Reeb graphs, Mapper graphs, and Metrics

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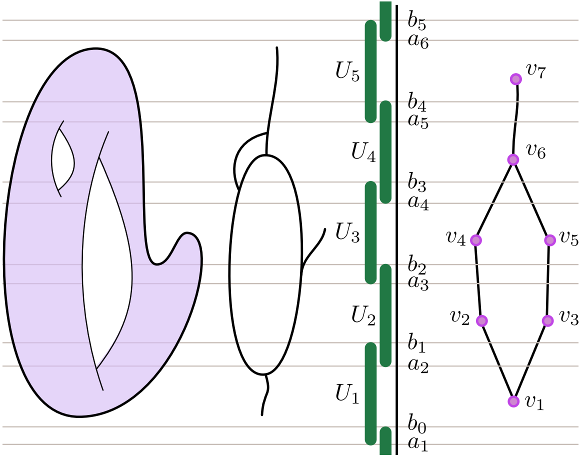
Reeb graphs and Mapper



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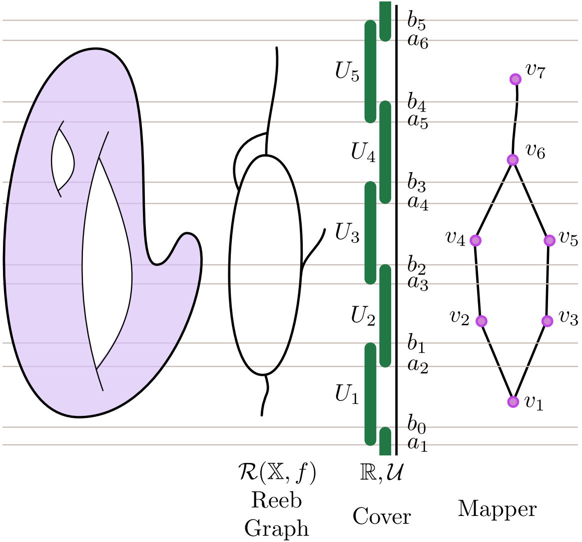

The point

- Useful for applications.
- Applications have noise.
- How do we understand distances and convergence?

Persistent Homology

$\mathbb{Z}$-parameterized
Given topological space $K = K_n$ and filtration

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$$

gives a sequence of maps on homology

$$H_k(K_0) \to H_k(K_1) \to \cdots \to H_k(K_n)$$

$\mathbb{R}$-parameterized
Given topological space K and filtration

$$\{K_a\}_{a \in \mathbb{R}} \text{ where } K_a \subseteq K_b \forall a \leq b$$

gives a collection of maps on homology

$$\varphi_a^b : H_k(K_a) \to H_k(K_b) \qquad \forall a \leq b$$

$$\varphi_b^c \circ \varphi_a^b = \varphi_a^c$$

Persistence Module

Definition

A persistence module $\mathcal{V} = (V_a, \varphi_a^b)$ is a collection of

- vector spaces V_a and
- linear maps $\varphi_a^b : V_a \to V_b$,
- such that $\varphi_a^a = \mathbb{1}_{V_a}$, and $\varphi_b^c \varphi_a^b = \varphi_a^c$.

Definition

A persistence module is a functor $\mathcal{V} : (\mathbf{R}, \leq) \to \mathbf{Vect}_k$.

Functorial Version

Equivalent definition

Persistence Modules

$\Leftrightarrow$

Functors

$$\begin{aligned} F : \quad \mathbb{R} &\to \mathbf{Vect} \\ t &\mapsto V_t \end{aligned}$$

$$\begin{aligned} G : \quad \mathbb{R} &\to \mathbf{Vect} \\ t &\mapsto W_t \end{aligned}$$

Functorial Version

Equivalent definition

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Morphisms

Natural transformations: φ: $F \Rightarrow G$

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d, "\varphi_1"] & V_2 \arrow[r] \arrow[d, "\varphi_2"] & V_3 \arrow[r] \arrow[d, "\varphi_3"] & \cdots \arrow[r] & V_k \arrow[d, "\varphi_k"] \\ W_1 \arrow[r] & W_2 \arrow[r] & W_3 \arrow[r] & \cdots \arrow[r] & W_k \end{tikzcd}$$

When are F and G the same?

Persistence Module Isomorphism

F and G are isomorphic if there exists a pair of natural transformations

$$\varphi\colon \mathrm{F} \Rightarrow \mathrm{G};\ \psi\colon \mathrm{G} \Rightarrow \mathrm{F}$$

such that each pair φ_a, ψ_a form an isomorphism of vector spaces.

$$\begin{tikzcd}
V_1 \arrow[r] \arrow[d, shift right, "\varphi_1"'] & V_2 \arrow[r] \arrow[d, shift right, "\varphi_2"'] & V_3 \arrow[r] \arrow[d, shift right, "\varphi_3"'] & \cdots \arrow[r] & V_k \arrow[d, shift right, "\varphi_k"'] \\
W_1 \arrow[r] \arrow[u, shift right, "\psi_1"'] & W_2 \arrow[r] \arrow[u, shift right, "\psi_2"'] & W_3 \arrow[r] \arrow[u, shift right, "\psi_3"'] & \cdots \arrow[r] & W_k \arrow[u, shift right, "\psi_k"']
\end{tikzcd}$$

When are F and G **almost** the same?

$$\begin{array}{rccc} S_\varepsilon : & \mathbb{R} & \to & \mathbb{R} \\ & a & \mapsto & a + \varepsilon \end{array}$$

$$\begin{array}{rccc} \mathcal{S}_\varepsilon : & \mathbf{Vect}^{\mathbb{R}} & \to & \mathbf{Vect}^{\mathbb{R}} \\ & \mathrm{F} & \mapsto & \mathrm{F}S_\varepsilon \end{array}$$

Persistence Module ε-interleaving

F and G are ε-interleaved if there exists a pair of natural transformations

$$\color{red}{\varphi}\color{black}{: \mathrm{F} \Rightarrow \mathcal{S}_\varepsilon(\mathrm{G});}\ \color{blue}{\psi}\color{black}{: \mathrm{G} \Rightarrow \mathcal{S}_\varepsilon(\mathrm{F})}$$

such that the diagram below commutes.

$$\begin{tikzcd}
V_1 \arrow[r] \arrow[drrr, red, "\varphi_1"] & V_2 \arrow[r] \arrow[drrr, red, "\varphi_2"] & \cdots \arrow[r] & V_{1+\varepsilon} \arrow[r] \arrow[drrr, red, "\varphi_{1+\varepsilon}"] & V_{2+\varepsilon} \arrow[r] \arrow[drrr, red, "\varphi_{2+\varepsilon}"] & \cdots \arrow[r] & V_{1+2\varepsilon} \arrow[r] & V_{2+2\varepsilon} \\
W_1 \arrow[r] \arrow[urrr, blue, "\psi_1"] & W_2 \arrow[r] \arrow[urrr, blue, "\psi_2"] & \cdots \arrow[r] & W_{1+\varepsilon} \arrow[r] \arrow[urrr, blue, "\psi_{1+\varepsilon}"] & W_{2+\varepsilon} \arrow[r] \arrow[urrr, blue, "\psi_{2+\varepsilon}"] & \cdots \arrow[r] & W_{1+2\varepsilon} \arrow[r] & W_{2+2\varepsilon}
\end{tikzcd}$$

Definition ($\mathbf{Vect}^{\mathbb{R}}$ interleaving)

Let $F, G : \mathbb{R} \to \mathbf{Vect}$ be given.
An ε-interleaving consists of two natural transformations

$$\varphi\colon F \Rightarrow \mathcal{S}_\varepsilon(G);\ \psi\colon G \Rightarrow \mathcal{S}_\varepsilon(F)$$

such that

$$\begin{tikzcd}
F \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "\varphi"] & \mathcal{S}_\varepsilon F \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "\mathcal{S}_\varepsilon[\varphi]"] & \mathcal{S}_{2\varepsilon} F \\
G \arrow[r, Rightarrow] \arrow[ur, Rightarrow, "\psi"'] & \mathcal{S}_\varepsilon G \arrow[r, Rightarrow] \arrow[ur, Rightarrow, "\mathcal{S}_\varepsilon[\psi]"'] & \mathcal{S}_{2\varepsilon} G
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commutes.

Definition ($\mathbf{Vect}^{\mathbb{R}}$ interleaving)

Let $\mathsf{F}, \mathsf{G} : \mathbb{R} \to \mathbf{Vect}$ be given.
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\mathsf{G} \arrow[rr, Rightarrow] \arrow[urr, Rightarrow, "\psi" near start] & & \mathcal{S}_\varepsilon \mathsf{G} \arrow[rr, Rightarrow] \arrow[urr, Rightarrow, "\mathcal{S}_\varepsilon[\psi]" near start] & & \mathcal{S}_{2\varepsilon} \mathsf{G}
\end{tikzcd}$$

commutes. The interleaving distance is defined to be

$$d_I(\mathsf{F}, \mathsf{G}) = \inf\{\varepsilon \mid \ \mathsf{F} \text{ and } \mathsf{G} \text{ are } \varepsilon\text{-interleaved}\}.$$

Properties of the Interleaving Distance for Pers

Theorem (Chazal et al. 2009, Lesnick 2015)

For pfd persistence modules,

$$d_B(Dgm(V), Dgm(W)) = d_I(V, W).$$

Corollary (Cohen-Steiner et al. 2007)

For nice enough functions $f, g : \mathbb{X} \to \mathbb{R}$,

$$d_I(\mathrm{Subl}(f), \mathrm{Subl}(g)) \leq \|f - g\|_\infty.$$

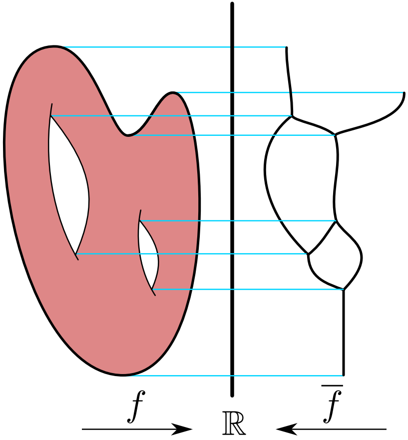
Section 1

Reeb graph Interleaving Distance

Reeb graph

Reeb Graph

- Given $f : \mathbb{X} \to \mathbb{R}$
- $x \sim y$ iff x and y in same (path) connected component of $f^{-1}(a)$.
- The Reeb graph of the function is the space $\mathbb{X}/\sim$ with the quotient topology.
- Denoted $\mathcal{R}(\mathbb{X}, f)$



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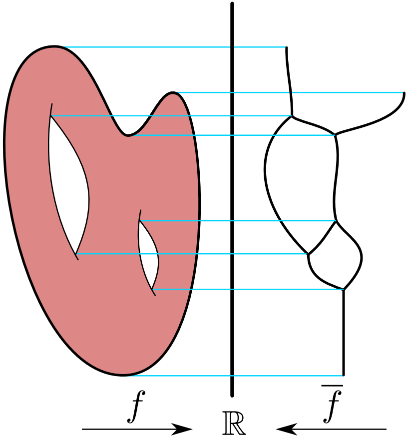
- The Reeb graph of a constructible $\mathbb{R}$-space is an $\mathbb{R}$-graph.
- A Reeb graph is itself an $\mathbb{R}$-space, so comes with a space *and* a function



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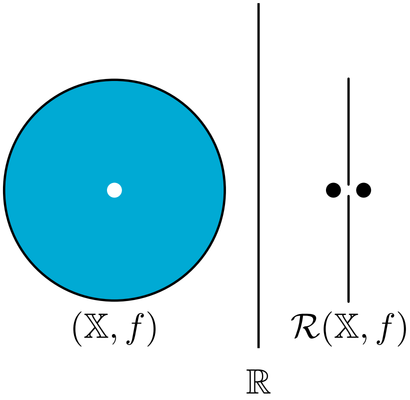

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$\mathbb{R}$-Spaces

Definition

An $\mathbb{R}$-space is a pair consisting of

- a topological space $\mathbb{X}$, and
- an $\mathbb{R}$ valued function $f : \mathbb{X} \to \mathbb{R}$.

This is denoted $(\mathbb{X}, f)$ or f.

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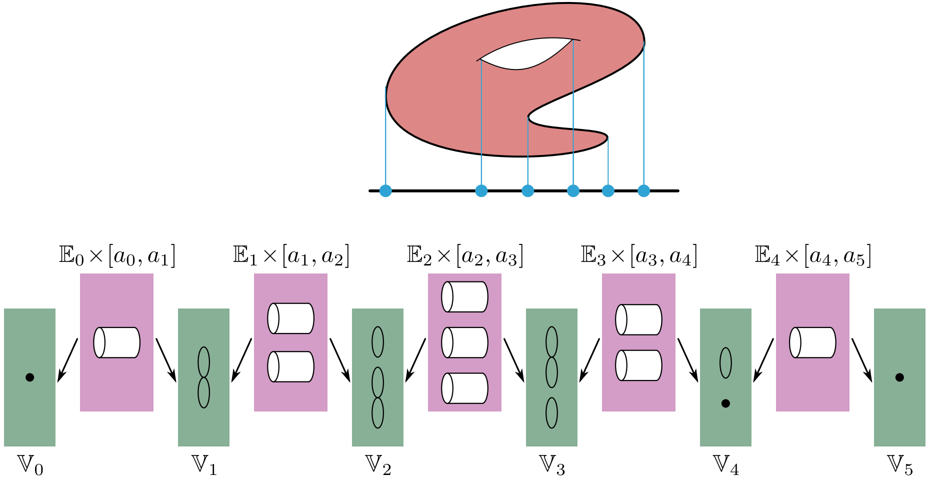
Definition

A **constructible** $\mathbb{R}$-space is an $\mathbb{R}$-space isomorphic to one constructed as follows:

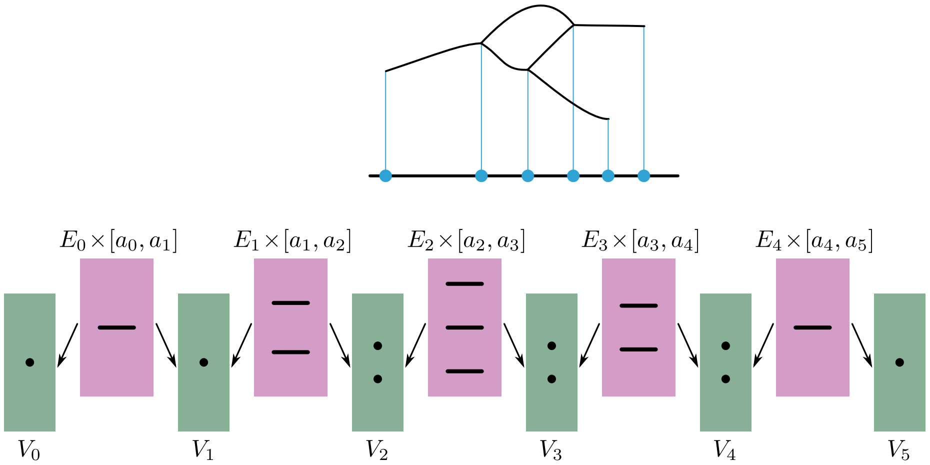
- $S = \{a_0, \cdots, a_n\}$ the set of critical points
- $0 \leq i \leq n$: $\mathbb{V}_i \times \{a_i\}$
- $0 \leq i \leq n-1$: $\mathbb{E}_i \times [a_i, a_{i+1}]$
- Attaching maps

$$\begin{tikzcd}
& \mathbb{E}_i \arrow[dl] \arrow[dr] & \\
\mathbb{V}_i & & \mathbb{V}_{i+1}
\end{tikzcd}$$

$\mathbb{R}$-Spaces

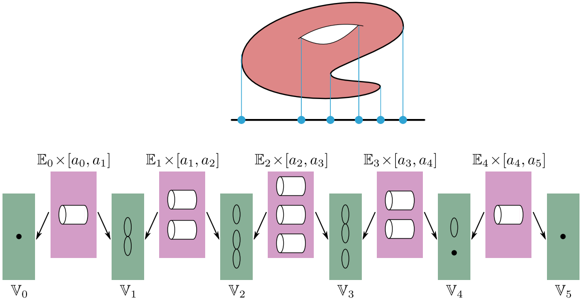


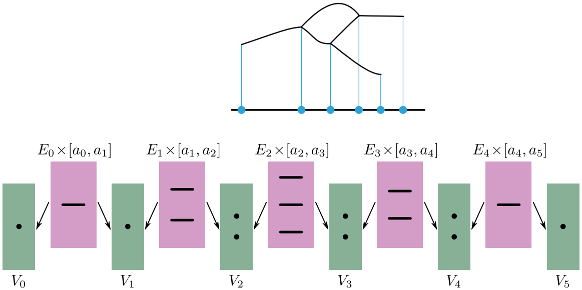
$\mathbb{R}$-Spaces



Definition

An $\mathbb{R}$-**graph** is a constructible $\mathbb{R}$-space where all $\mathbb{V}_i$ and $\mathbb{E}_i$ are 0-dimensional.



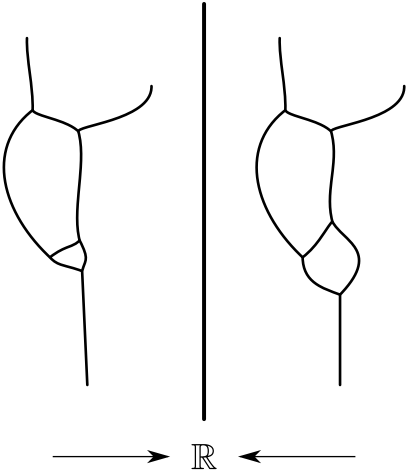


Definition

A *function preserving map* between two $\mathbb{R}$-spaces $(\mathbb{X}, f)$ and $(\mathbb{Y}, g)$ is a continuous map $\varphi : \mathbb{X} \to \mathbb{Y}$ such that

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & \mathbb{Y} \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd}$$

commutes.



Generalized Reeb Graphs

Definition

The set of

- Objects: $\mathbb{R}$-graphs $(\mathbb{X}, f)$
- Morphisms: Function preserving maps $\alpha : \mathbb{X} \to \mathbb{Y}$ such that

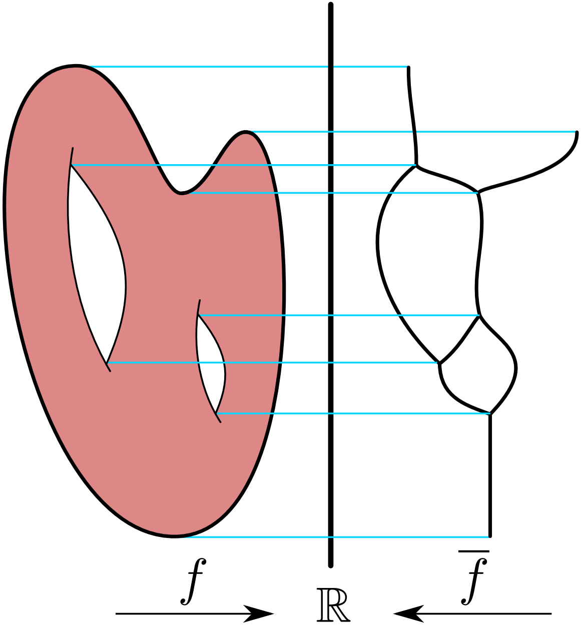
$$\begin{tikzcd} \mathbb{X} \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & \mathbb{Y} \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd}$$

  commutes.

is a category which we will call **Reeb**.

Want:

Categorify Reeb graphs



Definition

A **pre-cosheaf** is a functor

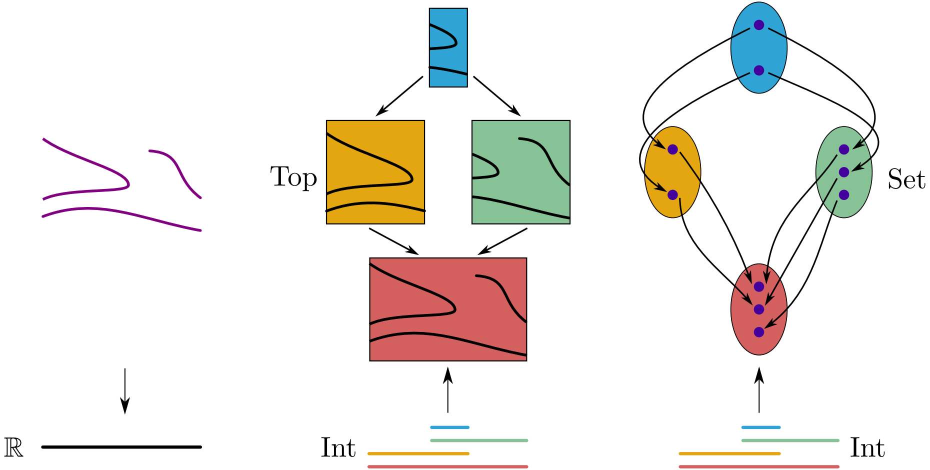
$$F : \mathbf{Int} \to \mathbf{Set}.$$

Definition

A pre-cosheaf $F : \mathbf{Int} \to \mathbf{Set}$ is a **cosheaf** if for all open $U \subset \mathbb{R}$ and covering $\{U_i\}$ of U, $F(U)$ is the colimit of the diagram

$$\coprod F(U_i \cap U_j) \rightrightarrows \coprod F(U_i)$$

Cosheaves



Constructible Cosheaves

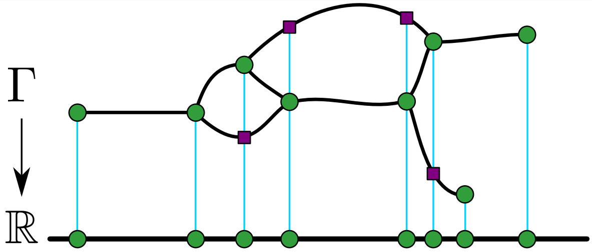
Definition

A cosheaf is S-constructible if it is compactly supported and

$$I \cap S = J \cap S$$

implies

$F[I \subset J] : F(I) \to F(J)$ is an isomorphism.



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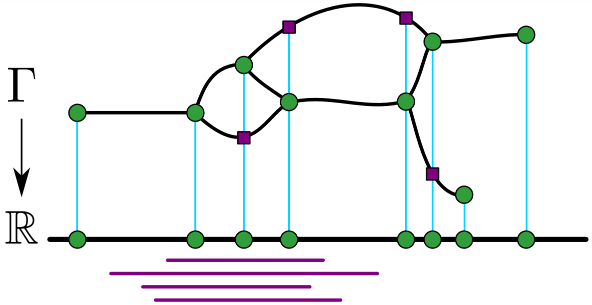


Categorical Reeb Graph

Definition

$\mathbf{Csh}^{\mathrm{c}}$ consists of

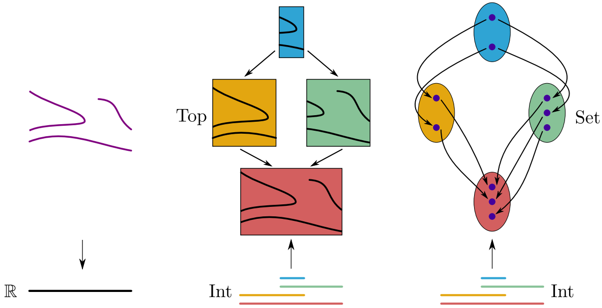
- Objects: Constructible cosheaves $F : \mathbf{Int} \to \mathbf{Set}$
- Morphisms: Natural transformations

The Reeb functor and construction

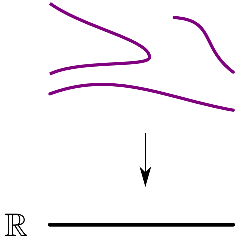
$$\mathbf{Reeb} \xrightarrow{\mathcal{C}} \mathbf{Csh}^c$$

$$\begin{tikzcd}
\mathbf{Int} \arrow[r, "f^{-1}"] \arrow[dr, "\mathcal{C}(f)"'] & \mathbf{Top} \arrow[d, "\pi_0"] \\
& \mathbf{Set}
\end{tikzcd}$$

Given Reeb graph $(\mathbb{X}, f)$,
$f : \mathbb{X} \to \mathbb{R}$



Equivalence of Categories

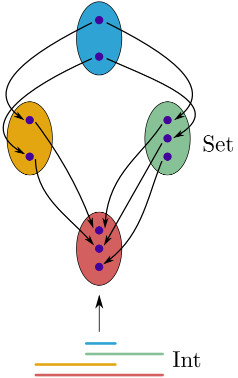


Theorem (Woolf; MacPherson; etc.)

The functor

$$\mathbf{Reeb} \xrightarrow{\mathcal{C}} \mathbf{Csh}^{c}$$

gives an equivalence of categories.



ε-Smoothing

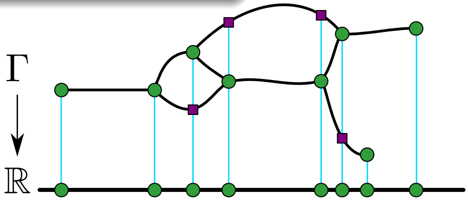
Definition

$$\begin{array}{cccc} \Omega_\varepsilon : & \mathbf{Int} & \longrightarrow & \mathbf{Int} \\ & J & \longmapsto & J^\varepsilon \\ & (a, b) & & (a-\varepsilon, b+\varepsilon) \end{array}$$

Definition

$$\begin{array}{cccc} \mathcal{S}_\varepsilon : & \mathbf{Csh}^c & \longrightarrow & \mathbf{Csh}^c \\ & F & \longmapsto & F\Omega_\varepsilon \end{array}$$

$$\begin{tikzcd}
F \arrow[d] \arrow[dr, red, "\varphi"' near start] & G \arrow[d] \arrow[dl, blue, "\psi" near start] \\
\mathcal{S}_\varepsilon F \arrow[d] \arrow[dr, red, "\varphi_\varepsilon"' near start] & \mathcal{S}_\varepsilon G \arrow[d] \arrow[dl, blue, "\psi_\varepsilon" near start] \\
\mathcal{S}_{2\varepsilon} F & \mathcal{S}_{2\varepsilon} G
\end{tikzcd}$$



$\mathbf{Csh}^{\mathrm{c}}$-interleaving

Definition

Let $\mathsf{F}, \mathsf{G} : \mathbf{Int} \to \mathbf{Set}$ be given.
An ε-interleaving consists of two natural transformations

$$\varphi\colon \mathsf{F} \Rightarrow \mathcal{S}_\varepsilon(\mathsf{G});\ \psi\colon \mathsf{G} \Rightarrow \mathcal{S}_\varepsilon(\mathsf{F})$$

such that

$$\begin{tikzcd}
\mathsf{F} \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "\varphi" near start] & \mathcal{S}_\varepsilon \mathsf{F} \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "\mathcal{S}_\varepsilon[\varphi]" near start] & \mathcal{S}_{2\varepsilon} \mathsf{F} \\
\mathsf{G} \arrow[r, Rightarrow] \arrow[ur, Rightarrow, "\psi"' near start] & \mathcal{S}_\varepsilon \mathsf{G} \arrow[r, Rightarrow] \arrow[ur, Rightarrow, "\mathcal{S}_\varepsilon[\psi]"' near start] & \mathcal{S}_{2\varepsilon} \mathsf{G}
\end{tikzcd}$$

commutes. The interleaving distance is defined to be

$$d_I(\mathsf{F}, \mathsf{G}) = \inf\{\varepsilon \mid \mathsf{F} \text{ and } \mathsf{G} \text{ are } \varepsilon\text{-interleaved}\}.$$

Definition (Reeb interleaving)

Let $f : \mathbb{X} \to \mathbb{R}$ and $g : \mathbb{Y} \to \mathbb{R}$ in **Reeb** be given.

Then
$(\mathbb{X}, f)$ and $(\mathbb{Y}, g)$ are ε-interleaved
iff
$\mathcal{C}(\mathbb{X}, f)$ and $\mathcal{C}(\mathbb{Y}, g)$ are ε-interleaved.

The interleaving distance is defined to be

$$d_I(f, g) = \inf\{\varepsilon \geq 0 \mid \ \mathcal{C}(f) \text{ and } \mathcal{C}(g) \text{ are } \varepsilon\text{-interleaved}\}.$$

$$\textbf{Reeb} \xrightarrow{\mathcal{C}} \textbf{Csh}^{\mathrm{c}}$$

Properties

Theorem (de Silva, EM, Patel 2016)

The interleaving distance is an extended metric.

$$d_I((\mathbb{X}, f), (\mathbb{Y}, g)) < \infty$$
$$\Leftrightarrow$$
$$\beta_0(\mathbb{X}) = \beta_0(\mathbb{Y})$$

Properties

Theorem (de Silva, EM, Patel 2016)

The interleaving distance is an extended metric.

$$d_I((\mathbb{X}, f), (\mathbb{Y}, g)) < \infty$$
$$\Leftrightarrow$$
$$\beta_0(\mathbb{X}) = \beta_0(\mathbb{Y})$$

Theorem (dS, EM, P 2016)

Given $f, g : \mathbb{X} \to \mathbb{R} \in \mathbb{R}\text{-}\mathbf{Top}^{\mathrm{c}}$,

$$d_I(\mathcal{R}(f), \mathcal{R}(g)) \leq \|f - g\|_\infty.$$

Section 2

Mapper

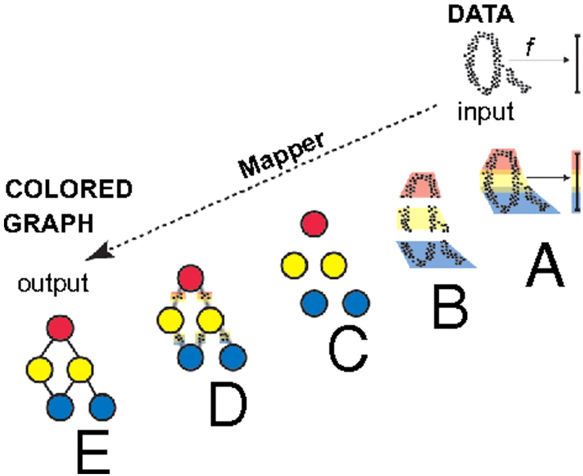
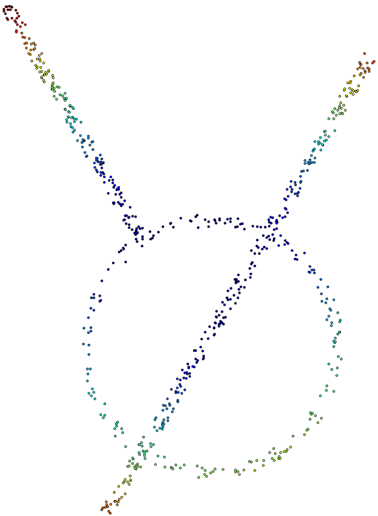
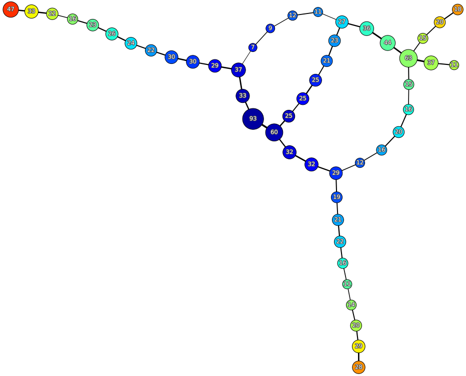


Image: Nicolau Levine Carlsson 2011

Bigger example





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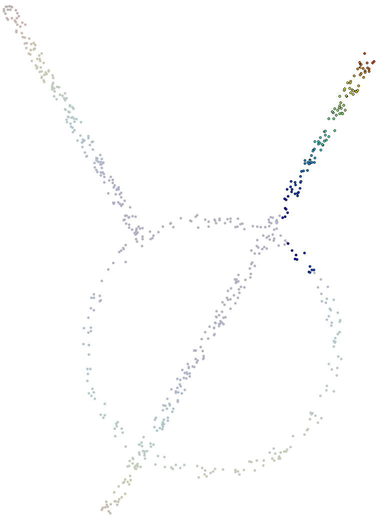

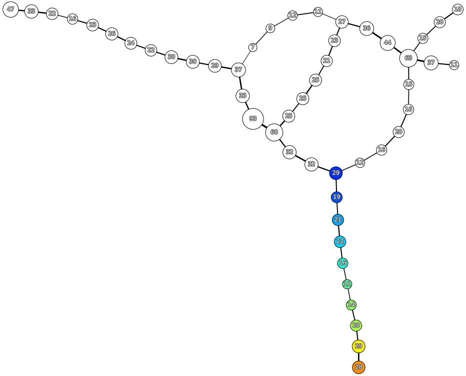

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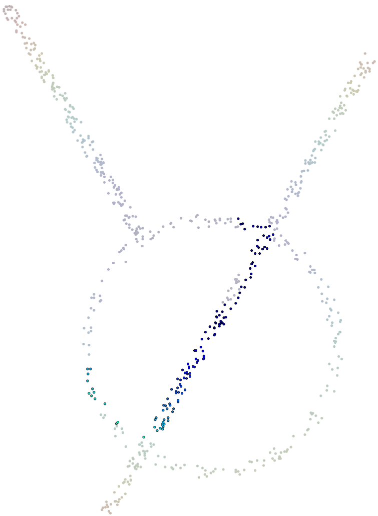
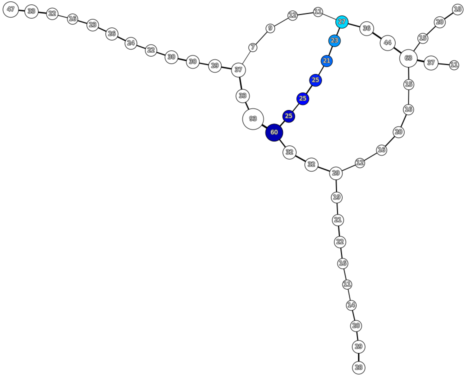




Ingredients

Data $+$ Function

Point cloud approximation
Original topological space

$\mathbb{R}$-valued
$\mathbb{R}^d$-valued

Ingredients

Data $+$ Function

Point cloud approximation
Original topological space

$\mathbb{R}$-valued
$\mathbb{R}^d$-valued

$+$ Cover Choice $+$ Clustering Choice

Ingredients

Data + Function

~~Point cloud approximation~~
Original topological space

$\mathbb{R}$-valued
$\mathbb{R}^d$-valued

+ Cover Choice + ~~Clustering Choice~~

Ingredients

Data + Function

~~Point cloud approximation~~
Original topological space

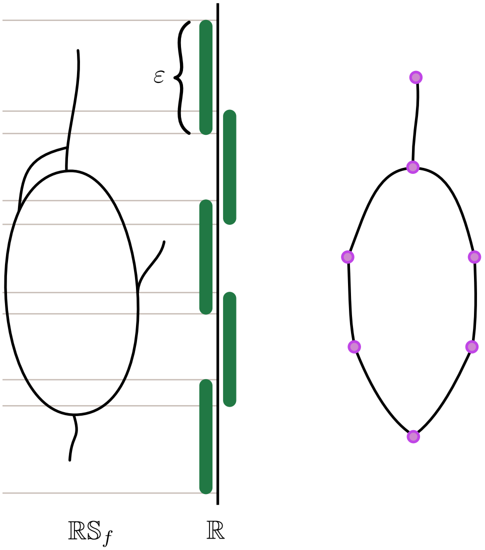
$\mathbb{R}$-valued
~~$\mathbb{R}^d$-valued~~

+ Cover Choice + ~~Clustering Choice~~

Mapper definition

Definition (Singh et al. 2007)

- Given $f : \mathbb{X} \to \mathbb{R}$.
- Fix a cover $\mathcal{U} = \{U_\alpha\}$ of $\mathbb{R}$.
- The collection $f^{-1}(\mathcal{U}) = \{f^{-1}(U_\alpha)\}$ is a cover of $\mathbb{X}$.
- Let $f^{-1}(\mathcal{U})^*$ be the cover which splits the sets into connected components.
- Then Mapper is the nerve of this cover.



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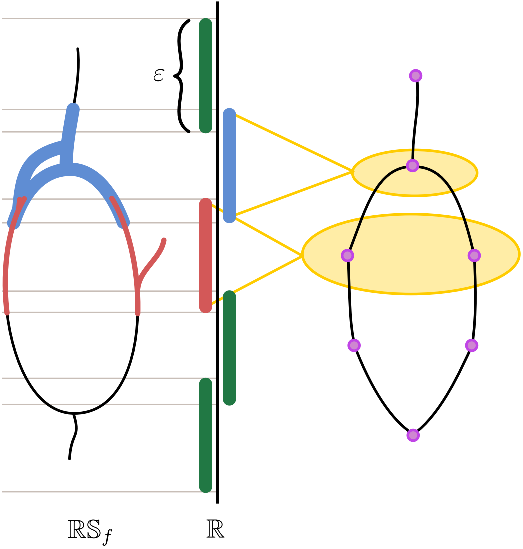
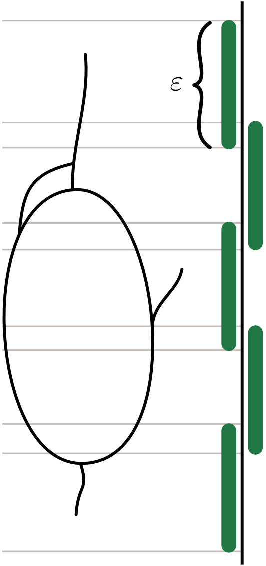
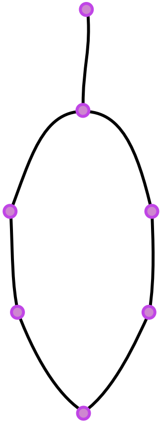
Definition (Singh et al. 2007)

- Given $f : \mathbb{X} \to \mathbb{R}$.
- Fix a cover $\mathcal{U} = \{U_\alpha\}$ of $\mathbb{R}$.
- The collection $f^{-1}(\mathcal{U}) = \{f^{-1}(U_\alpha)\}$ is a cover of $\mathbb{X}$.
- Let $f^{-1}(\mathcal{U})^*$ be the cover which splits the sets into connected components.
- Then Mapper is the nerve of this cover.



Mapper can be stored as data over nerve of cover





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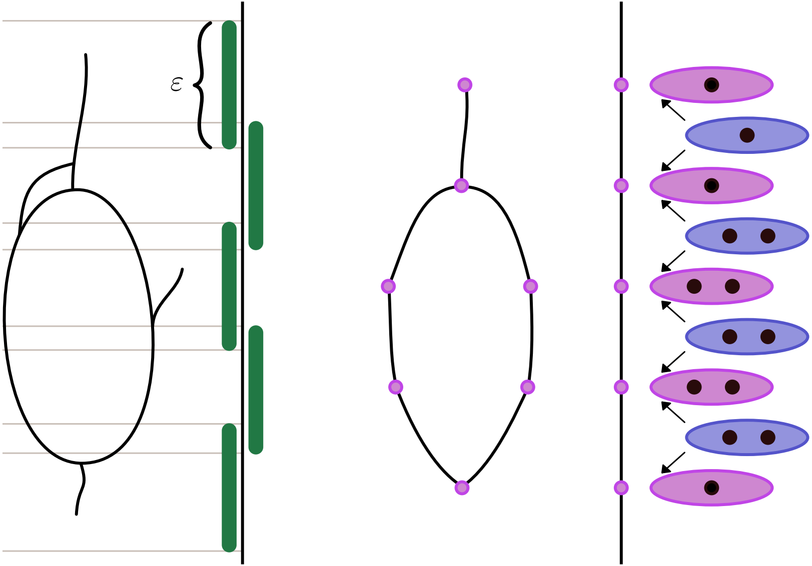

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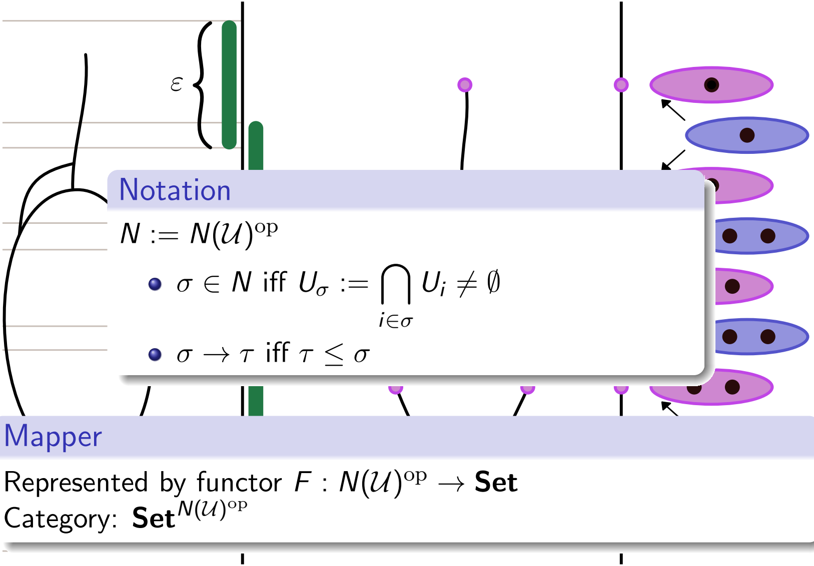


Notation

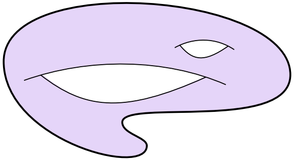
$N := N(\mathcal{U})^{\mathrm{op}}$

- $\sigma \in N$ iff $U_\sigma := \bigcap_{i \in \sigma} U_i \neq \emptyset$
- $\sigma \to \tau$ iff $\tau \leq \sigma$

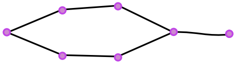
Mapper

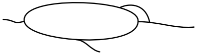
Represented by functor $F : N(\mathcal{U})^{\mathrm{op}} \to \mathbf{Set}$
Category: $\mathbf{Set}^{N(\mathcal{U})^{\mathrm{op}}}$

Big Picture

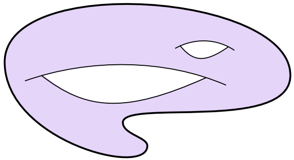


$$\begin{tikzcd} & \mathbb{R}\text{–Top} \arrow[dl] \arrow[dr] & \\ \text{Mapper} & & \text{Reeb} \end{tikzcd}$$

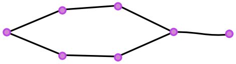


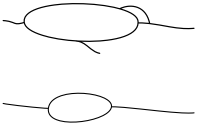


Big Picture



$$\begin{tikzcd} & \mathbb{R}-\mathbf{Top} \arrow[dl] \arrow[dr] & \\ \mathbf{Mapper} \arrow[rr] & & \mathbf{Reeb} \end{tikzcd}$$





Intuition

Mapper is an approximation of the Reeb graph.

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Question

How do we formalize this?

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Goal:

- Convergence
  - Turn Mapper into something that can be compared to the Reeb graph
  - Give bound on error for Mapper based on cover choice

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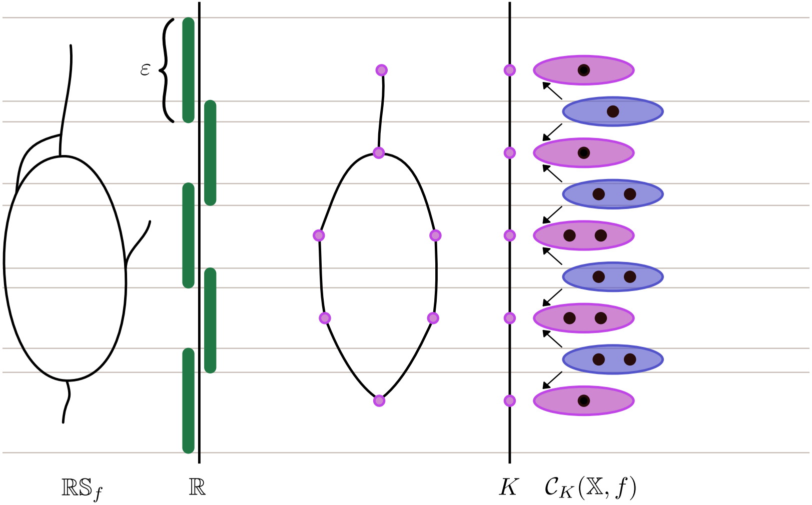
Question

How to do that?????

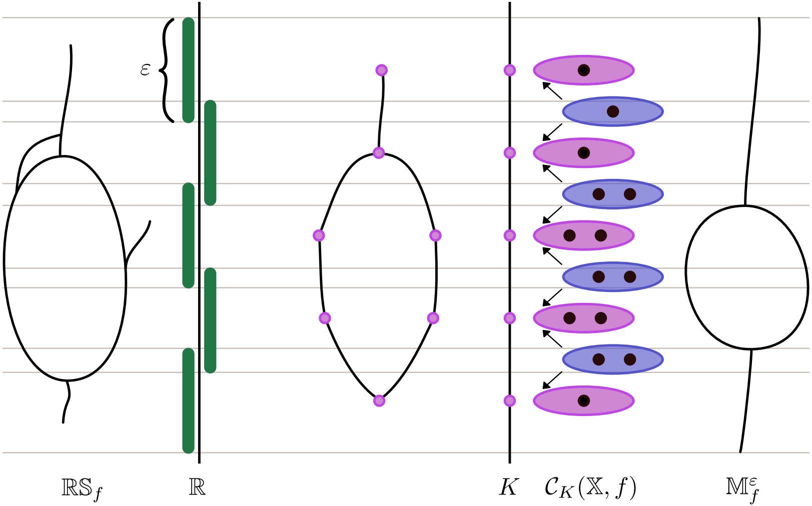
Answer:

Kan Extensions

Comparison requires continuous Mapper



Comparison requires continuous Mapper



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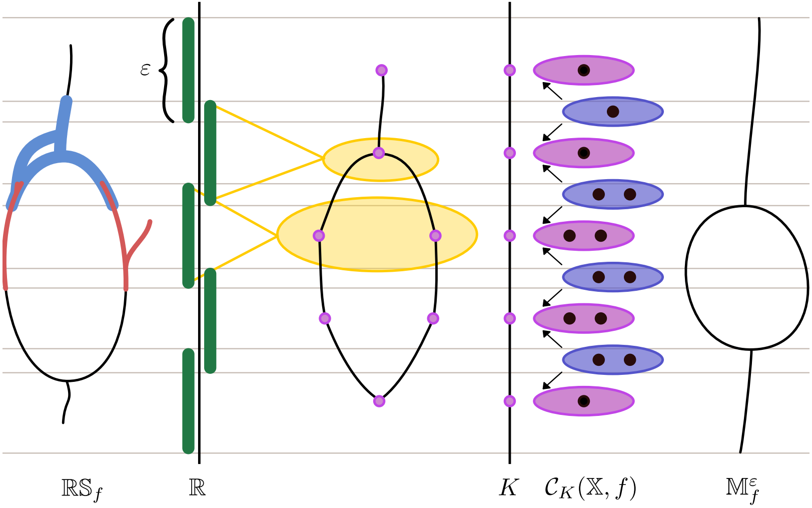

Building $\mathcal{S}$

$$
\begin{tikzcd}
& \mathbb{R}\text{−Top} \arrow[dl, "\mathcal{C}_{\mathcal{U}}"'] \arrow[dr, "\mathcal{C}"] & \\
\mathbf{Set}^{N(\mathcal{U})^{\mathrm{op}}} \arrow[rr, "\mathcal{S}"] & & \mathbf{Csh}^{\mathrm{c}}
\end{tikzcd}
$$

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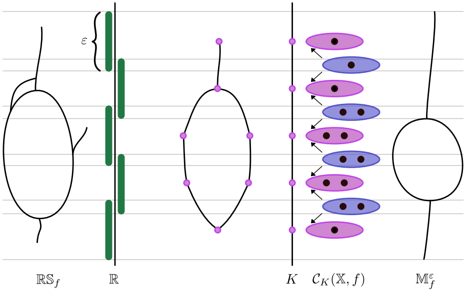

- $F : N(\mathcal{U})^{\mathrm{op}} \to \mathbf{Set}$

Building $\mathcal{S}$

$$\begin{tikzcd}
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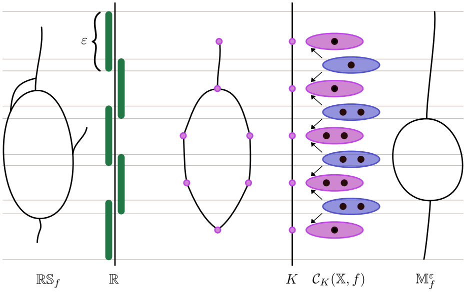

- $F : N(\mathcal{U})^{\mathrm{op}} \to \textbf{Set}$
- $N(\mathcal{U}) \cap A$

$$= \left\{ \sigma \in A \;\middle|\; \bigcap_{\alpha \in \sigma} U_\alpha \cap A \neq \emptyset \right\}$$

Building $\mathcal{S}$

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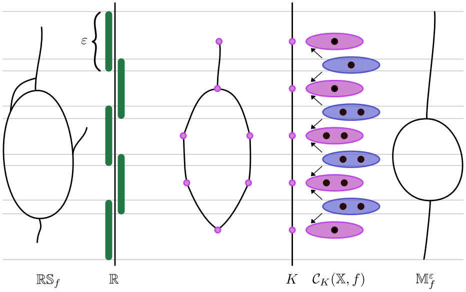


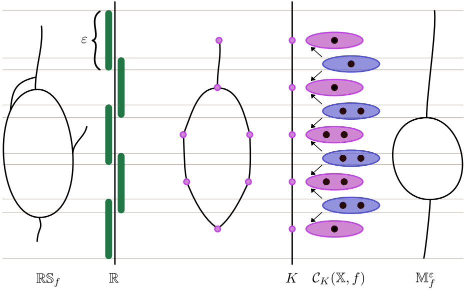
- $F : N(\mathcal{U})^{\mathrm{op}} \to \mathbf{Set}$
- $N(\mathcal{U}) \cap A$
  $$= \left\{ \sigma \in A \;\middle|\; \bigcap_{\alpha \in \sigma} U_\alpha \cap A \neq \emptyset \right\}$$
- $\mathcal{S}(F)(A) = \mathrm{colim}_{\sigma \in N(\mathcal{U}) \cap A} F(A)$

Results

Theorem (EM, B. Wang 2016)

Given a (nice enough) $\mathbb{R}$-space $f : \mathbb{X} \to \mathbb{R}$, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a good cover of $f(\mathbb{X}) \subseteq \mathbb{R}$, $\operatorname{res}(\mathcal{U}) = \max\{\operatorname{diam}(U_\alpha)\}$ Then

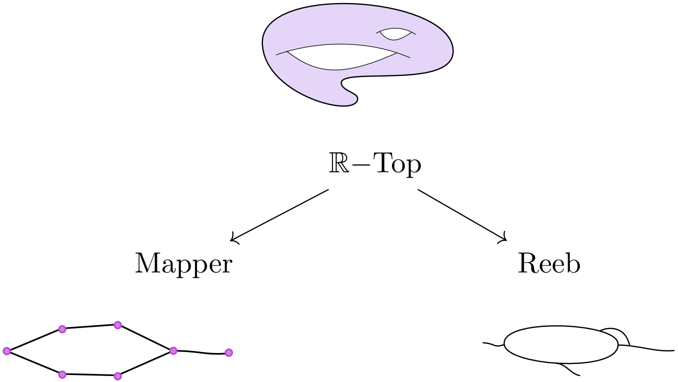
$$d_I(\mathcal{C}(f), \mathcal{SC}_{\mathcal{U}}(f)) \leq \operatorname{res}(\mathcal{U})$$



Section 3

Poset Interleavings

Big Picture



Reeb Graph Interleaving Distance

Definition (Reeb graph interleaving - de Silva, Patel, EM; Curry)

Let $F, G : \mathbf{Int} \to \mathbf{Set}$.
Let $S_\varepsilon : \mathbf{Int} \to \mathbf{Int}$, $U \mapsto U^\varepsilon := \{x \mid d(x, U) < \varepsilon\}$.
An ε-interleaving consists of natural transformations φ and ψ such that

$$
\begin{tikzcd}
F \arrow[r] \arrow[dr, "\varphi" near start] & FS_\varepsilon \arrow[r] \arrow[dr, "\varphi^\varepsilon" near start] & FS_{2\varepsilon} \\
G \arrow[r] \arrow[ur, "\psi" near start] & GS_\varepsilon \arrow[r] \arrow[ur, "\psi_\varepsilon" near start] & GS_{2\varepsilon}
\end{tikzcd}
$$

commutes. The interleaving distance is defined to be

$$d_I(F, G) = \inf\{\varepsilon \mid F \text{ and } G \text{ are } \varepsilon\text{-interleaved}\}.$$

$$\begin{tikzcd}
& \mathbb{R}\text{--Top} \arrow[dl, "\mathcal{C}_{\mathcal{U}}"'] \arrow[dr, "\mathcal{C}"] & \\
\mathbf{Set}^{N(\mathcal{U})^{\mathrm{op}}} \arrow[rr, "\mathcal{S}"] & & \mathbf{Csh}^{\mathrm{c}}
\end{tikzcd}$$

New vantage point

$$
\begin{tikzcd}
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\mathbf{Set}^{N(\mathcal{U})^{\mathrm{op}}} \arrow[rr, "\mathcal{S}"] & & \mathbf{Csh}^{\mathrm{c}}
\end{tikzcd}
$$

Definition (Mapper interleaving???)

Let $F, G : N(\mathcal{U})^{\mathrm{op}} \to \mathbf{Set}$ be given.
Let $T_\varepsilon : N(\mathcal{U})^{\mathrm{op}} \to N(\mathcal{U})^{\mathrm{op}}$ defined by ¯_(ツ)_/¯
An ε-interleaving consists of natural transformations φ and ψ such that

$$
\hat{F} \longrightarrow \hat{F}T_\varepsilon \longrightarrow \hat{F}T_{2\varepsilon}
$$

φ φ^ε

Extending poset using Alexandrov topology

$$\begin{aligned} D: \quad \mathbf{Poset} &\to \mathbf{Poset} \\ \mathcal{P} &\mapsto D(\mathcal{P}) := \left\{ X \subseteq \mathcal{P} \,\middle|\, \begin{array}{r} x \in X, y \le x \\ \Rightarrow y \in X \end{array} \right\} \end{aligned}$$

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\begin{array}{rccl}
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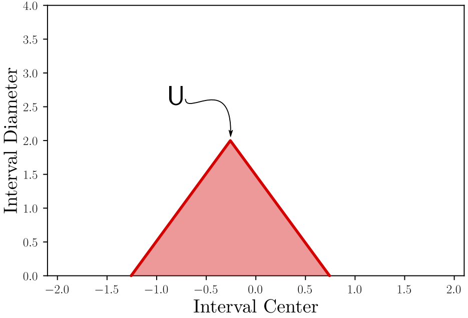
Example: $\mathcal{P} = \textbf{Open}(\mathbb{R})$



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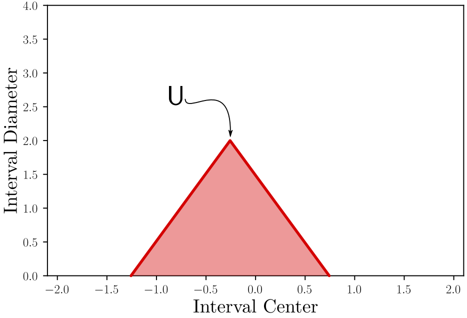
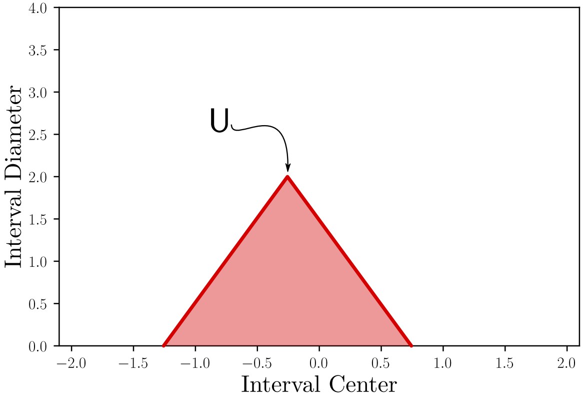


$$\begin{tikzcd} \mathcal{P} \arrow[r, "F"] \arrow[d, "\iota_N"'] & \mathsf{C} \\ D(\mathcal{P}) \arrow[ur, dashed, "\hat{F} := \mathrm{colim}_{V \in -} F(V)"'] & \end{tikzcd}$$

Definition of spreading function :: **Open**($\mathbb{R}$)

$$\begin{array}{rccc} T_\varepsilon : & D(\mathbf{Open}(\mathbb{R})) & \longrightarrow & D(\mathbf{Open}(\mathbb{R})) \\ & D(U) & \longmapsto & D(U^\varepsilon) = D(\{x \in \mathbb{R} \mid \|x - U\| < \varepsilon\}) \end{array}$$



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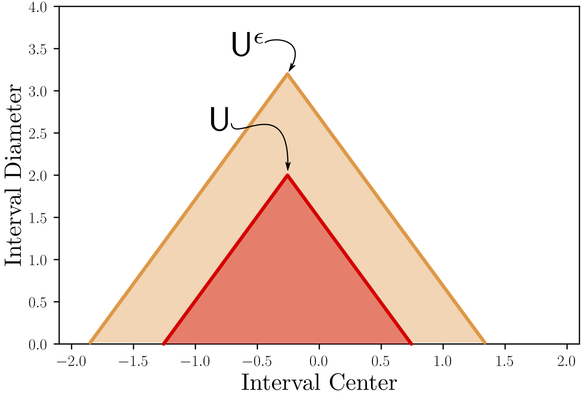
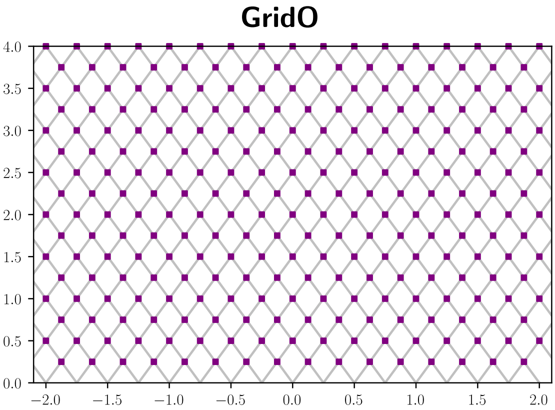
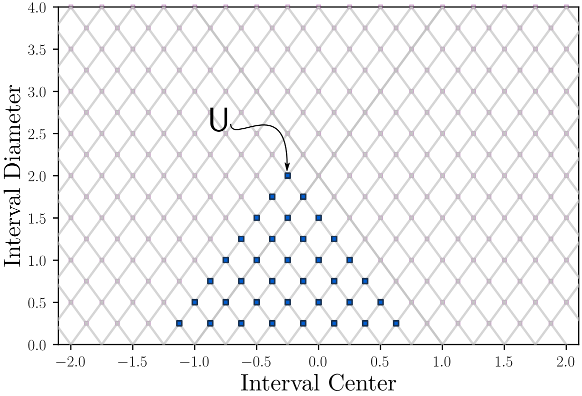


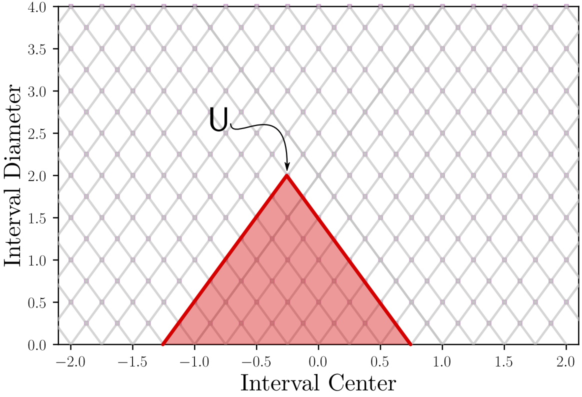
A nice cover



Definition of spreading function :: $\mathbf{GridO}_\delta$



$$
\begin{tikzcd}
D(\mathbf{GridO}) & D(\mathbf{Open}(\mathbb{R})) \arrow[l] \\
D(\mathbf{GridO}) \arrow[r] \arrow[u, dashed, "\hat{T}_\varepsilon"] & D(\mathbf{Open}(\mathbb{R})) \arrow[u, "T_\varepsilon"]
\end{tikzcd}
$$



$$
\begin{tikzcd}
D(\mathbf{GridO}) & D(\mathbf{Open}(\mathbb{R})) \arrow[l] \\
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$$

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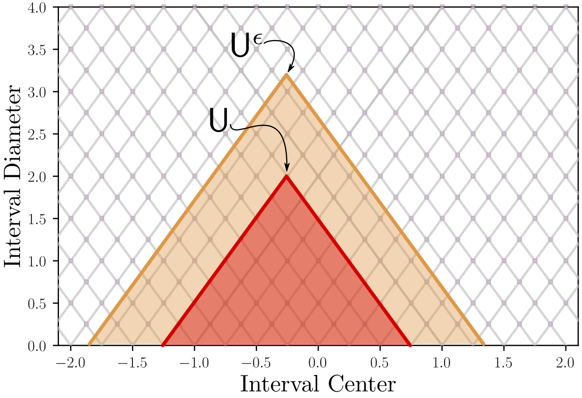

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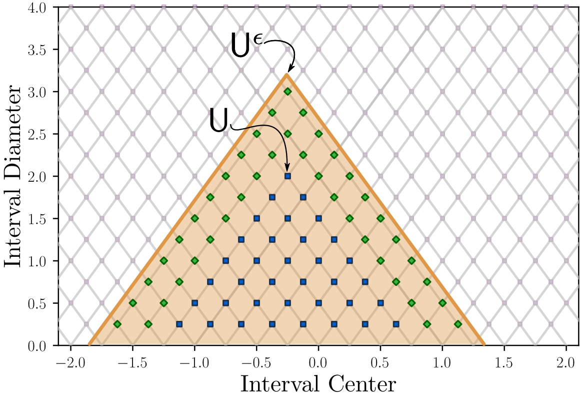

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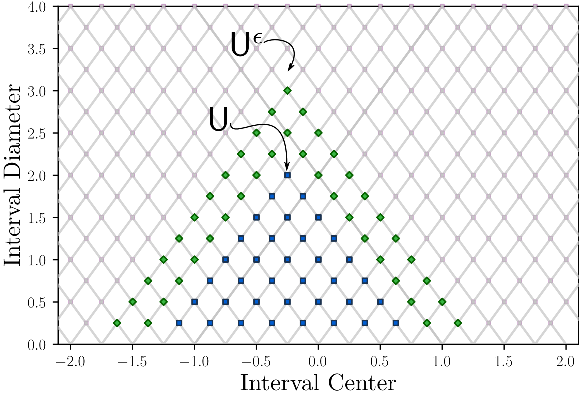

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D(\mathbf{GridO}) & D(\mathbf{Open}(\mathbb{R})) \arrow[l] \\
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\end{tikzcd}$$

Definition of the interleaving distance

Definition (Mapper interleaving)

Let $F, G : N \to \textbf{Set}$ be given. $\hat{F}, \hat{G} : D(N) \to \textbf{Set}$
An ε-interleaving consists of natural transformations $\varphi : \hat{F} \to \hat{F}T_\varepsilon$ and $\psi : \hat{G} \to \hat{G}T_\varepsilon$ such that

$$\begin{tikzcd}
\hat{F} \arrow[r] \arrow[dr, "\varphi"] & \hat{F}\hat{T}_\varepsilon \arrow[r] \arrow[dr, "\varphi^\varepsilon"] & \hat{F}\hat{T}_{2\varepsilon} \\
\hat{G} \arrow[r] \arrow[ur, "\psi"] & \hat{G}\hat{T}_\varepsilon \arrow[r] \arrow[ur, "\psi_\varepsilon"] & \hat{G}\hat{T}_{2\varepsilon}
\end{tikzcd}$$

commutes. The interleaving distance is defined to be

$$d_I(F, G) = \inf\{\varepsilon \mid F \text{ and } G \text{ are } \varepsilon\text{-interleaved}\}.$$

Mapper approximates Reeb: Answer #2

Theorem (Botnan, Curry, EM, 2018)

If $f : \mathcal{Q} \to \mathcal{P}$ is a δ-approximation that respects cosheaves and M and N are $\mathcal{P}$-modules, then

$$\left| d_I^T(M,N) - d_I^{\hat{T}}(f^*M, f^*N) \right| \leq \delta.$$

Mapper approximates Reeb: Answer #2

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$$\left| d_I^T(M,N) - d_I^{\hat{T}}(f^*M, f^*N) \right| \leq \delta.$$

Corollary (Botnan, Curry, EM, 2018)

Let $\mathcal{U}$ be a cover of $\mathbb{R}$ with $\delta = \sup\{\mathrm{diam}\,(U) \mid U \in \mathcal{U}\}$,
and f : $\mathbf{GridO} \to \mathbf{Open}(R)$.
Given F, G : $\mathbf{Int} \to \mathbf{Set}$ *(Reeb graph)*
and $F \circ f, G \circ f$: $\mathbf{GridO} \to \mathbf{Set}$. *(Mapper approximation)*

$$|d_I(F,G) - d_I(F \circ f, G \circ f)| \leq \delta.$$

Questions

- What can we figure out about convergence?
- Topological properties of the metric space $C^{\mathcal{Q}}$ vs $C^{\mathcal{P}}$ for $\mathcal{Q} \subseteq \mathcal{P}$
- Algorithms and computation
- What happens for incomparable covers
- Other cases where this framework is applicable
- Relationship with other stability ideas (Gromov-Hausdorff, Bottleneck distance)

Thank you!

Relevant papers

- VdS, AP, EM. *Categorified Reeb Graphs*, DCG 2016.
- EM, BW. *Convergence between Convergence between Categorical Representations of Reeb Space and Mapper*, SoCG 2016.
- EM, AS. *The ℓ^∞-Cophenetic Metric for Phylogenetic Trees as an Interleaving Distance*, arXiv:1803.07609, 2018.
- VdS, EM, AS. *Theory of interleavings on $[0,\infty)$-actegories*, arXiv:1706.04095, 2017.

Collaborators:

Teaspoon code available:

gitlab.msu.edu/TSAwithTDA/teaspoon

elizabethmunch.com
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CMSE
Dept. of Computational Mathematics, Science, and Engineering