the form $(S_1 \times S_2, R_1, R_2, (a_1, a_2))$ where R_i is a relation on S_i and where we have:

$$(x_1, x_2)R_1(y_1, y_2) \text{ iff } x_2 = y_2 \wedge x_1 R_1 y_1$$
$$(x_1, x_2)R_2(y_1, y_2) \text{ iff } x_1 y_1 \wedge x_2 R_2 y_2.$$

In general, charting the landscape for two modalities means looking at some natural interaction axioms and their corresponding semantic conditions.

Case of Fuzzy Logics
In this case the main immediate research problem is to axiomatise the combined system in terms of the axiomatisation of its components.

Case of Substructural Logics
We need to analyse the various ways of adding modality to a substructural implication and compare with the current systems (linear, relevant, etc.) in the literature. We should try to explain some existing peculiarities in terms of fibring. See (D'Agostino *et al.*, 1995; D'Agostino & Gabbay, 1996; Dörre *et al.*, to appear).

Case of Self Fibring of Predicate Logics
There is a lot to be done here:

- Apply the fibred semantics idea to intuitionistic logic. This should be straightforward.
- Check applications of the semantics to theories of self reference.
- Give fibred semantics to free logics and definite descriptions, and compare with existing solutions. Extend the method to intuitionistic logic.
- Investigate transfer theorems.

Case of the Conditional
The connection between the metalevel non-monotonic consequence relation and the object level subjunctive conditional are well-known. We know how to bring the consequence relation into the object level as a conditional, by using fibred semantics. We need to show systematically that the fibred semantics of the combined system is essentially the same as the traditional semantics of the conditional.

Case of Combining Metalevel with Object Level
Chart a methodology of bringing metalevel features into the object level in any reasonable system.

More General Fibring Problems
An alternative approach to combining logics and structures is in (Blackburn & de Rijke, to appear); the theory there evolves around the idea of explicitly controlling the communication between the structures (and logics) being combined through the use of modal operators to refer to the 'communication channels' between structures. The approach has clear links with out jump operator, but the precise connection remains to be determined.

For a discussion of combining general algebraic systems see (Baader & Schulz, 1995a; Baader & Schulz, 1995; Kepser & Schulz, 1996).

Also there is a connection with labelled deductive systems (Gabbay, 1995). It looks like the most natural environment for performing fibring and formulating the fibred semantics and proof system.

For topological fibring see (Pfalzgraf, 1987; Pfalzgraf, 1991; Pfalzgraf & Stokkermans, 1994).

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