# *Epidemic models 1*

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## *motivation*

- P & I data from Philadelphia 1918 flu:

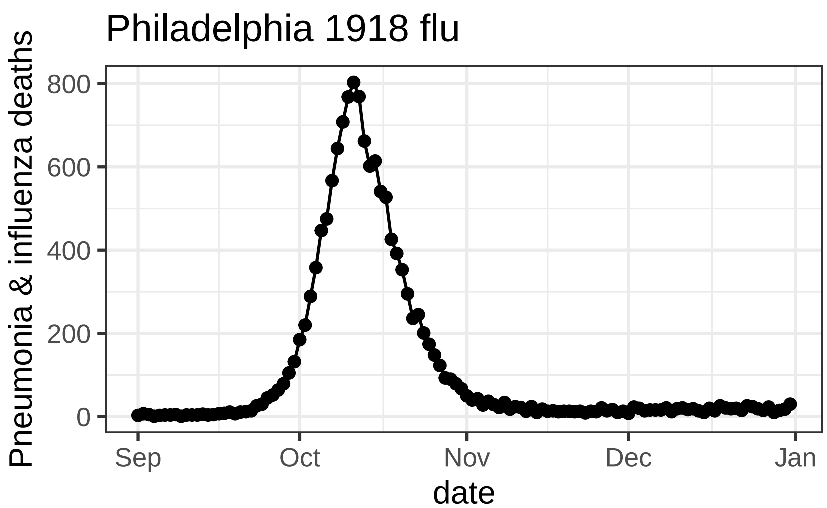


Figure 1: Phila. 1918 flu data

## *what do we want to figure out?*

## *what shall we assume?*

- classify individuals as $S$, $I$ (**compartmental** model; **microparasite** or **intensity-independent**)
- disease is transmitted from $S$ to $I$
- $S \to I$ instantaneously (zero latent period, no $E$)
- population is **homogeneous** (no heterogeneity in susceptibility, infectiousness, contact)
- fixed population size (birth = migration = 'natural' death = 0)
- transmission rate is time-invariant

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- assumption 2 is OK (Pasteur, Koch's postulates . . . )
- all the rest are approximations

  start simple!

- parsimony

- robustness?
- applicability/estimation?

Levins (1966) (also Orzack and Sober (1993), Levins (1993), Weisberg (2007))

## *exponential growth*

- one variable (=1D model)
- how does disease spread? → equation

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*what variables should we use?*

- time ($t$)
- state variable: incidence, prevalence, death rate, death toll (= cumulative death?)
- deaths loosely connected to transmission

but deaths are observed!

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when are deaths a good **proxy** for incidence?

- infection -> death time is fixed
- homogeneity? (might not matters?)
- mortality curve is shifted epidemic

(COVID context ... we observe case reports, number of tests, hospitalizations, and deaths)

- **incidence**: number of infections per unit time (rate or flow)
- **prevalence**: number of currently infected people (quantity or stock)

prevalence is closer to the **mechanism**

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model components:

- $I(t)$ (state variable: prevalence)
- $I(0)$ (initial conditions)
- $\beta$ (parameter) = avg contacts **per susceptible per infective per unit time**

$$I(t+\Delta t) \approx I(t) + \beta I(t)\Delta t$$

Take $\lim \Delta t \to 0$ (and solve):

$$\frac{dI}{dt} = \beta I \to I(t) = I(0) exp(\beta t)$$

*model criticism*

- Ignored discrete nature of individuals
- Ignored time-varying $\beta$ (e.g. **diurnal** fluctuations)
- Ignored finite infectious periods (recovery/death)

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**Next**: What if we make infectious periods finite? (i.e., including recovery (**clearance**) or death

$$dI/dt = \beta I - \gamma I$$

*mean infectious period*

$$I(t) = I(0)\exp(-\gamma t)$$
$$\text{proportion uninfected} = \exp(-\gamma t)$$
$$\text{proportion infected} = 1 - \exp(-\gamma t)(= \text{CDF} := C(t))$$
$$\text{PDF} := C'(t) = \gamma \exp(-\gamma t)$$
$$\text{substitute } x = \gamma t \quad \to \quad dx = \gamma\, dt$$
$$\text{mean} = E[t] = \int t \exp(-\gamma t)\, dt = \int x \exp(-x)\, dx/\gamma = 1/\gamma$$

*dimensional analysis*

rates and characteristic times/scales

- is $I$ a proportion or a density or a number ... ?
- what are the units of $\beta$, $\gamma$ ?

*nondimensionalization*

- standardize any values that can be eliminated **without loss of (mathematical) generality**
- what can we do here?
- $\gamma = 1$
- $I$ ? (depends on how we have defined it initially) $\to I/N$

*compare with data???*

Original scale:

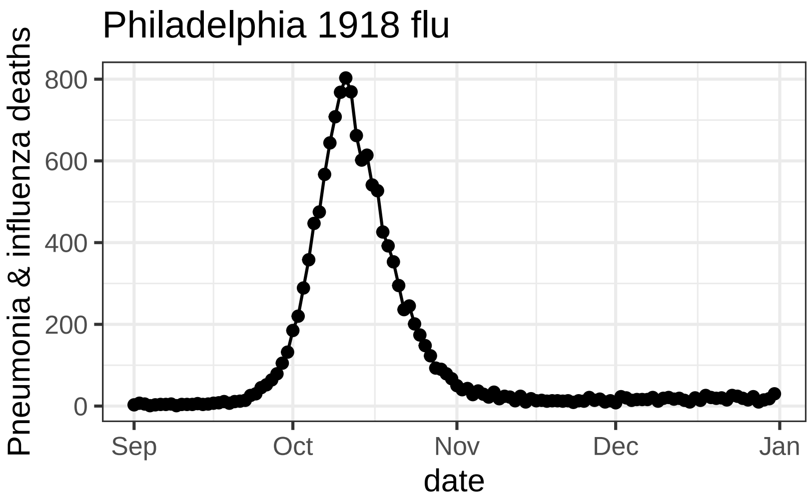


Figure 2: Philadelphia P&I

Log scale:

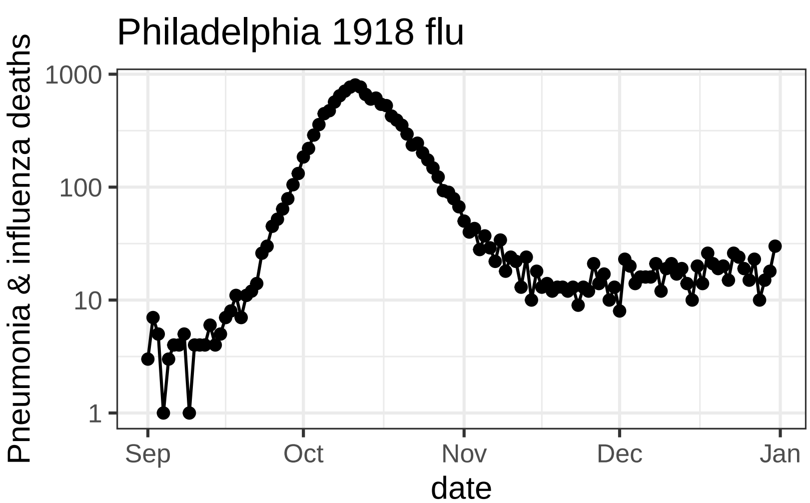


Figure 3: Philadelphia P&I, log scale

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- Fit a straight line through the straight part of the curve
- slope is $\beta N$
- "intercept" is $\log(I(0))$ (zero is defined in a tricky way)

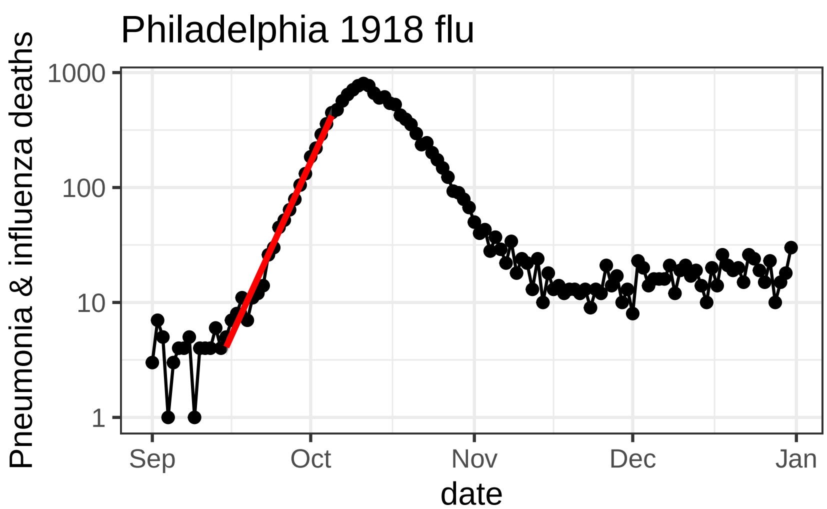


Figure 4: log-scale flu with regression

*model assessment*

- math is super-easy!
- clear, testable predictions
- parameter estimation is easy
- only consistent over a short time window
  - small $t$: arbitrarily close to zero
  - large $t$: ridiculous

*Simple (SI) epidemic*

- what are we missing?
- **depletion of susceptibles**
- let's take a step back and ignore death & recovery for now

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$$dS/dt = -\beta SI$$
$$dI/dt = \beta SI$$

This looks 2D **but** what if we assume $S + I = N$ is constant? Then $S = N - I$

$$dI/dt = \beta(N - I)I$$

How do we solve this? **Partial fractions**

$$
\begin{aligned}
\frac{dI}{\beta(N-I)I} &= dt \\
dI\left(\frac{A}{N-I}+\frac{B}{I}\right) &= dI\cdot\frac{A+B(N-I)}{I(N-I)} \\
A = B; &\quad B = 1/N \\
\frac{1}{\beta N}(-\log(N-I)+\log(I))\Big|_{I(0)}^{I} &= t - t_0 \\
(-\log(N-I)+\log(I))\Big|_{I(0)}^{I} &= (\beta N)(t-t_0) \quad (\text{set } t_0 = 0) \\
\log\left(\frac{I}{N-I}\right) - \log\left(\frac{I(0)}{N-I(0)}\right) &= \beta N t \\
\log\left(\frac{I}{N-I}\right) &= \beta N t + -\log\left(\frac{I(0)}{N-I(0)}\right) \\
\frac{I}{N-I} &= \exp(\beta N t)\frac{I(0)}{N-I(0)} \equiv Q \\
I &= Q(N-I) \\
I(t)(1+Q) &= QN \\
I(t) &= \frac{QN}{1+Q} = \frac{N}{1+\frac{1}{Q}} \\
&= \frac{N}{1+\left(\frac{N-I(0)}{I(0)}\right)\exp(-\beta N t)}
\end{aligned}
$$

?? $\equiv I(0)\exp(\beta N t)/(1+(I0/N)(\exp(\beta N t)-1))$ ??

*Qualitative analysis*

- $I \ll N$ ? exponential growth
- **per capita growth rate** ($(dI/dt)/I = d(\log(I))/dt$) decreases monotonically with increasing $I$
- asymptotic behaviour? equilibria? periodic orbits?
- periodic orbits impossible in 1D (uniqueness of flows)

*equilibrium analysis*

- $I = 0$, **disease free equilibrium** (DFE)
- $I = N$, **endemic equilibrium** (EE)

  Stability? (Assume $\beta > 0$)

- **local asymptotic stability**
- **global asymptotic stability** (Lyapunov functions)

*model criticism/conclusions*

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(Comparison to metapop, logistic growth model)

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## *SIR model*

### *Basic SIR model*

- put the pieces together

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = \beta SI - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

- really 2D (because $S + I + R = N$)
- rescale to $N = 1$ ($S$, $I$, $R$ as proportions)

Numerical solution (R version):

```
## define gradient function
SIRgrad <- function(t, y, parms) {
    g <- with(as.list(c(y,parms)), {
        c(-beta*S*I, beta*S*I-gamma*I, gamma*I)
    })
    return(list(g))
}
library(deSolve)
## initial conditions and parameters
y0 <- c(S=0.99, I=0.01, R=0)
p0 <- c(beta=4, gamma=1)
tvec <- seq(0,8,length=101)
## solve (LSODA by default)
sir_R <- ode(y=y0, times=tvec, parms=p0, func=SIRgrad)

## plot
par(las=1,bty="l") ## cosmetic
matplot(tvec, sir_R[,-1],
        type="l", lwd=2, ## solid lines, thicker
        xlab="time",ylab="proportion")
legend("right",names(y0), col=1:3, lty=1:3, lwd=2)
```

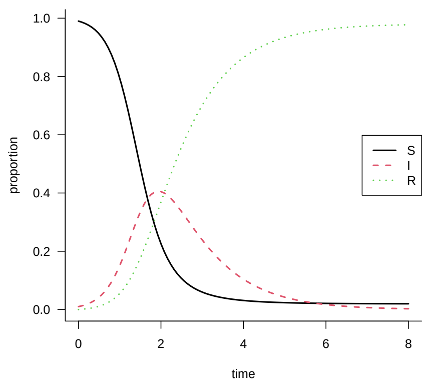



Figure 5: SIR model (R)

*Phase plane plot*

```
par(las=1,bty="l") ## cosmetic
plot(I~S,type="l",data=as.data.frame(sir_R))
with(as.data.frame(sir_R), points(S,I, cex=0.75,pch=16))
```

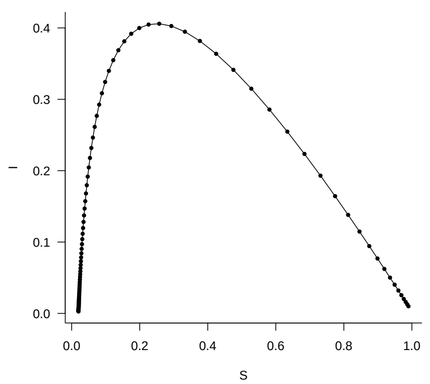



Figure 6: SIR phase plane (R)

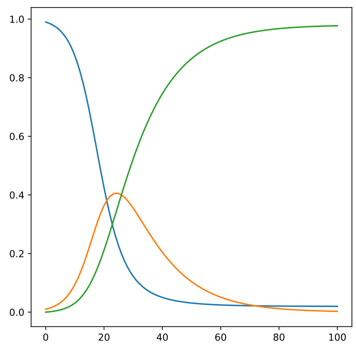
*Solve using Python*

```
import numpy as np
import scipy.integrate
def SIR_grad(x,t,params):
   """basic gradient definitions for SIR model"""
   beta,gamma = params   ## unpack parameters
   S,I,R = x             ## unpack state variables
   return(np.array([-beta*S*I, beta*S*I-gamma*I, gamma*I]))

t_vec = np.linspace(0,8,101)
params = (4,1) ## extra parameters (beta, gamma)
y0 = (0.99, 0.01, 0)
SIR_sol1 = scipy.integrate.odeint(SIR_grad,
                                  y0=y0,
```

```
                                  t=t_vec,
                                  args=(params,))

## https://community.rstudio.com/t/how-to-display-the-plot-in-the-python-chunk/22039/3
import matplotlib.pyplot as plt
fig, ax = plt.subplots()
ax.plot(SIR_sol1);
plt.show()
```




*dimensional analysis*

- initial growth rate ($\text{time}^{-1}$) $\beta - \gamma$
- mean infectious period $1/\gamma$ (time)
- basic reproduction number $\mathcal{R}_0 = \beta/\gamma$

*initial growth rate*

$$\begin{aligned}\frac{dI}{dt} &= \beta S - \gamma I \\ &= (\beta S - \gamma) I \\ &\approx (\beta - \gamma) I \quad \text{near } DFE\end{aligned}$$

or calculate **Jacobian** ($\partial X_i / \partial X_j$):

$$\begin{pmatrix} -\beta I & -\beta S & 0 \\ \beta I & \beta S - \gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}$$

Evaluate at DFE ($\{1, 0, 0\}$):

$$\begin{pmatrix} 0 & -\beta & 0 \\ 0 & \beta - \gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}$$

Eigenvalues of this are pretty boring! But useful approach.

### *Per capita rates*

In general we can express *per capita* gradients in $X$ as gradients of $\log(X)$:

$$
\begin{aligned}
\frac{dX}{dt} &= Xf(X,Y,Z,\ldots) \\
\frac{\frac{dX}{dt}}{X} &= f(X,Y,Z,\ldots) \\
\frac{d\log(X)}{dt} &= f(X,Y,Z,\ldots)
\end{aligned}
$$

Another way to see that $\beta - \gamma$ is the slope on the log scale.

### *Stability of DFE*

- $\beta > \gamma$ ($r > 0$)
- $\beta/\gamma > 1$ ($\mathcal{R}_0 > 1$)

Local asymptotic stability **or**

- $\frac{dI}{dt} = \beta SI - \gamma I$
- non-dimensionalize: $\gamma = 1$, $\beta = \mathcal{R}_0$
- $\frac{dI}{dt} = (\mathcal{R}_0 S - 1)I$
- $\frac{d\log I}{dt} = \mathcal{R}_0 S - 1$

Since $S \leq 1$, $\mathcal{R}_0 < 1 \rightarrow$ deriv of $\log I$ is always negative (don't really need the last step)

### *Automated analysis*

```r
library(phaseR)
```

```
## -------------------------------------------------------------------------------
## phaseR: Phase plane analysis of one- and two-dimensional autonomous ODE systems
## -------------------------------------------------------------------------------
##
## v.2.1: For an overview of the package's functionality enter: ?phaseR
##
## For news on the latest updates enter: news(package = "phaseR")
```

```r
par(las=1,bty="l",xaxs="i",yaxs="i") ## cosmetic
SIRgrad_2d <- function(t, y, parms) {
    g <- with(as.list(c(y,parms)), {
        c(-beta*S*I, beta*S*I-gamma*I)
    })
    return(list(g))
```

```
}
## plot(0:1,0:1,type="n",xlab="S",ylab="I")
f1 <- flowField(SIRgrad_2d,
          xlim=c(0,1),
          ylim=c(0,1),
          parameters=p0,
          state.names=c("S","I"),
          add=FALSE)
n1 <- nullclines(SIRgrad,
                 xlim=c(0,1),
                 ylim=c(0,1),
                 parameters=p0,
                 state.names=c("S","I"))
trajectory(SIRgrad_2d,parameters=p0,
           state.names=c("S","I"),
           ## n=10,
           y0=y0[1:2],
           tlim=c(0,5))
```

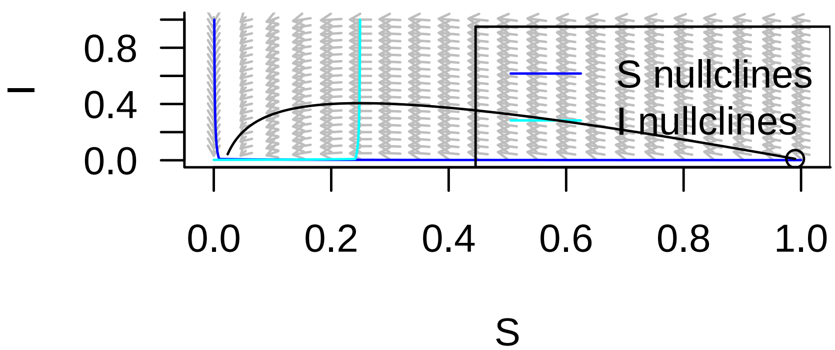



Figure 7: phase plane analysis in R

```
## $add
## [1] TRUE
##
## $col
## [1] "black"
##
## $deriv
## function(t, y, parms) {
##     g <- with(as.list(c(y,parms)), {
##         c(-beta*S*I, beta*S*I-gamma*I)
##     })
##     return(list(g))
## }
```

```
## <bytecode: 0x560fda701fb0>
##
## $n
## NULL
##
## $parameters
##  beta gamma
##     4     1
##
## $system
## [1] "two.dim"
##
## $t
##   [1] 0.00 0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.10 0.11 0.12 0.13 0.14
##  [16] 0.15 0.16 0.17 0.18 0.19 0.20 0.21 0.22 0.23 0.24 0.25 0.26 0.27 0.28 0.29
##  [31] 0.30 0.31 0.32 0.33 0.34 0.35 0.36 0.37 0.38 0.39 0.40 0.41 0.42 0.43 0.44
##  [46] 0.45 0.46 0.47 0.48 0.49 0.50 0.51 0.52 0.53 0.54 0.55 0.56 0.57 0.58 0.59
##  [61] 0.60 0.61 0.62 0.63 0.64 0.65 0.66 0.67 0.68 0.69 0.70 0.71 0.72 0.73 0.74
##  [76] 0.75 0.76 0.77 0.78 0.79 0.80 0.81 0.82 0.83 0.84 0.85 0.86 0.87 0.88 0.89
##  [91] 0.90 0.91 0.92 0.93 0.94 0.95 0.96 0.97 0.98 0.99 1.00 1.01 1.02 1.03 1.04
## [106] 1.05 1.06 1.07 1.08 1.09 1.10 1.11 1.12 1.13 1.14 1.15 1.16 1.17 1.18 1.19
## [121] 1.20 1.21 1.22 1.23 1.24 1.25 1.26 1.27 1.28 1.29 1.30 1.31 1.32 1.33 1.34
## [136] 1.35 1.36 1.37 1.38 1.39 1.40 1.41 1.42 1.43 1.44 1.45 1.46 1.47 1.48 1.49
## [151] 1.50 1.51 1.52 1.53 1.54 1.55 1.56 1.57 1.58 1.59 1.60 1.61 1.62 1.63 1.64
## [166] 1.65 1.66 1.67 1.68 1.69 1.70 1.71 1.72 1.73 1.74 1.75 1.76 1.77 1.78 1.79
## [181] 1.80 1.81 1.82 1.83 1.84 1.85 1.86 1.87 1.88 1.89 1.90 1.91 1.92 1.93 1.94
## [196] 1.95 1.96 1.97 1.98 1.99 2.00 2.01 2.02 2.03 2.04 2.05 2.06 2.07 2.08 2.09
## [211] 2.10 2.11 2.12 2.13 2.14 2.15 2.16 2.17 2.18 2.19 2.20 2.21 2.22 2.23 2.24
## [226] 2.25 2.26 2.27 2.28 2.29 2.30 2.31 2.32 2.33 2.34 2.35 2.36 2.37 2.38 2.39
## [241] 2.40 2.41 2.42 2.43 2.44 2.45 2.46 2.47 2.48 2.49 2.50 2.51 2.52 2.53 2.54
## [256] 2.55 2.56 2.57 2.58 2.59 2.60 2.61 2.62 2.63 2.64 2.65 2.66 2.67 2.68 2.69
## [271] 2.70 2.71 2.72 2.73 2.74 2.75 2.76 2.77 2.78 2.79 2.80 2.81 2.82 2.83 2.84
## [286] 2.85 2.86 2.87 2.88 2.89 2.90 2.91 2.92 2.93 2.94 2.95 2.96 2.97 2.98 2.99
## [301] 3.00 3.01 3.02 3.03 3.04 3.05 3.06 3.07 3.08 3.09 3.10 3.11 3.12 3.13 3.14
## [316] 3.15 3.16 3.17 3.18 3.19 3.20 3.21 3.22 3.23 3.24 3.25 3.26 3.27 3.28 3.29
## [331] 3.30 3.31 3.32 3.33 3.34 3.35 3.36 3.37 3.38 3.39 3.40 3.41 3.42 3.43 3.44
## [346] 3.45 3.46 3.47 3.48 3.49 3.50 3.51 3.52 3.53 3.54 3.55 3.56 3.57 3.58 3.59
## [361] 3.60 3.61 3.62 3.63 3.64 3.65 3.66 3.67 3.68 3.69 3.70 3.71 3.72 3.73 3.74
## [376] 3.75 3.76 3.77 3.78 3.79 3.80 3.81 3.82 3.83 3.84 3.85 3.86 3.87 3.88 3.89
## [391] 3.90 3.91 3.92 3.93 3.94 3.95 3.96 3.97 3.98 3.99 4.00 4.01 4.02 4.03 4.04
## [406] 4.05 4.06 4.07 4.08 4.09 4.10 4.11 4.12 4.13 4.14 4.15 4.16 4.17 4.18 4.19
## [421] 4.20 4.21 4.22 4.23 4.24 4.25 4.26 4.27 4.28 4.29 4.30 4.31 4.32 4.33 4.34
## [436] 4.35 4.36 4.37 4.38 4.39 4.40 4.41 4.42 4.43 4.44 4.45 4.46 4.47 4.48 4.49
## [451] 4.50 4.51 4.52 4.53 4.54 4.55 4.56 4.57 4.58 4.59 4.60 4.61 4.62 4.63 4.64
```

```
## [466] 4.65 4.66 4.67 4.68 4.69 4.70 4.71 4.72 4.73 4.74 4.75 4.76 4.77 4.78 4.79
## [481] 4.80 4.81 4.82 4.83 4.84 4.85 4.86 4.87 4.88 4.89 4.90 4.91 4.92 4.93 4.94
## [496] 4.95 4.96 4.97 4.98 4.99 5.00
##
## $tlim
## [1] 0 5
##
## $tstep
## [1] 0.01
##
## $x
##              [,1]
##   [1,] 0.99000000
##   [2,] 0.98959816
##   [3,] 0.98918443
##   [4,] 0.98875847
##   [5,] 0.98831992
##   [6,] 0.98786843
##   [7,] 0.98740363
##   [8,] 0.98692514
##   [9,] 0.98643258
##  [10,] 0.98592556
##  [11,] 0.98540366
##  [12,] 0.98486647
##  [13,] 0.98431356
##  [14,] 0.98374449
##  [15,] 0.98315882
##  [16,] 0.98255608
##  [17,] 0.98193581
##  [18,] 0.98129752
##  [19,] 0.98064071
##  [20,] 0.97996488
##  [21,] 0.97926952
##  [22,] 0.97855409
##  [23,] 0.97781805
##  [24,] 0.97706084
##  [25,] 0.97628190
##  [26,] 0.97548065
##  [27,] 0.97465650
##  [28,] 0.97380883
##  [29,] 0.97293704
##  [30,] 0.97204048
##  [31,] 0.97111851
##  [32,] 0.97017047
```

```
## [33,] 0.96919570
## [34,] 0.96819349
## [35,] 0.96716316
## [36,] 0.96610399
## [37,] 0.96501526
## [38,] 0.96389622
## [39,] 0.96274611
## [40,] 0.96156418
## [41,] 0.96034964
## [42,] 0.95910169
## [43,] 0.95781954
## [44,] 0.95650235
## [45,] 0.95514930
## [46,] 0.95375954
## [47,] 0.95233221
## [48,] 0.95086644
## [49,] 0.94936135
## [50,] 0.94781604
## [51,] 0.94622962
## [52,] 0.94460117
## [53,] 0.94292976
## [54,] 0.94121446
## [55,] 0.93945433
## [56,] 0.93764841
## [57,] 0.93579575
## [58,] 0.93389537
## [59,] 0.93194631
## [60,] 0.92994760
## [61,] 0.92789824
## [62,] 0.92579726
## [63,] 0.92364367
## [64,] 0.92143647
## [65,] 0.91917468
## [66,] 0.91685732
## [67,] 0.91448339
## [68,] 0.91205192
## [69,] 0.90956192
## [70,] 0.90701243
## [71,] 0.90440249
## [72,] 0.90173113
## [73,] 0.89899743
## [74,] 0.89620046
## [75,] 0.89333929
## [76,] 0.89041303
```

```
##  [77,] 0.88742081
##  [78,] 0.88436176
##  [79,] 0.88123505
##  [80,] 0.87803988
##  [81,] 0.87477545
##  [82,] 0.87144102
##  [83,] 0.86803586
##  [84,] 0.86455927
##  [85,] 0.86101062
##  [86,] 0.85738928
##  [87,] 0.85369468
##  [88,] 0.84992628
##  [89,] 0.84608361
##  [90,] 0.84216622
##  [91,] 0.83817374
##  [92,] 0.83410581
##  [93,] 0.82996218
##  [94,] 0.82574261
##  [95,] 0.82144695
##  [96,] 0.81707510
##  [97,] 0.81262702
##  [98,] 0.80810276
##  [99,] 0.80350241
## [100,] 0.79882614
## [101,] 0.79407421
## [102,] 0.78924695
## [103,] 0.78434474
## [104,] 0.77936807
## [105,] 0.77431749
## [106,] 0.76919365
## [107,] 0.76399727
## [108,] 0.75872915
## [109,] 0.75339019
## [110,] 0.74798137
## [111,] 0.74250373
## [112,] 0.73695843
## [113,] 0.73134671
## [114,] 0.72566988
## [115,] 0.71992935
## [116,] 0.71412660
## [117,] 0.70826321
## [118,] 0.70234084
## [119,] 0.69636123
## [120,] 0.69032619
```

```
## [121,] 0.68423761
## [122,] 0.67809748
## [123,] 0.67190783
## [124,] 0.66567079
## [125,] 0.65938854
## [126,] 0.65306333
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## [271,] 0.29650659
## [272,] 0.29452225
## [273,] 0.29253994
## [274,] 0.29056000
## [275,] 0.28858275
## [276,] 0.28660853
```

```
## [277,] 0.28463764
## [278,] 0.28267038
## [279,] 0.28070705
## [280,] 0.27874793
## [281,] 0.27679328
## [282,] 0.27484337
## [283,] 0.27289845
## [284,] 0.27095877
## [285,] 0.26902457
## [286,] 0.26709606
## [287,] 0.26517347
## [288,] 0.26325701
## [289,] 0.26134688
## [290,] 0.25944327
## [291,] 0.25754638
## [292,] 0.25565639
## [293,] 0.25377346
## [294,] 0.25189777
## [295,] 0.25002946
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## [297,] 0.24631565
## [298,] 0.24447042
## [299,] 0.24263315
## [300,] 0.24080398
## [301,] 0.23898303
## [302,] 0.23717041
## [303,] 0.23536624
## [304,] 0.23357061
## [305,] 0.23178364
## [306,] 0.23000542
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## [309,] 0.22472411
## [310,] 0.22298174
## [311,] 0.22124852
## [312,] 0.21952452
## [313,] 0.21780981
## [314,] 0.21610445
## [315,] 0.21440849
## [316,] 0.21272199
## [317,] 0.21104499
## [318,] 0.20937755
## [319,] 0.20771969
## [320,] 0.20607147
```

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## [322,] 0.20280405
## [323,] 0.20118492
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## [331,] 0.18858455
## [332,] 0.18705379
## [333,] 0.18553291
## [334,] 0.18402189
## [335,] 0.18252074
## [336,] 0.18102945
## [337,] 0.17954802
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## [340,] 0.17516280
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## [347,] 0.16527349
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## [355,] 0.15455192
## [356,] 0.15325463
## [357,] 0.15196677
## [358,] 0.15068830
## [359,] 0.14941920
## [360,] 0.14815942
## [361,] 0.14690894
## [362,] 0.14566771
## [363,] 0.14443569
## [364,] 0.14321285
```

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## [365,] 0.14199915
## [366,] 0.14079455
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## [368,] 0.13841248
## [369,] 0.13723492
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## [377,] 0.12813249
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## [379,] 0.12594358
## [380,] 0.12486189
## [381,] 0.12378864
## [382,] 0.12272378
## [383,] 0.12166727
## [384,] 0.12061907
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## [398,] 0.10678921
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## [404,] 0.10132468
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## [406,] 0.09956198
## [407,] 0.09869145
## [408,] 0.09782805
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## [409,] 0.09697176
## [410,] 0.09612250
## [411,] 0.09528024
## [412,] 0.09444494
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## [414,] 0.09279498
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## [417,] 0.09037097
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## [419,] 0.08878835
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## [421,] 0.08723201
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## [424,] 0.08494595
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## [426,] 0.08345367
## [427,] 0.08271691
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## [433,] 0.07842494
## [434,] 0.07773062
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## [438,] 0.07501171
## [439,] 0.07434635
## [440,] 0.07368666
## [441,] 0.07303260
## [442,] 0.07238410
## [443,] 0.07174114
## [444,] 0.07110368
## [445,] 0.07047166
## [446,] 0.06984505
## [447,] 0.06922380
## [448,] 0.06860788
## [449,] 0.06799724
## [450,] 0.06739184
## [451,] 0.06679165
## [452,] 0.06619661
```

```
## [453,] 0.06560669
## [454,] 0.06502184
## [455,] 0.06444204
## [456,] 0.06386724
## [457,] 0.06329739
## [458,] 0.06273246
## [459,] 0.06217242
## [460,] 0.06161721
## [461,] 0.06106681
## [462,] 0.06052118
## [463,] 0.05998027
## [464,] 0.05944405
## [465,] 0.05891248
## [466,] 0.05838552
## [467,] 0.05786315
## [468,] 0.05734531
## [469,] 0.05683197
## [470,] 0.05632310
## [471,] 0.05581866
## [472,] 0.05531861
## [473,] 0.05482293
## [474,] 0.05433156
## [475,] 0.05384448
## [476,] 0.05336166
## [477,] 0.05288305
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## [479,] 0.05193835
## [480,] 0.05147219
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## [482,] 0.05055207
## [483,] 0.05009805
## [484,] 0.04964801
## [485,] 0.04920192
## [486,] 0.04875974
## [487,] 0.04832145
## [488,] 0.04788700
## [489,] 0.04745638
## [490,] 0.04702953
## [491,] 0.04660645
## [492,] 0.04618709
## [493,] 0.04577141
## [494,] 0.04535941
## [495,] 0.04495103
## [496,] 0.04454625
```

```
## [497,] 0.04414504
## [498,] 0.04374737
## [499,] 0.04335321
## [500,] 0.04296253
## [501,] 0.04257531
##
## $y0
##         S    I
## [1,] 0.99 0.01
```

```
phasePlaneAnalysis(SIRgrad_2d,xlim=c(0,1),
                   parameters=p0,
                   state.names=c("S","I"),
                   ylim=c(0,1))
```

*Solution*

- can't get analytical solution for $S(t)$, $I(t)$
- **but**: we can solve for $I(S)$:

$$
\begin{aligned}
\frac{dI}{dS} &= \frac{dI/dt}{dS/dt} = -1 + \frac{1}{\mathcal{R}_0 S} \\
\int_{I(0)}^{I} (t)\, dI &= \int_{S(0)}^{S(t)} \left(-1 + \frac{1}{\mathcal{R}_0 S}\right) dS \\
I - I(0) &= -(S - S(0)) + \frac{1}{\mathcal{R}_0} \log(S/S(0)) \\
I + S - (I(0) + S(0)) &= \frac{1}{\mathcal{R}_0} \log(S/S(0))
\end{aligned}
$$

*Final size calculations*

- $t \to \infty$:

$$(I_\infty + S_\infty) - (I(0) + S(0)) = \frac{1}{\mathcal{R}_0} \log S_\infty / S(0)$$

- newly invading pathogen: $S \approx 1$, $I(0) \ll 1$ $(\approx 0)$, $I_\infty \to 0$
- in the limit $I(0) \to 0$:

$$S_\infty) - 1 = \frac{1}{\mathcal{R}_0} \log S_\infty$$

- "final size" $Z = 1 - S_\infty$
- $-Z = \frac{1}{\mathcal{R}_0} \log(1 - Z)$

*Lambert W functions*

- How do we solve this?

- Newton's method (or whatever)
- *Lambert W* (Corless et al. 1996): solves $W \exp(W) = Z$

$$Z = 1 + \frac{1}{\mathcal{R}_0} W(-\mathcal{R}_0 \exp(-\mathcal{R}_0))$$

*Epidemic threshold*

Assuming vaccination (or other perfect *prophylaxis* [protection]) at rate $p$

$$R_0 = 1 - 1/p$$

speed-based intervention:

$$\begin{aligned} \beta SI - (\gamma + \phi)I &< 0 \\ I(\beta - \gamma - \phi) &< 0 \\ \phi > (\beta - \gamma) &= r \end{aligned}$$

*Comparing Epidemic threshold vs. final size*

```
library(emdbook)
finalsize <- function(R0) {
    1+1/R0*lambertW(-R0*exp(-R0))
}
par(las=1,bty="l")
curve(finalsize(x),from=1,to=10,xlab=expression(R[0]),
      ylab="proportion")
curve(1-1/x, add=TRUE, col=2)
legend("bottomright",
       c("final size","herd immunity threshold"),
       col=1:2, lty=1)
```

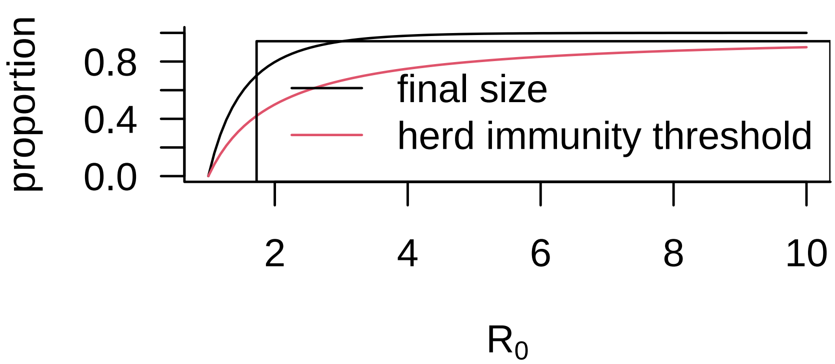


Figure 8: final size vs herd immunity

## *SIR with vital dynamics*

## *references*

Corless, Robert M., G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. 1996. "On the Lambert $W$ Function." *Advances in Computational Mathematics* 5 (4): 329–59.

Levins, R. 1966. "The Strategy of Model Building in Population Biology." *American Scientist* 54: 421–31. https://www.jstor.org/stable/27836590.

Levins, Richard. 1993. "A Response to Orzack and Sober: Formal Analysis and the Fluidity of Science." *Quarterly Review of Biology* 68 (4): 547–55.

Orzack, Steven Hecht, and Elliott Sober. 1993. "A Critical Assessment of Levins's the Strategy of Model Building in Population Biology (1966)." *Quarterly Review of Biology* 68 (4): 533–46.

Weisberg, Michael. 2007. "Forty Years of 'the Strategy': Levins on Model Building and Idealization." *Biology & Philosophy* 21 (5): 623–45. https://doi.org/10.1007/s10539-006-9051-9.