Many uses for simple dynamical models

Ben Bolker

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outline

- many applications of the logistic equation (single species)
- multispecies models

the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

- solve by separating variables, partial fractions

$$N(t) = \frac{K}{1 + \left(\frac{K}{N(0)} - 1\right)e^{-rt}}$$

- widely used in statistics (with $K = 1$) to describe sigmoidal patterns, especially of probability
- *non-dimensional form* (with $r = K = 1$, $N(0) = 1/2$):

$$f(x) = \frac{1}{1 + \exp(-x)}$$

ecology

- usual interpretation: r=exponential growth rate when rare (*per capita* birth-death); K=carrying capacity
$$\frac{d\,\log(N)}{dt} = \frac{dN/dt}{N} = r(1 - N/K)$$
- or could write it as
$$\frac{dN/dt}{N} = b - (d + \alpha N)$$
i.e., death rate increases linearly; α is sensitivity to competition.
- what is K in this parameterization?
- what's wrong with assuming birth rate decreases linearly?

metapopulation ecology

- instead of tracking birth & death of individuals, think about **patches** (Hanski and Gilpin 1991)
- patches either occupied or unoccupied (*separation of time scales*)
- infinite dispersal/all patches are equally far away from each other ("island model")
- *per capita* extinction probability (e) is constant
- *per capita* colonization probability decreases linearly with occupancy: can't colonize an already-colonized patch
$$\frac{dp}{dt} = cp(1 - p) - ep$$

- same as logistic with $K=?$

epidemic models

- now consider an epidemic
- everyone's initially susceptible
- *per capita* probability of infection $\propto$ number of infected individuals
- no-one ever recovers

$$\frac{dI}{dt} = S(\beta I) = (N-I)(\beta I)$$

- same equation, $K=?$
- can also consider individual hosts as patches (good for considering competition between diseases)

mathematical extensions

- we have the general form

$$\frac{dx}{dt} = bx + cx^2 = x \cdot (b + cx)$$

- what is K now?
- sign of b determines stability of $x=0$ equilibrium
- c usually < 0 (why?)
- what does it mean if we add a constant term $(a + bx + cx^2)$?
- what else could we do?

ecological extensions

- theta-logistic $((dN/dt)/N = r(1-(N/K)^{\theta}))$
- most useful to think about *per capita* term
- *Allee effects*
- constant terms (unrealistic)
- harvesting? maximum sustainable yield
- graphical analysis

epidemiological extensions

- also consider recovery (SIS model)
  - what is the equivalent model?
- can also frame the model as being about zombies (Smith? 2014), rumors, memes, …
- what does this change?

lazy person's math (single-population models)

- find equilibria
- assess stability of equilibria (maybe graphically)
- especially: when is 0 equilibrium stable/unstable?
- solve analytically???
- solve numerically (Excel, R, MATLAB, …)

multi-species models (ecology: competition)

- one equation per species, still quadratic
- *system of equations*
- can still find equilibria, compute stability

$$
\begin{aligned}
\frac{dN_i/dt}{N_i} &= r_i \left(1 - \left(\sum_j \alpha_{ij} N_j / K\right)\right) \\
&= b_i - (d_i + \sum_j \gamma_{ij} N_j)
\end{aligned}
$$

- can find equilibria, analyze stability, etc. for arbitrarily many species
- metapopulation equivalent

$$
\frac{dp_i}{dt} = c_i p_i \left(1 - \sum_j p_j\right) - e_i p_i
$$

  often make assumptions about *competitive dominance*

multi-species models (epidemiology: I)

- between-strain interactions as competition for patches (hosts): May and Nowak (1994)
- need to think about *superinfection* and *coinfection*
- helps us think about vaccine-induced *strain replacement* Murall, McCann, and Bauch (2014)

multi-species models (epidemiology: II)

- single strain of disease
- divide people into "species" according to disease status
- **S**usceptible, **I**nfected, **R**ecovered
- now disease takes off but dies away again
-

natural enemies

- predators, parasites …
- *Lotka-Volterra predator-prey model*

$$
\begin{aligned}
\frac{dV}{dt} &= rV - aVP \\
\frac{dP}{dt} &= acVP - dP
\end{aligned}
$$

- cycles (neutral)
- can add self-limitation, *functional response*
- phase plane analysis

ecological communities

- put together as many pieces as you need
  - many prey, many predators, diseases, many *trophic levels* …
  - maybe include nutrient dynamics
  - seasonality
  - evolution
  - hard to handle!

Estimating parameters

- direct measurement
- allometric scaling
- estimation from time series

Further resources

- Case (1999)
- Ellner and Guckenheimer (2006)
- Kokko (2007)
- Otto and Day (2007)
- Keeling and Rohani (2008)

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