INTERLEAVING DISTANCE OF REEB GRAPHS

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Abstract. This paper is going to talk about Graph Neural Networks and its possible applications to the Reeb Graph

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1. Introduction

1.1. **What is a Scalar Field?** Suppose we want to measure the temperature at of the surface of the Earth over a large region. We can think of this region as simply being a two-manifold ignoring the possible changes in height as we sample and only taking into consideration the geographic coordinates. Denote this space as $\mathbb{X}$. Then we also have a function $f : \mathbb{X} \to \mathbb{R}$ mapping from this two-manifold to the real numbers by associating the coordinate pair with the temperature at this point. The pair $(\mathbb{X}, f)$ is called a **scalar field**. We can often visualize well-behaved scalar fields as some sort of mountainous or hilly terrain. Just as we can use contour lines on a map to indicate regions that are the same elevation, we are sometimes interested in a subset $U \in \mathbb{X}$ such that all points in U have the same function value. These are called **level sets** or **fibers** of the scalar field.

1.2. **Comparing Scalar Fields.**

1.3. **Topological Evolution of a Scalar Field.** Let $(\mathbb{X}, f)$ be a scalar field. We can define an equivalence relation on $\mathbb{X}$ by defining $x \sim y$ if $f(x) = f(y) = a$ (they belong to the same a-fiber) and if x, y are in the same path connected component of the a-fiber. Then, the quotient space $\mathbb{X}_f := \mathbb{X}/\sim_f$ is called the **Reeb Graph** of the scalar field $(\mathbb{X}, f)$. We equip the Reeb Graph with the function $\tilde{f}$ which is defined as $\tilde{f} \circ \rho = f$, where $\rho : \mathbb{X} \to \mathbb{X}_f$ is the quotient map defined by our equivalence relation. The Reeb Graph tracks the evolution of the levelset topology of the scalar field. It shows how the fibers of the scalar field split and merge.

2. Categorifying Reeb Graphs and Scalar Fields

2.1. **Category Theory.**

Definition 2.1. A **category** is a collection $\mathbf{C}$ of **objects** with **morphisms** (also called **maps**) that map between these objects. The set of objects in $\mathbf{C}$ is denoted as $\mathsf{ob}(\mathbf{C})$ and the set of morphisms is denoted as $\mathsf{hom}(\mathbf{C})$. When referring to an object A in a category $\mathbf{C}$, we often denote this as $A \in \mathbf{C}$ instead of the more precise notation of $A \in \mathsf{ob}(\mathbf{C})$ when it is clear from the context that A is an object and not a morphism. Each morphism α has a source $A \in \mathbf{C}$ and a target $B \in \mathbf{C}$. Morphisms usually represent some sort of "structure" preserving idea, such as isomorphism when considering the category of vector spaces **Vect**, or homomorphisms in the category of groups **Grp**. Every object A in a category $\mathbf{C}$ has an identity morphsim $\mathbf{id}_A : A \to A$.

Definition 2.2. A **functor** F is a mapping between two categories $\mathbf{C}$ and $\mathbf{D}$ that satisfies the following properties:

- For each object $A \in \mathbf{C}$, there is a corresponding object $\mathsf{F}(A) \in \mathbf{D}$
- For each moprhism $\alpha : A \to B$ in $\mathbf{C}$, there is a corresponding morphism $\mathsf{F}[\alpha] : \mathsf{F}(A) \to \mathsf{F}(B)$ in $\mathbf{D}$.
- The functors respect composition: $\mathsf{F}[\alpha \circ \beta] = \mathsf{F}[\alpha] \circ \mathsf{F}[\beta]$
- The functors respect identities: $\mathsf{F}[\mathbf{id}_A] = \mathbf{id}_{\mathsf{F}[A]}$

A functor can be thought of as a way to map between categories in the same way that functions map between spaces (vector spaces, groups, etc.)

Definition 2.3. Let $\mathsf{F}, \mathsf{G} : \mathbf{A} \to \mathbf{B}$ be two functors. Let $\eta : \mathsf{F}(A) \to \mathsf{G}(C)$ and $\theta : \mathsf{F}(B) \to \mathsf{G}(D)$ be two morphisms in $\mathbf{B}$ and let $\alpha : A \to B$ and $\beta : C \to D$ be two morphisms in $\mathbf{A}$. Then we say that η and θ are **natural with respect to** α **and** β if the following diagram commutes:

$$\begin{tikzcd}
\mathsf{F}(A) \arrow[r, "\mathsf{F}[\alpha]"] \arrow[d, "\eta_A"'] & \mathsf{F}(B) \arrow[d, "\eta_B"] \\
\mathsf{G}(C) \arrow[r, "\mathsf{G}[\beta]"'] & \mathsf{G}(D)
\end{tikzcd}$$

Definition 2.4. A **natural transformation** is a map $\eta : \mathsf{F} \Rightarrow \mathsf{G}$ between two functors, $\mathsf{F}, \mathsf{G} : \mathbf{C} \to \mathbf{D}$. It consists of a family of morphisms $\eta_A : \mathsf{F}(A) \to \mathsf{G}(A)$, one for each object $A \in \mathbf{C}$, such that for each morphism $\alpha : A \to B$ in A, the following diagram commutes:

$$\begin{tikzcd} \mathsf{F}(A) \arrow[r, "\mathsf{F}[\alpha]"] \arrow[d, "\eta_A"'] & \mathsf{F}(B) \arrow[d, "\eta_B"] \\ \mathsf{G}(A) \arrow[r, "\mathsf{G}[\alpha]"'] & \mathsf{G}(B) \end{tikzcd}$$

In other words, we say that the family of maps η is natural with respect to every morphism α in $\mathbf{A}$.

Definition 2.5. We say that two functors $\mathsf{F}, \mathsf{G} : \mathbf{A} \to \mathbf{B}$ are **isomorphic** to each other if there exists a pair of natural transformations $\eta := \{\eta_A\}_{A \in \mathbf{A}}$ and $\theta := \{\theta_A\}_{A \in \mathbf{A}}$ where $\eta_A : \mathsf{F}(A) \to \mathsf{G}(A)$ and $\theta_A : \mathsf{G}(A) \to \mathsf{F}(A)$, such that $\eta_A \circ \theta_A = G[\mathbf{id}_A]$ and $\theta_A \circ \eta_A = F[\mathbf{id}_A]$. In otherwords, the maps η_A and θ_A are inverses of each other for each $A \in \mathbf{A}$.

Definition 2.6. Let $\mathbf{C}$ be a category. A **subcategory S** of $\mathbf{C}$ is a category consisting of a subcollection of objects of $\mathbf{C}$, denoted as $\mathsf{ob}(\mathbf{S})$ and a subcollection of morphisms of C, denoted as $\mathsf{hom}(\mathbf{S})$, such that

- for every X in $\mathsf{ob}(\mathbf{S})$, the identity morphism $\mathbf{id}_X$ is in $\mathsf{hom}(\mathbf{S})$
- for every morphism $\alpha : X \to Y$ in $\mathsf{hom}(\mathbf{S})$, both the source X and the target Y are in $\mathsf{ob}(\mathbf{S})$
- for every pair of morphisms α and β in $\mathsf{hom}(\mathbf{S})$, the composite $\alpha \circ \beta$ is in $\mathsf{hom}(\mathbf{S})$ (whenever the composition is defined).

2.2. **Category of Scalar Fields.**

Definition 2.7. We can construct the **category of scalar fields** (in related work also called the category of $\mathbb{R}$-spaces) denoted as $\mathbb{R}$-**Top** by stating that each object is a scalar field $(\mathbb{X}, f)$ and each morphism $\alpha : (\mathbb{X}, f) \to (\mathbb{Y}, g)$ is a continuous, **function-preserving** map $\alpha : \mathbb{X} \to \mathbb{Y}$ between topological spaces. A function preserving map means that $\alpha \circ g = f$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & \mathbb{Y} \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd}$$

Definition 2.8. We call the point pre-images $f^{-1}(a)$ of $\mathbb{X}$ the a-**levelset** or a-**fiber** of $\mathbb{X}$. In this way, we can say that the morphisms in the category of scalar fields preserve a-levelsets or a-fibers.

To make sure that we are dealing with well-behaved scalar fields, we define a new subcategory called the category of **constructible** scalar fields, denoted as $\mathbb{R}$-$\mathbf{Top}^{\mathbf{C}}$.

Let $S = \{a_1, \ldots, a_n\}$ be a finite set of real numbers, called "critical points". We define a **critical fiber**, $\mathbb{V}_i$, for each $0 \leq i \leq n$ and a **non-critical fiber**, $\mathbb{E}_i$ for each $0 \leq i \leq n-1$. Each critical and non-critical fiber is defined to a be a locally path-connected, compact space.

Now we define continuous attaching maps $\mathbb{l}_i : \mathbb{E}_i \to \mathbb{V}_i$ and $\mathbb{r}_i : \mathbb{E}_i \to \mathbb{V}_{i-1}$ for each $0 \leq i \leq n-1$. Finally, we define $\hat{\mathbb{X}}$ to be the quotient space obtained by the disjoint union of $\mathbb{V}_i \times \{a_i\}$ and $\mathbb{E}_i \times [a_i, a_{i+1}]$ by making the identifications $(l_i(x), a_i) \sim (x, a_i)$ and $(r_i(x), a_{i+1}) \sim (x, a_{i+1})$ for all i and all $x \in \mathbb{E}_i$. Now, define a function f such that $f(x, a) = a$. We can see that the pair $(\hat{\mathbb{X}}, f)$ defines a scalar field since $\hat{\mathbb{X}}$ is a topological space and $\hat{f}$ is simply a projection onto the second factor, which is a continuous function.

Definition 2.9. If $\mathbb{X}$ is isomoprhic to $\hat{\mathbb{X}}$, then $\mathbb{X}$ is a **constructible scalar field** and we say that $\hat{\mathbb{X}}$ is its **constructible base**. The intuition here is that the scalar fields are "well-behaved" if they have a finite set of critical points and if between critical points, their shape is homotopic to a cylinder. Then at the critical fibers, we have places where the cylindrical structures are able to "merge" or "split". Note that these attaching maps are always mapping *from* the non-critical fibers. So if our non-critical fiber has a cylinder that split into two cylinders, there must be a critical fiber which is a figure eight, exactly at the point where they start to split. We can always map a single cylinder to a figure eight or two cylinders to the figure eight. We never have to worry about actually "splitting" the figure eight into two cylinders (which is discontinuous) because the maps only go towards the critical-fibers.

2.3. **Category of Reeb Graphs and the Reeb Functor.**

Proposition 2.10. *The set of Reeb Graphs created from scalar fields defines a subcategory of* $\mathbb{R}$-***Top****.*

To prove this, we need to show that the Reeb Graphs are scalar fields and that there exists a subcollection of morphisms of $\mathsf{hom}(\mathbb{R}\text{-}\mathbf{Top})$ where the source and target are always Reeb Graphs. To help us with the following proofs, we refer to the lemma below:

Lemma 2.11. *(from Munkres, Theorem 22.2) Let $\rho : X \to Y$ be a quotient map. Let Z be a space and let $f : Y \to Z$ be a map that is constant on each set $\rho^{-1}(\{y\})$, for $y \in Y$. Then f induces a map $g : Y \to Z$ such that $g \circ \rho = f$. The induced map g is continuous if and only if f is continuous.*

Lemma 2.12. *Let $(\mathbb{X}, f)$ be a scalar field and $(\mathbb{X}_f, \tilde{f})$ denote its Reeb Graph. Then $(\mathbb{X}_f, \tilde{f})$ is a scalar field.*

Proof. By definition $\mathbb{X}_f$ is a topological space. What is left to show is that $\tilde{f}$ is a continuous function. Let ρ be the quotient map from $\mathbb{X}$ to $\mathbb{X}_f$. Since f is constant on point pre-images of $\mathbb{X}_f$, Munkres Theorem 22.2 tells us that there exists a continuous function $g \circ \rho = f$. By definition of a Reeb Graph, we have that $\tilde{f} \circ \rho = f$, so $\tilde{f} = g$ and $(\mathbb{X}_f, \tilde{f})$ is a scalar field. □

Lemma 2.13. *Let $(\mathbb{X}, f)$ and $(\mathbb{Y}, g)$ be scalar fields, let $(\mathbb{X}_f, \tilde{f})$ and $(\mathbb{Y}_g, \tilde{g})$ be their Reeb Graphs resepectively, and let ρ_f and ρ_g be the quotient maps from the scalar fields to the Reeb Graphs. If α is a morphism between $(\mathbb{X}, f)$ and $(\mathbb{Y}, g)$, then the map $\tilde{\alpha}$ defined by $\rho_g \circ \alpha = \tilde{\alpha} \circ \rho_f$ is a morphism in $\mathbb{R}$-**Top**.*

Proof. Consider the composition of maps $(\rho_g \circ \alpha) : \mathbb{X} \to \mathbb{Y}_g$. This map is continuous since both ρ_g and α are continuous. Now, consider a point pre-image $\rho_f^{-1}(\{y\}) \in \mathbb{X}$ where $\tilde{f}(y) = a$. Then this pre-image is the a-fiber of $\mathbb{X}$. Since $\alpha \in \mathsf{hom}(\mathbb{R}\text{-}\mathbf{Top})$, it preserves level sets, so $\alpha(\rho_f^{-1}(\{y\}))$ is the a-fiber of $\mathbb{Y}$. Then, since ρ_g is the quotient map turning all a-fibers into a single contracted point in $\mathbb{Y}_g$, the composition of maps $(\rho_g \circ \alpha)$ is constant on pre-images of ρ_f. Using Munkres Theorem 22.2 again, we have that the induced map $\tilde{\alpha}$ in the equation $\rho_g \circ \alpha = \tilde{\alpha} \circ \rho_f$ must be continuous, since $\rho_g \circ \alpha$ is continuous and constant on the point pre-images of the quotient map ρ_f.

To show that $\tilde{\alpha}$ is a function preserving map, we need to show that $\tilde{f} = \tilde{g} \circ \tilde{\alpha}$. We begin with the fact that $f = g \circ \alpha$:

$$
\begin{aligned}
f = g \circ \alpha \quad &\Rightarrow (\tilde{f} \circ \rho_f) = (\tilde{g} \circ \rho_g) \circ \alpha \\
&\Rightarrow (\tilde{f} \circ \rho_f) = \tilde{g} \circ (\rho_g \circ \alpha) \\
&\Rightarrow (\tilde{f} \circ \rho_f) = \tilde{g} \circ (\tilde{\alpha} \circ \rho_f) \\
&\Rightarrow \tilde{f} = \tilde{g} \circ \tilde{\alpha}
\end{aligned}
$$

□

Proof of Proposition 2.10 . Note that the set of Reeb Graphs is a collection of constructible scalar fields and that the maps we defined between Reeb Graphs are morphisms in $\mathbb{R}$-**Top**. Thus, the collection of Reeb Graphs constructed from scalar fields is a subcategory of $\mathbb{R}$-**Top** □

Definition 2.14. We define the **Reeb Functor** as the functor $\mathcal{R} : \mathbb{R}\text{-}\mathbf{Top} \to \mathbb{R}\text{-}\mathbf{Top}$ by the formulas $\mathcal{R}(\mathbb{X}, f) := (\mathbb{X}_f, \tilde{f})$ and $\mathcal{R}[\alpha] = \tilde{\alpha}$.

Proposition 2.15. *If $(\mathbb{X}, f)$ is constructible, then $\mathcal{R}(\mathbb{X}, f)$ is constructible as well. That is, when restricted to constructible scalar fields, the Reeb Functor maps to constructible scalar fields.*

Proof. Let $(\mathbb{X}, f) \in \mathbb{R}\text{-}\mathbf{Top}^{\mathbf{C}}$ and let $(\mathbb{X}_f, \tilde{f})$ be its Reeb Graph. Let $\hat{\mathbb{X}}$ be the constructible base of $(\mathbb{X}, f)$ with critical points $\mathcal{S} = \{a_0, \ldots, a_n\}$ and critical and non-critical fibers, $\mathbb{V}_i$ and $\mathbb{E}_i$. We create a new constructible scalar field $\mathbb{G}$ made up of critical fibers V_i and non-critical fibers E_i by using the following definitions:

$$V_i := \pi_0(\mathbb{V}_i) \qquad E_i := \pi_0(\mathbb{E}_i) \qquad l_i := \pi_0(\mathbb{l}_i) \qquad r_i := \pi_0(\mathbb{r}_i),$$

where π_0 denotes the set of path connected components. $\mathbb{G}$ is then defined as the quotient space of the disjoint union of spaces $V_i \times a_i$, for all $0 \leq i \leq n$ and $E_i \times [a_i, a_{i+1}]$ for all $0 \leq i \leq n-1$ using the identifications $(l_i(x), a_i) \sim (x, a_i)$ and $(r_i(x), a_{i+1}) \sim (x, a_{i+1})$. Note that this is exactly the same construction as in the definition of constructible scalar fields, except the critical and non-critical fibers here are 0-dimensional. Our goal is to show that $\mathbb{G}$ is a valid constructible base for $\mathbb{X}_f$. $\square$

2.4. **Pre-Cosheafs.** Pre-cosheafs allow us to abstract the data the that a Reeb Graph provides us. In this section, we define what a pre-cosheaf is and show how we can describe the pre-cosheaf of a Reeb Graph.

Definition 2.16. Let $\mathbf{Open}(\mathcal{X})$ be the cateogry of open sets on the topological space $\mathcal{X}$ where $I \to J$ if $I \subseteq J$. Then a **pre-cosheaf** is a functor $\mathsf{F} : \mathbf{Open}(\mathcal{X}) \to \mathbf{D}$, where D is some category.

In other words, **a pre-cosheaf is an assignment of data to open sets of a topological space $\mathcal{X}$.**

In this work, our pre-cosheaves will exclusively have the category of intervals on the real line **Int** as the domain space and the category of sets **Set** as the range space. Note that the morphisms in **Set** are functions between sets.

Definition 2.17. We define the **category of pre-cosheaves**, denoted as **Pre**, to be the set of functors from **Int** to **Set**. In other words, $\mathbf{Pre} := \mathbf{Set}^{\mathbf{Int}}$. The morphisms in **Pre** are then natural transformations.

Definition 2.18. Let $f = (\mathbb{X}, f)$ be a Reeb Graph. The **Reeb cosehaf functor** $\mathcal{C}$ is defined by the formulas

$$\mathsf{F}(I) = \pi_0(f^{-1}(I)), \qquad \mathsf{F}[I \subseteq J] = \pi_0[f^{-1}(I) \subseteq f^{-1}(J)].$$

This functor converts a Reeb Graph to its pre-cosheaf. Thus, $\mathcal{C} : \mathbf{Reeb} \to \mathbf{Pre}$.

3. Interleaving Distance

In this section, we begin the definition of "Interleaving Distance". We will see that the actual definition is quite abstract. However, we will also see that there is in fact a geometric realization of what this distance is computing. By the end of this section, we will introduce the idea of "thickening" a Reeb Graph, which will

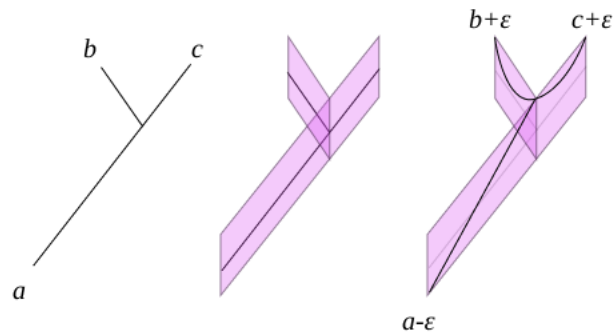

FIGURE 1. A reeb graph is displayed on the far left, with the thickened space in the middle. On the far right, we have superimposed the new reeb graph over the thickened space.

alter the critical points of the Reeb Graph, ultimately giving a more "coarse" view of the Reeb Graph. Figure 1 shows an example of a thickened Reeb Graph.

3.1. **Smoothing Functors and Interleaving of Pre-Cosheafs.** Recall that an isomorphism between two functors F, G is defined as two natrual transformations $\varphi = \{\varphi_I\}_{I \in \mathbf{C}}$ and $\psi = \{\psi_I\}_{I \in \mathbf{C}}$ such that $\varphi_I : \mathsf{F}(I) \to \mathsf{G}(I)$, $\psi_I : \mathsf{G}(I) \to \mathsf{F}(I)$ and φ_I, ψ_I are inverses of each other for each $I \in \mathbf{C}$. If we are not able to construct an isomorphism between two functors, we somehow want to quantify how far away we are from an isomorphism. **The Interleaving distance quantifies the distance between functors by finding the minimum value ε that we need to *alter* or *distort* the functors in order to make them isomorphic.**

Definition 3.1. Let $I = (a, b) \subseteq \mathbb{R}$ and $I^\varepsilon = (a - \varepsilon, b + \varepsilon)$. The ε-**expansion functor**, Ω_ε, where $\varepsilon > 0$, is defined as

$$\Omega_\varepsilon : \mathbf{Int} \to \mathbf{Int}; \quad I \mapsto I^\varepsilon$$

This is a well defined functor since when $I \subseteq J$ we have $I^\varepsilon \subseteq J^\varepsilon$. We say that an pre-cosheaf can be "ε-smoothed" to obtain a new pre-cosheaf by pre-composition of the smoothing functor:

$$\mathsf{F}\Omega_\varepsilon : \mathbf{Int} \to \mathbf{Set}; \quad \mathsf{F}\Omega_\varepsilon(I) = \mathsf{F}(I^\varepsilon)$$

Note that since $I \subseteq I^\varepsilon$ for all I, we can construct a natural transformation $\omega^\varepsilon : 1_{\mathbf{Int}} \Rightarrow \Omega_\varepsilon$. That is, the following diagram commutes:

$$\begin{tikzcd}
1_{\mathbf{Int}}(I) \arrow[r, "1_{\mathbf{Int}}[I \subseteq J]"] \arrow[d, "\omega^\varepsilon_I"'] & 1_{\mathbf{Int}}(J) \arrow[d, "\omega^\varepsilon_J"] \\
\Omega_\varepsilon(I) \arrow[r, "\Omega_\varepsilon[I \subseteq J]"'] & \Omega_\varepsilon(J)
\end{tikzcd}$$

Because of this, we can compose ω^ε with the pre-cosheaf F to get $\sigma^\varepsilon_{\mathsf{F}} = \mathsf{F}\omega^\varepsilon : \mathsf{F} \Rightarrow \mathsf{F}\Omega_\varepsilon$ defined explicitly by $(\sigma^\varepsilon_{\mathsf{F}})_I = \mathsf{F}[I \subseteq I^\varepsilon] : \mathsf{F}(I) \to \mathsf{F}(I^\varepsilon)$.

Remark 3.2. The functor Ω_ε is an example of a **pointed endofunctor**. An **endofunctor** is any functor that has the same source and target space. The term **pointed** refers to the fact that it induces this natural transformation ω^ε from the identity functor to itself.

Definition 3.3. We say that two pre-cosheafs F, G are **ε-interleaved** if there exists a pair of natural transformations $\varphi : \mathsf{F} \Rightarrow \mathsf{G}\Omega_\varepsilon$ and $\psi : \mathsf{G} \Rightarrow \mathsf{F}\Omega_\varepsilon$ such that the following diagrams commute:

$$\begin{tikzcd}
\mathsf{F} \arrow[dr, Rightarrow, "\varphi"] \arrow[dd, Rightarrow, "\sigma^{2\varepsilon}_{\mathsf{F}}"'] & \\
& \mathsf{G}\Omega_\varepsilon \arrow[dl, Rightarrow, "\psi\Omega_\varepsilon"] \\
\mathsf{F}\Omega_{2\varepsilon} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathsf{G} \arrow[dl, Rightarrow, "\psi"'] \arrow[dd, Rightarrow, "\sigma^{2\varepsilon}_{\mathsf{G}}"] \\
\mathsf{F}\Omega_\varepsilon \arrow[dr, Rightarrow, "\varphi\Omega_\varepsilon"'] & \\
& \mathsf{G}\Omega_{2\varepsilon}
\end{tikzcd}$$

If $\varepsilon = 0$, then this is exactly the definition of an isomoprhism between F and G. When two pre-cosheafs are ε-interleaved, we say that there exists an **ε-interleaving** between them.

To interpret what these diagrams mean, consider a single interval $I \subseteq \mathbb{R}$. The left diagram states that φ maps $\mathsf{F}(I)$ to $\mathsf{G}\Omega_\varepsilon(I) = \mathsf{G}(I^\varepsilon)$ and that $\psi\Omega_\varepsilon$ maps $\mathsf{G}\Omega_\varepsilon(I)$ to $\mathsf{F}\Omega_{2\varepsilon}(I)$. In other words, φ takes the data attached to I via F and carries it to the data attached to the larger interval I^ε via G. Then, $\psi\Omega_\varepsilon$ takes the data attached to I^ε via G and maps it to the data attached to an even larger interval $I^{2\varepsilon}$ via F. This way of mapping $\mathsf{F}(I)$ to $\mathsf{F}(I^{2\varepsilon})$ needs to be equivalent to the implicitly defined mapping $\sigma_{\mathsf{F}}^{2\varepsilon}$ between $\mathsf{F}(I)$ and $\mathsf{F}(I^{2\varepsilon})$. This implicit map $\sigma_{\mathsf{F}}^{2\varepsilon}$ is defined by noting that $I \subseteq I^{2\varepsilon}$, so the data attached to I can be mapped directly to the data attached to $I^{2\varepsilon}$.

When first defining the pre-cosheafs, we stated that the data attached to these intervals are just sets. Suppose that the data attached via the functors F, G are finite sets. An easy example for where these two routes to $\mathsf{F}(I^{2\varepsilon})$ differ is when $|\mathsf{F}(I)| = |\mathsf{F}(I^{2\varepsilon})| > |\mathsf{G}(I^{\varepsilon})|$. In this case, the map ϕ could be surjective but not injective. Then, the map $\psi\Omega_\varepsilon$ could be an injective map, but not surjective. However, the implictly defined map $\sigma^{2\varepsilon}$ could be both surjective and injective, meaning that mapping to $\mathsf{F}\Omega_{2\varepsilon}$ with these two routes would be different, therefore *not* defining a ε-interleaving.

Definition 3.4. The **interleaving distance** between two pre-cosheafs F and G is defined as

$$d_I(\mathsf{F}, \mathsf{G}) = \inf\{\varepsilon \geq 0 | \text{ there exists an } \varepsilon\text{-interleaving between } \mathsf{F}, \mathsf{G}\}$$

To find the interleaving distance between Reeb Graphs, we can convert them into pre-cosheafs and then find the interleaving distance between the cosheafs.

Definition 3.5. Let $f = (\mathbb{X}, f)$ and $g = (\mathbb{Y}, g)$ be two Reeb Graphs. Since we have a notion of distance between two pre-cosheafs, we can define the **interleaving distance between** f, g simply by

$$d_I(f, g) := d_I(\mathcal{C}(f), \mathcal{C}(g))$$

Recall that we convert Reeb Graphs to pre-cosheafs by essentially stating that the data attached to the interval I via F is the set of path-connected components on $f^{-1}(I)$. To reiterate the example before for when we *don't* have an ε-interleaving between two pre-cosheafs, two Reeb graphs f, g are *not* ε-interleaved if $|\pi_0(f^{-1}(I))| = |\pi_0(f^{-1}(I^{2\varepsilon}))| > |\pi_0(g^{-1}(I^{\varepsilon}))|$. Somehow, we need to avoid these situation. Suppose for example that $|\pi_0(f^{-1}(I))| > |\pi_0(g^{-1}(I))|$. We automatically know that $\varepsilon = 0$ will not suffice, so we need to choose a ε sufficiently large such that $|\pi_0(f^{-1}(I^{2\varepsilon}))| \leq |\pi_0(g^{-1}(I^{\varepsilon}))|$. In otherwords, ε has to be large enough so that the expansion from I to $I^{2\varepsilon}$ moves over a critical point where two or more path connected components merge, decreasing the cardinality. Figure 2 shows an example of this.

Definition 3.6. Let $\mathcal{S}_\varepsilon$ be defined by the formula $\mathcal{S}_\varepsilon(\mathsf{F}) := \mathsf{F}\Omega_\varepsilon$ and $\mathcal{S}_\varepsilon[\alpha] := \alpha\Omega_\varepsilon : \mathsf{F}\Omega_\varepsilon \Rightarrow \mathsf{G}\Omega_\varepsilon$ for all $\alpha \in \mathsf{hom}(\mathbf{Pre})$. We call $\mathcal{S}_\varepsilon$ the **Smoothing Functor**.

Definition 3.7. Let $f = (\mathbb{X}, f)$ and $g = (\mathbb{Y}, g)$ be two Reeb Graphs. We say that f, g are ε-interleaved if there exists a pair of natural transformations $\varphi : \mathcal{C}(f) \Rightarrow \mathcal{S}_\varepsilon\mathcal{C}(g)$ and $\psi : \mathcal{C}(g) \Rightarrow \mathcal{S}_\varepsilon\mathcal{C}(f)$ such that the following diagrams commute:

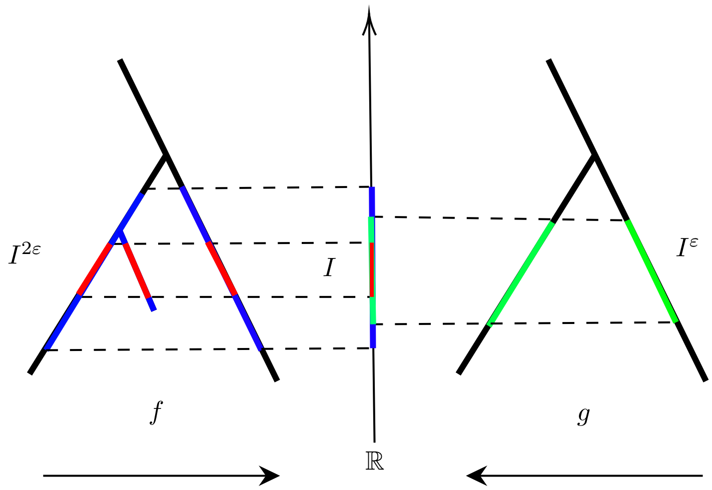

FIGURE 2. Here we have two Reeb Graphs with differing structure (missing leaf on the the Reeb Graph of g). In red, we have highlighted the three different path connected components of the $f^{-1}(I)$. In green we have highlighted the two different path connected components of $g^{-1}(I^{\varepsilon})$. These two graphs to be ε-interleaved, the ε has to be chosen such that the number of connected components of $f^{-1}(I^{2\varepsilon})$ is at most two, which is shown in blue.

$$
\begin{tikzcd}
\mathcal{C}(f) \arrow[dr, Rightarrow, "\varphi"] \arrow[dd, Rightarrow, "\sigma^{2\varepsilon}_{\mathcal{C}(f)}"'] & \\
& \mathcal{S}_{\varepsilon}\mathcal{C}(g) \arrow[dl, Rightarrow, "\mathcal{S}_{\varepsilon}[\psi]"] \\
\mathcal{S}_{2\varepsilon}\mathcal{C}(f) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{C}(g) \arrow[dl, Rightarrow, "\psi"'] \arrow[dd, Rightarrow, "\sigma^{2\varepsilon}_{\mathcal{C}(g)}"] \\
\mathcal{S}_{\varepsilon}\mathcal{C}(f) \arrow[dr, Rightarrow, "\mathcal{S}_{\varepsilon}[\varphi]"'] & \\
& \mathcal{S}_{2\varepsilon}\mathcal{C}(g)
\end{tikzcd}
$$

Remark 3.8. The natural transformation σ^{ε} was defined before as $\sigma^{\varepsilon}_{\mathsf{F}} : \mathsf{F} \Rightarrow \mathsf{F}\Omega_{\varepsilon}$. In the above diagram, the definition is the same except with a different functor $\mathcal{C}(f)$ and the target space has been rewritten. That is,

$$\sigma^{\varepsilon}_{\mathcal{C}(f)} : \mathcal{C}(f) \Rightarrow \mathcal{S}_{\varepsilon}\mathcal{C}(f)$$

In the following sections, we will leverage the above definition to show that interleaving distance can be defined without having to convert to pre-cosheafs at all.

3.2. **Thickening Functors.** Here we describe a **thickening functor** on Reeb Graphs that operates in parallel to the smoothing functors described previously. What we will show is that it is equivalent to work with thickened Reeb Graphs or smoothed pre-cosheafs.

Definition 3.9. For $\varepsilon \geq 0$, we define the thickening functor $\mathcal{T}_\varepsilon : \mathbb{R}$-**Top** $\to \mathbb{R}$-**Top** as follows:

- Let $(\mathbb{X}, f)$ be a scalar field, let $\mathbb{X}_\varepsilon = \mathbb{X} \times [-\varepsilon, \varepsilon]$, and let $f_\varepsilon(x,t) = f(x) + t$. Then $\mathcal{T}_\varepsilon(\mathbb{X}, f) = (\mathbb{X}_\varepsilon, f_\varepsilon)$
- Let $\alpha : (\mathbb{X}, f) \to (\mathbb{Y}, g)$ be a morphism in $\mathbb{R}$-**Top**. Then we define $\mathcal{T}_\varepsilon : (\mathbb{X}_\varepsilon, f_\varepsilon) \to (\mathbb{Y}_\varepsilon, g_\varepsilon)$ by $(x,t) \mapsto (\alpha(x), t)$.

Note that when $(\mathbb{X}, f)$ is a Reeb Graph, $\mathbb{X}_\varepsilon$ is a two dimensional space, so it is no longer a Reeb Graph.

Theorem 3.10. *The functors $\mathcal{CRT}_\varepsilon$ and $\mathcal{S}_\varepsilon\mathcal{C}$ are naturally isomoprhic.*

That is, looking at the cosheafs of an ε-thickened Reeb Graph is the same as ε-smoothing the pre-cosheafs. Note that on the left side, before finding the pre-cosheafs, we need to project $\mathcal{T}_\varepsilon$ back into **Reeb**. To prove this, we will use the following lemma:

Lemma 3.11. *Let $p : \mathbb{X}_\varepsilon \to \mathbb{X}$ note restriction on the first factor. The map p restricts to a homotopy equivalence $f_\varepsilon^{-1}(I) \xrightarrow{\sim} f^{-1}(I^\varepsilon)$ for each interval I.*

Proof of Theorem 3.10. We will define a natrual transformation $\rho : \mathcal{CT}_\varepsilon \Rightarrow \mathcal{S}_\varepsilon\mathcal{C}$ and show that ρ_f is an isomorphism for all $f \in \mathbb{R}$-**Top**. First, let $\rho_f : \mathcal{CT}_\varepsilon(f) \Rightarrow \mathcal{S}_\varepsilon\mathcal{C}(f)$ be defined by the formula $(\rho_f)_I = \pi_0[p_I]$, where p_I is the map defined in Lemma 3.11. Since p_I is a homotopy equivalence, $(\rho_f)_I$ must be an isomorphism. We begin by showing that the family of maps $(\rho_f)_I$ is natural with respect to the inclusions $I \subseteq J$. Consider the following two diagrams:

$$\begin{tikzcd}
f_\varepsilon^{-1}(I) \arrow[r] \arrow[d, "p_I"'] & f_\varepsilon^{-1}(J) \arrow[d, "p_J"] \\
f^{-1}(I^\varepsilon) \arrow[r] & f^{-1}(J^\varepsilon)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_0\left(f_\varepsilon^{-1}(I)\right) \arrow[r] \arrow[d, "\pi_0[p_I]"'] & \pi_0\left(f_\varepsilon^{-1}(J)\right) \arrow[d, "\pi_0[p_J]"] \\
\pi_0\left(f^{-1}(I^\varepsilon)\right) \arrow[r] & \pi_0\left(f^{-1}(J^\varepsilon)\right)
\end{tikzcd}$$

Since the left square commutes, the right square must commute as well. Thus, ρ_f is a natural transformation. Furthermore, since each $(\rho_f)_I$ is an isomoprhism, ρ_f defines an isomorphism between $\mathcal{CT}_\varepsilon$ and $\mathcal{S}_\varepsilon\mathcal{C}(f)$.

Now we need so show that the family of maps $\rho = (\rho_f)$ is natural with repsect to the morphisms in $\mathbb{R}$-**Top**. Again, consider the following diagrams

$$
\begin{tikzcd}
f_\varepsilon^{-1}(I) \arrow[r, "\alpha \times \mathbb{1}"] \arrow[d, "p_I^f"'] & g_\varepsilon^{-1}(I) \arrow[d, "p_I^g"] \\
f^{-1}(I^\varepsilon) \arrow[r, "\alpha"'] & g^{-1}(I^\varepsilon)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_0\left(f_\varepsilon^{-1}(I)\right) \arrow[r, "\pi_0[\alpha \times \mathbb{1}]"] \arrow[d, "\pi_0[p_I]"'] & \pi_0\left(g_\varepsilon^{-1}(I)\right) \arrow[d, "\pi_0\left[p_I^g\right]"] \\
\pi_0\left(f^{-1}(I^\varepsilon)\right) \arrow[r, "\pi_0[\alpha]"'] & \pi_0\left(g^{-1}(I^\varepsilon)\right)
\end{tikzcd}
$$

Here, $\alpha \in \mathsf{hom}(\mathbb{R}\text{-}\mathbf{Top})$. The morphism $\alpha \times \mathbb{1}$ is defined as $(x,t) \mapsto (\alpha(x), t)$. From this, we can see that ρ satisfies the naturality condition for each I. Thus, ρ defines an isomorphism between $\mathcal{CRT}_\varepsilon$ and $\mathcal{S}_\varepsilon\mathcal{C}$. $\square$

Without proof, we state another proposition concerning constructibility of the thickened Reeb Graphs:

Proposition 3.12. *If $(\mathbb{X}, f) \in$ **Reeb** or $(\mathbb{X}, f) \in \mathbb{R}$-$\boldsymbol{Top}^{\boldsymbol{C}}$, then $\mathcal{T}_\varepsilon(\mathbb{X}, f) \in \mathbb{R}$-$\boldsymbol{Top}^{\boldsymbol{C}}$.*

Because of this, we do not need to worry about our thickened scalar fields introducing complexity that we are not already equipped to deal with.

3.3. **Topological Smoothing of Reeb Graphs.** At this point, we are ready to give a geometric procedure for defining the interleaving distance.

Definition 3.13. Let $(\mathbb{X}, f)$ be a Reeb Graph. We define the **Reeb Smoothing Functor** $\mathcal{U}_\varepsilon$ as $\mathcal{U}_\varepsilon := \mathcal{RT}_\varepsilon$.

Remark 3.14. Each $\mathcal{U}_\varepsilon$ is a pointed endofunctor of **Reeb**. There exists a natural transformation $\zeta^\varepsilon : 1_{\mathbf{Reeb}} \Rightarrow \mathcal{U}_\varepsilon$ where $\zeta_f^\varepsilon : f \to \mathcal{U}_\varepsilon(f)$. We define ζ_f^ε as follows:

- Let $\tau_f^\varepsilon : \mathbb{X} \to \mathbb{X}_\varepsilon$ be inclusion map from the Reeb Graph $\mathbb{X}$ into the zero-section of $\mathbb{X}_\varepsilon = \mathbb{X} \times [-\varepsilon, \varepsilon]$
- Let $\rho_{f_\varepsilon} : \mathbb{X}_\varepsilon \to \mathbb{X}_\varepsilon / \sim$ be the quotient map from this new scalar field $\mathbb{X}_\varepsilon$ to its Reeb Graph

We define $\zeta_f^\varepsilon := \rho_{f_\varepsilon} \circ \tau_f^\varepsilon$.

Theorem 3.15. *If $f, g \in$ **Reeb**, then f, g are ε-interleaved if there exists maps*

$$\alpha : f \to \mathcal{U}_\varepsilon(g) \quad \textit{and} \quad \beta : g \to \mathcal{U}_\varepsilon(f)$$

such that the following diagrams

$$
\begin{tikzcd}
f \arrow[dd, "\zeta_f^{2\varepsilon}"'] \arrow[dr, "\alpha"] & & & & g \arrow[dl, "\beta"'] \arrow[dd, "\zeta_g^{2\varepsilon}"] \\
& \mathcal{U}_\varepsilon(g) \arrow[dl, "\beta_{2\varepsilon}^{\varepsilon}"] & \textit{and} & \mathcal{U}_\varepsilon(f) \arrow[dr, "\alpha_{2\varepsilon}^{\varepsilon}"'] & \\
\mathcal{U}_{2\varepsilon}(f) & & & & \mathcal{U}_{2\varepsilon}(g)
\end{tikzcd}
$$

commute.

This proposition tell us that we can find ε-interleavings by direclty relating the Reeb Graphs instead of converting them into pre-cosheafs – a useful result as it gives us a great geometric picture for computing the interleaving distance. To prove this, we start by stating and proving the following Lemma:

Lemma 3.16. *For $f \in$ **Reeb** and $\varepsilon \geq 0$, the diagram*

$$
\begin{tikzcd}
\mathcal{C}(f) \arrow[d, Rightarrow, "\mathcal{C}[\zeta]"'] \arrow[dr, Rightarrow, "\sigma"] & \\
\mathcal{C}\mathcal{U}_\varepsilon(f) \arrow[r, Rightarrow, "\rho_f"'] & \mathcal{S}_\varepsilon\mathcal{C}(f)
\end{tikzcd}
$$

commutes.

Proof. We need to show that this diagram commutes when evaluated on an arbitrary interval I. If we follow the definition of the Reeb Cosheaf functor, we get diagram on the right, below. On the left, we have the same diagram *before* applying the connected component function π_0:

$$\begin{tikzcd}
f^{-1}(I) \arrow[d, "p_I^*"'] \arrow[dr, "f^{-1}(I) \subseteq f^{-1}(I^\varepsilon)"] & & \pi_0\left(f^{-1}(I)\right) \arrow[d, "\mathcal{C}[\zeta]_I"'] \arrow[dr, "\sigma_I"] & \\
f_\varepsilon^{-1}(I) \arrow[r, "p_I"'] & f^{-1}(I^\varepsilon) \arrow[r, phantom, "\xrightarrow{\pi_0}"] & \pi_0\left(f_\varepsilon^{-1}(I)\right) \arrow[r, "\rho_I"'] & \pi_0\left(f^{-1}(I^\varepsilon)\right)
\end{tikzcd}$$

In the left square, the map p_I^* is defined as $x \mapsto (x, 0)$. Thus, based on the definition of p_I as defined in 3.11, the composite map $p_I \circ p_I^* : f^{-1}(I) \to f^{-1}(I^\varepsilon)$ maps x to x. On the other hand, the map defined by the inclusion $f^{-1}(I) \subseteq f^{-1}(I^\varepsilon$ works the same way: A point $x \in f^{-1}(I)$ gets mapped to itself in $f^{-1}(I^\varepsilon)$. Since this diagram commutes, the right diagram commutes as well. $\square$

Proof of Proposition 3.15. Suppose we have α, β such that the diagrams above commute. We can obtain a new diagram by applying the Reeb Cosheaf functor $\mathcal{C}$ to each of the objects and morphisms in the diagrams:

$$\begin{tikzcd}
\mathcal{C}(f) \arrow[dd, Rightarrow, "\mathcal{C}\left[\zeta_f^{2\varepsilon}\right]"'] \arrow[dr, Rightarrow, "\alpha"] & & & \mathcal{C}(g) \arrow[dl, Rightarrow, "\beta"'] \arrow[dd, Rightarrow, "\mathcal{C}\left[\zeta_g^{2\varepsilon}\right]"] \\
& \mathcal{C}\mathcal{U}_\varepsilon(g) \arrow[dl, Rightarrow, "\mathcal{C}\mathcal{U}_\varepsilon[\beta]"] & \mathcal{C}\mathcal{U}_\varepsilon(f) \arrow[dr, Rightarrow, "\mathcal{C}\mathcal{U}_\varepsilon[\alpha]"'] & \\
\mathcal{C}\mathcal{U}_{2\varepsilon}(f) & & & \mathcal{C}\mathcal{U}_{2\varepsilon}(g)
\end{tikzcd}$$

From Theorem 3.10 we know that ρ defines a natural transformation from $\mathcal{C}\mathcal{U}_\varepsilon$ to $\mathcal{S}_\varepsilon\mathcal{C}$. We can define $\varphi = \rho_g \circ \mathcal{C}[\alpha]$ and $\psi = \rho_f \circ \mathcal{C}[\beta]$. Then, using Lemma 3.16, we add new pieces to the diagram above, constructing the following diagram:

$$\begin{tikzcd}
 & & \mathcal{C}(f) \arrow[dll, Rightarrow, "\sigma"'] \arrow[dd, Rightarrow, "\mathcal{C}\left[\zeta_f^{2\varepsilon}\right]"'] \arrow[dr, Rightarrow, "\mathcal{C}[\alpha]"] \arrow[drr, Rightarrow, "\varphi"] & & \\
S_{2\varepsilon}\mathcal{C}(f) \arrow[drr, Leftrightarrow, "\mathcal{S}_\varepsilon[\rho_f]"'] & & & \mathcal{C}\mathcal{U}_\varepsilon(g) \arrow[r, Leftrightarrow, "\rho_g"'] \arrow[dl, Rightarrow, "\mathcal{C}\mathcal{U}_\varepsilon[\beta]"] & \mathcal{S}_\varepsilon\mathcal{C}(g) \arrow[dll, Rightarrow, "\psi^*"] \\
 & & \mathcal{C}\mathcal{U}_{2\varepsilon}(f) & &
\end{tikzcd}$$

Our ultimate goal is to show that this diagram commuting implies that the reference old diagram commutes. What needs to be shown is that $\mathcal{S}_\varepsilon[\rho_f] \circ \psi^* = \mathcal{S}_\varepsilon[\psi] = \mathcal{S}_\varepsilon\left[\rho_f \circ \mathcal{C}[\beta]\right]$. By definition of a functor, we have that $\mathcal{S}_\varepsilon\left[\rho_f \circ \mathcal{C}[\beta]\right] = \mathcal{S}_\varepsilon[\rho_f] \circ \mathcal{S}_\varepsilon\mathcal{C}[\beta]$. Thus, we need to show that $\mathcal{S}_\varepsilon\mathcal{C}[\beta] = \psi^* = \mathcal{C}\mathcal{U}_\varepsilon[\beta] \circ \rho_g^{-1}$. By definition, $\rho_g : \mathcal{C}\mathcal{U}_\varepsilon(g) \Rightarrow \mathcal{S}_\varepsilon\mathcal{C}(g)$ is a family of isomorphisms. Thus, $\mathcal{S}_\varepsilon\mathcal{C}[\beta] \circ \rho_g = \mathcal{C}\mathcal{U}_\varepsilon[\beta]$, which implies $\mathcal{S}_\varepsilon\mathcal{C}[\beta] = \mathcal{C}\mathcal{U}_\varepsilon[\beta] \circ \rho_g^{-1}$.

Finally, we have that original diagram commuting implies that the diagram above commutes, which further implies that there exists natural transformations $\varphi : \mathcal{C}(f) \Rightarrow \mathcal{S}_\varepsilon\mathcal{C}(g)$,$\psi : \mathcal{C}(g) \Rightarrow \mathcal{S}_\varepsilon\mathcal{C}(f)$ such that reference interleaveing dgm commutes. Thus, we have an ε-interleaving between f and g.

$\square$

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