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A Fragment of Intuitionistic Dynamic Logic

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Abstract. We present a fragment of Propositional Dynamic Logic based on the Intuitionistic Propositional Logic. We show that this logic has the finite model property, and therefore, is frame-complete.

Keywords: Modal Logic, Intuitionistic Modal Logic, Dynamic Logic.

1. Introduction

The Propositional Dynamic Logic (**PDL**) [3, 5, 6, 7] is one of the logics used in studies on reasoning about computer programs. The language of **PDL** is constructed by means of a set of atomic programs I, a set of connectives of programs, and a propositional language. The programing language and/or the propositional language might be modified according to the needs, thus opening up new logical systems. The fragment of **PDL** introduced by Vakarelov [9], where a cyclic repeating of a program can be interpreted, is an example of change of language of programming. The modal logics studied by D.Wijesekera [10] are examples of Dynamic Logics, or more precisely concurrent logics, defined on the Intuitionistic Propositional Logic (**Int**). In Wijesekera's work his main motivation lies on problems related to the study about states machines in Computer Science and about states of knowledge and belief in Artificial Intelligence based in *partial information*. However, modal logics based in classical logic are not the appropriate instrument for the analyses of problems where partial information has been considered, because the models in classical logic represent *complete states of information*.

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The relevance of **Int** is due to its ability to interpret partial information appropiatly. A natural way to model intuitionistic logic is by means of intuitionistic Kripke frames, that is, pairs $\langle X, \leq \rangle$ where X is a nonvoid set and $\leq$ is a partial order. The points of X can be interpreted as *incomplete states of information*, where $a \leq b$ can be interpreted as if the state b contains the information of the state a. So, when it is necessary to consider partial information in problems related to Computer Science or AI, modal logics based on intuitionistic logic are more appropiate. This would account for the research on different modal logic based on intuitionistic logic, particularly an intuitionistic version of Dynamic Logic.

In this paper we shall consider a simplified fragment of **PDL** (called $\mathbf{IK}^*_\Box$) based on intuitionistic logic (**Int**). We present a Kripke-style semantic for $\mathbf{IK}^*_\Box$, and we prove soundness and completeness with respect to this semantic. As in the case of the **PDL**, the completeness of $\mathbf{IK}^*_\Box$ is based on the construction of a filtration of a model of $\mathbf{IK}^*_\Box$.

In Section 2 we introduce the definitions and necessary notions to develop this paper. In Section 3 we define the calculus $\mathbf{IK}^*_\Box$ and we introduce the models for which $\mathbf{IK}^*_\Box$ is sound. In Section 4 we study the filtration of models for intuitionistic modal logic with only one operator. These results are applied in the following section to give the completeness of $\mathbf{IK}^*_\Box$.

2. Preliminaries

Let us consider a propositional language $\mathcal{L}$ with a denumerable set of propositional variables $Var = \{p_0, p_1, ..., p_n, ...\}$, the binary connectives $\vee, \wedge$, and $\rightarrow$, and the constant $\bot$. Define the negation $\neg$ by $\neg p = p \rightarrow \bot$, and the constant $\top = \neg\bot$. The multimodal language $\mathcal{L}_i$ is obtained extending $\mathcal{L}$ by means of the set of modal operators $\{\Box_i : i \in I\}$. If I has only one element, we write $\Box_1 = \Box$, and if I has two elements we write $\Box_1 = \Box$ and $\Box_2 = \triangle$. We denote by Fm the set of all well formed formulas in the language $\mathcal{L}_i$.

Let us consider the relational structure $\langle X, \leq \rangle$, where X is a set and $\leq$ is a reflexive and transitive relation on X. A subset $U \subseteq X$ is said to be *increasing* if for all $x, y \in X$ such that $x \in U$ and $x \leq y$, we have $y \in U$. The set of all increasing subsets of X is denoted by $\mathcal{IS}(X)$. For each $Y \subseteq X$, the increasing set generated by Y is $[Y) = \{x \in X : \exists y \in Y : y \leq x\}$.

Let X be a set and let R be a binary relation on X. We will use the symbol R^n for $n \geq 0$, to represent the n-power of a binary relation R and we will use the symbol R^* to represent the reflexive and transitive closure of R.

Now we define the adequate relational semantics for the modal logic extending intuitionistic logic. These relational structures should be a quasi-ordering $\leq$ for interpreting the intuitionistic formulas, and binary relations for interpreting the modal operators. As we need to consider only $\leq$-increasing valuations, we should have some connection between the quasi-ordering and the binary relations. Different conditions are given in [1] (see also [2]). For our purpose we will consider a simple condition that assures that the valuations are $\leq$-increasing.

Definition 2.1. A ordered-Kripke frame in the language $\mathcal{L}_i$ is a relational structure $\mathcal{F} = \left\langle X, \leq, \{R_i\}_{i \in I} \right\rangle$ where $\leq$ is a reflexive and transitive relation on X, R_i is a binary relation on X for each $i \in I$, and $\leq \circ R_i \subseteq R_i$ for each $i \in I$.

Let $\mathcal{F} = \langle X, \leq, \{R_i\}_{i\in I}\rangle$ be a frame. A valuation on $\mathcal{F}$ is a map $V : Var \to \mathcal{IS}(X)$. Any valuation V can be extended recursively to the set Fm by means of the following conditions:

1. $V(\perp) = \emptyset$,
2. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$,
3. $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,
4. $V(\varphi \to \psi) = \{x \in X : [x) \cap V(\varphi) \subseteq V(\psi)\}$,
5. $V(\Box_i\varphi) = \{x \in X : R_i(x) \subseteq V(\varphi)\}$,

By the conditions 1. and 4. above it follows:

$$V(\neg\varphi) = \{x \in X : [x) \cap V(\varphi) = \varnothing\}.$$

A model is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$ where $\mathcal{F}$ is a frame and V is a valuation on it. A formula φ is satisfied in an element $x \in X$, in symbols $\langle \mathcal{F}, V\rangle \vDash_x \varphi$, if $x \in V(\varphi)$. The formula φ is valid in $\mathcal{M}$, in symbols $\langle \mathcal{F}, V\rangle \vDash \varphi$, if $V(\varphi) = X$. The formula φ is valid in a frame $\mathcal{F}$, in symbols $\mathcal{F} \vDash \varphi$, if $\langle \mathcal{F}, V\rangle \vDash \varphi$ for any valuation V defined in $\mathcal{F}$.

Lemma 2.2. Let $\mathcal{M}$ be a model. Then for any $\varphi \in Fm$ and for any $i \in I$, $V(\varphi) \in \mathcal{IS}(X)$.

Proof. The proof is by induction on the complexity of formulas. We shall only consider the case $\Box_i\varphi$, for some $i \in I$. Suppose that $x \leq y$ and $x \in V(\Box_i\varphi)$. It follows that $R_i(x) \subseteq V(\varphi)$. Let $z \in X$ such that $(y, z) \in R_i$. By the condition $\leq \circ R_i \subseteq R_i$, we have that $(x, z) \in R_i$, and thus $z \in V(\varphi)$. So $y \in V(\Box_i\varphi)$. Therefore, $V(\Box_i\varphi) \in \mathcal{IS}(X)$. □

Let $\mathcal{F} = \langle X, \leq, \{R_i\}_{i\in I}\rangle$ a frame. For each $i \in I$, define the relation $R_{\Box_i} = R_i \circ \leq$. These relations are very important in this work. The following results are proved in [1, 2].

Lemma 2.3. Let $\mathcal{F} = \langle X, \leq, \{R_i\}_{i\in I}\rangle$ be a frame. Then for each $x \in X$, and for each $i \in I$, the maps $V^x_{\Box_i}(p) = R_{\Box_i}(x)$ are valuations

Lemma 2.4. Let $\mathcal{F} = \langle X, \leq, \{R_i\}_{i\in I}\rangle$ be a frame. Then for each $\varphi \in Fm$, and for each valuation V defined on $\mathcal{F}$ we have

$$V(\Box_i\varphi) = \{x \in X : R_{\Box_i}(x) \subseteq V(\varphi)\}.$$

Proof. Immediate by Lemma above (see [1, 2]). □

Lemma 2.5. Let $\mathcal{F} = \langle X, \leq, \{R_i\}_{i\in I}\rangle$ be a frame. Then

1. $(R_{\Box_i})^* = R_i^* \circ \leq$.
2. $\leq \circ (R_i)^* \subseteq (R_i)^*$.
3. $\leq \circ (R_{\Box_i})^* \subseteq (R_{\Box_i})^*$.

Proof. It is easy and left to the reader. □

Theorem 2.6. Let $\mathcal{F} = \left\langle X, \leq, \{R_i\}_{i \in I} \right\rangle$ be a frame. Then

1. $\mathcal{F} \vDash \Box_i \varphi \rightarrow \varphi$ iff $R_{\Box_i}$ is reflexive for each $i \in I$.
2. For each pair $i, j \in I$, $\mathcal{F} \vDash \Box_i \varphi \rightarrow \Box_j \Box_i$ iff $R_{\Box_j} \circ R_{\Box_i} \subseteq R_{\Box_j}$.

Proof. We only to check 2. Let any $i, j \in I$. Suppose that $\mathcal{F} \vDash \Box_i \varphi \rightarrow \Box_j \Box_i \varphi$ and let $x, y, z \in X$ such that $(x, z) \in R_{\Box_j}$ and $(z, y) \in R_{\Box_i}$. Let us consider the valuation

$$V^x_{\Box_i}(p) = R_{\Box_i}(x).$$

Since $x \in V^x_{\Box_i}(\Box_j p)$ and $\mathcal{F} \vDash \Box_i \varphi \rightarrow \Box_j \Box_i$, we get $x \in V^x_{\Box_i}(\Box_j \Box_i p)$. Then $z \in V^x_{\Box_i}(\Box_i p)$. Hence $(z, y) \in R_{\Box_i}$, we have $y \in V^x_{\Box_i}(p) = R_{\Box_i}(x)$. Thus $(x, y) \in R_{\Box_i}$.

The other direction is easy. □

3. The logic $\mathrm{IK}^*_\Box$

For the purpose of this paper, the term *logic* means a set of formulas in a suitable propositional language.

Let us consider the language $\mathcal{L}_2$ (i.e., with the modal operators $\Box$ and $\triangle$). The logic $\mathbf{IK}^*_\Box$ is the smallest set of formulas in the language $\mathcal{L}_2$, such that

1. Contain the Intuitionistic Propositional Calculus **Int**,
2. $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$,
3. $\triangle(p \rightarrow q) \rightarrow (\triangle p \rightarrow \triangle q)$,
4. $\triangle p \rightarrow p$,
5. $\triangle p \rightarrow \Box \triangle p$,
6. $\triangle(p \rightarrow \Box p) \rightarrow (p \rightarrow \triangle p)$,

   and closed under the following rules:

7. (Modus Pones) If $p, p \rightarrow q \in \mathbf{IK}^*_\Box$, then $q \in \mathbf{IK}^*_\Box$,
8. (Sustitutions) If $p \in \mathbf{IK}^*_\Box$, then $e(p) \in \mathbf{IK}^*_\Box$, for each substitution $e : Var \rightarrow Fm$,
9. (Necessity) If $p \in \mathbf{IK}^*_\Box$, then $\Box p \in \mathbf{IK}^*_\Box$ and $\triangle p \in \mathbf{IK}^*_\Box$.

The members of $\mathbf{IK}^*_\Box$ are called its theorems. We write $\vdash_{\mathbf{IK}^*_\Box} \varphi$ to mean that φ is a theorem of $\mathbf{IK}^*_\Box$.

The intended interpretation of the operator $\Box$ is that correspond to an one step of a program execution, and the interpretation of the operator $\triangle$ correspond to one repetition of finite number of times (iteration) of the a program execution. The operator $\triangle$ is a version of PDL's operator * applied to $\Box$.

As the propositional language considered in this section have only two modal operators we shall work with frames of the form $\mathcal{F} = \langle X, \leq, R_1, R_2 \rangle$, where R_1 is the binary relation that serve for interpreted the modal operator $\Box$ and R_2 is the binary relation that serve for interpreted the modal operator Δ. Let us recall that $R_\Box = R_1 \circ \leq$ and $R_\triangle = R_2 \circ \leq$.

Let $\mathbf{F}$ be a class of frames. The *theory* of $\mathbf{F}$ is

$$Th\left(\mathbf{F}\right) = \left\{\varphi \in Fm : \forall \mathcal{F} \in \mathbf{F}, \mathcal{F} \vDash \varphi\right\}.$$

It is easy to check that $Th\left(\mathbf{F}\right)$ is closed under the axioms 1., 2., and 3., and the rules 7., 8., and 9.

Let $\mathcal{IML}$ be a logic such that $\mathbf{IK}^*_\Box \subseteq \mathcal{IML}$. The class of frames of $\mathcal{IML}$ is denoted by $Fr\left(\mathcal{IML}\right)$, i.e., $Fr\left(\mathcal{IML}\right) = \left\{\mathcal{F} : \mathcal{F} \vDash \mathcal{IML}\right\}$. The logic $\mathcal{IML}$ is *frame-complete* if $\mathcal{IML} = Th\left(Fr\left(\mathcal{IML}\right)\right)$.

Definition 3.1. Let $\mathcal{F} = \langle X, \leq, R_1, R_2\rangle$ be a frame in the language $\mathcal{L}_2$. We say that $\mathcal{F}$ is a *-frame if $(R_\Box)^* = R_\triangle$, i.e., the interpretation of $\triangle$ is taken to mean iterated $\Box$.

Theorem 3.2. Let $\mathcal{F} = \langle X, \leq, R_1, R_2\rangle$ be a frame in the language $\mathcal{L}_2$. Then $\mathcal{F}$ is a *-frame if and only if $\mathcal{F} \in Fr\left(\mathbf{IK}^*_\Box\right)$.

*Proof***.** Let us consider the frame $\mathcal{F} = \langle X, \leq, R_1, R_2\rangle$. Suppose that $\mathcal{F} \in Fr\left(\mathbf{IK}^*_\Box\right)$. We shall first prove that for any pair $(x, y) \in R_\triangle$ there exists $n \geq 0$ such that $(x, y) \in (R_\Box)^n$, and thus we get $R_\triangle \subseteq (R_\Box)^*$.

Let $(x, y) \in R_\triangle$ and let us define the valuation V by

$$V\left(p\right) = \left\{y \in X : (x, y) \in (R_\Box)^n \text{ for some } n \geq 0\right\}$$

By Lemma 2.3, $V\left(p\right) \in \mathcal{IS}\left(X\right)$.

We shall first prove

$$\langle \mathcal{F}, V\rangle \vDash p \to \Box p. \tag{1}$$

Let $z, k, d \in X$ such that $z \leq k$, $k \in V\left(p\right)$ and $(k, d) \in R_\Box$. Then for some n, we get $(x, k) \in (R_\Box)^n$. Thus $(x, d) \in (R_\Box)^{n+1}$. So, $d \in V\left(p\right)$, i.e., $k \in V\left(\Box p\right)$. Therefore, $z \in V\left(p \to \Box p\right)$. This proves (1).

By (1) we have that $x \in V\left(\triangle\left(p \to \Box p\right)\right)$, and since $\mathcal{F} \in Fr\left(\mathbf{IK}^*_\Box\right)$, we get $[x) \subseteq V\left(p \to \triangle p\right)$. In particular, $x \in V\left(p \to \triangle p\right)$, because $\leq$ is reflexive. As $(x, x) \in (R_\Box)^0$, $x \in V\left(p\right)$. It follows that $x \in V\left(\triangle p\right)$, and since $(x, y) \in R_\triangle$, we get $y \in V\left(p\right)$. Then there exists $n \geq 0$ such that $(x, y) \in (R_\Box)^n$. Thus, by induction, we get $R_\triangle \subseteq (R_\Box)^*$.

The proof of the other inclusion is by induction. Let us recall, by Theorem 2.6, that $R_\triangle$ is reflexive and $R_\Box \circ R_\triangle \subseteq R_\triangle$.

For $n = 0$, $(R_\Box)^0 = R_1^0 \circ \leq$, and as $\leq$ is reflexive, we have $(R_\Box)^0 \subseteq R_\triangle$.

Suppose that the inclusion is satisfied for n and let $(x, y) \in (R_\Box)^{n+1}$. Then there exists $z \in X$ such that $(x, z) \in R_\Box$ and $(z, y) \in (R_\Box)^n$. By inductive hypothesis, $(z, y) \in R_\triangle$. Then $(x, y) \in R_\Box \circ R_\triangle$, and this implies that $(x, y) \in R_\triangle$. Thus, $(x, y) \in (R_\Box)^*$.

Reciprocally. Suppose that $\mathcal{F}$ is a *-frame, i.e., $(R_\Box)^* = R_\triangle$. Since $R_\triangle$ is reflexive, then $\mathcal{F} \vDash \triangle\varphi \to \varphi$, and since $R_\Box \circ (R_\Box)^* \subseteq (R_\Box)^*$, we get $\mathcal{F} \vDash \triangle\varphi \to \Box\triangle\varphi$.

We prove that $\mathcal{F} \vDash \triangle\left(\varphi \to \Box\varphi\right) \to \left(\varphi \to \triangle\varphi\right)$. Let V be a valuation on $\mathcal{F}$, and let us consider $x, y, z \in X$ such that $x \leq y$, $y \in V\left(\triangle\left(\varphi \to \Box\varphi\right)\right)$, $y \in V\left(\varphi\right)$, and $(y, z) \in (R_\Box)^*$. We prove that $z \in V\left(\varphi\right)$. Since $(y, z) \in (R_\Box)^*$, then there exist $y = y_1, y_2, ..., y_n = z$ such that $(y_i, y_{i+1}) \in R_\Box$, for

$2 \leq i \leq n-1$. As, $(y, y) \in (R_{\Box})^*$, $y \in V(\varphi \to \Box\varphi)$, and since $y \in V(\varphi)$, we get that $y \in V(\Box\varphi)$. It follows that $y_2 \in V(\varphi)$. So, it is easy to check that for any i such that $2 \leq i \leq n-1$, $y_i \in V(\varphi)$. In particular, $z \in V(\varphi)$. Therefore, $\mathcal{F} \vDash \triangle(\varphi \to \Box\varphi) \to (\varphi \to \triangle\varphi)$. □

4. The finite model property

A modal logic $\mathcal{ML}$ has the *finite model* (*frame*) *property* if for each formula φ such that $\varphi \notin \mathcal{ML}$ there exists a finite model $\mathcal{M}$ (a finite frame $\mathcal{F}$) such that $\mathcal{M} \nvDash \varphi$ ($\mathcal{F} \nvDash \varphi$). It is known that a modal logic extending classical logic has the finite model property if and only if it has the finite frame property. Moreover, if the logic $\mathcal{ML}$ has any of these properties, then it is frame complete, i.e., $\mathcal{ML} = Th(Fr(\mathcal{ML}))$ (see, for example [5] or [7]).

In this section we shall study the finite model property for intuitionistic modal logics with only one modal operator $\Box$. Let us consider the intuitionistic modal logic $\mathbf{IK}_{\Box}$ that has the only axiom $\Box(p \to q) \to (\Box p \to \Box q)$, and the rules of Modus Ponens, Necessity and Substitution. We shall prove that the finite model property is equivalent to the finite frame property. We shall apply this result in the next section when we study the completeness of the logic $\mathbf{IK}^*_{\Box}$. It is easy to check that the next results are also valid for multimodal languages.

Let $\mathcal{M} = \langle X, \leq, R, V\rangle$ be a model. For each $x \in X$ consider the sets

$$F_x = \{\varphi \in Fm : x \in V(\varphi)\}$$

and

$$\Box^{-1}(F_x) = \{\varphi \in Fm : x \in V(\Box\varphi)\}.$$

Definition 4.1. We shall say that a model $\mathcal{M}$ is *differentiated* if for any $x, y \in X$ if $F_x \subseteq F_y$ then $x \leq y$. So, in any differentiated model $\mathcal{M}$ the relation $\leq$ is an order. We shall say that $\mathcal{M}$ is *R-differentiated* if for any $x, y \in X$, if $\Box^{-1}(F_x) \subseteq F_y$ then $(x, y) \in R$. Two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are *equivalent* if $Th(\mathcal{M}_1) = Th(\mathcal{M}_2)$.

Let $\mathcal{M}$ be a model and let Σ be a set of formulas closed under subformulas. Define on X the equivalence relation $\underset{\Sigma}{\approx}$ as follows:

$$x \underset{\Sigma}{\approx} y \Leftrightarrow F_x \cap \Sigma = F_y \cap \Sigma.$$

Let $|x| = \left\{y \in X : x \underset{\Sigma}{\approx} y\right\}$ the equivalence class of x, and let $X_\Sigma = \{|x| : x \in X\}$ the quotient set. Define the order relation $\underset{\Sigma}{\preccurlyeq}$ in X_Σ by:

$$|x| \underset{\Sigma}{\preccurlyeq} |y| \Leftrightarrow F_x \cap \Sigma \subseteq F_y.$$

Note that if $x \leq y$, then $|x| \underset{\Sigma}{\preccurlyeq} |y|$. By last, we define the valuation $V_\Sigma : Var \to \mathcal{IS}(X_\Sigma)$ by:

$$V_\Sigma(p) = \{|x| : x \in V(p)\}.$$

If $\Sigma = Fm$, then we write $\approx$ ($\preccurlyeq$) instead of $\underset{\Sigma}{\approx}$ $\left(\underset{\Sigma}{\preccurlyeq}\right)$, and $X_{\approx}$ ($V_{\approx}$) instead of X_Σ (V_Σ).

Definition 4.2. Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a model and let Σ be a set of formulas closed under subformulas. The relation $R^{+} \subseteq (X_{\Sigma})^{2}$ is a Σ-**filtration** of R if and only if

F1 If $(x, y) \in R \circ \leq$, then $(|x|, |y|) \in R^{+} \circ \underset{\Sigma}{\preccurlyeq}$,

F2 If $(|x|, |y|) \in R^{+}$, then $\Box^{-1}(F_x \cap \Sigma) \subseteq F_y$,

F3 $\underset{\Sigma}{\preccurlyeq} \circ R^{+} \subseteq R^{+}$.

Given a model $\mathcal{M}$ there are at least two Σ-filtrations of R: the relation R_m defined by:

$$(|x|, |y|) \in R_m \Leftrightarrow \Box^{-1}(F_x \cap \Sigma) \subseteq F_y,$$

and the relation $\bar{R}$ given by:

$$(|x|, |y|) \in \bar{R} \Leftrightarrow \exists z\, \exists y' \left(|x| \underset{\Sigma}{\preccurlyeq} |z| \wedge (z, y') \in R \wedge y \underset{\Sigma}{\approx} y' \right).$$

It is easy to see that R_m satisfies the conditions of Definition above. We check that $\bar{R}$ is a filtration of R.

F1. Let $(x, z) \in R$ and $z \leq y$. We prove that $(|x|, |y|) \in \bar{R} \circ \underset{\Sigma}{\preccurlyeq}$. Since the relation $\leq$ is reflexive we get that $|x| \underset{\Sigma}{\preccurlyeq} |x|$, and as $z \underset{\Sigma}{\approx} z$, then $(|x|, |z|) \in \bar{R}$ and $|z| \underset{\Sigma}{\preccurlyeq} |y|$.

F2. Let $(|x|, |y|) \in \underset{\Sigma}{\preccurlyeq} \circ \bar{R}$. Then there exist $z, z', y' \in X$ such that $F_x \cap \Sigma \subseteq F_z$, $F_z \cap \Sigma \subseteq F_{z'}$, $(z', y') \in R$ and $F_{y'} \cap \Sigma = F_y \cap \Sigma$. Since $(z', y') \in R$, $\Box^{-1}(F_{z'} \cap \Sigma) \subseteq F_{y'}$. So, it is easy to check that $\Box^{-1}(F_x \cap \Sigma) \subseteq F_y$.

The proof for the condition **F3** is similar and left to the reader.

Definition 4.3. Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a model and let Σ be a set of formulas closed under subformulas. The structure $\langle \mathcal{F}_{\Sigma}, V_{\Sigma} \rangle = \left\langle X_{\Sigma}, \underset{\Sigma}{\preccurlyeq}, R^{+}, V_{\Sigma} \right\rangle$ is a Σ-**filtration** of $\mathcal{M}$ if $R^{+}$is a Σ-filtration of R.

Theorem 4.4. Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a model, Σ be a set of formulas closed under subformulas, and let $\langle \mathcal{F}_{\Sigma}, V_{\Sigma} \rangle$ be a Σ-filtration of $\mathcal{M}$. Then for each $x \in X$ and for each $\varphi \in \Sigma$,

$$\langle \mathcal{F}, V \rangle \vDash_x \varphi \Leftrightarrow \langle \mathcal{F}_{\Sigma}, V_{\Sigma} \rangle \vDash_{|x|} \varphi.$$

***Proof*.** The proof is by induction on the complexity of formulas. We only have to check the case of formulas of type $\Box\varphi$. Suppose that $x \in V(\Box\varphi)$ and let $y \in X$ such that $(|x|, |y|) \in R^{+} \circ \underset{\Sigma}{\preccurlyeq}$. Then there exists $|z| \in X_{\approx}$ such that $(|x|, |z|) \in R^{+}$ and $|z| \underset{\Sigma}{\preccurlyeq} |y|$. By **F2** and the definition of $\underset{\Sigma}{\preccurlyeq}$, $\Box^{-1}(F_x \cap \Sigma) \subseteq F_z$ and $F_z \cap \Sigma \subseteq F_y$. It follows that $y \in V(\varphi)$. Thus, by induction, $|y| \in V_{\Sigma}(\Box\varphi)$. The proof in the other direction is left to the reader. $\Box$

Lemma 4.5. For any model $\mathcal{M}$ there exists a model $\mathcal{M}'$ such that it is differentiated, R-differentiated, $Th(\mathcal{M}) = Th(\mathcal{M}')$, and $card(X') \leq card(X)$.

Proof. Let us consider the model $\mathcal{M}' = \langle X_{\approx}, \preccurlyeq, R_{\approx}, V_{\approx}\rangle$ where the relation $R_{\approx}$ is defined by $(|x|, |y|) \in R_{\approx}$ if and only if $\Box^{-1}(F_x) \subseteq F_y$. It is clear that the model $\mathcal{M}'$ is differentiated, R-differentiated, $card(X_{\approx}) \leq card(X)$, and by Theorem 4.4, $Th(\mathcal{M}) = Th(\mathcal{M}')$. □

Lemma 4.6. Let $\mathcal{M} = \langle \mathcal{F}, V\rangle$ be a finite and differentiated model. Then for each $x \in X$ there exists a formula φ such that $V(\varphi) = [x)$.

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$. Since $\mathcal{M}$ is differentiated, then for $x_i, x_j \in X$ such that $x_i \not\leq x_j$, $i \neq j$, there exists a formula $\varphi_{i,j}$ such that $\varphi_{i,j} \in F_{x_i}$ and $\varphi_{i,j} \notin F_{x_j}$. Define the formula $\bigwedge\limits_{\substack{\varphi_{i,j} \in F_{x_i} \\ i \neq j}} \varphi_{i,j} = \varphi_i$. So $x_i \in V(\varphi_i)$, and as $V(\varphi_i)$ is increasing, we have $[x) \subseteq V(\varphi_i)$. If $y \in V(\varphi_i)$ but $x_i \not\leq y$, then there is a formula φ_{ij} such that $x_i \in V(\varphi_{ij})$ and $y \notin V(\varphi_{ij})$, which is a contradiction. Therefore, $V(\varphi) = [x)$. □

Lemma 4.7. Let $\mathcal{IML}$ be an intuitionistic modal logic such that $\mathbf{IK}_{\Box} \subseteq \mathcal{IML}$. Let $\langle \mathcal{F}, V\rangle$ be a finite and differentiated model of $\mathcal{IML}$. Then $\mathcal{F} \vDash \mathcal{IML}$.

Proof. Let $\langle \mathcal{F}, V\rangle \vDash \mathcal{IML}$. Assume that $\mathcal{F} \nvDash \mathcal{IML}$. Then there exists a valuation V', there exists a formula $\varphi \in \mathcal{L}$, and there exists $x \in X$ such that $x \notin V'(\varphi)$. Let $p_n \in Var$ and consider the set $V'(p_n) = \left\{x_{i_1}^n, x_{i_2}^n, \ldots, x_{i_n}^n\right\}$. As $\langle \mathcal{F}, V\rangle$ is differentiated, then by the above Lemma, for each $x_{i_k}^n$, $k = 1, \ldots, n$ there is a formula $\psi_{i_k}^n$ such that $V\left(\psi_{i_k}^n\right) = \left[x_{i_k}^n\right)$. For each formula $\beta(p_0, p_1, ..., p_n)$ define $\beta^*(p_0/p_0^*, p_1/p_1^*, ..., p_n/p_n^*)$, where $p_k^* = \bigvee\limits_{k=1}^{n} \psi_{i_k}^n$ if $V'(p_k) \neq \varnothing$ and $p_k^* = \bot$ if $V'(p_k) = \varnothing$. By induction it is easy to see that for all formula β, $V'(\beta) = V(\beta^*)$. Since $x \notin V'(\varphi)$, $x \notin V(\varphi^*)$. It follows

$$\langle \mathcal{F}, V\rangle \nvDash \varphi^*. \tag{2}$$

On the other hand, since $\mathcal{IML}$ is closed under substitutions, and φ^* is an instance of substitution, then $\varphi^* \in \mathcal{IML}$, which is a contradiction to (2). Therefore, $\mathcal{F} \vDash \mathcal{IML}$. □

Theorem 4.8. The intuitionistic modal logic $\mathbf{IK}_{\Box}$ has the finite model property if and only if it has the finite frame property.

Proof. Suppose that $\mathbf{IK}_{\Box}$ has the finite model property. Let $\varphi \notin \mathbf{IK}_{\Box}$. Then there exists a finite model $\mathcal{M}$ of $\mathbf{IK}_{\Box}$ such that $\mathcal{M} \nvDash \varphi$. By Lemma 4.5 there exists a differentiated and finite model $\mathcal{M}' = \langle \mathcal{F}', V'\rangle$ such that $\mathcal{M}' \nvDash \varphi$. By Lemma 4.7 $\mathcal{F}' \nvDash \varphi$ and $\mathcal{F}' \in Fr(\mathbf{IK}_{\Box})$. Thus, $\mathbf{IK}_{\Box}$ has the finite frame property. The other direction is immediate. □

5. Filtrations of $\mathbf{IK}_{\Box}^*$. Completeness

In this section we shall give a construction of a finite model of $\mathbf{IK}_{\Box}^*$ which enables to prove that $\mathbf{IK}_{\Box}^*$ has the finite model property. As a consequence of this and the results of the previous section, we shall conclude that $\mathbf{IK}_{\Box}^*$ is complete with respect to its frames class.

Let $\Sigma \subseteq Fm$. We shall say that Σ is *closed* if and only if

1. Σ is a finite set,
2. Σ is closed under subformulas, and
3. Σ is closed under the following condition: if $\triangle\varphi \in \Sigma \Rightarrow \Box\triangle\varphi \in \Sigma$.

We note that the definition of closed set is really a version of the Fischer-Lander closure [3]

Let $\varphi \in Fm$, and let $Sf(\varphi)$ be the set of all subformulas of φ. Let Σ be the closure of $Sf(\varphi)$ under the condition 3. above. It is easy to prove that Σ is a finite set of formulas (see, for example, [5] page 113).

Let $\mathcal{F} = \langle X, \leq, R_1, R_2 \rangle$ be a frame and let V be a valuation on $\mathcal{F}$ such that $\langle \mathcal{F}, V \rangle$ be a model of $\mathbf{IK}_{\Box}^{*}$, i.e., if $\vdash_{\mathbf{IK}_{\Box}^{*}} \varphi$, then $\langle \mathcal{F}, V \rangle \vDash \varphi$. We note that $\mathcal{F}$ is not necessary a *-frame. Let us consider a Σ-filtration $R_f \subseteq X_\Sigma \times X_\Sigma$ of R. For example, we can take the relation $\bar{R}$ defined in the section above. Let $(R_f)^{*}$ be the reflexive and transitive closure of R_f. We shall prove that the structure $\langle \mathcal{F}_\Sigma, V_\Sigma \rangle = \left\langle X_\Sigma, \underset{\Sigma}{\preccurlyeq}, R_f, (R_f)^{*}, V_\Sigma \right\rangle$ is a model of $\mathbf{IK}_{\Box}^{*}$.

Lemma 5.1. Let us consider the model $\left\langle X_\Sigma, \underset{\Sigma}{\preccurlyeq}, R_f, (R_f)^{*}, V_\Sigma \right\rangle$. For each $x \in X$ there exists a formula φ_x such that

1. For all $y \in X$: $\quad y \in V(\varphi_x) \Leftrightarrow (|x|, |y|) \in (R_f)^{*} \circ \underset{\Sigma}{\preccurlyeq}$.
2. $x \in V(\triangle\varphi_x)$.

Proof. 1. For each $z \in X$ the set $F_z \cap \Sigma$ is finite, because Σ is finite. Define the formula $\varphi_z = \bigwedge (F_z \cap \Sigma)$. Then we have the following equivalences:

$$\begin{aligned} y \in V(\varphi_z) &\Leftrightarrow \forall\varphi \in F_z \cap \Sigma\,, y \in V(\varphi) \\ &\Leftrightarrow \forall\varphi \in F_z \cap \Sigma, \varphi \in F_y \\ &\Leftrightarrow F_z \cap \Sigma \subseteq F_y \\ &\Leftrightarrow |z| \underset{\Sigma}{\preccurlyeq} |y|\,. \qquad (**) \end{aligned}$$

Let us consider the set

$$H = \{|x| \in X_\Sigma : (|x|, |x_i|) \in (R_f)^{*}\}\,.$$

Since the set X_Σ is finite, then set H is finite. Define the formula

$$\varphi_x = \bigvee_{|x_i| \in H} \left(\bigwedge (F_{x_i} \cap \Sigma) \right)$$

Then,

$$\begin{aligned} y \in V(\varphi_x) &\Leftrightarrow y \in V\left(\bigwedge (F_{x_i} \cap \Sigma)\right) \text{ for some } i = 1, ..., n \\ &\Leftrightarrow |x_i| \underset{\Sigma}{\preccurlyeq} |y|\,, \text{ for some } i = 1, ..., n \text{ (by } (**)\text{)} \\ &\Leftrightarrow (|x|, |y|) \in (R_f)^{*} \circ \underset{\Sigma}{\preccurlyeq}\,, \text{ because } |x_i| \in H. \end{aligned}$$

Thus, 1. is valid.

For to prove 2., we first prove that

$$V(\varphi_x \to \Box\varphi_x) = X. \tag{3}$$

Let $y \leq z \in V(\varphi_x)$ and $(z,k) \in R_\Box$. By 1., $(|x|, |z|) \in (R_f)^* \circ \underset{\Sigma}{\preccurlyeq}$. Since R_f is Σ-filtration of $R_\Box$, then $(|z|, |k|) \in (R_f)^* \circ \underset{\Sigma}{\preccurlyeq}$. Thus, $(|x|, |k|) \in (R_f) \circ \underset{\Sigma}{\preccurlyeq}$. It follows that $k \in V(\varphi_x)$. Thus, (3) is valid. Since $\langle \mathcal{F}, V\rangle \vDash \triangle(\varphi_x \to \Box\varphi_x) \to (\varphi_x \to \triangle\varphi_x)$, by (3) we have $V(\triangle(\varphi_x \to \Box\varphi_x)) = X$. As $x \in V(\varphi_x)$, because $(|x|, |x|) \in (R_f)^* \circ \underset{\Sigma}{\preccurlyeq}$, then $x \in V(\triangle\varphi_x)$. □

Theorem 5.2. Let $\mathcal{M} = \langle X, \leq, R_1, R_2, V\rangle$ a model of $\mathbf{IK}^*_\Box$. Let Σ be a closed set, and let R_f be a Σ-filtration of R_1. Then the model $\left\langle X_\Sigma, \underset{\Sigma}{\preccurlyeq}, R_f, (R_f)^*, V_\Sigma \right\rangle$ is a Σ-filtration.

*Proof***.** We shall prove that $(R_f)^*$ is a Σ-filtration of R_2.

F1. Let $(x,y) \in R_2 \circ \leq$. From Lemma 5.1 there exists a formula φ_x such that $(|x|, |z|) \in (R_f)^* \circ \underset{\Sigma}{\preccurlyeq}$ if and only if $z \in V(\varphi_x)$. So, to prove **F1** is equivalent to prove that $(x,y) \in R_2 \circ \leq$ implies $y \in V(\varphi_x)$. By Lemma 5.1, $x \in V(\triangle\varphi_x)$. Thus, $y \in V(\varphi_x)$.

F2. Suppose that $(|x|, |y|) \in (R_f)^* \circ \underset{\Sigma}{\preccurlyeq}$. We prove that $\triangle^{-1}(F_x \cap \Sigma) \subseteq F_y$. Let $\triangle\varphi \in \Sigma$ and $x \in V(\triangle\varphi)$. Since $(|x|, |y|) \in (R_f)^* \circ \underset{\Sigma}{\preccurlyeq}$, then there exist $|x| = |x_1|, |x_2|, \ldots, |x_{n-1}|, |x_n| = |z|$ such that

$$(|x|, |x_i|) \in R_f \text{ and } |z| \underset{\Sigma}{\preccurlyeq} |y|.$$

As R_f is a Σ-filtration of R_1, then for each $i = 1, ..., n-1$, we have

$$\Box^{-1}(F_{x_i} \cap \Sigma) \subseteq F_{x_{i+1}}.$$

So, for each $i = 1, ..., n-1$, if $\triangle\varphi \in \Sigma$, then

$$\begin{aligned} x_i \in V(\triangle\varphi) \quad &\Rightarrow \quad x_i \in V(\Box\triangle\varphi) \text{ (because } \triangle\varphi \to \Box\triangle\varphi) \\ &\Rightarrow \quad x_{i+1} \in V(\triangle\varphi). \end{aligned}$$

In particular, $z \in V(\triangle\varphi)$. As $|z| \underset{\Sigma}{\preccurlyeq} |y|$, then $y \in V(\triangle\varphi)$. Since $V(\triangle\varphi) \subseteq V(\varphi)$, we get $y \in V(\varphi)$. Therefore $\triangle^{-1}(F_x \cap \Sigma) \subseteq F_y$.

F3. As R_f is a Σ-filtration of R_1, then it satisfies the condition **F3**. Thus, by Lemma 2.5, $(R_f)^*$ also satisfies **F3.** □

Corollary 5.3. The logic $\mathbf{IK}^*_\Box$ has the finite model property, and thus has the finite frame property.

Let $\mathcal{IML}$ be a modal logic such that $\mathbf{IK}^*_\Box \subseteq \mathcal{IML}$. In the following result we need the notion of canonical model. Let X_c the set of all prime and consistent theories of $\mathcal{IML}$ (for these definitions see [1]). On X_c we define two relations R_1^c and R_2^c as follows:

$$(P,Q) \in R_1^c \Leftrightarrow \Box^{-1}(P) \subseteq Q$$
$$(P,Q) \in R_2^c \Leftrightarrow \triangle^{-1}(P) \subseteq Q.$$

Let V_c be the valuation defined by:

$$V_c(p) = \{P \in X_c : p \in P\}.$$

It easy to check that (see [1]) that the canonical model $\mathcal{M}_c = \langle X_c, R_1^c, R_2^c, V_c \rangle$ is a model of $\mathcal{IML}$.

Theorem 5.4. The logic $\mathbf{IK}_\Box^*$ is frame complete.

Proof. We prove that $\mathbf{IK}_\Box^* = Th(Fr(\mathbf{IK}_\Box^*))$. It is clear that $\mathbf{IK}_\Box^* \subseteq Th(Fr(\mathbf{IK}_\Box^*))$. Let $\varphi \notin \mathbf{IK}_\Box^*$. Then $\mathcal{M}_c = \langle X_c, R_c, R_c^\bullet, V_c \rangle \nvDash \varphi$. Let Σ be the closure of $Sf(\varphi)$ by the rule: if $\triangle\alpha \in \Sigma \Rightarrow \Box\triangle\alpha \in \Sigma$. Let us consider any Σ-filtration $\langle \mathcal{F}_\Sigma, V_\Sigma \rangle$ of $\mathcal{M}_c$. By Lemma 4.4, $\langle \mathcal{F}_\Sigma, V_\Sigma \rangle \nvDash \varphi$. Then $\mathcal{F}_\Sigma \nvDash \varphi$. But since $\mathcal{F}_\Sigma \in Fr(\mathbf{IK}_\Box^*)$, we get $\varphi \notin Th(Fr(\mathbf{IK}_\Box^*))$. Therefore, $\mathbf{IK}_\Box^* = Th(Fr(\mathbf{IK}_\Box^*))$. $\Box$

6. Conclusions

In this paper we showed that the Intuitionistic Modal logic $\mathbf{IK}_\Box^*$ has the finite model property, and consequently is frame complete. Some of the ideas presented in this work also have been applied to study an intuitionistic extension of the logic studied by D. Vakarelov in [9]. These results will be presented in a future paper. On the other hand, it would be interesting to see if the given results can be extended to full language of PDL, i.e., logics extending the intuitionistic logic with countably many operators.

References

[1] M. Božić, K. Dožen, *Models for Normal Intuitionistic Modal Logics*, **Studia Logica**, vol. 43 (1984), pp. 217-245.

[2] K. Dožen, *Models for Stronger Normal Intuitionistic Modal Logics*, **Studia Logica**, vol. 44 (1985), pp. 39-70.

[3] Fischer, M. J., and R. E. Ladner. *Propositional Dynamic Logic of Regular Programs*. **Journal of Computer and Systems Sciences** 18 (1979), 194-211.

[4] Goldblatt, R., *The semantics of Hoare's Iteration Rule*. **Studia Logica**, vol. 41 (1982), 141-158.

[5] Goldblatt, R., **Logics of Time and Computation**, CSLI, Lectures Notes No. 7,1992.

[6] Harel, D., *Dynamic logic*, In **Handbook of Philosophical Logic**, vol. II, D. Reidel and Co., 1984.

[7] Popkorn, Sally., **First steps in Modal Logic**, Cambridge Univ. Press,1994.

[8] Sotirov, V. H., *Modal theories with intuitionistic logic*. Proceedings of the Conference dedicated to Markov, Sofia (1984), 139-172.

[9] Vakarelov, D., *A Modal Logic for Cyclic Repeating*. **Information and Computation** 101 (1992), 103-122.

[10] Wijesekera D., *Constructive modal logics I*. **Ann. Pure Appl. Logic** 50 (1990) 271-301.