

Epidemic models 1

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motivation

- P & I data from Philadelphia 1918 flu:

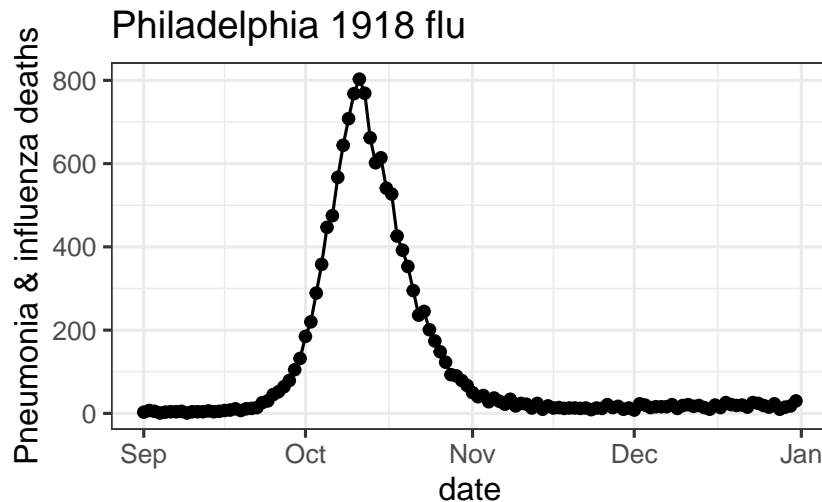


Figure 1: Phila. 1918 flu data

what do we want to figure out?

what shall we assume?

- classify individuals as S , I (**compartmental** model; **microparasite** or **intensity-independent**)
- disease is transmitted from S to I
- $S \rightarrow I$ instantaneously (zero latent period, no E)
- population is **homogeneous** (no heterogeneity in susceptibility, infectiousness, contact)
- fixed population size (birth = migration = 'natural' death = 0)
- transmission rate is time-invariant

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- assumption 2 is OK (Pasteur, Koch's postulates ...)
 - all the rest are approximations

start simple!

- parsimony

- robustness?
- applicability/estimation?

Levins (1966) (also Orzack and Sober (1993), Levins (1993), Weisberg (2007))

exponential growth

- one variable (=1D model)
 - how does disease spread? → equation
-

what variables should we use?

- time (t)
- state variable: incidence, prevalence, death rate, death toll (= cumulative death?)
- deaths loosely connected to transmission

but deaths are observed!

when are deaths a good **proxy** for incidence?

- infection → death time is fixed
- homogeneity? (might not matters?)
- mortality curve is shifted epidemic

(COVID context ... we observe case reports, number of tests, hospitalizations, and deaths)

- **incidence**: number of infections per unit time (rate or flow)
- **prevalence**: number of currently infected people (quantity or stock)

prevalence is closer to the **mechanism**

model components:

- $I(t)$ (state variable: prevalence)
- $I(0)$ (initial conditions)
- β (parameter) = avg contacts **per susceptible per infective per unit time**

$$I(t + \Delta t) \approx I(t) + \beta I(t) \Delta t$$

Take $\lim \Delta t \rightarrow 0$ (and solve):

$$\frac{dI}{dt} = \beta I \rightarrow I(t) = I(0) \exp(\beta t)$$

model criticism

- Ignored discrete nature of individuals
- Ignored time-varying β (e.g. **diurnal** fluctuations)
- Ignored finite infectious periods (recovery/death)

Next: What if we make infectious periods finite? (i.e., including recovery (**clearance**) or death

$$dI/dt = \beta I - \gamma I$$

mean infectious period

$$I(t) = I(0) \exp(-\gamma t)$$

proportion uninfected = $\exp(-\gamma t)$

proportion infected = $1 - \exp(-\gamma t)$ (= CDF := $C(t)$)

$$\text{PDF} := C'(t) = \gamma \exp(-\gamma t)$$

$$\text{substitute } x = \gamma t \rightarrow dx = \gamma dt$$

$$\text{mean} = E[t] = \int t \exp(-\gamma t) dt = \int x \exp(-x) dx / \gamma = 1/\gamma$$

dimensional analysis

rates and characteristic times/scales

- is I a proportion or a density or a number ... ?
- what are the units of β, γ ?

nondimensionalization

- standardize any values that can be eliminated **without loss of (mathematical) generality**
- what can we do here?
- $\gamma = 1$
- I ? (depends on how we have defined it initially) $\rightarrow I/N$

compare with data???

Original scale:

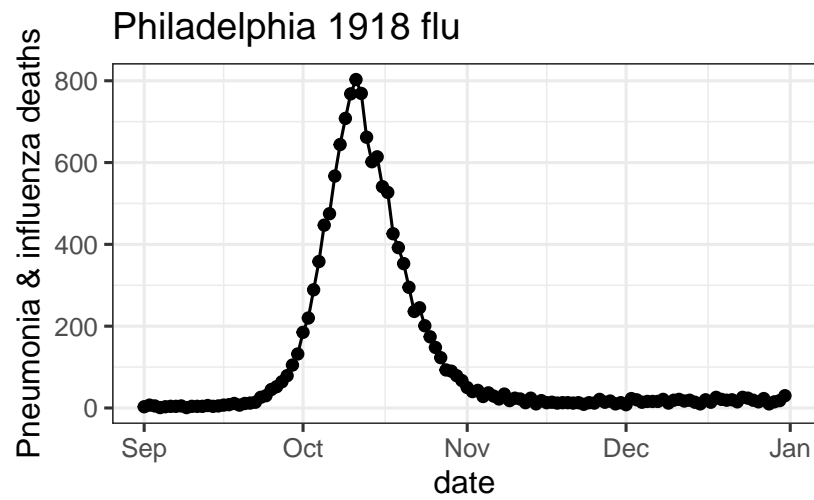


Figure 2: Philadelphia P&I

Log scale:

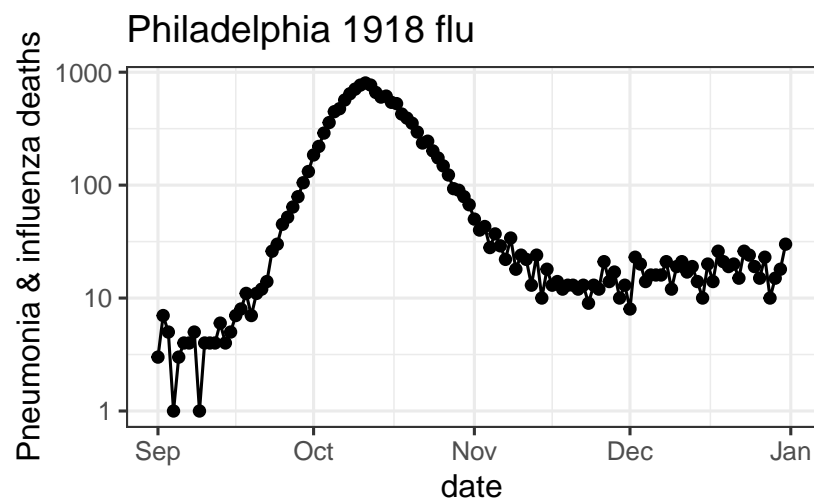


Figure 3: Philadelphia P&I, log scale

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- Fit a straight line through the straight part of the curve
 - slope is βN
 - “intercept” is $\log(I(0))$ (zero is defined in a tricky way)

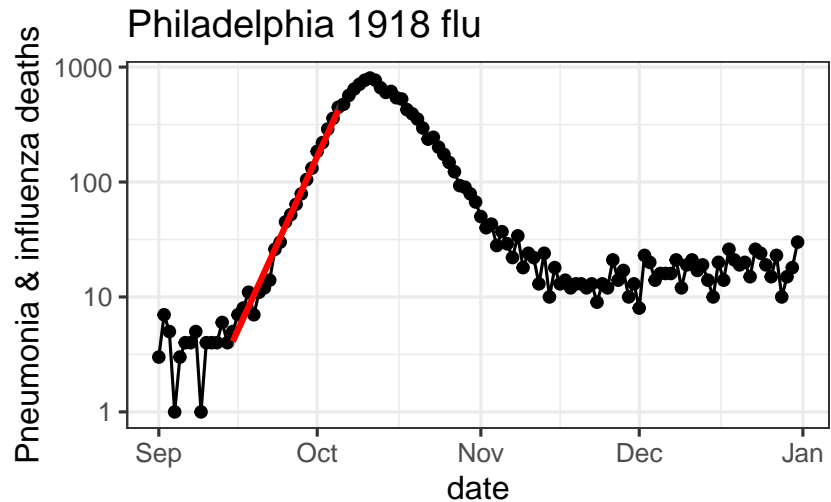


Figure 4: log-scale flu with regression

model assessment

- math is super-easy!
- clear, testable predictions
- parameter estimation is easy
- only consistent over a short time window
 - small t : arbitrarily close to zero
 - large t : ridiculous

Simple (SI) epidemic

- what are we missing?
- **depletion of susceptibles**
- let's take a step back and ignore death & recovery for now

$$dS/dt = -\beta SI$$

$$dI/dt = \beta SI$$

This looks 2D **but** what if we assume $S + I = N$ is constant? Then $S = N - I$

$$dI/dt = \beta(N - I)I$$

How do we solve this? **Partial fractions**

$$\frac{dI}{\beta(N-I)I} = dt$$

$$dI \left(\frac{A}{N-I} + \frac{B}{I} \right) = dI \cdot \frac{A + B(N-I)}{I(N-I)}$$

$$A = B; \quad B = 1/N$$

$$\frac{1}{\beta N} (-\log(N-I) + \log(I)) \Big|_{I(0)}^I = t - t_0$$

$$(-\log(N-I) + \log(I)) \Big|_{I(0)}^I = (\beta N)(t - t_0) \quad (\text{set } t_0 = 0)$$

$$\log \left(\frac{I}{N-I} \right) - \log \left(\frac{I(0)}{N-I(0)} \right) = \beta N t$$

$$\log \left(\frac{I}{N-I} \right) = \beta N t + -\log \left(\frac{I(0)}{N-I(0)} \right)$$

$$\frac{I}{N-I} = \exp(\beta N t) \frac{I(0)}{N-I(0)} \equiv Q$$

$$I = Q(N-I)$$

$$I(t)(1+Q) = QN$$

$$I(t) = \frac{QN}{1+Q} = \frac{N}{1+\frac{1}{Q}}$$

$$= \frac{N}{1 + \left(\frac{N-I(0)}{I(0)} \right) \exp(-\beta N t)}$$

$$?? \equiv I(0) \exp(\beta N t) / (1 + (I(0)/N)(\exp(\beta N t) - 1)) ??$$

Qualitative analysis

- $I \ll N$? exponential growth
- **per capita growth rate** $((dI/dt)/I = d(\log(I))/dt)$ decreases monotonically with increasing I
- asymptotic behaviour? equilibria? periodic orbits?
- periodic orbits impossible in 1D (uniqueness of flows)

equilibrium analysis

- $I = 0$, **disease free equilibrium** (DFE)
- $I = N$, **endemic equilibrium** (EE)

Stability? (Assume $\beta > 0$)

- **local asymptotic stability**
- **global asymptotic stability** (Lyapunov functions)

model criticism/conclusions

(Comparison to metapop, logistic growth model)

references

- Levins, R. 1966. "The Strategy of Model Building in Population Biology." *American Scientist* 54: 421–31. <https://www.jstor.org/stable/27836590>.
- Levins, Richard. 1993. "A Response to Orzack and Sober: Formal Analysis and the Fluidity of Science." *Quarterly Review of Biology* 68 (4): 547–55.
- Orzack, Steven Hecht, and Elliott Sober. 1993. "A Critical Assessment of Levins's the Strategy of Model Building in Population Biology (1966)." *Quarterly Review of Biology* 68 (4): 533–46.
- Weisberg, Michael. 2007. "Forty Years of 'the Strategy': Levins on Model Building and Idealization." *Biology & Philosophy* 21 (5): 623–45. <https://doi.org/10.1007/s10539-006-9051-9>.