Basic SIR fitting - Details

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This vignette provides technical details of the `fitsir` package.

1 Sensitivity equations

Let $x_i(t, \theta)$ be the states of the SIR model and $\theta_{j,u}$ and $\theta_{j,c}$ be unconstrained and constrained parameters of the model. In order to employ gradient-based optimizatino algorithms (e.g. `BFGS`), we must solve for $dx_i(t, \theta)/d\theta_{j,u}$. To find the sensitivity equations, `fitsir` integrates the following set of differential equations along with the basic SIR model:

$$\begin{aligned}\frac{d}{dt}\frac{dx_i(t,\theta)}{d\theta_{j,u}} &= \left(\frac{d}{dt}\frac{dx_i(t,\theta)}{d\theta_{j,c}}\right)\frac{d\theta_{j,c}}{d\theta_{j,u}}\\ &= \left(\frac{\partial f_x}{\partial\theta_{j,c}} + \sum_i \frac{\partial f_x}{\partial x_i}\frac{dx_i(t,\theta)}{d\theta_{j,c}}\right)\frac{d\theta_{j,c}}{d\theta_{j,u}}\end{aligned}$$

Essentially, we integrate sensitivity equations with respect to constrained parameters for simplicity but multiply $d\theta_{j,c}/d\theta_{j,u}$ after to obtain sensitiviy equations with respect to unconstrained parameters because optimization is done using unconstrained parameters.

For clarity, we write ν_{x_i,θ_j} to represent sensitivity equations with respect to constrained parameters. Then, we write

$$\nu_{x,\theta}(t; x, \theta) = \begin{bmatrix}\nu_{S,\beta} & \nu_{S,N} & \nu_{S,\gamma} & \nu_{S,I_0}\\ \nu_{I,\beta} & \nu_{I,N} & \nu_{I,\gamma} & \nu_{I,I_0}\end{bmatrix}$$

So the sensitivity equations of the SIR model is given by

$$\frac{d}{dt}\nu_{x,\theta}(t;\cdot) = \begin{bmatrix}-\beta I/N & -\beta S/N\\ \beta I/N & \beta S/N - \gamma\end{bmatrix}\nu_{x,\theta}(t;\cdot) + \begin{bmatrix}-SI/N & \beta SI/N^2 & 0 & 0\\ SI/N & -\beta SI/N^2 & -I & 0\end{bmatrix}$$

The following additional equations completes the sensitivity equations:

$$\begin{aligned}\nu_{x,\theta}(0; x(0), \theta) &= \begin{bmatrix}0 & 1 - I_0 & 0 & -N\\ 0 & I_0 & 0 & N\end{bmatrix}\\ \left[\frac{d\theta_{j,c}}{d\theta_{j,u}}\right]_{j=1,2,3,4} &= \begin{bmatrix}\beta & \gamma & N & I_0^2\exp(-\text{qlogis}(I_0))\end{bmatrix}\end{aligned}$$

2 Starting function