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THE DISPLAY LOCALE OF A COSHEAF

by J. FUNK

Résumé On montre que la catégorie des antifaisceaux sur un locale X arbitraire est équivalente à la sous-catégorie réflexive pleine de la catégorie des locales localement connexes sur X . Il s'agit de la sous-catégorie des *complete spreads* (cf. [5]) sur X . On accorde une attention particulière au cas d'un espace topologique, et l'on montre comment construire l'antifaisceau associé à un préantifaisceau arbitraire dans un espace métrique complet. On donne un contre-exemple qui montre qu'en général, les antifaisceaux ne constituent pas un topos.[1]

0 INTRODUCTION

Partial results (cf. [1]) have been obtained concerning the representation of a cosheaf on a topological space X as a locally connected space over X . The principal aim of this paper is to show that an arbitrary cosheaf on a locale X can be uniquely represented as the cosheaf of connected components of a complete spread over X with locally connected domain. This is accomplished by exhibiting a (fully faithful) functor from cosheaves on X to locally connected locales over X which is right adjoint to the 'connected components' functor, and then by recognizing those objects in the essential image as the complete spreads over X .

[1] M. Bunge (cf. [2]) has shown that the category of cosheaves on an arbitrary site is the category of *points* of a topos. This topos is called the *symmetric* topos in [2] where it is shown to exist as the classifying topos of the notion of cosheaf.

The explanation of the right adjoint centers on a locale called the display locale of a precosheaf. This locale is defined in §1. After some preliminaries on locally connected locales are presented in §2, the main adjunction is established in §3. The first part of the main result, that the right adjoint is fully faithful, can then be proved.

Thus, the category of cosheaves on a locale X is equivalent to a full reflective subcategory of locally connected locales over X , i.e., those for which the unit is an isomorphism. Let us call such a locale a cosheaf locale over X . We show in §4 that the notion of a cosheaf locale over X is equivalent to the notion of a complete spread over X . The latter notion comes from [5], and was introduced there as an encompassing notion of branched and folded covering spaces (cf. [18]).

The spatial case is given special consideration in §5. We describe the display *space*, its relation to the display locale, and the connection with (spatial) complete spreads. This prepares us for the concluding section which considers the case of a complete metric space. It is shown (as stated in [1], p. 204) that on a complete metric space X , every cosheaf arises as the cosheaf of connected components of a locally connected space over X . This result can be combined with the locally connected coclosure to give a construction of the associated cosheaf of an arbitrary precosheaf on a complete metric space.

To summarize, the category of cosheaves on an arbitrary locale is equivalent to the category of complete spreads over the locale. This equivalence cannot in general be expected to hold in the spatial context; however, when the base space is a complete metric space the category of cosheaves is equivalent to the category of T_1 complete spreads over the base space.

Throughout, FRM denotes the category of *frames*.[2] The category of *locales* is by definition the opposite of FRM, and is denoted by LOC. If X denotes an object of LOC, then the same object considered as a frame is denoted by $\mathcal{O}(X)$. A morphism of locales $X \xrightarrow{f} Y$ is written as $\mathcal{O}(Y) \xrightarrow{f^*} \mathcal{O}(X)$ when regarded in FRM. An element of $\mathcal{O}(X)$ is referred

[2] A comprehensive account of the theory of frames can be found in [10].

to as an *open* of X . Opens are typically denoted by U, with 0 and X being reserved for the bottom and top elements of $\mathcal{O}(X)$.[3] A *cover* of an open U of a locale X is a down-closed subset $\{U_\alpha \leq U\}$ such that the supremum $\bigvee U_\alpha$ is equal to U . A *precosheaf* D on a locale X is a functor

$$D : \mathcal{O}(X) \rightarrow Set \ .$$

A precosheaf D is a *cosheaf* (*coseparated precosheaf*) if D has the property that for any open $U \in \mathcal{O}(X)$ and any cover $\{U_\alpha \leq U\}$, $\{DU_\alpha \rightarrow DU\}$ is a colimiting cone (epimorphic family).[4]

Let $coSh(X)$ denote the full subcategory of $Set^{\mathcal{O}(X)}$ determined by those objects which are cosheaves. Let $Cocts(Sh(X), Set)$ denote the category of Set-valued *cocontinuous* (i.e., small colimit preserving) functors on the topos of sheaves on the locale X (with all natural transformations as morphisms).[5] The following important fact is well known (cf. [16]).

0.1 Theorem. *For any locale X , composition with the Yoneda embedding $Yon : \mathcal{O}(X) \longrightarrow Sh(X)$ yields an equivalence*

$$Cocts(Sh(X), Set) \cong coSh(X) \ .$$

0.2 Corollary. *Let $X \xrightarrow{f} Y$ be an arbitrary locale morphism, and C a cosheaf on X . Then $C \cdot f^*$ is a cosheaf on Y .*

Proof. By Theorem 0.1, there is a cocontinuous $\varphi : Sh(X) \longrightarrow Set$ such that $C \simeq \varphi \cdot \mathrm{Yon}_X$. Hence,

$$C \cdot f^* \simeq \varphi \cdot \mathrm{Yon}_X \cdot f^* \simeq \varphi \cdot \mathbf{f}^* \cdot \mathrm{Yon}_Y \ ,$$

where $\mathbf{f}$ denotes the geometric morphism determined by f . But $\varphi \cdot \mathbf{f}^*$ is a cocontinuous functor (on $\mathrm{Sh}(Y)$), so $C \cdot f^*$ is a cosheaf. □

[3] These conventions follow [12]. See also [11].

[4] One can speak of cosheaves on an arbitrary site, i.e., on a small category equipped with a Grothendieck topology. In this paper, we shall restrict our attention to locales; however, the reader is referred to [2] where the study of cosheaves on a site is reduced to that of cosheaves on a locale plus the action of a localic groupoid. This is by analogy with the structure theorem of [11] for sheaves on a site.

[5] The objects of this category have also been referred to as *distributions* (cf. [2]).

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1 The Display Locale

We need some preliminary notation before proceeding to the definition of the display locale. If P is a partially ordered set, then we shall denote by $\mathbf{P}$ the locale corresponding to the frame $\mathcal{O}(\mathbf{P})$ of down-closed subsets of P . The principal down-closed sets $\downarrow p$ constitute a base for $\mathbf{P}$. A poset map $P \xrightarrow{f} Q$ induces a locale morphism $f : \mathbf{P} \to \mathbf{Q}$ such that for a down-closed subset A of Q ,

$$f^*A = \{p \mid f(p) \in A\} .$$

We will also use f^{-1} , but meaning inverse image in the context of posets.

1.1 *Remark.* We distinguish between a *discrete opfibration* on P , which can be realized as a functor $P \longrightarrow Set$, and a *precosheaf* on $\mathbf{P}$, i.e., a functor $\mathcal{O}(\mathbf{P}) \longrightarrow Set$. Composition with the poset map $P \to \mathcal{O}(\mathbf{P})$, $p \mapsto \downarrow p$, gives an equivalence $coSh(\mathbf{P}) \cong Set^P$. One way to prove this is to use Theorem 0.1, and the fact that $Sh(\mathbf{P}) \cong Set^{P^{op}}$

Let X denote an arbitrary locale. The *total poset* of a precosheaf D on X has as its elements all pairs (U, d), $d \in DU$, and $(U, d) \leq (V, e)$ if $U \leq V$ and $d \mapsto e$ under $DU \to DV$. This poset will also be denoted by D . The locale $\mathbf{D}$ will be referred to as the *total locale* of the precosheaf D . A base for $\mathbf{D}$ is the set of principal down-closed sets $\downarrow (U, d)$. If $D \xrightarrow{t} E$ is a natural transformation between precosheaves, then there is a morphism of total locales $t : \mathbf{D} \to \mathbf{E}$ induced by the poset map $(U, d) \mapsto (U, t_U(d))$.

Let T_X denote the terminal precosheaf on X . There is a unique natural transformation $D \to T_X$.

1.2 Definition. The *display locale* of a precosheaf D will be denoted by $disD$, and is defined to be the pullback

$$\begin{tikzcd} disD \arrow[r, "\pi"] \arrow[d, "\gamma_D"'] & \mathbf{D} \arrow[d] \\ X \arrow[r] & \mathbf{T}_X \end{tikzcd}$$

in LOC, where $\mathbf{T}_X$ is the total locale of T_X .

The frame $\mathcal{O}(\mathbf{T}_X)$ is the set of down-closed subsets of $\mathcal{O}(X)$, and the frame morphism $\mathcal{O}(\mathbf{T}_X) \to \mathcal{O}(X)$ which corresponds to $X \to \mathbf{T}_X$ carries a down-closed subset to its supremum. X is a sublocale of $\mathbf{T}_X$ (cf. [3]), and therefore $disD$ is a sublocale of $\mathbf{D}$. This inclusion will always be denoted by π . The structure map of $disD$ over X will be denoted by γ_D , as indicated in the above diagram. The display locale construction obviously constitutes a functor

$$\gamma : Set^{\mathcal{O}(X)} \longrightarrow \mathrm{LOC}/X \ .$$

A description of the display locale in terms of generators and relations now follows. Some facts will become apparent from this description which will be used in §3.

Let D denote an arbitrary precosheaf. For $R \in \mathcal{O}(\mathbf{T}_X)$ and $d \in D(\vee R)$, let

$$R^{-1}d = \{(V, e) \mid V \in R,\ e \in DV \text{ such that } e \mapsto d \text{ under } DV \to D(\vee R)\} \,.$$

Then $R^{-1}d \in \mathcal{O}(\mathbf{D})$, and the congruence relation generated by the set of pairs

$$\Big\{ \, (R^{-1}d, \downarrow(\vee R, d)) \mid R \in \mathcal{O}(\mathbf{T}_X), d \in D(\vee R) \Big\}$$

yields the frame $\mathcal{O}(disD)$ as the set (cf. [11], p. 24)

$$\Big\{ A \in \mathcal{O}(\mathbf{D}) \mid \forall (R, d) \text{ with } d \in D(\vee R),\ R^{-1}d \subseteq A \Rightarrow (\vee R, d) \in A \Big\} \,.$$

Also, π^* is calculated as

$$\pi^* B = \bigcap \{ A \in \mathcal{O}(disD) \mid B \subseteq A \} \ ,$$

and the right adjoint π_* is inclusion.

A locale morphism $X \xrightarrow{f} Y$ is said to be *dense* if $\forall U, f^*U = 0 \Rightarrow U = 0$. This is equivalent to the condition $f_* 0 = 0$.

1.3 Theorem. *Let D denote an arbitrary precosheaf. If D is coseparated, then $disD \xrightarrow{\pi} \mathbf{D}$ is dense. If D is a cosheaf, then π_* preserves disjoint suprema (and 0 since cosheaves are coseparated).*

Proof. The 0 of $\mathcal{O}(\mathbf{D})$ is the empty set $\emptyset$. If D is coseparated, then for any $R \in \mathcal{O}(\mathbf{T}_X)$ and $d \in D(\vee R)$, we have $R^{-1}d \neq \emptyset$. In fact, that $\{DV \to D(\vee R) \mid V \in R\}$ is an epimorphic family means that there is a $V_0 \in R$ and a $d_0 \in V_0$ such that $d_0 \mapsto d$ under $DV_0 \to D(\vee R)$. Therefore, $\emptyset \in \mathcal{O}(disD)$ $(R^{-1}d \subseteq \emptyset \ \Rightarrow \ (\vee R, d) \in \emptyset$ is vacuously satisfied). Consequently, $\emptyset$ is the bottom element of $\mathcal{O}(disD)$ as well, and hence, $\pi_* 0 = 0$.

Assume now that D is a cosheaf. We will show that π_* preserves disjoint *binary* suprema, leaving the general case to the reader. It must be shown that for any $A, A' \in \mathcal{O}(disD)$ with $A \cap A' = \emptyset$,

$$\bigcap \{ B \in \mathcal{O}(disD) \mid A \cup A' \subseteq B \} = A \cup A' \ . \tag{1}$$

Note that (1) follows if $A \cup A' \in \mathcal{O}(disD)$. So given $R \in \mathcal{O}(\mathbf{T}_X)$ and $d \in D(\vee R)$, it will be shown that

$$R^{-1}d \subseteq A \cup A' \Rightarrow (\vee R, d) \in A \cup A' \ .$$

Since D is a cosheaf, $R^{-1}d \neq \emptyset$ and $R^{-1}d$ is connected in the sense that for any (V, e), $(V', e') \in R^{-1}d$, there is in $R^{-1}d$ a finite diagram

$$\begin{tikzcd}
& \cdot \arrow[dl] \arrow[dr] & & \cdots & & \cdot \arrow[dl] \arrow[dr] & \\
DV & & \cdot & & \cdot & & DV'
\end{tikzcd}$$

which 'connects' e to e' . Since A and A' are down-closed and disjoint, it follows that either $R^{-1}d \subseteq A$ or $R^{-1}d \subseteq A'$. Hence, that either $(\vee R, d) \in A$ or $(\vee R, d) \in A'$. □

We include in this section two results which will not be used in this paper, but are of independent interest. The first of these concerns the pullback stability of the display locale, and the second the preservation of products.

A locale morphism $X \xrightarrow{f} Y$ is said to be *essential* if f^* has a left adjoint $f_!$. Intuitively, this means that for every open U in X , there is a smallest open set containing the image set fU . If, for example, f is an open map, then f is essential. If $X \xrightarrow{f} Y$ is essential, and if D is a precosheaf on Y , then $Df_!$ is a precosheaf on X .

1.4 Theorem. *Let $X \xrightarrow{f} Y$ be an essential morphism of locales which in addition is a surjection. Then for any precosheaf D on Y , there is a canonical morphism $dis(Df_!) \to disD$, which makes*

$$\begin{tikzcd} dis(Df_!) \arrow[r] \arrow[d, "\gamma_{Df_!}"'] & disD \arrow[d, "\gamma_D"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

a pullback in Loc.

We will use the following lemma in the proof of Theorem 1.4.

1.5 Lemma. *Let $P \xrightarrow{s} Q$ be a poset map, and assume that s has a right adjoint r with $s \cdot r = 1_Q$. Assume also that P has finite meets. Let $D \xrightarrow{\delta} Q$ be an arbitrary discrete opfibration. Denote by*

$$\begin{tikzcd} Ds \arrow[r, "S"] \arrow[d, "\delta s"'] & D \arrow[d, "\delta"] \\ P \arrow[r, "s"'] & Q \end{tikzcd}$$

the pullback in posets of δ along s . The map S is $(p,x) \mapsto (sp,x)$, for $x \in \delta^{-1}(sp)$. Then

$$\begin{tikzcd} \mathbf{Ds} \arrow[r, "S"] \arrow[d, "\delta s"'] & \mathbf{D} \arrow[d, "\delta"] \\ \mathbf{P} \arrow[r, "s"'] & \mathbf{Q} \end{tikzcd}$$

is a pullback in Loc.

Proof. Let Z denote an arbitrary locale, and assume there is given a commutative square

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "g"'] & \mathbf{D} \arrow[d, "\delta"] \\ \mathbf{P} \arrow[r, "s"'] & \mathbf{Q} \end{tikzcd}$$

of locale morphisms. Define $\rho : Z \to \mathbf{Ds}$ on principal down-closed subsets as

$$\rho^* \downarrow (p,x) = h^* \downarrow (sp,x) \wedge g^* \downarrow p \ ; \ x \in \delta^{-1}(sp) \ .$$

For an arbitrary down-closed subset G, let

$$\rho^* G = \bigvee_{(p,x)\in G} \rho^* \downarrow (p,x) \ .$$

It is not hard to see that these definitions are consistent. The non-trivial fact concerning the definition of ρ is that

$$\rho^*(\downarrow (p,y_0) \wedge \downarrow (p',z_0)) = \rho^* \downarrow (p,y_0) \wedge \rho^* \downarrow (p',z_0) \ . \qquad (2)$$

It is clear that ρ^* is order preserving, and therefore the left side of (2) is less than or equal to the right side. To see the reverse inequality, first observe that for any $q \in Q$, $g^*(\downarrow rq) = g^* s^* \downarrow q = h^* \delta^* \downarrow q$. Then, since

$$\delta^* \downarrow q = \bigsqcup_{x \in \delta^{-1} q} \downarrow (q,x) \ ,$$

we have

$$g^*(\downarrow rq) = h^*(\bigsqcup_{x\in\delta^{-1}q} \downarrow(q,x)) = \bigsqcup_{x\in\delta^{-1}q} h^* \downarrow(q,x) , \tag{3}$$

where $\bigsqcup$ indicates that the supremum is disjoint. In particular, (3) holds for $q = sp$. Hence,

$$g^* \downarrow p \leq g^* \downarrow rs(p) = \bigsqcup_{\delta^{-1}(sp)} h^* \downarrow(sp,x) .$$

From this is obtained the 'involution' formula

$$g^* \downarrow p = (\bigsqcup_{\delta^{-1}(sp)} h^* \downarrow(sp,x)) \wedge g^* \downarrow p = \bigsqcup_{\delta^{-1}(sp)} \rho^* \downarrow(p,x) . \tag{4}$$

From this follows

$$\begin{aligned}\bigsqcup_{\delta^{-1}(s(p\wedge p'))} \rho^* \downarrow(p\wedge p',x) &= g^* \downarrow(p\wedge p') = (g^* \downarrow p) \wedge (g^* \downarrow p') \\ &= (\bigsqcup_{\delta^{-1}(sp)} \rho^* \downarrow(p,y)) \wedge (\bigsqcup_{\delta^{-1}(sp')} \rho^* \downarrow(p',z)) \\ &= \bigsqcup_{\delta^{-1}(sp)\times\delta^{-1}(sp')} \rho^* \downarrow(p,y) \wedge \rho^* \downarrow(p',z) .\end{aligned}$$

To summarize,

$$\bigsqcup_{\delta^{-1}(s(p\wedge p'))} \rho^* \downarrow(p\wedge p',x) = \bigsqcup_{\delta^{-1}(sp)\times\delta^{-1}(sp')} \rho^* \downarrow(p,y) \wedge \rho^* \downarrow(p',z) . \tag{5}$$

Next, let us calculate the left side of (2). By definition,

$$\rho^*(\downarrow(p,y_0)\wedge \downarrow(p',z_0)) = \rho^*(\bigsqcup_{x\mapsto y_0,z_0} \downarrow(p\wedge p',x)) = \bigsqcup_{x\mapsto y_0,z_0} \rho^* \downarrow(p\wedge p',x) ,$$

where the subscript $x \mapsto y_0, z_0$ means those $x \in \delta^{-1}(s(p\wedge p'))$ that are carried both to y_0 and z_0 under Ds . Finally, observe that if $x \mapsto y_0, z_0$, then

$$\rho^* \downarrow(p\wedge p',x) \leq \rho^* \downarrow(p,y_0) \wedge \rho^* \downarrow(p',z_0) . \tag{6}$$

But conversely if $\rho^* \downarrow (p \wedge p', x) \neq 0$ and (6) holds, then by the disjointness of (5) it must be that $x \mapsto y_0, z_0$. This observation together with the equality (5) gives (2). Thus, ρ^* preserves finite infima of basic opens, and therefore extends to a frame morphism (morphisms defined in this manner which preserve finite meets of principal down-closed subsets of a poset extend uniquely to frame morphisms).

Let us now verify that $(\delta s) \cdot \rho = g$ and that $S \cdot \rho = h$. The former follows from the involution formula and since for all $p \in P$,

$$(\delta s)^* \downarrow p = \bigsqcup_{\delta^{-1}(sp)} \downarrow (p, x) \; .$$

That $S \cdot \rho = h$ is as follows. Observe first that $S^* \downarrow (q, y) = \downarrow (rq, y)$ and $s^* \downarrow q = \downarrow rq$, for $y \in \delta^{-1} q = \delta^{-1}(srq)$. Then $\rho^* S^* \downarrow (q, y) = \rho^* \downarrow (rq, y)$, which by definition is equal to $h^* \downarrow (srq, y) \wedge g^* \downarrow rq$. This is equal to

$$h^* \downarrow (q, y) \wedge g^*(s^* \downarrow q) = h^* \downarrow (q, y) \wedge h^* \delta^* \downarrow q$$

$$= h^*(\downarrow (q, y) \wedge \delta^* \downarrow q) = h^* \downarrow (q, y) \; .$$

To conclude the proof of the lemma, observe that

$$\downarrow (p, x) = S^*(\downarrow (sp, x)) \wedge (\delta s)^*(\downarrow p), \; x \in \delta^{-1}(sp) \; .$$

This forces the uniqueness of ρ . □

Proof of Theorem 1.4. One can verify immediately that

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathbf{T}_X \arrow[d, "f_!"] \\ Y \arrow[r] & \mathbf{T}_Y \end{tikzcd}$$

commutes, where $f_!$ denotes the locale morphism induced by the poset map $U \mapsto f_! U$. Thus, by the definition of the display locale it suffices to show that the commutative square

$$\begin{tikzcd} \mathbf{Df}_! \arrow[r, "F_!"] \arrow[d] & \mathbf{D} \arrow[d] \\ \mathbf{T}_X \arrow[r, "f_!"'] & \mathbf{T}_Y \end{tikzcd}$$

is a pullback in Loc, where $F_!$ is induced by the poset map $(U,x) \mapsto (f_!U, x)$, $x \in Df_!(U)$. This square is a pullback by Lemma 1.5. $\square$

1.6 Theorem. *The functor γ preserves finite products, i.e., for any precosheaves D, E on a locale X ,*

$$\begin{tikzcd} dis(D \times E) \arrow[r] \arrow[d] & disD \arrow[d, "\gamma_D"] \\ disE \arrow[r, "\gamma_E"'] & X \end{tikzcd}$$

is a pullback in Loc, *where $D \times E$ is the precosheaf product of D and E .*

Theorem 1.6 is a direct consequence of the definition of γ and the following.

1.7 Proposition. *Let Q denote an arbitrary poset, and assume that Q has finite meets* (this assumption can be dropped - see Remark 1.8 following the proof). *Let $D \xrightarrow{\delta} Q$ and $E \xrightarrow{\varphi} Q$ be discrete opfibrations. Then*

$$\begin{tikzcd} \mathbf{D} \times \mathbf{E} \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] & \mathbf{D} \arrow[d, "\delta"] \\ \mathbf{E} \arrow[r, "\varphi"'] & \mathbf{Q} \end{tikzcd}$$

is a pullback in Loc, *where $\mathbf{D} \times \mathbf{E}$ is the total locale of the precosheaf product $D \times E$.*

Proof. A closer examination of the proof of Lemma 1.4 with φ replacing s , reveals the following. First, given a commutative square

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "g"'] & \mathbf{D} \arrow[d, "\delta"] \\ \mathbf{E} \arrow[r, "\varphi"'] & \mathbf{Q} \end{tikzcd}$$

in Loc, the definition of $\rho : Z \to \mathbf{D} \times \mathbf{E}$ becomes

$$\rho^* \downarrow (q, y, x) = h^* \downarrow (q, x) \wedge g^* \downarrow (q, y) \; ; \quad x \in \delta^{-1} q, \; y \in \varphi^{-1} q \, .$$

Note that if δ is put into the role of s , then the definition of ρ remains the same. Also note that although E may not have finite meets, if Q does have finite meets, then the argument used for Lemma 1.5 can be adapted to show that ρ^* preserves finite meets.

Second, the involution formula (see (4))

$$g^* \downarrow (q, y) = \bigsqcup_{\delta^{-1} q} \rho^* \downarrow (q, y, x) \; ; \quad y \in \varphi^{-1} q$$

remains valid. From this one obtains $\pi_1 \cdot \rho = g$. If δ is put into the role of s , then in a similar manner $\pi_0 \cdot \rho = h$ is obtained. □

1.8 *Remark.* Proposition 1.7 remains valid without the assumption that the poset Q has finite meets. Simply regard Q as the locale $\mathbf{Q}$, and apply (the now established) Theorem 1.6. See also Remark 1.1.

2 The Connected Components Functor

An open of a locale is said to be *connected* if it cannot be non-trivially partitioned. A locale is said to be *locally connected* if the set of all connected opens forms a base. This set could be partitioned into equivalence classes where two connected opens U, V would be declared equivalent if there were connected opens $U = U_0, U_1, \ldots, U_{n-1}, U_n = V$ such that $U_{i-1} \cap U_i \neq 0$, $i = 1, \ldots, n$. One would say in this case that U and V are *chained together*. For a locally connected locale X , let $\Lambda(X)$ denote the set of equivalence classes under chaining. Note that the supremum of such an equivalence class is again a connected open. Hence, we will often identify $\Lambda(X)$ as a subset of $\mathcal{O}(X)$. The set $\Lambda(X)$ will be referred to as the set of *connected components* of X .

For any morphism of locales $L \xrightarrow{l} X$, define a precosheaf λ_l such that $\lambda_l(U) = \Lambda(l^\sharp U)$. The locale $l^\sharp U$ denotes the pullback

$$\begin{tikzcd}
l^\sharp U \arrow[r] \arrow[d] & U \arrow[d] \\
L \arrow[r, "l"'] & X
\end{tikzcd}$$

in LOC, where the frame corresponding to the open sublocale U is $\downarrow U = \{V \mid V \leq U\}$. It follows that $\mathcal{O}(l^\sharp U) = \downarrow l^* U$.

2.1 Proposition. *λ_l is a cosheaf.*

We will use the following fact (cf. [17], p. 30) to prove Proposition 2.1.

2.2 Lemma. *An open U of a locale is connected if and only if every cover R of U has the property that every pair of opens $V, W \in R$ can be chained together by opens in R .*

Proof of Proposition 2.1. First note that $\lambda_l = \lambda_{1_L} \cdot l^*$. Hence, by Corollary 0.2, we can assume that X is locally connected, and take $L \xrightarrow{l} X$ to be $X \xrightarrow{1} X$. Let there be given a cover $R = \{U_\alpha \leq U\}$ of an open $U \neq 0$ of X . We assume that U is connected (the general case immediately reduces to this), and thus, $\Lambda U \simeq 1$. To be shown is that $\varinjlim_R \Lambda U_\alpha \simeq 1$. Let $c_\alpha \in \Lambda U_\alpha$ and $c_\beta \in \Lambda U_\beta$ be arbitrary, $U_\alpha, U_\beta \in R$. We want to show that c_α and c_β are equal in the colimit. The (down-closure of the) set $\{c \in R \mid c \text{ is connected }\}$ is a cover R' of U satisfying

$c_\alpha,\ c_\beta \in R' \subseteq R$. Since U is connected, by Lemma 2.2 there are opens (which we can assume to be connected)

$$\{c_\alpha = c_0, c_1 \ldots, c_{n-1}, c_n = c_\beta\} \subseteq R'$$

such that $c_{i-1} \wedge c_i \neq 0,\ i = 1, \ldots, n$. We have

$$\begin{tikzcd}[column sep=small] & \Lambda(c_0 \wedge c_1) \arrow[dl] \arrow[dr] & & \cdots & & \Lambda(c_{n-1} \wedge c_n) \arrow[dl] \arrow[dr] & \\ \Lambda U_\alpha & & \Lambda c_1 & & \Lambda c_{n-1} & & \Lambda U_\beta \end{tikzcd}$$

which 'connects' c_α to c_β . Note: we can choose any $a_i \in \Lambda(c_{i-1} \wedge c_i),\ i = 1, \ldots, n$, since $\Lambda c_i \simeq 1,\ i = 1, \ldots, n-1$. This shows that c_α and c_β are equal in the colimit. □

Let LCLOC denote the full subcategory of LOC determined by the locally connected locales. For any locale X , there is the *connected components functor*

$$\lambda : \mathrm{LCLOC}/X \longrightarrow coSh(X)$$

given by

$$L \xrightarrow{l} X \ \mapsto\ (\lambda_l : U \mapsto \Lambda(l^\sharp U))\ .$$

Alternative Description of Connected Components:

For a locale Y and any precosheaf D on Y , define a precosheaf

$$D^+ : U \mapsto \varprojlim_{JU} \varinjlim_{R} D$$

where the colimit runs over the opens of a given cover R of U, and the limit runs over the collection JU of covers of U .

2.3 Proposition. *There is a natural transformation $D^+ \to D$ through which every natural transformation $C \to D$, with C a cosheaf, factors uniquely.*

No proof will be given here of this result, but see Example 5.6 for the case $D = T_Y^{+}$ (this precosheaf sends $U \neq 0 \mapsto 1$ and $0 \mapsto \emptyset$). Cosheaves fall within the general framework of [15] as shown in [7], and the above construction and result can be established in that setting.

2.4 Definition. Denote by $c(D)$ the precosheaf D^{++} .

2.5 *Example.* A typical element of $c(T_Y)(V)$ is a consistent vector (b_R) , where R runs over the covers of V and b_R denotes an equivalence class of opens in R under chaining in R . 'Consistent' means that if $R \subseteq R'$, then any non-0 $U \in b_R$, $U' \in b_{R'}$ are chained together in R' . □

2.6 Proposition. *For any* $L \xrightarrow{l} X \in$ LCLOC$/X$, $\lambda_l \simeq c(T_L) \cdot l^*$.

Proof. Since $\lambda_l = \lambda_{1_L} \cdot l^*$, it suffices to show that $\lambda_{1_L} \simeq c(T_L)$. Let $V \in \mathcal{O}(L)$. We want to show $\Lambda(V) \simeq c(T_L)(V)$. Send $d \in \Lambda(V)$ to the consistent vector (d_R) (in the notation of Example 2.5) such that for any cover R of V , d_R is the unique equivalence class such that $d \leq \bigvee d_R$. This map is isomorphism because every consistent vector (b_R) is uniquely determined by b_{R_c} , where R_c denotes the cover of V given by the connected components of V . To see this, observe that for any R , $\bigvee b_{R_c} \leq \bigvee b_R$. □

2.7 *Remark.* λ extends to a functor LOC$/X \longrightarrow Set^{\mathcal{O}(X)}$ by defining for any $Y \xrightarrow{f} X$, $\lambda_f = c(T_Y) \cdot f^*$. In general, the c construction more resembles *quasi*-components than it does connected components. In fact, let λ^c denote connected components and let λ^q denote quasi-components. Then for any map $Y \xrightarrow{f} X$ of topological spaces, there are natural transformations of precosheaves

$$\lambda_f^c \longrightarrow \lambda_f^q \hookrightarrow \lambda_f \ .$$

When Y is locally connected these three precosheaves coincide and that precosheaf is a cosheaf. See Example 5.6, where the display *space* of $c(T_X)$ is computed for an arbitrary topological space X .

3 THE ADJOINTNESS $\lambda \dashv \gamma$

3.1 Theorem. *The connected components functor λ is a partial left adjoint to γ . For any $L \xrightarrow{l} X \in$ LCLOC$/X$, and any precosheaf D on X , there is a natural bijection*

$$\frac{\textit{locale morphisms } l \to \gamma_D}{\textit{natural transformations } \lambda_l \to D \ .}$$

For a cosheaf C , $disC$ is a locally connected locale, and the counit $\lambda\gamma_C \to C$ is an isomorphism.

The above adjointness is obtained as follows. Observe that for a precosheaf D , $\gamma_D^*(U)$ is the disjoint supremum over DU ,

$$\gamma_D^*(U) = \bigsqcup_{DU} \pi^* \downarrow (U, d) \ ,$$

where π is the locale inclusion $disD \to \mathbf{D}$. Hence, given $f : L \to disD$ such that $\gamma_D \cdot f = l$, we have

$$l^*U = \bigsqcup_{DU} f^*\pi^* \downarrow (U, d) \ .$$

Define a natural transformation $\bar{f}$ such that for $U \in \mathcal{O}(X)$ and c a component of $l^\sharp U$,

$$\bar{f}_U(c) = \text{ the unique } d_0 \in DU \text{ such that } c \leq f^*\pi^* \downarrow (U, d_0) \ .$$

The return passage associates to a natural transformation $t : \lambda_l \to D$ the morphism $\gamma_t \cdot \eta_l$, where the unit

$$\eta_l : L \to dis(\lambda_l)$$

is obtained as the universal morphism arising from the commutative square

$$\begin{tikzcd} L \arrow[r, "\nu"] \arrow[d, "l"'] & \boldsymbol{\lambda}_l \arrow[d] \\ X \arrow[r] & \mathbf{T}_X \ . \end{tikzcd}$$

By definition, $\nu^* \downarrow(U,c) = c$, for $c \in \Lambda(l^\sharp U)$. These two passages give the expressed bijection.

In the case of a cosheaf C , $disC$ is locally connected. In fact, by Theorem 1.3, the inclusion $disC \xrightarrow{\pi} \mathbf{C}$ satisfies the hypothesis of the following lemma.

3.2 Lemma. *Suppose a morphism $X \xrightarrow{f} Y$ of locales has the property that f_* preserves 0 and disjoint finite suprema. Then f^* takes connected opens to connected opens.*

Proof. Let B be an arbitrary connected open of Y . To see that f^*B is connected, write

$$f^*B = U \vee V,\ U \wedge V = 0 \ .$$

Then

$$B \leq f_*f^*B = f_*U \vee f_*V,\ f_*U \wedge f_*V = 0 \ .$$

Therefore (say) $B \leq f_*V$, whence $f^*B \leq V$. □

Now observe that for any poset P, the basic opens $\downarrow p$ of the locale $\mathbf{P}$ are connected. In particular, the basic opens $\downarrow(U,d)$ of the total locale $\mathbf{C}$ of the cosheaf C are connected. Thus, by Lemma 3.2, $disC$ is locally connected.

Now that we know $disC$ is locally connected, we can define the counit $\varepsilon_C : \lambda\gamma_C \to C$, and show that it is an isomorphism. For any open U in X , observe that

$$\Lambda(\gamma_C^*(U)) = \coprod_{d \in CU} \Lambda(\pi^* \downarrow(U,d)) \ .$$

Hence, define

$$\varepsilon_C(U) : \Lambda(\gamma_C^*(U)) \to CU$$

as

$$\varepsilon_C(U)(c) = \text{ the unique } d \text{ such that } c \in \Lambda(\pi^* \downarrow(U,d)) \ .$$

This definition is natural in U and in C . ε_C is a monomorphism (in precosheaves) because the basic opens $\pi^* \downarrow(U,d)$ are connected. Since

π is dense (Theorem 1.3), ε_C is an epimorphism in precosheaves. This concludes the proof of Theorem 3.1.

4 COMPLETE SPREADS

It was seen in the previous section that the category of cosheaves on an arbitrary locale can be represented as a full reflective subcategory of LCLOC$/X$. Let us call a locally connected locale $L \to X$ a *cosheaf locale* if it arises as the display locale of its cosheaf of connected components. By definition, $L \to X$ is a cosheaf locale if

$$\begin{tikzcd} L \arrow[r, "\nu"] \arrow[d] & \boldsymbol{\lambda}_l \arrow[d] \\ X \arrow[r] & \mathbf{T}_X \end{tikzcd}$$

is a pullback in LOC, where $\nu^* \downarrow (U, c) = c$.

The notion of cosheaf locale will be examined in this section. We shall see that it is equivalent to the (localic version of the) notion of *complete spread* (cf. [5]).

4.1 Definition. An object $L \xrightarrow{l} X \in$ LCLOC$/X$ will be called a *spread* if the components of the open sets $\{l^*U \mid U \in \mathcal{O}(X)\}$ constitute a base for L .

Let $L \xrightarrow{l} X \in$ LCLOC$/X$. Let $\mathcal{C}_s$ denote the following covering system of the poset λ_l, i.e., of the total poset of the cosheaf λ_l. Recall that a typical element of the poset λ_l is a pair (U, c), where c is a component of l^*U . Declare $\{(U_\alpha, c_\alpha) \le (U, c)\}$ to be a cover if $\bigvee c_\alpha = c$. Let us refer to $\mathcal{C}_s$ as the *spread coverage* of λ_l .

4.2 Proposition. *The image of ν coincides with the sublocale of $\boldsymbol{\lambda}_l$ determined by $\mathcal{C}_s$.*

Proof. Denote the surjection-inclusion factorization in LOC of ν by $L \twoheadrightarrow I \hookrightarrow \boldsymbol{\lambda}_l$. The local operator, or nucleus, given by ν is $\nu_* \nu^*$, and the sublocale I is given as the fixed-point set of this operator. The value of this operator on a basic open $\downarrow (U, c_0)$ is the down-closed set $\{(V, c) \mid c \leq c_0\}$. It is clear that I coincides with the sublocale determined by the coverage $\mathcal{C}_s$. □

4.3 Proposition. *For any $L \xrightarrow{l} X \in$ LCLOC$/X$, the following are equivalent.*

1. *l is a spread.*
2. *L is the sublocale of $\boldsymbol{\lambda}_l$ determined by $\mathcal{C}_s$.*
3. *L is a sublocale of a cosheaf locale.*
4. *The unit η_l is an inclusion.*
5. *The morphism ν is an inclusion.*
6. *L is a dense sublocale of a cosheaf locale.*

Proof. *3, 4* and *5* are clearly equivalent. Observe that $\nu_* W = \{(V, c) \mid c \leq W\}$ for any open W of L . Then $\nu^* \nu_* W = \bigvee_{c \leq W} \nu^* \downarrow (V, c) = \bigvee c$. If l is a spread, then this supremum gives back W. Thus, *1* implies *5*. Conversely, if this supremum gives back W, i.e., if ν is an inclusion, then this says that l is a spread. Next, that *2* implies *5* is a triviality, and the converse implication follows from Proposition 4.2. The equivalence of *6* to the other conditions is a consequence of the fact that η_l is dense. To see that η_l is dense, observe that ν is dense; if $\nu^* \downarrow (U, c) = 0$, then $c = 0$, whence $\downarrow (U, c) = \emptyset$. Thus, η_l is dense since $\nu = \pi \cdot \eta_l$ and since π is an inclusion. □

Let $\mathcal{C}_d$ denote the covering system on λ_l that yields the display locale. This covering system was described in §1. In general, the spread coverage is finer than $\mathcal{C}_d$, i.e., $\mathcal{C}_d \subseteq \mathcal{C}_s$.

4.4 Proposition. *The following are equivalent for* $L \xrightarrow{l} X \in \text{LCLOC}/X$.

1. $C_d = C_s$.

2. *The unit* η_l *is a surjection.*

3. *For any* $(U, c) \in \lambda_l$ *, and any* $V \leq U$ *such that* $c \leq l^*V$ *, we have* $\pi^* \downarrow (V, c) = \pi^* \downarrow (U, c)$.

Proof. It is clear that *1* and *2* are equivalent. Assume η is a surjection, and that there is given $(U, c) \in \lambda_l$ and $V \leq U$ such that $c \leq l^*V$. Then

$$\eta^*\pi^* \downarrow (V, c) = \nu^* \downarrow (V, c) = c = \eta^*\pi^* \downarrow (U, c) .$$

Hence, $\pi^* \downarrow (V, c) = \pi^* \downarrow (U, c)$ since η is a surjection. Finally, assume *3*. It will be shown that η is a surjection, i.e., that for any $W \leq Z$, $\eta^*W = \eta^*Z \Rightarrow W = Z$. To be a surjection, it suffices that this property be satisfied on a base. Recall that the opens $\pi^* \downarrow (U, c)$ are a base for $dis(\lambda_l)$. Assume there is given $\pi^* \downarrow (V, d) \leq \pi^* \downarrow (U, c)$ such that under η^* these opens are identified. Since $\pi \cdot \eta = \nu$ one obtains that $d = c$. By the hypothesis *3*,

$$\pi^* \downarrow (V, d) = \pi^* \downarrow (V \wedge U, c) = \pi^* \downarrow (U, c) .$$

Note that c is a component of $l^*(V \wedge U)$. □

4.5 Definition. $L \xrightarrow{l} X \in \text{LCLOC}/X$ is said to be *complete* if any one of the equivalent conditions of Proposition 4.4 is satisfied.

Propositions 4.3 and 4.4 (and Theorem 3.1) imply the following.

4.6 Theorem. *The notions of cosheaf locale and complete spread are equivalent. For any locale* X *, the display locale construction establishes an equivalence between* $coSh(X)$ *and the category of complete spreads over* X *.*

4.7 *Remark.* If $L \xrightarrow{l} X$ is a complete spread, it has been shown that η_l^* is an isomorphism of frames. Its inverse is given, at least on connected opens c which appear as a component of an l^*U (since l is a spread there is a base for L of such c), as $c \mapsto \pi^* \downarrow (U, c)$. Note that if V is any other open of X such that c is a component of l^*V, then $\pi^* \downarrow (V, c) = \pi^* \downarrow (U, c)$ by the completeness of l .

This discussion of complete spreads (as cosheaf locales) will be continued in the spatial context in the next section; however, we include here the following basic property of complete spreads. Naturally, the proof will avail itself of Theorem 4.6 (first statement).

4.8 Proposition. *Let $M \xrightarrow{m} L \xrightarrow{l} X$ be morphisms of locales with M and L locally connected.*

1. *If l and m are complete spreads, then so is lm .*
2. *If lm and l are complete spreads, then so is m .*

The proof of Proposition 4.8 will use the following lemma (compare Lemma 1.5).

4.9 Lemma. *Let $P \xrightarrow{s} Q$ be an arbitrary poset map, and let $D \xrightarrow{\delta} Q$ denote an arbitrary discrete opfibration. Then the commutative square*

$$\begin{tikzcd} \mathbf{Ds} \arrow[r, "S"] \arrow[d, "\delta s"'] & \mathbf{D} \arrow[d, "\delta"] \\ \mathbf{P} \arrow[r, "s"'] & \mathbf{Q} \end{tikzcd}$$

is a pullback in LCLOC.

Proof. The geometric morphism $\mathrm{s} : Sh(\mathbf{P}) \longrightarrow Sh(\mathbf{Q})$ induced by s is *essential*, i.e., the inverse image functor s^* has a left adjoint $\mathrm{s}_!$. Therefore (by Theorem 0.1), composition with s^*

$$\_ \cdot s^* : coSh(\mathbf{P}) \longrightarrow coSh(\mathbf{Q})$$

has right adjoint

$$\_ \cdot s_! : coSh(\mathbf{Q}) \longrightarrow coSh(\mathbf{P}) \ ,$$

where

$$s_! : \mathcal{O}(\mathbf{P}) \to \mathcal{O}(\mathbf{Q}) \, ; \ W \mapsto \downarrow \{ sp \mid p \in W \} \ .$$

For any $L \xrightarrow{l} \mathbf{P}$ with L locally connected, there are natural bijections

$$\begin{array}{c} sl \to \delta = \gamma\lambda_\delta \text{ over } \mathbf{Q} \\ \hline \lambda_{sl} \to \lambda_\delta \text{ in } coSh(\mathbf{Q}) \\ \hline \lambda_l \cdot s^* \to \lambda_\delta \text{ in } coSh(\mathbf{Q}) \\ \hline \lambda_l \to \lambda_\delta \cdot s_! \text{ in } coSh(\mathbf{P}) \\ \hline \lambda_l \to \lambda_{\delta s} \text{ in } coSh(\mathbf{P}) \\ \hline l \to \gamma\lambda_{\delta s} = \delta s \text{ over } \mathbf{P}. \end{array}$$

Note that for any $q \in Q$,

$$\lambda_{sl}(\downarrow q) = \Lambda l^* s^* (\downarrow q) = \lambda_l \cdot s^*(\downarrow q) \ ,$$

and that therefore, $\lambda_{sl} = \lambda_l \cdot s^*$ since the principal down-closed subsets are a base of $\mathbf{Q}$. Also, for any $p \in P$,

$$\lambda_\delta \cdot s_!(\downarrow p) = \lambda_\delta(\downarrow sp) = \delta^{-1}s(p) = \lambda_{\delta s}(\downarrow p) \ ,$$

and therefore, $\lambda_\delta \cdot s_! = \lambda_{\delta s}$. □

Proof of Proposition 4.8. *1.* Consider the following commutative diagram in Loc.

$$\begin{tikzcd}
M \arrow[r, "\nu_{lm}"] \arrow[d, "m"'] \arrow[dr, phantom, "1"] & \boldsymbol{\lambda}_{lm} \arrow[r, "\mu"] \arrow[d, "p"] \arrow[dr, phantom, "2"] & \boldsymbol{\lambda}_m \arrow[d] \\
L \arrow[r, "\nu_l"] \arrow[d, "l"'] \arrow[dr, phantom, "3"] & \boldsymbol{\lambda}_l \arrow[r, "q"'] \arrow[d] & \mathbf{T}_L \\
X \arrow[r] & \mathbf{T}_X &
\end{tikzcd}$$

The map p sends (U, d), d a component of m^*l^*U, to (U, c) where c is the component of l^*U containing the image $m(d)$. Under the same terminology, by definition $\mu(U,d) = (c,d)$ and $q(U,c) = c$. It is clear that square 2 commutes. It is not clear that $\nu_m = \mu \cdot \nu_{lm}$ (and in general it will not be true), but it holds in this case because l is a spread. That $q \cdot \nu_l$ is equal to the canonical inclusion $L \to \mathbf{T}_L$ also follows from the fact that l is a spread. To be shown is that 1-3 is a pullback in LOC. By hypothesis, it suffices to show that 1 is a pullback in LOC. It is not hard to check that the discrete opfibrations $\lambda_m q$ and p are identical. This puts us in a position to apply Lemma 4.9 (with q as s). We conclude that 2 is a pullback in LCLOC. Included in the hypotheses is that 1-2 is a pullback in LOC, and hence in LCLOC. Therefore, 1 is a pullback in LCLOC. But the pullback in LOC of 1 is $dis(\lambda_{lm})$, and this locale is locally connected. Hence, $dis(\lambda_{lm})$ is the pullback of 1 in LCLOC as well. Thus, M and $dis(\lambda_{lm})$ must coincide.

2. This is similar to *1*. □

4.10 *Remark.* Proposition 4.8 has the immediate consequence that for any cosheaf $C \in coSh(X)$, the slice category $coSh(X)/C$ is equivalent to $coSh(disC)$.

5 THE SPATIAL CASE

Let LCTSP$/X$ denote the full subcategory of TSP$/X$ determined by those objects $L \to X$ with L locally connected. The inclusion of this full subcategory has a right adjoint as will now be shown. The reader is reminded of the fact that a quotient space of an locally connected space is locally connected (cf. [4], p. 125).

5.1 Proposition. *The intersection of a set of locally connected topologies is again a locally connected topology.*

Proof. First observe that if X and Y are two locally connected spaces, then the coproduct $X + Y$, which has as its underlying set the disjoint union of X and Y , is locally connected. Next, let T and T' be two locally connected topologies on a set X . Then $(X,T)+(X,T')$ is locally connected as just remarked. Give X the quotient topology with respect to the codiagonal $(X,T)+(X,T')\to X$. This topology on X is locally connected as remarked above. Also, observe that it coincides with the intersection of T and T' . This establishes the proposition for two locally connected topologies. The general case can be established in the same way. □

Given any space (X,T), by Proposition 5.1 there exists a smallest locally connected topology larger than T. Denote this space by $\hat{X}$ and denote the identity map $\hat{X}\to X$ by ϵ_X . Let $L \xrightarrow{l} X$ be an arbitrary locally connected space over X , and let I denote the image of l . Then there is a commutative diagram

$$\begin{tikzcd} L \arrow[r] \arrow[dr, "l"'] & L+ \mid X - I \mid \arrow[r] \arrow[d, "p"] & \hat{X} \arrow[dl, "\epsilon_X"] \\ & X & \end{tikzcd}$$

where the p is a surjection, and $\mid X - I \mid$ denotes the complement of I carrying the discrete topology. Note that the coproduct $L+ \mid X - I \mid$ is locally connected. Therefore, the quotient topology on X induced by p is locally connected, and hence finer than the topology on $\hat{X}$. Consequently, p factors through ϵ_X . This gives a unique (because ϵ_X is a monomorphism) factorization of l through ϵ_X . Thus, $X \mapsto \hat{X}$ is right adjoint to the inclusion of LCTSP into TSP.

5.2 Definition. $\hat{X}$ will be referred to as the *locally connected coclosure* of X .

5.3 *Example.* The locally connected coclosure of a totally disconnected space (i.e., no connected subsets other than the singletons) is discrete. □

Let *loc* : TSP$\longrightarrow$LOC denote the functor which carries a topological space X to the locale associated to the frame of open sets of X . Let

$| \cdot |$ denote its right adjoint, i.e., the 'points' functor. A locale Y is said to be *spatial* (cf. [10, 12]) if the counit $loc \mid Y \mid \to Y$ is an isomorphism. Observe that a space Y is locally connected if and only if $locY$ is. Therefore, we have $loc : \text{LCTSP} \longrightarrow \text{LCLOC}$, and this functor has right adjoint $\widehat{| \cdot |}$. If a locale Y is spatial and locally connected, then $| Y |$ is locally connected, and therefore $loc|\widehat{Y}| = loc \mid Y \mid \simeq Y$. Conversely, if $loc|\widehat{Y}| \to loc \mid Y \mid \to Y$ is an isomorphism, then Y is: spatial, being the locale of opens of *some* topological space (namely $|\widehat{Y}|$), and locally connected, being the locale of opens of a locally connected space. But then $| Y |$ must be locally connected, hence $|\widehat{Y}| = | Y |$. The following fact has been established.

5.4 Proposition. *For any locale Y , the following are equivalent.*

1. *Y is spatial and locally connected.*
2. *$loc|\widehat{Y}| \to loc \mid Y \mid \to Y$ is an isomorphism.*
3. *Y is spatial and $| Y |$ is locally connected.*

Let X denote an arbitrary topological space. The composite functor

$$\text{LCTSP}/X \xrightarrow{loc/X} \text{LCLOC}/locX \xrightarrow{\lambda} coSh(X)$$

will also be denoted by λ , and will be referred to as the *connected components functor for spaces*. For a locally connected space $L \xrightarrow{l} X$ over X , $\lambda_l(U)$ is the set of connected components of $l^{-1}U$ in the usual sense. This functor has a right adjoint since it is a composite of functors with right adjoints. Note: The right adjoint of loc/X is the locally connected coclosure of the pullback along the unit $X \to \mid locX \mid$. An explicit description of the right adjoint of λ for spaces follows. Naturally, this discussion will center on the display *space* of a precosheaf. We remind the reader that for any poset P , the points $| \mathbf{P} |$ of the locale $\mathbf{P}$ can be identified with the up-closed, down-directed subsets of P .

5.5 Definition. Associated to any precosheaf D on X is a topological space $disD$ over X which will be referred to as *display space* of D . By definition, it is the pullback

$$\begin{tikzcd} disD \arrow[r, "\pi_D"] \arrow[d, "\gamma_D"'] & \mid \mathbf{D} \mid \arrow[d] \\ X \arrow[r, "\pi_X"'] & \mid \mathbf{T}_X \mid \end{tikzcd}$$

is TSP, where $\pi_X(x) = \{U \mid x \in U\}$.

It will be helpful to write down an explicit description of the space $disD$. The underlying set of $disD$ is

$$\mid disD \mid = \{ \text{ pairs } (x,t) \mid x \in X,\ \lambda_x \xrightarrow{t} D \},$$

where t is a natural transformation, and λ_x is the 'point-cosheaf' defined by the point $1 \xrightarrow{x} X$,

$$\lambda_x : U \mapsto \begin{cases} 1 & \text{if } x \in U \\ \emptyset & \text{otherwise.} \end{cases}$$

The set $\mid disD \mid$ can also be described as the disjoint union over X of the *costalks* $S_x = \varprojlim DU$, where this limit is taken over those U containing x . The elements of S_x are the *cogerms* over x . The map γ_D is the projection $(x,t) \mapsto x$, and π_D sends a cogerm (x,t) to the up-closed, down-directed set $\{(U,d) \mid (x,t) \in (U,d)\}$. If $\mathcal{B}$ denotes a base for the topology on X , then the sets

$$(B,b) = \{(x,t) \mid x \in B \text{ and } t_B = b\}\,;\ B \in \mathcal{B},\ b \in DB\ ,$$

constitute a base for $disD$. In particular, a base for the topology on $disD$ can be taken to be $\{(U,b) \mid U \in \mathcal{O}(X)\ , b \in DU\ \}$.

5.6 *Example. The display space of the precosheaf* $c(T_X)$ (see Example 2.5). Let X denote a topological space. It will be shown in this example that $dis(c(T_X))$ *has* X *as its underlying set, but topologized by taking as basic opens the quasi-components of the open sets of* X . To digress for a moment, for $x \in X$, let us denote by $\tilde{x}$ the intersection of all closed-open sets containing x . Then declaring x and y equivalent if $\tilde{x} = \tilde{y}$ is an equivalence relation, and the equivalence classes of this relation are called the quasi-components of X .

Let us simplify the notation and write cT for $c(T_X)$. First observe that for any cosheaf C , there is a unique map $C \to cT$ (see Proposition 2.3). Existence: For an open U, $u \in CU$ and $R \in JU$, let $t_U(u)_R$ denote the R^{th}-component of the consistent vector $t_U(u)$. Choose $V \in R$ such that there is $v \in CV$ with $v \mapsto_C u$ (i.e., v is carried to u under the map $CV \to CU$). Let $t_U(u)_R$ be the equivalence class of V under chaining in R . This gives a well-defined natural transformation. Uniqueness: Denote by t the natural transformation just constructed. Suppose there is $C \xrightarrow{s} cT$ which is not equal to t . Then there is an open U and a $u \in CU$ such that the consistent vector $s_U(u) \neq t_U(u)$. Hence, there is a cover R of U such that $s_U(u)_R \neq t_U(u)_R$. Let $U_\alpha \neq 0$ represent $t_U(u)_R$ and let $U_\beta \neq 0$ represent $s_U(u)_R$. Note that by the definition of t one can choose U_α such that there is a $v \in CU_\alpha$ with $v \mapsto_C u$. Since $s_U(u)_R \neq t_U(u)_R$, U_β and U_α are not chained together in R . This violates at v the commutativity of

$$\begin{tikzcd} CU_\alpha \arrow[r] \arrow[d, "s_{U_\alpha}"'] & CU \arrow[d, "s_U"] \\ cT(U_\alpha) \arrow[r] & cT(U) \end{tikzcd}.$$

Thus, the underlying set of $dis(cT)$ is in bijection with X , i.e., $\gamma : dis(cT) \to X$ is a bijection. For $x \in X$, $\gamma^{-1}x$ is the unique morphism $\lambda_x \to cT$. Let us use x again to refer to this morphism. Recall that a base for the topology on $dis(cT)$ is the collection of sets $\{(U, b) \mid U \in \mathcal{O}(X),\ b \in cT(U)\}$. Identifying the underlying set of $dis(cT)$ with X , these are the sets

$$(U, b) = \{\ x \in U \mid x_U = b\ \}\ .$$

It will be shown that these sets are precisely the quasi-components of the open set U . First, note that for any $z \in (U,b)$, $(U,b) \subseteq \tilde{z}$. Next, note that

$$(U,b) = \bigcap_{JU} P_R \ , \quad P_R = \bigcup_{b_R} V \ ,$$

where $b = (b_R)$, as R runs over the covers JU of U , and where b_R denotes an equivalence class of opens in R under chaining in R . Each P_R is closed-open in U because it is a member of an open partition of U . Thus, (U,b) is equal to $\widetilde{(U,b)}$, i.e., to the intersection of all the closed-open sets containing (U,b) . Whence $\tilde{z} \subseteq (U,b)$, for any $z \in (U,b)$. Thus, (U,b) is a quasi-component of U . □

For the following, we shall temporarily denote the display space by $dis^s D$, to distinguish it from the display locale $disD$.

5.7 Theorem. *The points of the display locale of a precosheaf give its display space. More precisely, for any precosheaf D , there is a morphism τ which makes the diagram*

$$\begin{tikzcd} dis^s D \arrow[r, "\tau"] \arrow[d, "\gamma_D"'] & \mid disD \mid \arrow[d, "\mid \gamma_D \mid"] \\ X \arrow[r] & \mid locX \mid \end{tikzcd}$$

a pullback of spaces, where the bottom arrow is the unit of $loc \dashv \mid \cdot \mid$ at X . If X is sober, then $dis^s D \simeq \mid disD \mid$.

Proof. The canonical *locale* morphism $locX \to \mathbf{T}_X$ gives rise to a commutative triangle

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "\pi_X"] & \\ \mid locX \mid \arrow[r] & \mid \mathbf{T}_X \mid \end{tikzcd}$$

in TSP. The theorem follows as

$$\begin{tikzcd} \mid disD \mid \arrow[r, "\mid \pi \mid"] \arrow[d] & \mid \mathbf{D} \mid \arrow[d] \\ \mid locX \mid \arrow[r] & \mid \mathbf{T}_X \mid \end{tikzcd}$$

is a pullback in TSP (since $|\cdot|$ preserves limits). □

5.8 *Remark.* The map τ of Theorem 5.7 has the following description. There is a locale morphism μ making

$$\begin{tikzcd} loc(dis^s D) \arrow[r, "\mu"] \arrow[d, "loc(\gamma_d)"'] & \mathbf{D} \arrow[d] \\ locX \arrow[r] & \mathbf{T}_X \end{tikzcd}$$

commute. The frame morphism μ^* sends the basic open set $\downarrow(U,d)$ to the open set (U,d) of $dis^s D$. Hence, there is an induced morphism $\bar{\mu} : loc(dis^s D) \to disD$ of which τ is the transpose under $loc \dashv |\cdot|$.

For a natural transformation $D \xrightarrow{s} E$, define

$$\gamma_s : disD \to disE;\ (x,t) \mapsto (x, st)\ .$$

The function γ_s is easily seen to be continuous. The functor γ followed by the locally connected coclosure gives a functor (which should be denoted by $\hat{\gamma}$, but will be denoted again by γ)

$$\gamma : Set^{\mathcal{O}(X)} \longrightarrow \text{LCTSP}/X\ ,\ D \mapsto \widehat{disD} \xrightarrow{\gamma_D} X\ .$$

Let i denote the full inclusion of $coSh(X)$ into $Set^{\mathcal{O}(X)}$.

5.9 Theorem. *γ is right adjoint to $i\lambda$* (compare Theorem 3.1).

Proof. Maps $L \rightarrow \widehat{dis^s D}$ over X are in bijective correspondence with maps $L \rightarrow dis^s D$ over X . By Theorem 5.7, such maps correspond to commutative squares

$$\begin{tikzcd} L \arrow[r] \arrow[d, "l"'] & \mid disD \mid \arrow[d] \\ X \arrow[r] & \mid locX \mid \end{tikzcd}$$

which correspond to commutative triangles

$$\begin{tikzcd} locL \arrow[r] \arrow[dr, "loc(l)"'] & disD \arrow[d] \\ & locX \end{tikzcd}$$

These correspond, by Theorem 3.1, to natural transformations $\lambda_l \rightarrow D$.
□

5.10 *Remark.* We will later need concrete descriptions of the unit and counit of the adjointness $\lambda \dashv \gamma$ for spaces. These descriptions follow.
The unit η : For a locally connected space $L \xrightarrow{l} X$, to define a map

$$\eta_l : L \rightarrow \widehat{dis\lambda_l}$$

it suffices (since L is locally connected) to define a map

$$\eta_l : L \rightarrow dis\lambda_l \ .$$

(In any case, $dis\lambda_l$ is locally connected - see Proposition 5.14). For $a \in L$, let

$$\eta_l(a) = (la, t^a) \ ,$$

where $t^a : \lambda_{la} \rightarrow \lambda_l$ is the natural transformation such that for $la \in U \in \mathcal{O}(X)$,

$$t^a_U : 1 \rightarrow \Lambda(l^{-1}U)$$

selects the connected component of $l^{-1}U$ which contains a . Observe that η_l is continuous. In fact, let (B, b) is a basic open of $dis\lambda_l$, where $b \in \lambda_l B$, i.e., b is a connected component of $l^{-1}B$. Then $\eta_l^{-1}(B, b) = b$, which is an open set since L is locally connected.
The counit ε : Given a precosheaf D , we want to define a natural transformation

$$\varepsilon_D : \lambda\gamma_D \to D \, .$$

For $U \in \mathcal{O}(X)$, define

$$\varepsilon_D U : \Lambda(\gamma_D^{-1}U) \to DU$$

as follows. Note: $\Lambda(\gamma_D^{-1}U)$ is referring to connected components in $\widehat{disD}$. Send an $(x, t) \in \gamma_D^{-1}U$, i.e., $\lambda_x \xrightarrow{t} D$, $x \in U$, to $t_U \in DU$. This defines a map $\gamma_D^{-1}U \to DU$ which will be shown to factor through connected components. Let (x, t) and (y, s) be in the same connected component of $\gamma_D^{-1}U$ (in $\widehat{disD}$). Observe that $\gamma_D^{-1}U$ is the disjoint union of opens

$$\gamma_D^{-1}U = \bigsqcup_{b \in DU} (U, b) \, .$$

Thus, if (x, t) and (y, s) are in the same connected component, then there is a unique $b_0 \in DU$ such that (U, b_0) contains both (x, t) and (y, s) . In other words, $t_U = b_0 = s_U$. This defines $\varepsilon_D U$. It is left to the reader to see that ε_D is natural in U , and that ε is natural in D .

Our analysis allows us to extract the following information about the locally connected coclosure.

5.11 Proposition. *The quasi-components of the open sets of a space X are open in its locally connected coclosure $\hat{X}$.*

Proof. There is a unique map $\lambda_\epsilon \to cT$ (see Example 5.6), where $\hat{X} \xrightarrow{\epsilon} X$. Thus, we obtain by adjointness a map $\hat{X} \to dis(cT)$ over X . It was shown in Example 5.6 that if q is a quasi-component of an open of X , then $\gamma_{cT}^{-1}q$ is open (in fact, such sets constitute a base) in $dis(cT)$. Thus, $\epsilon^{-1}q$ is open in $\hat{X}$ as well. □

5.12 Definition. A precosheaf D will be said to be a *spatial cosheaf* if the counit ε_D is an isomorphism.

It is shown in the next section that on a complete metric space, every cosheaf is a spatial cosheaf.

By Theorem 5.9, there is an equivalence between the category of spatial cosheaves and the category of those $L \xrightarrow{l} X$ such that η_l is a homeomorphism. We analyse these conditions.

5.13 Proposition. *For any precosheaf D , ε_D is a monomorphism if and only if the basic opens (B, b) of $disD$ are connected (in which case, $disD$ is locally connected and is therefore equal to $\widehat{disD}$).*

Proof. It was observed in Remark 5.10 that $\gamma_D^{-1}B$ is partitioned by the open sets (B, b) , $b \in DB$. If the (B, b)'s are connected, then the non-0 ones are the connected components of $\gamma_D^{-1}B$. That is, $\lambda\gamma_D B$ is in bijection (via $\varepsilon_D B$) with those $b \in DB$ such that $(B, b) \neq 0$. Conversely, if some (B, b) is not connected, then $\varepsilon_D B$ will identify its connected components, i.e., $\varepsilon_D B$ will not be a monomorphism. □

5.14 Proposition. *For any $L \xrightarrow{l} X \in$ LCTSP , $dis\lambda_l$ is locally connected, and the cosheaf λ_l is spatial.*

Proof. It will be shown that ε_{λ_l} is both an epimorphism and a monomorphism in $Set^{\mathcal{O}(X)}$. The (U, c)'s of $dis\lambda_l$ are connected. In fact, given a (U, c) observe that the closure of

$$\eta_l c = \eta c = \{\eta a \mid a \in c\}$$

in $\gamma^{-1}U$ is equal to (U, c) , where η is the unit. To see this, let $\overline{\eta c}$ denote the closure of ηc in $\gamma^{-1}U$. (U, c) is closed in $\gamma^{-1}U$ since its complement in $\gamma^{-1}U$ is a union of basic opens and hence open. Therefore, $\overline{\eta c} \subseteq (U, c)$. For the reverse inclusion, let $(x, t) \in (U, c)$ and let $(V, d) \subseteq \gamma^{-1}U$ be an arbitrary basic open containing (x, t) . To be shown is that (V, d) meets ηc . Observe that

$$d = \eta^{-1}(V, d) \subseteq \eta^{-1}\gamma^{-1}U = l^{-1}U \ .$$

Therefore, d is a component of $l^{-1}(U \cap V)$ and $t_{U \cap V} = d$. But then the naturality of t forces $d \subseteq c$. Hence, for any $y \in d$, ηy is in both (V, d) and ηc . Thus, $dis\lambda_l$ is locally connected and ε_{λ_l} is a monomorphism. ε_{λ_l} is an epimorphism because for any open U and any connected component $c \in \Lambda(l^{-1}U)$, one can choose an $a \in c$. Then $\varepsilon_{\lambda_l}(U)$ sends the component of $\eta_l(a)$ to c . □

Let us turn to an analysis of cosheaf spaces. This analysis parallels and extends the discussion begun in §4 with cosheaf locales.

5.15 Definition. $L \xrightarrow{l} X \in \mathrm{LCT{\small SP}}/X$ is said to be a *cosheaf space* if the unit $\eta_l : L \to dis\lambda_l$ is a homeomorphism. (Note: the locally connected coclosure does not play a role - see Proposition 5.14).

5.16 *Example.* *The terminal cosheaf space.* If X is locally connected, then clearly the unit $X \to dis\lambda_X$ is a homeomorphism. This follows because $dis\lambda_X$ is locally connected. More generally, for any topological space X , the locally connected coclosure $\hat{X} \xrightarrow{\epsilon} X$ is the terminal cosheaf space. Again, this follows because $dis\lambda_\epsilon$ is locally connected. One uses the universal properties of both ϵ and γ_ϵ to obtain that η_ϵ is a homeomorphism. □

Recall from §4 the notion of a *complete spread* over X . The spatial version of the notion of spread remains unchanged; however, completeness will be defined[6] as the requirement that for all $x \in X$, every consistent choice of components $c_U \in l^{-1}U$ over all neighbourhoods U of x satisfy $\bigcap c_U \neq \emptyset$. Here, 'consistent' means $U \subseteq V \Rightarrow c_U \subseteq c_V$.

5.17 Theorem. *Assume that the base space X is T_1 . For any $L \xrightarrow{l} X \in \mathrm{LCT{\small SP}}/X$, the following are equivalent.*

1. *l is a complete spread and L is T_1 .*

[6]This is the original definition given in [5]. Fox constructs the completion of a spread $l : L \to X$, which in our notation is $\eta : L \to dis\lambda_l$.

2. l is a cosheaf space.

Proof. Assume *1*. For any given natural transformation $\lambda_z \xrightarrow{t} \lambda_l$, completeness says there exists $a \in \bigcap t_U$. Every open containing z must contain la and so $z = la$ since X is T_1 . Thus, $\eta a = (z, t)$, i.e., η is onto. To see that η is injective, let $\eta a = \eta b = (x, s)$. Then $la = lb = x$, and, in the notation of Remark 5.10, for every open U of X containing x , we have

$$\text{comp. of } l^{-1}U \text{ containing } a = t_U^a = t_U^b = \text{comp. of } l^{-1}U \text{ containing } b\,.$$

The spread property of l says that there is a base $\{c_\alpha\}$ for L such that every c_α appears as a component of some $l^{-1}U$. Hence, for all $V \in \mathcal{O}(L)$, $a \in V$ if and only if $b \in V$. Since L is assumed to be T_1 , we conclude that $a = b$. Finally, observe that η is an open map. In fact, since η is bijective, $\eta c_\alpha = (U, c_\alpha)$, which is an open set. Therefore, η is open on the base $\{c_\alpha\}$, and consequently is an open map. Thus, η is a homeomorphism.

Assume that l is a cosheaf space. First, it is not hard to see that display spaces are T_1 if the base space is. Second, the definition of the display space says clearly that such spaces are spreads. Finally, a consistent choice of components $c_U \in l^{-1}U$ over all U containing x obviously defines a cogerm (x, c). Then there is an $a \in L$ such that $(x, c) = \eta a$. Hence, $a \in \bigcap c_U$. □

A fundamental fact is that covering spaces are cosheaf spaces. Indeed, it is immediate that a covering space is a spread, but we are adopting a 'cosheaf space' approach, so we will give a direct explanation of the relationship between covering spaces and cosheaf spaces (i.e., one *not* depending on Theorem 5.17). For the following definitions let $L \xrightarrow{p} X$ denote an arbitrary map of topological spaces, with L locally connected. An open U of X is said to be *admissible* (with respect to p) if $p^{-1}U$ has a partition $\{V_\alpha\}$ of open sets such that p maps each V_α homeomorphically onto U . $L \xrightarrow{p} X$ is said to be a *covering space*[7] if X has a base of

[7] Generalizations of this notion will not be considered here, but the interested reader may wish to see [6].

admissible open sets. Note that a covering space $L \xrightarrow{p} X$ is necessarily a surjection, and X has a base of connected opens such that for each U of that base, each component of $p^{-1}U$ is mapped homeomorphically onto U .

5.18 Proposition. *Let $L \xrightarrow{p} X$ denote an arbitrary covering space. Then the unit η_p is a homeomorphism, i.e., p is a cosheaf space.*[8]

Proof. Let t denote an arbitrary cogerm over $x \in X$. Choose a connected admissible open set B containing x . Then there exists a unique $y \in t_B$ such that $py = x$. It follows that $\eta y = (x, t)$, which shows that η is surjective. Assume now that $y, z \in L$ determine the same cogerm over x , i.e., that $\eta y = \eta z$. Let U be an admissible neighborhood of x . y and z must be in the same component of $p^{-1}U$, and therefore, since that component maps homeomorphically onto U, must coincide. Thus, η is a bijection. It is clear that L has a base of connected sets c that appear as components of opens of the form $p^{-1}U$, i.e., that p is a spread. For such c, $\eta c = (U, c)$. Thus, η is an open map, whence a homeomorphism. □

We are thus presented with the following picture when the base space X is locally connected.

[8] If in the definition of covering space, 'X has a base of admissible open sets' is replaced with 'for every $a \in L$ and every open V in X such that $pa \in V$, there is an admissible open set U satisfying $pa \in U \subseteq V$ ', then the conclusion of Proposition 5.18 becomes 'η_p is an open, dense inclusion'. In this case, the completion $dis\lambda_p$ is not in general a covering space. In fact, it may not even be a local homeomorphism as the following simple example shows. Take X to be the real plane, and $p : L \rightarrow X$ the complement of the y-axis as a subspace. The completion $dis\lambda_p$ is the *cut* (see footnote 10) along the y-axis. This consists of two closed half-planes mapped to X by 'gluing' them along the y-axis.

$$\begin{tikzcd} & \mathrm{LCTSP}/X \arrow[dl] \arrow[dr] & \\ \mathrm{coSheaf}/X \arrow[ur] \arrow[dr] & & \mathrm{Sheaf}/X \arrow[ul] \arrow[dl] \\ & \mathrm{Cov}/X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Cosheaf spaces over X are a full reflective subcategory of LCTSP/X (whereas sheaf spaces are a full coreflective subcategory). Covering spaces are both cosheaf and sheaf spaces.[9] Examples of cosheaf spaces that are not sheaf spaces are not difficult to produce. A simple such example is the projection of a 2-sphere onto a disk. *Cuts* also provide such examples.[10]

6 Complete Metric Spaces

6.1 Theorem. *Let X denote an arbitrary complete metric space. The display locale of a cosheaf on X is spatial. The display space of a cosheaf on X is locally connected. Every cosheaf on X is spatial* (see Definition 5.12).

Proof. (sketch) Let C denote an arbitrary cosheaf on X . Start with $W, Z \in \mathcal{O}(disC)$, i.e., $W, Z \in \mathcal{O}(\mathbf{C})$ and closed under covers, such that $W \nleq Z$. To be exhibited is a point $1 \xrightarrow{p} disC$ such that $p^*W = 1$ and $p^*Z = 0$. To do this, build a sequence

$$(B_{n+1}, c_{n+1}) \leq (B_n, c_n) \in W, \; n = 1, 2, 3, \ldots$$

[9]If we wish to include non-surjective covering spaces, then we must enlarge coSheaf$/X$ so as to include open, dense subspaces of cosheaf spaces - see footnote 8.

[10]Cuts were introduced by Michael (cf. [14]). See footnote 8 for an example of a cut.

with radius$(B_n) \to 0$, and also such that every $(B_n, c_n) \notin Z$. This can be done since Z is closed under covers and since C is a cosheaf. The intersection of the B_n consists of a single point z . The following square can now be completed with q making it commute and thus giving the desired point p .

$$\begin{tikzcd} 1 \arrow[r, "q"] \arrow[d, "z"'] & \mathbf{C} \arrow[d] \\ X \arrow[r] & \mathbf{T}_X \end{tikzcd}$$

Define, for a basic open $\downarrow(U,d)$ of $\mathcal{O}(\mathbf{C})$,

$$q^* \downarrow(U,d) = \begin{cases} 1 & \text{if there is a } B_n \leq U \text{ with } c_n \mapsto d \\ 0 & \text{otherwise.} \end{cases}$$

This proves the first statement of the theorem.

The second statement follows from the first because the display locale of a cosheaf is always locally connected. See Theorem 5.7 (metric spaces are sober) and Proposition 5.4. The last statement follows from the first because in the context of locales, the counit $\lambda_{\gamma C} \to C$ is always an isomorphism for any cosheaf C . □

Theorems 5.17 and 6.1 combine to give the following.

6.2 Theorem. *On a complete metric space X , the display space construction yields an equivalence between the category of cosheaves on X and the category of T_1 complete spreads over X .*

Since all cosheaves on a complete metric space are spatial, the associated cosheaf of an arbitrary precosheaf can be constructed.

6.3 Theorem. *On a complete metric space, the associated cosheaf of an arbitrary precosheaf D exists, and can be constructed as the cosheaf of connected components of the locally connected coclosure of the display space of D .*

6.4 *Remark.* The general existence of the associated cosheaf is a fact. For any site, i.e., for any small category equipped with a Grothendieck topology, the inclusion of the category of cosheaves on that site into its corresponding category of precosheaves has a right adjoint. This can be established as follows. The category of cosheaves on an arbitrary site has all small colimits and is locally small. It is also cowell-powered as shown in [8], and has a small generating family. The existence of a generating set can be established as an application of the downward Löwenheim-Skolem theorem (for infinitary logic).[11] (The interested reader is referred to the monograph [13]). The Special Adjoint Functor Theorem now gives the right adjoint since the inclusion of cosheaves into precosheaves is colimit preserving.

This section is concluded with a counterexample. It depends on the following.

6.5 Proposition. *If X is a complete metric space, then the set of point-cosheaves $\{\lambda_x \mid x \in X\}$ generate coSh(X) .*

Proof. The points $1 \xrightarrow{x} X$ generate $\mathrm{LCT{\small SP}}/X$. □

6.6 *Remark.* We will in the following example make use of the observation that a space X has the T_1 separation property if and only if for all $1 \xrightarrow{x} X$, the unit $\eta_x : x \to \gamma\lambda_x$ is an isomorphism (see the description of η in Remark 5.10).

6.7 *Example.* *Let X denote a complete metric space which is locally connected and not discrete* (e.g., the real numbers). *Then coSh(X) is not a Grothendieck topos.* Indeed, if in this case *coSh*(X) were a topos, then the full subcategory determined by any generating set could be taken as a site of definition for *coSh*(X) (cf. [9]), and *coSh*(X) would be a full subcategory of the category of presheaves on that (small) full subcategory. The full subcategory of *coSh*(X) determined by the generating set

[11] This argument was shown to the author by Bill Boshuck.

$\{\lambda_x \mid x \in X\}$ is discrete, i.e., there are no morphisms $\lambda_x \to \lambda_y$, for x and y distinct. In fact, if there were a morphism $\lambda_x \to \lambda_y$, then there would be $x \simeq \gamma\lambda_x \to \gamma\lambda_y \simeq y$ (see Remark 6.6) as spaces over X, i.e., $x = y$. Thus, $\mid X \mid \simeq \{\lambda_x \mid x \in X\}$, and one concludes that $coSh(X)$ is a full subcategory of $Set^{|X|}$ via the functor

$$\Phi : C \mapsto (\ x \mapsto \hom(\lambda_x, C)\ ) .$$

Let $\mid X \mid \xrightarrow{id} X$ denote the discrete space over X. Then the morphism λ_{id} is not an isomorphism because X is assumed not discrete; however, $\Phi(\lambda_{id})$ is an isomorphism. To see this, observe that for any x there is exactly one natural transformation $\lambda_x \to \lambda_{|X|}$ (Existence: $\eta_{id}(x)$. Uniqueness: X is T_1 and γ is faithful). But λ_X is the terminal object in $coSh(X)$ - in particular, there is exactly one natural transformation $\lambda_x \to \lambda_X$. Thus, $\Phi(\lambda_{id})$ is an isomorphism - a contradiction. □

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