A family of metrics from the truncated smoothing of Reeb graphs [*]

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Abstract

In this paper, we introduce an extension of smoothing on Reeb graphs, which we call truncated smoothing; this in turn allows us to define a new family of metrics which generalize the interleaving distance for Reeb graphs. Intuitively, we "chop off" parts near local minima and maxima during the course of smoothing, where the amount cut is controlled by a parameter τ. After formalizing truncation as a functor, we show that when applied after the smoothing functor, this prevents extensive expansion of the range of the function, and yields particularly nice properties (such as maintaining connectivity) when combined with smoothing for $0 \leq \tau \leq 2\varepsilon$, where ε is the smoothing parameter. Then, for the restriction of $\tau \in [0, \varepsilon]$, we have additional structure which we can take advantage of to construct a categorical flow for any choice of slope $m \in [0, 1]$. Using the infrastructure built for a category with a flow, this then gives an interleaving distance for every $m \in [0, 1]$, which is a generalization of the original interleaving distance, which is the case $m = 0$. While the resulting metrics are not stable, we show that any pair of these for $m, m' \in [0, 1)$ are strongly equivalent metrics, which in turn gives stability of each metric up to a multiplicative constant. We conclude by discussing implications of this metric within the broader family of metrics for Reeb graphs.

1   Introduction

The Reeb graph, originally defined in the context of Morse theory [44], represents a portion of the underlying structure of a topological space $\mathbb{X}$ through the lens of a real valued function $h : \mathbb{X} \to \mathbb{R}$; the pair of data $(\mathbb{X}, h)$ is known as an $\mathbb{R}$-space. Specifically, points in the Reeb graph correspond to connected components in the levelsets of the function; as such, the Reeb graph inherits a real valued function from the original input data. For nice enough inputs, the resulting object is a finite graph. So, at its core, we focus our study on objects of the form (G, f) where G is a graph and $f : G \to \mathbb{R}$ is a function given on vertices and interpolated linearly on the edges. See Fig. 1 for an example.

Reeb graphs have become increasingly useful in a wide range of applications, including settings such as shape comparison [29, 34], denoising [54], shape understanding [25, 33], reconstructing non-linear 1-dimensional structure in data [19, 31, 42, 51], summarizing collections of trajectory data [16], and allowing for informed exploration of otherwise hard-to-visualize high-dimensional data [32, 53]; see [5] for a survey of these and more topics. As a result, there is interest in defining metrics on these objects, to evaluate their quality in the face of noisy input data as well as to allow for more accurate shape comparison and analysis. In this setting, we are generally working with metrics that incorporate both the graph and function information: so $d((G, f_1), (G, f_2))$ should be non-zero if $f_1 \neq f_2$ even though they are defined on the same underlying graphs.

Several metrics have arisen recently to do this, taking inspiration from different mathematical backgrounds [1–4, 17, 23, 27, 28, 48]. In this paper, we focus on the Reeb graph interleaving distance [23]. The basic
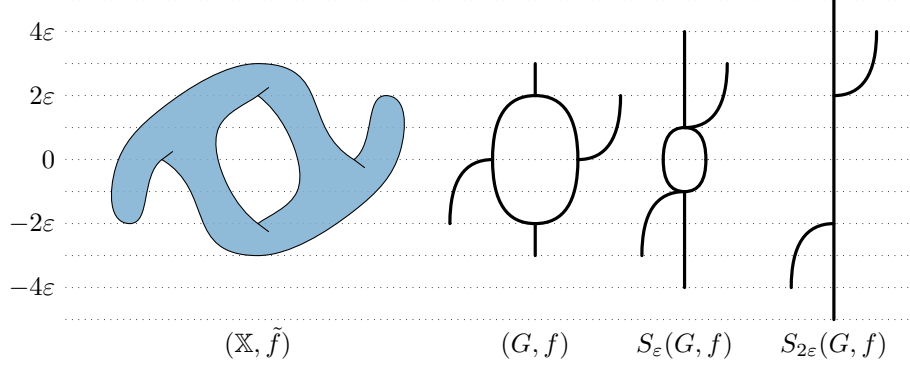

Figure 1: From left to right: an $\mathbb{R}$-space $(\mathbb{X}, \tilde{f})$, its Reeb graph (G, f), smoothings are shown for two parameters, ε and 2ε. Function values are shown by height.

idea is to work with a notion of smoothing, which returns a parameterized family of Reeb graphs, $S_\varepsilon(G, f)$ for every $\varepsilon \geq 0$, starting with $\varepsilon = 0$ which leaves the input unchanged. This procedure simplifies the loop structures and stretches tails [56]; see Fig. 1 for an example. Then the goal is to find an ε-interleaving, which is a pair of families of maps which make the diagram

$$
\begin{array}{ccc}
(G, f) \longrightarrow S_\varepsilon(G, f) \longrightarrow S_{2\varepsilon}(G, f) \\
\\
(H, h) \longrightarrow S_\varepsilon(H, h) \longrightarrow S_{2\varepsilon}(H, h)
\end{array}
$$

commute. If $\varepsilon = 0$, this diagram simplifies down to finding an isomorphism between the two Reeb graphs; increasing ε provides more flexibility to find such pairs of maps. Then we can give a metric by defining $d_I((G, f), (H_h))$ to be the infimum over the set of ε for which such a diagram exists.

This metric takes root in the interleaving distance defined for persistence modules [18], and is largely inspired by the subsequent category theoretic treatment [13, 14]. This viewpoint comes from encoding the data of a Reeb graph in a constructible set-valued cosheaf [21, 22]. It was later shown that these metrics are special cases of a more general theory of interleaving distances given on a so-called *category with a flow* [20, 24, 49]. This framework encompasses common metrics including ℓ_∞ distance on points or functions [49], regular Hausdorff distance [49], and the Gromov-Hausdorff distance [15]. Using this framework, interleaving metrics have been studied in the context of $\mathbb{R}$-spaces [9], multiparameter persistence modules [37], merge trees [38], and formigrams [35, 36], and on more general category theoretic constructions [11, 46]. There are also interesting restrictions to labeled merge trees, where one can pass to a matrix representation and show that the interleaving distance is equivalent to the point-wise ℓ_∞ distance [30, 39, 50, 55].

On the negative side, it has been shown that Reeb graph interleaving is graph isomorphism complete [7, 23], and that many other variants are also NP-hard [7, 8]. All of this means that these metrics, while mathematically interesting, may not lead to feasible algorithms for comparison and analysis. However, a glimmer of hope arises with work investigating fixed parameter tractable algorithms [50, 52]. Despite the issues of computational complexity, notions of similarity for graphs in general, and Reeb graphs in particular, are of pressing interest due to their extensive use in data analysis; in many such settings, we are concerned with questions of quality in the face of noise, and understanding convergence of approximations to a true underlying structure. For example, the interleaving distance has been used in evaluating the quality of the mapper graph [47], which can be viewed as a approximation of the Reeb graph [12, 40]. Furthermore, there is considerable interest in unifying the interleaving distance with the emerging collection of other Reeb graph metrics.

In this paper, we introduce truncation, which intuitively cuts off portions of the Reeb graph near local extrema with respect to f; this operation is easy to compute for any Reeb graphs. We show that truncation

is a functor, and when combined with the smoothing functor, it defines a flow on the the category of Reeb graphs. We investigate and show particularly desirable geometric and topological properties of *truncated smoothing* for certain ranges of the two parameters controlling the functors. We then introduce a new family of metrics for Reeb graphs, called truncated interleaving distances. They are parameterized by $m \in [0, 1]$, and generalize the interleaving distance while making the actual graph structure simpler, with the case of $m = 0$ returning the original interleaving distance. We show that they are all strongly equivalent to each other for $m \in [0, 1)$ and, while these metrics are not stable in the sense of [3], strong equivalence gives that they are at least stable up to a constant.

When combined with preliminary work on geometric implications of smoothing [56], truncated smoothing is interesting in its own right, as it provides a collection of paths for Reeb graph space to be studied in terms of the resulting persistence diagrams. It also is useful when considering algorithms to test planarity for Reeb graphs, or find planar representations of them. The new family of metrics also provide the possibility for new approaches for approximation algorithms for the interleaving distance, as well as new avenues for further unification of the broader family of Reeb graph metrics.

Outline: We begin by summarizing the main results and contributions of this paper in Section 2. In order to maintain readability, we reserve the more detailed category theoretic background to Section 3. We give several equivalent definitions of truncation in Section 4 and provide the full categorical viewpoint of truncated smoothing in Section 5 and Section 6. Finally, we show that the family of metrics constructed are equivalent in Section 7, and discuss some broader implications and future work in Section 8. Technical details of some proofs are omitted due to space constraints; these are presented more in full in Appendices A to C.

2 Overview of our results

In this section, we give an overview of the main definitions (both new and old), and the results of this paper. We reserve the full constructions and proofs for later sections, with the most technical included in the appendix.

2.1 Background: Reeb graphs, smoothing, and interleaving

Given a topological space $\mathbb{X}$ along with a continuous $\mathbb{R}$-valued function $f \colon \mathbb{X} \to \mathbb{R}$, we call the pair $(\mathbb{X}, f)$ an $\mathbb{R}$-*space*. For two $\mathbb{R}$-spaces $(\mathbb{X}, f)$ and $(\mathbb{X}', g)$, we call a continuous map $\varphi \colon \mathbb{X} \to \mathbb{X}'$ *function-preserving* if $f = g \circ \varphi$, and write $\varphi \colon (\mathbb{X}, f) \to (\mathbb{X}', g)$ in that case.

For an $\mathbb{R}$-space $(\mathbb{X}, f)$, we define an equivalence relation $\sim_f$ on the points of $\mathbb{X}$, such that $x \sim_f x'$ if and only if x and x' lie in the same path-connected component of $f^{-1}(y)$ for some $y \in \mathbb{R}$. For sufficiently nice[1] functions, the quotient space $\mathbb{X}/\sim_f$ is a graph, called a *Reeb graph*, and we denote the quotient map by $q_f \colon (\mathbb{X}, f) \to (\mathbb{X}/\sim_f, g)$. Since $f(x) = f(x')$ whenever $x \sim_f x'$, we can treat the Reeb graph as an $\mathbb{R}$-space $(X/\sim_f, g)$ by defining $g(q_f(x)) = f(x)$, so that q_f is function-preserving. Most but not all functions in this paper are function preserving. See Fig. 1 for an example.

For the purposes of this work, we will largely divorce the idea of the Reeb graph from the need for a starting space that was used to construct it. Thus for our purposes, a *Reeb graph* is a pair (G, f) where $G = (V_G, E_G)$ is a finite multigraph and $f \colon G \to \mathbb{R}$, referred to as the *height function*, is a continuous map that is linearly interpolated along edges of G, and for which no two neighboring vertices have the same function value. The function can equivalently be stored by defining $f \colon V_G \to \mathbb{R}$ as a function on the vertices, and extending it to the edges implicitly. We treat G as a topological space, so that a point $x \in G$ lies either on a vertex of G, or interior to an edge of G. For succinctness, we also write $x \in (G, f)$ to mean $x \in G$. Since no two adjacent vertices have the same function value, a level sets $f^{-1}(y)$ for $y \in \mathbb{R}$ is a finite set of points in G which could be vertices and/or points in the interior edges.

Together, the collection of Reeb graphs (treated as $\mathbb{R}$-spaces) with function-preserving maps as morphisms forms a category, **Reeb**. For the reader without a background in category theory, the basic idea is that that

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[1]A Morse function on a manifold, or a constructible space and function [23], or a space with a levelset-tame function [26].

this collection of objects and morphisms satisfy some basic axiomatic structures that make their analysis easier to view as a collection. It also makes available the viewpoint of *functors* between categories, which are essentially structure preserving maps. For now, we will largely hand-wave past the categorical constructions, and defer the technicalities to Section 3.

Define a *path* from x to x' in (G, f) to be a continuous map $\pi\colon [0,1] \to G$ such that $\pi(0) = x$ and $\pi(1) = x'$. A path is called an *up-path* if it is monotone-increasing with respect to the function, i.e. $f(\pi(t)) \leq f(\pi(t'))$ for $t \leq t'$. Symmetrically, a path is a *down-path* if it is monotone-decreasing. In the case of an up- or down-path π, we call $|f(\pi(0)) - f(\pi(1))|$ the *height* of the path.

In a Reeb graph (G, f), let the *up-paths* of a point x be the set of f-monotone paths that have x as minimum. The *up-set* of a point x is the set of points reachable from x by an up-path, including x itself. Define an *up-fork* to be a vertex x whose up-set contains at least two edges adjacent to x. We define *down-paths*, *down-sets*, and *down-forks* symmetrically. Call the up-set of a point x an *up-tree* if it contains no down-forks of (G, f), and say that x *roots* an up-tree in such case. The concept of rooting a *down-tree* is defined symmetrically. See Fig. 2.

Definition 2.1. *Fix a Reeb graph $R = (G, f)$ and $\varepsilon \geq 0$. Define the ε-thickening of G to be the space $G \times [-\varepsilon, \varepsilon]$ with the product topology, and define $(f + \mathrm{Id})\colon G \times [-\varepsilon, \varepsilon] \to \mathbb{R}$ by $(f + \mathrm{Id})(x, t) = f(x) + t$. We define the ε-smoothing $S_\varepsilon(G, f)$ to be the Reeb graph of $(f + \mathrm{Id})$, and denote the corresponding quotient map by $q\colon G \times [-\varepsilon, \varepsilon] \to S_\varepsilon(G, f)$.*

Figure 2: Left: the up-set (red) and down-set (blue) of a point. Although the up-set is a tree, it is not an up-tree as it contains down-forks of the ambient graph. Right: the sets U_δ and D_δ of points with no length δ up-path or down-path, respectively. The leftmost component of D_δ does not contain the down-fork.
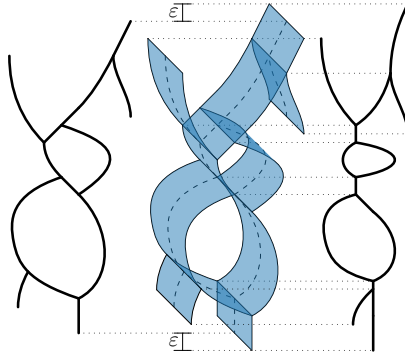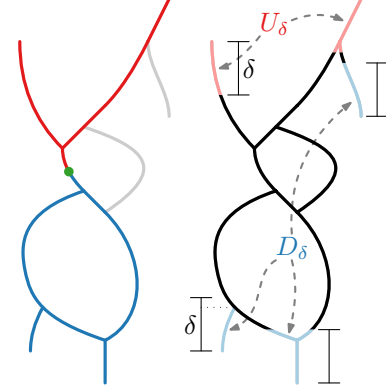

Figure 3: From left to right: a Reeb graph (G, f), its ε-thickening $(G \times [-\varepsilon, \varepsilon], f + \mathrm{Id})$, and the Reeb graph $S_\varepsilon(G, f)$ of the ε-thickening. The product of an edge with an interval is drawn to reflect the function value at a given height.

See Fig. 3 for an example. In essence, smoothing eliminates small cycles whose height is $\leq 2\varepsilon$, and shrinks all other cycles; it also moves every local maxima up and every local minima down. Under the lens of studying the topology of the graph (and in turn the original space), this serves as a functor that can be used to remove noise and simplify topology in a parameterized fashion.

The smoothing construction, S_ε, holds quite a bit more useful structure as not only is it a functor, it is an example of a flow [24]. In particular, there is a function preserving map $\eta\colon (G, f) \to S_\varepsilon(G, f)$ which sends a point x in f to $q(x, 0)$. We will reserve the full investigation of η until Section 3.2, but will use the following property of categories with a flow, with further details provided in Section 3.1.

Theorem 2.2 ([24, Thm. 2.7]). *A category with a flow gives rise to an interleaving distance on the objects of the category; specifically, this construction is an extended pseudometric.*

This construction is quite useful since simply by finding some relatively easy to check structure on a category, we can immediately get out a distance measure on the objects. Depending on the category and flow, this construction encompases many standard metrics such as the Hasudorff distance; and with a choice of other categories and flows we can construct new metrics. So, we denote by d_I the interleaving distance for Reeb graphs defined by the smoothing S_ε. In the construction on this category, d_I is an extended metric since the interleaving distance between Reeb graphs with different numbers of connected components is ∞ [23].
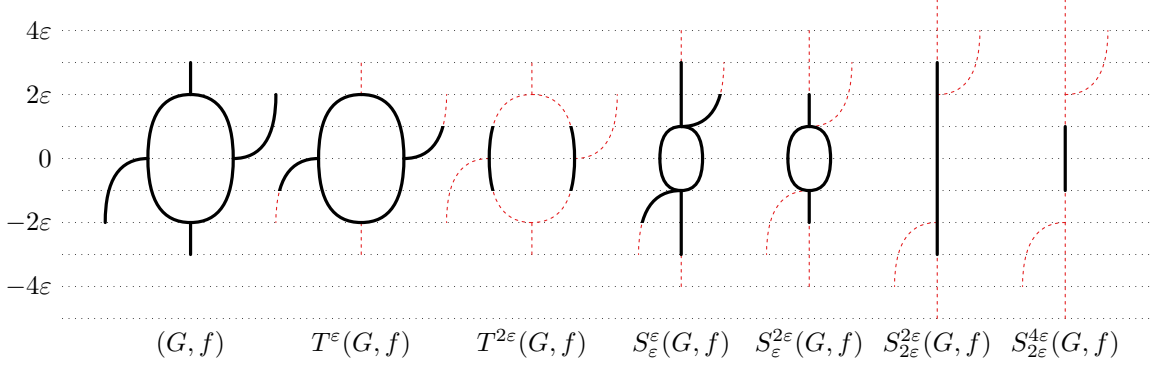
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Figure 4: Example of smoothing and truncating for a range of values, on the graph from Fig. 1.

One particularly useful property we will make use of is understanding how the image of the smoothed Reeb graph, $\text{Im}(S_\varepsilon(G, f)) := f(G) \subseteq \mathbb{R}$, changes under smoothing. Note that if G is connected, $\text{Im}(G, f)$ is connected so it is an interval.

Proposition 2.3. *For a connected Reeb graph (G, f) with $\text{Im}(G, f) = [a, b]$, $\text{Im}(S_\varepsilon(G, f)) = [a - \varepsilon, b + \varepsilon]$*

Proof. If $c \in \text{Im}(S_\varepsilon(G, f))$, then there is an $x \in S_\varepsilon(G, f)$ with $f_\varepsilon(x) = c$. Then there is a $(y, t) \in G \times [-\varepsilon, \varepsilon]$ with $f(y) + t = c$. Combining $a \leq f(y) \leq b$ and $-\varepsilon \leq t \leq \varepsilon$ gives that $c \in a - \varepsilon \leq c \leq b + \varepsilon$.

For the other direction, let $c \in [a - \varepsilon, b + \varepsilon]$. Then because $c - \varepsilon \leq b$, there is a $y \in f^{-1}(c - \varepsilon) \subseteq G$. So $x = q(y, \varepsilon)$ is a point in $S_\varepsilon(G, f)$ with $f(x) = c$, and hence $c \in \text{Im}(S_\varepsilon(G, f))$. $\square$

2.2 Truncated smoothing

We can now introduce our new, modified smoothing of Reeb graphs. Notice from Proposition 2.3 that as the Reeb graph is smoothed, the image becomes larger. The basic idea of truncated smoothing is to cut off some of those expanding tails in a well-defined way.

Let $U_\tau(G, f)$ be the set of points of G that do not have a length τ up-path, and define $D_\tau(G, f)$ symmetrically for down-paths. Note that for any point $x \in U_\tau(G, f)$, all up-paths from x also lie in $U_\tau(G, f)$; the symmetric property is true for $D_\tau(G, f)$. Both $U_\tau(G, f)$ and $D_\tau(G, f)$ are open subsets of (G, f). See Fig. 2 for an example. With this, we can define truncation as follows.

Definition 2.4. *The τ-truncation of (G, f),*

$$T^\tau(G, f) := (G, f) \setminus (U_\tau(G, f) \cup D_\tau(G, f)),$$

is the subgraph of (G, f) consisting of the points that have both an up-path and a down-path of height τ.

This operation can be seen in the second and third graphs of Fig. 4. Notice that $T^0(G, f) = (G, f)$, and that for large enough τ, it is entirely possible to disconnect the graph, or even to be left with an empty graph. Utilizing the truncation operation in conjunction with the Reeb graph smoothing operation is what we call truncated smoothing.

Definition 2.5. *Let (G, f), $\varepsilon \geq 0$ and $\tau \geq 0$ be given. Then the truncated smoothing of (G, f) is defined by $S_\varepsilon^\tau(G, f) = T^\tau S_\varepsilon(G, f)$.*

If $\tau = 0$, $S_\varepsilon^0(G, f) = T^0(S_\varepsilon(G, f)) = S_\varepsilon(G, f)$. So S_ε^0 is the same as S_ε, and thus the truncated smoothing can be thought of as a generalization of the smoothing definition.

Consider Fig. 4, which shows why we smooth before truncating and more generally, why we will soon want to place restrictions on the relationship between τ and ε. Namely, for this example, we have drawn

$T^\varepsilon(G, f)$ and $T^{2\varepsilon}(G, f)$. In the second case in particular, it is clear that truncation has massive detrimental effects on the topology as evidenced by the fact that $T^{2\varepsilon}(G, f)$ has two connected components. However, we can avoid these issues when we smooth first. In the last four examples, smoothing serves to move cycles away from the extrema, so that for a limited amount of truncation, no cycles are broken. We will quantify this 'safe' amount of truncation in Section 2.3. So, while the smoothing parameter still gets rid of the center circle, the truncation only gets rid of expanding tails.

Algorithm The τ-truncation of a Reeb graph (G, f) can be computed by first storing the length of the longest up-path and down-path of each vertex. This can be done in linear time using a topological sort of the graph based on the ordering of vertices induced by their function value. We can for each local maximum store that it has a 0-length up-path, and for the remaining vertices, use a dynamic program that processes vertices in the order given by the topological sort, and stores the length of their up-path based on the stored length of all previously processed neighbors. We store the length of the longest down-path for each vertex symmetrically. Now, we can compute for each edge how much of it remains in the truncation, and subdivide the edges if necessary. Finally, remove all vertices and edges that do not have a sufficiently long up-path or down-path. This procedure takes $O(n + m)$ time on a graph with n vertices at m edges. The truncated smoothing can be computed by first computing the smoothing [23] in $O(m \log(m + n))$ time, giving a total running time of $O(m \log(m + n))$.

2.3 Properties of truncated smoothing

We can visualize the relationship between τ and ε as drawn in Fig. 5. For this figure, we assume we start with a connected Reeb graph (G, f) and study properties of $S_\varepsilon^\tau(G, f)$ which is represented by the point (ε, τ) in the plane. In the remainder of this section, we state the properties of S_ε^τ in different regions of the ε-τ-plane, culminating in the parameter space labeling of Fig. 7. We will focus in this section on the case where G is a connected graph, although some results can be modified to incorporate disconnected inputs. We will leave these results, as well as many of the more technical proofs, to Appendix A.
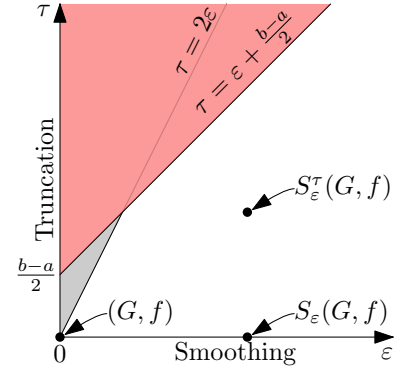

Figure 5: Parameters for which $S_\varepsilon^\tau(G, f)$ are definitely empty is given in red, where $\mathrm{Im}(G, f) = [a, b] \subset \mathbb{R}$.

We first study the values of ε and τ for which the truncated smoothing is empty. For the purposes of notation, define $\mathrm{Im}(G, f) = f(G) \subset \mathbb{R}$.

Consider the following simple example: Let $L_{[a,b]}$ be a Reeb graph consisting of a single edge with image $[a, b] \subseteq \mathbb{R}$, and for an interval $I \subseteq A$, let $L_I \subseteq L_{[a,b]}$ be the unique subgraph with image I. Then $T^\tau(L_{[a,b]}) = L_{[a+\tau, b-\tau]}$ if $2\tau \leq b - a$, and is the empty Reeb graph for $2\tau > b - a$. On the other hand, $S_\varepsilon(L_{[a,b]})$ is isomorphic to $L_{[a-\varepsilon, b+\varepsilon]}$. In particular, T^τ and S_ε transform any monotone path with image $[a, b]$ into a monotone path with image $[a + \tau, b - \tau]$, $[a - \varepsilon, b + \varepsilon]$, respectively. Finally, smoothing or truncating the empty Reeb graph again yields the empty Reeb graph.

We can build this intuition into the following proposition. Note that in the case of a connected input graph G, $\mathrm{Im}(G, f)$ is connected, and thus it is an interval.

Proposition 2.6. *For connected (G, f) with $\mathrm{Im}(G, f) = [a, b]$, and for $\varepsilon \leq 2\tau$,*

$$\mathrm{Im}(S_\varepsilon^\tau(G, f)) = \begin{cases} [a - (\varepsilon - \tau), b + (\varepsilon - \tau)] & \text{if } b - a \geq 2(\tau - \varepsilon) \\ \emptyset & \text{otherwise.} \end{cases}$$

Sketch proof. Note that by Proposition 2.3, $\mathrm{Im}(S_\varepsilon(G, f)) = [a - \varepsilon, b + \varepsilon]$. We show in Proposition A.10 that $T^\tau(G, f)$ is empty if $b - a < 2\tau$. Then if we assume $b - a < 2(\tau - \varepsilon)$, we have $(b + \varepsilon) - (a - \varepsilon) \leq 2\tau$, so $\mathrm{Im}(S_\varepsilon^\tau(G, f)) = \mathrm{Im}(T^\tau(S_\varepsilon(G, f))) = \emptyset$.

Now, we can assume $b - a \geq 2(\tau - \varepsilon)$. One direction of containment is easy since by Proposition 2.3,

$$\mathrm{Im}(S_\varepsilon^\tau(G, f)) = \mathrm{Im}(T^\tau(S_\varepsilon(G, f))) \subseteq [a - (\varepsilon - \tau), b + (\varepsilon - \tau)].$$

6

Thus, it remains to show that $[a - (\varepsilon - \tau), b + (\varepsilon - \tau)] \subseteq \text{Im}(S_\varepsilon^\tau(G, f))$. The basic idea is to take two points $s, t \in S_\varepsilon(G, f)$ with $f(s) = a - \varepsilon$ and $f(t) = b + \varepsilon$, and show that they are connected by a path π in $S_\varepsilon(G, f)$ for which the only portions that get truncated are the endpoints. This is simple if π is itself a monotone path; otherwise we must use the fact that G has already been smoothed to show that the parts of the path which are not monotone still have long enough up- and down-paths to not be removed. The full proof is provided in Appendix A. □

This proposition gives us that $T^\tau S_\varepsilon(G, f)$ is an empty graph if and only if (ε, τ) is interior to the pink region of Fig. 7, and a single point if and only if (ε, τ) lies on the boundary. Further, setting $\varepsilon = 0$, we know that $T^\tau(G, f)$ is also empty in that region, and thus $S_\varepsilon T^\tau(G, f)$ is also empty. If we continue and consider the commuted version, where we truncated and then smooth, we cannot make promises about the non-emptiness of $S_\varepsilon T^\tau(G, f)$ outside of the pink region; hence, we know these two functors do NOT commute in general. For an example, see Fig. 6. In this case, despite the fact that $\|\text{Im}(G, f)\| \geq 2(\tau - \varepsilon)$, each position in the graph has either a small up-path or a small down-path, and hence the truncated graph is empty.

Our next goal is to understand when truncation preserves the topology, in particular the connectivity, of the input. For this, we introduce two properties, *t-tailed* and *s-safe*, and study how they are affected by smoothing and truncation.

Figure 6: A Reeb graph (G, f) for which $T^\tau(G, f)$ is empty, although the image of f has diameter greater than 2τ.

Definition 2.7. *A Reeb graph is t-tailed if it has a height t up-path at every down-fork and a length t down-path at every up-fork. A Reeb graph is weakly s-safe if each component has a point with both an up-path and a down-path of height at least s. A Reeb graph is s-safe if it is both s-tailed and weakly s-safe.*

Note that every non-empty Reeb graph is 0-safe. For example, the graph drawn in Fig. 2 is not δ-tailed because the bottommost up-fork has no down-path of height δ; in addition, the topmost down-fork has no up-path of height δ.

We next have two results, proved in Appendix A, which show how the •-tailed and •-safe properties are maintained under smoothing and truncating, albeit with modified parameters.

Proposition 2.8. *If (G, f) is t-tailed, then $S_\varepsilon(G, f)$ is $(t + 2\varepsilon)$-tailed. If (G, f) is s-safe, then $S_\varepsilon(G, f)$ is $(s + \varepsilon)$-safe. Putting these together, if (G, f) is nonempty, then $S_\varepsilon(G, f)$ is 2ε-tailed and ε-safe.*

Lemma 2.9. *Fix $0 \leq \tau \leq \varepsilon$. If (G, f) is ε-tailed or safe, then $T^\tau(G, f)$ is $(\varepsilon - \tau)$-tailed or safe, respectively.*

Combining Proposition 2.8 and Lem. 2.9, we can see that outside the pink and grey regions of Fig. 7, we know that $S_\varepsilon^\tau(G, f)$ is $(t + 2\varepsilon - \tau)$-tailed and $(s + \varepsilon - \tau)$-safe.

Proposition 2.10. *Fix $0 \leq \varepsilon$ and $0 \leq \tau$, and assume (G, f) is t-tailed and s-safe. If $\tau \leq t + 2\varepsilon$ and $\tau \leq \varepsilon + \|\text{Im}(G, f)\|/2$, then $S_\varepsilon^\tau(G, f)$ is $(t + 2\varepsilon - \tau)$-tailed and $(s + \varepsilon - \tau)$-safe.*

Proof. Because (G, f) is t-tailed, $S_\varepsilon(G, f)$ is $(t + 2\varepsilon)$-tailed by the first statement of Proposition 2.8. Then since $\tau \leq t + 2\varepsilon$, $S_\varepsilon^\tau(G, f) = T^\tau S_\varepsilon(G, f)$ is $(t + 2\varepsilon - \tau)$-tailed by Lem. 2.9. Similarly, since (G, f) is s-safe, $S_\varepsilon(G, f)$ is $(t + \varepsilon)$-safe by the second statement of Proposition 2.8. Then since $\tau \leq t + 2\varepsilon$, $S_\varepsilon^\tau(G, f) = T^\tau S_\varepsilon(G, f)$ is $(s + \varepsilon - \tau)$-safe by Lem. 2.9. □

We next ask when the connectivity is maintained. As seen in Fig. 4, clearly just truncating the graph can disconnect an originally connected graph. However, what is interesting is that smoothing first and not truncating too much relative to the smoothing will maintain the connectivity, as shown in Proposition 2.11. Note that we show a generalized version of this statement for disconnected G in Corollary A.8.

Proposition 2.11. *If (G, f) is connected and $\tau \in [0, 2\varepsilon]$, then $S_\varepsilon^\tau(G, f)$ is also connected.*

Sketch proof. We can show that for a connected, t-tailed graph, $T^t(G, f)$ is connected by ensuring that the portion of the graph G removed because it is lacking an up-path, and that which is removed because it is lacking a down-path, are disjoint (Proposition A.7). The result is then a corollary of Proposition 2.8. □
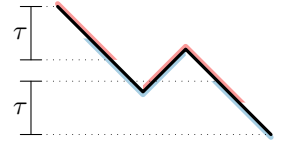
7

We finally investigate the commutativity of smoothing and truncating. The example of Fig. 4 shows why we must be careful with order of operations since $T^\tau S_\varepsilon(G, f)$ is not necessarily the same as $S_\varepsilon T^\tau(G, f)$. Specifically, $S_\varepsilon T^\tau(G, f)$, is not necessarily the same as $S_\varepsilon T^\tau(G, f)$. Specifically, $S_{2\varepsilon}^{2\varepsilon}(G, f) = T^{2\varepsilon} S_{2\varepsilon}(G, f)$ has one connected component, but any smoothing of $T^{2\varepsilon}(G, f)$ has two connected components. However, as we will see in the next two results, this issue does not arise if we sufficiently smooth before truncating.

Proposition 2.12. *If (G, f) is τ-safe, then $S_\varepsilon T^\tau(G, f) \cong T^\tau S_\varepsilon(G, f)$.*

This proof is provided in Appendix B.2. Combining the proposition with Lem. 2.9 and Proposition 2.8 gives surprising result that the functors T and S do commute in the green region of Fig. 7. We can next use this result to show that for certain choices of ε and τ, we can additively combine the parameters for truncated smoothing.

Figure 7: For connected, t-tailed, and s-safe (G, f), properties of $S_\varepsilon^\tau(G, f) = T^\tau S_\varepsilon(G, f)$ and $S_\varepsilon T^\tau(G, f)$, parameterized by τ and ε.

Theorem 2.13. *If (1) (G, f) is empty or (2) $\tau_1 \leq 2\varepsilon_1$ and (G, f) is weakly $(\tau_1 - \varepsilon_1)$-safe, then*

$$S_{\varepsilon_2}^{\tau_2} S_{\varepsilon_1}^{\tau_1}(G, f) \cong S_{\varepsilon_1 + \varepsilon_2}^{\tau_1 + \tau_2}(G, f).$$

Proof. Both smoothing and truncating the empty Reeb graph yields the empty Reeb graph. So we are done if (G, f) is the empty Reeb graph, and we obtain not only an isomorphism but an equality. Now suppose that (G, f) is not empty. Then $S_{\varepsilon_1}(G, f)$ is $2\varepsilon_1$-tailed and weakly $(\tau_1 - \varepsilon_1 + \varepsilon_1)$-safe, and by definition $\min(2\varepsilon_1, \tau_1) \geq \tau_1$-safe. Therefore $S_{\varepsilon_2} T^{\tau_1} S_{\varepsilon_1}(G, f) \cong T^{\tau_1} S_{\varepsilon_2} S_{\varepsilon_1}(G, f)$, and hence using Proposition 2.12,

$$S_{\varepsilon_2}^{\tau_2} S_{\varepsilon_1}^{\tau_1}(G, f) = T^{\tau_2} S_{\varepsilon_2} T^{\tau_1} S_{\varepsilon_1}(G, f) \cong T^{\tau_2} T^{\tau_1} S_{\varepsilon_2} S_{\varepsilon_1}(G, f) \cong S_{\varepsilon_1 + \varepsilon_2}^{\tau_1 + \tau_2}(G, f). \qquad \square$$

In particular, the assumptions of the theorem are satisfied if $\tau_1 \leq \varepsilon_1$ since every non-empty graph is 0-safe.

2.4   Truncated interleaving distance

In this section, we survey the results related to defining the family of truncated interleaving distances, proving that certain linear subspaces of our two parameter functor space (shown in Fig. 7) form a categorical flow. Since any category with a flow gives an interleaving distance, we then use truncated smoothing to build a new family of metrics for Reeb graphs.

The whole idea behind building a category with a flow is that the flow itself must be functorial, which means we must have knowledge of how it acts both on objects and morphisms. So far, the results discussed in Section 2.3 only correspond to the object information. In Section 3, we will describe how to explicitly build the morphisms $S_\varepsilon^\tau(G, f) \to S_{\varepsilon'}^\tau(G, f)$ (i.e., function preserving maps). However, these morphisms are only available for certain choices of parameters (Table 1). Restricting our view only to (ε, τ) pairs for which these morphisms exist gives us that for any choice of $m \in [0, 1]$ we can set $\tau = m\varepsilon$ to get a flow.

Theorem 2.14. *For any $m \in [0, 1]$, the map $S^m \colon ([0, \infty), \leq) \to \mathbf{End}(\mathbf{Reeb}); \varepsilon \mapsto S_\varepsilon^{m\varepsilon}$ is a functor and defines a categorical flow on* **Reeb**.

Essentially, this m can be thought of as defining the slope of a line based at the origin in the parameter space of Fig. 7, and thus using Theorem 2.2, we have an interleaving distance for any line with slope less than 1.

Corollary 2.15. *For any $m \in [0, 1]$, S^m gives rise to an interleaving-type distance*

$$d_I^m((G, f), (H, h)) := \inf\{\varepsilon \geq 0 \mid \text{there exists a } \varepsilon\text{-interleaving with respect to } S^m\}.$$

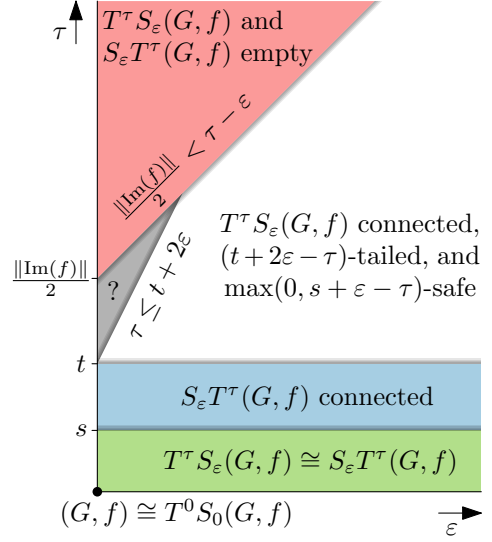*Specifically, d_I^m is an extended pseudo-metric.*

In the next theorem, we show that with the exception of $m = 1$, all the metrics created are closely related in the following sense. Two metrics d_A and d_B are said to be *strongly equivalent* if there are positive constants α_1 and α_2 such that $\alpha_1 d_A \leq d_B \leq \alpha_2 d_A$. In the following theorem, we show that d_I^m and $d_I^{m'}$ are strongly equivalent if (m, m') is contained in the white region of Fig. 8.

Theorem 2.16. *For any pair $0 \leq m \leq m' < 1$ with $0 \leq m' - m \leq 1 - m'$, the metrics d_I^m and $d_I^{m'}$ are equivalent. Specifically, given Reeb graphs (G, f) and (H, h),*

$$\frac{1 - 2m' + m}{1 - m} d_I^{m'}((G, f), (H, h)) \leq d_I^m((G, f), (H, h)) \leq \frac{1 - m'}{1 - 2m' + m} d_I^{m'}((G, f), (H, h)).$$

The proof of this theorem is contained in Section 7. Of course, as long as we are willing to loosen the bounds, we can extend this result to apply to any pair of $m, m' \in [0, 1)$.

Corollary 2.17. *For all pairs $0 \leq M \leq M' < 1$, there exist positive constants C_1 and C_2 dependent on M and M' such that*

$$C_1 d_I^M((G, f), (H, h)) \leq d_I^{M'}((G, f), (H, h)) \leq C_2 d_I^M((G, f), (H, h)),$$

and thus d_I^M and $d_I^{M'}$ are strongly equivalent metrics.

Proof. Say M, M' given, without loss of generality assume $M \leq M'$ and $M' \geq \frac{1+M}{2}$. Then d_I^M is equivalent to d_I^α for $\alpha = \frac{1+M}{2}$ and $d_I^{M'}$ is equivalent to d_I^β for $\beta = 2M' - 1$. Then there is a zigzag like the example in Fig. 8 between α and β which remains in the white region and for which each adjacent pair are strongly equivalent metrics. Equivalence of metrics is transitive, so this implies d_I^M and $d_I^{M'}$ are equivalent. □

2.5 Properties of the metrics

As noted, d_I^m is an extended pseudometric, which means that it is possible for $d_I^m((G, f), (H, h))$ to be infinite. However, it turns out this is not the case for broad classes of graphs. In fact, in order to take infinite value, there must be no ε-interleaving with respect to S^m between the two Reeb graphs. That being said, there are very specific instances where this metric takes on infinite value.

The easiest to case to handle is when $m \in [0, 1)$, since we can use the characterization given in [23] in conjunction with the equivalence of metrics Corollary 2.17.

Proposition 2.18. *Let $m \in [0, 1)$. Then $d_I^m((G, f), (H, h)) < \infty$ iff G and H have the same number of path-connected components.*
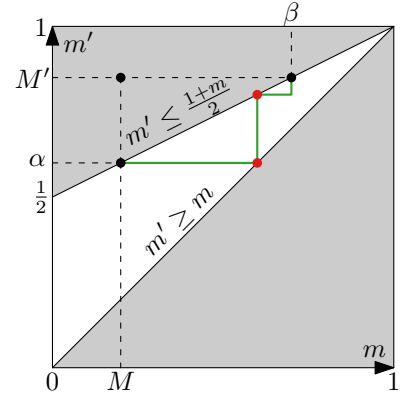

Figure 8: Parameter space for comparing metrics d_I^m and $d_I^{m'}$. The white region is allowable pairs for Theorem 2.16. The vertices of the zigzag give pairs of strongly equivalent metrics which show that M and M' are strongly equivalent in Corollary 2.17.

Proof. By [23, Prop. 4.5], $d_I((G, f), (H, h))$ is finite if and only if G and H have the same number of path connected components. This combined with Corollary 2.17 gives the proposition. □

The characterization of when d_I^m is infinite for $m = 1$ is more complicated. Consider a connected graph (G, f) with $\text{Im}(G, f) = [a, b]$. When $m = 1$, we are interested in understanding the behavior of $S_\varepsilon^\varepsilon(G, f)$. By Proposition 2.6, we see that $b - a \geq 2(\tau - \varepsilon) = 0$, so $S_\varepsilon^\varepsilon(G, f) = [a, b]$. That is to say that the image of (G, f) is unchanged by $S_\varepsilon^\varepsilon$. Now, if we wanted to determine the interleaving distance d_I^1 for a given (G, f) and (H, h), one requirement is always that we must smooth the given graphs enough for there to be a morphism $(G, f) \rightarrow S_\varepsilon^\varepsilon(H, h)$. However, because $S_\varepsilon^\varepsilon$ doesn't change the image, the function preserving requirement of morphisms mean that if the graphs did not start with the same image no choice of ε will make this possible. With this example in mind, we can characterize when d_I^m takes on infinite values for $m = 1$, and provide a strenghtened version of this proposition for disconnected graphs in Proposition A.9.

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Proposition 2.19. *Let $m = 1$ and assume G and H are connected. Then $d_I^m((G, f), (H, h)) < \infty$ if and only if $\mathrm{Im}(G, f) = \mathrm{Im}(H, h)$.*

Proof. Note that by Proposition 2.6, for any connected G' with $\mathrm{Im}(G', f') = [a, b]$, $S_\varepsilon^\varepsilon(G', f') = [a, b]$. So the smoothing maintains the image for every connected component, and thus for the union of the connected components. Thus, we have $\mathrm{Im}(G, f) = \mathrm{Im}(S_\varepsilon^\varepsilon(G, f))$ and $\mathrm{Im}(H, h) = \mathrm{Im}(S_\varepsilon^\varepsilon(H, h))$ for any choice of ε.

Assume we have an $S_\varepsilon^\varepsilon$ interleaving $\varphi \colon (G, f) \to S_\varepsilon^\varepsilon(H, h)$ and $\psi \colon (H, h) \to S_\varepsilon^\varepsilon(G, f)$. Because φ and ψ are function preserving, $\varphi(G) = \mathrm{Im}(G, f) \subseteq \mathrm{Im}(S_\varepsilon^\varepsilon(H, h))$ and $\psi(H) = \mathrm{Im}(H, h) \subseteq \mathrm{Im}(S_\varepsilon^\varepsilon(G, f))$. But since $S_\varepsilon^\varepsilon$ leaves the images unchanged, this implies that $\mathrm{Im}(G, f) = \mathrm{Im}(H, h)$.

Now assume $\mathrm{Im}(G, f) \neq \mathrm{Im}(H, h)$ or $\beta_0(G) \neq \beta_0(H)$. If $\mathrm{Im}(G, f) \neq \mathrm{Im}(H, h)$, without loss of generality there is $c \in \mathrm{Im}(G, f) \setminus \mathrm{Im}(H, h)$. Then the set $h_\varepsilon^{-1}(c)$ is empty for every $\varepsilon \geq 0$, so because any function preserving map $\varphi \colon (G, f) \to S_\varepsilon^\varepsilon(H, h)$ must send $x \in f^{-1}(c)$ to a point with $h_\varepsilon(\varphi(x)) = c$, no φ can exist. Thus no interleaving is possible, so $d_I^1((G, f), (H, h)) = \infty$. $\square$

We next investigate stability, or lack thereof, for this collection of metrics.

Definition 2.20. *Let $(\mathbb{X}, f)$ and $(\mathbb{X}, g)$ be $\mathbb{R}$-spaces with the same total space $\mathbb{X}$, and let $R(\mathbb{X}, f)$ and $R(\mathbb{Y}, g)$ be the respective Reeb graphs. A metric d is said to be stable if*

$$d(R(\mathbb{X}, f), R(\mathbb{X}, g)) \leq \|f - g\|_\infty.$$

The original Reeb interleaving distance, $m = 0$, is stable [23, Thm 4.4]. Unfortunately, ours is not; to see why, consider the following simple example. Given line graphs, (L, f_1) and (L, f_2) where $\mathrm{Im}(L, f_1) = [-a, a]$ and $\mathrm{Im}(L, f_2) = [-b, b]$ for $a < b$. Then $\|f_1 - f_2\|_\infty = b - a$. However, the interleaving distance requires that we smooth at least until $[-b, b] = \mathrm{Im}(L, f_2) \subseteq \mathrm{Im}(S_\varepsilon(L, f_1))$. But by Proposition 2.6, $\mathrm{Im}(S_\varepsilon(L, f_1)) = [a - (\varepsilon - m\varepsilon), a + (\varepsilon - m\varepsilon)]$. Thus $d_I^m(f_1, f_2) \geq \frac{b-a}{1-m} \geq b - a$, and is strictly greater if $m \neq 0$. This means that $b - a = \|f_1 - f_2\|_\infty < d_I^m((L, f_1), (L, f_2))$, and thus d_I^m is not stable.

We can regain at least partial control of the distance, however, as it is still Lipschitz with respect to the choice of m.

Proposition 2.21. *Let $m \in [0, 1)$. Assume $(\mathbb{X}, g_1)$ and $(\mathbb{X}, g_2)$ are given for a connected space $\mathbb{X}$ and denote the associated Reeb graphs by (G, f) and (H, h) respectively. Then there is a positive constant C dependent on m for which*

$$d_I^m((G, f), (H, h)) \leq C\|g_1 - g_2\|_\infty.$$

Proof. By Corollary 2.17, d_I^0 and d_I^m are strongly equivalent metrics, so there is a positive constant C for which $d_I^m \leq Cd_I^0$. Then because the Reeb interleaving distance d_I^0 is stable, we have

$$d_I^m((G, f), (H, h)) \leq Cd_I^0((G, f), (H, h)) \leq C\|g_1 - g_2\|. \qquad \square$$

We conclude by connecting our extended pseudometric to two other metrics for Reeb graphs, the functional distortion distance [2] and the bottleneck distance [43]. We simply state these, as the proof is a relatively straightforward implication of inequalities, so (due to space constraints) we do not formally define either. We refer the interested reader to [2] and [43] for further details on the metrics.

Proposition 2.22. *The truncated interleaving distance is strongly equivalent to the functional distortion distance and the bottleneck distance of level set persistent homology.*

Proof. The interleaving distance, d_I^0, is strongly equivalent to the functional distortion distance by [4, Thm 16] and the bottleneck distance of level set persistent homology by [6, 10]. So by Corollary 2.17 and transitivity of strong equivalence, they are each strongly equivalent to d_I^m for any $m \in [0, 1)$. $\square$

3 Categories and interleavings

A key tool in our operations on Reeb graphs comes from a category theory perspective, so we begin filling in the holes left behind in Section 2.4 by not fully describing the categorical aspects of the constructions discussed. We briefly review some essential concepts, but refer the reader to [45] for a background in category theory, as well as to prior work on category theory for Reeb graphs [40] and categories with a flow [24] for more details.

3.1 Categories, flows, and interleaving distances

A category $\mathcal{C}$ is a collection of objects, a collection of morphisms between the objects. We further require an associative composition operator which can compose any two of the morphisms, and that every object $c \in \mathcal{C}$ has an identity morphism $\mathrm{Id}_c \colon c \to c$. Mathematics is full of examples, from sets to vector spaces, as well as constructible $\mathbb{R}$-spaces. Denote by **Reeb** the category of Reeb graphs with function preserving maps as the morphisms.

A functor $F \colon \mathcal{C} \to \mathcal{D}$ is a map between any two categories sending objects to objects: $F(c) = d$; and morphisms to morphisms: $F[\varphi] \colon F(c) \to F(d)$ for $\varphi \colon c \to d$. This collection of data must preserve composition and identities, so $F[\varphi \circ \psi] = F[\varphi] \circ F[\psi]$ and $F[\mathrm{Id}_c] = \mathrm{Id}_{F(c)}$. Some examples of useful functors are homology $H_k \colon \mathrm{Top} \to \mathrm{Vect}$ from topological spaces to vector spaces (assuming field coefficients), or the functor $\pi_0 \colon \mathrm{Top} \to \mathrm{Set}$ sending a topological space to the set of its path-connected components.

We can treat the collection of functors from $\mathcal{C}$ to $\mathcal{D}$ as a category in itself, where the morphisms are given by *natural transformations*. Specifically, given $F, G \colon \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta \colon F \Rightarrow G$ is a collection of morphisms $\eta_c \colon F(c) \to G(c)$ such that

$$
\begin{array}{ccc}
F(c) & \xrightarrow{\eta_c} & G(c) \\
{\scriptstyle F[\varphi]}\downarrow & & \downarrow{\scriptstyle G[\varphi]} \\
F(c') & \xrightarrow{\eta_{c'}} & G(c')
\end{array}
$$

commutes for any morphism $\varphi \colon c \to c'$ in $\mathcal{C}$. This functor category with objects as functors and morphisms given by natural transformations is denoted $\mathcal{D}^{\mathcal{C}}$. A natural transformation is a *natural isomorphism* if every map η_c is an isomorphism. When we have a natural isomorphism between functors we write $F \cong G$. A special case of the functor category is when $\mathcal{D} = \mathcal{C}$. A functor from a category to itself, $F \colon \mathcal{C} \to \mathcal{C}$, is called an endomorphism and the category of all such functors with natural transformations is denoted $\mathbf{End}(\mathcal{C})$.

The end goal of this paper is to study flows of Reeb graphs, where the idea is to have a 1-parameter varying collection of Reeb graphs satisfying nice properties. For this, we look to the definition of a category with a flow given in [24].

Definition 3.1. *Let $[0, \infty)$ denote the poset category of positive real numbers with morphisms given by $\leq$. Given a category $\mathcal{C}$, a* categorical flow *is a functor $F \colon [0, \infty) \to \mathbf{End}(\mathcal{C})$, $\varepsilon \mapsto F_\varepsilon$, with $F_0 \cong \mathbb{1}_{\mathcal{C}}$ and $F_a F_b \cong F_{a+b}$ for all $a, b \geq 0$.*

Note that this definition is hiding quite a bit of infrastructure. In particular, F_ε is a functor, so it gives rise to a morphism $F_\varepsilon[\varphi]$ for every morphism φ of $\mathcal{C}$. The fact that F is a functor means we get a natural transformation $F[\varepsilon \leq \varepsilon'] \colon F_\varepsilon \Rightarrow F_{\varepsilon'}$ for every $\varepsilon \leq \varepsilon'$. That this is a natural transformation means that

$$
\begin{array}{ccc}
F_\varepsilon(c) & \xrightarrow{F[\varepsilon \leq \varepsilon']_c} & F_{\varepsilon'}(c) \\
{\scriptstyle F_\varepsilon[\varphi]}\downarrow & & \downarrow{\scriptstyle F_{\varepsilon'}[\varphi]} \\
F_\varepsilon(c') & \xrightarrow[F[\varepsilon \leq \varepsilon']_{c'}]{} & F_{\varepsilon'}(c')
\end{array}
$$

commutes for every morphism φ of $\mathcal{C}$. The final requirement, $F_a F_b \cong F_{a+b}$, checks that flowing by a and then b is at least closely related to flowing by the total amount $a + b$ all at once.

This definition is particularly useful since it can be used to provide an interleaving distance for any category with a given flow.

Definition 3.2. *Given a category with a (categorical) flow $(\mathcal{C}, F)$ and two objects $X, Y \in \mathcal{C}$, an ε-interleaving of X and Y is a pair of morphisms $\varphi : X \to F_\varepsilon Y$ and $\psi : Y \to F_\varepsilon X$ such that*

$$
\begin{array}{ccccc}
X & \xrightarrow{\ \ F[0 \leq \varepsilon]\ \ } & F_\varepsilon X & \xrightarrow{\ \ F[\varepsilon \leq 2\varepsilon]\ \ } & F_{2\varepsilon} X \\
& {}^{\varphi} & {}_{F_\varepsilon[\varphi]} & & \\
Y & \xrightarrow[\ \ F_\varepsilon[0 \leq \varepsilon]\ \ ]{} & F_\varepsilon Y & \xrightarrow[\ \ F[\varepsilon \leq 2\varepsilon]\ \ ]{F_\varepsilon[\psi]} & F_{2\varepsilon} Y
\end{array}
\tag{3.3}
$$

commutes. Then, the interleaving distance is given by

$$
d_{(\mathcal{C},F)}(X,Y) = \inf\{\varepsilon \geq 0 \mid X, Y \text{ are } \varepsilon\text{-interleaved}\}.
$$

Note that $d_{\mathcal{C},F}$ is an extended pseudometric on the objects of $\mathcal{C}$ [24, Thm. 2.7]; i.e. it can take infinite value, and $d(X,Y) = 0$ does not imply that $X = Y$.

Remark 3.4. *It is necessary to now point out that we are consciously abusing notation from here on out. This categorical flow is a special case of the definition of flow given in [24, 49]; specifically, what we have defined is called a strong flow in that work. In [24], the flow comes with additional notation to encode the isomorphisms of $T_0 \cong \mathbb{1}_\mathcal{C}$ and $T_a T_b \cong T_{a+b}$, and to ensure that they interact accordingly. Then, when giving the interleaving definition, the interleaving diagram is expanded to essentially be comprised of linked pentagons rather than the large scale triangles seen in Eq. (3.3). In this paper, we will do our best to point out when it happens, but we will suppress the isomorphism since it does not serve to illuminate the work, but rather invariably results in exponential growth of the size of the required commutative diagrams.*

3.2 Smoothing and the interleaving distance for Reeb graphs

In this section, we give further specifics of the original smoothing definition from [23], given as Defn. 2.1. While the idea comes from the equivalence of categories between **Reeb** and a particular category of cosheaves, we will not need that construction here so we will focus on the geometric definition of smoothing.

We construct the thickening of the graph, $(G \times [-\varepsilon, \varepsilon], f + \mathrm{Id})$, and the smoothing $S_\varepsilon(G, f)$ is the Reeb quotient of this space, where we denote the quotient map by q. However, this leaves out the important collection of morphisms that come with the smoothing construction. Specifically, we have an inclusion $(\mathrm{Id}, 0): G \to G \times [-\varepsilon, \varepsilon]$ given by $x \mapsto (x, 0)$; see Fig. 3 for an illustration, where this inclusion is shown in the thickened space as a dotted copy of the original graph from the left. Let $\eta = q_\varepsilon \circ (\mathbb{1} \times 0): G \to G_\varepsilon$, so that $\eta(x) = q_\varepsilon(x, 0)$. The process can be summarized in the diagram

$$
\begin{array}{ccc}
& (G \times [-\varepsilon, \varepsilon], f + \mathrm{Id}) & \\
{}^{(\mathrm{Id},0)} \nearrow & & \searrow {}^{q} \\
(G, f) & \xrightarrow[\ \ \ \ \ \ \ \ \eta \ \ \ \ \ \ \ \ ]{} & S_\varepsilon(G, f).
\end{array}
\tag{3.5}
$$

Note that η, q, and $(\mathrm{Id}, 0)$ are all function preserving maps.

Given a morphism $\varphi : (G, f) \to (H, h)$, i.e. a function preserving map $\varphi : G \to H$, it can be checked that

there is an induced morphism $S_\varepsilon[\varphi] : S_\varepsilon(G, f) \to S_\varepsilon(H, h)$ making the diagram

$$
\begin{array}{c}
(G \times [-\varepsilon, \varepsilon], f + \mathrm{Id}) \\
\end{array}
$$

(3.6)

commute. With these $S_\varepsilon[\varphi]$ maps, S_ε is an endofunctor on **Reeb**; see [23, Sec 4.4] for details.

Further, note that replacing (H, f) with $S_\varepsilon(G, f)$ in Eq. (3.6) gives a map $S_\varepsilon[\eta] : S_\varepsilon(G, f) \to S_\varepsilon S_\delta(G, f)$. As noted in [23, Obs. 4.30], there is a natural isomorphism $S_{\varepsilon_2} S_{\varepsilon_1}(G, f) \to S_{\varepsilon_1 + \varepsilon_2}(G, f)$. For this reason and in the spirit of Remark 3.4, we abuse notation and write $\eta : S_{\varepsilon_1}(G, f) \to S_{\varepsilon_1 + \varepsilon_2}(G, f)$ for the composition of maps

$$
\begin{array}{ccc}
S_{\varepsilon_1}(G, f) & & \\
{\scriptstyle S_\varepsilon[\eta]}\downarrow & \searrow{\scriptstyle \eta} & \\
S_{\varepsilon_2}(S_{\varepsilon_1}(G, f)) & \xrightarrow{\cong} & S_{\varepsilon_1 + \varepsilon_2}(G, f).
\end{array}
$$

In particular, for any $0\varepsilon \leq \varepsilon'$, we write $\eta : S_\varepsilon(G, f) \to S_{\varepsilon'}(G, f)$ without changing the notation η unless it is necessary for clarity. A full discussion of this suppressed homeomorphism is in Appendix B.1.

With this notation, we have that the η's compose, in the sense that

$$
\begin{array}{ccccc}
S_{\varepsilon_1}(G, f) & \xrightarrow{\eta} & S_{\varepsilon_2}(G, f) & \xrightarrow{\eta} & S_{\varepsilon_3}(G, f) \\
 & \searrow & & \nearrow & \\
 & & {\scriptstyle \eta} & &
\end{array}
$$

commutes for all $\varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_3$. Again suppressing the homeomorphism, the η maps also interact with the Reeb quotient map in the sense that

$$
\begin{array}{ccc}
G \times [-\varepsilon, \varepsilon] & \hookrightarrow & G \times [-\varepsilon', \varepsilon'] \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\
S_\varepsilon(G, f) & \xrightarrow{\eta} & S_{\varepsilon'}(G, f)
\end{array}
$$

(3.7)

commutes for all $\varepsilon \leq \varepsilon'$. Another useful diagram to note is that with this supression of the homeomorphism, the diagram

$$
\begin{array}{ccc}
S_\varepsilon(G, f) & \xrightarrow{\eta} & S_{\varepsilon'}(G, f) \\
{\scriptstyle S_\varepsilon[\psi]}\downarrow & & \downarrow{\scriptstyle S_{\varepsilon'}[\psi]} \\
S_\varepsilon(H, h) & \xrightarrow{\eta} & S_{\varepsilon'}(H, h)
\end{array}
$$

(3.8)

commutes.

All of this bookkeeping can be summarized by $S_0(G, f) \cong (G, f)$ and $S_b S_a(G, f) \cong S_{a+b}(G, f)$, and thus $S : \varepsilon \mapsto S_\varepsilon$ with $S_\varepsilon \in \mathbf{End}(\mathbf{Reeb})$ is a strong flow. Finally, since S defines a flow, we have an interleaving distance (Defn. 3.2) on **Reeb** which we will call simply the interleaving distance, $d_I = d_{\mathbf{Reeb}, S}$.

Proposition 3.9 ([23, Props. 4.3, 4.5, and 4.6])**.** *The Reeb interleaving distance d_I is an extended pseudometric. It can take infinite value if and only if the Reeb graphs have different numbers of path components. It is 0 if and only if the Reeb graphs are isomorphic (i.e. if their graphs are isomorphic and that isomorphism is function preserving).*
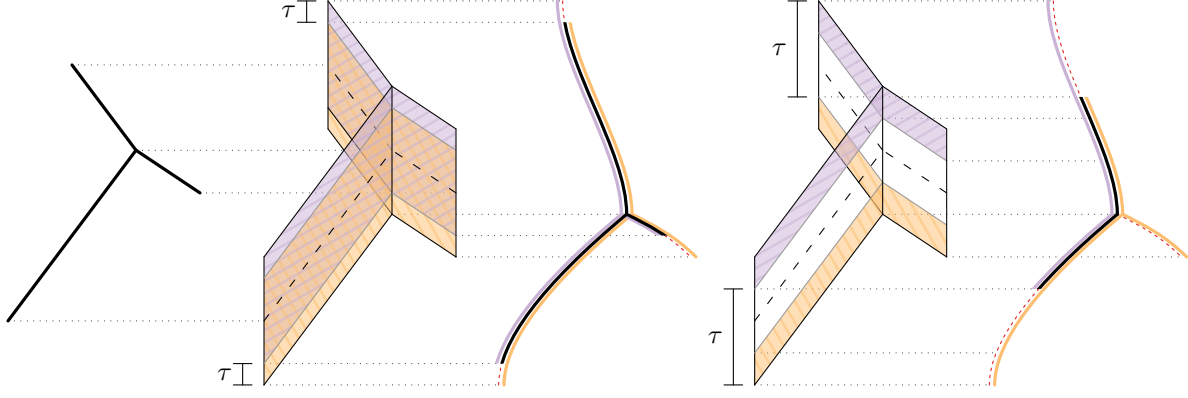
Figure 9: Left: a Reeb graph (G, f). Middle: its thickening and truncated smoothing $T^\tau S_\varepsilon(G, f)$ for $\tau \in [0, \varepsilon]$. Right: the same for $\tau \in (\varepsilon, 2\varepsilon]$.

4 Equivalent definitions of truncated smoothing

We have defined truncated smoothing by removing the portion of $S_\varepsilon(G, f)$ corresponding to growing tails (Defn. 2.5). In this section, we show three other equivalent definitions (Proposition 4.2 and Corollaries 4.4 and 4.5) which will prove useful later when for calculations or proving mathematical properties.

4.1 Definition in terms of quotient maps

We begin by presenting a definition of truncated smoothing that more closely tied to the thickening definition used to construct $S_\varepsilon(G, f)$ in the first place by restricting the values of τ that can be used. Given $\varepsilon \geq 0$ and $\tau \in [0, 2\varepsilon]$, we have the commutative diagram

$$
\begin{array}{c}
(G \times [\tau - \varepsilon, \varepsilon], f + \mathrm{Id}) \\
\downarrow \qquad \overset{q}{\dashrightarrow} \\
(G \times [-\varepsilon, \varepsilon], f + \mathrm{Id}) \xrightarrow{\ q\ } S_\varepsilon(G, f) \\
\uparrow \qquad \underset{q}{\dashrightarrow} \\
(G \times [-\varepsilon, \varepsilon - \tau], f + \mathrm{Id})
\end{array}
$$

where the middle map q is the Reeb quotient map, and the diagonal arrows are its restrictions. We can then study the intersection

$$
q\Big(G \times [\tau - \varepsilon, \varepsilon] \Big) \cap q\Big(G \times [-\varepsilon, \varepsilon - \tau] \Big) \tag{4.1}
$$

with function given by the restriction of the function from $S_\varepsilon(G, f)$. See Fig. 9 for an example with two different choices of τ relative to ε. The following proposition shows that this intersection gives an equivalent definition for S_ε^τ to that of Defn. 2.5.

Proposition 4.2. *For $\tau \in [0, 2\varepsilon]$,*

$$
S_\varepsilon^\tau(G, f) = q\Big(G \times [\tau - \varepsilon, \varepsilon] \Big) \cap q\Big(G \times [-\varepsilon, \varepsilon - \tau] \Big).
$$

Proof. For the sake of notation, denote $A = q\Big(G \times [\tau - \varepsilon, \varepsilon] \Big) \cap q\Big(G \times [-\varepsilon, \varepsilon - \tau] \Big)$ for the remainder of the proof. We first show that any point $x \in A$ has both an up- and a down-path of height τ in $S_\varepsilon(G, f)$, and hence $x \in T^\tau S_\varepsilon(G, f) = S_\varepsilon^\tau(G, f)$. Let $(y, t) \in G \times [\tau - \varepsilon, \varepsilon]$ and $(y', t') \in G \times [-\varepsilon, \varepsilon - \tau]$ such that

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$x = q(y, t) = q(y', t')$. Then we have paths $\gamma, \gamma' : [0, \tau] \rightsquigarrow S_\varepsilon(G)$ given by $\gamma(s) = (y, t-s)$ and $\gamma'(s) = (y', t'+s)$. Therefore $q \circ \gamma$ is a down-path and $q \circ \gamma'$ is an up-path of x in $S_\varepsilon(G, f)$ of height τ, so $S_\varepsilon^\tau(G, f) \subseteq T^\tau S_\varepsilon(G, f)$.

For the other direction, we show that any point $x \in S_\varepsilon(G, f) \setminus A$ has no up-path or no down-path of height at least τ in at least one direction. Since $x \notin A$, we have $q^{-1}(x) \cap (G \times [\tau - \varepsilon, \varepsilon]) = \emptyset$ or $q^{-1}(x) \cap (G \times [-\varepsilon, \varepsilon - \tau]) = \emptyset$ by definition. Without loss of generality, assume that $q^{-1}(x) \cap (G \times [-\varepsilon, \varepsilon - \tau]) = \emptyset$ as the other case is symmetric.

First, consider the superlevelset $\{p \in G \times [-\varepsilon, \varepsilon] \mid f(p) \geq f(x)\}$ and let C be its component containing $q^{-1}(x)$. For a point $p = (y, \lambda) \in C$, we define $\beta(p)$ to be the point $(y, \lambda') \in C$ that minimizes λ'. Specifically, we set $\beta(p) = (y, \max\{f(x) - f(y), -\varepsilon\})$, and note that $f(\beta(p)) = \max\{f(x), f(y) - \varepsilon\}$; see Fig. 10 for a visual of the notation.

We claim that for every $p \in C$, we have $f(\beta(p)) = f(x)$, which will later imply that no up-path of sufficient length exists. Seeking a contradiction, suppose instead that there is a $p \in C$ with $f(\beta(p)) > f(x)$. This implies that $f(\beta(p)) = f(y) - \varepsilon > f(x)$. Clearly $\beta(p) \notin q^{-1}(x)$, otherwise $f(\beta(p)) = f(x)$. Because C is path-connected, there exists a path $\pi : [0, 1] \to C$ from p to a point $p' \in C$ with $\beta(p') \in q^{-1}(x)$; again, see Fig. 10. Denote $\pi(t) = (y(t), \lambda(t))$. Without loss of generality, assume that $\beta(\pi(t)) \notin q^{-1}(x)$ for all $t < 1$. If $f(\beta\pi(t)) = f(x)$ for all t, then $\beta(p)$ is in the same level-set connected component as $q^{-1}(x)$, contradicting that $\beta(p) \notin q^{-1}(x)$. The assumption that $\beta\pi(t) \notin q^{-1}(x)$ for $t < 1$ thus further implies that there is a δ for which $f(\beta\pi(t)) > f(x)$ for $t \in [1 - \delta, 1)$.

Figure 10: The notation used in the proof of Proposition 4.2.

By definition of β, this means that $f(\beta\pi(t)) = f(y(t)) - \varepsilon$ for $t \in [1 - \delta, 1)$. However, by continuity of $t \mapsto f(y(t)) - \varepsilon$, we have $f(\beta\pi(1)) = f(x) = f(y(1)) - \varepsilon$. Therefore $\beta\pi(1) = (y, -\varepsilon) \in G \times [-\varepsilon, \varepsilon - \tau]$, and $\beta\pi(1) = \beta(p') \in q^{-1}(x)$, contradicting that $q^{-1}(x) \cap (G \times [-\varepsilon, \varepsilon - \tau]) = \emptyset$.

Now, for any $p \in C$, we have $\beta(\pi(p)) \in q^{-1}(x)$, so the image of C under f is a subset of $[f(x), f(x) + \tau]$. The preimage of any up-path in $S_\varepsilon(G, f)$ starting at x lies in C, so no such path can have height at least τ, completing the proof. □

4.2 Alternative definitions for $\tau \in [0, \varepsilon]$

Note that if $\tau \in (\varepsilon, 2\varepsilon]$, then $[-\varepsilon, \varepsilon - \tau] \cap [\tau - \varepsilon, \varepsilon]$ is empty. In this case, $G \times [-\varepsilon, \varepsilon - \tau] \cap G \times [\tau - \varepsilon, \varepsilon]$ is empty, which is the reason for to passing to the image of q before intersection in Eq. (4.1). However, if $\tau \in [0, \varepsilon]$, we have no such issue, which leads to an alternative definition for $S_\varepsilon^\tau(G)$ which is a corollary to the following lemma.

Lemma 4.3. *Given (G, f), if $\tau \in [0, \varepsilon]$,*

$$q\Big(G \times [-\varepsilon, \varepsilon - \tau]\Big) \cap q\Big(G \times [\tau - \varepsilon, \varepsilon]\Big) = q\Big(G \times [-\varepsilon + \tau, \varepsilon - \tau]\Big)$$

Proof. First, note that $q(G \times [-\varepsilon, \varepsilon - \tau] \cap G \times [\tau - \varepsilon, \varepsilon]) = q(G \times [-\varepsilon + \tau, \varepsilon - \tau])$. The left inclusion of the lemma is immediate since given any $x \in q(G \times [-\varepsilon + \tau, \varepsilon - \tau])$ there is $(y, t) \in G \times [-\varepsilon + \tau, \varepsilon - \tau]$ such that $q(y, t) = x$. Thus (y, t) is in both $G \times [-\varepsilon, \varepsilon - \tau]$ and $G \times [\tau - \varepsilon, \varepsilon]$, and so $x \in q(G \times [-\varepsilon, \varepsilon - \tau]) \cap q(G \times [-\varepsilon + \tau, \varepsilon])$.

For the right inclusion, let $x \in q(G \times [-\varepsilon, \varepsilon - \tau]) \cap q(G \times [\tau - \varepsilon, \varepsilon])$. Then there is $(y, t) \in G \times [-\varepsilon, \varepsilon - \tau]$ and $(y', t') \in G \times [-\varepsilon + \tau, \varepsilon]$ such that $x = q(y, t) = q(y', t')$. If either t or t' are contained in $[-\varepsilon + \tau, \varepsilon - \tau]$, then we are done, so we can assume $t \in [\varepsilon + \tau, \varepsilon]$ and $t' \in [-\varepsilon, \varepsilon + \tau]$. Because they both have the same image under q, there is a path $(\gamma_1, \gamma_2) = \gamma : (y, t) \rightsquigarrow (y', t')$ with $f(\gamma_1(s)) + \gamma_2(s)$ constant; specifically, $q(\gamma(s)) = x$ for all $s \in [0, 1]$. As γ_2 is a continuous map, there must be an $s \in [0, 1]$ for which $\gamma_2(s) \in [-\varepsilon + \tau, \varepsilon - \tau]$. Then $\gamma(s) \in G \times [-\varepsilon + \tau, \varepsilon - \tau]$ and $q(\gamma(s)) = x$, completing the proof. □

Combined with Defn. 2.5, this gives us an immediate corollary that can be viewed as an equivalent definition for the truncated smoothing whenever τ is small enough.

Notation	Direction	Slope ($\frac{d\tau}{d\varepsilon}$)	Requirements for existence $S_{\varepsilon_1}^{\tau_1}(G,f) \to S_{\varepsilon_2}^{\tau_2}(G,f)$	Notes	Location of definition
ν	$\downarrow$	vertical	$\varepsilon_1 = \varepsilon_2,\ \tau_1 \geq \tau_2$	inclusion map	Defn. 5.1
η	$\rightarrow$	horizontal	$\varepsilon_1 \leq \varepsilon_2,\ \tau_1 = \tau_2$	smoothing map	Defn. 5.2
ω	$\searrow$	$m \in [-\infty, 0]$	$\varepsilon_1 \leq \varepsilon_2,\ \tau_1 \geq \tau_2$	$\omega = \nu\eta = \eta\nu$ (Lemma 5.3)	Defn. 5.4
ρ	$\nearrow$	$m \in [0,1]$	$0 \leq \tau_2 - \tau_1 \leq \varepsilon_2 - \varepsilon_1$	equals η when $\tau_1 = \tau_2$	Defn. 6.1

Table 1: Maps and when they exist.

Corollary 4.4. *Given (G,f) and $\tau \in [0,\varepsilon]$,*

$$S_\varepsilon^\tau(G,f) = q(G \times [\tau - \varepsilon, \varepsilon - \tau])$$

where q is a restriction of the quotient map $q : G \times [-\varepsilon, \varepsilon] \to S_\varepsilon(G,f)$, and the Reeb graph function is given by the restriction of the function from $S_\varepsilon(G,f)$.

Another way of viewing the truncated definition is by looking backward in the flow by τ. Because flows are only defined for $\varepsilon \geq 0$, we can use this viewpoint only if τ is small enough that $\varepsilon - \tau$ is non-negative, thus we have the following equivalent definition for truncated smoothing with small enough τ.

Corollary 4.5. *Given (G,f) and $\tau \in [0,\varepsilon]$, then*

$$S_\varepsilon^\tau(G,f) = \eta(S_{\varepsilon-\tau}(G,f)) := \mathrm{Im}\{\eta \colon S_{\varepsilon-\tau}(G,f) \to S_\varepsilon(G,f)\}.$$

Proof. This comes from combining Lem. 4.3 with the commutative diagram

$$
\begin{array}{ccc}
G \times [-\varepsilon + \tau, \varepsilon - \tau] & \lhook\joinrel\longrightarrow & G \times [-\varepsilon, \varepsilon] \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
S_{\varepsilon-\tau}(G) & \xrightarrow{\ \eta\ } & S_\varepsilon(G).
\end{array}
$$

$\square$

5 Maps and their properties

We next build several maps, closely related to the map $\eta : S_\varepsilon(G,f) \to S_{\varepsilon'}(G,f)$ coming from the smoothing, which we will use to eventually build a new flow for our category **Reeb**. We then show that truncation itself is functorial and relate its definition to these maps.

5.1 The maps η, ν, and ω

We will define several maps that relate $S_\varepsilon^\tau(G,f)$ for various values of τ and ε. These maps can be visualized in the ε-τ plane; see Table 1 for a handy reference of the maps and their requirements and properties.

The easiest map to define is ν, which is simply an inclusion of an untruncated graph into the original. That is, we have immediate inclusions maps $S_\varepsilon^\tau(G,f) \hookrightarrow S_\varepsilon(G,f)$ since $S_\varepsilon^\tau(G,f)$ is defined as a subspace of $S_\varepsilon(G,f)$. Given $\tau \leq \tau'$, this generalizes to map $\nu \colon S_\varepsilon^{\tau'}(G,f) \to S_\varepsilon^\tau(G,f)$ on the truncated smoothings. The easiest way to see this is using the fact that any point with an up- and down-path of height τ' certainly has each of height τ.

Definition 5.1. *For any $0 \leq \tau_i \leq 2\varepsilon_i$ and $\tau_2 \leq \tau_1$, the map $\nu \colon S_\varepsilon^{\tau_1}(G,f) \hookrightarrow S_\varepsilon^{\tau_2}(G,f)$ is given by inclusion.*

The next map is the restriction of $\eta\colon S_\varepsilon(G,f) \to S_{\varepsilon'}(G,f)$ to the truncated graphs. It is immediate that the following diagram

$$
\begin{array}{c}
\text{(commutative diagram)}
\end{array}
$$

Diagram with vertices:
$G \times [-\varepsilon + \tau, \varepsilon] \hookrightarrow G \times [-\varepsilon' + \tau, \varepsilon']$,
$S_\varepsilon(G) \xrightarrow{\ \eta\ } S_{\varepsilon'}(G)$,
$G \times [-\varepsilon, \varepsilon] \hookrightarrow G \times [-\varepsilon', \varepsilon']$,
$G \times [-\varepsilon, \varepsilon - \tau] \hookrightarrow G \times [-\varepsilon' + \tau, \varepsilon']$,
with maps labeled q.

commutes since the black square was shown to be commutative in Eq. (3.7), and all other maps are either inclusions or defined by composition. From this diagram, we can extend the definition of η to the truncated graphs.

Definition 5.2. *For $0 \le \varepsilon \le \varepsilon'$ and $0 \le \tau \le 2\varepsilon$, the map $\eta\colon S_\varepsilon^\tau(G,f) \to S_{\varepsilon'}^\tau(G,f)$ is given by the restriction of $\eta\colon S_\varepsilon(G,f) \to S_{\varepsilon'}(G,f)$ to $\nu(S_\varepsilon^\tau(G,f)) \subseteq S_\varepsilon(G,f)$.*

The maps η and ν commute as shown in the following lemma.

Lemma 5.3. *For any $0 \le \tau \le \tau' \le 2\varepsilon \le 2\varepsilon'$, the following diagram commutes:*

$$
\begin{array}{ccc}
S_\varepsilon^{\tau'}(G,f) & \xrightarrow{\ \eta\ } & S_{\varepsilon'}^{\tau'}(G,f) \\[4pt]
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\[4pt]
S_\varepsilon^{\tau}(G,f) & \xrightarrow{\ \eta\ } & S_{\varepsilon'}^{\tau}(G,f)
\end{array}
$$

Proof. The proof comes from diagram chasing in the following commutative diagram.

Diagram with vertices:
$G \times [-\varepsilon + \tau, \varepsilon] \hookrightarrow G \times [-\varepsilon' + \tau, \varepsilon']$,
$G \times [-\varepsilon + \tau', \varepsilon] \hookrightarrow G \times [-\varepsilon' + \tau', \varepsilon']$,
$S_\varepsilon(G,f) \xrightarrow{\ \eta\ } S_{\varepsilon'}(G,f)$,
$G \times [-\varepsilon, \varepsilon] \hookrightarrow G \times [-\varepsilon', \varepsilon']$,
$G \times [-\varepsilon, \varepsilon - \tau'] \hookrightarrow G \times [-\varepsilon', \varepsilon' - \tau']$,
$G \times [-\varepsilon, \varepsilon - \tau] \hookrightarrow G \times [-\varepsilon', \varepsilon' - \tau]$,
with maps labeled q.

$\square$

We will use the square of Lem. 5.3 repeatedly, so we name the diagonal map as follows.

Definition 5.4. *For $0 \leq \tau_i \leq 2\varepsilon_i$ $i = 1, 2$, $\varepsilon_1 \leq \varepsilon_2$ and $\tau_2 \leq \tau_1$, $\omega\colon S_{\varepsilon_1}^{\tau_1}(G, f) \to S_{\varepsilon_2}^{\tau_2}(G, f)$ is defined by is the diagonal of the square*

$$
\begin{array}{ccc}
S_{\varepsilon_1}^{\tau_1}(G, f) & \xrightarrow{\eta} & S_{\varepsilon_2}^{\tau_1}(G, f) \\
\downarrow{\scriptstyle\nu} & \searrow{\scriptstyle\omega} & \downarrow{\scriptstyle\nu} \\
S_{\varepsilon_1}^{\tau_2}(G, f) & \xrightarrow{\eta} & S_{\varepsilon_2}^{\tau_2}(G, f).
\end{array}
$$

That is, $\omega = \eta\nu = \nu\eta$ when those maps exist.

See Fig. 11 for a visualization. Note that this definition implies that $\omega = \eta$ if $\tau_1 = \tau_2$; and $\omega = \nu$ if $\varepsilon_1 = \varepsilon_2$. Notice that we can also define ω by looking backwards by using Corollary 4.5.

Figure 11: Visualization of the commutative diagram of Lem. 5.3.

Lemma 5.5. *Assume τ_1, τ_2, ε_1 and ε_2 chosen so that the map $\omega\colon S_{\varepsilon_1}^{\tau_1}(G, f) \to S_{\varepsilon_2}^{\tau_2}(G, f)$ is defined; i.e. $0 \leq \tau_i \leq 2\varepsilon_i$, $\varepsilon_1 \leq \varepsilon_2$ and $\tau_2 \leq \tau_1$.*

i The map ω is equal to the restriction of $\eta\colon S_{\varepsilon_1}(G, f) \to S_{\varepsilon_2}(G, f)$ to $\nu(S_{\varepsilon_1}^{\tau_1}(G, f)) \subseteq S_{\varepsilon_1}(G, f)$.

ii If $0 \leq \tau_i \leq \varepsilon_i$, then ω is the restriction of $\eta\colon S_{\varepsilon_1}(G, f) \to S_{\varepsilon_2}(G, f)$ to $\eta(S_{\varepsilon_1 - \tau_1}) \subseteq S_{\varepsilon_1}(G, f)$.

Proof. The first statement is saying that the diagram

$$
\begin{array}{ccc}
S_{\varepsilon_1}^{\tau_1}(G, f) & & \\
\downarrow{\scriptstyle\nu} & \searrow{\scriptstyle\omega} & \\
S_{\varepsilon_1}^{\tau_2}(G, f) & \xrightarrow{\eta} & S_{\varepsilon_2}^{\tau_2}(G, f) \\
\downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\nu} \\
S_{\varepsilon_1}(G, f) & \xrightarrow{\eta} & S_{\varepsilon_2}(G, f)
\end{array}
$$

commutes. The second statement is immediate from combining Corollary 4.5, (which requires $0 \leq \tau_i \leq \varepsilon_i$) with the first statement. $\qquad\square$

5.2 A categorical view of truncation

We next investigate the properties of truncation that arises from our combinatorial interpretation, Proposition 4.2, separating the truncation operation from the smoothing functor and considering it as an operation on a graph equipped with a height function in its own right.

From Defn. 2.4, it is immediate that on the objects, $T^0(G, f) = (G, f)$. It is also easy to see that $T^{\tau_1}(G, f) \subseteq T^{\tau_2}(G, f)$ for $\tau_1 \geq \tau_2$. To make it clear which graph we are working with, we will again use ν to denote the relevant inclusion.

Definition 5.6. *For $\tau_1 \geq \tau_2$, we write $\nu\colon T^{\tau_1}(G, f) \hookrightarrow T^{\tau_2}(G, f)$ for the inclusion.*

Proposition 5.7. *For $0 \leq \tau_1, \tau_2$, $T^{\tau_2}T^{\tau_1}(G, f) = T^{\tau_1 + \tau_2}(G, f)$.*

Proof. Let $x \in T^{\tau_2}T^{\tau_1}(G, f)$. Then x has an up-path of height τ_2 in $T^{\tau_1}(G, f)$, $\pi_2\colon [0, 1] \to G$. Since the endpoint, $\pi(1)$ is in $T^{\tau_1}(G, f)$, it has an up-path π_1 of height τ_1 in G. Concatenating these gives an up-path in G of height $\tau_1 + \tau_2$, so $x \in T^{\tau_1 + \tau_2}(G, f)$.

For the other direction, set $\tau = \tau_1 + \tau_2$ and assume $x \in T^\tau(G, f)$. Then x has an up-path of height τ, reparameterized as $\pi\colon [0, \tau] \to G$ with $f(\pi(t)) = f(x) + t$. So $x \in T^{\tau_1}(G, f)$ since it has a path of height $\tau_1 \leq \tau$ in G by simply restricting π to $[0, \tau_1]$. Further, the path π restricted to $[0, \tau_2]$ is an up-path of height τ_2 starting from x which is entirely contained in $T^{\tau_1}(G, f)$. This is because every point $\pi(t)$ for $t \in [0, \tau_2]$ has the up-path of height τ_1 given by $\pi\colon [t, t + \tau_1]$. Taken together, this means that $x \in T^{\tau_2}T^{\tau_1}(G, f)$. $\qquad\square$
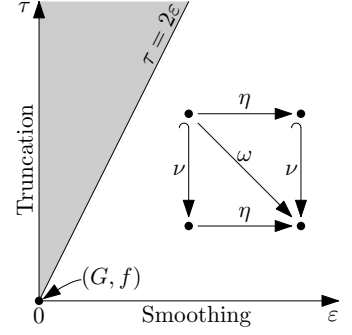
We next show that truncation is a functor on Reeb.

Lemma 5.8. *For any fixed $\tau \geq 0$, $T^\tau \in \mathbf{End}(\mathbf{Reeb})$.*

Proof. We need to show that for a fixed τ, T^τ can be viewed as an endofunctor on **Reeb** which requires construction of $T^\tau[\psi]$ for a given morphism $\psi \colon (G, f) \to (H, h)$ in **Reeb**. By definition, T^τ is a subgraph of its argument, so let $T^\tau[\psi]$ be the restriction of ψ to $T^\tau(G, f)$, and let $\nu \colon T^\tau(G, f) \to (G, f)$ and $\nu' \colon T^\tau(H, h) \to (H, h)$ be the maps induced by the inclusion. Because ψ is function-preserving, it maps every monotone path in (G, f) to a monotone path of the same length, so the image of $T^\tau[\psi]$ lies in $T^\tau(H, h)$. Hence, the diagram

$$
\begin{array}{ccc}
T^\tau(G, f) & \overset{\nu}{\longhookrightarrow} & (G, f) \\
{\scriptstyle T^\tau[\psi]}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\
T^\tau(H, h) & \overset{\nu'}{\longhookrightarrow} & (H, h)
\end{array}
$$

commutes, and this diagram can be used to check that $T^\tau[\mathrm{Id}] = \mathrm{Id}$ and that $T^\tau[\varphi \circ \psi] = T^\tau[\varphi] \circ T^\tau[\psi]$. $\square$

Now, we can expand our view of S_ε^τ to be a functor since it is the composition of two functors $T^\tau S_\varepsilon$. In particular, we have the following definition.

Definition 5.9. *For a morphism $\varphi \colon (G, f) \to (H, h)$, define $S_\varepsilon^\tau[\varphi] \colon S_\varepsilon^\tau(G, f) \to S_\varepsilon^\tau(H, h)$ to be the restriction of $S_\varepsilon[\varphi] \colon S_\varepsilon(G, f) \to S_\varepsilon(H, h)$ (given by Eq. (3.6) restricted to $\nu(S_\varepsilon^\tau[\varphi])$).*

This definition is well defined since, as with $T^\tau[\varphi]$, the image of a monotone path under φ is a monotone path, and thus $S_\varepsilon^\tau[\varphi](S_\varepsilon^\tau(G, f)) \subseteq S_\varepsilon^\tau(H, h)$. With this definition for morphisms, the following lemma is immediate.

Lemma 5.10. *For any fixed ε and τ for which $0 \leq \tau \leq 2\varepsilon$, $S_\varepsilon^\tau \in \mathbf{End}(\mathbf{Reeb})$.*

Our final lemma shows how S_ε^τ interacts with the inclusion maps ν.

Lemma 5.11. *For $0 \leq \varepsilon$ and $0 \leq \tau' \leq \tau \leq 2\varepsilon$,*

$$
\begin{array}{ccccc}
(G, f) & & S_\varepsilon^\tau(G, f) & \overset{\nu}{\longhookrightarrow} & S_\varepsilon^{\tau'}(G, f) \\
\big\downarrow{\scriptstyle \varphi} & & {\scriptstyle S_\varepsilon^\tau[\varphi]}\big\downarrow & & \big\downarrow{\scriptstyle S_\varepsilon^{\tau'}[\varphi]} \\
(H, h) & & S_\varepsilon^\tau(H, h) & \overset{\nu}{\longhookrightarrow} & S_\varepsilon^{\tau'}(H, h)
\end{array}
$$

Proof. The lemma follows from the fact that both $S_\varepsilon^\tau[\varphi]$ and $S_\varepsilon^{\tau'}[\varphi]$ are given by restricting $S_\varepsilon[\varphi]$ to the relevant subset of $S_\varepsilon(G, f)$. That is, the remaining faces of the triangular prism

commute, hence the left square commutes. $\square$

6 Categorical flow for $\tau \in [0, \varepsilon]$

Intuitively, we would like to be able to define a new categorical flow which incorporates the truncation parameter. That is, a flow given increasing ε parameter and non-decreasing τ. We have two barriers to deal with.

First, for some choices of parameters truncating and then smoothing is not the same as smoothing and then truncating; i.e. $S_\varepsilon^\tau S_\varepsilon^\tau \not\cong S_{2\varepsilon}^{2\tau}$. For example, in Fig. 4, $S_0^{2\varepsilon} S_{2\varepsilon}^0(G, f)$ has one connected component, but $S_{2\varepsilon}^0 S_0^{2\varepsilon}(G, f) = S_{2\varepsilon} T^{2\varepsilon}(G, f)$ has two. The second roadblock is that the forward morphisms $S_\varepsilon^\tau \Rightarrow S_{\varepsilon'}^{\tau'}$ required of a categorical flow corresponding to $\varepsilon \leq \varepsilon'$ might not be available since we require morphisms to be function preserving, and the truncation can erode the height of the codomain enough to leave no available points to map to. Luckily, both of these issues can be overcome if we restrict the amount of truncation done to $\tau \in [0, \varepsilon]$. So, working in this restricted space, we will first investigate properties of S_ε^τ, build the map ρ and prove that S_ε^τ can be used to build a flow on **Reeb**.
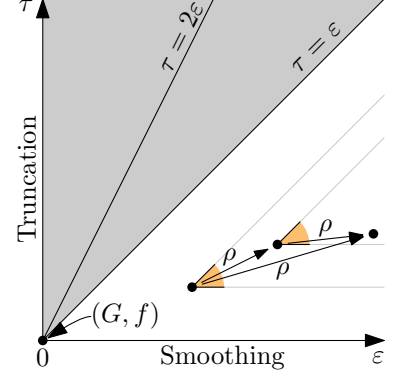
Figure 12: An illustration where ρ is defined: given $S_{\varepsilon_1}^{\tau_1}$ and $S_{\varepsilon_2}^{\tau_2}$, the map ρ exists if (ε_2, τ_2) is in the wedge that originates at point (ε_1, τ_1) with slope between 0 and 1 (shown in orange), and these maps commute whenever available.

6.1 The map ρ

So far, all maps we have used go down and/or right in the ε-τ plane. In general, we do not have a way to map up-right, that is for both increasing smoothing and truncating parameters. This in fact makes sense: truncated graphs naturally include downward, as they are subsets of less-truncated versions, and it is not clear how to map from the larger, less-truncated/smoothed version to something that is more smoothed and more truncated. However, we will see in this section that we do have such a map in a restricted setting.

Assume $0 \leq \varepsilon \leq \varepsilon'$, $0 \leq \tau \leq 2\varepsilon$ and $0 \leq \tau' \leq 2\varepsilon'$ so that $S_\varepsilon^\tau(G)$ and $S_{\varepsilon'}^{\tau'}(G)$ exist. We further assume $0 \leq \tau' - \tau \leq \varepsilon' - \varepsilon$; that is, the slope of the line connecting $S_\varepsilon^\tau(G)$ and $S_{\varepsilon'}^{\tau'}(G)$ in parameter space is in $[0, 1]$. There are two equivalent ways to find the map $\rho \colon S_\varepsilon^\tau(G) \to S_{\varepsilon'}^{\tau'}(G)$. First, with the assumptions we have that $0 \leq \varepsilon - \tau \leq \varepsilon' - \tau'$. Using Corollary 4.5, we have that

$$S_\varepsilon^\tau(G) = \eta(S_{\varepsilon - \tau}(G)) := \mathrm{Im}\{\eta \colon S_{\varepsilon - \tau}(G) \to S_\varepsilon(G)\}.$$

so we have the commutative diagram

$$
\begin{array}{ccc}
S_{\varepsilon - \tau}(G) & \xrightarrow{\quad \eta \quad} & S_{\varepsilon' - \tau'}(G) \\
\downarrow^\eta \quad \nearrow^\eta & & \searrow^\eta \\
S_\varepsilon(G) & \xrightarrow{\quad \eta \quad} & S_{\varepsilon'}(G)
\end{array}
$$

where the middle η arrow points towards the larger of ε or $\varepsilon' - \tau'$. Thus, we can define ρ as follows, see also the visualization of Fig. 12.

Definition 6.1. *For $0 \leq \tau_i \leq \varepsilon_i$, $i = 1, 2$, and $0 \leq \varepsilon_2 - \varepsilon_1 \leq \tau_2 - \tau_1$, we have a map $\rho \colon S_{\varepsilon_1}^{\tau_1}(G) \to S_{\varepsilon_2}^{\tau_2}(G)$ which is the restriction of $\eta \colon S_{\varepsilon_1}(G) \to S_{\varepsilon_2}(G)$ to $\eta(S_{\varepsilon_1 - \tau_1}(G)) \subseteq S_\varepsilon(G)$.*

An equivalent construction of ρ comes from utilizing Corollary 4.4 where $S_\varepsilon^\tau(G) = q(G \times [-\varepsilon + \tau, \varepsilon - \tau])$. Because of the constraints,

$$-\varepsilon' + \tau' \leq -\varepsilon + \tau \leq 0 \leq \varepsilon - \tau \leq \varepsilon' - \tau'$$

and thus $G\times[-\varepsilon+\tau,\varepsilon-\tau]\subseteq G\times[-\varepsilon'+\tau',\varepsilon'-\tau']$. So, we can view ρ as the restriction of η to $q(G\times[-\varepsilon+\tau,\varepsilon-\tau])$ in the commutative diagram

$$
\begin{array}{ccc}
G\times[-\varepsilon+\tau,\varepsilon-\tau] & \hookrightarrow & G\times[-\varepsilon'+\tau',\varepsilon'-\tau'] \\
\downarrow & & \downarrow \\
G\times[-\varepsilon,\varepsilon] & \hookrightarrow & G\times[-\varepsilon',\varepsilon'] \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
S_\varepsilon(G) & \xrightarrow{\ \eta\ } & S_{\varepsilon'}(G).
\end{array}
$$

The assumptions on ε and τ mean that in terms of the visualization of Fig. 12, ρ is available at slope between 0 and 1, denoted by the yellow wedge. Further, these maps compose whenever available as shown below.

Lemma 6.2. *Whenever the required spaces and maps exist, the diagrams*

commute.

Proof. The result arises from using the fact that both ρ and ω are restrictions of the η maps to the relevant subspaces of $S_\varepsilon(G,f)$. $\qquad\square$

While we leave the full details to Appendix B.3, we note that with the suppression of the homeomorphism discussed in Remark 3.4, we have the following notation.

Notation 6.3. *Up to homeomorphism, $S_\varepsilon^\tau[\eta]=\eta$, $S_\varepsilon^\tau[\omega]=\omega$, and $S_\varepsilon^\tau[\rho]=\rho$.*

That is, for an $\eta\colon S_{\varepsilon_1}^{\tau_1}(G,f)\to S_{\varepsilon_2}^{\tau_2}(G,f)$, $S_\varepsilon^\tau[\eta]\colon S_{\varepsilon_1+\varepsilon}^{\tau_1}(G,f)\to S_{\varepsilon_2+\varepsilon}^{\tau_2}(G,f)$ is the standard map η from Defn. 5.2; the others are defined similarly from Defns. 5.4 and 6.1.

6.2 The categorical flow

We can use this information to construct a truncated flow based on any line with slope at most 1. In particular, we can now prove our main result.

Theorem 2.14. *For any $m\in[0,1]$, the map $S^m\colon([0,\infty),\leq)\to\mathbf{End}(\mathbf{Reeb}); \varepsilon\mapsto S_\varepsilon^{m\varepsilon}$ is a functor and defines a categorical flow on* **Reeb**.

Proving this theorem amounts to checking that S^m satisfies quite a few properties encased in Defn. 3.1, largely handeld in the previous sections. The last bits necessary are the following two lemmas.

Lemma 6.4. *For any $0\leq\varepsilon\leq\varepsilon'$, $0\leq\tau\leq2\varepsilon$ and $0\leq\tau'\leq2\varepsilon'$ satisfying $0\leq\tau'-\tau\leq\varepsilon'-\varepsilon$ there is a natural transformation given by $\rho\colon S_\varepsilon^\tau\Rightarrow S_{\varepsilon'}^{\tau'}$; in particular the diagram*

$$
\begin{array}{ccc}
G & S_\varepsilon^\tau(G) \xrightarrow{\ \rho\ } S_{\varepsilon'}^{\tau'}(G) \\
\downarrow{\scriptstyle\psi} & \quad S_\varepsilon^\tau[\psi]\downarrow\qquad\qquad\downarrow{\scriptstyle S_{\varepsilon'}^{\tau'}[\psi]} \\
H & S_\varepsilon^\tau(H) \xrightarrow{\ \rho\ } S_{\varepsilon'}^{\tau'}(H)
\end{array}
$$

commutes.

Proof. Combining the definition given by Corollary 4.4 Eq. (3.8), we have that the diagram

commutes, and thus the inner square commutes. □

Proof of Theorem 2.14. Lem. 5.10 and Lem. 6.4 combine to show that S^m is a functor. By Proposition 2.12, $S^{m\varepsilon'}_{\varepsilon'} S^{m\varepsilon}_\varepsilon \cong S^{m(\varepsilon+\varepsilon')}_{\varepsilon+\varepsilon'}$ so S^m is additive. The combination means that S^m is a categorical flow on **Reeb**. □

7 Equivalence of Metrics

We have seen so far that for any $0 \le m \le 1$, the map $\varepsilon \mapsto S^{m\varepsilon}_\varepsilon$ gives rise to a flow on **Reeb**. This in turn means that we have an interleaving distance d^m_I for every $m \in [0,1]$ (Corollary 2.15). In particular, when $m = 0$, we have the original Reeb graph interleaving distance. In this section, we prove Theorem 2.16, showing the equivalence of these metrics for $m \in [0,1)$.

Theorem 2.16. *For any pair $0 \le m \le m' < 1$ with $0 \le m' - m \le 1 - m'$, the metrics d^m_I and $d^{m'}_I$ are equivalent. Specifically, given Reeb graphs (G,f) and (H,h),*

$$\frac{1-2m'+m}{1-m} d^{m'}_I((G,f),(H,h)) \le d^m_I((G,f),(H,h)) \le \frac{1-m'}{1-2m'+m} d^{m'}_I((G,f),(H,h)).$$

Sketch proof. We give a sketch of the proof of the left inequality here, as the idea of the proof is the same for the right inequality with slight modifications to the algebra. Full details are contained in Appendix C.

If we wish to show that $d^{m'}_I \le C d^m_I$ for some constant C, we need to take an ε-interleaving using $S^{m\varepsilon}_\varepsilon$, and show that we can build a $\Delta = (C\varepsilon)$-interleaving using $S^{m'\Delta}_\Delta$. Then, because d^m_I is the infimum of all such ε, we obtain the desired inequality.

Say we have an ε-interleaving using m given by $\alpha\colon (G,f) \to S^{m\varepsilon}_\varepsilon(H,h)$ and $\beta\colon (H,h) \to S^{m\varepsilon}_\varepsilon(G,f)$. Then we know for any choice of $\Delta \ge \varepsilon$, the diagram

$$(G,f) \dashrightarrow^\rho S^{m\varepsilon}_\varepsilon(G,f) \dashrightarrow^\rho S^{m\Delta}_\Delta(G,f) \dashrightarrow^\rho S^{m(\Delta+\varepsilon)}_{\Delta+\varepsilon}(G,f)$$
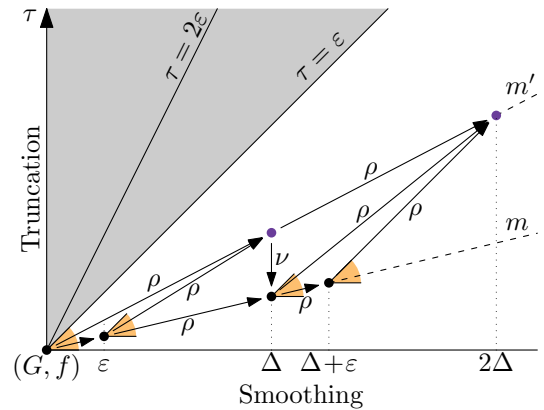
Figure 13: The top half of diagram Eq. (7.1).

$$(H,h) \dashrightarrow^\rho S^{m\varepsilon}_\varepsilon(H,h) \dashrightarrow^\rho S^{m\Delta}_\Delta(H,h) \dashrightarrow^\rho S^{m(\Delta+\varepsilon)}_{\Delta+\varepsilon}(H,h)$$

with diagonal maps α, β and $S^{m\Delta}_\Delta[\alpha]$, $S^{m\Delta}_\Delta[\beta]$.

commutes. Focusing on the top row of this diagram, we see that these Reeb graphs are represented as points on the line of slope m in the right of Fig. 13. Then the goal becomes finding Δ such that we have the drawn ρ maps to spaces on the line with slope m' for Δ and 2Δ as shown. Given such a Δ, we build the commutative diagram

$$
\begin{array}{c}
(7.1)
\end{array}
$$

to build a Δ interleaving using

$$\alpha' := \rho\alpha \colon (G, f) \to S_\Delta^{m\Delta}(H, h)$$
$$\beta' := \rho\beta \colon (H, h) \to S_\Delta^{m\Delta}(G, f).$$

Working with the restriction that ρ maps must have slope at most 1 leads to the calculation that $\Delta = \frac{1-m}{1-2m'+m}\varepsilon$, which gives the left inequality of the theorem. See Appendix C for details. $\qquad\square$

8 Conclusion and discussion

Our primary aim has been to introduce the concept of truncated smoothing and establish properties and connections of this operation. We have several reasons for considering this as a similarity measure on Reeb graphs. First, it has potential for providing bounds for the stable interleaving distance via the equivalence of metrics. Second, we came to this definition while investigating drawings of Reeb graphs and when planarity is achievable (that, is whether a Reeb graph has a planar drawing which respects the function in the y-coordinate). In a subsequent paper, we will show that while traditional smoothing does not maintain planarity, the truncated smoothing does for $\varepsilon \leq \tau \leq 2\varepsilon$.

We suspect additional potential applications of truncated smoothing in comparing geometric or planar graphs, since it simplifies the graph's topology (via smoothing) without suffering from extensive expansion of the co-domain or destruction of desirable combinatorial properties like level planarity. Truncated smoothing also allows for interesting manipulation of the extended persistence diagram of the Reeb graph and computation of morphs between Reeb graphs; again, we defer details to future work, as a full classification of that manipulation is necessary.

We suspect that the loss of stability discussed in Section 2.5 is not as dire as it seems. All our examples which break stability do so by comparing Reeb graphs with different images. Thus, we conjecture that we do have stability for functions with the same image.

While we are able to connect our collection of metrics to several Reeb metrics (Proposition 2.22), we have not investigated further connections to other metrics as of yet. One particularly interesting future direction is to determine whether this collection of metrics is also equivalent to the universal distance of [3]. Perhaps this broader collection of metrics will help to provide stronger bounds between the various metrics on Reeb graphs, since strong equivalence with one is strong equivalence with all.

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A Properties of truncation

In this section, we provide proofs of the main results stated in Section 2.3. We show the main results pertaining to connectedness of $S^\tau_\varepsilon(G, f)$ in Appendix A.1, and give results for when $S^\tau_\varepsilon(G, f)$ is empty in Appendix A.2

A.1 Connectedness

Truncation does not necessarily preserve connectedness. In fact, for the (0-tailed) graph of Fig. 6, for any $\tau > 0$, each edge of $T^\tau(G, f)$ is a separate component. In this section, we utilze the notions of s-safe and t-tailed (Defn. 2.7) to show that for a τ-tailed graph, $T^\tau(G, f)$ remains connected (Proposition A.7) Finally, we use this to show in Proposition 2.11 and Corollary A.8 that for certain ranges of τ relative to ε, $S^\tau_\varepsilon(G, f)$ maintains its connected components.

Proposition 2.8. *If (G, f) is t-tailed, then $S_\varepsilon(G, f)$ is $(t + 2\varepsilon)$-tailed. If (G, f) is s-safe, then $S_\varepsilon(G, f)$ is $(s + \varepsilon)$-safe. Putting these together, if (G, f) is nonempty, then $S_\varepsilon(G, f)$ is 2ε-tailed and ε-safe.*

Proof. For the first statement, assume (G, f) is t-tailed. A point $y \in S^\varepsilon(G, f)$ is an up-fork only if it has multiple interior-disjoint positive-length up-paths. For such paths to be interior-disjoint, the preimage of y must contain a point $(x, \varepsilon) \in (G, f) \times [-\varepsilon, \varepsilon]$ where x is an up-fork in G. Therefore, x has a down-path γ of length t in G. S^ε transforms γ into a length $t + 2\varepsilon$ down-path of y in $S_\varepsilon(G, f)$. Therefore, every up-fork has a down-path $t + 2\varepsilon$ long down-path. Symmetrically, every down-fork has a $t + 2\varepsilon$ long up-path, so $S_\varepsilon(G, f)$ is $(t + 2\varepsilon)$-tailed.

For the second statement, assume (G, f) is s-safe, and thus by definition, s-tailed. By the first statement, $S_\varepsilon(G, f)$ is $(s + 2\varepsilon)$-tailed and therefore $(s + \varepsilon)$-tailed. Moreover, some $x \in (G, f)$ has an up-path and down-path of length s. S_ε transforms their union to a monotone path of length $2s + 2\varepsilon$, so $S_\varepsilon(G, f)$ contains a point with both an up-path and down-path of length $s + \varepsilon$.

For the final statement, note that the empty Reeb graph is not s-safe for any s, whereas every nonempty Reeb graph is at least 0-safe. Moreover, every Reeb graph, including the empty Reeb graph is at least 0-tailed. So, setting $t = s = 0$ in the first two statements gives the following corollary. $\qquad\square$

The next lemma shows that even without smoothing first, truncation can preserve •-safe and •-tailed properties for decreased parameters.

Lemma 2.9. *Fix $0 \le \tau \le \varepsilon$. If (G, f) is ε-tailed or safe, then $T^\tau(G, f)$ is $(\varepsilon - \tau)$-tailed or safe, respectively.*

Proof. Let x be a downfork in $T^\tau(G, f) \subseteq (G, f)$. Then it is also a downfork in (G, f), so if (G, f) is ε-tailed, then x has an up-path of height ε in (G, f). Assuming the path is parameterized with respect to function value, the portion of this up-path defined on $[0, \varepsilon - \tau]$ consists of points who all have an up-path of height τ in (G, f), so x has an up path of height $\varepsilon - \tau$ in $T^\tau(G, f)$. Showing up forks have a down path of height τ is a symmetric argument, so $T^\tau(G, f)$ is $(\varepsilon - \tau)$-safe.

If (G, f) is ε-safe, then it just remains to show that $T^\tau(G, f)$ has a point with an up- and down-path of height $\varepsilon - \tau$. Since (G, f) has a point x with an up- and down-path of height ε, a similar argument to the above shows that the portion of these parameterized paths inside $[0, \varepsilon - \tau]$ remain in $T^\tau(G, f)$, so $T^\tau(G, f)$ is $(\varepsilon - \tau)$-safe. $\qquad\square$

We will now characterize the structure of the points removed by truncating, as well as the structure of the remainder. In the following lemmas and utilizing the notation of Defn. 2.4, let $U = U_\tau(G, f)$ be the set of points in G with no up-path of height τ and $D = D_\tau(G, f)$ be those without a height τ down-path.

Lemma A.1. *If (G, f) is τ-tailed, then any point in U roots an up-tree of height less than τ, and any point in D roots a down-tree of height less than τ.*

Proof. Pick $x \in U$, and let C be the set of points reachable from x by (possibly length 0) up-paths. As x has no up-path of length τ, C has height less than τ. It remains to show that C is an up-tree. Suppose not, then C contains a down-fork x' of (G, f). Since (G, f) is τ-tailed, x' has an up-path of length τ that also lies in C, contradicting that C has height less than τ. So any point in U roots an up-tree of height less than τ. Symmetrically, any point in D roots a down-tree of height less than τ. $\qquad\square$

Lemma A.2. *If (G, f) is connected and τ-safe, then $U \cup D$ contains no path from U to D.*

Proof. Suppose that U and D intersect, then there exists some $x \in U \cap D$. As (G, f) is τ-safe, it contains a point $y \notin U \cup D$. Since (G, f) is connected, there is a simple path from x to y. Because x roots both an up-tree and a down-tree, this path must be monotone. Therefore, y lies in the up-tree or down-tree rooted at x, so $y \in U \cup D$, which is a contradiction. So U and D are disjoint, and since U and D are open, there is no path inside $U \cup D$ from U to D. $\qquad\square$

For a component of C of U or D, call a point $x \in T^\tau(G, f)$ a *root* of C if it lies in the closure of C.

Lemma A.3. *If (G, f) is τ-tailed, then any component C of $U \cup D$ has at most one root.*

Proof. If (G, f) is not τ-safe, then $T^\tau(G, f)$ is empty, so $U \cup D$ has no root and we are done. So assume that (G, f) is τ-safe. By Lem. A.2, $C \subseteq U$ or $C \subseteq D$. Without loss of generality, assume that $C \subseteq U$, and that C has multiple roots $x, x' \notin U \cup D$. Let π be a simple path in C connecting x and x'. Because C contains no down-forks and C contains all points reachable from C by up-paths, π has length 0. Because G is a Hausdorff space, this means that $x = x'$, so C has at most one root. $\qquad\square$

Lemma A.4. *Let γ be a simple path in (G, f) that starts and ends in $T^\tau(G, f)$. If C is a component of $U \cup D$ with at most one root, then γ does not intersect C.*

Proof. Suppose instead that $\gamma(t)$ lies in C, and γ starts and ends in $T^\tau(G, f)$, which is disjoint from C, there is some $t' < t$ and $t'' > t$ for which $\gamma(t')$ and $\gamma(t'')$ is the unique root of C, contradicting that γ is simple. $\qquad\square$

Corollary A.5. *If (G, f) is τ-tailed, then any simple path γ in (G, f) that starts and ends in $T^\tau(G, f)$ lies completely inside $T^\tau(G, f)$.*

Lemma A.6. *If (G, f) is connected and τ-safe, then any component C of $U \cup D$ has a root.*

Proof. Assume that $U \cup D$ is nonempty and $\tau > 0$ (otherwise we are done). Then U and D are both nonempty. By Lemma A.2, U is disconnected from D, so $U \cup D$ has at least two components. Since (G, f) is connected, there is a path from C to a different component of $U \cup D$, so C has at least one root. $\qquad\square$

We show that connectivity is preserved for τ-tailed graphs.

Proposition A.7. *If (G, f) is connected and τ-tailed, then $T^\tau(G, f)$ is also connected.*

Proof. Suppose not, then there is a simple path in (G, f) connecting two components of $T^\tau(G, f)$. This path must enter and exit a component C of $U \cup D$. Since C has a single root, and the path must revisit it, contradicting that the path is simple. $\qquad\square$

Note that the previous collection of lemmas was independent of the smoothing operation. We can combine Proposition A.7 with Proposition 2.8 to get the main result of this section.

Proposition 2.11. *If (G, f) is connected and $\tau \in [0, 2\varepsilon]$, then $S_\varepsilon^\tau(G, f)$ is also connected.*

Proof. Because G is connected, by Proposition 2.8, $S_\varepsilon(G, f)$ is 2ε-tailed. This further implies that $S_\varepsilon(G, f)$ is τ tailed for every $\tau \in [0, 2\varepsilon]$, so by Proposition A.7, $S_\varepsilon^\tau(G, f)$ is connected. $\qquad\square$

We also note that since taking path components π_0 is a functor, we immediately have that the induced map $\pi_0[\eta]$ preserves connected components, as stated in the following corollary, which is the other main result of this section.

Corollary A.8. *For $\tau \in [0, 2\varepsilon]$, $\pi_0[\eta]\colon \pi_0(G, f) \cong \pi_0(S_\varepsilon^\tau(G, f))$.*

We can use this corollary to extend the result of Proposition 2.19 on when the interleaving distance can take infinite values to the case of disconnected G.

Proposition A.9. *Let $m = 1$, and consider graphs (G, f) and (H, h) with connected components $G = \bigsqcup_{i=1}^{K} G_i$ and $H = \bigsqcup_{i=1}^{L} H_i$. Write f_i for f restricted to G_i and h_i for h restricted to H_i. Then $d_I^m((G, f), (H, h)) < \infty$ iff $K = L$ and there is a permutation $\sigma\colon \{1, \ldots, K\} \to \{1, \ldots, L\}$ with $\mathrm{Im}(G_i, f_i) = \mathrm{Im}(H_{\sigma(i)}, f_{\sigma(i)})$ for every component.*

Proof. First, assume that the interleaving distance is finite. Interleavings preserve connected components since by Corollary A.8, $\pi_0(G, f) \cong \pi_0(S_\varepsilon^\varepsilon(G, f))$ and $\pi_0(H, h) \cong \pi_0(S_\varepsilon^\varepsilon(H, h))$ where the isomorphisms are given by $\pi_0[\eta]$. The pair of maps φ and ψ also constitute isomorphisms on path components, $\pi_0[\varphi]\colon \pi_0(G, f) \to \pi_0(S_\varepsilon^\varepsilon(H, h))$ and $\pi_0[\psi]\colon \pi_0(H, h) \to \pi_0(S_\varepsilon^\varepsilon(G, f))$, and thus $\pi_0(G, f) \cong \pi_0(H, h)$. Further, for there to be an interleaving using $S_\varepsilon^\varepsilon$ on connected components, the images must be the same, and thus the permutation simply encodes this association.

Now assume that we have the permutation σ. If we set $\varepsilon = \max\{\mathrm{diam}(\mathrm{Im}(G, f)), \mathrm{diam}(H, h)\}$. Assume $\mathrm{Im}(G_i, f_i) = [a_i, b_i]$ Consider any pair of points $(x, t), (x', t') \in G_i \times [-\varepsilon, \varepsilon]$ with $f(x) + t = f(x') + t' = c$ and $c \in [a_i, b_i]$. There is a path $\pi\colon x \rightsquigarrow x'$ in G_i, so the path $\overline{\pi} = (\pi(t), c - f(\pi(t)))$ is a function constant path in $(G \times [-\varepsilon, \varepsilon], f + \mathrm{Id})$. This implies that the portion of $S_\varepsilon^\varepsilon(G_i, f_i)$ in $[a_i, b_i]$ is a line graph, and further that this graph is unchanged for any $\varepsilon' \geq \varepsilon$. Then we define the interleaving by setting φ to be the isomorphism of line graphs given by $(H_i, h_i) \cong S_\varepsilon(G_i, f_i)$ and symmetrically define φ. This constitutes an interleaving, so the $m = 1$ interleaving distance is finite. $\qquad\square$

A.2 Emptiness

We conclude this section with a results on how smoothing and truncation can affect the image of the Reeb graph $\mathrm{Im}(G, f) := f(G) \subseteq \mathbb{R}$. It is relatively immediate to see that if the diameter of the image $\|\mathrm{Im}(G, f)\| < 2\tau$, then $T^\tau(G, f)$ is empty. However, when $\|\mathrm{Im}(G, f)\| \geq 2\tau$, it is still possible for the τ truncation to be empty; see, for example, Fig. 6.

Proposition A.10. *Fix a Reeb graph (G, f) with $f(G) = [a, b] \subseteq \mathbb{R}$. If $b - a < 2\tau$, then $\mathrm{Im}(T^\tau(G, f)) = \emptyset$. Otherwise, $\mathrm{Im}(T^\tau(G, f)) \subseteq [a + \tau, b - \tau]$.*

Proof. First, if $b - a < 2\tau$, then we will show that $T^\tau(G, f)$ is empty. Indeed, since any point in $T^\tau(G, f)$ is a point in x with up-and down-paths of height τ in G, the projection of these paths imply that $[f(x) - \tau, f(x) + \tau] \subseteq \mathrm{Im}(G, f)$, but this is impossible if $b - a < 2\tau$.

Now we can assume that $b - a \geq 2\tau$ and need to show that $\mathrm{Im}(T^\tau(G, f)) \subseteq [a + \tau, b - \tau]$. Let $c \in \mathrm{Im}(T^\tau(G, f))$, so there is an $x \in G$ with an up-path π_+ and a down path π_-, each of height τ, with $\pi_+(0) = \pi_-(0) = x$. Then $f(\pi_+(1)) = c + \tau$, and since $f(\pi_+(1)) \leq b$, $c \leq b - \tau$. The symmetric argument gives us that $f(\pi_-(1)) = c - \tau$ so $c \geq a + \tau$. $\qquad\square$

We can mitigate the undesirable properties of truncation by smoothing first. In the case that we start with a connected Reeb graph, we can make a stronger statement about the change in the image when smoothing and truncating.

Proposition 2.6. *For connected (G, f) with $\mathrm{Im}(G, f) = [a, b]$, and for $\varepsilon \leq 2\tau$,*

$$\mathrm{Im}(S_\varepsilon^\tau(G, f)) = \begin{cases} [a - (\varepsilon - \tau), b + (\varepsilon - \tau)] & \text{if } b - a \geq 2(\tau - \varepsilon) \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Note that by Proposition 2.3, $\mathrm{Im}(S_\varepsilon(G, f)) = [a - \varepsilon, b + \varepsilon]$. If we assume $b - a < 2(\tau - \varepsilon)$, then $(b + \varepsilon) - (a - \varepsilon) \leq 2\tau$, so by Proposition A.10, $\mathrm{Im}(S_\varepsilon^\tau(G, f)) = \mathrm{Im}(T^\tau(S_\varepsilon(G, f))) = \emptyset$.

Now, we can assume $b - a \geq 2(\tau - \varepsilon)$. One direction of containment is easy since by Proposition 2.3,

$$\mathrm{Im}(S_\varepsilon^\tau(G, f)) = \mathrm{Im}(T^\tau(S_\varepsilon(G, f))) \subseteq [a - (\varepsilon - \tau), b + (\varepsilon - \tau)].$$

Thus, it remains to show that $[a - (\varepsilon - \tau), b + (\varepsilon - \tau)] \subseteq \mathrm{Im}(S_\varepsilon^\tau(G, f))$. There exist points $s, t \in S_\varepsilon(G, f)$ with $f(s) = a - \varepsilon$ and $f(t) = b + \varepsilon$ that are connected by a simple (not necessarily monotone path) π. If π is monotone, then truncating retains its monotone subpath with image $[a - (\varepsilon - \tau), b + (\varepsilon - \tau)]$, and we are done. If on the other hand π is not monotone, consider its maximal monotone subpaths. The subpath containing s ends at a downfork d, and the other subpath containing d ends at an upfork u with $f(d) \geq f(u)$. By Proposition 2.8, every down-fork of $S_\varepsilon(G, f)$ has a 2ε long up-path, so $f(d) \leq b - \varepsilon$ and symmetrically $f(u) \geq a + \varepsilon$. Let d' and u' be the points reachable from d and u by a 2ε long up-path and down-path, respectively. Then, $f(d') \geq f(d) + \varepsilon \geq f(u) \geq f(u') - \varepsilon \geq a = f(s)$, so the first monotone subpath of π has length at least 4ε, and truncating it retains a point p with $f(p) = a - (\varepsilon - \tau)$. Symmetrically, the monotone subpath of π containing t retains a point q with $f(q) = b + (\varepsilon - \tau)$ after truncation. By Proposition 2.11, p and q are connected, so $[a - (\varepsilon - \tau), b + (\varepsilon - \tau)] \subseteq \mathrm{Im}(S_\varepsilon^\tau(G, f))$ by the intermediate value theorem. □

B Technical proofs for morphisms

In this section we fully unpack the various isomorphisms which were glossed over in the main body. We begin with the related work, and in particular Remark 3.4, which involves prior work. We then follow these issues into our own proof that smoothing and truncation commute for τ-safe graphs, which in turn implies that truncated smoothing can be applied to (as a functor) the morphims we presented in Table 1.

B.1 Isomorphism for smoothing

The first technicality we note is the isomorphism issue noted in Remark 3.4. The problem is that $S_\varepsilon S_\delta(G, f)$ is not exactly the same thing as $S_{\varepsilon+\delta}(G, f)$, although they are isomorphic. Basically, points in $S_{\varepsilon+\delta}(G, f)$ are equivalence classes of points from $G \times [-\varepsilon - \delta, \varepsilon + \delta]$, while points of $S_\varepsilon S_\delta(G, f)$ are equivalence classes of points in $S_\delta(G, f) \times [-\varepsilon, \varepsilon]$.

As noted in [23, Obs. 4.30], there is a natural isomorphism $S_{\varepsilon_2} S_{\varepsilon_1}(G, f) \to S_{\varepsilon_1 + \varepsilon_2}(G, f)$ given as follows. Setting $\varepsilon = \varepsilon_1 + \varepsilon_2$, a point in $S_{\varepsilon_2} S_{\varepsilon_1}(G, f)$ is represented by $q_2(q_1(x, \varepsilon_1), \varepsilon_2)$ and is sent to $q_3(x, \varepsilon) \in S_\varepsilon(G, f)$ making the diagram

commute.

If we use the notation $S_\varepsilon(G, f) =: (G_\varepsilon, f_\varepsilon)$, constructing Eq. (3.6) with $\varphi = \eta : (G, f) \to S_\delta(G, f)$ gives

the following commutative diagram

So, we abuse notation and write $\eta : S_{\varepsilon_1}(G, f) \to S_{\varepsilon_1 + \varepsilon_2}(G, f)$ for the composition of maps

giving a map $\eta : S_\varepsilon(G, f) \to S_{\varepsilon'}(G, f)$ for every $\varepsilon \le \varepsilon'$.

B.2 Safe truncation and smoothing commute

We next construct a map $\psi \colon S_\varepsilon T^\tau(G, f) \to T^\tau S_\varepsilon(G, f)$ in Lem. B.1 which is continuous and function-preserving; we then show that if (G, f) is τ-safe, ψ is an isomorphism, proving that truncation and smoothing commute (Proposition 2.12) and smoothing and truncate combine additively (Theorem 2.13).

Let $\sim$ and $\sim_T$ respectively be the relevant equivalence relations on $(G, f) \times [-\varepsilon, \varepsilon]$ and $T^\tau(G, f) \times [-\varepsilon, \varepsilon]$, and let q and q_T be their quotient maps.

Lemma B.1. *Let ν and ν_S be the inclusion maps of the relevant truncations. For any $0 \le \tau$ and $0 \le \varepsilon$, there exists a unique map $\psi \colon S_\varepsilon T^\tau(G, f) \to T^\tau S_\varepsilon(G, f)$ that makes the diagram*

commute, and ψ is continuous and function-preserving.

Proof. Since q_T is surjective and ν_S is injective, the map ψ, if it exists, is uniquely determined by the above diagram. Any function ψ making this diagram commute is automatically continuous and function-preserving.

Let $\psi(q_T(x, \lambda)) := q(x, \lambda)$. We first show that the image of ψ lies in $T^\tau S_\varepsilon(G, f)$. By surjectivity of q_T, it suffices to show for any $x \in T^\tau(G, f)$ and $\lambda \in [-\varepsilon, \varepsilon]$, that $q(x, \lambda) \in T^\tau S_\varepsilon(G, f)$. We have some length τ up-path π in G starting at x. Then $s \mapsto q(\pi(s), \lambda)$ is a length τ up-path of $q(x, \lambda)$ in $S_\varepsilon(G, f)$, and $q(x, \lambda)$ similarly has a length τ down-path, so $q(x, \lambda) \in T^\tau S_\varepsilon(G, f)$. It remains to show that ψ is well-defined by showing that whenever $q(x, \lambda) = q(x', \lambda')$, we have $q_T(x, \lambda) = q_T(x', \lambda')$. This follows from the inclusion of any equivalence class of $\sim_T$ into one of $\sim$. Explicitly, any path with constant function value in $T^\tau(G) \times [-\varepsilon, \varepsilon]$ is contained in $G \times [-\varepsilon, \varepsilon]$, where it also has constant function value. $\square$

We will show that $S_\varepsilon^\tau[\eta]$ is interchangeable with η when after accounting for the isomorphism.

Theorem B.2. *If $0 \leq \tau$ and $0 \leq \varepsilon$, and η and η_T are the natural maps into the smoothings in the following diagram, the diagram commutes.*

$$
T^\tau(G,f) \xrightarrow[\eta_T]{\overset{T^\tau[\eta]}{\Longrightarrow}} S_\varepsilon T^\tau(G,f) \xrightarrow{\psi} T^\tau S_\varepsilon(G,f)
$$

$$
(G,f) \xrightarrow{\quad\quad\eta\quad\quad} S_\varepsilon(G,f).
$$

Proof. Recall that $T[\eta]$ is simply the restriction of η to the truncation. Commutativity of the diagram below is evident from Lemma B.1, and the theorem follows.

Proposition 2.12. *If (G,f) is τ-safe, then $S_\varepsilon T^\tau(G,f) \cong T^\tau S_\varepsilon(G,f)$.*

We will break up the proof into several lemmas. Assume that (G,f) is τ-safe. It is a standard result of point set topology that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism [41, Thm 26.6]. Because ψ is continuous and function-preserving, it remains to show that it is a bijection. For injectivity, we use Corollary A.5 of τ-safe graphs that simple paths that start and end in $S_\varepsilon T^\tau(G,f)$ lie completely inside $S_\varepsilon T^\tau(G,f)$.

Lemma B.3. *If (G,f) is τ-safe, then ψ is injective.*

Proof. Suppose it is not, then there exist points $y \neq y' \in S_\varepsilon T^\tau(G,f)$ with $\psi(y) = \psi(y')$. Without loss of generality, let $(x,\lambda),(x',\lambda') \in T^\tau(G,f) \times [-\varepsilon,\varepsilon]$ with $y =: q_T(x,\lambda)$ and $y' =: q_T(x',\lambda')$. As $y \neq y'$ and $\psi(y) = \psi(y')$, we have

$$
\begin{aligned}
(x,\lambda) &\not\sim_T (x',\lambda'), and\\
(x,\lambda) &\sim (x',\lambda').
\end{aligned}
$$

There is a simple function preserving path $\pi \colon (x,\lambda) \rightsquigarrow (x',\lambda')$ in $(G,f) \times [-\varepsilon,\varepsilon]$. Let $(\gamma,h) := \pi$. We show that its first component γ is also simple. Otherwise $\gamma(t) = \gamma(t')$ for some $t \neq t'$, and since π is function preserving, $f(\pi(t)) = f(\pi(t'))$, then

$$
f(\gamma(t)) + h(t) = f(\gamma(t')) + h(t') = f(\gamma(t)) + h(t'),
$$

so $h(t) = h(t')$ and hence $\pi(t) = \pi(t')$ contradicting that π is simple. So γ is a simple path.

Moreover, γ starts and ends in $T^\tau(G,f)$, so because (G,f) is τ-safe, γ lies completely inside $T^\tau(G,f)$ by Corollary A.5. So π is a function preserving path in $T^\tau(G,f) \times [-\varepsilon,\varepsilon]$, and therefore $(x,\lambda) \sim_T (x',\lambda')$, which is a contradiction, so ψ is injective. $\square$

For surjectivity, we use the fact that truncated points lie in an up-tree or down-tree of height at most τ.

Lemma B.4. *If (G, f) is τ-safe, then ψ is surjective.*

Proof. Recall that $\psi(q_T(x, \lambda)) := q(x, \lambda)$. Suppose for a contradiction that ψ is not surjective. Then a point y does not lie in the image of ψ, so $q^{-1}(y)$ does not intersect $T(G, f) \times [-\varepsilon, \varepsilon]$. Denote by X_y the set of points x with $(x, \lambda) \in q^{-1}(y)$ (for some $\lambda \in [-\varepsilon, \varepsilon]$). Then all points in X_y are truncated. Because (G, f) is τ-safe, each connected component of $G \setminus T$ is a union of up-trees or a union of down-trees of height less than τ. Since X_y is connected, it lies completely inside one such component, and assume without loss of generality that it is a union of uptrees U.

As (G, f) is τ-safe, the closure of U contains exactly one point of T, call it r. We have $f(r) < f(u) \leq f(r+\tau)$ for any $u \in U$. Consider a point $x \in X_y$. Since $x \in U$, there is a down-path $\gamma \colon x \rightsquigarrow r$. We have $q(x, f(y) - f(x)) = y$ and as $f(\gamma(t))$ is nonincreasing, $q(\gamma(t), f(y) - f(\gamma(t))) = y$ as long as $f(\gamma(t)) \geq f(y) - \varepsilon$. As $r \notin X_y$, we therefore have $f(r) < f(y) - \varepsilon$.

As $y \in T^\tau S_\varepsilon(G, f)$, y has an up-path of length τ in $S_\varepsilon(G, f)$ to a point y'. Then $f(y') = f(y) + \tau > f(r) + \tau + \varepsilon > f(u) + \varepsilon$ for any $u \in U$. For $x' \in X_{y'}$, we have $f(x') \geq f(y') - \varepsilon$, so $X_{y'}$ does not intersect U. Therefore, any path from X_y to $X_{y'}$ passes through r. In particular, the up-path from y to y' contains a point y'' with $f(y) \leq f(y'')$ and $r \in X_{y''}$, but then $f(y) \leq f(y'') \leq f(r) + \varepsilon$, contradicting that $f(r) < f(y) - \varepsilon$. So ψ is surjective. $\qquad\square$

Proof of Proposition 2.12. We have constructed a continuous function-preserving map $\varphi \colon S_\varepsilon T^\tau(G, f) \to T^\tau S_\varepsilon(G, f)$. Moreover, we have shown in Lems. B.3 and B.4 that it is a bijection if (G, f) is τ-safe.

A continuous bijection from a compact space to a Hausdorff space is a homeomorphism [41, Thm 26.6]. A constructible space in the sense of [23] is compact and Hausdorff, and both Reeb graphs and the smoothing of a Reeb graph are constructible. Similarly, truncation preserves constructibility as it removes an open subset. Hence, the continuous bijection ψ is a homeomoprhism. As it is also function preserving, we have that $S_\varepsilon T^\tau(G, f) \cong T^\tau S_\varepsilon(G, f)$ as Reeb graphs. $\qquad\square$

B.3 Applying S_ε^τ to morphisms

We will next use this map ψ to show that truncated smoothing can be applied to morphisms from Table 1. For example, $S_\varepsilon^\tau[\eta]$ is equivalent to a different version of η (up to suppressing our isomorphism), and similar statements hold for ω and ρ. That is, this section is meant to justify the abuse of notation used throughout the paper writing $S_\varepsilon^\tau[\eta] = \eta'$, since in reality, $S_\varepsilon^\tau[\eta]$ is defined on $S_{\varepsilon'}^{\tau'}(G, f)$ while η' is defined on $S_{\varepsilon+\varepsilon'}^{\tau+\tau'}(G, f)$, Since the two spaces are homeomorphic, we need to ensure that these maps are indeed equal, respecting that homeomorphism. Luckily, this is relatively easy, since replacing (G, f) in Theorem B.2 by $S_{\varepsilon_1}(G, f)$ and τ by τ_1, we obtain the following lemma.

Lemma B.5. *For $0 \leq \tau_1, \tau_2$ and $0 \leq \varepsilon_1 \leq \varepsilon_2$ and natural inclusions η and η', the diagram*

$$
\begin{array}{ccc}
S_{\varepsilon_1}^{\tau_1}(G, f) & S_\varepsilon^\tau(S_{\varepsilon_1}^{\tau_1}(G, f)) \xrightarrow{\ \cong\ } S_{\varepsilon+\varepsilon_1}^{\tau+\tau_1}(G, f) \\
\ \downarrow{\scriptstyle\eta} & \quad S_\varepsilon^\tau[\eta]\downarrow\qquad\qquad\qquad\downarrow{\scriptstyle\eta'} \\
S_{\varepsilon_2}^{\tau_1}(G, f) & S_\varepsilon^\tau(S_{\varepsilon_2}^{\tau_1}(G, f)) \xrightarrow{\ \cong\ } S_{\varepsilon+\varepsilon_2}^{\tau+\tau_1}(G, f)
\end{array}
$$

commutes. That is, abusing notation by suppressing ψ, we have $S_\varepsilon^\tau[\eta] = \eta'$.

Since the ω and ρ maps are simply restrictions of η, we can apply this results to those maps as well.

Corollary B.6. *For $\omega \colon S_{\varepsilon_1}^{\tau_1}(G, f) \to S_{\varepsilon_2}^{\tau_2}(G, f)$, the diagram*

$$
\begin{array}{ccc}
S_{\varepsilon_1}^{\tau_1}(G, f) & S_\varepsilon^\tau(S_{\varepsilon_1}^{\tau_1}(G, f)) \xrightarrow{\ \cong\ } S_{\varepsilon+\varepsilon_1}^{\tau+\tau_1}(G, f) \\
\ \downarrow{\scriptstyle\omega} & \quad S_\varepsilon^\tau[\omega]\downarrow\qquad\qquad\qquad\downarrow{\scriptstyle\omega} \\
S_{\varepsilon_2}^{\tau_2}(G, f) & S_\varepsilon^\tau(S_{\varepsilon_1}^{\tau_1}(G, f)) \xrightarrow{\ \cong\ } S_{\varepsilon+\varepsilon_1}^{\tau+\tau_1}(G, f)
\end{array}
$$

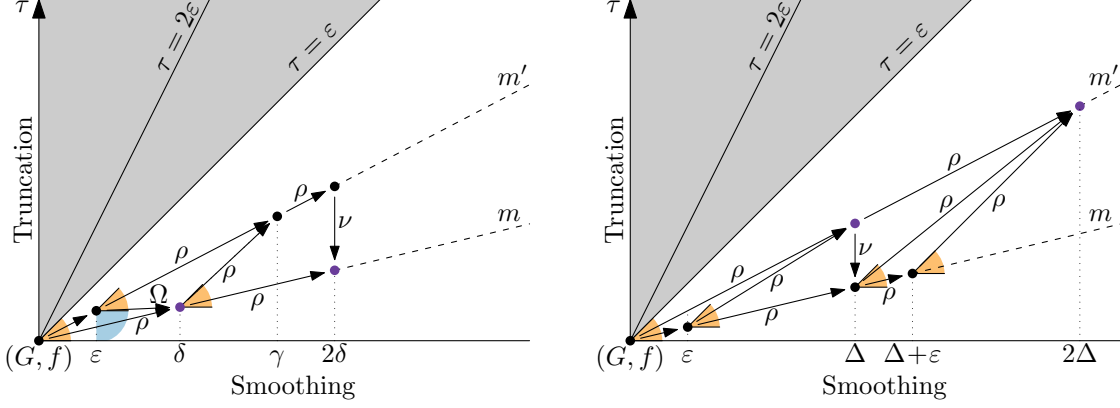Figure 14: The top half of the diagrams Eq. (C.1) (left) and Eq. (C.2) (right). Note that for $m' \geq \frac{1}{2}$, $\Omega = \rho$ since it has a positive slope, and for $m' \leq \frac{1}{2}$, Ω has a negative slope, so $\Omega = \omega$.

so long as all spaces and maps exist. That is, abusing notation by suppressing the homeomorphism, we have $S_\varepsilon^\tau[\omega] = \omega$.

Proof. This is immediate from combining Lem. B.5 with Lem. 5.5. □

Corollary B.7. *For* $\rho\colon S_{\varepsilon_1}^{\tau_1}(G, f) \to S_{\varepsilon_2}^{\tau_2}(G, f)$, *the diagram*

$$
\begin{array}{ccc}
S_{\varepsilon_1}^{\tau_1}(G, f) & S_\varepsilon^\tau(S_{\varepsilon_1}^{\tau_1}(G, f)) \xrightarrow{\;\cong\;} S_{\varepsilon+\varepsilon_1}^{\tau+\tau_1}(G, f) \\
\Big\downarrow \rho & S_\varepsilon^\tau[\rho]\Big\downarrow \qquad\qquad\qquad \Big\downarrow \rho \\
S_{\varepsilon_2}^{\tau_2}(G, f) & S_\varepsilon^\tau(S_{\varepsilon_1}^{\tau_1}(G, f)) \xrightarrow{\;\cong\;} S_{\varepsilon+\varepsilon_1}^{\tau+\tau_1}(G, f)
\end{array}
$$

so long as all spaces and maps exist. That is, abusing notation by suppressing the homeomorphism, we have $S_\varepsilon^\tau[\rho] = \rho$.

Proof. This is immediate from combining Lem. B.5 with Defn. 6.1. □

C Full proof of strong equivalence of metrics

Our final appendix is dedicated to the specifics of the proof of strong equivalence of metrics for some choices of m and m', with the result for all pairs $0 \leq m \leq m' < 1$ given as Corollary 2.17.

Theorem 2.16. *For any pair* $0 \leq m \leq m' < 1$ *with* $0 \leq m' - m \leq 1 - m'$, *the metrics* d_I^m *and* $d_I^{m'}$ *are equivalent. Specifically, given Reeb graphs* (G, f) *and* (H, h),

$$
\frac{1 - 2m' + m}{1 - m} d_I^{m'}((G, f), (H, h)) \leq d_I^m((G, f), (H, h)) \leq \frac{1 - m'}{1 - 2m' + m} d_I^{m'}((G, f), (H, h)).
$$

Proof. We first show that

$$
d_I^m((G, f), (H, h)) \leq \frac{1 - m'}{1 - 2m' + m} d_I^{m'}((G, f), (H, h)).
$$

Assume we have an ε-interleaving using m' given by

$$
\alpha\colon (G, f) \to S_\varepsilon^{m'\varepsilon}(H, h)
$$
$$
\beta\colon (H, h) \to S_\varepsilon^{m'\varepsilon}(G, f).
$$

35

Our goal is then to construct a δ-interleaving using m by, in essence, making δ large enough to make space for maps ρ and ω that go between the lines $y = mx$ and $y = m'x$. These can then be concatenated with the α and β maps to build an interleaving. See Fig. 13 for a visual representation of this.

Set $\delta = \frac{1-m'}{1-2m'+m}\varepsilon$ and $\gamma = \frac{1-m}{1-m'}\delta$. Note that $\gamma + \varepsilon = 2\delta$. Also, by the assumptions on m and m', $\frac{1-m'}{1-2m'+m} \geq 1$ because

$$1 - 2m' + m = (1 - m') - (m' - m) \leq 1 - m'$$

as $m' - m \geq 0$. This implies $\delta \geq \varepsilon$. So, we have $0 \leq \varepsilon \leq \delta \leq \gamma \leq 2\delta$. Then consider the diagram

$$(\text{C.1})$$

We first need to check, following Table 1, that $\rho\colon S_\delta^{m\delta} \to S_\gamma^{m'\gamma}$ and $\rho\colon S_\delta^{m\delta} \to S_{2\delta}^{2m'\delta}$ exists, and that there is a map $\Omega\colon S_\varepsilon^{m'\varepsilon}(G, f) \to S_\delta^{m\delta}$. The existence of the other maps are immediate. First, $\rho\colon S_\delta^{m\delta} \to S_\gamma^{m'\gamma}$ exists because $0 \leq m'\gamma - m\delta = \frac{m'-m}{1-m}\delta = \gamma - \delta$. Second, $\rho\colon S_\delta^{m\delta} \to S_{2\delta}^{2m'\delta}$ exists because $0 \leq (2m' - m)\delta \leq 2\delta - \delta$ as $0 \leq (m' - m) + m' \leq m' + (1 - m') = 1$.

Now we will show that there is a map $\Omega\colon S_\varepsilon^{m'\varepsilon}(G, f) \to S_\delta^{m\delta}$. If $m' \leq 1/2$, then $\Omega = \omega$. Indeed, we need to show that $m\delta \leq m'\varepsilon$. We calculate

$$m'\varepsilon - m\delta = \frac{(m' - m)(1 - 2m')}{1 - 2m' + m}\varepsilon.$$

Since $m' - m$, $1 - 2m'$, and $(1 - 2m' + m)$ are all positive, we have that $m'\varepsilon - m\delta \geq 0$, so ω exists.

If $1/2 < m' < 1$, then we will show that $\Omega = \rho$. This means checking that $0 \leq m\delta - m'\varepsilon \leq \delta - \varepsilon$. Indeed, since $0 \leq 2m' - 1 < 1$,

$$0 \leq \frac{(m' - m)(2m' - 1)}{1 - 2m' + m}\varepsilon = m\delta - m'\varepsilon$$

and

$$m\delta - m'\varepsilon = \frac{(m' - m)(2m' - 1)}{1 - 2m' + m}\varepsilon \leq \frac{m' - m}{1 - 2m' + m}\varepsilon = \delta - \varepsilon.$$

Note that by Lem. 6.2, all triangles in the top and bottom half of Eq. (C.1) commute both when $\Omega = \rho$ and $\Omega = \omega$. By functoriality of $S_\gamma^{m'\gamma}$, the middle strip commutes. Thus, the entire diagram commutes.

We will build a δ interleaving using

$$\alpha' := \Omega\alpha\colon (G, f) \to S_\delta^{m\delta}(H, h)$$
$$\beta' := \Omega\beta\colon (H, h) \to S_\delta^{m\delta}(G, f)$$

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We need to check that $S_\delta^{m\delta}[\alpha'] = \nu \circ S_\gamma^{m'\gamma}[\alpha] \circ \rho$ and $S_\delta^{m\delta}[\beta'] = \nu \circ S_\gamma^{m'\gamma}[\beta] \circ \rho$. Indeed, consider the diagram

By Corollaries B.6 and B.7, $S_\delta^{m\delta}[\Omega] = \Omega$. By functoriality of $S_\delta^{m\delta}$, the top triangle commutes. The left square commutes by Lem. 6.4 and the right triangle commutes by Lem. 6.2. As the whole diagram commutes, $S_\delta^{m\delta}[\alpha'] = \nu \circ S_\gamma^{m'\gamma}[\alpha] \circ \rho$ as required. The β version is symmetric. Then commutativity of Eq. (C.1) proves that α' and β' form a δ-interleaving, and hence

$$d_I^m((G,f),(H,h)) \le \delta d_I^{m'}((G,f),(H,h)).$$

For the other direction, we need to show

$$d_I^{m'}((G,f),(H,h)) \le \frac{1-m}{1-2m'+m} d_I^m((G,f),(H,h)).$$

Assume we have an ε-interleaving using m given by

$$\alpha\colon (G,f) \to S_\varepsilon^{m\varepsilon}(H,h)$$
$$\beta\colon (H,h) \to S_\varepsilon^{m\varepsilon}(G,f)$$

We will use these maps to build a $\Delta = \frac{1-m}{1-2m'+m}\varepsilon$ interleaving using $S_\varepsilon^{m'\varepsilon}$. First, note that because $1-2m'+m = (1-m')-(m'-m) \le 1-m' \le 1-m$, we have that $\varepsilon \le \Delta$. Also,

$$\Delta + \varepsilon = \frac{2(1-m')}{1-2m'+m}\varepsilon \le \frac{2(1-m)}{1-2m'+m}\varepsilon = 2\Delta.$$

Putting this together, we have $0 \le \varepsilon \le \Delta \le \Delta + \varepsilon \le 2\Delta$. So, we can consider the diagram

(C.2)

As before, we need to check for the existence of the maps; the non-obvious ones are $\rho\colon S_\varepsilon^{m\varepsilon}(G,f) \to S_\Delta^{m'\Delta}(G,f)$, and $\rho\colon S_{\Delta+\varepsilon}^{m(\Delta+\varepsilon)}(G,f) \to S_{2\Delta}^{2m'\Delta}(G,f)$. For $\rho\colon S_\varepsilon^{m\varepsilon}(G,f) \to S_\Delta^{m'\Delta}(G,f)$, we see that it exists since

$$0 \le \frac{(m'-m)(1+m)}{1-2m'+m}\varepsilon = m'\Delta - m\varepsilon$$

as all terms in the middle are positive, and

$$m'\Delta - m\varepsilon = \frac{(m'-m)(1+m)}{1-2m'+m}\varepsilon \le \frac{2(m'-m)}{1-2m'+m}\varepsilon = \Delta - \varepsilon$$

since $1 + m < 2$. To check $\rho\colon S_{\Delta+\varepsilon}^{m(\Delta+\varepsilon)}(G,f) \to S_{2\Delta}^{2m'\Delta}(G,f)$, we have that

$$0 \le 2m'\Delta - m(\Delta + \varepsilon) = \frac{2(m'-m)}{1-2m'+m}\varepsilon = 2\Delta - (\Delta + \varepsilon).$$

Again using Lem. 6.2, all triangles in the top and bottom half of Eq. (C.2). Functoriality of $S_\Delta^{m\Delta}$ gives commutativity of the middle strip, so the entire diagram commutes.

We will build a Δ interleaving using

$$\alpha' := \rho\alpha\colon (G,f) \to S_\Delta^{m\Delta}(H,h)$$
$$\beta' := \rho\beta\colon (H,h) \to S_\Delta^{m\Delta}(G,f).$$

We need to check $S_\Delta^{m'\Delta}[\alpha'] = \rho \circ S_\Delta^{m\Delta}[\alpha] \circ \nu$ and $S_\Delta^{m'\Delta}[\beta'] = \rho \circ S_\Delta^{m\Delta}[\beta] \circ \nu$. Consider the diagram

$$
\begin{array}{ccccc}
 & & S_\Delta^{m'\Delta}[\alpha'] & & \\
 & \nearrow & & \searrow & \\
S_\Delta^{m'\Delta}(G,f) & \xrightarrow{S_\Delta^{m'\Delta}[\alpha]} & S_{\Delta+\varepsilon}^{m'\Delta+m\varepsilon}(H,h) & \xrightarrow[\rho]{S_\Delta^{m'\Delta}[\rho]} & S_{2\Delta}^{2m'\Delta}(H,h) \\
\downarrow\nu & & \downarrow\nu & \nearrow \rho & \\
S_\Delta^{m\Delta}(G,f) & \xrightarrow{S_\Delta^{m\Delta}[\alpha]} & S_{\Delta+\varepsilon}^{m(\Delta+\varepsilon)}(H,h). & &
\end{array}
$$

The top triangle commutes by functoriality of $S_\Delta^{m'\Delta}$”, and the bottom right triangle commutes by Lem. 6.2. The left square commutes by Lem. 5.11. Thus, the entire diagram commutes, implying $S_\Delta^{m'\Delta}[\alpha'] = \rho \circ S_\Delta^{m\Delta}[\alpha] \circ \nu$. The β version is symmetric. Then commutativity of Eq. (C.2) proves that α' and β' form a Δ-interleaving, and hence

$$d_I^{m'}((G,f),(H,h)) \le \Delta d_I^m((G,f),(H,h)).$$

$\square$