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The Edit Distance for Reeb Graphs of Surfaces

Barbara Di Fabio[1,2] · **Claudia Landi**[1,2]

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Abstract Reeb graphs are structural descriptors that capture shape properties of a topological space from the perspective of a chosen function. In this work, we define a combinatorial distance for Reeb graphs of orientable surfaces in terms of the cost necessary to transform one graph into another by edit operations. The main contributions of this paper are the stability property and the optimality of this edit distance. More precisely, the stability result states that changes in the Reeb graphs, measured by the edit distance, are as small as changes in the functions, measured by the maximum norm. The optimality result states that the edit distance discriminates Reeb graphs better than any other distance for Reeb graphs of surfaces satisfying the stability property.

Keywords Shape similarity · Graph edit distance · Morse function · Natural stratification

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Barbara Di Fabio
barbara.difabio@unimore.it

Claudia Landi
claudia.landi@unimore.it

1 Dipartimento di Scienze e Metodi dell'Ingegneria, Università di Modena e Reggio Emilia, Via Amendola 2, Pad, 42122 Morselli, Reggio Emilia, Italy

2 ARCES, Università di Bologna, Via Toffano 2/2, 40125 Bologna, Italy

1 Introduction

In shape comparison, a widely used scheme is to measure the dissimilarity between descriptors associated with each shape rather than to match shapes directly. Reeb graphs describe shapes from topological and geometrical perspectives. In this framework, a shape is modeled as a topological space X endowed with a scalar function $f : X \to \mathbb{R}$. The role of f is to explore geometrical properties of the space X. The Reeb graph of f is obtained by shrinking each connected component of a level set of f to a single point [22]. Scientists in Computer Graphics have used Reeb graphs as an effective tool for shape analysis and description tasks since [26,27].

One of the most important questions is whether Reeb graphs are robust against perturbations that may occur because of noise and approximation errors in the data acquisition process. In the past, researchers dealt with this problem by developing heuristics so that Reeb graphs would be resistant to connectivity changes caused by simplification, subdivision and remesh, and robust against noise and certain changes due to deformation [4,15]. In the last years, several authors have started investigating the question of Reeb graph stability from the theoretical point of view. In [12], the authors of this paper introduced an edit distance between Reeb graphs of curves endowed with Morse functions, and showed that it yields stability. Importantly, despite the combinatorial nature of this distance, it coincides with the natural pseudo-distance between shapes [13], therefore showing the maximal discriminative power for this sort of distances. More recently, a functional distortion distance between Reeb graphs has been proposed in [2], and further studied in [3], with proven stable and discriminative properties. The functional distortion distance is based on continuous maps between the topological spaces realizing the Reeb graphs, so that it is not combinatorial in its definition. Noticeably, it allows for comparison of non-homeomorphic spaces, meaning that it is possible to deal also with artifacts that change the homotopy type of the space. Moreover, it yields stability of Reeb graphs of a broad class of functions called *tame functions*. Simple Morse functions on a surface are a particular case of tame functions. On the other hand, it may be the case that the functional distortion distance does not fully discriminate shapes, even restricting to simple Morse functions on a surface.

In this paper, we deal with the comparison problem for Reeb graphs of surfaces. Indeed, the case of surfaces represent an interesting area of application of the Reeb graph as a shape descriptor [1,21]. As a tradeoff between generality and simplicity, we confine ourselves to the case of smooth compact orientable surfaces without boundary endowed with simple Morse functions.

The basic properties we consider important for a distance between Reeb graphs are the robustness to perturbations of the input functions, the ability to discriminate functions on the same manifold, and the deployment of the combinatorial nature of graphs. These correspond to properties that are viewed as important in applications to 3D shape comparison [11,28]. For this reason, we apply to the case of surfaces the same underlying ideas as used in [12] for curves. More precisely, starting from Reeb graphs labeled on the vertices by the function values, we carry out the following steps. First, we detect a set of admissible edit operations to transform a labeled Reeb graph into another. Next, a suitable cost is associated with each edit operation. Finally, a combinatorial dissimilarity measure between labeled Reeb graphs, called an *edit* *distance*, is defined in terms of the least cost necessary to transform one graph into another by edit operations. In conclusion, the edit distance between Reeb graphs belongs to the family of Graph Edit Distances [14], widely used in pattern analysis.

Despite the similarity of the approach we use here for surfaces and that used in the case of curves, the passage from curves to surfaces is not automatic. Indeed, Reeb graphs of surfaces are structurally different from those of curves. For example, the degree of vertices is different for Reeb graphs of curves and surfaces. Therefore, the set of edit operations as well as their costs cannot be directly imported from the case of curves but need to be suitably defined.

Our first main result is that changes in the functions, measured by the maximum norm, imply not greater changes in this edit distance, yielding the stability property under function perturbations. To prove this result, we track the changes in the Reeb graphs as the function varies along a linear path. From the stability property, we deduce that the edit distance between the Reeb graphs of two functions f and g defined on a surface is a lower bound for the natural pseudo-distance between f and g obtained by minimizing the change in the functions due to the application of a self-diffeomorphism of the manifold, with respect to the maximum norm. We can think of the natural pseudo-distance as a way to compare f and g directly, while the edit distance provides an indirect comparison between f and g through their Reeb graphs. Thus, by virtue of the stability result, the edit distance provides a combinatorial tool to estimate the natural pseudo-distance.

Our second contribution is the proof that the edit distance between Reeb graphs of surfaces actually coincides with the natural pseudo-distance. We prove this by showing that for every edit operation on a Reeb graph there is a self-diffeomorphism of the surface whose cost is not greater than that of the considered edit operation. This result implies that the edit distance is actually a metric and not only a pseudo-metric. Moreover, it implies the optimality of the edit distance for Reeb graphs of surfaces in that the edit distance turns out to have the maximum discriminative power among all the distances between Reeb graphs of surfaces with the stability property.

In conclusion, this paper shows that the results of [12] for curves also hold in the more interesting case of surfaces.

The paper is organized as follows. In Sect. 2 we recall the basic properties of labeled Reeb graphs of orientable surfaces. In Sect. 3 we define the edit deformations between labeled Reeb graphs, and show that through a finite sequence of these deformations we can always transform a Reeb graph into another. In Sect. 4 we define the cost associated with each type of edit deformation and the edit distance in terms of these costs. Section 5 illustrates the robustness of Reeb graphs with respect to the edit distance. Eventually, Sect. 6 provides relationships between the edit distance and other stable distances: the natural pseudo-distance, the bottleneck distance and the functional distortion distance.

2 Labeled Reeb Graphs of Orientable Surfaces

Hereafter, $\mathscr{M}$ denotes a connected, closed (i.e. compact and without boundary), orientable, smooth surface, and $\mathscr{F}(\mathscr{M})$ the set of C^∞ real functions on $\mathscr{M}$.

For $f \in \mathscr{F}(\mathscr{M})$, we denote by K_f the set of its critical points. If $p \in K_f$, then the real number $f(p)$ is called a *critical value* of f, and the set $f^{-1}(f(p)) \subset \mathscr{M}$ is called a *critical level* of f. Moreover, a critical point p is called *non-degenerate* if the Hessian matrix of f at p is non-singular. The *index* of a non-degenerate critical point p of f is the dimension of the largest subspace of the tangent space to $\mathscr{M}$ at p on which the Hessian matrix is negative definite. In particular, the index of a point $p \in K_f$ is equal to 0, 1, or 2 depending on whether p is a minimum, a saddle, or a maximum point of f.

A function $f \in \mathscr{F}(\mathscr{M})$ is called a *Morse function* if all its critical points are non-degenerate. Besides, a Morse function is said to be *simple* if each critical level contains exactly one critical point. The set of simple Morse functions will be denoted by $\mathscr{F}^0(\mathscr{M})$, as a reminder that it is a sub-manifold of $\mathscr{F}(\mathscr{M})$ of co-dimension 0 (see also Sect. 5).

Definition 1 Let $f \in \mathscr{F}^0(\mathscr{M})$, and define on $\mathscr{M}$ the following equivalence relation: for every $p, q \in \mathscr{M}$, $p \sim_f q$ whenever p, q belong to the same connected component of $f^{-1}(f(p))$. The quotient space $\mathscr{M}/\sim_f$ is the *Reeb graph* associated with f.

Throughout this paper, we regard Reeb graphs as combinatorial graphs rather than topological spaces because commonly edit distances are defined to deal with problems of combinatorial pattern matching (see, e.g., [14]). To avoid confusion, we will write $\mathscr{M}/\sim_f$ when we refer to Reeb graphs as topological spaces, and Γ_f when we refer to Reeb graphs as combinatorial graphs. This is justified by the following result.

Proposition 2 [22] *The Reeb graph Γ_f associated with $f \in \mathscr{F}^0(\mathscr{M})$ is a finite and connected simplicial complex of dimension* 1. *A vertex of Γ_f has degree equal to* 1 *if it corresponds to a critical point of f of index* 0 *or* 2*, while it has degree equal to* 2, 3*, or* 4 *if it corresponds to a critical point of f of index* 1.

Hence, in the following, we will often identify vertices of Γ_f with the corresponding critical points in K_f. Our assumptions that $\mathscr{M}$ is orientable, compact and without boundary ensure that there are no vertices of degree 2 or 4. The vertex set of Γ_f will be denoted by $V(\Gamma_f)$, and the edge set by $E(\Gamma_f)$. Moreover, if $v_1, v_2 \in V(\Gamma_f)$ are adjacent vertices, i.e., connected by an edge, we will write $e(v_1, v_2) \in E(\Gamma_f)$. Also, for $m \geq 2$, we call a *length-m cycle* any cycle containing exactly m edges (or vertices) in the graph.

Denoting by $\mathfrak{g}$ the genus of $\mathscr{M}$, by χ the Euler characteristic of $\mathscr{M}$, and by $\mathtt{p}$, $\mathtt{q}$, $\mathtt{r}$ the number of minima, maxima, and saddle points of f, respectively, the following statements hold true.

Remark 3 The cardinality of $V(\Gamma_f)$ is even. In particular, if $\mathtt{p} = \mathtt{q} = 1$, then $\mathtt{r} = 2\mathfrak{g}$.

Indeed, the relationship between χ and $\mathtt{p}$, $\mathtt{q}$, $\mathtt{r}$, i.e. $\chi = \mathtt{p}+\mathtt{q}-\mathtt{r}$, and the relationship between χ and $\mathfrak{g}$, i.e. $\chi = 2 - 2\mathfrak{g}$, imply that the cardinality of $V(\Gamma_f)$, which is $\mathtt{p} + \mathtt{q} + \mathtt{r}$, is also equal to $2(\mathtt{p} + \mathtt{q} + \mathfrak{g} - 1)$.

Remark 4 [10, Lem. A] If $\mathscr{M}$ has genus $\mathfrak{g}$, then Γ_f has exactly $\mathfrak{g}$ linearly independent cycles.

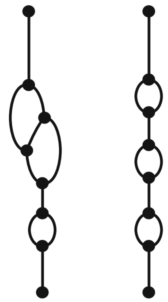


Fig. 1 Examples of minimal Reeb graphs. The graph on the *right* is also canonical

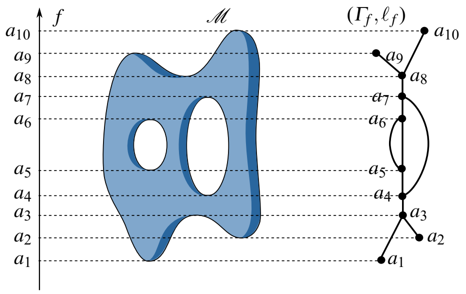


Fig. 2 *Left* the height function $f : \mathscr{M} \to \mathbb{R}$; *center* the surface $\mathscr{M}$ of genus $\mathfrak{g} = 2$; *right* the associated labeled Reeb graph (Γ_f, ℓ_f). The vertices equipped with labels a_3, a_6 and a_7 correspond to merging saddles, while those at the heights a_4, a_5 and a_8 to splitting saddles

These remarks motivate the following definition.

Definition 5 We say that the Reeb graph Γ_f of a function f having $\mathtt{p} = \mathtt{q} = 1$ is *minimal*. Moreover, we say that Γ_f is *canonical* if it is minimal and all its cycles, if any, are of length 2.

We underline that our definition of canonical Reeb graph slightly differs from that of [17]. The reason is to simplify the proof of Proposition 23. We show some examples of minimal and canonical Reeb graphs in Fig. 1.

In what follows, we label the vertices of Γ_f by equipping each of them with the value of f at the corresponding critical point.

Definition 6 We call a *labeled graph* any pair (Γ, ℓ), where Γ is a graph and $\ell : V(\Gamma) \to \mathbb{R}$ is a function. For $f \in \mathscr{F}^0$, we call the pair (Γ_f, ℓ_f), where Γ_f is the Reeb graph associated with f and $\ell_f : V(\Gamma_f) \to \mathbb{R}$ is the function induced by restricting $f : \mathscr{M} \to \mathbb{R}$ to K_f, the *labeled Reeb graph* of f.

We observe that ℓ_f is injective because f is simple. Moreover, each vertex $v \in V(\Gamma_f)$ of degree 3 has at least two of its adjacent vertices, say v_1, v_2, such that $\ell_f(v_1) < \ell_f(v) < \ell_f(v_2)$. In particular, v corresponds to a *splitting saddle* (a *merging saddle*, respectively) if v has two adjacent vertices with higher (lower, respectively) labels. An example of labeled Reeb graph is displayed in Fig. 2.

Remark 7 If (Γ_f, ℓ_f) is a minimal labeled Reeb graph, then the number of splitting (respectively, merging) saddles is equal to the genus $\mathfrak{g}$ of $\mathscr{M}$.

In fact, for $\mathtt{p} = \mathtt{q} = 1$, the splitting saddles are as many as the merging saddles, and, by Remark 3, their total number is $2\mathfrak{g}$.

Let us consider the realization problem, i.e. the problem of constructing a smooth surface and a simple Morse function on it from a graph on an even number of vertices, all of which are of degree 1 or 3, appropriately labeled. We need the following definition.

Definition 8 We say that two labeled Reeb graphs (Γ_f, ℓ_f), (Γ_g, ℓ_g) are *isomorphic*, and we write $(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, if there exists a graph isomorphism $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ such that, for every $v \in V(\Gamma_f)$, $\ell_f(v) = \ell_g(\Phi(v))$ (i.e. Φ preserves edges and vertex labels).

Proposition 9 (Realization Theorem) *Let Γ be a graph with k linearly independent cycles, and an even number of vertices, all of which of degree* 1 *or* 3. *Let $\ell : V(\Gamma) \to \mathbb{R}$ be an injective function such that, for any vertex $v \in V(\Gamma)$ of degree* 3, *at least two among its adjacent vertices, say w, w', are such that $\ell(w) < \ell(v) < \ell(w')$. Then an orientable closed surface $\mathscr{M}$ of genus $\mathfrak{g} = k$, and a simple Morse function $f : \mathscr{M} \to \mathbb{R}$ exist such that $(\Gamma_f, \ell_f) \cong (\Gamma, \ell)$.*

Proof Let us orient each edge $e(v_1, v_2)$ of Γ from v_1 to v_2 if $\ell(v_1) < \ell(v_2)$. We observe that the so-obtained directed labeled graph has no oriented closed cycles (according to the terminology of [25], also called loops in [19]). Hence, we can construct $\mathscr{M}$ and f as in the proof of Theorem 2.1 in [19]. □

We now deal with the uniqueness problem, up to isomorphism of labeled Reeb graphs. First of all we consider the following two equivalence relations on $\mathscr{F}^0(\mathscr{M})$.

Definition 10 Let $\mathscr{D}(\mathscr{M})$ be the set of self-diffeomorphisms of $\mathscr{M}$. Two functions $f, g \in \mathscr{F}^0(\mathscr{M})$ are called *right-equivalent* (briefly, *R-equivalent*) if there exists $\xi \in \mathscr{D}(\mathscr{M})$ such that $f = g \circ \xi$, i.e. the following diagram

$$\begin{tikzcd} \mathscr{M} \arrow[rr, "\xi"] \arrow[dr, "f"'] & & \mathscr{M} \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd}$$

commutes. Moreover, f, g are called *right-left equivalent* (briefly, *RL-equivalent*) if there exist $\xi \in \mathscr{D}(\mathscr{M})$ and an orientation preserving self-diffeomorphism η of $\mathbb{R}$ such that $f = \eta \circ g \circ \xi$, i.e. the following diagram

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\xi"] \arrow[d, "f"'] & \mathscr{M} \arrow[d, "g"] \\ \mathbb{R} & \mathbb{R} \arrow[l, "\eta"] \end{tikzcd}$$

commutes.

These equivalence relations on functions are mirrored by Reeb graphs isomorphisms, as formally stated in Proposition 13. This fact has already been observed by other authors (see, for example, [18,24]). Here we give our own proof of it in order to keep the paper as self-contained as possible. Moreover, it will also turn out to be useful later to prove Lemma 35. We start with a preliminary proposition and a corollary.

Proposition 11 *Let f and g in $\mathscr{F}^0(\mathscr{M})$ be such that their Reeb graphs Γ_f and Γ_g are isomorphic by an isomorphism $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ that preserves the order of adjacent vertices, i.e., for every $e(v, w) \in E(\Gamma_f)$, $\ell_f(v) < \ell_f(w)$ if and only if $\ell_g(\Phi(v)) < \ell_g(\Phi(w))$. Then there is a diffeomorphism $\xi : \mathscr{M} \to \mathscr{M}$ such that*

$$\max_{p\in\mathscr{M}} |f(p) - g \circ \xi(p)| = \max_{v\in V(\Gamma_f)} |\ell_f(v) - \ell_g(\Phi(v))|.$$

Proof The main idea of the proof is to cut $\mathscr{M}$ along level curves into disks, pants and cylinders, and define suitable diffeomorphisms on each of these submanifolds. The key property of these partial diffeomorphisms is to preserve level sets, up to a shift, and gradient lines of f and g. Thus, they can be glued together to obtain a global homeomorphism that can be subsequently smoothed.

Because Γ_f and Γ_g are isomorphic, the functions f and g have the same number of critical points of the same type: if $K_f = \{p_1, \dots, p_n\}$, $K_g = \{p'_1, \dots, p'_n\}$, we can assume $\Phi(p_i) = p'_i$, with p_i, p'_i being of the same index. Moreover, because Φ preserves the order of adjacent vertices, Φ sends splitting saddles of f to splitting saddles of g, and likewise for merging saddles. We set $c_i = f(p_i)$ and $c'_i = g(p'_i)$ for $1 \le i \le n$.

Let us consider the submanifolds of $\mathscr{M}$ obtained by cutting $\mathscr{M}$ along level curves $f^{-1}(c_i \pm \varepsilon)$ and $g^{-1}(c'_i \pm \varepsilon)$, $i = 1, \dots, n$, for a small enough value of $\varepsilon > 0$ so that each submanifold contains at most one critical point in its interior. According to whether these submanifolds contain points of maximum, minimum, saddle points, or no critical points at all, we treat the cases differently. In all the considered cases, we assume $\mathscr{M}$ endowed with a Riemannian metric so that the gradient vector fields of f and g are defined.

Claim 1. Let p_i, p'_i be two points of minimum or two points of maximum of f and g, respectively. Let $\varepsilon > 0$ be sufficiently small so that the connected component $D = D_i$ of $f^{-1}([c_i - \varepsilon, c_i + \varepsilon])$ (resp. $D' = D'_i$ of $g^{-1}([c'_i - \varepsilon, c'_i + \varepsilon])$) that contains p_i (resp. p'_i) is contained in a Morse chart. There exists a diffeomorphism $\xi^D : D \to D'$ such that:

(a_1) ξ^D maps level curves of f in D onto level curves of g in D' with a constant displacement equal to $c'_i - c_i$: $g(\xi^D(p)) = f(p) + c'_i - c_i$ for every $p \in D$;

(b_1) ξ^D maps gradient lines of f onto gradient lines of g.

Proof of Claim 1. We prove the claim for minima, the proof for maxima being analogous.

Let $B_{\sqrt{\varepsilon}}$ denote the closed disk in $\mathbb{R}^2$ centered at $(0, 0)$ with radius $\sqrt{\varepsilon}$. By the Morse lemma, there are diffeomorphisms $\varphi : D \to B_{\sqrt{\varepsilon}}$ and $\varphi' : D' \to B_{\sqrt{\varepsilon}}$, such that $f \circ \varphi^{-1}(x, y) = c_i + x^2 + y^2$ and $g \circ \varphi'^{-1}(x, y) = c'_i + x^2 + y^2$. Let us consider the model function $h : B_{\sqrt{\varepsilon}} \to \mathbb{R}$, $h(x, y) = x^2 + y^2$. The diffeomorphism φ takes gradient lines of f to gradient lines of h, and level sets of f to level sets of h with a shift equal to $-c_i$. Indeed, $h(\varphi(p)) = f(p) - c_i$ for every $p \in D$. Analogously for φ' and g. Hence, it is sufficient to take $\xi^D = \varphi'^{-1} \circ \varphi : D \to D'$ to get a diffeomorphism that maps gradient lines of f to gradient lines of g, and level sets of f to level sets of g with a shift equal to $c'_i - c_i$.

Claim 2. Let p_i, p'_i be two merging saddle points or two splitting saddle points of f and g, respectively. Let $\varepsilon > 0$ be sufficiently small so that there are a Morse chart $\varphi : U \to \mathbb{R}^2$ around p_i with $f^{-1}([c_i - \varepsilon, c_i + \varepsilon]) \subseteq f^{-1}(f(U))$, and a Morse chart $\varphi' : U' \to \mathbb{R}^2$ around p'_i with $g^{-1}([c'_i - \varepsilon, c'_i + \varepsilon]) \subseteq g^{-1}(g(U'))$. We also assume $\varepsilon < \frac{5}{12}$. Let $P = P_i$ be the connected component of $f^{-1}([c_i - \varepsilon, c_i + \varepsilon])$ that contains p_i. Analogously, let $P' = P'_i$ the connected component of $g^{-1}([c'_i - \varepsilon, c'_i + \varepsilon]))$ that contains p'_i. There exists a diffeomorphism $\xi^P : P \to P'$ such that:

(a_2) ξ^P maps level curves of f in P onto level curves of g in P' with a constant displacement equal to $c'_i - c_i$: $g(\xi^P(p)) = f(p) + c'_i - c_i$ for every $p \in P$;
(b_2) ξ^P maps gradient lines of f onto gradient lines of g.

Proof of Claim 2. We prove the claim for merging saddles, the proof for splitting saddles being analogous. In this case P and P' are pairs of pants.

Following [16, pp. 194–198], we consider the model function $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x, y) = \frac{x^4}{4} - \frac{x^3}{3} - x^2 + y^2$, and its interlevel set $h^{-1}([-\varepsilon, \varepsilon])$ with $0 < \varepsilon < \frac{5}{12}$. Because $\varepsilon < \frac{5}{12}$, the only critical point of h in $h^{-1}([-\varepsilon, \varepsilon])$ is a merging saddle point at $(0, 0)$. Level sets and gradient lines of h are shown in [16, Fig. 9-2] and reproduced here in Fig. 3. P and P' are diffeomorphic to $h^{-1}([-\varepsilon, \varepsilon])$ via diffeomorphisms $\varphi : P \to h^{-1}([-\varepsilon, \varepsilon])$ and $\varphi' : P' \to h^{-1}([-\varepsilon, \varepsilon])$ that map level sets of f and g, respectively, to level sets of h with a shift equal to $-c_i$ and $-c'_i$, respectively. Moreover, gradient lines of f and g are mapped by φ and φ', respectively, to gradient lines of h. Thus, the claim follows by taking $\xi^P = \varphi'^{-1} \circ \varphi$.

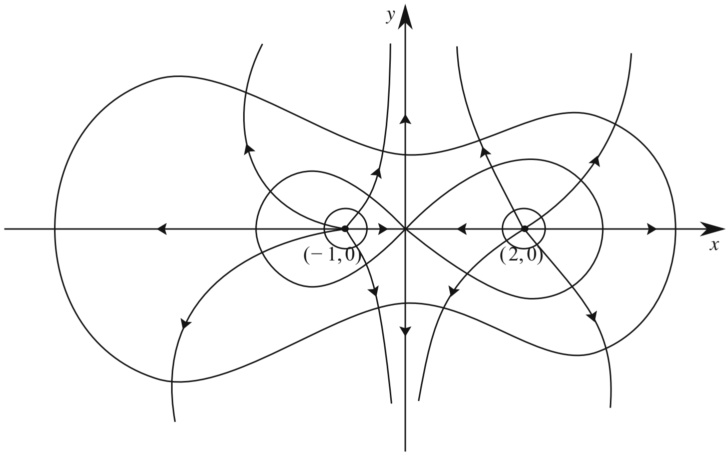


Fig. 3 Level sets and gradient lines of the model function $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x, y) = \frac{x^4}{4} - \frac{x^3}{3} - x^2 + y^2$

Claim 3. Let p_i, p_j (resp. p'_i, p'_j) be critical points of f (resp. g) connected by an edge in the Reeb graph of f (resp. g). Assume $c_i < c_j$ (resp. $c'_i < c'_j$). Let $\varepsilon > 0$ be sufficiently small so that the cylinders $C = C_{ij} = \{p \in \mathscr{M} : [p]_{\sim_f} \in e(p_i, p_j),\ c_i + \varepsilon \le f(p) \le c_j - \varepsilon\}$ and $C' = C'_{ij} = \{p \in \mathscr{M} : [p]_{\sim_g} \in e(p'_i, p'_j),\ c'_i + \varepsilon \le g(p) \le c'_j - \varepsilon\}$ are non-empty. For any diffeomorphism $\sigma : f^{-1}(c_i+\varepsilon)\cap C \to g^{-1}(c'_i+\varepsilon)\cap C'$, there exists a diffeomorphism $\xi^C : C \to C'$ such that:

(a_3) ξ^C maps level curves of f in C onto level curves of g in C' with a maximal displacement given by $\max\{|c'_i - c_i|, |c'_j - c_j|\}$:

$$\max_{p\in C} |f(p) - g \circ \xi^C(p)| = \max\{|c'_i - c_i|, |c'_j - c_j|\};$$

(b_3) ξ^C maps gradient lines of f onto gradient lines of g;
(c_3) the restriction of ξ^C to $f^{-1}(c_i + \varepsilon) \cap C$ is equal to σ.

Proof of Claim 3. We consider the smooth vector field $X = -\frac{\nabla f}{\|\nabla f\|^2}$ on C, and the smooth vector field $X' = \frac{\nabla g}{\|\nabla g\|^2}$ on C'. Let us denote by $\psi_t(p)$, $t \in \mathbb{R}$, the integral curve defined by X on C that passes through p and satisfies $f(\psi_t(p)) = f(p) - t$. Analogously, let us denote by $\psi'_t(p')$ the integral curve defined by X' on C' that passes through p' and satisfies $g(\psi'_{p'}(t)) = g(p') + t$. For every $p \in C$, we set λ_p equal to the only value in [0, 1] such that $f(p) = (1 - \lambda_p)(c_i + \varepsilon) + \lambda_p(c_j - \varepsilon)$. Finally, we define $\xi^C(p) = p'$, with $p' = \psi'_{\lambda_p(c'_j - c'_i - 2\varepsilon)} \circ \sigma \circ \psi_{\lambda_p(c_j - c_i - 2\varepsilon)}(p)$.

The map ξ^C is a diffeomorphism because C and C' do not contain critical points. It satisfies properties (b_3) and (c_3) by construction. To prove property (a_3), it is sufficient to observe that, for every $p \in C$,

$$\begin{aligned}|f(p) - g(\xi^C(p))| &= |(1 - \lambda_p)(c_i + \varepsilon) + \lambda_p(c_j - \varepsilon) - (c'_i + \varepsilon + \lambda_p(c'_j - c'_i - 2\varepsilon))|\\ &= |(1 - \lambda_p)(c_i - c'_i) + \lambda_p(c_j - c'_j)|.\end{aligned}$$

Having Claims 1, 2, and 3, we now proceed to construct the required diffeomorphism $\xi : \mathscr{M} \to \mathscr{M}$. Let us take $\varepsilon > 0$ sufficiently small so that Claims 1, 2, and 3 hold for all the critical points of f and g. We define a homeomorphism $\hat{\xi} : \mathscr{M} \to \mathscr{M}$ by gluing the diffeomorphisms constructed for proving Claims 1, 2, and 3, as follows. We start defining $\hat{\xi}$ on the disks D_i corresponding to minima: for $p \in D_i$, we set $\hat{\xi}(p) = \xi^{D_i}(p)$, with ξ^{D_i} as in Claim 1. Next, if $p_i \in D_i$ is adjacent in Γ_f to a maximum point (resp. a saddle point) p_j in D_j (resp. P_j), we extend $\hat{\xi}$ to cylinders C_{ij} using Claim 3. To this end, we observe that the diffeomorphism σ needed to apply Claim 3 can be taken as the restriction of ξ^{D_i} to ∂D_i. Then, we proceed applying Claim 1 (resp. Claim 2) and define $\hat{\xi}$ equal to ξ^{D_j} on D_j (resp. ξ^{P_j} on P_j). We observe that $\hat{\xi}$ is well defined on $C_{ij} \cap D_j$ (resp. $C_{ij} \cap P_j$). Indeed, the maps $\xi^{C_{ij}}$ and ξ^{D_j} (resp. ξ^{P_j}) map level sets of f to level sets of g, and gradient lines of f to gradient lines of g by properties (a_k) and (b_k), $k = 1, 2, 3$. Iterating these arguments we get a well-defined homeomorphism $\hat{\xi} : \mathscr{M} \to \mathscr{M}$. It is even a diffeomorphism on the disks D_i, the pants P_i and the cylinders C_{ij}.

We now smooth $\hat{\xi}$ to obtain a global diffeomorphism ξ. Let V be the union of the boundaries of all the submanifolds D_i, P_i and C_{ij} defined by f. Analogously let V' be the union of the boundaries of all the submanifolds D'_i, P'_i and C'_{ij} defined by g. By the Smoothing Theorem (see, e.g., [16, Theorem 8.1.9]), there is a diffeomorphism $\xi : \mathscr{M} \to \mathscr{M}$ such that $\xi(D_i) = D'_i$, $\xi(P_i) = P'_i$, $\xi(C_{ij}) = C'_{ij}$ for any possible i and j, $\xi_{|V} = \hat{\xi}_{|V}$, and ξ coincides with $\hat{\xi}$ outside a given neighborhood of V. If this neighborhood is chosen sufficiently small, then

$$\begin{aligned}\max_{p\in\mathscr{M}} |f(p) - g\circ\xi(p)| &= \max_{p\in\mathscr{M}} |f(p) - g\circ\hat{\xi}(p)| = \max_{i=1,\ldots,n} \{|c'_i - c_i|\}\\ &= \max_{v\in V(\Gamma_f)} |\ell_g(\Phi(v)) - \ell_f(v)|\end{aligned}$$

by properties (a_1), (a_2) and (a_3). In conclusion ξ is the required diffeomorphism. □

Corollary 12 *Let f and g in $\mathscr{F}^0(\mathscr{M})$ be such that their Reeb graphs Γ_f and Γ_g are isomorphic by an isomorphism $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ that preserves the order of all vertices, i.e., for every $v, w \in V(\Gamma_f)$, $\ell_f(v) < \ell_f(w)$ if and only if $\ell_g(\Phi(v)) < \ell_g(\Phi(w))$. Then there is a diffeomorphism $\xi : \mathscr{M} \to \mathscr{M}$ and an orientation preserving diffeomorphism $\eta : \mathbb{R} \to \mathbb{R}$ such that*

$$f = \eta \circ g \circ \xi.$$

Moreover, if Φ also preserves the vertex labels, i.e. Φ is an isomorphism of labeled graphs, then η can be taken to be the identity.

Proof Because Γ_f and Γ_g are isomorphic, the functions f and g have the same number of critical points of the same type: $K_f = \{p_1, \ldots, p_n\}$, $K_g = \{p'_1, \ldots, p'_n\}$. We set $c_i = f(p_i)$ and $c'_i = g(p'_i)$, for $1 \le i \le n$.

Because Φ preserves the order of all the vertices, there exists an orientation preserving diffeomorphism $\lambda : \mathbb{R} \to \mathbb{R}$ such that $\lambda(c_i) = c'_i$. In particular, if Φ preserves not only the order but also the values of all the vertices, then λ can be taken to be the identity.

We now observe that $\lambda \circ f$ and g have isomorphic Reeb graphs and the same critical values. Hence, we can apply Proposition 11 to $\lambda \circ f$ and g. Thus, we obtain a diffeomorphism $\xi : \mathscr{M} \to \mathscr{M}$ such that $\max_{p\in\mathscr{M}} |\lambda\circ f(p) - g\circ\xi(p)| = 0$, implying $\lambda \circ f = g \circ \xi$. It is now sufficient to take $\eta = \lambda^{-1}$. □

Proposition 13 (Uniqueness Theorem) *If $f, g \in \mathscr{F}^0(\mathscr{M})$, then*

(1) *f and g are RL-equivalent if and only if their Reeb graphs Γ_f and Γ_g are isomorphic by an isomorphism $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ that preserves the order of all vertices*;
(2) *f and g are R-equivalent if and only if their labeled Reeb graphs (Γ_f, ℓ_f) and (Γ_g, ℓ_g) are isomorphic.*

Proof The converse implications of statements (1) and (2) hold by Corollary 12.

In order to prove the direct implication of statement (1), let us assume for a moment that the direct implication of statement (2) holds. We observe that, if $f = \eta \circ g \circ \xi$ with η and ξ diffeomorphisms, then $\eta^{-1} \circ f = g \circ \xi$. Thus, statement (2) applies to $\eta^{-1} \circ f$ and g, implying that the Reeb graphs of $\eta^{-1} \circ f$ and g are isomorphic as labeled graphs, by an isomorphism preserving the vertex values. On the other hand, the Reeb graphs of $\eta^{-1} \circ f$ and f are isomorphic by an isomorphism that preserves the vertex order, yielding the claim.

We now prove the direct implication of statement (2). If $f = g \circ \xi$ and ξ is a self-diffeomorphism of $\mathscr{M}$, then ξ bijectively maps the critical points of f to those of g, also preserving their values. Thus, it is sufficient to prove that Γ_f and Γ_g are isomorphic, or, equivalently, that $\mathscr{M}/\sim_f$ and $\mathscr{M}/\sim_g$ are homeomorphic. Let $\widetilde{\xi}$: $\mathscr{M}/\sim_f \to \mathscr{M}/\sim_g$ be the map that makes the following diagram commute:

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "\xi"] \arrow[d, "\pi_f"'] & \mathscr{M} \arrow[d, "\pi_g"] \\
\mathscr{M}/\sim_f \arrow[r, "\widetilde{\xi}"'] & \mathscr{M}/\sim_g
\end{tikzcd}$$

In other words, $\widetilde{\xi}$ maps the connected component of the f level set containing p to the connected component of the g level set containing $\xi(p)$. The map $\widetilde{\xi}$ exists because we are assuming $f = g \circ \xi$. Moreover, $\widetilde{\xi}$ is invertible because ξ is a diffeomorphism and $g = f \circ \xi^{-1}$: $\widetilde{\xi}^{-1}([p]_{\sim_g}) = [\xi^{-1}(p)]_{\sim_f}$. Moreover, $\widetilde{\xi}$ is also continuous because ξ and π_g are continuous, and π_f is closed. Indeed, $\mathscr{M}/\sim_f$ and $\mathscr{M}/\sim_g$ are Hausdorff compact spaces, implying that π_f is closed by the closed map lemma. Finally, $\widetilde{\xi}^{-1}$ is continuous, again by the closed map lemma. □

3 Edit Deformations Between Labeled Reeb Graphs

In this section, f and g always denote simple Morse functions on the same surface $\mathscr{M}$. We define the edit deformations admissible to transform the labeled Reeb graph of f into that of g. We introduce at first elementary deformations, and later the deformations obtained by their composition. A brief description and an illustration accompany the definitions of elementary deformations in order to provide the reader with the underlying idea. We use the convention of denoting by $]a, b[$ the open interval with endpoints $a, b \in \mathbb{R}$.

The first two types of elementary deformations are the *deformation of birth type* and *deformation of death type*, respectively. They allow us, under certain conditions on labels, to insert or delete a vertex of degree 1 together with the adjacent vertex of degree 3 and the edge connecting them, as illustrated in Table 1.

Definition 14 T is an *elementary deformation* of (Γ_f, ℓ_f) *of birth type* (briefly, B-*type*) if, for a fixed edge $e(v_1, v_2) \in E(\Gamma_f)$, with $\ell_f(v_1) < \ell_f(v_2)$, $T(\Gamma_f, \ell_f)$ is a labeled graph (Γ, ℓ) such that

– $V(\Gamma) = V(\Gamma_f) \cup \{u_1, u_2\}$;

Table 1 A schematization of the elementary deformations of B- and D-type introduced in Definition 14

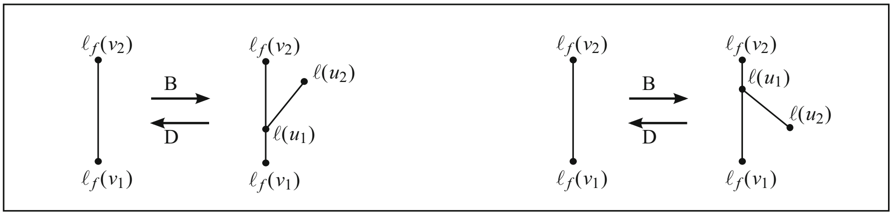
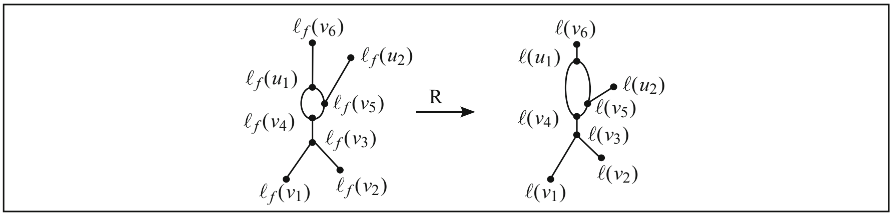


Table 2 A schematization of the elementary deformation of R-type introduced in Definition 15



- $E(\Gamma) = (E(\Gamma_f) - \{e(v_1, v_2)\}) \cup \{e(v_1, u_1), e(u_1, u_2), e(u_1, v_2)\}$;
- $\ell_{|V(\Gamma_f)} = \ell_f$ and either $\ell_f(v_1) < \ell(u_1) < \ell(u_2) < \ell_f(v_2)$, with $\ell^{-1}(]\ell(u_1), \ell(u_2)[) = \emptyset$, or $\ell_f(v_1) < \ell(u_2) < \ell(u_1) < \ell_f(v_2)$, with $\ell^{-1}(]\ell(u_2), \ell(u_1)[) = \emptyset$.

T is an *elementary deformation* of (Γ_f, ℓ_f) *of death type* (briefly, D-*type*) if, for fixed edges $e(v_1, u_1)$, $e(u_1, u_2)$, $e(u_1, v_2) \in E(\Gamma_f)$, with u_2 of degree 1 and either $\ell_f(v_1) < \ell_f(u_1) < \ell_f(u_2) < \ell_f(v_2)$, with $\ell_f^{-1}(]\ell_f(u_1), \ell_f(u_2)[) = \emptyset$, or $\ell_f(v_1) < \ell_f(u_2) < \ell_f(u_1) < \ell_f(v_2)$, with $\ell_f^{-1}(]\ell_f(u_2), \ell_f(u_1)[) = \emptyset$, $T(\Gamma_f, \ell_f)$ is a labeled graph (Γ, ℓ) such that

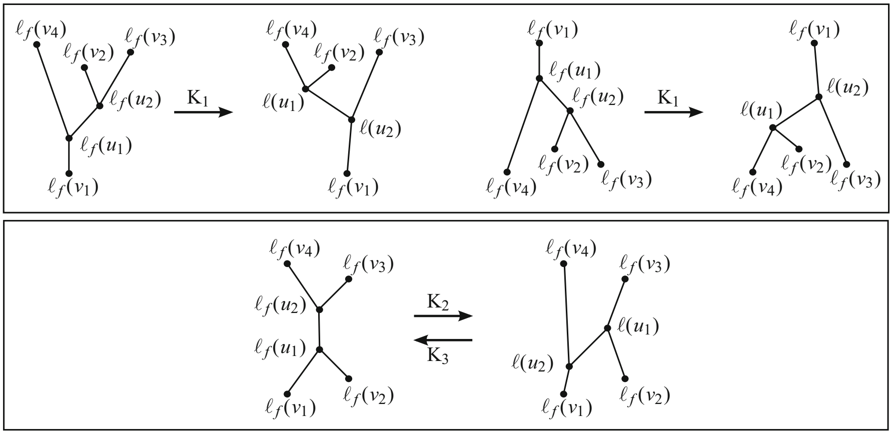
- $V(\Gamma) = V(\Gamma_f) - \{u_1, u_2\}$;
- $E(\Gamma) = (E(\Gamma_f) - \{e(v_1, u_1), e(u_1, u_2), e(u_1, v_2)\}) \cup \{e(v_1, v_2)\}$;
- $\ell = \ell_{f\,|V(\Gamma_f)-\{u_1,u_2\}}$.

The third type of elementary deformation is the *deformation of relabeling type*. It preserves vertices and edges while changing the vertex labels. It is also allowed to exchange the label order of at most two non-adjacent vertices, as illustrated in Table 2.

Definition 15 T is an *elementary deformation* of (Γ_f, ℓ_f) *of relabeling type* (briefly, R-*type*) if $T(\Gamma_f, \ell_f)$ is a labeled graph (Γ, ℓ) such that

- $\Gamma = \Gamma_f$;
- $\ell : V(\Gamma) \to \mathbb{R}$ induces the same vertex-order as ℓ_f except for at most two non-adjacent vertices, say u_1, u_2, with $\ell_f(u_1) < \ell_f(u_2)$, $\ell_f^{-1}(]\ell_f(u_1), \ell_f(u_2)[) = \emptyset$, and $\ell(u_1) > \ell(u_2)$ and $\ell^{-1}(]\ell(u_2), \ell(u_1)[) = \emptyset$.

Table 3 A schematization of the elementary deformations of K_i-type, $i = 1, 2, 3$, introduced in Definition 16



We underline that the definitions of the deformations of B-, D- and R-type are essentially different from the definitions of the deformations with the same names for Reeb graphs of curves as given in [12], even if the associated cost will be the same (see Sect. 4). This is because the degree of the involved vertices is equal to 2 for Reeb graphs of closed curves, whereas it is equal to 1 or 3 for Reeb graphs of surfaces.

The last three types of elementary deformations we consider have been introduced by Kudryavtseva [17]. For this reason, we call them *deformations of* K_1-, K_2-, K_3-*type*. They maintain the same vertices while changing some adjacencies and labels, as illustrated in Table 3.

Definition 16 T is an *elementary deformation* of (Γ_f, ℓ_f) *of* K_1-*type* if, for fixed edges $e(v_1, u_1), e(u_1, u_2), e(u_1, v_4), e(u_2, v_2), e(u_2, v_3) \in E(\Gamma_f)$, with two among the vertices v_2, v_3, v_4 possibly coincident, and either $\ell_f(v_1) < \ell_f(u_1) < \ell_f(u_2) < \ell_f(v_2), \ell_f(v_3), \ell_f(v_4)$ with $\ell_f^{-1}(]\ell_f(u_1), \ell_f(u_2)[) = \emptyset$, or $\ell_f(v_2), \ell_f(v_3), \ell_f(v_4) < \ell_f(u_2) < \ell_f(u_1) < \ell_f(v_1)$ with $\ell_f^{-1}(]\ell_f(u_2), \ell_f(u_1)[) = \emptyset$, $T(\Gamma_f, \ell_f)$ is a labeled graph (Γ, ℓ) such that:

- $V(\Gamma) = V(\Gamma_f)$;
- $E(\Gamma) = (E(\Gamma_f) - \{e(v_1, u_1), e(u_2, v_2)\}) \cup \{e(v_1, u_2), e(u_1, v_2)\}$;
- $\ell(v) = \ell_f(v)$ for every $v \in V(\Gamma_f) - \{u_1, u_2\}$, and either $\ell_f(v_1) < \ell(u_2) < \ell(u_1) < \ell_f(v_2), \ell_f(v_3), \ell_f(v_4)$ with $\ell^{-1}(]\ell(u_2), \ell(u_1)[) = \emptyset$, or $\ell_f(v_2), \ell_f(v_3), \ell_f(v_4) < \ell(u_1) < \ell(u_2) < \ell_f(v_1)$ with $\ell^{-1}(]\ell(u_1), \ell(u_2)[) = \emptyset$.

T is an *elementary deformation* of (Γ_f, ℓ_f) *of* K_2-*type* if, for fixed edges $e(v_1, u_1)$, $e(v_2, u_1), e(u_1, u_2), e(u_2, v_3), e(u_2, v_4) \in E(\Gamma_f)$, with u_1, u_2 vertices of degree 3, v_2, v_3 possibly coincident with v_1, v_4, respectively, and $\ell_f(v_1), \ell_f(v_2) < \ell_f(u_1) < \ell_f(u_2) < \ell_f(v_3), \ell_f(v_4)$ with $\ell_f^{-1}(]\ell_f(u_1), \ell_f(u_2)[) = \emptyset$, $T(\Gamma_f, \ell_f)$ is a labeled graph (Γ, ℓ) such that:

– $V(\Gamma) = V(\Gamma_f)$;
– $E(\Gamma) = (E(\Gamma_f) - \{e(v_1, u_1), e(u_2, v_3)\}) \cup \{e(v_1, u_2), e(u_1, v_3)\}$;
– $\ell(v) = \ell_f(v)$ for every $v \in V(\Gamma_f) - \{u_1, u_2\}$, and $\ell_f(v_1), \ell_f(v_2) < \ell(u_2) < \ell(u_1) < \ell_f(v_3), \ell_f(v_4)$, with $\ell^{-1}(]\ell(u_2), \ell(u_1)[) = \emptyset$.

T is an *elementary deformation* of (Γ_f, ℓ_f) *of* K_3*-type* if, for fixed edges $e(v_1, u_2)$, $e(u_1, u_2)$, $e(v_2, u_1)$, $e(u_1, v_3)$, $e(u_2, v_4) \in E(\Gamma_f)$, with u_1, u_2 vertices of degree 3, v_2, v_3 possibly coincident with v_1, v_4, respectively, and $\ell_f(v_1), \ell_f(v_2) < \ell_f(u_2) < \ell_f(u_1) < \ell_f(v_3), \ell_f(v_4)$, with $\ell_f^{-1}(]\ell_f(u_2), \ell_f(u_1)[) = \emptyset$, $T(\Gamma_f, \ell_f)$ is a labeled graph (Γ, ℓ) such that:

– $V(\Gamma) = V(\Gamma_f)$;
– $E(\Gamma) = (E(\Gamma_f) - \{e(v_1, u_2), e(u_1, v_3)\}) \cup \{e(v_1, u_1), e(u_2, v_3)\}$;
– $\ell(v) = \ell_f(v)$ for every $v \in V(\Gamma_f) - \{u_1, u_2\}$, and $\ell_f(v_1), \ell_f(v_2) < \ell(u_1) < \ell(u_2) < \ell_f(v_3), \ell_f(v_4)$, with $\ell^{-1}(]\ell(u_1), \ell(u_2)[) = \emptyset$.

The following result is an immediate consequence of Proposition 9 (Realization Theorem).

Corollary 17 *Let $f \in \mathscr{F}^0(\mathscr{M})$ and $(\Gamma, \ell) = T(\Gamma_f, \ell_f)$ for T an elementary deformation of B-, D-, R-, or K_i-type, i=1,2,3. There exists $g \in \mathscr{F}^0(\mathscr{M})$ such that $(\Gamma_g, \ell_g) \cong (\Gamma, \ell)$.*

We now introduce the concept of inverse of an elementary deformation.

Definition 18 Let T be an elementary deformation such that $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$ and let Φ be the labeled graph isomorphism between $T(\Gamma_f, \ell_f)$ and (Γ_g, ℓ_g). Let us identify $T(\Gamma_f, \ell_f)$ with (Γ_g, ℓ_g) via Φ. We denote by T^{-1}, and call it the *inverse* of T, the elementary deformation acting on the vertices, edges, and labels of (Γ_g, ℓ_g) as follows:

– if T is an elementary deformation of D-type deleting two vertices, then T^{-1} is of B-type inserting the same vertices, with the same labels, and vice versa if T is of B-type;
– if T is an elementary deformation of R-type relabeling the vertices of $V(\Gamma_f)$, then T^{-1} is again of R-type relabeling these vertices in the inverse way;
– if T is an elementary deformation of K_1-type changing adjacencies and relabeling two vertices, then T^{-1} is again of K_1-type changing adjacencies and relabeling the same vertices in the inverse way;
– if T is an elementary deformation of K_2-type changing adjacencies and relabeling two vertices, then T^{-1} is of K_3-type changing adjacencies and relabeling the same vertices in the inverse way, and vice versa if T is of K_3-type.

Remark 19 We observe that from Definition 18, if $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, then $T^{-1}(\Gamma_g, \ell_g) \cong (\Gamma_f, \ell_f)$.

Applying Corollary 17, we can apply elementary deformations and their inverses iteratively. We use this fact in the next Definition 20. Given an elementary deformation T of (Γ_f, ℓ_f) and an elementary deformation S of $T(\Gamma_f, \ell_f)$, the juxtaposition ST means applying first T and then S.

Definition 20 We call *deformation* of (Γ_f, ℓ_f) any finite ordered sequence $T = (T_1, T_2, \ldots, T_r)$ of elementary deformations such that T_1 is an elementary deformation of (Γ_f, ℓ_f), and for every $2 \le k \le r$, T_k is an elementary deformation of $T_{k-1}T_{k-2}\cdots T_1(\Gamma_f, \ell_f)$. We denote by $T(\Gamma_f, \ell_f)$ the result of the deformation $T_r T_{r-1} \cdots T_1$ applied to (Γ_f, ℓ_f). Moreover, if $T = (T_1, \ldots, T_r)$ is such that $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, then the *inverse deformation* of T is $T^{-1}(\Gamma_g, \ell_g) \cong (\Gamma_f, \ell_f)$, where $T^{-1} = (T_r^{-1}, \ldots, T_1^{-1})$.

In the rest of the paper we write $\mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ to denote the set of deformations turning (Γ_f, ℓ_f) into (Γ_g, ℓ_g) up to isomorphisms:

$$\mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = \{T = (T_1, \ldots, T_n), n \ge 1 : T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)\}.$$

From the fact that $T^{-1}T(\Gamma_f, \ell_f) \cong (\Gamma_f, \ell_f)$ it follows that the set $\mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, when non-empty, always contains infinitely many deformations. We end the section proving that it is always non-empty. We first need two lemmas that are widely inspired by Lemma 1 and Theorem 1 of [17], respectively.

Lemma 21 *Let (Γ_f, ℓ_f) be a labeled Reeb graph. The following statements hold*:

(i) *For any $u, v \in V(\Gamma_f)$ corresponding to two minima (maxima, respectively) of f, there exists a deformation T such that u and v are minima (maxima, respectively) adjacent to the same vertex w in $T(\Gamma_f, \ell_f)$.*
(ii) *Let $\mathfrak{g} > 1$ and $u \in V(\Gamma_f)$ belong to a length-m cycle C in Γ_f, $m \ge 2$, with $u = \operatorname{argmin}\{\ell_f(w) : w \in C\}$. There exists a deformation T such that u belongs to a length-2 cycle C' in $T(\Gamma_f, \ell_f) = (\Gamma, \ell)$ and $u = \operatorname{argmin}\{\ell(w) : w \in C'\}$.*

Proof Let us prove statement (i) assuming that in (Γ_f, ℓ_f) two vertices u, v exist that correspond to two minima of f. The case of maxima is analogous.

Let us consider a path γ on Γ_f having u, v as endpoints, whose length is $m \ge 2$, and the finite sequence of vertices through which it passes is $(w_0, w_1, \ldots, w_m)$, with $w_0 = u$, $w_m = v$, and $w_i \ne w_j$ for $i \ne j$. We aim at showing that there exists a deformation T such that, in $T(\Gamma_f, \ell_f)$, the vertices u, v are still minima and become adjacent to the same vertex w with $w \in \{w_1, \ldots, w_{m-1}\}$. Thus, the path γ is transformed by T into a path γ' that is of length 2 and passes through the vertices u, w, v.

If $m = 2$, then it is sufficient to take T as the deformation of R-type such that $T(\Gamma_f, \ell_f) = (\Gamma_f, \ell_f)$ since γ already coincides with γ'. If $m > 2$, let $w_i = \operatorname{argmax}\{\ell_f(w_j) : w_j \text{ with } 0 \le j \le m\}$. It holds that $w_i \ne u, v$ because u, v are minima of f and it is unique because f is simple. It is easy to observe that, in a neighborhood of w_i, possibly after a finite sequence of deformations of R-type, the graph gets one of the configurations shown on the left of Fig. 4a–e.

Figure 4 also shows that a finite sequence of deformations of K_1-, K_3-, and, possibly, R-type transforms the simple path γ, which has length m, into a simple path of length $m-1$. Iterating this procedure, in the last step we are necessarily in one of the situations illustrated on the right of Fig. 4a or b, with $w_{i-2} = u$, $w_{i+1} = v$. Hence, u and v remain minima after applying T, and statement (i) is proved.

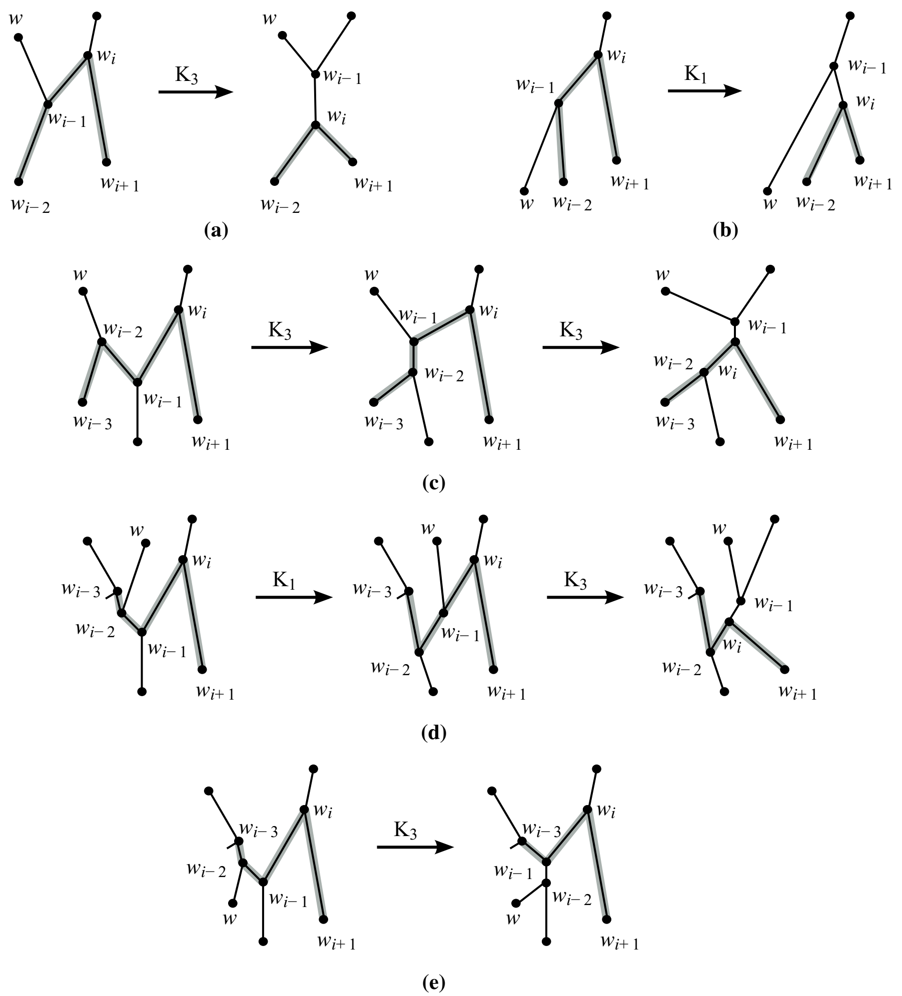


Fig. 4 Possible configurations of a simple path on a labeled Reeb graph in a neighborhood of its maximum point, and elementary deformations that reduce its length. To facilitate the reader, f has been represented as the height function, so that $\ell_f(w_a) < \ell_f(w_b)$ if and only if w_a is lower than w_b in the pictures

The proof of statement (ii) is analogous to that of statement (i), provided that γ is taken to be a length-m cycle with $u \equiv v$ of degree 3, and $u = \mathrm{argmin}\{\ell_f(w_j) : w_j \text{ with } 0 \leq j \leq m-1\}$. □

Lemma 22 *Every labeled Reeb graph (Γ_f, ℓ_f) can be transformed into a canonical one through a finite sequence of elementary deformations.*

Proof Our proof is in two steps. First, we show how to transform an arbitrary Reeb graph into a minimal one. Then, we show how to reduce a minimal Reeb graph to the canonical form.

The first step is by induction on the number $\mathtt{s}$ of degree 1 vertices: $\mathtt{s} = \mathtt{p} + \mathtt{q}$, with $\mathtt{p}$ and $\mathtt{q}$ denoting the number of minima and maxima of f. If $\mathtt{s} = 2$, then $\varGamma_f$ is already minimal by definition. Let us assume that any Reeb graph with $\mathtt{s} \geq 2$ vertices of degree 1 can be transformed into a minimal one through a certain deformation. Let $\varGamma_f$ have $\mathtt{s} + 1$ vertices of degree 1. Thus, at least one between $\mathtt{p}$ and $\mathtt{q}$ is greater than one. Let $\mathtt{p} > 1$ (the case when $\mathtt{q} > 1$ is analogous). By Lemma 21(i), if u, v correspond to two minima of f, we can construct a deformation T such that in $T(\varGamma_f, \ell_f)$ these vertices are still minima and both adjacent to a certain vertex w of degree 3. Let $T(\varGamma_f, \ell_f) = (\varGamma, \ell)$, with $\ell(u) < \ell(v) < \ell(w)$. When there is a vertex w' in $\ell^{-1}(]\ell(v), \ell(w)[)$, we can apply a deformation of R-type relabeling only v, because v, w' cannot be adjacent, and get a new labeling ℓ' such that $\ell'(w')$ is not contained in $]\ell'(v), \ell'(w)[$. Possibly repeating this procedure finitely many times, we get a new labeling, that for simplicity we still denote by ℓ, such that $\ell^{-1}(]\ell(v), \ell(w)[) = \emptyset$. Hence, we are in the position of applying a deformation of D-type that deletes v, w. The resulting labeled Reeb graph has $\mathtt{s}$ vertices of degree 1. By the inductive hypothesis, it is deformable into a minimal Reeb graph.

Now we prove the second step. Let $\varGamma_f$ be a minimal Reeb Graph, i.e. with $\mathtt{p} = \mathtt{q} = 1$. Denoting by $\mathfrak{g}$ the genus of $\mathscr{M}$, the total number of splitting saddles (i.e. vertices of degree 3 such that there are two higher adjacent vertices) of $\varGamma_f$ is $\mathfrak{g}$ by Remark 7. By Remark 4, if $\mathfrak{g} = 0$, then $\varGamma_f$ is already canonical. Let us consider the case $\mathfrak{g} \geq 1$. Let $v \in V(\varGamma_f)$ be a splitting saddle such that, for every cycle C containing v, $\ell_f(v) = \min_{w \in C}\{\ell_f(w)\}$, and let C be one of these cycles. By Lemma 21(ii), there exists a deformation T that transforms C into a length-2 cycle, still having v as the lowest vertex. Let v' be the highest vertex in this length-2 cycle. We observe that no other cycles of $T(\varGamma_f, \ell_f)$ contain v and v', otherwise the initial assumption on $\ell_f(v)$ would be contradicted. Hence, v, v' and the edges adjacent to them are not touched when applying again Lemma 21(ii) to reduce the length of another cycle. Therefore, iterating the same argument on different splitting saddles at most $\mathfrak{g}$ times, $\varGamma_f$ is transformed into a canonical Reeb graph. □

Proposition 23 *Let $f, g \in \mathscr{F}^0(\mathscr{M})$. The set $\mathscr{T}((\varGamma_f, \ell_f), (\varGamma_g, \ell_g))$ is non-empty.*

Proof By Lemma 22 we can find two deformations T_f and T_g transforming $(\varGamma_f, \ell_f)$ and $(\varGamma_g, \ell_g)$, respectively, into canonical Reeb graphs. Apart from the labels, $T_f(\varGamma_f, \ell_f)$ and $T_g(\varGamma_g, \ell_g)$ are isomorphic because associated with the same surface $\mathscr{M}$. Hence, $T_f(\varGamma_f, \ell_f)$ can be transformed into a graph isomorphic to $T_g(\varGamma_g, \ell_g)$ through an elementary deformation of R-type, say T_{R}. Thus, $(\varGamma_g, \ell_g) \cong T_g^{-1} T_{\mathrm{R}} T_f(\varGamma_f, \ell_f)$, i.e. $T_g^{-1} T_{\mathrm{R}} T_f \in \mathscr{T}((\varGamma_f, \ell_f), (\varGamma_g, \ell_g))$. □

We show a simple example illustrating the above proof in Fig. 5.

4 The Edit Distance Between Labeled Reeb Graphs

In this section, we introduce an edit distance between labeled Reeb graphs, in terms of the cost necessary to transform Reeb graphs into one another by using deformations. We begin by defining the cost of a deformation.

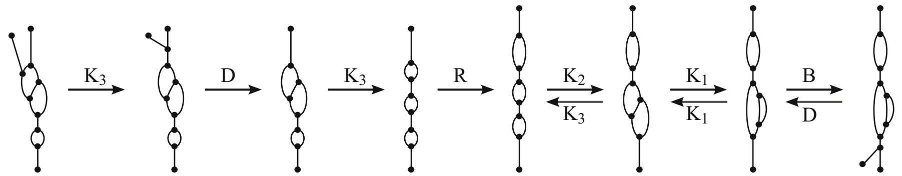


Fig. 5 The leftmost labeled Reeb graph is transformed into the rightmost one passing through canonical Reeb graphs, as in the proof of Proposition 23

For the sake of simplicity, whenever $(\Gamma_g, \ell_g) \cong (\Gamma, \ell) = T(\Gamma_f, \ell_f)$, we identify $V(\Gamma_g)$ with $V(\Gamma)$, and ℓ_g with ℓ. The notation concerning elementary deformations is the same as in Definitions 14–16.

Definition 24 Let $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ be a deformation.

– For T elementary of B-type, inserting the vertices $u_1, u_2 \in V(\Gamma_g)$, the associated cost is

$$c(T) = \frac{|\ell_g(u_1) - \ell_g(u_2)|}{2}.$$

– For T elementary of D-type, deleting the vertices $u_1, u_2 \in V(\Gamma_f)$, the associated cost is

$$c(T) = \frac{|\ell_f(u_1) - \ell_f(u_2)|}{2}.$$

– For T elementary of R-type, relabeling the vertices $v \in V(\Gamma_f) = V(\Gamma_g)$, the associated cost is

$$c(T) = \max_{v \in V(\Gamma_f)} |\ell_f(v) - \ell_g(v)|.$$

– For T elementary of K_i-type, with $i = 1, 2, 3$, changing adjacencies and relabeling the vertices $u_1, u_2 \in V(\Gamma_f)$, the associated cost is

$$c(T) = \max\{|\ell_f(u_1) - \ell_g(u_1)|, |\ell_f(u_2) - \ell_g(u_2)|\}.$$

– For $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, with $T = (T_1, \ldots, T_r)$, the associated cost is

$$c(T) = \sum_{i=1}^{r} c(T_i).$$

Proposition 25 *For every deformation $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, $c(T^{-1}) = c(T)$.*

Proof It is sufficient to observe that, for every deformation $T = (T_1, \ldots, T_r)$, $r \geq 1$, such that $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, the following three equalities follow from Definition 24, 18 and 20, respectively:

$$c(T) = \sum_{i=1}^{r} c(T_i) = \sum_{i=1}^{r} c(T_i^{-1}) = c(T^{-1}).$$

□

Theorem 26 *For every $f, g \in \mathscr{F}^0(\mathscr{M})$, we set*

$$d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = \inf_{T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))} c(T).$$

It holds that d_E is a pseudo-metric on isomorphism classes of labeled Reeb graphs.

Proof By Proposition 23, d_E is a real number. The coincidence property can be verified by observing that the deformation of R-type such that $T(\Gamma_f, \ell_f) = (\Gamma_f, \ell_f)$ has a cost equal to 0; the symmetry property is a consequence of Proposition 25; the triangle inequality can be proved in the standard way. □

In order to say that d_E is actually a metric, we need to prove that if $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ vanishes then $(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$. We postpone this until Sect. 6. Nevertheless, for simplicity, we already refer to d_E as to the *edit distance*.

The following proposition shows that we can obtain the same effect of a finite sequence of deformations of D-type also through a cheaper deformation that involves a relabeling of vertices. Analogous results hold for other types of deformations. These results yield, in some cases, sharper estimates of the edit distance between labeled Reeb graphs.

Proposition 27 *Let $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, with $T = (T_1, \ldots, T_n)$ and T_i an elementary deformation of D-type for $i = 1, \ldots, n$. Then, for every sufficiently small $\varepsilon > 0$, there exists a deformation $S^\varepsilon \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, with $S^\varepsilon = (S_0^\varepsilon, S_1^\varepsilon, \ldots, S_n^\varepsilon)$ such that S_0^ε is an elementary deformation of R-type, $S_1^\varepsilon, \ldots, S_n^\varepsilon$ are elementary deformations of D-type, and $\inf_\varepsilon c(S^\varepsilon) \leq \max_{i=1,\ldots,n} c(T_i)$. Hence, when $n > 1$, there exists a value $\varepsilon > 0$ such that $c(S^\varepsilon) < c(T)$.*

Proof Let $T = (T_1, \ldots, T_n)$, with each T_i of D-type, and let v_i, w_i be the vertices of Γ_f deleted by T_i. It is not restrictive to assume that $\ell_f(v_i) < \ell_f(w_i)$. We prove the claim by induction on n. For $n = 1$, it is sufficient to take S_0^ε as the elementary deformation of R-type such that $S_0^\varepsilon(\Gamma_f, \ell_f) = (\Gamma_f, \ell_f)$ and $S_1^\varepsilon = T_1$. For $n > 1$, for every i, j with $1 \leq i, j \leq n$, let us set $T_i \preceq T_j$ if and only if $[\ell_f(v_i), \ell_f(w_i)] \subseteq [\ell_f(v_j), \ell_f(w_j)]$. Let us denote by $T_{r_1}, \ldots, T_{r_m}$ the maximal elements of the poset $(\{T_1, \ldots, T_n\}, \preceq)$.

We observe that, for $1 \leq i \leq n$, there exists exactly one value k, with $1 \leq k \leq m$, such that $[\ell_f(v_i), \ell_f(w_i)] \subseteq [\ell_f(v_{r_k}), \ell_f(w_{r_k})]$. Moreover, $[\ell_f(v_i), \ell_f(w_i)] \cap [\ell_f(v_{r_h}), \ell_f(w_{r_h})] = \emptyset$ for every $h \neq k$ because T_i is an elementary deformation of D-type.

Let

$$0 < \varepsilon < \min_{k=1,\ldots,m} \frac{\ell_f(w_{r_k}) - \ell_f(v_{r_k})}{2}.$$

To define S_0^ε, we take $\ell : V(\Gamma_f) \to \mathbb{R}$ as follows. For $1 \le k \le m$, we set

$$\ell(v_{r_k}) = \frac{\ell_f(w_{r_k}) + \ell_f(v_{r_k})}{2} - \varepsilon$$

and

$$\ell(w_{r_k}) = \frac{\ell_f(w_{r_k}) + \ell_f(v_{r_k})}{2} + \varepsilon.$$

Next, for $1 \le i \le n$, assuming $[\ell_f(v_i), \ell_f(w_i)] \subseteq [\ell_f(v_{r_k}), \ell_f(w_{r_k})]$, we let $\lambda_i, \mu_i \in [0,1]$ be the unique values such that $\ell_f(v_i) = (1-\lambda_i)\ell_f(v_{r_k}) + \lambda_i \ell_f(w_{r_k})$ and $\ell_f(w_i) = (1-\mu_i)\ell_f(v_{r_k}) + \mu_i \ell_f(w_{r_k})$, and we set $\ell(v_i) = (1-\lambda_i)\ell(v_{r_k}) + \lambda_i \ell(w_{r_k})$ and $\ell(w_i) = (1-\mu_i)\ell(v_{r_k}) + \mu_i \ell(w_{r_k})$. We observe that ℓ preserves the vertex order induced by ℓ_f and, therefore, S_0^ε defined by setting $S_0^\varepsilon(\Gamma_f, \ell_f) = (\Gamma_f, \ell)$ is an elementary deformation of R-type. By Definition 24, the cost of S_0^ε is

$$c(S_0^\varepsilon) = \max_{i=1,\dots,n} \Big\{ \max \Big\{ \big|\ell_f(v_i) - \ell(v_i)\big|, \big|\ell_f(w_i) - \ell(w_i)\big| \Big\}\Big\}.$$

A direct computation shows that $\ell(v_i) - \ell_f(v_i) \le \ell(v_{r_k}) - \ell_f(v_{r_k})$ and $\ell_f(v_i) - \ell(v_i) \le \ell_f(w_{r_k}) - \ell(w_{r_k})$. In the same way, $\ell(w_i) - \ell_f(w_i) \le \ell(v_{r_k}) - \ell_f(v_{r_k})$ and $\ell_f(w_i) - \ell(w_i) \le \ell_f(w_{r_k}) - \ell(w_{r_k})$. Hence

$$\begin{aligned} c(S_0^\varepsilon) &= \max_{k=1,\dots,m} \Big\{ \max \Big\{ \ell(v_{r_k}) - \ell_f(v_{r_k}), \ell_f(w_{r_k}) - \ell(w_{r_k}) \Big\}\Big\} \\ &= \max_{k=1,\dots,m} \frac{\ell_f(w_{r_k}) - \ell_f(v_{r_k})}{2} - \varepsilon = \max_{k=1,\dots,m} c(T_{r_k}) - \varepsilon. \end{aligned} \tag{4.1}$$

Now, for $i = 1, \dots, n$, we set S_i^ε to be the elementary deformation of D-type that deletes the vertices v_i, w_i from $S_0^\varepsilon(\Gamma_f, \ell_f)$. If $[\ell_f(v_i), \ell_f(w_i)] \subseteq [\ell_f(v_{r_k}), \ell_f(w_{r_k})]$, then

$$c(S_i^\varepsilon) = \frac{\ell(w_i) - \ell(v_i)}{2} \le \frac{\ell(w_{r_k}) - \ell(v_{r_k})}{2} = \varepsilon. \tag{4.2}$$

Setting $S^\varepsilon = (S_0^\varepsilon, S_1^\varepsilon, \dots, S_n^\varepsilon)$, we have $S^\varepsilon \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, and by formulas (4.1) and (4.2) it holds that

$$c(S^\varepsilon) = c(S_0^\varepsilon) + \sum_{i=1}^n c(S_i^\varepsilon) \le \max_{k=1,\dots,m} c(T_{r_k}) - \varepsilon + n \cdot \varepsilon.$$

Therefore,

$$\inf_\varepsilon c(S^\varepsilon) \le \max_{k=1,\dots,m} c(T_{r_k}) \le \max_{i=1,\dots,n} c(T_i) \le \sum_{i=1}^n c(T_i).$$

The last inequality is strict when $n > 1$. $\square$

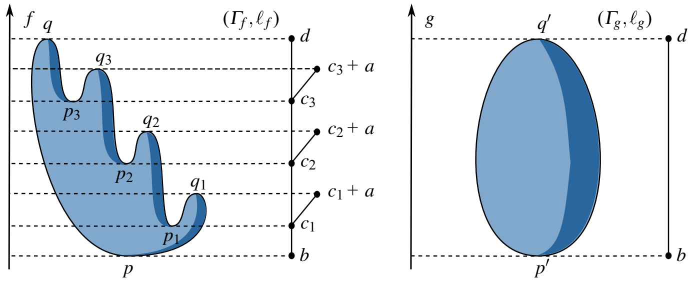


Fig. 6 The functions $f, g \in \mathscr{F}^0(\mathscr{M})$ considered in Example 29

5 Stability

This section is devoted to proving that Reeb graphs of simple Morse functions on orientable surfaces are stable under function perturbations. More precisely, we will show that arbitrary changes in the space of simple Morse functions with respect to the C^0-norm (also known as the L_∞-norm or maximum norm) imply not greater changes in the edit distance between the associated labeled Reeb graphs. Formally:

Theorem 28 *For every $f, g \in \mathscr{F}^0(\mathscr{M})$, letting $\|f - g\|_{C^0} = \max_{p\in\mathscr{M}} |f(p) - g(p)|$, we have*

$$d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \|f - g\|_{C^0}.$$

We observe that such result holds because of the way the cost of an elementary deformation of R-type was defined, as the following Example 29 shows.

Example 29 Let $f, g : \mathscr{M} \to \mathbb{R}$ with $f, g \in \mathscr{F}^0(\mathscr{M})$ as illustrated in Fig. 6. Let $f(q_i) - f(p_i) = a$, $i = 1, 2, 3$. Up to re-parameterization of $\mathscr{M}$, we have $\|f - g\|_{C^0} = \frac{a}{2}$. The deformation T that deletes the three edges $e(p_i, q_i) \in E(\Gamma_f)$ has cost $c(T) = 3\cdot\frac{a}{2}$, implying $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq 3\cdot\|f - g\|_{C^0}$. On the other hand, applying Proposition 27, we see that actually $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \|f - g\|_{C^0}$. Indeed, for every $0 < \varepsilon < \frac{a}{2}$, we can apply to (Γ_f, ℓ_f) a deformation of R-type S_0^ε that relabels the vertices $p_i, q_i, i = 1, 2, 3$, in such a way that $\ell_f(p_i)$ is increased by $\frac{a}{2} - \varepsilon$ and $\ell_f(q_i)$ is decreased by $\frac{a}{2} - \varepsilon$, composed with deformations S_i^ε of D-type that delete p_i, q_i and the edge $e(p_i, q_i)$, for $i = 1, 2, 3$. The deformation $S^\varepsilon = (S_0^\varepsilon, S_1^\varepsilon, S_2^\varepsilon, S_3^\varepsilon)$ has cost $c(S^\varepsilon) = \frac{a}{2} + 2\varepsilon$, implying that $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \frac{a}{2}$ by the arbitrariness of ε.

In order to prove Theorem 28, we consider the set $\mathscr{F}(\mathscr{M})$ of smooth real-valued functions on $\mathscr{M}$ endowed with the C^2 topology defined as follows. Let $\{U_\alpha\}$ be a coordinate covering of $\mathscr{M}$ with coordinate maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$, and let $\{C_\alpha\}$ be a compact refinement of $\{U_\alpha\}$. For every positive constant $\delta > 0$ and every $f \in \mathscr{F}(\mathscr{M})$, define $N(f, \delta)$ as the subset of $\mathscr{F}(\mathscr{M})$ consisting of all maps g such that, denoting $f_\alpha = f \circ \varphi_\alpha^{-1}$ and $g_\alpha = g \circ \varphi_\alpha^{-1}$, it holds that $\max_{i+j\leq 2} \left|\frac{\partial^{i+j}}{\partial x^i \partial y^j}(f_\alpha - g_\alpha)\right| < \delta$ at all

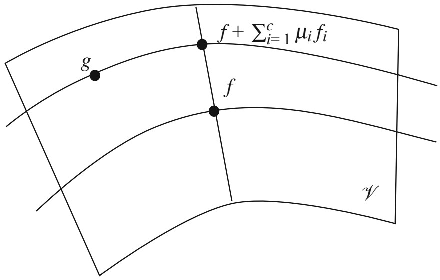


Fig. 7 The natural stratification is trivial in a neighborhood $\mathscr{V}$ of any function f with finite codimension: If g belongs to $\mathscr{V}$, then g is in the orbit of $f+\sum_{i=1}^{c}\mu_i f_i$ for some $(\mu_1,\ldots,\mu_c)\in\mathbb{R}^c$ and $\{f_1,\ldots,f_c\}$ a basis of the complementary of the orbit of f

points of $\varphi_\alpha(C_\alpha)$. The C^2 topology is the topology obtained by taking the sets $N(f,\delta)$ as a base of neighborhoods.

Next we consider the natural stratification of $\mathscr{F}(\mathscr{M})$, as presented by Cerf in [7] and further studied by Sergeraert in [23]. It is based on the codimension of a function. To define the codimension $c(f)$ of a function $f\in\mathscr{F}(\mathscr{M})$, consider the map $\mathscr{D}_+(\mathbb{R})\times\mathscr{D}(\mathscr{M})\to\mathscr{F}(\mathscr{M})$ defined by $(\eta,\xi)\mapsto\eta\circ f\circ\xi$, where η is a self-diffeomorphism of $\mathbb{R}$ preserving the orientation, and ξ is a self-diffeomorphism of $\mathscr{M}$. The codimension of f is set to be the codimension of the image of the differential of this map in $\mathscr{F}(\mathscr{M})$. For each integer number $c\geq 0$, $\mathscr{F}^c(\mathscr{M})$ denotes the subspace of $\mathscr{F}(\mathscr{M})$ consisting of functions of codimension c. The family $(\mathscr{F}^c(\mathscr{M}))$ is called the *natural stratification* of $\mathscr{F}(\mathscr{M})$. In particular:

- The stratum $\mathscr{F}^0(\mathscr{M})$ is the set of simple Morse functions. Therefore, for $f\in\mathscr{F}^0(\mathscr{M})$ and p one of its critical points, there exists a coordinate chart in a neighborhood of p where f can be written as $f=f(p)+x^2+y^2$, $f=f(p)\pm x^2\mp y^2$, or $f=f(p)-x^2-y^2$ depending on whether the index of p is 0, 1 or 2.
- The stratum $\mathscr{F}^1(\mathscr{M})$ is the disjoint union of two sets $\mathscr{F}^1_\alpha(\mathscr{M})$ and $\mathscr{F}^1_\beta(\mathscr{M})$, where
  - $\mathscr{F}^1_\alpha(\mathscr{M})$ is the set of functions f whose critical levels contain exactly one critical point, and the critical points are all non-degenerate, except exactly one, say p, such that the following property holds: in a neighborhood of p, f can be written as $f=f(p)\pm x^2+y^3$ (cf. [7, p. 23]).
  - $\mathscr{F}^1_\beta(\mathscr{M})$ is the set of Morse functions whose critical levels contain at most one critical point, except for one level containing exactly two critical points.

Let us consider the local behavior of the action of $\mathscr{D}_+(\mathbb{R})\times\mathscr{D}(\mathscr{M})$ on $\mathscr{F}(\mathscr{M})$. In [23] Sergeraert shows that this action is locally trivial. In particular, for any function $f\in\mathscr{F}(\mathscr{M})$ with finite codimension c, if g is sufficiently close to f, then it is RL-equivalent to $f+\sum_{i=1}^{c}\mu_i f_i$ for some $(\mu_1,\ldots,\mu_c)\in\mathbb{R}^c$, where $\{f_1,\ldots,f_c\}$ is a basis of the complementary of the orbit of f. A picture illustrating this fact is given in Fig. 7.

The global action of $\mathscr{D}_+(\mathbb{R})\times\mathscr{D}(\mathscr{M})$ on $\mathscr{F}(\mathscr{M})$ follows its infinitesimal behavior: The orbit of f in $\mathscr{F}(\mathscr{M})$ with respect to the action of the group $\mathscr{D}_+(\mathbb{R})\times\mathscr{D}(\mathscr{M})$ is a submanifold of $\mathscr{F}(\mathscr{M})$ of codimension $c(f)$. Hence, given two functions $f,g\in\mathscr{F}^0(\mathscr{M})$, we can always find $\widehat{f},\widehat{g}\in\mathscr{F}^0(\mathscr{M})$ arbitrarily close to f,g, respectively, such that

- $\widehat{f},\widehat{g}$ are RL-equivalent to f, g, respectively, and

- the path $h(\lambda) = (1-\lambda)\widehat{f} + \lambda\widehat{g}$, with $\lambda \in [0,1]$, is such that
  - $h(\lambda)$ belongs to $\mathscr{F}^0(\mathscr{M}) \cup \mathscr{F}^1(\mathscr{M})$ for every $\lambda \in [0,1]$;
  - if $h(\lambda)$ intersects $\mathscr{F}^1(\mathscr{M})$, then it does so transversally and for finitely many values of λ.

With these preliminaries set, we will prove the stability theorem by considering a path that connects f to g via $\widehat{f}$, $h(\lambda)$, and $\widehat{g}$ as aforementioned. This path can be split into a finite number of linear sub-paths whose endpoints are such that the stability theorem holds on them, as will be shown in some preparatory lemmas. In conclusion, Theorem 28 will be proven by applying the triangle inequality of the edit distance.

In the following lemmas, f and g belong to $\mathscr{F}^0(\mathscr{M})$ and $h : [0,1] \to \mathscr{F}(\mathscr{M})$ denotes their convex linear combination: $h(\lambda) = (1-\lambda)f + \lambda g$.

Lemma 30 $\|h(\lambda') - h(\lambda'')\|_{C^0} = |\lambda' - \lambda''| \cdot \|f - g\|_{C^0}$ *for every* $\lambda', \lambda'' \in [0,1]$.

Proof

$$\begin{aligned}\|h(\lambda') - h(\lambda'')\|_{C^0} &= \|(1-\lambda')f + \lambda' g - (1-\lambda'')f - \lambda'' g\|_{C^0}\\ &= \|(\lambda''-\lambda')f - (\lambda''-\lambda')g\|_{C^0} = |\lambda'-\lambda''| \cdot \|f-g\|_{C^0}.\end{aligned}$$

□

Lemma 31 *If* $h(\lambda) \in \mathscr{F}^0(\mathscr{M})$ *for every* $\lambda \in [0,1]$, *then* $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \le \|f - g\|_{C^0}$.

Proof The claim can be proved in the same way as [12, Prop. 5.4]. We review here the sequence of facts that, taken together, yield the proof. First, we have the stability of critical values of simple Morse functions under sufficiently small perturbations. Formally:

Statement 1 (cf. [12, Lem. 5.1]): Let $f \in \mathscr{F}^0(\mathscr{M})$ and let c be a critical value of f. Then, there exists a real number $\delta(f,c) > 0$ such that each $g \in \mathscr{F}^0$ with $\|f - g\|_{C^0} \le \delta(f,c)$ admits at least one critical value c' with $|c - c'| \le \|f - g\|_{C^0}$.

Next, we observe that Morse functions on $\mathscr{M}$ have finitely many critical values because $\mathscr{M}$ is compact. Moreover, all functions in a sufficiently small neighborhood of a simple Morse function are RL-equivalent so that we can apply Proposition 13(1). These observations lead to the second result.

Statement 2 (cf. [12, Lem. 5.2]): For $f \in \mathscr{F}^0(\mathscr{M})$, a real number $\delta(f) > 0$ exists such that, for every $g \in N(f, \delta(f))$, there is an edge and vertex-order preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ with $\max_{v \in V(\Gamma_f)} |\ell_f(v) - \ell_g(\Phi(v))| \le \|f - g\|_{C^0}$.

In other words, for functions in a sufficiently small neighborhood of a simple Morse function, the labeled Reeb graphs can be obtained from each other by deformations of R-type whose cost is also small. Hence, the third fact follows.

Statement 3 (cf. [12, Thm. 5.3]): For $f \in \mathscr{F}^0(\mathscr{M})$, there exists a real number $\delta(f) > 0$ such that, for every $g \in N(f, \delta(f))$, it holds that $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \le \|f - g\|_{C^0}$.

Finally, let us now consider a linear path $h : [0,1] \to \mathscr{F}^0(\mathscr{M})$. By the compactness of $[0,1]$, we can decompose the path h into sufficiently small arcs so that each of them belongs to some neighborhood where Statement 3 holds. Thus, we can apply Statement 3 to the functions at the endpoints of these arcs. In conclusion, the claim follows by the linearity of h and Lemma 30. □

Lemma 32 *Let $h(\lambda)$ intersect $\mathscr{F}^1(\mathscr{M})$ only at $h(\overline{\lambda})$, $0 < \overline{\lambda} < 1$, and transversely. Then, for every real value $\delta > 0$, two real numbers λ', λ'' exist, with $0 < \lambda' < \overline{\lambda} < \lambda'' < 1$, such that*

$$d_E((\Gamma_{h(\lambda')}, \ell_{h(\lambda')}), (\Gamma_{h(\lambda'')}, \ell_{h(\lambda'')})) \leq \delta.$$

Proof We denote $h(\overline{\lambda})$ simply by $\overline{h}$. The proof is in three steps.

Step 1. The first step is to construct a deformation $G : \mathbb{R} \times \mathscr{M} \to \mathbb{R}$ of $\overline{h}$, such that $G(0, \cdot) = \overline{h}$ and, for each $\delta > 0$,

$$d_E((\Gamma_{G(-s,\cdot)}, \ell_{G(-s,\cdot)}), (\Gamma_{G(s,\cdot)}, \ell_{G(s,\cdot)})) \leq \delta/3$$

for any s with $|s|$ sufficiently small.

To this end, we consider separately the case when $\overline{h} \in \mathscr{F}^1_\alpha(\mathscr{M})$ and the case when $\overline{h} \in \mathscr{F}^1_\beta(\mathscr{M})$.

Case $\overline{h} \in \mathscr{F}^1_\alpha(\mathscr{M})$: Let $\overline{p}$ be the sole degenerate critical point of $\overline{h}$. Let (x, y) be a coordinate system around $\overline{p}$ such that the canonical expression of $\overline{h}$ is $\overline{h}(x, y) = \overline{h}(\overline{p}) \pm x^2 + y^3$.

Let $\omega : \mathscr{M} \to [0, 1]$ be a smooth function equal to 1 in a neighborhood of $\overline{p}$, which decreases moving from $\overline{p}$, and whose support is contained in the chosen coordinate chart around $\overline{p}$. Finally, for $s \in \mathbb{R}$, let $G(s, \cdot) : \mathscr{M} \to \mathbb{R}$ be the function equal to $\overline{h}$ away from $\overline{p}$, and defined by $G(s, (x, y)) = \overline{h}(x, y) - s \cdot \omega(x, y) \cdot y$ near $\overline{p}$. For $s < 0$, $G(s, \cdot)$ has no critical points in the support of ω and is equal to $\overline{h}$ everywhere else; for $s > 0$, $G(s, \cdot)$ has exactly two critical points in the support of ω, precisely $p_1 = \left(0, -\sqrt{\frac{s}{3}}\right)$ and $p_2 = \left(0, \sqrt{\frac{s}{3}}\right)$, and is equal to $\overline{h}$ everywhere else (see Fig. 8). Therefore, for every $s > 0$ sufficiently small, the labeled Reeb graph of $G(-s, \cdot)$ can be transformed into that of $G(s, \cdot)$ by an elementary deformation T of B-type. Analogously, the labeled Reeb graph of $G(s, \cdot)$ can be transformed into that of $G(-s, \cdot)$ by an elementary deformation T of D-type. In any case, by Definition 24 and Proposition 25, the cost of T is

$$\begin{aligned} c(T) &= \frac{\left|\overline{h}(\overline{p}) + \left(\sqrt{\frac{s}{3}}\right)^3 - s \cdot \sqrt{\frac{s}{3}} - \left(\overline{h}(\overline{p}) + \left(-\sqrt{\frac{s}{3}}\right)^3 + s \cdot \sqrt{\frac{s}{3}}\right)\right|}{2} \\ &= 2 \cdot \left(\frac{s}{3}\right)^{3/2}. \end{aligned} \tag{5.1}$$

Case $\overline{h} \in \mathscr{F}^1_\beta(\mathscr{M})$: Let $\overline{p}$ and $\overline{q}$ be the critical points of $\overline{h}$ such that $\overline{h}(\overline{p}) = \overline{h}(\overline{q})$. Since $\overline{p}$ is non-degenerate, there exists a suitable local coordinate system (x, y) around $\overline{p}$ in which the canonical expression of $\overline{h}$ is $\overline{h}(x, y) = \overline{h}(\overline{p}) + x^2 + y^2$ if $\overline{p}$ is a minimum, or $\overline{h}(x, y) = \overline{h}(\overline{p}) - x^2 - y^2$ if $\overline{p}$ is a maximum, or $\overline{h}(x, y) = \overline{h}(\overline{p}) \pm x^2 \mp y^2$ if $\overline{p}$ is a saddle point. Let $\omega : \mathscr{M} \to [0, 1]$ be a smooth function equal to 1 in a

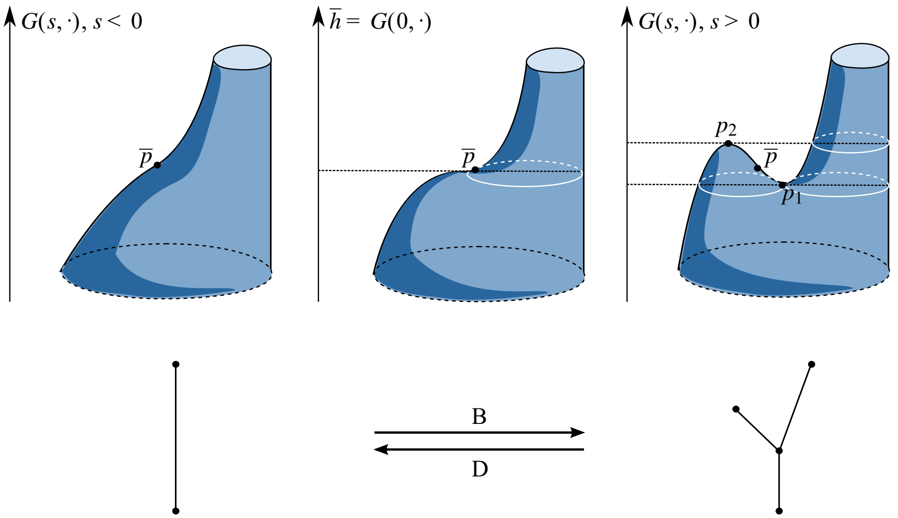


Fig. 8 *Center* a function $\overline{h}\in\mathscr{F}^1_\alpha(\mathscr{M})$; *left–right* the deformation $G(s,\cdot)$ for $s<0$ and $s>0$ and, *below*, the associated Reeb graphs

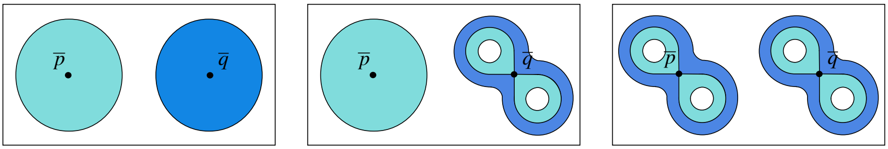


Fig. 9 Two critical points in different connected components of the same critical level. The *dark* (resp. *light*) *regions* correspond to points below (resp. above) this critical level. Possibly inverting the *colors* of one or both the components, we have all the possible cases

neighborhood of $\overline{p}$, which decreases moving from $\overline{p}$, and whose support is contained in the coordinate chart around $\overline{p}$ in which $\overline{h}$ has one of the above expressions. Finally, for $s\in\mathbb{R}$, let $G(s,\cdot):\mathscr{M}\to\mathbb{R}$ be the function equal to $\overline{h}$ away from $\overline{p}$, and defined by $G(s,(x,y))=\overline{h}(x,y)+s\cdot\omega(x,y)$ near $\overline{p}$. For every $s\in\mathbb{R}$, with $|s|$ sufficiently small, $G(s,\cdot)$ has the same critical points, with the same indices, as $\overline{h}$. As for critical values, they are the same as well, apart from the value taken at $\overline{p}$: $G(s,\overline{p})=\overline{h}(\overline{p})+s$. We now distinguish the following two cases:

(1) the points $\overline{p}$ and $\overline{q}$ belong to two different connected components of $\overline{h}^{-1}(\overline{h}(\overline{p}))$;
(2) the points $\overline{p}$ and $\overline{q}$ belong to the same connected component of $\overline{h}^{-1}(\overline{h}(\overline{p}))$.

Case (1) is illustrated in Fig. 9. The points $\overline{p}$ and $\overline{q}$ can be of index 0, 1, or 2, and the corresponding vertices in the Reeb graph are not adjacent. Hence, for every $s>0$ sufficiently small, the labeled Reeb graphs of $G(-s,\cdot)$ and $G(s,\cdot)$ can be obtained one from the other through an elementary deformation T of R-type (see, e.g., Fig. 10).

Case (2) is illustrated in Fig. 11. Necessarily, the points $\overline{p}$ and $\overline{q}$ are saddle points. For $\varepsilon>0$ sufficiently small, the connected component of $\overline{h}^{-1}([\overline{h}(\overline{p})-\varepsilon,\overline{h}(\overline{p})+\varepsilon])$ containing $\overline{p}$ and $\overline{q}$ is obtained by attaching strips to suitable neighborhoods of $\overline{p}$

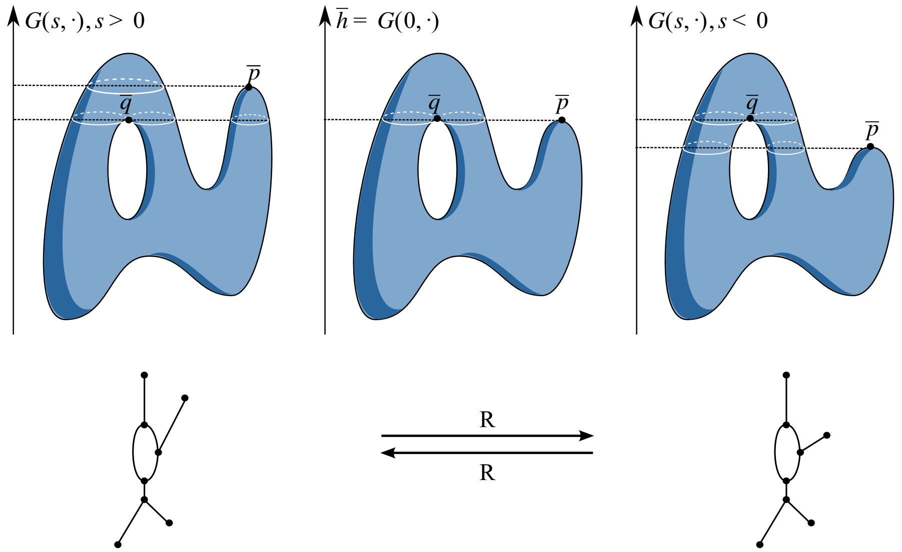


Fig. 10 *Center* a function $\overline{h} \in \mathscr{F}^1_\beta(\mathscr{M})$ as in case (1) of the proof of Lemma 32. *Left–right* the deformation $G(s,\cdot)$ for $s > 0$ and $s < 0$, and, *below*, the associated Reeb graphs

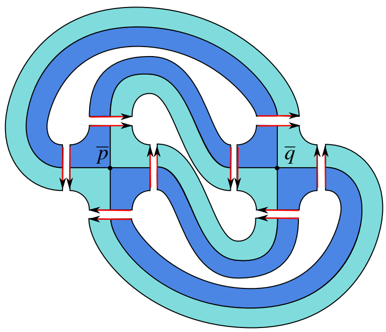


Fig. 11 If $\overline{p}$ and $\overline{q}$ belong to the same connected component of $\overline{h}^{-1}(\overline{h}(\overline{p}))$, then the cross-shaped neighborhoods of $\overline{p}$ and $\overline{q}$ defined by $|x^2 - y^2| \leq \varepsilon$ and $x^2y^2 \leq \varepsilon$ are connected by strips in such a way that the increasing direction of $\overline{h}$ (represented by *arrows*) is preserved

and $\overline{q}$. Doing so in such a way that the resulting surface is compact and orientable gives rise to a finite list of possible configurations, up to diffeomorphisms. This list is summarized in Fig. 12 (see also the orientable cases of complexity 2 in [5, Table 2.1]). Hence, the following elementary deformations need to be analyzed:

– If $\overline{p}$ and $\overline{q}$ are as in Fig. 12a, then, for every $s > 0$ sufficiently small, the labeled Reeb graphs of $G(-s,\cdot)$ and $G(s,\cdot)$ can be obtained one from the other through an elementary deformation T of K_1-type (see, e.g., Fig. 13).
– If $\overline{p}$ and $\overline{q}$ are as in Fig. 12b, then, for every $s > 0$ sufficiently small, the labeled Reeb graphs of $G(-s,\cdot)$ and $G(s,\cdot)$ can be obtained one from the other through an elementary deformation T of K_3- or K_2-type (see, e.g., Fig. 14).
– If $\overline{p}$ and $\overline{q}$ are as in Fig. 12c or d, then, for every $s > 0$ sufficiently small, the labeled Reeb graphs of $G(-s,\cdot)$ and $G(s,\cdot)$ can be obtained one from the other through an elementary deformation T of R-type (see, e.g., Figs. 15, 16).

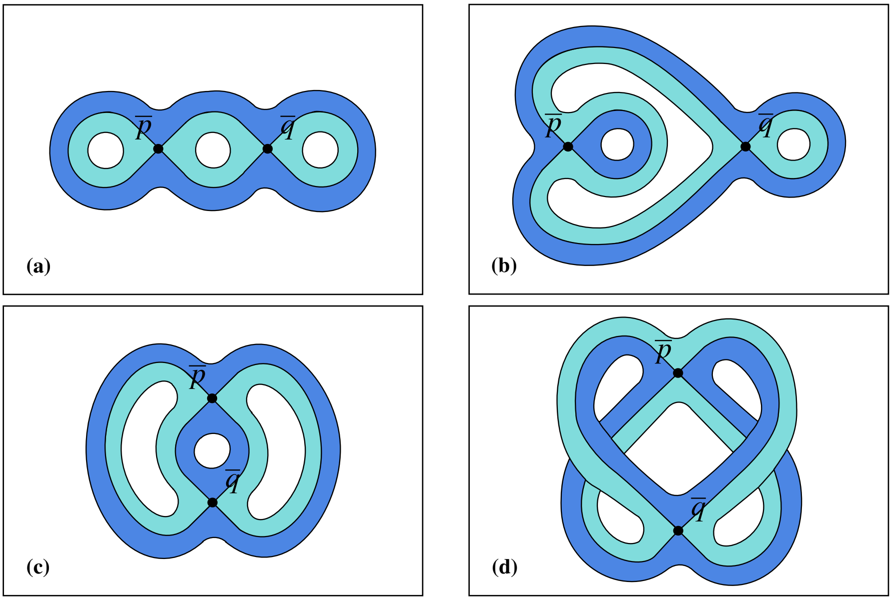


Fig. 12 Two critical points in the same connected component of the same critical level. The *dark* (resp. *light*) *regions* correspond to points below (resp. above) this critical level. Possibly inverting the *colors* of this component, we have all the possible cases

In all the cases, for every $s > 0$ sufficiently small, the cost of the considered deformation T is:

$$c(T) = |\overline{h}(\overline{p}) - s - (\overline{h}(\overline{p}) + s)| = 2s. \tag{5.2}$$

In conclusion, from equalities (5.1) and (5.2), for every $s > 0$ sufficiently small, we get

$$d_E((\Gamma_{G(-s,\cdot)}, \ell_{G(-s,\cdot)}), (\Gamma_{G(s,\cdot)}, \ell_{G(s,\cdot)})) \le \max\left\{2 \cdot \left(\frac{s}{3}\right)^{3/2}, 2s\right\}.$$

Thus, for every $\delta > 0$, we can always take a value $|s|$ sufficiently small to imply the following inequality:

$$d_E((\Gamma_{G(-s,\cdot)}, \ell_{G(-s,\cdot)}), (\Gamma_{G(s,\cdot)}, \ell_{G(s,\cdot)})) \le \delta/3. \tag{5.3}$$

Step 2. As a second step, we consider a different deformation $F : \mathbb{R} \times \mathscr{M} \to \mathbb{R}$ of $\overline{h}$ defined by $F(\mu, \cdot) = \overline{h} + \mu\big(\widetilde{h} - \overline{h}\big)$, with $\widetilde{h} = h(\widetilde{\lambda})$ for some fixed $\widetilde{\lambda} \neq \overline{\lambda}$ sufficient close to $\overline{\lambda}$, and prove that, for each $\delta > 0$, there exists $\mu_s \in \mathbb{R}$ such that

$$d_E((\Gamma_{G(s,\cdot)}, \ell_{G(s,\cdot)}), (\Gamma_{F(\mu_s,\cdot)}, \ell_{F(\mu_s,\cdot)})) \le \delta/3. \tag{5.4}$$

for any s with $|s|$ small enough.

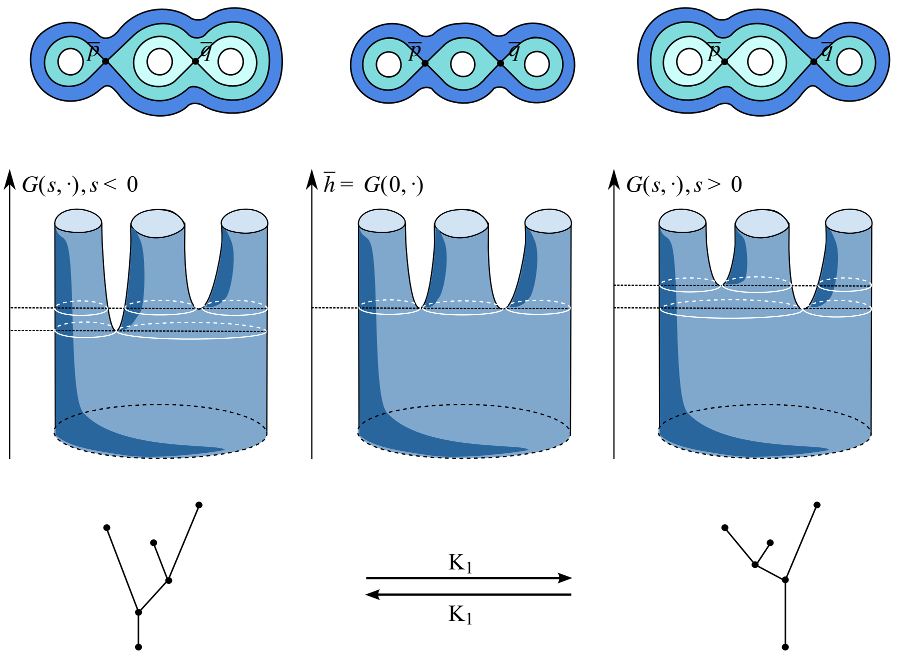


Fig. 13 *Center* a function $\overline{h} \in \mathscr{F}^1_\beta(\mathscr{M})$ as in case (2) with $\overline{p}, \overline{q}$ as in Fig. 12a. *Left–right* the deformation $G(s,\cdot)$ for $s<0$ and $s>0$, and, *below*, the associated Reeb graphs

Since $h(\lambda)$ intersects $\mathscr{F}^1(\mathscr{M})$ transversely at $\overline{h}$, for any fixed $\widetilde{\lambda} \neq \overline{\lambda}$ sufficient close to $\overline{\lambda}$, $\widetilde{h}-\overline{h}$ is a basis for the complementary of the orbit of $\overline{h}$ in $\mathscr{F}(\mathscr{M})$. Recall that F is set to be the deformation of $\overline{h}$ defined by $F(\mu,\cdot)=\overline{h}+\mu(\widetilde{h}-\overline{h})$ and that the natural stratification is locally trivial. Thus, for any g sufficiently close to $\overline{h}$, there exist a value $\mu \in \mathbb{R}$, a diffeomorphism ξ of $\mathscr{M}$, and an orientation preserving diffeomorphism η of $\mathbb{R}$, such that $g=\eta \circ F(\mu,\cdot)\circ \xi$. In particular, taking $g=G(s,\cdot)$, for $|s|$ sufficiently small, we deduce that a value $\mu_s \in \mathbb{R}$ exists such that $G(s,\cdot)=\eta\circ F(\mu_s,\cdot)\circ\xi$. Hence, by Proposition 13, for any $s\in\mathbb{R}$ with $|s|$ sufficiently small, $F(\mu_s,\cdot)$ has a Reeb graph isomorphic to that of $G(s,\cdot)$. Moreover, the difference of the labels at pairs of corresponding vertices in the Reeb graphs of $F(\mu_s,\cdot)$ and $G(s,\cdot)$ continuously depends on s, and is 0 for $s=0$. Therefore, for every $\delta>0$, taking $|s|$ sufficiently small, it is possible to transform the labeled Reeb graph of $F(\mu_s,\cdot)$ into that of $G(s,\cdot)$, or vice versa, by a deformation of R-type whose cost is not greater than $\delta/3$. This proves inequality (5.4).

Step 3. As a final step, we prove that, for each $\delta>0$, setting $\lambda'=\overline{\lambda}+\mu_s(\widetilde{\lambda}-\overline{\lambda})$ and $\lambda''=\overline{\lambda}+\mu_{-s}(\widetilde{\lambda}-\overline{\lambda})$ with $|s|$ sufficiently small, we have

$$d_E((\Gamma_{h(\lambda')},\ell_{h(\lambda')}),(\Gamma_{h(\lambda'')},\ell_{h(\lambda'')}))\le\delta.$$

Indeed, from inequalities (5.3) and (5.4), and applying the triangle inequality, we see that, for each $\delta>0$, there exists a sufficiently small $s>0$ such that the distance between the labeled Reeb graphs of $F(\mu_s,\cdot)$ and $F(-\mu_s,\cdot)$ is not greater than δ.

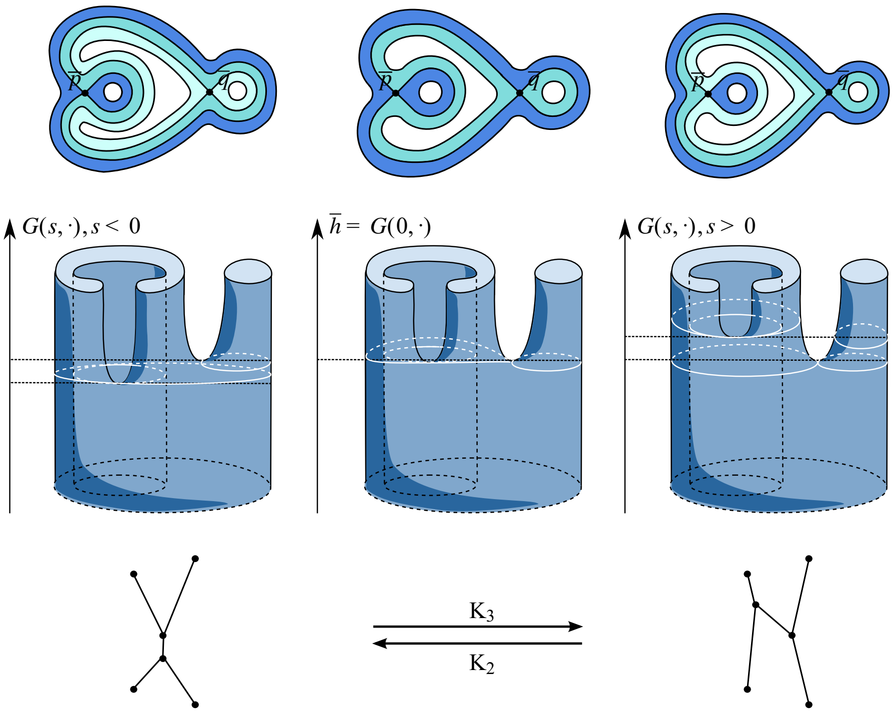


Fig. 14 *Center* a function $\overline{h} \in \mathscr{F}^1_\beta(\mathscr{M})$ as in case (2) with $\overline{p}, \overline{q}$ as in Fig. 12b. *Left–right* the deformation $G(s,\cdot)$ for $s<0$ and $s>0$, and, *below*, the associated Reeb graphs

Recalling that $F(s,\cdot) = \overline{h} + s(\widetilde{h} - \overline{h})$, by the linearity of h we get $h(\lambda') = F(\mu_s,\cdot)$ and $h(\lambda'') = F(\mu_{-s},\cdot)$, yielding the claim. □

Lemma 33 *If* $h(\lambda)$ *belongs to* $\mathscr{F}^0(\mathscr{M})$ *for every* $\lambda \in [0,1]$ *except for one value* $0 < \overline{\lambda} < 1$ *at which* h *transversely intersects* $\mathscr{F}^1(\mathscr{M})$*, then* $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \leq \|f-g\|_{C^0}$.

Proof By Lemma 32, for every real number $\delta > 0$, we can find two values $0 < \lambda' < \overline{\lambda} < \lambda'' < 1$ such that $d_E((\Gamma_{h(\lambda')},\ell_{h(\lambda')}),(\Gamma_{h(\lambda'')},\ell_{h(\lambda'')})) \leq \delta$.

Applying the triangle inequality, we have:

$$\begin{aligned} d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \leq\ & d_E((\Gamma_f,\ell_f),(\Gamma_{h(\lambda')},\ell_{h(\lambda')})) \\ & + d_E((\Gamma_{h(\lambda')},\ell_{h(\lambda')}),(\Gamma_{h(\lambda'')},\ell_{h(\lambda'')})) \\ & + d_E((\Gamma_{h(\lambda'')},\ell_{h(\lambda'')}),(\Gamma_g,\ell_g)). \end{aligned}$$

Moreover, we have

$$d_E((\Gamma_f,\ell_f),(\Gamma_{h(\lambda')},\ell_{h(\lambda')})) \leq \|f-h(\lambda')\|_{C^0} = \lambda' \cdot \|f-g\|_{C^0},$$

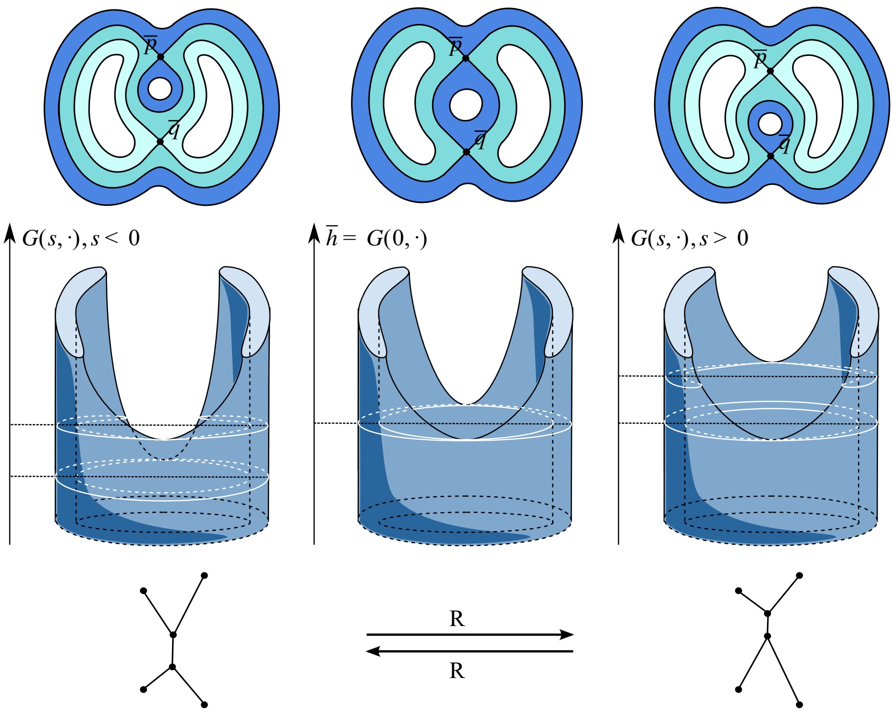


Fig. 15 *Center* a function $\overline{h} \in \mathscr{F}^1_\beta(\mathscr{M})$ as in case (2) with $\overline{p}, \overline{q}$ as in Fig. 12c. *Left–right* the deformation $G(s,\cdot)$ for $s<0$ and $s>0$, and, *below*, the associated Reeb graphs

and

$$d_E((\Gamma_{h(\lambda'')}, \ell_{h(\lambda'')}), (\Gamma_g, \ell_g)) \le \|h(\lambda'') - g\|_{C^0} = (1-\lambda'') \cdot \|f-g\|_{C^0},$$

where the inequalities follow from Lemma 31, and equalities from Lemma 30 with $f = h(0)$, $g = h(1)$. Hence,

$$d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \le (1+\lambda'-\lambda'') \cdot \|f-g\|_{C^0} + \delta.$$

In conclusion, since $0 \le 1+\lambda'-\lambda'' \le 1$, we get the inequality $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \le \|f-g\|_{C^0} + \delta$. This yields the claim by the arbitrariness of δ. □

We are now ready to prove the stability Theorem 28.

Proof of Theorem 28 Recall from [16, p. 147] that $\mathscr{F}^0(\mathscr{M})$ is open in $\mathscr{F}(\mathscr{M})$ endowed with the C^2 topology. Thus, for every sufficiently small real number $\delta > 0$, the neighborhoods $N(f,\delta)$ and $N(g,\delta)$ are contained in $\mathscr{F}^0(\mathscr{M})$. Take $\widehat{f} \in N(f,\delta)$ and $\widehat{g} \in N(g,\delta)$ such that the path $h : [0,1] \to \mathscr{F}(\mathscr{M})$, with $h(\lambda) = (1-\lambda)\widehat{f} + \lambda\widehat{g}$, belongs to $\mathscr{F}^0(\mathscr{M})$ for every $\lambda \in [0,1]$, except for at most a finite number n of values, $\mu_1, \mu_2, \ldots, \mu_n$, at which h transversely intersects $\mathscr{F}^1(\mathscr{M})$. We begin by proving our statement for $\widehat{f}$ and $\widehat{g}$, and then show its validity for f and g. We proceed by induction on n. If $n = 0$ or $n = 1$, the inequality $d_E((\Gamma_{\widehat{f}}, \ell_{\widehat{f}}), (\Gamma_{\widehat{g}}, \ell_{\widehat{g}})) \le \|\widehat{f} - \widehat{g}\|_{C^0}$

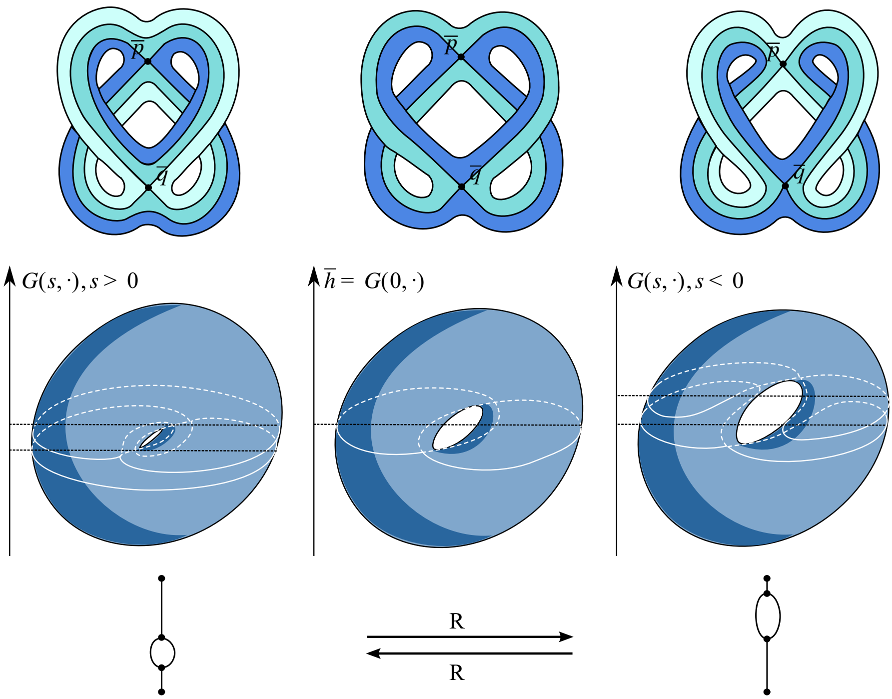


Fig. 16 *Center* a function $\overline{h}\in\mathscr{F}^1_\beta(\mathscr{M})$ as in case (2) with $\overline{p},\overline{q}$ as in Fig. 12d. *Left–right* the deformation $G(s,\cdot)$ for $s<0$ and $s>0$, and, *below*, the associated Reeb graphs

holds because of Lemma 31 or 33, respectively. Let us assume the claim is true for $n\geq 1$, and prove it for $n+1$. Let $0<\mu_1<\lambda_1<\mu_2<\lambda_2<\cdots<\mu_n<\lambda_n<\mu_{n+1}<1$, with $h(0)=\widehat{f}$, $h(1)=\widehat{g}$, $h(\mu_i)\in\mathscr{F}^1(\mathscr{M})$, for every $i=1,\ldots,n+1$, and $h(\lambda_j)\in\mathscr{F}^0(\mathscr{M})$, for every $j=1,\ldots,n$. We consider h as the concatenation of the paths $h^1,h^2:[0,1]\to\mathscr{F}(\mathscr{M})$, defined, respectively, as $h^1(\lambda)=(1-\lambda)\widehat{f}+\lambda h(\lambda_n)$, and $h^2(\lambda)=(1-\lambda)h(\lambda_n)+\lambda\widehat{g}$. The path h^1 transversally intersect $\mathscr{F}^1(\mathscr{M})$ at n values $\mu_1,\ldots,\mu_n$. Hence, by the inductive hypothesis, we have $d_E((\Gamma_{\widehat{f}},\ell_{\widehat{f}}),(\Gamma_{h(\lambda_n)},\ell_{h(\lambda_n)}))\leq\|\widehat{f}-h(\lambda_n)\|_{C^0}$. Moreover, the path h^2 transversally intersect $\mathscr{F}^1(\mathscr{M})$ only at the value μ_{n+1}. Consequently, by Lemma 33, we have $d_E((\Gamma_{h(\lambda_n)},\ell_{h(\lambda_n)}),(\Gamma_{\widehat{g}},\ell_{\widehat{g}}))\leq\|h(\lambda_n)-\widehat{g}\|_{C^0}$. Using the triangle inequality and Lemma 30, we can conclude that:

$$\begin{aligned} d_E((\Gamma_{\widehat{f}},\ell_{\widehat{f}}),(\Gamma_{\widehat{g}},\ell_{\widehat{g}})) &\leq d_E((\Gamma_{\widehat{f}},\ell_{\widehat{f}}),(\Gamma_{h(\lambda_n)},\ell_{h(\lambda_n)}))\\ &\quad+d_E((\Gamma_{h(\lambda_n)},\ell_{h(\lambda_n)}),(\Gamma_{\widehat{g}},\ell_{\widehat{g}}))\\ &\leq\lambda_n\|\widehat{f}-\widehat{g}\|_{C^0}+(1-\lambda_n)\|\widehat{f}-\widehat{g}\|_{C^0}=\|\widehat{f}-\widehat{g}\|_{C^0}. \end{aligned}\tag{5.5}$$

Let us now estimate $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$. By the triangle inequality, we have:

$$\begin{aligned} d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) &\leq d_E((\Gamma_f,\ell_f),(\Gamma_{\widehat{f}},\ell_{\widehat{f}}))\\ &\quad+d_E((\Gamma_{\widehat{f}},\ell_{\widehat{f}}),(\Gamma_{\widehat{g}},\ell_{\widehat{g}}))+d_E((\Gamma_{\widehat{g}},\ell_{\widehat{g}}),(\Gamma_g,\ell_g)). \end{aligned}$$

Since $\widehat{f} \in N(f,\delta) \subset \mathscr{F}^0(\mathscr{M})$ and $\widehat{g} \in N(g,\delta) \subset \mathscr{F}^0(\mathscr{M})$, the following facts hold: (a) for every $\lambda \in [0,1]$, $(1-\lambda)f+\lambda\widehat{f}$, $(1-\lambda)g+\lambda\widehat{g} \in \mathscr{F}^0(\mathscr{M})$; (b) $\|f-\widehat{f}\|_{C^0} \le \delta$ and $\|\widehat{g}-g\|_{C^0} \le \delta$. Hence, from (a) and Lemma 31, we get $d_E((\Gamma_f,\ell_f),(\Gamma_{\widehat{f}},\ell_{\widehat{f}})) \le \|f-\widehat{f}\|_{C^0}$, and $d_E((\Gamma_g,\ell_g),(\Gamma_{\widehat{g}},\ell_{\widehat{g}})) \le \|\widehat{g}-g\|_{C^0}$. Using inequality (5.5) and the fact that

$$\|\widehat{f}-\widehat{g}\|_{C^0} \le \|\widehat{f}-f\|_{C^0} + \|f-g\|_{C^0} + \|g-\widehat{g}\|_{C^0},$$

we deduce that

$$\begin{aligned} d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) &\le \|f-\widehat{f}\|_{C^0} + \|\widehat{f}-\widehat{g}\|_{C^0} + \|\widehat{g}-g\|_{C^0} \\ &\le \|f-g\|_{C^0} + 2(\|f-\widehat{f}\|_{C^0} + \|\widehat{g}-g\|_{C^0}). \end{aligned}$$

Hence, from (b), we have $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \le \|f-g\|_{C^0} + 4\delta$. This yields the conclusion by the arbitrariness of δ.

6 Relationships with Other Stable Distances

In this section, we consider relationships between the edit distance and other distances for shape comparison: the natural pseudo-distance between functions [13], the functional distortion distance between Reeb graphs [2] and the bottleneck distance between persistence diagrams [9]. More precisely, the main result we are going to show states that the natural pseudo-distance between two simple Morse functions f and g and the edit distance between the corresponding Reeb graphs actually coincide (Theorem 38). Therefore, we deduce that the edit distance is a metric (Corollary 39), and that it is more discriminative than the bottleneck distance between persistence diagrams (Corollary 40) and the functional distortion distance between Reeb graphs (Corollary 41), whenever the comparison applies.

The natural pseudo-distance is a dissimilarity measure between any two functions defined on the same compact manifold. It is obtained by minimizing the difference in the functions via a re-parameterization of the manifold [13]. In general, the natural pseudo-distance is only a pseudo-metric. However, it turns out to be a metric in some particular cases such as the case of simple Morse functions on a smooth closed connected surface, considered up to R-equivalence, as proved in [6]. We give the definition in this context.

Definition 34 The *natural pseudo-distance* between R-equivalence classes of simple Morse functions f, g on the same surface $\mathscr{M}$ is defined as

$$d_N([f],[g]) = \inf_{\xi \in \mathscr{D}(\mathscr{M})} \|f - g\circ\xi\|_{C^0},$$

where $\mathscr{D}(\mathscr{M})$ is the set of self-diffeomorphisms of $\mathscr{M}$.

The following Lemmas 35–37 state that the cost of each elementary deformation upper-bounds the natural pseudo-distance.

Lemma 35 *For every elementary deformation $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ of R-type, $c(T) \geq d_N([f], [g])$.*

Proof Since T is of R-type, there exists an edge preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ that also preserves the order of adjacent vertices. Hence, we can apply Proposition 11 to deduce that there is a diffeomorphism $\xi : \mathscr{M} \to \mathscr{M}$ such that

$$\max_{p \in \mathscr{M}} |f(p) - g \circ \xi(p)| = \max_{v \in V(\Gamma_f)} |\ell_f(v) - \ell_g(\Phi(v))|.$$

On one hand we have $d_N([f], [g]) \leq \max_{p \in \mathscr{M}} |f(p) - g \circ \xi(p)|$, on the other hand $c(T) = \max_{v \in V(\Gamma_f)} |\ell_f(v) - \ell_g(\Phi(v))|$. Hence, the claim. □

Lemma 36 *For every elementary deformation $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ of B- or D-type, $c(T) \geq d_N([f], [g])$.*

Proof We prove the assertion only for the case when T is of D-type, because the other case will then follow from $c(T^{-1}) = c(T)$ and the symmetry property of d_N.

By definition of elementary deformation of D-type, T transforms (Γ_f, ℓ_f) into a labeled Reeb graph that differs from (Γ_f, ℓ_f) in that two adjacent vertices, say $p_1, p_2 \in K_f$, have been deleted together with their connecting edge. Otherwise, they have the same vertices, adjacencies and labels. Assuming $f(p_1) = c_1$, $f(p_2) = c_2$, with $c_1 < c_2$, we have $c(T) = \frac{c_2 - c_1}{2}$. We recall that $f^{-1}([c_1, c_2]) \cap K_f = \{p_1, p_2\}$. By Proposition 27, there exists a deformation $S = (S_0, S_1) \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, S_0 being of R-type, S_1 of D-type, such that $c(S) = c(T)$. In particular, as shown in the proof of the same proposition (formulas (4.1) and (4.2)), for every $\varepsilon > 0$ sufficiently small, S_0 and S_1 can be built so that $c(S_0) = \frac{c_2 - c_1}{2} - \varepsilon$ and $c(S_1) = \varepsilon$.

For any h_ε such that $S_0(\Gamma_f, \ell_f) \cong (\Gamma_{h_\varepsilon}, \ell_{h_\varepsilon})$, by Lemma 35 we have $d_N([f], [h_\varepsilon]) \leq c(S_0) = \frac{c_2 - c_1}{2} - \varepsilon$. Thus,

$$d_N([f], [g]) \leq d_N([f], [h_\varepsilon]) + d_N([h_\varepsilon], [g]) \leq \frac{c_2 - c_1}{2} - \varepsilon + d_N([h_\varepsilon], [g]).$$

Therefore, proving that $d_N([h_\varepsilon], [g]) \leq 4\varepsilon$ will yield the claim, by the arbitrariness of $\varepsilon > 0$.

Let W_ε be the connected component of $h_\varepsilon^{-1}([\frac{c_1+c_2}{2} - 2\varepsilon, \frac{c_1+c_2}{2} + 2\varepsilon])$ containing p_1, p_2, and let us assume that ε is so small that $h_\varepsilon^{-1}([\frac{c_1+c_2}{2} - 2\varepsilon, \frac{c_1+c_2}{2} + 2\varepsilon])$ does not contain other critical points of h_ε. By the Cancellation Theorem in [20, Sect. 5], it is possible to define a new simple Morse function $h'_\varepsilon : \mathscr{M} \to \mathbb{R}$ that coincides with h_ε on $\mathscr{M} \setminus W_\varepsilon$, and has no critical points in W_ε. In particular, $(\Gamma_{h'_\varepsilon}, \ell_{h'_\varepsilon}) \cong (\Gamma_g, \ell_g)$, implying that h'_ε and g are R-equivalent. It necessarily holds that

$$d_N([h_\varepsilon], [h'_\varepsilon]) \leq \max_{p \in \mathscr{M}} |h_\varepsilon(p) - h'_\varepsilon(p)| = \max_{p \in W_\varepsilon} |h_\varepsilon(p) - h'_\varepsilon(p)| \leq 4\varepsilon.$$

Moreover, by the R-equivalence of h'_ε and g, we have $d_N([h'_\varepsilon], [g]) = 0$, so that $d_N([h_\varepsilon], [g]) \leq 4\varepsilon$ by the triangle inequality property of d_N. □

Lemma 37 *For every elementary deformation* $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ *of* K_i*-type,* $i = 1, 2, 3$*,* $c(T) \geq d_N([f], [g])$.

Proof For an elementary deformation T of K$_i$-type, $i = 1, 2, 3$, the sets K_f and K_g have the same cardinality, and all but at most two of the critical values of f and g coincide. Let $K_f = \{p_1, \ldots, p_n\}$ and $K_g = \{p'_1, \ldots, p'_n\}$, with $f(p_k) = c_k$, $g(p'_k) = c'_k$ for every $k = 1, \ldots, n$.

Assuming that the points p_1, p_2 correspond to the vertices u_1, u_2 of Γ_f shown in Table 3, it holds that $c_1 < c_2$ if and only if $c'_1 > c'_2$, and $c_k = c'_k$ for $k = 3, \ldots, n$. Let us consider the case $c_1 < c_2$. By Definition 16, $K_f \cap f^{-1}([c_1, c_2]) = \{p_1, p_2\}$ and $K_g \cap g^{-1}([c'_2, c'_1]) = \{p'_1, p'_2\}$.

Since $f, g \in \mathscr{F}^0(\mathscr{M})$, there exist $a, b \in \mathbb{R}$, with $a < b$, such that c_1, c_2 and c'_1, c'_2 are the sole critical values of f and g, respectively, that belong to the interval $[a, b]$. Let us denote by W the connected component of $f^{-1}([a, b])$ containing p_1, p_2. Under our assumptions, we can apply the Preliminary Rearrangement Theorem [20, Thm. 4.1], and deduce that, for some choice of a gradient-like vector field X for f, there exists a Morse function $h : W \to \mathbb{R}$ that has the same gradient-like vector field as f, coincides with $f_{|W}$ near ∂W and is equal to f plus a constant in some neighborhood of p_1 and in some neighborhood of p_2. Moreover, $K_h = K_{f_{|W}}$, $h(p_1) = c'_1$, $h(p_2) = c'_2$. We can extend h to the whole surface by defining

$$\widehat{h}(p) = \begin{cases} f(p) & \text{if } p \in \mathscr{M} \setminus W, \\ h & \text{if } p \in W. \end{cases}$$

Hence, $\widehat{h} \in \mathscr{F}^0(\mathscr{M})$ and $(\Gamma_{\widehat{h}}, \ell_{\widehat{h}}) \cong T(\Gamma_f, \ell_f)$, implying that $\widehat{h}$ is R-equivalent to g. Therefore, by Definition 10, $d_N([f], [g]) = d_N([f], [\widehat{h}])$.

Let us prove that $d_N([f], [\widehat{h}]) \leq c(T)$. We observe that, by the definitions of d_N and $\widehat{h}$, we get:

$$d_N([f], [\widehat{h}]) \leq \|f - \widehat{h}\|_{C^0} = \max_{p \in \mathscr{M}} |f(p) - \widehat{h}(p)| = \max_{p \in W} |f(p) - h(p)|. \quad (6.1)$$

To estimate the value of $\max_{p \in W} |f(p) - h(p)|$, we review the construction of the function h, as given in [20]. Let $\mu : W \to [a, b]$ be a smooth function that is constant on each trajectory of X, zero near the set of points on trajectories going to or from p_1, and one near the set of points on trajectories going to or from p_2. Then the function h can be defined as $h(p) = G(f(p), \mu(p))$, where $G : [a, b] \times [0, 1] \to [a, b]$ is a smooth function defined as $G(x, t) = (1-t) \cdot G(x, 0) + t \cdot G(x, 1)$, with the following properties (see also Fig. 17):

- $\frac{\partial G}{\partial x}(x, 0) = 1$ for x in a neighborhood of c_1 (in particular $G(x, 0) = x + c'_1 - c_1$ for x in a neighborhood of c_1), $\frac{\partial G}{\partial x}(x, 1) = 1$ for x in a neighborhood of c_2 (in particular $G(x, 1) = x + c'_2 - c_2$ for x in a neighborhood of c_2);
- For all x and t, $G(x, t)$ monotonically increases from a to b as x increases from a to b;
- $G(x, t) = x$ for x near to a or b and for every $t \in [0, 1]$.

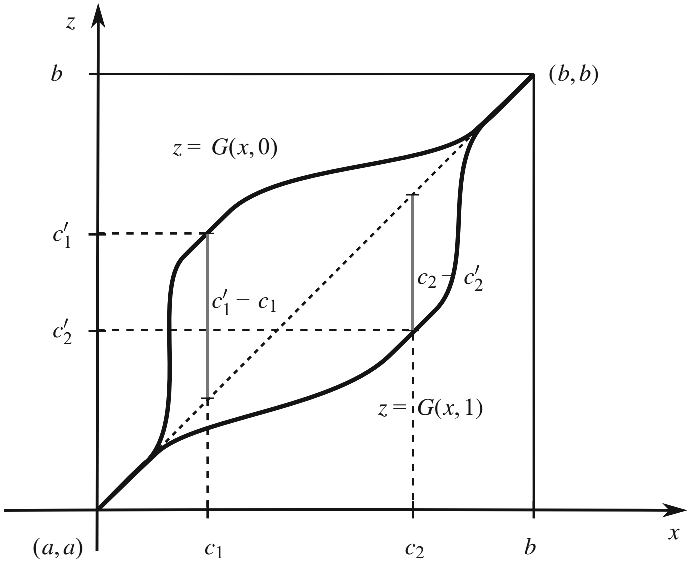


Fig. 17 The function G introduced in [20] and used in the proof of Lemma 37

By the construction of h and the inequality (6.1), we have:

$$\begin{aligned} d_N([f],[\widehat{h}]) &\le \max_{p\in W} |f(p)-G(f(p),\mu(p))| \\ &= \max\{|f(p)-G(f(p),0)|,|f(p)-G(f(p),1)|\} \\ &= \max\{|c_1-c_1'|,|c_2-c_2'|\}=c(T). \end{aligned}$$

□

Theorem 38 *Let $f,g\in\mathscr{F}^0(\mathscr{M})$, and (Γ_f,ℓ_f), (Γ_g,ℓ_g) be the associated labeled Reeb graphs. Then $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))=d_N([f],[g])$.*

Proof The inequality $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))\ge d_N([f],[g])$ holds because, for every deformation $T\in\mathscr{T}((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$, $c(T)\ge d_N([f],[g])$. To see this, let $T=(T_1,\ldots,T_n)$, and set $T_i\cdots T_1(\Gamma_f,\ell_f)\cong(\Gamma_{f^{(i)}},\ell_{f^{(i)}})$, $f=f^{(0)}$, $g=f^{(n)}$. From Lemmas 35–37 and the triangle inequality property of d_N, we get

$$c(T)=\sum_{i=1}^n c(T_i)\ge\sum_{i=1}^n d_N([f^{(i-1)}],[f^{(i)}])\ge d_N([f],[g]).$$

Conversely, by Theorem 28, $d_E((\Gamma_f,\ell_f),(\Gamma_{g\circ\xi},\ell_{g\circ\xi}))\le\|f-g\circ\xi\|_{C^0}$ for every $\xi\in\mathscr{D}(\mathscr{M})$. Therefore, from the equality $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))=d_E((\Gamma_f,\ell_f),(\Gamma_{g\circ\xi},\ell_{g\circ\xi}))$, we can conclude that $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))\le\inf_{\xi\in\mathscr{D}(\mathscr{M})}\|f-g\circ\xi\|_{C^0}=d_N([f],[g])$. □

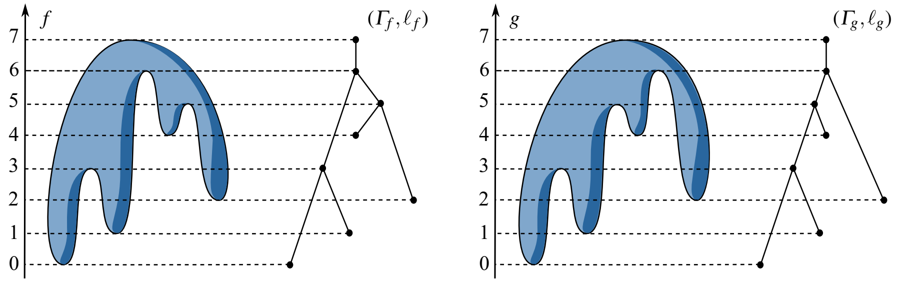


Fig. 18 The example used in the proof of Corollary 40 to show that the edit distance between labeled Reeb graphs can be more discriminative than the bottleneck distance between persistence diagrams whenever the same functions are considered

Corollary 39 *For every $f, g \in \mathscr{F}^0(\mathscr{M})$, the edit distance between the associated labeled Reeb graphs is a metric on the set of isomorphism classes of labeled Reeb graphs.*

Proof The claim is an immediate consequence of Theorem 38 together with [6, Thm. 4.2], which states that the natural pseudo-distance is actually a metric on the space $\mathscr{F}^0(\mathscr{M})$. □

Corollary 40 *For every $f, g \in \mathscr{F}^0(\mathscr{M})$, $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \geq d_B(D_f, D_g)$, where d_B denotes the bottleneck distance between the persistence diagrams D_f and D_g of f and g. In some cases, this inequality is strict.*

Proof The inequality $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \geq d_B(D_f, D_g)$ holds because of Theorem 38 and the fact that the bottleneck distance is a lower bound for the natural pseudo-distance (cf. [8]).

As for the second statement, an example showing that the edit distance between the labeled Reeb graphs of two functions $f, g \in \mathscr{F}^0(\mathscr{M})$ can be strictly greater than the bottleneck distance between the persistence diagrams of f and g is displayed in Fig. 18.

Indeed, f and g have the same persistence diagrams for any homology degree implying that $d_B(D_f, D_g) = 0$, whereas the labeled Reeb graphs are not isomorphic, implying that $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) > 0$. □

Corollary 41 *For every $f, g \in \mathscr{F}^0(\mathscr{M})$, $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \geq d_{FD}(R_f, R_g)$, where d_{FD} denotes the functional distortion distance between the spaces $R_f = \mathscr{M}/\sim_f$ and $R_g = \mathscr{M}/\sim_g$. In some cases, this inequality is strict.*

Proof The inequality $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \geq d_{FD}(R_f, R_g)$ is a consequence of the stability of Reeb graphs with respect to d_{FD} [2, Thm. 4.1], and can be seen in the same way as the second inequality shown in the proof of Theorem 38.

As for the second statement, an example showing that, for two functions $f, g \in \mathscr{F}^0(\mathscr{M})$, $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ can be strictly greater than $d_{FD}(R_f, R_g)$ is displayed in Fig. 19.

In this case, $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = a$, because a is both the cost of the deformation T of R-type that changes the vertex label c_i into $c_i + a$, $i = 1, 2$, and the value of

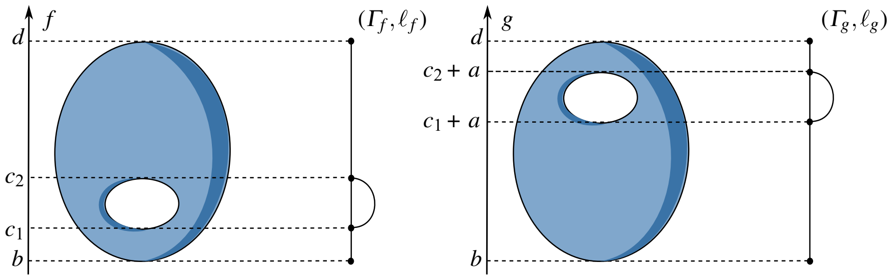


Fig. 19 The example used in the proof of Corollary 41 to show that the edit distance between labeled Reeb graphs can be more discriminative than the functional distortion distance between Reeb graphs whenever the same functions are considered

the bottleneck distance between the 1st homology degree (ordinary) persistence diagrams of f and g. On the other hand, $d_{FD}(R_f, R_g) \leq (c_2 - c_1)/4$ as can be seen by considering any continuous map $\Phi : R_f \to R_g$ that takes each point of R_f to a point of R_g with the same function value, together with any continuous map $\Psi : R_g \to R_f$ that takes each point of R_g to a point of R_f with the same function value. □

7 Conclusion

We have presented a new metric to measure the similarity of Reeb graphs of simple Morse functions on a surface. It is based on edit operations on the vertices, edges and labels to transform Reeb graphs into each other. For this reason, it is called an edit distance. In particular, we have defined the cost of each type of edit operation so that the resulting edit distance between two Reeb graphs is equal to the infimum, over all the finite sequences of edits taking one graph to the other one, of the sum of the edit costs.

We have proved a number of properties for this metric. We have proved that it yields the stability of Reeb graphs with respect to perturbations of the functions. We have also proved that it coincides with the natural pseudo-distance defined between simple Morse functions. This implies that the edit distance is optimal in discriminating functions on surfaces using Reeb graphs. In particular, in the case considered here, it is strictly more discriminative than the functional distorsion distance.

A number of questions remain open and we have not treated them in this paper. The most important one is how to compute the edit distance. Indeed, whereas in some particular cases we can deduce the value of the edit distance from the lower bounds provided by the bottleneck distance of persistence diagrams or the functional distortion distance of Reeb graphs, in general we do not know how to compute it. A second open problem is to which extent the theory developed in this paper for the smooth category can be transported to the piecewise linear category. A third question that would deserve investigation is how to generalize the edit distance to compare functions on non-homeomorphic surfaces as well, and the relationship with the functional distortion distance in that case.

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