# *Epidemic models 1*

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## *motivation*

- P & I data from Philadelphia 1918 flu:

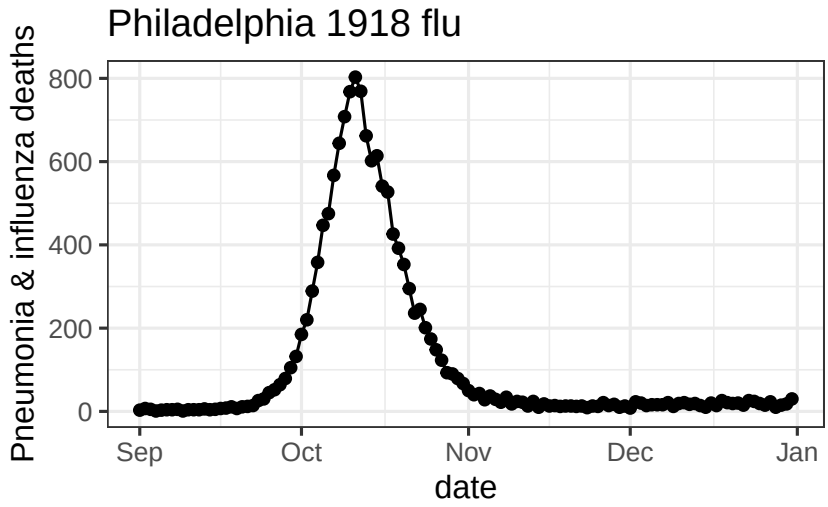


Figure 1: Phila. 1918 flu data

## *what do we want to figure out?*

## *what shall we assume?*

- classify individuals as $S$, $I$ (**compartmental** model; **microparasite** or **intensity-independent**)
- disease is transmitted from $S$ to $I$
- $S \to I$ instantaneously (zero latent period, no $E$)
- population is **homogeneous** (no heterogeneity in susceptibility, infectiousness, contact)
- fixed population size (birth = migration = 'natural' death = 0)
- transmission rate is time-invariant

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- assumption 2 is OK (Pasteur, Koch's postulates . . . )
- all the rest are approximations

  start simple!

- parsimony

- robustness?
- applicability/estimation?

Levins (1966) (also Orzack and Sober (1993), Levins (1993), Weisberg (2007))

## *exponential growth*

- one variable (=1D model)
- how does disease spread? $\rightarrow$ equation

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*what variables should we use?*

- time ($t$)
- state variable: incidence, prevalence, death rate, death toll (= cumulative death?)
- deaths loosely connected to transmission

but deaths are observed!

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when are deaths a good **proxy** for incidence?

- infection -> death time is fixed
- homogeneity? (might not matters?)
- mortality curve is shifted epidemic

(COVID context ... we observe case reports, number of tests, hospitalizations, and deaths)

- **incidence**: number of infections per unit time (rate or flow)
- **prevalence**: number of currently infected people (quantity or stock)

prevalence is closer to the **mechanism**

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model components:

- $I(t)$ (state variable: prevalence)
- $I(0)$ (initial conditions)
- $\beta$ (parameter) = avg contacts **per susceptible per infective per unit time**

$$I(t + \Delta t) \approx I(t) + \beta I(t) \Delta t$$

Take $\lim \Delta t \to 0$ (and solve):

$$\frac{dI}{dt} = \beta I \to I(t) = I(0) exp(\beta t)$$

*model criticism*

- Ignored discrete nature of individuals
- Ignored time-varying $\beta$ (e.g. **diurnal** fluctuations)
- Ignored finite infectious periods (recovery/death)

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**Next**: What if we make infectious periods finite? (i.e., including recovery (**clearance**) or death

$$dI/dt = \beta I - \gamma I$$

*mean infectious period*

$$I(t) = I(0) \exp(-\gamma t)$$

$$\text{proportion uninfected} = \exp(-\gamma t)$$

$$\text{proportion infected} = 1 - \exp(-\gamma t) (= \text{CDF} := C(t))$$

$$\text{PDF} : = C'(t) = \gamma \exp(-\gamma t)$$

$$\text{substitute } x = \gamma t \quad \to \quad dx = \gamma \, dt$$

$$\text{mean} = E[t] = \int t \exp(-\gamma t) \, dt = \int x \exp(-x) \, dx/\gamma = 1/\gamma$$

*dimensional analysis*

rates and characteristic times/scales

- is $I$ a proportion or a density or a number ... ?
- what are the units of $\beta$, $\gamma$ ?

*nondimensionalization*

- standardize any values that can be eliminated **without loss of (mathematical) generality**
- what can we do here?
- $\gamma = 1$
- $I$ ? (depends on how we have defined it initially) $\to I/N$

*compare with data???*

Original scale:

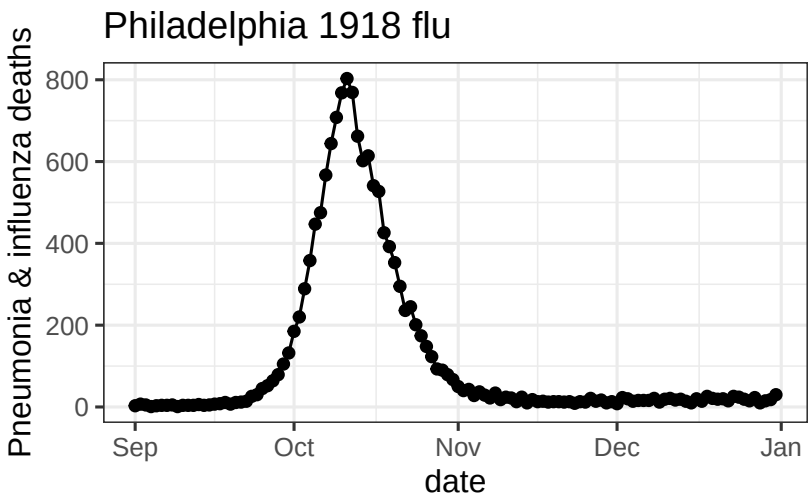


Figure 2: Philadelphia P&I

Log scale:

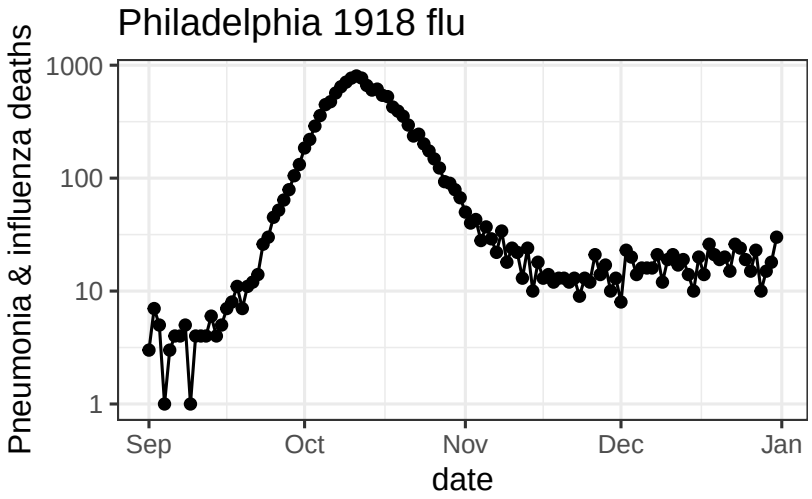


Figure 3: Philadelphia P&I, log scale

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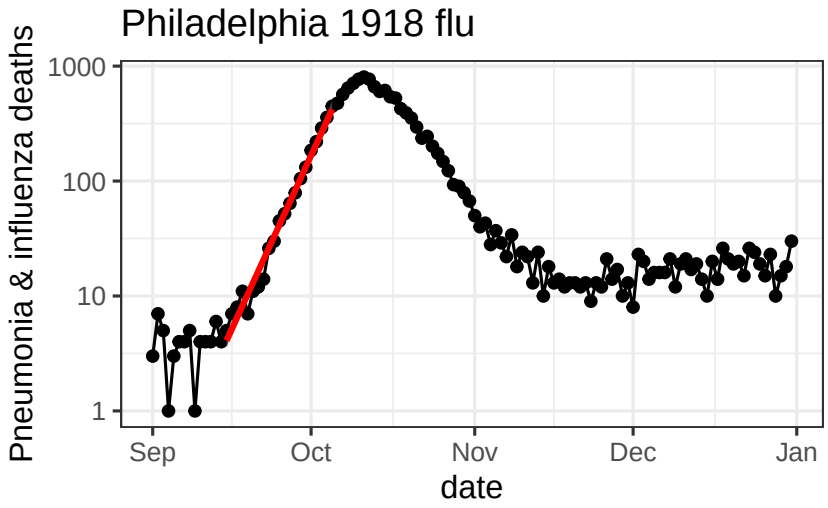
- Fit a straight line through the straight part of the curve
- slope is $\beta N$
- "intercept" is $\log(I(0))$ (zero is defined in a tricky way)




*model assessment*

- math is super-easy!
- clear, testable predictions
- parameter estimation is easy
- only consistent over a short time window
  - small $t$: arbitrarily close to zero
  - large $t$: ridiculous

## *Simple (SI) epidemic*

- what are we missing?
- **depletion of susceptibles**
- let's take a step back and ignore death & recovery for now

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$$dS/dt = -\beta SI$$
$$dI/dt = \beta SI$$

This looks 2D **but** what if we assume $S + I = N$ is constant? Then $S = N - I$

$$dI/dt = \beta(N - I)I$$

How do we solve this? **Partial fractions**

$$\frac{dI}{\beta(N-I)I} = dt$$

$$dI\left(\frac{A}{N-I} + \frac{B}{I}\right) = dI \cdot \frac{A + B(N-I)}{I(N-I)}$$

$$A = B; \quad B = 1/N$$

$$\frac{1}{\beta N}(-\log(N-I) + \log(I))\Big|_{I(0)}^{I} = t - t_0$$

$$(-\log(N-I) + \log(I))\Big|_{I(0)}^{I} = (\beta N)(t - t_0) \quad (\text{set } t_0 = 0)$$

$$\log\left(\frac{I}{N-I}\right) - \log\left(\frac{I(0)}{N-I(0)}\right) = \beta N t$$

$$\log\left(\frac{I}{N-I}\right) = \beta N t + -\log\left(\frac{I(0)}{N-I(0)}\right)$$

$$\frac{I}{N-I} = \exp(\beta N t)\frac{I(0)}{N-I(0)} \equiv Q$$

$$I = Q(N-I)$$

$$I(t)(1+Q) = QN$$

$$I(t) = \frac{QN}{1+Q} = \frac{N}{1+\frac{1}{Q}}$$

$$= \frac{N}{1 + \left(\frac{N-I(0)}{I(0)}\right)\exp(-\beta N t)}$$

(Comparison to metapop, logistic growth model)

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*references*


Levins, R. 1966. "The Strategy of Model Building in Population Biology." *American Scientist* 54: 421–31. https://www.jstor.org/stable/27836590.

Levins, Richard. 1993. "A Response to Orzack and Sober: Formal Analysis and the Fluidity of Science." *Quarterly Review of Biology* 68 (4): 547–55.

Orzack, Steven Hecht, and Elliott Sober. 1993. "A Critical Assessment of Levins's the Strategy of Model Building in Population Biology (1966)." *Quarterly Review of Biology* 68 (4): 533–46.

Weisberg, Michael. 2007. "Forty Years of 'the Strategy': Levins on Model Building and Idealization." *Biology & Philosophy* 21 (5): 623–45. https://doi.org/10.1007/s10539-006-9051-9.