Univariate nonlinear discrete-time models

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Logistic model

Impose bounds on an otherwise ridiculous growth process. Begin with the geometric difference equation, $N(t+1) - N(t) = RN(t)$. Set R equal to a decreasing linear function of $N(t)$ with x-intercept, N_{max}, and y-intercept R_{max}. This yields the logistic difference equation,

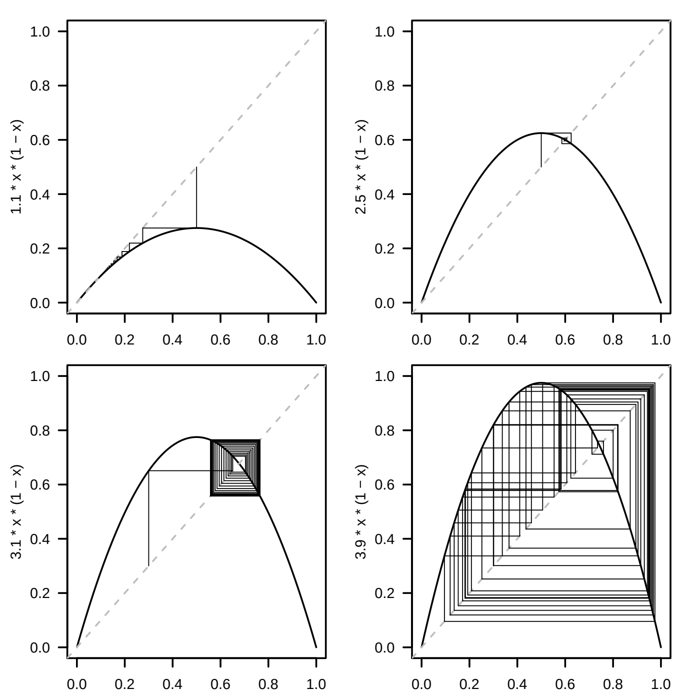
$$N(t+1) - N(t) = R_{\text{max}} N(1 - N(t)/N_{\text{max}});$$

Fixed points: $N(t+1) - N(t) = 0 = RN^*(1 - N^*/N_{\text{max}})$ has two solutions, $N^* = 0$ and $N^* = N_{\text{max}}$. Mathematically, we can set $N_{\text{max}} = 1$ without loss of generality (non-dimensionalization).

Stability

The geometric recursion, $N(t+1) = f(N(t)) = RN(t)$, is stable at the fixed point $N^* = 0$, whenever $|R| < 1$. For general scalar function, f, and fixed point N^*, this criterion becomes $|f'(N)|_{N=N^*} < 1$, where $f'(N)$ is the first derivative of f with respect to N. Note that this is a true generalization because $f'(N) = R$ for the geometric model.

The derivative of the function defining the logistic recursion, $f(N) = N + R_{\text{max}} N(1 - N/N_{\text{max}})$, is $f'(N) = 1 + R_{\text{max}} - 2NR_{\text{max}}/N_{\text{max}}$. When are the equilibria stable? Bifurcation diagrams



[1, 2]

Alternative parameterizations

An ecologist or other normal person might choose to parameterize the discrete logistic model as above. A mathematician would choose $x(t+1) = Rx(1-x)$. The mathematician has chosen $R = r/K \to K = 1 - 1/R$. Mathematically equivalent parameterizations often have quite different meanings (or statistical properties), as well as cultural connotations. Get used to it.

More nonlinear models

Other 1-D discrete nonlinear models: *Ricker* model ($N = rNe^{-bN}$); population genetics; approximations of continuous models. Epidemic models (SI) (equivalent to discrete logistic). Metapopulation (Levins) models. (notes)

$$\begin{aligned}
S(t+1) =& m(N-S) - bSI + gI \\
=& m(N-S) - bS(N-S) + g(N-S) \\
=& m(1-S) - bS(1-S) + g(1-S) \\
=& mI - bI(1-I) + gI \\
=& (m+g-b)I + bI^2
\end{aligned} \tag{1}$$

$$\begin{aligned}
N(t+1) =& N + rN(1-N/K) \\
=& (1+r)N - (r/K)N^2 \\
=& (1+r)N - rN^2
\end{aligned} \tag{2}$$

Graphical approaches, continued: *Allee effects*. Bistability, multiple stable states.

References

[1] R. M. May (June 1976) Simple mathematical models with very complicated dynamics. *Nature*, **261**(5560):459–467.

[2] R. M. May and G. F. Oster (1976) Bifurcations and dynamic complexity in simple ecological models. *The American Naturalist*, **110**(974):573–599.