Epidemic models 1

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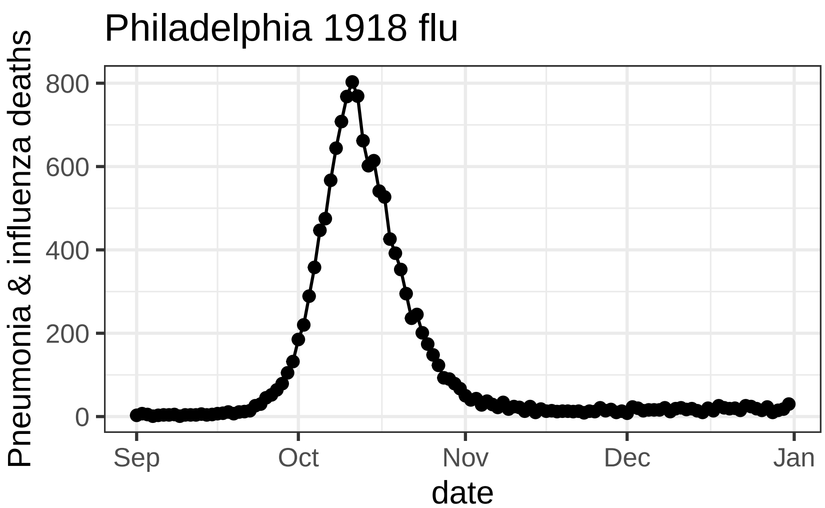
motivation

- P & I data from Philadelphia 1918 flu:



what do we want to figure out?

what shall we assume?

- classify individuals as S, I (**compartmental** model; **microparasite** or **intensity-independent**)
- disease is transmitted from S to I
- $S \to I$ instantaneously (zero latent period, no E)
- population is **homogeneous** (no heterogeneity in susceptibility, infectiousness, contact)
- fixed population size (birth = migration = 'natural' death = 0)
- transmission rate is time-invariant

- assumption 2 is OK (Pasteur, Koch's postulates ...)
- all the rest are approximations

  start simple!

- parsimony
- robustness?
- applicability/estimation?

Levins (1966) (also Orzack and Sober (1993), Levins (1993), Weisberg (2007))

exponential growth

- one variable (=1D model)
- how does disease spread? $\rightarrow$ equation

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what variables should we use?

- time (t)
- state variable: incidence, prevalence, death rate, death toll (= cumulative death?)
- deaths loosely connected to transmission

but deaths are observed!

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when are deaths a good **proxy** for incidence?

- infection -> death time is fixed
- homogeneity? (might not matters?)
- mortality curve is shifted epidemic

(COVID context . . . we observe case reports, number of tests, hospitalizations, and deaths)

- **incidence**: number of infections per unit time (rate or flow)
- **prevalence**: number of currently infected people (quantity or stock)

prevalence is closer to the **mechanism**

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model components:

- $I(t)$ (state variable: prevalence)
- $I(0)$ (initial conditions)
- β (parameter) = avg contacts **per susceptible per infective per unit time**

$$I(t + \Delta t) \approx I(t) + \beta I(t)\Delta t$$

Take lim $\Delta t \rightarrow 0$ (and solve):

$$\frac{dI}{dt} = \beta I \rightarrow I(t) = I(0)exp(\beta t)$$

model criticism

- Ignored discrete nature of individuals
- Ignored time-varying β (e.g. **diurnal** fluctuations)
- Ignored finite infectious periods (recovery/death)

Next: What if we make infectious periods finite? (i.e., including recovery (**clearance**) or death

$$dI/dt = \beta I - \gamma I$$

mean infectious period

$$\begin{aligned} I(t) &= I(0)\exp(-\gamma t) \\ \text{proportion uninfected} &= \exp(-\gamma t) \\ \text{proportion infected} &= 1 - \exp(-\gamma t) (= \text{CDF} := C(t)) \\ \text{PDF} : &= C'(t) = \gamma \exp(-\gamma t) \\ \text{substitute } x &= \gamma t \quad \to \quad dx = \gamma\, dt \\ \text{mean} &= E[t] = \int t \exp(-\gamma t)\, dt = \int x \exp(-x)\, dx/\gamma = 1/\gamma \end{aligned}$$

dimensional analysis

rates and characteristic times/scales

- is I a proportion or a density or a number … ?
- what are the units of β, γ ?

nondimensionalization

- standardize any values that can be eliminated **without loss of (mathematical) generality**
- what can we do here?
- $\gamma = 1$
- I ? (depends on how we have defined it initially) $\to I/N$

Simple (SI) epidemic

- what are we missing?
- **depletion of susceptibles**
- let's take a step back and ignore death & recovery for now

$$dS/dt = -\beta SI$$
$$dI/dt = \beta SI$$

This looks 2D **but** what if we assume $S + I = N$ is constant? Then $S = N - I$

$$dI/dt = \beta(N - I)I$$

How do we solve this? Partial fractions

$$\frac{dI}{(N-I)I} = \beta\, dt$$
$$dI(A/(N-I) + B/I) = dI(IA + B(N-I))/(I(N-I))$$
$$A = B; B = 1/N$$
$$(1/N)(-\log(N-I) + \log(I))\Big|_0^t = t - t_0$$

(Comparison to metapop, logistic growth model)

references

Levins, R. 1966. "The Strategy of Model Building in Population Biology." *American Scientist* 54: 421–31. https://www.jstor.org/stable/27836590.

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Weisberg, Michael. 2007. "Forty Years of 'the Strategy': Levins on Model Building and Idealization." *Biology & Philosophy* 21 (5): 623–45. https://doi.org/10.1007/s10539-006-9051-9.