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Reeb graphs of curves are stable under function perturbations

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Reeb graphs provide a method to combinatorially describe the shape of a manifold endowed with a Morse function. One question deserving attention is whether Reeb graphs are robust against function perturbations. Focusing on one-dimensional manifolds, we define an editing distance between Reeb graphs of curves, in terms of the cost necessary to transform one graph into another through editing moves. Our main result is that changes in Morse functions induce smaller changes in the editing distance between Reeb graphs of curves, implying stability of Reeb graphs under function perturbations. We also prove that our editing distance is equal to the natural pseudo-distance and, moreover, that it is lower bounded by the bottleneck distance of persistent homology. Copyright © 2012 John Wiley & Sons, Ltd.

Keywords: shape similarity; editing distance; Morse function; natural stratification; natural pseudo-distance

1. Introduction

The shape similarity problem has been studied for a long time by the computer vision community for dealing with shape classification and retrieval tasks. Comparison of 2D images is often performed by taking into account only the silhouette or contour curve of the studied object. Shape properties, such as curvature, are encoded in compact representations of shapes, namely, shape descriptors. In this framework, shape similarity can be measured by defining an appropriate distance on the set of the chosen shape descriptors.

A question that deserves attention is the choice of the distance used to compare shape descriptors. Indeed, it is clear that any data acquisition is subject to perturbations, noise, and approximation errors, and without stability, distinct computational investigations of the same object could produce completely different results. So, a major problem in shape comparison concerns stability against data perturbations.

In this paper, we focus on the Reeb graph shape descriptor for curves. Reeb graphs have been used as an effective tool for shape analysis and description tasks since [1, 2], in the case of surfaces, even if they were introduced in [3] as a topological construct for manifolds of any dimension.

Reeb graphs of curves endowed with simple Morse functions, that is, smooth real-valued functions with only non-degenerate critical points, each belonging to a different critical level, are simply cycle graphs with an even number of vertices corresponding alternatively to the maxima and minima of the function. We also equip vertices of Reeb graphs with the value taken by the function at the corresponding critical points.

Our main contribution is the construction of a combinatorial distance between Reeb graphs of curves such that changes in functions imply smaller changes in the distance:

Main Result (Theorem 5.6). *For two simple Morse functions $f, g : S^1 \to \mathbb{R}$, the editing distance between the associated labeled Reeb graphs is never greater than $\|f-g\|_{C^0}$, where $\|f-g\|_{C^0} = \max_{p\in S^1}|f(p)-g(p)|$.*

This proves the stability of Reeb graphs of curves under perturbations.

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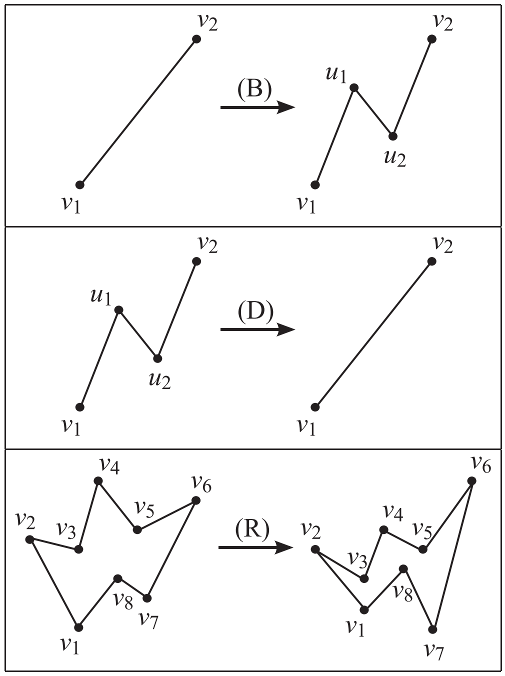
Table I. The upper two figures schematically show the elementary deformations of types (B) and (D), respectively; the third figure shows an example of elementary deformation of type (R).



Our distance is based on an adaptation of the well-known notion of editing distance between graphs [4]. We introduce three basic types of editing operations, represented in Table I, corresponding to the insertion (birth) of a new pair of adjacent points of maximum and minimum, the deletion (death) of such a pair, and the relabeling of the vertices. A cost is associated with each of these operations. Then, our distance is given by the infimum of the costs necessary to transform a graph into another by using these editing operations.

The main idea of the proof is to consider the linear path $\lambda g+(1-\lambda)f$ between f and g in the space of smooth real functions on S^1 and the corresponding one-parameter family of Reeb graphs. Assuming genericity, the changes that the functions undergo along the linear path can be translated into editing operations (insertions, deletions, and relabelings) on the corresponding Reeb graphs. Some care must be taken in order to reduce to a situation in which genericity of the path can be assumed. By appropriately taking a discretization of the path, we show that each editing operation has a cost that is not greater than the C^0-norm evaluated at the difference between the corresponding functions. In particular, this requires a stability result for critical values of Morse functions.

As a further contribution of this paper, our editing distance is compared with other distances that can be used to measure shape similarity of curves: the natural pseudo-distance [5] and the bottleneck distance of persistent homology [6]. These distances share the stability under function perturbations property. We prove that our editing distance coincides with the natural pseudo-distance, thus obtaining a tool for its study. Moreover, we prove that the bottleneck distance is never greater than the editing distance, and we exhibit an example in which it is strictly smaller. Hence, the bottleneck distance does not discriminate shapes as thoroughly as the editing distance.

The paper is organized as follows. After recalling the basic properties of labeled Reeb graphs of closed curves in Section 2, in Section 3, we give the definition of the admissible deformations transforming a Reeb graph into another, the cost associated with each kind of deformation, and the definition of an editing distance in terms of this cost. Section 4 is mainly devoted to proving that our editing distance is actually a metric. In Section 5, we first show that our distance is locally stable; that is, each simple Morse function f has a neighborhood consisting of functions g such that, for f and g, the main result holds. Then, we show that our distance is globally stable; that is, the main result holds for any two simple Morse functions f, g. We end the paper by comparing the editing distance with the natural pseudo-distance and the bottleneck distance in Section 6. For the reader's convenience, a proof of the stability of critical values of simple Morse functions (for a manifold of any dimension), obtained through homological arguments, is given in Appendix A.

2. Labeled Reeb graphs of closed curves

Throughout the paper, $\mathcal{F}$ denotes the set of C^∞ real functions on S^1. For $f \in \mathcal{F}$, we denote by $K(f)$ the set of its critical points. If $p \in K(f)$, then the real number $f(p)$ is called a *critical value* of f, and the set $\{q \in S^1 : q \in f^{-1}(f(p))\}$ is called a *critical level* of f. Otherwise, if $p \in S^1 \setminus K(f)$, then $f(p)$ is called a *regular value*. Moreover, a critical point p is called *non-degenerate* if and only if the second derivative of f at p is non-zero.

A function $f \in \mathcal{F}$ is called a *Morse function* if all its critical points are non-degenerate. Besides, a Morse function is said to be *simple* if each critical level contains exactly one critical point. The set of simple Morse functions will be denoted by $\mathcal{F}^0$, as a reminder that it is a sub-manifold of $\mathcal{F}$ of co-dimension 0 (see also Section 5).

Let $f \in \mathcal{F}^0$. The Reeb graph Γ_f associated with f is a cycle graph on an even number of vertices, corresponding, alternatively, to the minima and maxima of f on S^1 [7].

The vertex set of Γ_f will be denoted by $V(\Gamma_f)$, and its edge set by $E(\Gamma_f)$. Moreover, if $v_1, v_2 \in V(\Gamma_f)$ are adjacent vertices, that is, connected by an edge, we will write $e(v_1, v_2) \in E(\Gamma_f)$.

We label the vertices of Γ_f, by equipping each of them with the value of f at the corresponding critical point. We denote such a labeled graph by $(\Gamma_f, f_|)$, where $f_| : V(\Gamma_f) \to \mathbb{R}$ is the restriction of $f : S^1 \to \mathbb{R}$ to $K(f)$. A simple example is displayed in Figure 1(a)–(c).

To facilitate the reader, in all the figures of this paper, we shall adopt the convention of representing f as the height function so that $f_|(v_a) < f_|(v_b)$ if and only if v_a is lower than v_b in the picture. Moreover, we will often identify each $v \in V(\Gamma_f)$ with the corresponding $p \in K(f)$.

Definition 2.1. Two functions $f, g \in \mathcal{F}^0$ are called *topologically equivalent* if there exists a diffeomorphism $\xi : S^1 \to S^1$ and an orientation preserving diffeomorphism $\eta : \mathbb{R} \to \mathbb{R}$ such that $g(\xi(p)) = \eta(f(p))$ for every $p \in S^1$.

Given two topologically equivalent functions $f, g \in \mathcal{F}^0$, it is well-known that the associated Reeb graphs, Γ_f and Γ_g, are isomorphic graphs; that is, there exists an edge-preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$. Beyond that, an even stronger result holds. Two functions $f, g \in \mathcal{F}^0$ are topologically equivalent if and only if such a bijection Φ also preserves the vertex order; that is, for every $v, w \in V(\Gamma_f)$, $f(v) < f(w)$ if and only if $g(\Phi(v)) < g(\Phi(w))$. The preceding result has been used by Arnold in [8] to classify simple Morse functions up to topological equivalence.

The natural definition of isomorphism between labeled Reeb graphs is the following one.

Definition 2.2. We shall say that two labeled Reeb graphs $(\Gamma_f, f_|)$, $(\Gamma_g, g_|)$ are *isomorphic*, and we write $(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$, if there exists an edge-preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ such that $f_| = g_| \circ \Phi$.

The following Proposition 2.5 provides a necessary and sufficient condition in order that two labeled Reeb graphs are isomorphic. It is based on the next definition of re-parameterization equivalent functions.

Definition 2.3. Let $\mathcal{H}(S^1)$ be the set of homeomorphisms on S^1. We shall say that two functions $f, g \in \mathcal{F}^0$ are *re-parameterization equivalent* if there exists $\tau \in \mathcal{H}(S^1)$ such that $f = g \circ \tau$.

Lemma 2.4
Let $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$ be labeled Reeb graphs. If an edge-preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ exists, then there also exists $\tau \in \mathcal{H}(S^1)$ such that $\tau_{|V(\Gamma_f)} = \Phi$. If moreover $f_| = g_| \circ \Phi$, then τ can be chosen such that $f = g \circ \tau$.

Proof
The proof of the first statement is inspired by [5, Lemma 4.2]. Let us construct τ by extending Φ to S^1 as follows. Let us recall that $V(\Gamma_f) = K(f)$ and $V(\Gamma_g) = K(g)$, and, by abuse of notation, for every pair of adjacent vertices $p', p'' \in V(\Gamma_f)$, let us identify the edge $e(p', p'') \in E(\Gamma_f)$ with the arc of S^1 having endpoints p' and p'', and not containing any other critical point of f. For every $p \in K(f)$, let $\tau(p) = \Phi(p)$. Now, let us define $\tau(p)$ for every $p \in S^1 \setminus K(f)$. Given $p \in S^1 \setminus K(f)$, we observe that there always exist $p', p'' \in V(\Gamma_f)$ such that $p \in e(p', p'')$. Because Φ is edge-preserving, there exists $e(\Phi(p'), \Phi(p'')) = e(\tau(p'), \tau(p'')) \in E(\Gamma_g)$. Hence, we can define $\tau(p)$ as the unique point of $e(\tau(p'), \tau(p''))$ such that, if $f(p) = (1-\lambda_p)f(p') + \lambda_p f(p'')$, with $\lambda_p \in [0, 1]$, then $g(\tau(p)) = (1-\lambda_p)g(\tau(p')) + \lambda_p g(\tau(p''))$. Clearly, τ belongs to $\mathcal{H}(S^1)$.

As for the second statement, it is sufficient to observe that, if $f_| = g_| \circ \Phi$, then clearly $f_|(p) = g_|(\tau(p))$ for every $p \in K(f)$ because $\tau(p) = \Phi(p)$ for every $p \in K(f)$. Moreover, for every $p \in S^1 \setminus K(f)$, by the construction of τ, it holds that $g(\tau(p)) = (1 - \lambda_p) g(\Phi(p')) + \lambda_p g(\Phi(p'')) = (1 - \lambda_p)f(p') + \lambda_p f(p'') = f(p)$. In conclusion, $f(p) = g(\tau(p))$ for every $p \in S^1$, and, hence, f, g are re-parameterization equivalent. □

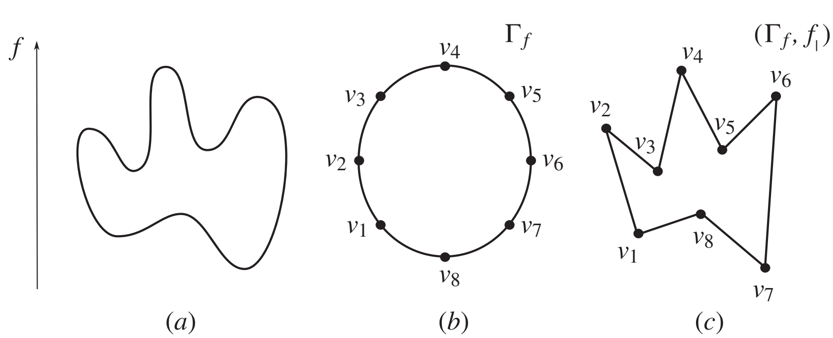


Figure 1. (a) $f : S^1 \to \mathbb{R}$ is the height function; (b) the associated Reeb graph Γ_f; (c) the associated labeled Reeb graph $(\Gamma_f, f_|)$. Here, labels are represented by the heights of the vertices.

Proposition 2.5 (Uniqueness theorem)
Let $(\Gamma_f, f_|)$, $(\Gamma_g, g_|)$ be labeled Reeb graphs. Then, $(\Gamma_f, f_|)$ is isomorphic to $(\Gamma_g, g_|)$ if and only if f and g are re-parameterization equivalent.

Proof
The direct statement is a trivial consequence of Lemma 2.4.

As for the converse statement, it is sufficient to observe that any $\tau \in \mathcal{H}(S^1)$ such that $f = g \circ \tau$, as well as its inverse τ^{-1}, takes the minima of f to the minima of g and the maxima of f to the maxima of g. Hence, $\Phi : V(\Gamma_f) \to V(\Gamma_g)$, with $\Phi = \tau_{|V(\Gamma_f)}$, is an edge-preserving bijection such that $f_| = g_| \circ \Phi$. □

The following Proposition 2.6 ensures that, for every cycle graph on an even number of vertices, appropriately labeled, there exists a unique (up to re-parameterization) $f \in \mathcal{F}^0$, having such a graph as the associated labeled Reeb graph.

Proposition 2.6 (Realization theorem)
Let (G, ℓ) be a labeled graph, where G is a cycle graph on an even number of vertices and $\ell : V(G) \to \mathbb{R}$ is an injective function such that, for any vertex v_2 adjacent (that is connected by an edge) to the vertices v_1 and v_3, either both $\ell(v_1)$ and $\ell(v_3)$ are smaller than $\ell(v_2)$, or both $\ell(v_1)$ and $\ell(v_3)$ are greater than $\ell(v_2)$. Then, there exists a simple Morse function $f : S^1 \to \mathbb{R}$ such that $(\Gamma_f, f_|) \cong (G, \ell)$.

Proof
Assume G has n vertices, and let us fix n distinct points $p_1, p_2, \ldots, p_n \in S^1$, ordered by adjacency. Let $\Phi : \{p_1, p_2, \ldots, p_n\} \to V(G)$ be an adjacency preserving bijection.

Let us now consider an open covering $\{U_i\}_{i=1,\ldots,n}$ of S^1, where each U_i is connected and $p_i \in U_i \setminus \bigcup_{j \neq i} U_j$, and a partition of unity $\{\lambda_i\}_{i=0,\ldots,n}$ subordinate to it. The considered covering $\{U_i\}_{i=1,\ldots,n}$ of S^1 has the property that each U_i intersects only U_{i-1} and U_{i+1}, with the convention that U_0 is equal to U_n and U_{n+1} is equal to U_1.

By viewing S^1 as obtained from the interval $[0, 1] \subset \mathbb{R}$ by identification of its endpoints, each U_i can be thought of as an interval. Thus, on each U_i, we can consider a second-degree polynomial function f_i whose graph is an arc of parabola with vertex at height $\ell_i = \ell(\Phi(p_i))$ and concavity up or down according to whether $\Phi(p_i)$ has a smaller label than its two adjacent vertices or not. We can choose the parabolas so that they overlap as in Figure 2.

We can glue all the parabolas together using the partition of unity. The setting $f(x) = \sum_{i=1}^{n} \lambda_i(x) f_i(x)$ defines a simple Morse function $f : S^1 \to \mathbb{R}$ whose only critical points are the points p_i with values ℓ_i. Indeed, for every $i = 1, \ldots, n$, on $U_i \setminus (U_{i-1} \cup U_{i+1})$, we have $f = f_i$. Hence, $p \in U_i \setminus (U_{i-1} \cup U_{i+1})$ is a critical point if and only if $p = p_i$. Moreover, the critical points p_i are non-degenerate and $f(p_i) = \ell_i$. For every $i = 1, \ldots, n$, on $U_i \cap U_{i+1}$, we have $f = \lambda_i f_i + \lambda_{i+1} f_{i+1}$, with $\lambda_i, \lambda_{i+1} \geq 0$, $\lambda_i + \lambda_{i+1} = 1$. Thus,

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}\lambda_i}{\mathrm{d}x}(x)\,(f_i(x) - f_{i+1}(x)) + \lambda_i(x)\frac{\mathrm{d}f_i}{\mathrm{d}x}(x) + (1 - \lambda_i(x))\frac{\mathrm{d}f_{i+1}}{\mathrm{d}x}(x).$$

Let us consider the case when p_i is a maximum and p_{i+1} is a minimum. Then, by construction, $f_i(x) - f_{i+1}(x) > 0$, $\frac{\mathrm{d}\lambda_i}{\mathrm{d}x}(x) < 0$, $\frac{\mathrm{d}f_i}{\mathrm{d}x}(x) < 0$, $\frac{\mathrm{d}f_{i+1}}{\mathrm{d}x}(x) < 0$, $0 \leq \lambda_i(x) \leq 1$, implying that $\frac{\mathrm{d}f}{\mathrm{d}x}(x) < 0$. The case when p_i is a minimum and p_{i+1} is a maximum can be treated analogously. Hence, the only critical points are the points p_i. □

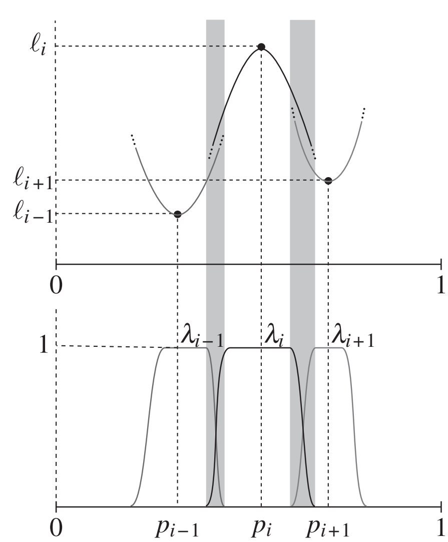


Figure 2. The parabolas and the partition of unity used in the proof of Proposition 2.6

3. Editing distance between labeled Reeb graphs

We now define the editing deformations admissible to transform a labeled Reeb graph of a closed curve into another. We introduce at first elementary deformations and then the deformations obtained by their composition. Next, we associate a cost with each type of deformation, and define a distance between labeled Reeb graphs in terms of such a cost.

Definition 3.1. Let $(\Gamma_f, f_|)$ be a labeled Reeb graph with $2n$ vertices, $n \geq 1$. We call T an *elementary deformation* of $(\Gamma_f, f_|)$ if T transforms $(\Gamma_f, f_|)$ in one and only one of the following ways:

(B) (Birth): Fix an edge $e(v_1, v_2) \in E(\Gamma_f)$, with $f_|(v_1) < f_|(v_2)$. Then, T transforms $(\Gamma_f, f_|)$ into a labeled graph (G, ℓ) according to the following rule: G is the new graph on $2n+2$ vertices, obtained by deleting the edge $e(v_1, v_2)$ and inserting two new vertices u_1, u_2 and the edges $e(v_1, u_1), e(u_1, u_2), e(u_2, v_2)$; moreover, $\ell : V(G) \to \mathbb{R}$ is defined by extending $f_|$ from $V(\Gamma_f)$ to $V(G) = V(\Gamma_f) \cup \{u_1, u_2\}$ in such a way that $\ell_{|V(\Gamma_f)} \equiv f_|$, and $f_|(v_1) < \ell(u_2) < \ell(u_1) < f_|(v_2)$.

(D) (Death): Assume $n \geq 2$, and fix edges $e(v_1, u_1), e(u_1, u_2), e(u_2, v_2) \in E(\Gamma_f)$, with $f_|(v_1) < f_|(u_2) < f_|(u_1) < f_|(v_2)$. Then, T transforms $(\Gamma_f, f_|)$ into a labeled graph (G, ℓ) according to the following rule: G is the new graph on $2n-2$ vertices, obtained by deleting u_1, u_2 and the edges $e(v_1, u_1)$, $e(u_1, u_2)$, $e(u_2, v_2)$ and inserting an edge $e(v_1, v_2)$; moreover, $\ell : V(G) \to \mathbb{R}$ is defined as the restriction of $f_|$ to $V(\Gamma_f) \setminus \{u_1, u_2\}$.

(R) (Relabeling): T transforms $(\Gamma_f, f_|)$ into a labeled graph (G, ℓ) according to the following rule: $G = \Gamma_f$, and $\ell : V(G) \to \mathbb{R}$ induces the same order on adjacent vertices as f.

We shall denote by $T(\Gamma_f, f_|)$ the result of the elementary deformation T applied to $(\Gamma_f, f_|)$.

Table I schematically illustrates the elementary deformations described in Definition 3.1.

Proposition 3.2

Let T be an elementary deformation of $(\Gamma_f, f_|)$, and let $(G, \ell) = T(\Gamma_f, f_|)$. Then, (G, ℓ) is isomorphic to a labeled Reeb graph $(\Gamma_g, g_|)$, and $g \in \mathcal{F}^0$ is unique up to re-parameterization equivalence.

Proof

The claim follows from Propositions 2.6 and 2.5. □

As a consequence of Proposition 3.2, we can apply elementary deformations iteratively. This fact is used in Definition 3.3.

Given an elementary deformation T of $(\Gamma_f, f_|)$ and an elementary deformation S of $T(\Gamma_f, f_|)$, the juxtaposition ST means applying first T and then S.

Definition 3.3. We shall call *deformation* of $(\Gamma_f, f_|)$ any finite ordered sequence $T = (T_1, T_2, \ldots, T_r)$ of elementary deformations such that T_1 is an elementary deformation of $(\Gamma_f, f_|)$, T_2 is an elementary deformation of $T_1(\Gamma_f, f_|)$, ..., T_r is an elementary deformation of $T_{r-1}T_{r-2}\cdots T_1(\Gamma_f, f_|)$. We shall denote by $T(\Gamma_f, f_|)$ the result of the deformation T applied to $(\Gamma_f, f_|)$.

Let us define the cost of a deformation.

Definition 3.4. Let T be an elementary deformation such that $T(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$.

- If T is of type (B) inserting the vertices $u_1, u_2 \in V(\Gamma_g)$, then we define the associated cost as

$$c(T) = \frac{|g_|(u_1) - g_|(u_2)|}{2}.$$

- If T is of type (D) deleting the vertices $u_1, u_2 \in V(\Gamma_f)$, then we define the associated cost as

$$c(T) = \frac{|f_|(u_1) - f_|(u_2)|}{2}.$$

- If T is of type (R) relabeling the vertices $v \in V(\Gamma_f) = V(\Gamma_g)$, then we define the associated cost as

$$c(T) = \max_{v \in V(\Gamma_f)} |f_|(v) - g_|(v)|.$$

Moreover, if $T = (T_1, \ldots, T_r)$ is a deformation such that $T_r \cdots T_1(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$, we define the associated cost as

$$c(T) = \sum_{i=1}^{r} c(T_i).$$

We now introduce the concept of inverse deformation.

Definition 3.5. Let T be a deformation such that $T(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$. Then we denote by T^{-1}, and call it the *inverse* of T, the deformation such that $T^{-1}(\Gamma_g, g_|) \cong (\Gamma_f, f_|)$ defined as follows:

- If T is elementary of type (B) inserting two vertices, then T^{-1} is of type (D) deleting the same vertices.
- If T is elementary of type (D) deleting two vertices, then T^{-1} is of type (B) inserting the same vertices, with the same labels.
- If T is elementary of type (R) relabeling vertices of $V(\Gamma_f)$, then T^{-1} is again of type (R) relabeling these vertices in the inverse way.
- If $T = (T_1, \dots, T_r)$, then $T^{-1} = (T_r^{-1}, \dots, T_1^{-1})$.

Proposition 3.6
For every deformation T such that $T(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$, $c(T^{-1}) = c(T)$.

Proof
It is sufficient to observe that, for every deformation $T = (T_1, \dots, T_r)$ such that $T(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$, Definitions 3.4 and 3.5 imply the following equalities:

$$c(T) = \sum_{i=1}^{r} c(T_i) = \sum_{i=1}^{r} c(T_i^{-1}) = c(T^{-1}).$$

□

We prove that, for every two labeled Reeb graphs, a finite number of elementary deformations always allows us to transform any of them into the other one, up to isomorphism. We first need a lemma, stating that in any labeled Reeb graph with at least four vertices, we can find two adjacent vertices that can be deleted.

Lemma 3.7
Let $(\Gamma_f, f_|)$ be a labeled Reeb graph with at least four vertices. Then, there exist $e(v_1, u_1), e(u_1, u_2), e(u_2, v_2) \in E(\Gamma_f)$, with $f_|(v_1) < f_|(u_2) < f_|(u_1) < f_|(v_2)$.

Proof
It is sufficient to take adjacent vertices u_1 and u_2 such that $f_|(u_1) - f_|(u_2)$ is equal to $\min_{e(u,v) \in E(\Gamma_f)} |f_|(u) - f_|(v)|$. □

Proposition 3.8
Let $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$ be two labeled Reeb graphs. Then, the set of all the deformations T such that $T(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$ is non-empty. This set of deformations will be denoted by $\mathcal{T}((\Gamma_f, f_|), (\Gamma_g, g_|))$.

Proof
Let m and n be the number of vertices of Γ_f and Γ_g, respectively. If $m = n$, then it is sufficient to take an elementary deformation T of type (R) in order that $T(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$. Otherwise, let us suppose $m > n$. Then, $m \geq 4$ and, by Lemma 3.7, we can apply a finite sequence of elementary deformations of type (D) to $(\Gamma_f, f_|)$ so that in the resulting labeled Reeb graph, $(\Gamma_h, h_|)$, Γ_h has only n vertices. Now, $(\Gamma_h, h_|)$ can be transformed into a graph isomorphic to $(\Gamma_g, g_|)$ through an elementary deformation of type (R). For the case $m < n$, the same proof applies with deformations of type (B) instead of deformations of type (D). □

A simple example explaining the aforementioned proof is given in Figure 3.

We point out that the deformation constructed in the proof of Proposition 3.8 is not necessarily the cheapest one, as can be deduced from Example 4.5.

We now introduce an editing distance between labeled Reeb graphs, in terms of the cost necessary to transform one graph into another.

Theorem 3.9
For every two labeled Reeb graphs $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$, we set

$$d((\Gamma_f, f_|), (\Gamma_g, g_|)) = \inf_{T \in \mathcal{T}((\Gamma_f, f_|), (\Gamma_g, g_|))} c(T).$$

Then, d is a distance on isomorphism classes of labeled Reeb graphs.

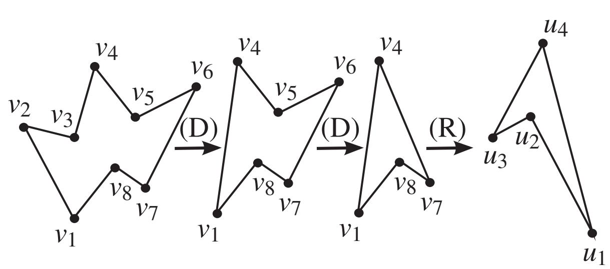


Figure 3. The leftmost labeled Reeb graph is transformed into the rightmost one applying first two elementary deformations of type (D), then one elementary deformation of type (R).

The proof of the aforementioned theorem will be postponed to the end of the following section. Indeed, even if the properties of symmetry and triangular inequality can be easily verified, the property of the positive definiteness of d is not straightforward because the set of all possible deformations transforming $(\Gamma_f, f_|)$ into $(\Gamma_g, g_|)$ is not finite. In order to prove the positive definiteness of d, we will need a further result concerning the connection between the editing distance between two labeled Reeb graphs, $(\Gamma_f, f_|)$, $(\Gamma_g, g_|)$, and the natural pseudo-distance between the associated functions f, g.

4. A lower bound for the editing distance

The natural pseudo-distance (cf. [5, 9, 10]) is a measure of the dissimilarity between any two continuous real-valued functions φ, ψ defined on the same compact topological space X.

Definition 4.1. The *natural pseudo-distance* between $\varphi : X \to \mathbb{R}$ and $\psi : X \to \mathbb{R}$ is defined as

$$D(\varphi, \psi) = \inf_{\tau \in \mathcal{H}(X)} \|\varphi - \psi \circ \tau\|_{C^0},$$

where $\mathcal{H}(X)$ is the set of homeomorphisms on X.

Theorem 4.2 states that the natural pseudo-distance computed between the simple Morse functions $f : S^1 \to \mathbb{R}$ and $g : S^1 \to \mathbb{R}$ is a lower bound for the editing distance between the associated labeled Reeb graphs.

Such a lower bound is useful for achieving two different results: firstly, the proof of Theorem 3.9, that is, that d is a distance (see Corollary 4.3); secondly, the computation, in certain simple cases, of the value of d (see, e.g., Examples 4.4 and 4.5).

In Section 6, we will show that the editing distance is actually equal to the natural pseudo-distance.

Theorem 4.2
Let $f, g \in \mathcal{F}^0$, and $(\Gamma_f, f_|)$, $(\Gamma_g, g_|)$ be the associated labeled Reeb graphs. Then, $d((\Gamma_f, f_|), (\Gamma_g, g_|)) \geq D(f, g)$.

Proof
Let us prove that, for every $T \in \mathcal{T}((\Gamma_f, f_|), (\Gamma_g, g_|))$, $c(T) \geq \inf_{\tau \in \mathcal{H}(S^1)} \|f - g \circ \tau\|_{C^0}$.

First of all, assume that T is an elementary deformation such that $T(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$. For conciseness, slightly abusing notations, we will identify arcs of S^1 having as endpoints two critical points $p', p'' \in V(\Gamma_f)$ and not containing other critical points of f, with the edges $e(p', p'') \in E(\Gamma_f)$.

(i) Let T be of type (R) relabeling vertices of $V(\Gamma_f)$. By Definition 3.1 (R), $\Gamma_f = \Gamma_g$, and thus we can apply Lemma 2.4, considering Φ as the identity map, to obtain a homeomorphism τ on S^1 such that $\tau(p) = p$ for every $p \in K(f)$. As far as non-critical points are concerned, following the proof of Lemma 2.4, for every $p \in S^1 \setminus K(f)$, $\tau(p)$ is defined as that point on S^1 such that, if $p \in e(p', p'') \in E(\Gamma_f)$, with $f(p) = (1 - \lambda_p) f(p') + \lambda_p f(p'')$, $\lambda_p \in [0, 1]$, then $\tau(p) \in e(p', p'')$ with $g(\tau(p)) = (1 - \lambda_p) g(p') + \lambda_p g(p'')$. Therefore, by substituting to $f(p)$ and $g(\tau(p))$ the aforementioned expressions, we see that $\max_{p \in S^1} |f(p) - g(\tau(p))| = \max_{p \in V(\Gamma_f)} |f_|(p) - g_|(p)| = c(T)$.

(ii) Let T be of type (D) deleting $q_1, q_2 \in V(\Gamma_f)$, the edges $e(p_1, q_1), e(q_1, q_2), e(q_2, p_2)$, and inserting the edge $e(p_1, p_2)$. Thus, for every $p \in K(f) \setminus \{q_1, q_2\}$, $f(p) = g(p)$. It is not restrictive to assume that $f(p_1) < f(q_2) < f(q_1) < f(p_2)$. Then, we can define a sequence (τ_n) of homeomorphisms on S^1 approximating this elementary deformation. Let $\tau_n(p) = p$ for every $p \in V(\Gamma_f) \setminus \{q_1, q_2\} = V(\Gamma_g)$ and $n \in \mathbb{N}$. Moreover, let $\overline{q}$ be the point of $e(p_1, p_2) \in E(\Gamma_g)$ such that $g(\overline{q}) = \frac{f(q_1) + f(q_2)}{2}$ (such a point $\overline{q}$ exists because $g(p_1) = f(p_1) < f(q_2) < f(q_1) < f(p_2) = g(p_2)$, and it is unique because we are assuming that no critical points of g occur in the considered arc). Let us fix a positive real number $c < \min\{g(p_2) - g(\overline{q}), g(\overline{q}) - g(p_1)\}$. For every $n \in \mathbb{N}$, let us define $\tau_n(q_1)$ (resp. $\tau_n(q_2)$) as the only point on S^1 belonging to the arc with endpoints $p_1, \overline{q}$ (resp. $\overline{q}, p_2$) contained in $e(p_1, p_2)$, such that $g(\tau_n(q_1)) = g(\overline{q}) - \frac{c}{n}$ (resp. $g(\tau_n(q_2)) = g(\overline{q}) + \frac{c}{n}$) as shown in Figure 4. Now, let us linearly extend τ_n to all S^1 in the following way. For every $p \in S^1 \setminus K(f)$, if p belongs to the arc with endpoints $p', p'' \in K(f)$ not containing any other critical point and is such that $f(p) = (1 - \lambda_p) f(p') + \lambda_p f(p'')$, $\lambda_p \in [0, 1]$, then $\tau_n(p)$ belongs to the arc with endpoints $\tau_n(p'), \tau_n(p'')$ not containing any other critical point and is such that $g(\tau_n(p)) = (1 - \lambda_p) g(\tau_n(p')) + \lambda_p g(\tau_n(p''))$. Hence, τ_n is a homeomorphism on S^1 for every $n \in \mathbb{N}$, and $\lim_{n \to \infty} \max_{p \in S^1} |f(p) - g(\tau_n(p))| = \lim_{n \to \infty} \max_{p \in V(\Gamma_f)} |f(p) - g(\tau_n(p))| = \lim_{n \to \infty} \max\{f(q_1) - g(\tau_n(q_1)), f(q_2) - g(\tau_n(q_2))\} = |f(q_1) - g(\overline{q})| = \frac{f_|(q_1) - f_|(q_2)}{2} = c(T)$.

(iii) Let T be of type (B), deleting $e(p_1, p_2) \in E(\Gamma_f)$ and inserting two vertices q_1, q_2 and the edges $e(p_1, q_1), e(q_1, q_2), e(q_2, p_2)$. Then, we can apply the same proof as (ii), by considering the inverse deformation T^{-1} that, by Definition 3.5, is of type (D) and, by Proposition 3.6, has the same cost of T.

Therefore, observing that in (i), the homeomorphism τ can be clearly replaced by the constant sequence (τ_n), with $\tau_n = \tau$ for every $n \in \mathbb{N}$, we can assert that, for every elementary deformation T, there exists a sequence of homeomorphisms on S^1, (τ_n), such that $c(T) = \lim_{n \to \infty} \|f - g \circ \tau_n\|_{C^0} \geq \inf_{\tau \in \mathcal{H}(S^1)} \|f - g \circ \tau\|_{C^0}$.

Now, let $T = (T_1, \ldots, T_r) \in \mathcal{T}((\Gamma_f, f_|), (\Gamma_g, g_|))$ and prove that, also in this case, $c(T) \geq \inf_{\tau \in \mathcal{H}(S^1)} \|f - g \circ \tau\|_{C^0}$. Let us set $T_i \cdots T_1(\Gamma_f, f_|) \cong (\Gamma_{f^{(i)}}, f_|^{(i)})$, $f = f^{(0)}$, $g = f^{(r)}$. For $i = 1, \ldots, r$, let $(\tau_n^{(i)})_n$ be a sequence of homeomorphisms on S^1 for which it

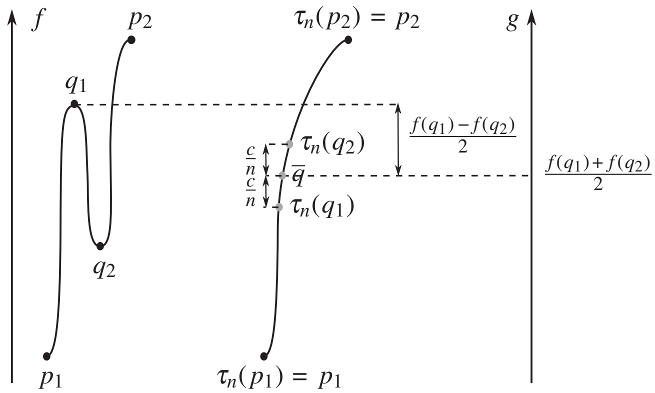


Figure 4. The construction of the homeomorphism τ_n as described in step (ii) of the proof of Theorem 4.2. The arc $e(p_1, q_1)$ ($e(q_1, q_2)$, and $e(q_2, p_2)$, respectively) is homeomorphically taken to the arc having $\tau_n(p_1), \tau_n(q_1)$ ($\tau_n(q_1), \tau_n(q_2)$ and $\tau_n(q_2), \tau_n(p_2)$, respectively) as endpoints.

holds that $c(T_i) = \lim_{n\to\infty} \|f^{(i-1)} - f^{(i)} \circ \tau_n^{(i)}\|_{C^0}$, and let $(\tau_n^{(0)})_n$ be the constant sequence such that $\tau_n^{(0)}$ is the identity map for every $n \in \mathbb{N}$. Then

$$\begin{aligned}
c(T) = \sum_{i=1}^{r} c(T_i) &= \lim_{n\to\infty} \|f^{(0)} - f^{(1)} \circ \tau_n^{(1)}\|_{C^0} + \sum_{i=1}^{r-1} \lim_{n\to\infty} \|f^{(i)} - f^{(i+1)} \circ \tau_n^{(i+1)}\|_{C^0} \\
&= \lim_{n\to\infty} \|f^{(0)} - f^{(1)} \circ \tau_n^{(1)}\|_{C^0} + \sum_{i=1}^{r-1} \lim_{n\to\infty} \|f^{(i)} \circ \tau_n^{(i)} \circ \cdots \circ \tau_n^{(0)} \\
&\quad - f^{(i+1)} \circ \tau_n^{(i+1)} \circ \tau_n^{(i)} \circ \cdots \circ \tau_n^{(0)}\|_{C^0} \\
&\geq \lim_{r\to\infty} \|f^{(0)} - f^{(r)} \circ \tau_n^{(r)} \circ \tau_n^{(r-1)} \circ \cdots \circ \tau_n^{(0)}\|_{C^0} \geq \inf_{\tau \in \mathcal{H}(S^1)} \|f - g \circ \tau\|_{C^0},
\end{aligned}$$

where the third equality is obtained by observing that

$$f^{(i)} \circ \tau_n^{(i)} \circ \cdots \circ \tau_n^{(0)} - f^{(i+1)} \circ \tau_n^{(i+1)} \circ \tau_n^{(i)} \circ \cdots \circ \tau_n^{(0)} = (f^{(i)} - f^{(i+1)} \circ \tau_n^{(i+1)}) \circ \tau_n^{(i)} \circ \cdots \circ \tau_n^{(0)}$$

for every $i \in \{1, \ldots, r-1\}$ and that $\|\cdot\|_{C^0}$ is invariant under re-parameterization; the first inequality is consequent to the triangular inequality. □

Corollary 4.3
If $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = 0$ then $(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$.

Proof
From Theorem 4.2, $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = 0$ implies that $D(f, g) = 0$. In [11], it has been proved that when $D(f, g) = 0$, with $f, g : X \to \mathbb{R}$, X being a closed curve of class at least C^2, a homeomorphism $\bar{\tau} \in \mathcal{H}(X)$ exists such that $f = g \circ \bar{\tau}$. Therefore, the claim follows from Proposition 2.5. □

Proof of Theorem 3.9
The positive definiteness of d has been proved in Corollary 4.3; the symmetry is a consequence of Proposition 3.6; the triangular inequality can be easily verified in the standard way. □

Now, we describe two simple examples showing how it is possible to compute the editing distance between two labeled Reeb graphs, $(\Gamma_f, f_|), (\Gamma_g, g_|)$, by exploiting the knowledge of the natural pseudo-distance value between f and g. In particular, Example 4.4 provides a situation in which the infimum cost over all the deformations transforming $(\Gamma_f, f_|)$ into $(\Gamma_g, g_|)$ is actually a minimum. In Example 4.5, this infimum is obtained by applying a passage to the limit.

Example 4.4. Let us consider $f, g : S^1 \to \mathbb{R}$, with $f, g \in \mathcal{F}^0$, depicted in Figure 5. We can show that, in this case, $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = \frac{1}{2}(f(q_1) - f(p_1))$.

The proof consists in showing these two claims:

- $D(f, g) = \frac{1}{2}(f(q_1) - f(p_1))$, which implies that $d((\Gamma_f, f_|), (\Gamma_g, g_|)) \geq \frac{1}{2}(f(q_1) - f(p_1))$ by Theorem 4.2.
- A deformation T such that $T(\Gamma_f, f_|) \cong (\Gamma_g, g_|)$ and $c(T) = \frac{1}{2}(f(q_1) - f(p_1))$ exists, implying that $d((\Gamma_f, f_|), (\Gamma_g, g_|)) \leq \frac{1}{2}(f(q_1) - f(p_1))$.

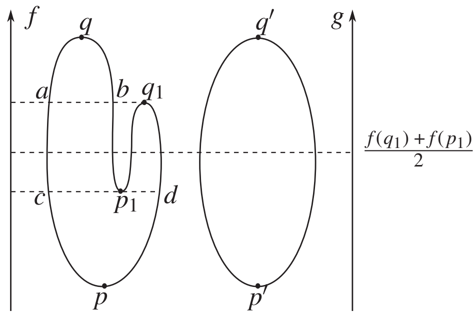


Figure 5. The functions $f, g \in \mathcal{F}^0$ considered in Example 4.4. In this case, $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = D(f, g) = \frac{1}{2}(f(q_1) - f(p_1))$.

The equality $D(f,g) = \frac{1}{2}(f(q_1) - f(p_1))$ has been proved in [5] by constructing a sequence of homeomorphisms (τ_n) such that $\lim_{n\to\infty} \|f - g \circ \tau_n\|_{C^0} = \frac{1}{2}(f(q_1) - f(p_1))$, as the one used in step (ii) of the proof of Theorem 4.2, and showing that $\|f - g \circ \tau\|_{C^0} \leq \frac{1}{2}(f(q_1) - f(p_1))$ for no homeomorphism $\tau \in \mathcal{H}(S^1)$. Indeed, if such a homeomorphism existed, then, for every $s \in S^1$, we would have $|f(s) - g(\tau(s))| \leq \frac{1}{2}(f(q_1) - f(p_1))$. By replacing s with p_1, q_1, a, b, c, and d, respectively, where $f(p_1) = f(a) = f(b)$, and $f(q_1) = f(c) = f(d)$, we obtain that $g(\tau(p_1)), g(\tau(a)), g(\tau(b)) \leq \frac{1}{2}(f(q_1) + f(p_1))$ and $\frac{1}{2}(f(q_1) + f(p_1)) \leq g(\tau(q_1)), g(\tau(c)), g(\tau(d))$, respectively. This yields a contradiction with the assumption that τ is a homeomorphism.

As for the second claim, the deformation T of type (D) that deletes the vertices $p_1, q_1 \in V(\Gamma_f)$, the edges $e(p, q_1), e(q_1, p_1), e(p_1, q)$ and inserts the edge $e(p, q)$ transforms $(\Gamma_f, f_|)$ into $(\Gamma_g, g_|)$, up to isomorphism, with cost $c(T) = \frac{1}{2}(f(q_1) - f(p_1))$.

Hence, $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = \frac{1}{2}(f(q_1) - f(p_1))$.

Example 4.5. Let us consider now $f, g : S^1 \to \mathbb{R}$, with $f, g \in \mathcal{F}^0$, illustrated in Figure 6. Let $f(q_1) - f(p_1) = f(q_2) - f(p_2) = a$. Then, by the same argument as in Example 4.4, $D(f, g) = a/2$. Let us show that the editing distance between $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$ is $a/2$, too. For every $0 < \epsilon < a/2$, we can apply to $(\Gamma_f, f_|)$ a deformation of type (R) that relabels p_1, p_2, q_1, q_2 in such a way that $f(p_i)$ is increased by $(a/2) - \epsilon$, and $f(q_i)$ is decreased by $(a/2) - \epsilon$ for $i = 1, 2$, composed with two deformations of type (D) that delete p_i with q_i, $i = 1, 2$. Thus, because the total cost is equal to $(a/2) - \epsilon + 2\epsilon$, by the arbitrariness of ϵ, it holds that $d((\Gamma_f, f_|), (\Gamma_g, g_|)) \leq a/2$. Applying Theorem 4.2, we deduce that $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = a/2$.

5. Stability

This section is devoted to proving that Reeb graphs of closed curves are stable under arbitrary function perturbations. More precisely, it will be shown that arbitrary changes in simple Morse functions imply smaller changes in the editing distance between Reeb graphs.

5.1. Preliminaries

First of all, let us recall that the C^r-norm of a function $f \in \mathcal{F}$ is defined as

$$\|f\|_{C^r} = \max\{\max_{p\in S^1}|f(p)|, \max_{p\in S^1}|f'(p)|, \ldots, \max_{p\in S^1}|f^{(r)}(p)|\}$$

whenever $0 \leq r < \infty$ (see, e. g., [7, 12]) and that for $r = 0$, the C^0-norm is the maximum norm. The C^r-norm defines a topology on $\mathcal{F}$, known as the C^r *topology* (or *weak topology*), with $0 \leq r < \infty$ (cf. [13, chap. 2]). The C^∞ *topology* is simply the union of the C^r topologies on $\mathcal{F}$ for every $0 \leq r < \infty$.

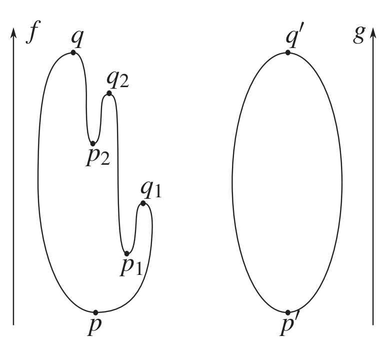


Figure 6. The functions $f, g \in \mathcal{F}^0$ considered in Example 4.5. Even in this case $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = D(f, g) = \frac{1}{2}(f(q_1) - f(p_1))$.

Let us endow $\mathcal{F}$ with the C^{∞} topology, and consider the strata $\mathcal{F}^0$ and $\mathcal{F}^1$ of the *natural stratification* of $\mathcal{F}$, as presented by Cerf in [14] (see also [15]).

- The stratum $\mathcal{F}^0$ is the set of simple Morse functions.
- The stratum $\mathcal{F}^1$ is the disjoint union of two sets $\mathcal{F}^1_\alpha$ and $\mathcal{F}^1_\beta$ open in $\mathcal{F}^1$, where
  - $\mathcal{F}^1_\alpha$ is the set of functions whose critical levels contain exactly one critical point, and the critical points are all non-degenerate, except exactly one. In a neighborhood of such a point, say p, a local coordinate system x can be chosen such that $f = f(p) + x^3$.
  - $\mathcal{F}^1_\beta$ is the set of Morse functions whose critical levels contain, at most, one critical point, except for one level containing exactly two critical points.

$\mathcal{F}^0$ is open and dense in the space $\mathcal{F}$ endowed with the C^r topology, $2 \le r \le \infty$ (cf. [13, chap. 6, Thm. 1.2]). Therefore, any function of $\mathcal{F}^1$ can be turned into a simple Morse function by arbitrarily small perturbations. Degenerate critical points can be split into non-degenerate singularities, with different critical values (Figure 7(a)). Moreover, when more than one critical point occur at the same level, they can be moved to close but different levels (Figure 7(b)).

It is well-known that two simple Morse functions are topologically equivalent if and only if they belong to the same arcwise connected component (or *co-cellule*) of $\mathcal{F}^0$ [14, p. 25].

$\mathcal{F}^1$ is a sub-manifold of co-dimension 1 of $\mathcal{F}^0 \cup \mathcal{F}^1$, and the complement of $\mathcal{F}^0 \cup \mathcal{F}^1$ in $\mathcal{F}$ is of co-dimension greater than 1. Consequently, given two functions $f, g \in \mathcal{F}^0$, we can always find $\widehat{f}, \widehat{g} \in \mathcal{F}^0$ arbitrarily near to f, g, respectively, for which

- $\widehat{f}, \widehat{g}$ are topologically equivalent to f, g, respectively,

and the path $h(\lambda) = (1-\lambda)\widehat{f} + \lambda\widehat{g}$, with $\lambda \in [0,1]$, is such that

- $h(\lambda)$ belongs to $\mathcal{F}^0 \cup \mathcal{F}^1$ for every $\lambda \in [0,1]$;
- $h(\lambda)$ is transversal to $\mathcal{F}^1$.

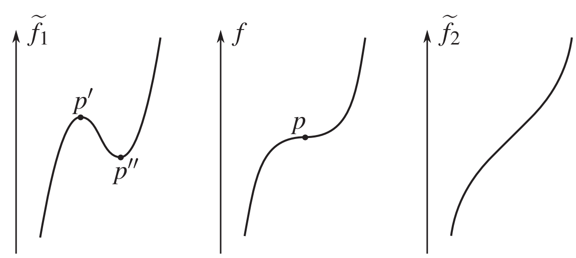
As a consequence, $h(\lambda)$ belongs to $\mathcal{F}^1$ for at most a finite collection of values λ and does not traverse strata of co-dimension greater than 1 (see, e.g., [16]).

5.2. Local stability

We now prove the local stability of labeled Reeb graphs of closed curves. More precisely, we prove that each simple Morse function f has a neighborhood consisting of simple Morse functions g such that the editing distance between the labeled Reeb graphs of f and g is never greater that the C^0-norm of $f - g$.

We first need some lemmas.

Lemma 5.1
Let $f \in \mathcal{F}^0$ and let c be a critical value of f. Then, there exists a real number $\delta(f,c) > 0$ such that each $g \in \mathcal{F}^0$ verifying $\|f-g\|_{C^0} \le \delta(f,c)$ admits at least one critical value c' for which $|c - c'| \le \|f-g\|_{C^0}$.



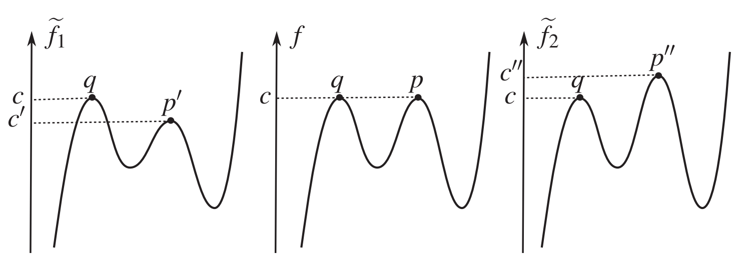


Figure 7. (a) A function $f \in \mathcal{F}^1_\alpha$ admitting one degenerate critical point p (center) can be perturbed into a simple Morse function $\widetilde{f}_1$ with two non-degenerate critical points p', p'' (left), or into a simple Morse function $\widetilde{f}_2$ without critical points around p (right); (b) a function $f \in \mathcal{F}^1_\beta$ (center) can be turned into two simple Morse functions $\widetilde{f}_1, \widetilde{f}_2$, that are not topologically equivalent (left–right).

The proof of the aforementioned result will be given in Appendix A for manifolds of arbitrary dimension.

Lemma 5.2
Let $f \in \mathcal{F}^0$. Then, there exists a positive real number $\delta(f)$ such that, for every $g \in \mathcal{F}^0$, with $\|f-g\|_{C^2} \le \delta(f)$, an edge and vertex-order preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ exists for which $\max_{v \in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| \le \|f-g\|_{C^0}$.

Proof
Let $p_1, \ldots, p_n$ be the critical points of f, and $c_1, \ldots, c_n$ the respective critical values, with $c_i < c_{i+1}$ for $i = 1 \ldots, n-1$. Because $\mathcal{F}^0$ is open in $\mathcal{F}$, endowed with the C^2 topology, there always exists a sufficiently small $\delta(f) > 0$, such that the closed ball with center f and radius $\delta(f)$, $\overline{B_2(f, \delta(f))}$, is contained in $\mathcal{F}^0$. Moreover, $\delta(f)$ can be chosen so small that, for every $i = 1, \ldots, n-1$, the intervals $[c_i - \delta(f), c_i + \delta(f)]$ and $[c_{i+1} - \delta(f), c_{i+1} + \delta(f)]$ are disjoint and $\delta(f) < \min_{i=1,\ldots,n} \delta(f, c_i)$, with $\delta(f, c_i)$ as in Lemma 5.1.

Fix such a $\delta(f)$. For every $g \in \mathcal{F}^0$ such that $\|f-g\|_{C^2} \le \delta(f)$, f and g belong to the same arcwise connected component of $\mathcal{F}^0$ endowed with the C^∞ topology and, therefore, are topologically equivalent functions. Consequently, there exists an edge and vertex-order preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ (Section 2).

Let us prove that Φ is such that $\max_{v \in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| \le \|f-g\|_{C^0}$. Because f and g are topologically equivalent, it follows that g has exactly n critical points, $p'_1, \ldots, p'_n$. Let $c'_1 = g(p'_1), \ldots, c'_n = g(p'_n)$. We can assume $c'_i < c'_{i+1}$, for $i = 1, \ldots, n-1$. The assumption $\|f-g\|_{C^2} \le \delta(f)$ implies that $\|f-g\|_{C^0} \le \delta(f)$. Therefore, recalling that $\delta(f) < \min_{i=1,\ldots,n} \delta(f, c_i)$, by Lemma 5.1, for every critical value c_i of f, there exists at least one critical value c''_i of g of the same index of c_i with $|c_i - c''_i| \le \|f-g\|_{C^0}$. Moreover, because $[c_i - \delta(f), c_i + \delta(f)] \cap [c_{i+1} - \delta(f), c_{i+1} + \delta(f)] = \emptyset$ for every $i = 1, \ldots, n-1$, and $\|f-g\|_{C^0} \le \delta(f)$, it follows that $c''_i \in [c_i - \delta(f), c_i + \delta(f)]$ for every $i = 1, \ldots, n$. Hence, because Φ preserves the order of the vertices, necessarily $\Phi(p_i) = p'_i$ and $c''_i = c'_i$, yielding that $\max_{v \in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| = \max_{p_i \in K(f)} |f_|(p_i) - g_|(\Phi(p_i))| = \max_{1 \le i \le n} |c_i - c''_i| \le \|f-g\|_{C^0}$. □

Theorem 5.3 (Local stability)
Let $f \in \mathcal{F}^0$. Then, there exists a positive real number $\delta(f)$ such that, for every $g \in \mathcal{F}^0$ with $\|f-g\|_{C^2} \le \delta(f)$, it holds that

$$d((\Gamma_f, f_|), (\Gamma_g, g_|)) \le \|f-g\|_{C^0}.$$

Proof
By Lemma 5.2, there exists $\delta(f) > 0$ such that, for every $g \in \mathcal{F}^0$, with $\|f-g\|_{C^2} \le \delta(f)$, an edge and vertex-order preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ exists for which $\max_{v \in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| \le \|f-g\|_{C^0}$. Necessarily Φ takes minima into minima and maxima into maxima. Therefore, $(\Gamma_f, g_| \circ \Phi) \cong T(\Gamma_f, f_|)$, with T an elementary deformation of type (R), relabeling vertices of $V(\Gamma_f)$, having cost $c(T) = \max_{v \in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| \le \|f-g\|_{C^0}$. Moreover, let us observe that $(\Gamma_f, g_| \circ \Phi)$ is isomorphic to $(\Gamma_g, g_|)$ as labeled Reeb graph (see Definition 2.2). Thus, $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = d((\Gamma_f, f_|), (\Gamma_f, g_| \circ \Phi)) = \inf_{T \in \mathcal{T}((\Gamma_f, f_|), (\Gamma_g, g_|))} c(T) \le \|f-g\|_{C^0}$. □

5.3. Global stability

To prove the global stability of Reeb graphs, we proceed by steps: Proposition 5.4 shows such a stability property when the functions defined on S^1 belong to the same arcwise connected component of $\mathcal{F}^0$; Proposition 5.5 proves the same result in the case that the linear convex combination of two simple Morse functions traverses the stratum $\mathcal{F}^1$ at most in one point; Theorem 5.6 extends the result to two arbitrary functions in $\mathcal{F}^0$.

Proposition 5.4
Let $f, g \in \mathcal{F}^0$ and let us consider the path $h : [0,1] \to \mathcal{F}$ defined by $h(\lambda) = (1-\lambda)f + \lambda g$. If $h(\lambda) \in \mathcal{F}^0$ for every $\lambda \in [0,1]$, then $d((\Gamma_f, f_|), (\Gamma_g, g_|)) \le \|f-g\|_{C^0}$.

Proof
Let $\delta(h(\lambda)) > 0$ be the fixed real number obtained by applying Theorem 5.3 to $h(\lambda)$. For conciseness, let us denote it by $\delta(\lambda)$ and $\|f-g\|_{C^2}$ by a. If $a = 0$, then the claim trivially follows. If $a > 0$, let C be the open covering of $[0,1]$ constituted of open intervals $I_\lambda = \left(\lambda - \frac{\delta(\lambda)}{2a}, \lambda + \frac{\delta(\lambda)}{2a}\right)$. Let C' be a finite minimal (i.e., such that, for every i, $I_{\lambda_i} \not\subseteq \bigcup_{j \ne i} I_{\lambda_j}$) sub-covering of C, with $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ the middle points of its intervals. Because C' is minimal, for every $i \in \{1, \ldots, n-1\}$, $I_{\lambda_i} \cap I_{\lambda_{i+1}}$ is non-empty. This implies that

$$\lambda_{i+1} - \lambda_i < \frac{\delta(\lambda_i)}{2a} + \frac{\delta(\lambda_{i+1})}{2a} \le \frac{\max\{\delta(\lambda_i), \delta(\lambda_{i+1})\}}{a}. \tag{5.1}$$

Moreover, by the definition of h and the linearity of derivatives, it can be deduced that

$$\|h(\lambda_{i+1}) - h(\lambda_i)\|_{C^2} = (\lambda_{i+1} - \lambda_i) \cdot \|f-g\|_{C^2}. \tag{5.2}$$

Now, substituting (5.1) in (5.2), we obtain

$$\|h(\lambda_{i+1})-h(\lambda_i)\|_{C^2}<\frac{\max\{\delta(\lambda_i),\delta(\lambda_{i+1})\}}{a}\cdot\|f-g\|_{C^2}=\max\{\delta(\lambda_i),\delta(\lambda_{i+1})\}.$$

Let us consider the labeled Reeb graphs $(\Gamma_{h(\lambda_i)},h(\lambda_i)_|)$ with $i=1,\dots,n$.

Let $i\in\{1,\dots,n-1\}$. If $\max\{\delta(\lambda_i),\delta(\lambda_{i+1})\}=\delta(\lambda_i)$, then using Theorem 5.3, with f replaced by $h(\lambda_i)$ and g by $h(\lambda_{i+1})$, it holds that

$$d((\Gamma_{h(\lambda_i)},h(\lambda_i)_|),(\Gamma_{h(\lambda_{i+1})},h(\lambda_{i+1})_|))\le\|h(\lambda_{i+1})-h(\lambda_i)\|_{C^0}. \tag{5.3}$$

The same inequality holds when $\max\{\delta(\lambda_i),\delta(\lambda_{i+1})\}=\delta(\lambda_{i+1})$, as can be analogously checked.

Now, setting $\lambda_0=0$, $\lambda_{n+1}=1$, it can be verified that (5.3) also holds for $i=0,n$. Consequently, because $\Gamma_f=\Gamma_{h(\lambda_0)}$, and $\Gamma_g=\Gamma_{h(\lambda_{n+1})}$, we have

$$\begin{aligned}d((\Gamma_f,f_|),(\Gamma_g,g_|))&\le\sum_{i=0}^{n}d((\Gamma_{h(\lambda_i)},h(\lambda_i)_|),(\Gamma_{h(\lambda_{i+1})},h(\lambda_{i+1})_|))\le\sum_{i=0}^{n}\|h(\lambda_{i+1})-h(\lambda_i)\|_{C^0}\\&=\sum_{i=0}^{n}(\lambda_{i+1}-\lambda_i)\cdot\|f-g\|_{C^0}=\|f-g\|_{C^0},\end{aligned}$$

where the first inequality is due to the triangular inequality, the second one to (5.3), the first equality holds because of (5.2), the second one because $\sum_{i=0}^{n}(\lambda_{i+1}-\lambda_i)=1$. □

Proposition 5.5

Let $f,g\in\mathcal{F}^0$ and let us consider the path $h:[0,1]\to\mathcal{F}$ defined by $h(\lambda)=(1-\lambda)f+\lambda g$. If $h(\lambda)\in\mathcal{F}^0$ for every $\lambda\in[0,1]\setminus\{\overline{\lambda}\}$, with $0<\overline{\lambda}<1$, and h transversely intersects $\mathcal{F}^1$ at $\overline{\lambda}$, then $d((\Gamma_f,f_|),(\Gamma_g,g_|))\le\|f-g\|_{C^0}$.

Proof

We begin proving the following claim.

Claim. For every $\delta>0$ there exist two real numbers $\lambda',\lambda''\in[0,1]$, with $\lambda'<\overline{\lambda}<\lambda''$, such that $d((\Gamma_{h(\lambda')},h(\lambda')_|),(\Gamma_{h(\lambda'')},h(\lambda'')_|))\le\delta$.

To prove this claim, let us first assume that $h(\overline{\lambda})$ belongs to $\mathcal{F}^1_\alpha$. To simplify the notation, we denote $h(\overline{\lambda})$ simply by $\overline{h}$. Let $\overline{p}$ be the sole degenerate critical point for $\overline{h}$. It is well known that there exists a suitable local coordinate system x around $\overline{p}$ in which the canonical expression of $\overline{h}$ is $\overline{h}=\overline{h}(\overline{p})+x^3$ (see Figure 7(a) with $\overline{h}$ replaced by f).

Let us take a smooth function $\omega:S^1\to\mathbb{R}$ whose support is contained in the coordinate chart around $\overline{p}$ in which $\overline{h}=\overline{h}(\overline{p})+x^3$; moreover, let us assume that ω is equal to 1 in a neighborhood of $\overline{p}$ and decreases moving from $\overline{p}$. Let us consider the family of smooth functions $\overline{h}_t$ obtained by locally modifying $\overline{h}$ near $\overline{p}$ as follows: $\overline{h}_t=\overline{h}+t\cdot\omega\cdot x$. There exists $\overline{t}>0$ sufficiently small such that (*i*) for $0<t\le\overline{t}$, $\overline{h}_t$ has no critical points in the support of ω and is equal to $\overline{h}$ everywhere else (see Figure 7(a) with $\overline{h}_t$ replaced by $\widetilde{f}_2$), and (*ii*) for $-\overline{t}\le t<0$, $\overline{h}_t$ has exactly two critical points in the support of ω whose values difference tends to vanish as t tends to 0, and $\overline{h}_t$ is equal to $\overline{h}$ everywhere else (see [14] and Figure 7(a) with $\overline{h}_t$ replaced by $\widetilde{f}_1$).

Because $\overline{h}_t$ is a universal deformation of $\overline{h}=h(\overline{\lambda})$ and h intersects $\mathcal{F}^1$ transversely at $\overline{\lambda}$, either the maps $h(\lambda)$ with $\lambda<\overline{\lambda}$ are topologically equivalent to h_t with $t>0$ or to h_t with $t<0$ (cf. [14,15,17]). Analogously for the maps $h(\lambda)$ with $\lambda>\overline{\lambda}$. Let us assume that $h(\lambda)$ is topologically equivalent to h_t with $t<0$ when $\lambda<\overline{\lambda}$, whereas $h(\lambda)$ is topologically equivalent to h_t with $t>0$ when $\lambda>\overline{\lambda}$. Hence, for every $\delta>0$, there exist λ', with $0\le\lambda'<\overline{\lambda}$, and λ'', with $\overline{\lambda}<\lambda''\le1$, such that $h(\lambda')$ and $h(\lambda'')$ have the same critical points, with the same values, except for two critical points of $h(\lambda')$, whose values difference is smaller than δ, that are non-critical for $h(\lambda'')$. Therefore, $(\Gamma_{h(\lambda')},h(\lambda')_|)$ can be transformed into a graph isomorphic to $(\Gamma_{h(\lambda'')},h(\lambda'')_|)$ by an elementary deformation of type (D) whose cost is not greater than δ. In the case when $h(\lambda)$ is topologically equivalent to h_t with $t>0$ when $\lambda<\overline{\lambda}$ whereas $h(\lambda)$ is topologically equivalent to h_t with $t<0$ when $\lambda>\overline{\lambda}$, the claim can be proved similarly, applying an elementary deformation of type (B).

Let us now prove the claim when $\overline{h}=h(\overline{\lambda})$ belongs to $\mathcal{F}^1_\beta$. Let us denote by $\overline{p}$ and $\overline{q}$ the critical points of $\overline{h}$ such that $\overline{h}(\overline{p})=\overline{h}(\overline{q})$. Because $\overline{p}$ is non-degenerate, there exists a suitable local coordinate system x around $\overline{p}$ in which the canonical expression of $\overline{h}$ is $\overline{h}=\overline{h}(\overline{p})\pm x^2$ (see Figure 7(b) with $\overline{h}$ replaced by f). Let us take ω as before, whose support is contained in such a coordinate chart. Let us locally modify $\overline{h}$ near $\overline{p}$ as follows: $\overline{h}_t=\overline{h}+t\cdot\omega$. There exists $\overline{t}>0$ sufficiently small such that for $|t|\le\overline{t}$, $\overline{h}_t$ has exactly the same critical points as $\overline{h}$. As for critical values, they are the same as well, apart from the value taken at $\overline{p}$: $\overline{h}_t(\overline{p})<\overline{h}(\overline{p})$, for $-\overline{t}\le t<0$ (see Figure 7(b) with $\overline{h}_t$ replaced by $\widetilde{f}_1$), whereas $\overline{h}_t(\overline{p})>\overline{h}(\overline{p})$, for $0<t\le\overline{t}$ (see Figure 7(b) with $\overline{h}_t$ replaced by $\widetilde{f}_2$), and $\overline{h}_t(\overline{p})$ tends to $\overline{h}(\overline{p})$ as t tends to 0 (cf. [14]). Because $\overline{h}_t$ is a universal deformation of $\overline{h}=h(\overline{\lambda})$, and h intersects $\mathcal{F}^1$ transversely at $\overline{\lambda}$, we deduce that for every $\delta>0$, there exist λ', with $0\le\lambda'<\overline{\lambda}$ and λ'', with $\overline{\lambda}<\lambda''\le1$, such that $(\Gamma_{h(\lambda')},h(\lambda')_|)$ can be transformed into a graph isomorphic to $(\Gamma_{h(\lambda'')},h(\lambda'')_|)$ by an elementary deformation of type (R) whose cost is not greater than δ. Therefore, the initial claim is proved.

Let us now estimate $d((\Gamma_f,f_|),(\Gamma_g,g_|))$. By the claim, for every $\delta>0$, there exist $0<\lambda'<\lambda''<1$ such that, applying the triangular inequality,

$$\begin{aligned}d((\Gamma_f,f_|),(\Gamma_g,g_|))&\le d((\Gamma_f,f_|),(\Gamma_{h(\lambda')},h(\lambda')_|))+d((\Gamma_{h(\lambda')},h(\lambda')_|),(\Gamma_{h(\lambda'')},h(\lambda'')_|))\\&\quad+d((\Gamma_{h(\lambda'')},h(\lambda'')_|),(\Gamma_g,g_|))\\&\le d((\Gamma_f,f_|),(\Gamma_{h(\lambda')},h(\lambda')_|))+d((\Gamma_{h(\lambda'')},h(\lambda'')_|),(\Gamma_g,g_|))+\delta.\end{aligned}$$

By Proposition 5.4,

$$d((\Gamma_f, f_{|}), (\Gamma_{h(\lambda')}, h(\lambda')_{|})) \le \|f - h(\lambda')\|_{C^0} = \lambda' \cdot \|f - g\|_{C^0},$$

and

$$d((\Gamma_{h(\lambda'')}, h(\lambda'')_{|}), (\Gamma_g, g_{|})) \le \|h(\lambda'') - g\|_{C^0} = (1 - \lambda'') \cdot \|f - g\|_{C^0}.$$

Hence, $d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \le \|f - g\|_{C^0} + \delta$, yielding the conclusion by the arbitrariness of δ. □

Theorem 5.6 (Global stability)
Let $f, g \in \mathcal{F}^0$. Then

$$d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \le \|f - g\|_{C^0}.$$

Proof
As observed at the end of Section 5.1, for every sufficiently small $\delta > 0$ such that the balls with center f and g and radius δ with respect to the C^2-norm, $B_2(f, \delta), B_2(g, \delta)$, are contained in $\mathcal{F}^0$, there exist $\widehat{f} \in B_2(f, \delta)$ and $\widehat{g} \in B_2(g, \delta)$ such that the path $h : [0, 1] \to \mathcal{F}$, with $h(\lambda) = (1 - \lambda)\widehat{f} + \lambda\widehat{g}$, belongs to $\mathcal{F}^0$ for every $\lambda \in [0, 1]$, except for at most a finite number n of values $0 < \mu_1 < \mu_2 < \cdots < \mu_n < 1$ at which h transversely intersects $\mathcal{F}^1$. If $n = 0$ ($n = 1$, respectively), then the claim immediately follows from Proposition 5.4 (Proposition 5.5, respectively). If $n > 1$, let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{2n-1} < 1$, with $\lambda_{2i-1} = \mu_i$ for $i = 1, \ldots, n$. Then, $h(\lambda_{2i-1}) \in \mathcal{F}^1$ for $i = 1, \ldots, n$, $h(\lambda_{2i}) \in \mathcal{F}^0$ for $i = 1, \ldots, n-1$. Set $\lambda_0 = 0$ so that $\widehat{f} = h(\lambda_0)$, and $\lambda_{2n} = 1$ so that $\widehat{g} = h(\lambda_{2n})$ (a schematization of this path is illustrated in Figure 8). Then, by Proposition 5.5, we have

$$d((\Gamma_{h(\lambda_{2i})}, h(\lambda_{2i})_{|}), (\Gamma_{h(\lambda_{2i+2})}, h(\lambda_{2i+2})_{|})) \le \|h(\lambda_{2i}) - h(\lambda_{2i+2})\|_{C^0}$$

for every $i = 0, \ldots, n-1$. Therefore,

$$\begin{aligned} d((\Gamma_{\widehat{f}}, \widehat{f}_{|}), (\Gamma_{\widehat{g}}, \widehat{g}_{|})) &\le \sum_{i=0}^{n-1} d((\Gamma_{h(\lambda_{2i})}, h(\lambda_{2i})_{|}), (\Gamma_{h(\lambda_{2i+2})}, h(\lambda_{2i+2})_{|})) \\ &\le \sum_{i=0}^{n-1} \|h(\lambda_{2i}) - h(\lambda_{2i+2})\|_{C^2} \le \|\widehat{f} - \widehat{g}\|_{C^0}. \end{aligned}$$

Then, recalling that $\widehat{f} \in B_2(f, \delta)$ implies $\|\widehat{f} - f\|_{C^2} \le \delta$, and $B_2(f, \delta) \subset \mathcal{F}^0$ implies that $(1 - \lambda)f + \lambda\widehat{f} \in \mathcal{F}^0$ for every $\lambda \in [0, 1]$, we can apply Proposition 5.4 to state that

$$d((\Gamma_f, f_{|}), (\Gamma_{\widehat{f}}, \widehat{f}_{|})) \le \|\widehat{f} - f\|_{C^0} \le \|\widehat{f} - f\|_{C^2} \le \delta.$$

It is analogous for g and $\widehat{g}$. Thus, from the triangular inequality, we have

$$\begin{aligned} d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) &\le d((\Gamma_f, f_{|}), (\Gamma_{\widehat{f}}, \widehat{f}_{|})) + d((\Gamma_{\widehat{f}}, \widehat{f}_{|}), (\Gamma_{\widehat{g}}, \widehat{g}_{|})) + d((\Gamma_{\widehat{g}}, \widehat{g}_{|}), (\Gamma_g, g_{|})) \\ &\le 2\delta + \|\widehat{f} - \widehat{g}\|_{C^0}. \end{aligned}$$

Now, because by the triangular inequality, $\|\widehat{f} - \widehat{g}\|_{C^0} \le \|\widehat{f} - f\|_{C^0} + \|f - g\|_{C^0} + \|g - \widehat{g}\|_{C^0}$, with $\|\widehat{f} - f\|_{C^0} \le \delta$, and $\|g - \widehat{g}\|_{C^0} \le \delta$, it follows that $d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \le 4\delta + \|f - g\|_{C^0}$. Finally, because of the arbitrariness of δ, we can let δ tend to zero and obtain the claim. □

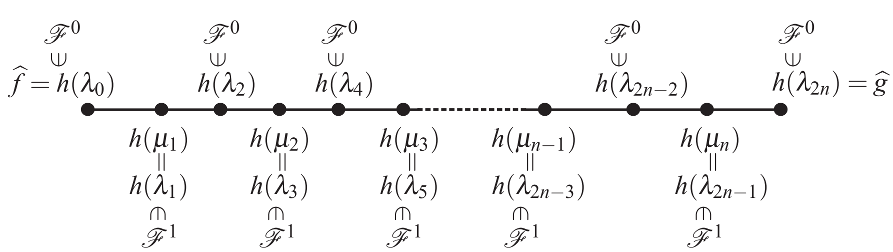


Figure 8. The linear path used in the proof of Theorem 5.6.

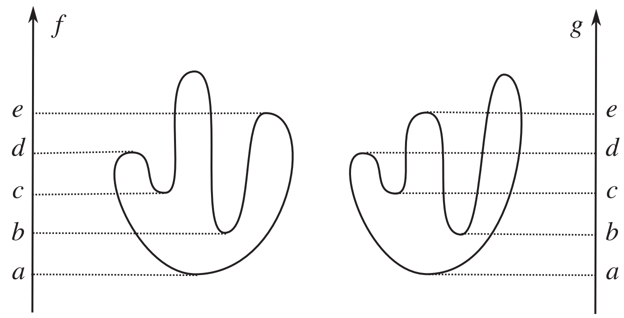


Figure 9. Two curves with the same persistence diagrams for any homology degree, whose natural pseudo-distance is non-zero.

6. Relationships with other distances

In this section, we consider the relationships between the editing distance and other two metrics: the natural pseudo-distance (see Section 4) and the bottleneck distance of persistent homology (see [6] for the definition).

Corollary 6.1
For every $f, g \in \mathcal{F}^0$, $d((\Gamma_f, f_|), (\Gamma_g, g_|))$ is equal to the natural pseudo-distance between f and g.

Proof
By Theorem 5.6, for every $\tau \in \mathcal{H}(S^1)$, $d((\Gamma_f, f_|), (\Gamma_{g\circ\tau}, g \circ \tau_|)) \leq \|f - g \circ \tau\|_{C^0}$. Moreover, for every $\tau \in \mathcal{H}(S^1)$, $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = d((\Gamma_f, f_|), (\Gamma_{g\circ\tau}, g \circ \tau_|))$. Hence, $d((\Gamma_f, f_|), (\Gamma_g, g_|)) \leq \inf_{\tau \in \mathcal{H}(S^1)} \|f - g \circ \tau\|_{C^0} = D(f, g)$. Recalling that, by Theorem 4.2, also the inequality $d((\Gamma_f, f_|), (\Gamma_g, g_|)) \geq D(f, g)$ holds, the claim follows. □

Corollary 6.2
For every $f, g \in \mathcal{F}^0$, $d((\Gamma_f, f_|), (\Gamma_g, g_|))$ is greater than or equal to the bottleneck distance between the persistence diagrams of f and g.

Proof
The claim immediately follows from Corollary 6.1 and the fact that the bottleneck distance is a lower bound for the natural pseudo-distance (cf. [18]). □

Remark 6.3. The editing distance between Reeb graphs is not equal to the bottleneck distance between the corresponding persistence diagrams.

To see this fact, we exhibit in Figure 9 an example in which the editing distance between Reeb graphs is strictly greater than the bottleneck distance between the corresponding persistence diagrams. Indeed, the two curves have the same persistence diagrams for any homology degree, but their natural pseudo-distance is non-zero because a homeomorphism $\tau : S^1 \to$ ï£¡S^1 such that $f = g \circ \tau$ should take critical points of f into critical points of g preserving their values and adjacencies, which is clearly impossible.

APPENDIX A. Stability of simple Morse functions' critical values

In this section, we give a proof that critical values of simple Morse functions are stable. We only use the homological properties of the lower level sets of a simple Morse function f on a manifold $\mathcal{M}$, and its validity does not depend on the dimension of $\mathcal{M}$. Therefore, it will be given for any smooth compact manifold without boundary.

For every C^∞ function $f : \mathcal{M} \to \mathbb{R}$, and for every $a \in \mathbb{R}$, let us denote by f^a the lower level set $f^{-1}((-\infty, a])$. Let us recall the existing link between the topology of a pair of lower level sets (f^b, f^a), with $a, b \in \mathbb{R}$, $a < b$, regular values of f, and the critical points of f lying between a and b (cf. [19]):

- *(St. 1)* If $f^{-1}([a, b])$ contains no critical points, then f^a is a deformation retract of f^b so that the inclusion map $f^a \to f^b$ is a homotopy equivalence.
- *(St. 2)* If $f^{-1}([a, b])$ contains exactly one critical point of index $\bar{k}$, then, denoting by G the homology coefficient group, it holds that

$$H_k(f^b, f^a) = \begin{cases} G, & \text{if } k = \bar{k} \\ 0, & \text{otherwise.} \end{cases}$$

In the remainder of this section, we require f to be a simple Morse function. Accordingly, it makes sense to use the terminology *critical value of index k* to indicate a critical value that is the image of a critical point of index k.

Lemma A.1
Let $f : \mathcal{M} \to \mathbb{R}$ be a simple Morse function, and let $a, b \in \mathbb{R}$, $a < b$, be regular values of f. If there exists $\bar{k} \in \mathbb{Z}$ such that $H_{\bar{k}}(f^b, f^a) \neq 0$, then $[a, b]$ contains at least one critical value of index $\bar{k}$.

Proof

From *(St. 1)*, the absence of critical values in $[a,b]$ implies that the homomorphism induced by inclusion $i_k : H_k(f^a) \to H_k(f^b)$ is an isomorphism for each $k \in \mathbb{Z}$. Consequently, by using the long exact sequence of the pair:

$$\cdots \longrightarrow H_k(f^a) \xrightarrow{i_k} H_k(f^b) \xrightarrow{j_k} H_k(f^b, f^a) \xrightarrow{\partial_k} H_{k-1}(f^a) \xrightarrow{i_{k-1}} H_{k-1}(f^b) \longrightarrow \cdots,$$

it is easily seen that, for every $k \in \mathbb{Z}$, the surjectivity of i_k and the injectivity of i_{k-1} imply the triviality of $H_k(f^b, f^a)$. This proves that if there exists $\bar{k} \in \mathbb{Z}$ such that $H_{\bar{k}}(f^b, f^a) \neq 0$, then $[a,b]$ contains at least one critical value of f. Moreover, as a consequence of *(St. 2)*, the index of at least one of the critical values of f contained in $[a,b]$ is exactly $\bar{k}$. Indeed, let $c_1, \ldots, c_m$ be the critical values of f belonging to $[a,b]$, and let $s_0, \ldots, s_m$ be $m+1$ regular values such that $a = s_0 < c_1 < s_1 < c_2 < \cdots < s_{m-1} < c_m < s_m = b$. Because it holds that $\mathrm{rank} H_{\bar{k}}(f^b, f^a) \leq \sum_{i=1}^{m} \mathrm{rank} H_{\bar{k}}(f^{s_i}, f^{s_{i-1}})$, and we are assuming $\mathrm{rank} H_{\bar{k}}(f^b, f^a) \geq 1$, there exists at least one index $i \in \{1, \ldots, m\}$ such that $H_{\bar{k}}(f^{s_i}, f^{s_{i-1}}) \neq 0$. Now, applying *(St. 2)* with a replaced by s_{i-1} and b replaced by s_i, we deduce that c_i is a critical value of f of index $\bar{k}$. $\square$

The aforementioned statements *(St. 1–2)*, Lemma A.1, together with the following lemma, that is a reformulation of Lemma 4.1 in [20], provide the tools for proving the stability of critical values under small function perturbations (Theorem A.3).

Lemma A.2

Let $X_1, X_2, X_3, X'_1, X'_2, X'_3$ be topological spaces such that $X_1 \subseteq X_2 \subseteq X_3 \subseteq X'_1 \subseteq X'_2 \subseteq X'_3$. Let $H_k(X_3, X_1) = 0$, $H_k(X'_3, X'_1) = 0$ for every $k \in \mathbb{Z}$. Then, the homomorphism induced by inclusion $H_k(X'_1, X_1) \to H_k(X'_2, X_2)$ is injective for every $k \in \mathbb{Z}$.

Theorem A.3 (Stability of critical values)

Let $f : \mathcal{M} \to \mathbb{R}$ be a simple Morse function and let c be a critical value of index $\bar{k}$ of f. Then, there exists a real number $\delta(f,c) > 0$ such that each simple Morse function $g : \mathcal{M} \to \mathbb{R}$ verifying $\|f - g\|_{C^0} \leq \delta(f,c)$ admits at least one critical value c' of index $\bar{k}$ for which $|c - c'| \leq \|f - g\|_{C^0}$.

Proof

Because f is Morse, we can choose a real number $\delta(f,c) > 0$ such that $[c - 3 \cdot \delta(f,c), c + 3 \cdot \delta(f,c)]$ does not contain any critical value of f besides c. Let g be a simple Morse function such that $\|f - g\|_{C^0} \leq \delta(f,c)$. If $g = f$, then the claim immediately follows. Assume $g \neq f$ and denote $\|f - g\|_{C^0}$ by δ'. Then, for every $n \in \mathbb{N}$,

$$f^{c-\delta' \cdot \frac{2n+1}{n}} \subseteq g^{c-\delta' \cdot \frac{n+1}{n}} \subseteq f^{c-\delta'/n} \subseteq f^{c+\delta'/n} \subseteq g^{c+\delta' \cdot \frac{n+1}{n}} \subseteq f^{c+\delta' \cdot \frac{2n+1}{n}}.$$

Because $[c - \delta' \cdot \frac{2n+1}{n}, c - \delta'/n]$ and $[c + \delta'/n, c + \delta' \cdot \frac{2n+1}{n}]$ do not contain any critical value of f for any $n \in \mathbb{N}$, both $H_k(f^{c-\delta'/n}, f^{c-\delta' \cdot \frac{2n+1}{n}})$ and $H_k(f^{c+\delta' \cdot \frac{2n+1}{n}}, f^{c+\delta'/n})$ are trivial for every $k \in \mathbb{Z}$, and $n \in \mathbb{N}$. Consequently, from Lemma A.2, the homomorphism induced by inclusion $H_k(f^{c+\delta'/n}, f^{c-\delta' \cdot \frac{2n+1}{n}}) \to H_k(g^{c+\delta' \cdot \frac{n+1}{n}}, g^{c-\delta' \cdot \frac{n+1}{n}})$ is injective for each $k \in \mathbb{Z}$, and $n \in \mathbb{N}$. Moreover, because, for every $n \in \mathbb{N}$, $[c - \delta' \cdot \frac{2n+1}{n}, c + \delta'/n]$ contains c, that is a critical value of index $\bar{k}$ of f, from *(St. 2)*, it holds that $H_{\bar{k}}(f^{c+\delta'/n}, f^{c-\delta' \cdot \frac{2n+1}{n}}) \neq 0$ for every $n \in \mathbb{N}$. This fact, together with the injectivity of the above map, implies that also $H_{\bar{k}}(g^{c+\delta' \cdot \frac{n+1}{n}}, g^{c-\delta' \cdot \frac{n+1}{n}}) \neq 0$ for every $n \in \mathbb{N}$. So, by Lemma A.1, for every $n \in \mathbb{N}$, there exists at least one critical value c'_n of index $\bar{k}$ of g with $c'_n \in (c - \delta' \cdot \frac{n+1}{n}, c + \delta' \cdot \frac{n+1}{n})$. Now, by contradiction, let us assume that $[c - \delta', c + \delta']$ contains no critical values of index $\bar{k}$ of g. Then, because g is Morse and $\mathcal{M}$ is compact, there would exist a sufficiently small real number $\varepsilon > 0$ such that $(c - \delta' - \varepsilon, c + \delta' + \varepsilon)$ does not contain critical values of index $\bar{k}$ of g either, yielding a contradiction. $\square$

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