Multivariate linear (and affine) discrete-time deterministic models (1)

Ben Bolker and Steve Walker

Annual plant example

Parameters: seed production γ, overwinter survival σ, first-year germination α, second-year germination β. Homogeneous second-order linear model: $N(t) = \gamma\alpha\sigma N(t-1) + \gamma\sigma^2(1-\alpha)\beta N(t-2)$.

For homogeneous linear equations of any order, 0 is an equilibrium. Find other eq. by (1) plugging in trial solution $C\lambda^t$; (2) dividing through by $C\lambda^{n-1}$; (3) solve for λ by finding roots of *characteristic equation*; (4) possibly plugging in initial conditions ($N(0)$, $N(-1)$, ...) to solve for constants c_i in particular solution $\sum c_i\lambda_i^t$ (ignoring repeated-root case).

In this case with $a = \gamma\alpha\sigma$, $b = \gamma\sigma^2(1-\alpha)\beta$, we have $\lambda^2 - a\lambda - b = 0 \to \lambda = (a \pm \sqrt{a^2+4b})/2$. Given that $\lambda > 1 \leftrightarrow a + b > 1$, population grows if $\gamma > 1/(\alpha\sigma + \beta(1-\alpha)\sigma^2)$.

Or we can set this up as a matrix equation:

$$\begin{pmatrix} P(t+1) \\ S(t+1) \end{pmatrix} = \begin{pmatrix} \gamma\alpha\sigma & \sigma\beta \\ \gamma\sigma(1-\alpha) & 0 \end{pmatrix} \begin{pmatrix} P(t) \\ S(t) \end{pmatrix}$$

Basic model: $\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t)$. For example, juvenile/adult model: fractions $\{s_J, s_A\}$ of juveniles and adults survive; adults have f offspring each (on average); surviving juveniles become adults. So $A(t+1) = s_A A(t) + s_J J(t)$, $J(t+1) = fA(t)$ *or* $A(t+1) = s_A A(t) + s_J f A(t-1)$. This can be written as a matrix equation,

$$\begin{bmatrix} J(t+1) \\ A(t+1) \end{bmatrix} = \begin{bmatrix} 0 & f \\ s_J & s_A \end{bmatrix} \begin{bmatrix} J(t) \\ A(t) \end{bmatrix}$$

Fixed points

$\boldsymbol{x}\star$ is a fixed point if $\boldsymbol{x}\star = \boldsymbol{A}\boldsymbol{x}\star$, $\boldsymbol{0} = (\boldsymbol{A} - \boldsymbol{I})\boldsymbol{x}\star$ where $\boldsymbol{I}$ is the identity matrix. The null space of $\boldsymbol{A} - \boldsymbol{I}$ has all the fixed points. If $\boldsymbol{A} - \boldsymbol{I}$ is invertible, we find $\boldsymbol{0} = (\boldsymbol{A} - \boldsymbol{I})^{-1}(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{x}\star$, which implies $\boldsymbol{0} = \boldsymbol{I}\boldsymbol{x}\star$, or $\boldsymbol{x}\star = \boldsymbol{0}$. However, if $\boldsymbol{A} - \boldsymbol{I}$ is not invertible, there is an $n - r$ dimensional space of fixed-points, where n is the number of rows/columns in $\boldsymbol{A} - \boldsymbol{I}$ and r is the rank of that matrix. A helpful trick is that a matrix is invertible if its determinant is non-zero. For example, the determinant of the juvenile-adult model is $-fs_J$, which is not zero and so the only fixed point is at the origin.

In Python, the rank, inverse, and determinant of a matrix `B` are given by `numpy.linalg.matrix_rank(B)`, `numpy.linalg.inv(B)`, and `numpy.linalg.det(B)`.

Time-dependent solution

Four approaches:

recursion $\boldsymbol{x}(1) = \boldsymbol{A}\boldsymbol{x}(0)$, then $\boldsymbol{x}(2) = \boldsymbol{A}\boldsymbol{A}\boldsymbol{x}(0)$, and in general $\boldsymbol{x}(t) = \underbrace{\boldsymbol{A}...\boldsymbol{A}}_{t-\text{times}}\boldsymbol{x}(0)$.

matrix powers We can define matrix powers, so that $\boldsymbol{x}(t) = \boldsymbol{A}^t\boldsymbol{x}(0)$. However, this method doesn't provide much insight.

diagonalization Gain insight by *diagonalizing* $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}$, where $\boldsymbol{S}$ is a matrix whose columns are the eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{D}$ is a matrix with eigenvalues on the diagonal and zeros everywhere else. Substituting into the matrix power equation, $\boldsymbol{x}(t) = \left(\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}\right)^t\boldsymbol{x}(0) = \underbrace{\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}...\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}}_{t-\text{times}}\boldsymbol{x}(0) = \underbrace{\boldsymbol{S}\boldsymbol{D}^t\boldsymbol{S}^{-1}}_{t-\text{times}}\boldsymbol{x}(0)$, because the $\boldsymbol{S}^{-1}\boldsymbol{S}$ terms cancel.

series Let $\boldsymbol{c} = \boldsymbol{S}^{-1}\boldsymbol{x}(0)$. This allows us to write $\boldsymbol{x}(t) = \sum_i c_i d_i^t \boldsymbol{v}_i$, where c_i, d_i, and $\boldsymbol{v}_i$ is the ith element of $\boldsymbol{c}$, eigenvalue, and eigenvector respectively. This form also lets us see the importance of the dominant eigenvalue (i.e. eigenvalue with largest absolute value) – because all the eigenvalues get raised to the power of time, as time increases all other terms except for the dominant become neglible. Therefore, for t sufficiently large, $\boldsymbol{x}(t) \approx c_1 d_1^t v_1$, where d_1 is the dominant eigenvalue. What happens when $d_1 = 1$? What happens when $d_1 = d_2$? Try it out in `R`.

change of variables Let $\boldsymbol{y}(t) = \boldsymbol{S}^{-1}\boldsymbol{x}(t)$. Then the model becomes $\boldsymbol{y}(t+1) = \boldsymbol{D}\boldsymbol{y}(t)$. But since $\boldsymbol{D}$ is diagonal, this model is exceptionally simple. It is actually just a bunch of decoupled univariate models (Why?) and you know how to handle those.

Eigen-tips (mostly for the 2 by 2 case)

If $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, determinant is $\Delta = a_{11}a_{22} - a_{12}a_{21}$, trace is $T = a_{11} + a_{22}$, and eigen values obey $d_1 + d_2 = T$ and $d_1 d_2 = \Delta$. This leads to the characteristic polynomial $d_i^2 - T d_i + \Delta$. And so the eigenvalues obey $d_i = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}$. Finally, if $\boldsymbol{v}_i$ and d_i are an eigenvector/eigenvalue pair for $\boldsymbol{A}$, then $\boldsymbol{A}\boldsymbol{v}_i = d_i \boldsymbol{v}_i$ (i.e. a matrix and a single scalar value to the same thing to an eigenvector!).

Example: for the juvenile-adult model, we have $d_i = \frac{s_A \pm \sqrt{s_A^2 + 4 s_J f}}{2}$. For each eigenvalue, solve $\begin{bmatrix} 0 & f \\ s_J & s_A \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = d \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ to find the eigen vectors. For the dominant eigenvalue, this is,

$$
\begin{aligned}
f v_2 &= \frac{s_A + \sqrt{s_A^2 + 4 s_J f}}{2} v_1 \\
s_J v_1 + s_A v_2 &= \frac{s_A + \sqrt{s_A^2 + 4 s_J f}}{2} v_2
\end{aligned}
$$

Could keep going but you get the idea. Simplify this system. Do the same for the other eigenvalue. Write down a time-dependent solution for this model with your computations. What are the conditions for stability of the fixed point at the origin?

Affine model

Multivariate bucket/line-up: $\boldsymbol{x}(t+1) = \boldsymbol{b} + \boldsymbol{A}\boldsymbol{x}(t)$. For fixed points solve $\boldsymbol{x}\star = \boldsymbol{b} + \boldsymbol{A}\boldsymbol{x}\star$. If $\boldsymbol{A} - \boldsymbol{I}$ is invertible, then the solution is $\boldsymbol{x}\star = (\boldsymbol{A} - \boldsymbol{I})^{-1}\boldsymbol{b}$. Same stability conditions as in the linear case. Can you reparameterize this model such that the fixed point is a parameter? It would be nice to just read off the fixed point woudn't it?