# Interleaving Distance between Scalar Fields

## 1 Category of Scalar Fields and Reeb Graphs

Our goal in this section is to define some preliminary definitions and content. We hope to give a general overview of the cateory theoretic approach, while not getting too invovled in details that do not immediately concern the scalar fields that we are looking into.

**Definition 1.1:** A smooth, real-valued function on a manifold $\mathbb{M}$ is a *Morse Function* if it has no degenerate critical points. A critical point is said to be degenerate if the matrix of second partial derivatives (the Hessian Matrix) is singular (non-invertible). Another interpretation of this is that every critical point $b$ of our function is locally quadratic. See the Morse Theory article from Wikipedia.org for more details.

**Definition 1.2:** Suppose we have a topological space $\mathbb{X}$ equipped with a continuous map $f : \mathbb{X} \to \mathbb{R}$. We call this a *scalar field* and denote it as $(\mathbb{X}, f)$.

In our work, we often use the example of a compact, differentiable, 2-manifold as our topological space and our function $f$ is a morse function. This case creates a specific scalar field which is easier to work with than just general scalar fields

**Definition 1.3:** The category of scalar fields is denoted $\mathbb{R} - \textbf{Top}$ and the morphisms between scalar fields $\phi : (\mathbb{X}, f) \to (\mathbb{Y}, g)$ is a continuous map $\phi : \mathbb{X} \to \mathbb{Y}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\phi"] \arrow[dr, "f"'] & \mathbb{Y} \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

We call this function $\phi$ a *function preserving map.*

**Definition 1.4:** The point-preimage $f^{-1}(a)$ is called a *levelset* or a *fiber* of our scalar field. More specifically, we can call $f^{-1}(a)$ an $a$-fiber of $f$. Since $\phi$ is a function preserving map, it must carry $a$-fibers of $f$ to $a$-fibers of $g$. In other words, it "preserves levelsets".

> **Question 1: How do we deal two scalar fields which don't have the same range?**
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> Suppose we have two scalar fields $f$ and $g$ such that they are defined on the same domain (so their scalar functions are the only things discerning them). Now suppose that there exists no point $x \in \mathbb{X}$ such that $g(x) = a$ but there exists some subset $U \subseteq \mathbb{X}$ such that $f(x) = a$ for all $x \in U$. Then, the $a$-fiber of $f$ is $U$ while the $a$-fiber of $g$ is the empty set. So how does $phi$ work in this case? If there is multiple values of $\mathbb{R}$ such that this situation arises, we run into the problem of $\phi$ mapping from $U_1$ to the empty set and $U_2$ mapping to the empty set as well, which would break commutativity because $g$ on the empty set cannot map to two different values. Note that this situation can arise even if the domains differ, of course.
>
> **Possible Solution:** I believe that $\phi$ only needs to really be function preserving on values of $a \in \mathbb{R}$ such that the $a$-fibers of both $f$ and $g$ are non-empty subsets of $\mathbb{X}$.

We now want to restrict our category $\mathbb{R} - \mathbf{Top}$ a bit to scalar fields that are well behaved. We define this new category to be the category of *constructible scalar fields*, denoted by $\mathbb{R} - \mathbf{Top}^{\mathbf{C}}$. An object in this category is any scalar field that is isomorphic to another scalar field $(\mathbb{X}, f)$ which is constructed in the following way: First, let there be a finite set $S = \{a_1, \ldots, a_n\}$ of "critical points". We use the term "critical points" here because in our example (two-manifold with a morsefunction), that is exactly what these points will be (maxima, minima, and saddles). Then:

- For $0 \leq i \leq n$ we have a locally path-connected compact space $\mathbb{V}_i$ (critical fiber)
- For $0 \leq i \leq n-1$ we have a locally path-connected compact space $\mathbb{E}_i$ (non-critical fiber)
- For $0 \leq i \leq n-1$ we have continuous maps $\mathrm{l}_i : \mathbb{E}_i \to \mathbb{V}_i$ and $\mathrm{r}_i : \mathbb{E}_i \to \mathbb{V}_{i+1}$ (attaching maps)

Now we let $\mathbb{X}$ be the quotient space obtained by the disjoint union of spaces $\mathbb{V}_i \times \{a_i\}$ and $\mathbb{E}_\mathrm{i} \times [a_i, a_{i+1}]$ by making the identifiactions $(\mathrm{l}_i(x), a_i) \sim (x, a_i)$ and $(\mathrm{r}_i(x), a_{i+1}) \sim (x, a_{i+1})$ for all $i$ and for all $x \in \mathbb{E}_i$.

Let's consider the example where our topological space is a 2-manifold $\mathbb{M}$ that is compact and differentiable at every point, and that the function $f$ is a morse function. We can think of this as being a hilly landscape, where the manifold is the base and the height of the peaks tell us what $f$ does to each point of the manifold. Furthemore, suppose there are no critical points that are shared. On the intervals between critical points, we select a single $a$-fiber, $a \in (a_i, a_{i+1})$ for every $0 \leq i \leq n-1$ and call it $\mathbb{E}_i$, the non-critical fiber of $i$. Now, for every critical point $a_i$ we look at the critical $a_i$-fibers and label them as $\mathbb{V}_i$. Critical points in this case are exactly minima, maxima, and saddles. Since $f$ is a morse function, we don't have any monkey saddles or higher order critical points. **just double check this**. Now, we should be able to construct a new scalar field from these fibers by elongating the non-critical fibers across the interval $[a_i, a_{i+1}]$ and then assigning the attaching maps above. What we are essentially achieving are homotopy equivalences between the non-critical fibers $\mathbb{E}_i$ and $f^{-1}(a_i, a_i + 1)$. **Go into more detail about why this creates a homotopy equivalence**

**Definition 1.5:** A constructible scalar field such that each of the spaces $\mathbb{V}_i$ and $\mathbb{E}_i$ are all 0-dimensional is called a *Reeb Graph* and is an object of the category **Reeb**. We can think of these spaces $\mathbb{V}_i$ and $\mathbb{E}_i$ as vertices and edges, respectively. **Go into detail about morphisms**

**You can probably go more in depth about notation.** Given a constructible scalar field, we can construct the corresponding Reeb Graph by assigning $V_i = \pi_0(\mathbb{V}_i)$ to be the vertices of the Reeb graph and $E_i = \pi_0(\mathbb{E}_i)$ to be the edges, where $\pi_0$ maps a topological space to its set of path-connected components.

## 2 Cosheaves

**Definition 2.6:** Let **Int** be the category whose objects are open intervals $I \subseteq \mathbb{R}$ and whose morphisms are inclusions $I \subseteq J$. Let the category **Set** be the category whose objects are sets and whose morphisms are total functions (functions that use ever element in the domain set) from on set to the other. Then, we define the category $\mathbf{Pre} = \mathbf{Set}^{\mathbf{Int}}$ as being the category of functors $\mathbf{Int} \to \mathbf{Set}$. This category is called the *category of pre-cosheaves.*

In other works, similar structures have been created and refered to as "indexed diagrams". For example, we could have the category $\mathbf{Top}^{(\mathbb{R},\leq)}$ which is the category of functors $(\mathbb{R}, \leq) \to \mathbf{Top}$ where $(R, \leq)$ is the category of elements of the real line whose morphisms are ordering, and **Top** is the category of topological spaces. While seemingly strange, we can interpret this as topological spaces that are "indexed" by values on the real line. When we think about persistent homology, we think of homology groups that are dependent on some real number (the birth time for a persistence diagram).

We can also talk about other pre-cosheaves, which are just depending on the target category of the functor. For example, we can have the category of pre-cosheaves **Pre(Top)** which are functors $\mathbf{Int} \to \mathbf{Top}$. Then, we can do a post composition of this functor to move the topological spaces to sets. In this work, we consider the elements of **Pre** to be functors defined as $F(I) = \pi_0(f^{-1}(I))$, where $\pi_0$ is the set of path connected components of the topological space, and $f$ is the function defined on our scalar field. So, we look at the connected components of the pull back of our scalar field.

Now, we want to convert our Reeb Graphs into these (pre)cosheafs. We define the *Reeb Cosheaf functor* $\mathcal{C}$ as follows: Let $f = (\mathbb{X}, f)$ be a scalar field. Then, $\mathcal{C}(f) = F$ is the pre-cosheaf defined by

$$F(I) = \pi_0(f^{-1}(I)), \quad F[I \subseteq J] = \pi_0[f^{-1}(I) \subseteq f^{-1}(J)]$$

While the definition of cosheaf is pretty simple, it is seemingly useless without a nice geometric realization. If we don't understand how the cosheafs are mapped to each other, we won't know how the Reeb Graphs are being compared. Recall that any morphism of scalar fields $\alpha : (\mathbb{X}, f) \to (\mathbb{Y}, g)$ is function preserving, meaning it preserves level sets. So, $\alpha$ automatically maps $f^{-1}(I)$ onto $g^{-1}(I)$. **This is extremely important for computation, as well**.

Recall that a natural transformation between two functors is a family of maps $\eta_X : F(X) \to G(X)$ where $X \in \mathbf{C}$. Furthermore, it must be true that $\eta_Y \circ F(f) = G(f) \circ \eta_X$ where $f$ maps $X$ to $Y$ (which implies $F(f)$ maps $F(X)$ to $F(Y)$ and $G(f)$ maps $G(X)$ to $G(Y)$) and $\eta_Y$ is the mapping from $F(Y)$ to $G(Y)$. This composition property is referred to as "natural". **This can be drawn with a commutative diagram easily** Our map $\alpha$ is completely independent of the choice of I. Thus, we can create a natrual transformation between pre-cosheafs $F, G$ by defining the family of maps between $F, G$ to be $\mathcal{C}[\alpha]_I = \pi_0[f^{-1}(I) \to g^{-1}(I)]$. That is, for every choice of $I$, we already have a map $\alpha$ between $f^{-1}(I)$ and $g^{-1}(I)$. So, applying $\pi_0$ to the map (meaning the map now acts from $\pi_0(f^{-1}(I))$ to $\pi_0(g^{-1}(I))$) moves it to acting on cosheafs. But in this, case, it is now a family of maps (becasue it works for every interval $I$ in the initial category).

## 3 Interleaving Distance

Recall that a natural transformation between two functors is a family of maps $\eta_X : F(X) \to G(X)$ where $X \in \mathbf{C}$. Furthermore, it must be true that $\eta_Y \circ F(f) = G(f) \circ \eta_X$ where $f$ maps $X$ to $Y$ (which implies $F(f)$ maps $F(X)$ to $F(Y)$ and $G(f)$ maps $G(X)$ to $G(Y)$) and $\eta_Y$ is the mapping from $F(Y)$ to $G(Y)$. This composition property is referred to as "natural". Let $F, G : \mathbf{Int} \to \mathbf{Set}$ be pre-cosheaves. An isomorphism between two functors consists of a family of maps

$$\varphi_I : F(I) \to G(I) \text{ and } \psi_I : G(I) \to F(I)$$

for each $I \subseteq \mathbb{R}$, such that $\varphi_I$ and $\psi_I$ are inverses for each $I$ and are natural with respect to the inclusions $I \subseteq J$. That is, $\varphi_J \circ F[I \subseteq J] = G[I \subseteq J] \circ \varphi_I$.

**Definition 3.7:** If $I = (a, b)$, then let $I^\varepsilon = (a - \varepsilon, b + \varepsilon)$. An $\varepsilon$ - *interleaving* between $F, G$ is given by two families of maps

$$\varphi_I : F(I) \to G(I^\epsilon) \text{ and } \psi_I : G(I) \to F(I^\epsilon)$$

that are natural with respect ot inclusions $I \subseteq J$ and such that

$$\psi_{I^\varepsilon} \circ \varphi_I = F[I \subseteq I^{2\varepsilon}] \text{ and } \varphi_{I^\varepsilon} \circ \psi_I = G[I \subseteq I^{2\varepsilon}]$$