

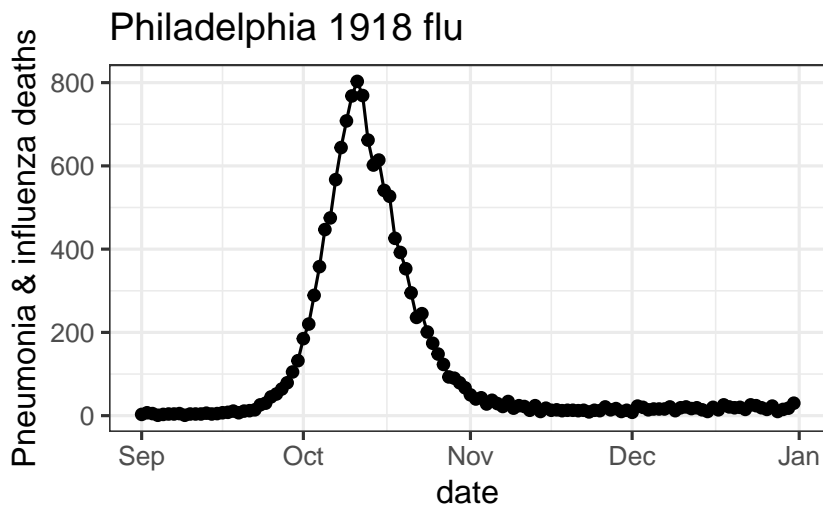
Epidemic models 1

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motivation

- P & I data from Philadelphia 1918 flu:



what do we want to figure out?

what shall we assume?

- classify individuals as S , I (**compartmental** model; **microparasite** or **intensity-independent**)
- disease is transmitted from S to I
- $S \rightarrow I$ instantaneously (zero latent period, no E)
- population is **homogeneous** (no heterogeneity in susceptibility, infectiousness, contact)
- fixed population size (birth = migration = 'natural' death = 0)
- transmission rate is time-invariant

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- assumption 2 is OK (Pasteur, Koch's postulates ...)
 - all the rest are approximations

start simple!

- parsimony
- robustness?
- applicability/estimation?

Levins (1966) (also Orzack and Sober (1993), Levins (1993), Weisberg (2007))

exponential growth

- one variable (=1D model)
 - how does disease spread? → equation
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what variables should we use?

- time (t)
- state variable: incidence, prevalence, death rate, death toll (= cumulative death?)
- deaths loosely connected to transmission

but deaths are observed!

when are deaths a good **proxy** for incidence?

- infection → death time is fixed
- homogeneity? (might not matters?)
- mortality curve is shifted epidemic

(COVID context . . . we observe case reports, number of tests, hospitalizations, and deaths)

- **incidence**: number of infections per unit time (rate or flow)
- **prevalence**: number of currently infected people (quantity or stock)

prevalence is closer to the **mechanism**

model components:

- $I(t)$ (state variable: prevalence)
- $I(0)$ (initial conditions)
- β (parameter) = avg contacts **per susceptible per infective per unit time**

$$I(t + \Delta t) \approx I(t) + \beta I(t) \Delta t$$

Take $\lim \Delta t \rightarrow 0$ (and solve):

$$\frac{dI}{dt} = \beta I \rightarrow I(t) = I(0) \exp(\beta t)$$

model criticism

- Ignored discrete nature of individuals
- Ignored time-varying β (e.g. **diurnal** fluctuations)
- Ignored finite infectious periods (recovery/death)

Next: What if we make infectious periods finite? (i.e., including recovery (**clearance**) or death

$$dI/dt = \beta I - \gamma I$$

mean infectious period

$$I(t) = I(0) \exp(-\gamma t)$$

$$\text{proportion uninfected} = \exp(-\gamma t)$$

$$\text{proportion infected} = 1 - \exp(-\gamma t) (= \text{CDF} := C(t))$$

$$\text{PDF} := C'(t) = \gamma \exp(-\gamma t)$$

$$\text{substitute } x = \gamma t \rightarrow dx = \gamma dt$$

$$\text{mean} = E[t] = \int t \exp(-\gamma t) dt = \int x \exp(-x) dx / \gamma = 1/\gamma$$

dimensional analysis

rates and characteristic times/scales

- is I a proportion or a density or a number ... ?
- what are the units of β, γ ?

nondimensionalization

- standardize any values that can be eliminated **without loss of (mathematical) generality**
- what can we do here?
- $\gamma = 1$
- I ? (depends on how we have defined it initially) $\rightarrow I/N$

Simple (SI) epidemic

- what are we missing?
 - **depletion of susceptibles**
 - let's take a step back and ignore death & recovery for now
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$$dS/dt = -\beta SI$$

$$dI/dt = \beta SI$$

This looks 2D **but** what if we assume $S + I = N$ is constant? Then
 $S = N - I$

$$dI/dt = \beta(N - I)I$$

How do we solve this? Partial fractions

$$\frac{dI}{(N - I)I} = \beta dt$$

$$dI \left(\frac{A}{N - I} + \frac{B}{I} \right) = dI \frac{A + B(N - I)}{I(N - I)}$$

$$A = B; B = 1/N$$

$$\frac{1}{\beta N} (-\log(N - I) + \log(I)) \Big|_{I(0)}^I = t - t_0$$

$$(-\log(N - I) + \log(I)) \Big|_{I(0)}^I = (\beta N)(t - t_0)$$

$$\log \left(\frac{I}{N - I} \right) - \log \left(\frac{I(0)}{N - I(0)} \right) = \beta N t$$

$$\log \left(\frac{I}{N - I} \right) = \beta N t + -\log \left(\frac{I(0)}{N - I(0)} \right)$$

$$\frac{I}{N - I} = \exp(\beta N t) \frac{I(0)}{N - I(0)} = Q$$

$$I = Q(N - I)$$

$$I(t)(1 + Q) = QN$$

$$I(t) = \frac{QN}{1 + Q} = \frac{N}{1 + \frac{1}{Q}}$$

$$\frac{N}{1 + \left(\frac{N - I(0)}{I(0)} \right) \exp(-\beta N t)}$$

(Comparison to metapop, logistic growth model)

references

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