

The Reeb Graph Edit Distance is Universal*

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Abstract

We consider the setting of Reeb graphs of piecewise linear functions and study distances between them that are stable, meaning that functions which are similar in the supremum norm ought to have similar Reeb graphs. We define an edit distance for Reeb graphs and prove that it is stable and universal, meaning that it provides an upper bound to any other stable distance. In contrast, via a specific construction, we show that the interleaving distance and the functional distortion distance on Reeb graphs are not universal.

1 Introduction

The concept of Reeb graphs of a Morse function first appeared in [12] and was subsequently applied to the problems in shape analysis in [13, 9]. The literature on Reeb graphs in the computational geometry and computational topology is ever growing (see, e.g., [3, 4] for a discussion and references). The Reeb graph plays a central role in topological data analysis, not least because of the success of Mapper [14], a method providing a discretization of the Reeb graph for a function defined on a point cloud.

A recent line of work has concentrated on questions about identifying suitable notions of distance between Reeb graphs: These include the so called *functional distortion distance* [3], the *interleaving distance* [6], and various *graph edit distances* [8, 7, 2]. There is of course interest in understanding the connection between different existing distances. In this regard, it has been shown in [4] that the functional distortion and the interleaving distances are bi-Lipschitz equivalent. The edit distances defined in [8, 7] for Reeb graphs of curves and surfaces, respectively, are shown to be universal in their respective setting, so the functional distortion and interleaving distances restricted to the same settings are a lower bound for those distances. Moreover, an example in [7] shows that the functional distortion distance can be strictly smaller than the edit distance considered in that paper.

In this paper we concentrate on the setting of PL functions on compact triangulable spaces and in this realm we study the properties of *stability* and *universality* of distances between Reeb graphs. Inspired by a construction of distance between filtered spaces [11], we first construct a novel distance δ_{PL} based on considering joint pullbacks of two given Reeb graphs and prove it satisfies both stability and universality. Via analyzing a specific construction we then prove that neither the functional distortion nor the interleaving distances are universal. Finally, we define two edit-like additional distances between Reeb graphs that reinterpret those appearing in [8, 7, 2] and prove that both are stable and universal. As a consequence, both distances agree with δ_{PL} .

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2 Topological and categorical aspects of Reeb graphs

We start by exploring some topological ideas behind the definition of Reeb graphs. All maps and functions considered in this paper will be assumed to be continuous. Otherwise, we call them set maps and set functions.

2.1 Reeb graphs as quotient spaces

The classical construction of a Reeb graph [12] is given via an equivalence relation as follows:

Definition 2.1. For $f : X \rightarrow \mathbb{R}$ a Morse function on a compact smooth manifold, the *Reeb graph of f* is the quotient space X/\sim_f , with $x \sim_f y$ if and only if x and y belong to the same connected component of some level set $f^{-1}(t)$ (implying $t = f(x) = f(y)$).

While this definition was originally considered in the setting of Morse theory, it does not make explicit use of the smooth structure, and so it can be applied to a quite broad setting. However, some additional assumptions of X and f are justified in order to maintain some of the characteristic properties of Reeb graphs in a generalized setting. With this motivation in mind, we revisit the definition in terms of quotient maps and functions with discrete fibers.

A *quotient map* $p : X \rightarrow Y$ is a surjection such that a set U is open in Y if and only if $p^{-1}(U)$ is open in X . In particular, a surjection between compact Hausdorff spaces is a quotient map by the closed map lemma. A quotient map $p : X \rightarrow Y$ is characterized by the universal property that a set map $\Phi : Y \rightarrow Z$ into any topological space Z is continuous if and only if $\Phi \circ p$ is continuous.

The motivation for considering quotient maps and functions with discrete fibers is explained by the following fact.

Proposition 2.2. Let $f : X \rightarrow \mathbb{R}$ be a function with locally connected fibers, and let $q : X \rightarrow X/\sim_f$ be the canonical quotient map. Then the induced function $\tilde{f} : X/\sim_f \rightarrow \mathbb{R}$ with $f = \tilde{f} \circ q$ has discrete fibers.

Proof. To see that the fibers of \tilde{f} are discrete, we show that any subset S of $\tilde{f}^{-1}(t)$ is closed. Let $T = \tilde{f}^{-1}(t) \setminus S$. Then $q^{-1}(T)$ is a disjoint union of connected components of $f^{-1}(t)$. Since $\tilde{f}^{-1}(t)$ is locally connected, each of its connected components is open in the fiber, and so $q^{-1}(T)$ is open in $f^{-1}(t)$, implying that $q^{-1}(S)$ is closed in $f^{-1}(t)$ and hence in X . Since q is a quotient map, $q^{-1}(S)$ is closed if and only if S is closed, yielding the claim. \square

2.2 Reeb quotient maps and Reeb graphs of piecewise linear functions

We now define a class of quotient maps that leave Reeb graphs invariant up to isomorphism. The main goal is to provide a natural construction for lifting functions $f : X \rightarrow \mathbb{R}$ to spaces Y through a quotient map $Y \rightarrow X$ in a way that yields isomorphic Reeb graphs. To this end, we will define two categories, the category of Reeb domains and the category of Reeb graphs.

Definition 2.3. We define the category **PLReebDom** of (compact triangulable) Reeb domains as follows:

- The objects of **PLReebDom** (Reeb domains) are connected compact triangulable spaces.
- The morphisms of **PLReebDom** (Reeb quotient maps) are surjective piecewise linear maps with connected fibers.

The fact that this is indeed a category will be established in Theorem 2.13.

Definition 2.4. The category of *Reeb graphs*, denoted by **PLReebGrph**, is the category whose objects are Reeb domains R_f endowed with PL functions $\tilde{f} : R_f \rightarrow \mathbb{R}$ with discrete fibers called *Reeb functions*, and whose morphisms between Reeb domains R_f and R_g respectively endowed with Reeb functions \tilde{f} and \tilde{g} are PL maps $\Phi : R_f \rightarrow R_g$ such that $\tilde{g} \circ \Phi = \tilde{f}$.

In particular, the isomorphisms between Reeb graphs are PL homeomorphisms that preserve the function values of the associated Reeb functions. A Reeb graph is actually a *finite topological graph* (a compact triangulable space of dimension at most 1).

Theorem 2.5. Any Reeb graph R_f in **PLReebGrph** is a finite topological graph.

Proof. By definition, \tilde{f} is (simplexwise) linear for some triangulation of R_f . If there were a simplex σ of dimension at least 2 in the triangulation of R_f , then for any x in the interior of σ , the intersection $\sigma \cap \tilde{f}^{-1}(\tilde{f}(x))$ would have to be of at least dimension 1. But this would contradict the assumption that \tilde{f} has discrete fibers. \square

Definition 2.6. Generalizing the classical definition (Definition 2.1), we say that a Reeb graph R_f is the *Reeb graph of* $f : X \rightarrow \mathbb{R}$ if there is a Reeb quotient map $p : X \rightarrow R_f$ such that $f = \tilde{f} \circ p$, where $\tilde{f} : R_f \rightarrow \mathbb{R}$ is the Reeb function of R_f .

The following lemma shows how a transformation $g = \xi \circ f$ of a function f lifts to a Reeb quotient map ζ between the corresponding Reeb graphs.

Lemma 2.7. Assume that $\tilde{f} : R_f \rightarrow \mathbb{R}$, $\tilde{g} : R_g \rightarrow \mathbb{R}$ are Reeb functions, $p_f : X \rightarrow R_f$, $p_g : X \rightarrow R_g$ are Reeb quotient maps, $f = \tilde{f} \circ p_f$, $g = \tilde{g} \circ p_g$, and $\xi : \text{im } f \rightarrow \text{im } g$ is a piecewise linear function such that $g = \xi \circ f$. Then $\zeta = p_g \circ p_f^{-1}$ is a Reeb quotient map from R_f to R_g .

In particular, if ξ is a PL homeomorphism, then so is ζ .

Proof. Let $x \in R_f$, and let $t = \tilde{f}(x)$. Then $C = p_f^{-1}(x)$ is a connected component of $f^{-1}(t)$ by the assumption that p_f is a Reeb quotient map. By commutativity, we have $f^{-1} \subseteq \tilde{f}^{-1} \circ \xi^{-1} \circ \xi = g^{-1} \circ \xi$, and since C is connected, there must be a single $y \in R_g$ with $p_g(C) = \{y\}$. Hence, $\zeta = p_g \circ p_f^{-1}$ is a set map. Moreover, since p_g is continuous and p_f is closed, the map ζ is continuous; since p_g and p_f are PL, the map ζ is PL as well.

Now let $y \in R_g$ and let $s = \tilde{g}(y)$. Similarly to above, $C = p_g^{-1}(y)$ is a connected component of $g^{-1}(s)$. By commutativity, there is $x \in p_f(C) \subseteq R_f$ with $t = \tilde{f}(x) \in \xi^{-1}(s)$. Hence ζ is surjective and the fiber $\zeta^{-1}(y) = p_f(C)$ is connected. \square

Remark 2.8. By Proposition 2.2 and Lemma 2.7, the Reeb graph R_f of $f : X \rightarrow \mathbb{R}$ is isomorphic to X/\sim_f . As a consequence, the Reeb graph R_f together with the Reeb quotient map p is unique up to a unique isomorphism, turning the Reeb graph into a universal property.

We now proceed to prove that Reeb quotient maps are closed under composition. We start by showing that not only the fibers, but more generally all preimages of closed connected sets are connected.

Proposition 2.9. If $p : X \rightarrow Y$ is a Reeb quotient map, then the preimage $p^{-1}(K)$ of a closed connected set $K \subseteq Y$ is connected.

Proof. Assume that K is nonempty; otherwise, the claim holds trivially. Let $p^{-1}(K) = U \cup V$, with U, V nonempty and closed in $p^{-1}(K)$. To show that $p^{-1}(K)$ is connected, it suffices to show that $U \cap V$ is necessarily nonempty.

Because $p^{-1}(K)$ is closed in X , the sets U and V are also closed in X . The images $p(U)$ and $p(V)$ are closed by the closed map lemma, and their union is K . By connectedness of K , their intersection is nonempty. Let $y \in p(U) \cap p(V)$. We have

$$p^{-1}(y) = (p^{-1}(y) \cap U) \cup (p^{-1}(y) \cap V).$$

The subspaces $(p^{-1}(y) \cap U)$ and $(p^{-1}(y) \cap V)$ are closed in $p^{-1}(y)$, and by connectedness of the fiber $p^{-1}(y)$, their intersection must be nonempty. In particular, $U \cap V$ is nonempty. \square

Corollary 2.10. *If $p : X \rightarrow Y$ and $q : Y \rightarrow Z$ are Reeb quotient maps, then the composition $q \circ p : X \rightarrow Z$ is a Reeb quotient map too.*

As mentioned before, the main purpose of Reeb quotient maps is to lift Reeb functions to larger domains while maintaining the same Reeb graph. The following property is a consequence of the above statement:

Corollary 2.11. *Let $f : X \rightarrow \mathbb{R}$ be a function with Reeb graph R_f , and let $q : Y \rightarrow X$ be a Reeb quotient map. Then R_f is also the Reeb graph of $f \circ q : Y \rightarrow \mathbb{R}$.*

Proof. Let $\tilde{f} : R_f \rightarrow \mathbb{R}$ be the Reeb function of R_f and $p : X \rightarrow R_f$ be the Reeb quotient map factoring $f = \tilde{f} \circ p$. Then by Corollary 2.10, R_f is also a Reeb graph for $f \circ q = \tilde{f} \circ (p \circ q) : Y \rightarrow \mathbb{R}$ via the Reeb quotient map $p \circ q : Y \rightarrow R_f$. \square

We now show that Reeb quotient maps are stable under pullbacks.

Proposition 2.12. *Consider the pullback diagram*

$$\begin{array}{ccc} & Y & \\ p_1 \nearrow & & \swarrow p_2 \\ X_1 & & X_2 \\ q_1 \nwarrow & & \nearrow q_2 \\ & X_1 \times_Y X_2 & \end{array}$$

If the map p_1 (resp. p_2) is a Reeb quotient map, then so is the map q_2 (resp. q_1).

Proof. First note that the category of compact triangulable spaces has all pullbacks [15]. For $x_2 \in X_2$, by surjectivity of p_1 there is some $x_1 \in X_1$ such that $p_1(x_1) = p_2(x_2)$. Thus $(x_1, x_2) \in X_1 \times_Y X_2$ and $q_2(x_1, x_2) = x_2$, proving that q_2 is surjective. Moreover, for $x_2 \in X_2$, we have $q_2^{-1}(x_2) = p_1^{-1}(p_2(x_2)) \times \{x_2\}$. By assumption, $p_1^{-1}(p_2(x_2))$ is connected being a fiber of p_1 , implying that $p_1^{-1}(p_2(x_2)) \times \{x_2\}$ is connected. Finally, applying Proposition 2.9 to q_2 , we obtain that the pullback space $X_1 \times_Y X_2$ is connected. The proof for q_1 is analogous. \square

Theorem 2.13. *The Reeb domains and Reeb quotient maps form a finitely complete category, i.e., every finite diagram has a limit.*

Proof. By Corollary 2.10, the Reeb quotient maps are closed under composition and contain the identity maps of Reeb domains, so they form a category. This category has all pullbacks by Proposition 2.12, and the one-point space is a terminal object, so equivalently it has all finite limits [1, Prop. 5.14 and 5.21]. \square

3 Stable and universal distances

Throughout this paper, we will use the term *distance* to describe an extended pseudo-metric $d : X \times X \rightarrow [0, \infty]$ on some collection X . Our main goal is the introduction of a distance between Reeb graphs that is stable and universal in the following sense.

Definition 3.1. We say that a distance d_S on the objects of **PLReebGrph** is *stable* if and only if given any two Reeb graphs R_f and R_g respectively endowed with Reeb functions \tilde{f} and \tilde{g} , for any Reeb domain X with Reeb quotient maps $p_f : X \rightarrow R_f$ and $p_g : X \rightarrow R_g$ we have

$$d_S(R_f, R_g) \leq \|\tilde{f} \circ p_f - \tilde{g} \circ p_g\|_\infty. \quad (\text{S})$$

Note that stability implies that isomorphic Reeb graphs have distance 0. Indeed, an isomorphism of Reeb graphs $\gamma : R_f \rightarrow R_g$ yields $d_S(R_f, R_g) \leq \|\tilde{f} \circ \text{id} - \tilde{g} \circ \gamma\|_\infty = 0$.

Moreover, we say that a stable distance d_U on the objects of **PLReebGrph** is *universal* if and only if for any other stable distance d_S on **PLReebGrph**, we have

$$d_S(R_f, R_g) \leq d_U(R_f, R_g). \quad (\text{U})$$

Remark 3.2. By connectedness of R_f and R_g , there is at least one space X with maps p_f, p_g as needed to define the stability property: $X = R_f \times R_g$, with p_f, p_g the canonical projections. The resulting functions $f = \tilde{f} \circ p_f, g = \tilde{g} \circ p_g : R_f \times R_g \rightarrow \mathbb{R}$ then satisfy $\|f - g\|_\infty = \max(\sup f, \sup g) - \min(\inf f, \inf g)$. In particular, for compact Reeb graphs a stable distance is always finite.

The definition of stability yields the following canonical universal distance.

Definition 3.3. For any two Reeb graphs R_f and R_g endowed with Reeb graph functions \tilde{f} and \tilde{g} , let

$$\delta_{PL}(R_f, R_g) := \inf_{p_f : R_f \leftarrow X \rightarrow R_g : p_g} \|\tilde{f} \circ p_f - \tilde{g} \circ p_g\|_\infty,$$

where X is any Reeb domain, and p_f, p_g are Reeb quotient maps.

Proposition 3.4. The distance δ_{PL} is the largest stable distance on **PLReebGrph**. Hence, δ_{PL} is universal.

Proof. To see that δ_{PL} is a distance, the only non-trivial part is showing the triangle inequality. To this end, given diagrams $p_f : R_f \leftarrow X \rightarrow R_g : p_g$ and $p'_g : R_g \leftarrow Y \rightarrow R_h : p_h$, we can pullback the diagram $p_g : X \rightarrow R_g \leftarrow Y : p'_g$ to obtain the diagram $q_X : X \leftarrow X \times_{R_g} Y \rightarrow Y : q_Y$, where $X \times_{R_g} Y$ is a Reeb domain and q_X, q_Y are Reeb quotient maps by Proposition 2.12. Defining $f = \tilde{f} \circ p_f \circ q_X$, $g = \tilde{g} \circ p_g \circ q_X = \tilde{g} \circ p'_g \circ q_Y$, and $h = \tilde{h} \circ p_h \circ q_Y$, we have

$$\begin{aligned} \delta_{PL}(R_f, R_h) &\leq \|f - h\|_\infty \leq \|f - g\|_\infty + \|g - h\|_\infty \\ &\leq \|\tilde{f} \circ p_f - \tilde{g} \circ p_g\|_\infty + \|\tilde{g} \circ p'_g - \tilde{h} \circ p_h\|_\infty, \end{aligned}$$

where the last inequality holds because $\text{im } q_X \subseteq X$ and $\text{im } q_Y \subseteq Y$. Hence, $\delta_{PL}(R_f, R_h) \leq \delta_{PL}(R_f, R_g) + \delta_{PL}(R_g, R_h)$. By definition of stability, $d_S \leq \delta_{PL}$ for any stable distance d_S defined on the objects of **PLReebGrph**, implying that δ_{PL} is universal. \square

Example 3.5. Consider the one point Reeb graph $*_c$ endowed with the function identical to $c \in \mathbb{R}$. Then, for any Reeb graph R_f endowed with the function \tilde{f} , $\delta_{PL}(R_f, *_c) = \|\tilde{f} - c\|_\infty$.

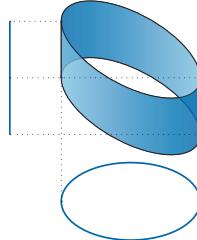
We now consider an example where we can explicitly determine the value of the distance $\delta_{PL}(R_f, R_g)$ between two specific simple Reeb graphs $R_f = \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with $\tilde{f}(x, y) = x$ and $R_g = [-1, 1]$ with $\tilde{g}(t) = t$. The example demonstrates the non-universality of certain distances proposed in the literature. We prove:

Proposition 3.6. $\delta_{PL}(R_f, R_g) = 1$.

The proof of this proposition will be obtained from the two claims below.

Claim 3.7. $\delta_{PL}(R_f, R_g) \leq 1$.

Proof. Consider the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, |2z - x| \leq 1\}$ together with functions $f(x, y, z) = x$ and $g(x, y, z) = z$ defined on C . Then R_f is a Reeb graph of f via the Reeb quotient map



$(x, y, z) \mapsto (x, y)$, and R_g is a Reeb graph of g via the Reeb quotient map $(x, y, z) \mapsto z$. Since we have $|f(c) - g(c)| \leq 1$ for all $c \in C$, this implies that $\delta_{PL}(R_f, R_g) \leq 1$. \square

Claim 3.8. $\delta_{PL}(R_f, R_g) \geq 1$.

Proof. Assume for a contradiction that there is a diagram $p_f : R_f \leftarrow Z \rightarrow R_g : p_g$ of Reeb quotient maps such that, letting $\hat{f} = \tilde{f} \circ p_f$ and $\hat{g} = \tilde{g} \circ p_g$, we have $\|\hat{f} - \hat{g}\|_\infty = \delta < 1$. We then observe the following:

- $\hat{g}^{-1}(0) \subseteq \hat{f}^{-1}([- \delta, + \delta])$.
- $\hat{f}^{-1}([- \delta, + \delta])$ consists of two circular arcs homeomorphic by \tilde{f} to $[- \delta, + \delta]$, and thus, by Proposition 2.9, $\hat{f}^{-1}([- \delta, + \delta])$ consists of two connected components C_+ and C_- as well.
- For both components we have $\hat{f}(C_\pm) = [- \delta, + \delta]$, and so $\|\hat{f} - \hat{g}\|_\infty = \delta$ implies that $0 \in \hat{g}(C_\pm)$. Thus $\hat{g}^{-1}(0) \cap C_- \neq \emptyset$ and $\hat{g}^{-1}(0) \cap C_+ \neq \emptyset$.

But since $\hat{g}^{-1}(0) \subseteq C_- \sqcup C_+$, this would contradict the assumption that the fiber $\hat{g}^{-1}(0)$ is connected. \square

The current example illustrates that the *functional distortion distance* introduced in [3] and the *interleaving distance* introduced in [6] both fail to be universal. We first recall the definition of the former. For any Reeb graph R_f with Reeb function \tilde{f} , consider the metric on R_f given by

$$d_f(x, y) = \inf\{b - a \mid x, y \text{ are in the same connected component of } \tilde{f}^{-1}([a, b])\}.$$

Given maps $\phi : R_f \rightarrow R_g$ and $\psi : R_g \rightarrow R_f$, we write

$$G(\phi, \psi) = \{(p, \phi(p)) : p \in R_f\} \cup \{(\psi(q), q) : q \in R_g\}$$

for the correspondences induced by the two maps. The functional distortion distance is

$$d_{FD}(R_f, R_g) = \inf_{\phi, \psi} (\max \{ \sup_{(p, q), (p', q') \in G(\phi, \psi)} \frac{1}{2} |d_f(p, p') - d_g(q, q')|, \|f - g \circ \phi\|_\infty, \|f \circ \psi - g\|_\infty \}).$$

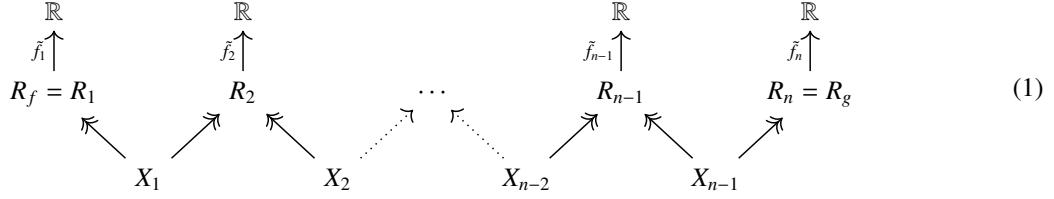
To see that neither the functional distortion distance nor the interleaving distance are universal we establish:

Proposition 3.9. $d_I(R_f, R_g) \leq d_{FD}(R_f, R_g) \leq \frac{1}{2}$.

Proof. By [4, Lemma 8], the functional distortion distance is an upper bound on the interleaving distance on Reeb graphs [6], and so it is enough to prove that $d_{FD}(R_f, R_g) \leq \frac{1}{2}$. To this end consider the maps $\phi : R_f \rightarrow R_g$, $(x, y) \mapsto x$ and $\psi : R_g \rightarrow R_f$, $t \mapsto (t, \sqrt{1-t^2})$. For every pair $p, p' \in R_f$ one can verify that $|\tilde{f}(p) - \tilde{f}(p')| \leq d_f(p, p') \leq |\tilde{f}(p) - f(p')| + 1$, while for every pair $q, q' \in R_g$, we have $d_g(q, q') = |\tilde{g}(q) - \tilde{g}(q')|$. This implies that for any two corresponding pairs $(p, q), (p', q') \in G(\phi, \psi)$, we have $|d_f(p, p') - d_g(q, q')| \leq 1$, and thus $D(\phi, \psi) \leq \frac{1}{2}$. Moreover, both maps preserve function values, so $d_{FD}(R_f, R_g) \leq \frac{1}{2}$. \square

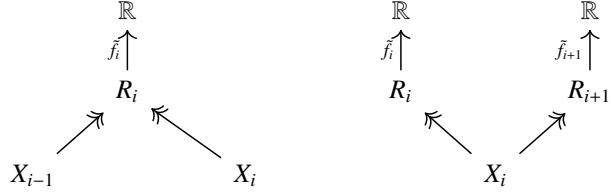
4 The topological and graph edit distances

Given a pair of Reeb graphs R_f, R_g , consider a diagram of the form



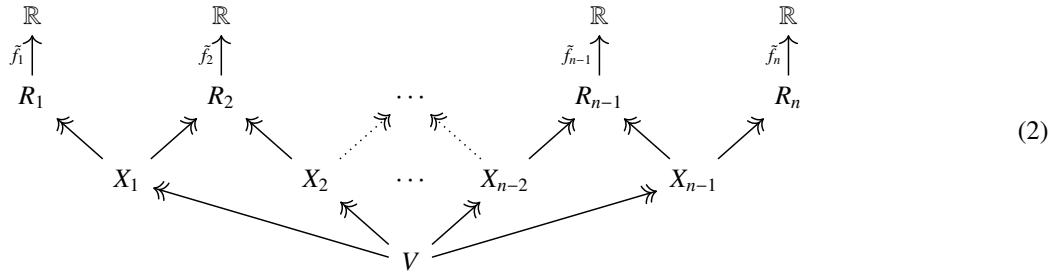
where for $n \in \mathbb{N}$ $\tilde{f}_1, \dots, \tilde{f}_n$ are Reeb functions with $\tilde{f}_1 = \tilde{f}$ and $\tilde{f}_n = \tilde{g}$, and the maps $X_i \rightarrow R_i, R_{i+1}$ for $i = 1, \dots, n-1$, are Reeb quotient maps. We call the diagram a *Reeb zigzag diagram* between R_f and R_g . Observe that, by Remark 3.2, between any two Reeb graphs R_f and R_g there exists a Reeb zigzag diagram.

A Reeb zigzag diagram can be regarded as being composed of the following elementary diagrams:



This way, we may think of a Reeb zigzag diagram as a sequence of operations transforming the R_f into R_g . The elementary diagram on the left corresponds to an *edit* operation: the space X_{i-1} , together with a function $X_{i-1} \rightarrow \mathbb{R}$ with Reeb graph R_i , is transformed to another space X_i , with a function $X_i \rightarrow \mathbb{R}$ having the same Reeb graph R_i . The elementary diagram on the right corresponds to a *relabel* operation: the function on X_i with Reeb graph R_i is transformed to another function with Reeb graph R_{i+1} . The idea of edit and relabel operations is inspired by previous work on edit distances for Reeb graphs [7, 2].

In order to define an edit distance using Reeb zigzag diagrams, we need to assign a cost to a given Reeb zigzag diagram between R_f and R_g . To that end, we can consider a cone from a space V by Reeb quotient maps $V \rightarrow R_i$:



We call this diagram a *Reeb cone*. Any Reeb zigzag diagram admits such a cone. Indeed, the category **PLReebDom** has all finite limits by Theorem 2.13, and the limit over the lower part of diagram (1), consisting of Reeb quotient maps, yields a limit over the whole diagram. In a Reeb cone, by commutativity, each of the Reeb functions \tilde{f}_i induces a unique function $f_i : V \rightarrow \mathbb{R}$. By Corollary 2.11, the Reeb graph of f_i is isomorphic to R_i . This way, we pull back the individual functions \tilde{f}_i to functions f_i on a common space with the same Reeb graphs, where they can be compared using the supremum norm.

Using these ideas, we can now introduce distances on the objects of **PLReebGrph**, and proceed to prove that they are stable and universal.

Definition 4.1. Given a Reeb cone from a space V as in (2), we define the *spread* of the functions $(f_i)_{i=1,\dots,n} : V \rightarrow \mathbb{R}$, as the function $s^V : V \rightarrow \mathbb{R}$, $x \mapsto \max_{i=1,\dots,n} f_i(x) - \min_{j=1,\dots,n} f_j(x)$. Moreover, for a Reeb zigzag diagram Z between R_f and R_g as in (1), consider the limit of Z , denoted by L . The *cost* of the Reeb zigzag diagram Z is the supremum norm of the spread s^L ,

$$c_Z := \|s^L\|_\infty = \sup_{x \in L} \left(\max_i f_i(x) - \min_j f_j(x) \right).$$

Definition 4.2. We define the *(PL) edit distance* δ_{ePL} between Reeb graphs R_f and R_g in **PLReebGrph** as the infimum cost of all Reeb zigzag diagrams Z in **PLReebDom** between R_f and R_g :

$$\delta_{ePL}(R_f, R_g) = \inf_Z c_Z.$$

Moreover, we define the *graph edit distance* δ_{eGraph} between Reeb graphs R_f and R_g in **PLReebGrph** analogously by restricting the infimum to Reeb zigzag diagrams Z where all the spaces X_i and R_i are finite topological graphs, and all the maps are PL.

Thus, on **PLReebGrph** we have two edit distances, satisfying

$$\delta_{ePL} \leq \delta_{eGraph}. \quad (3)$$

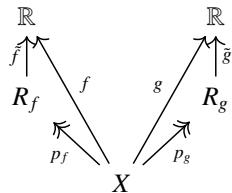
The Reeb graph edit distance δ_{eGraph} is a categorical reformulation of the definition given in [2]. The main goal is to prove that these distances have the stability and universality properties (Propositions 4.4 and 4.5, Theorem 5.6, and Corollary 5.7). As a consequence, whenever applicable, they actually coincide with the canonical universal distance δ_{PL} defined in Definition 3.3:

Corollary 4.3. $\delta_{PL} = \delta_{ePL} = \delta_{eGraph}$.

The proofs of stability and universality for δ_{ePL} are straightforward and are given next. The verification of stability and universality for δ_{eGraph} follows in Section 5.

Proposition 4.4. δ_{ePL} is a stable distance.

Proof. Let R_f, R_g be Reeb graphs with Reeb functions \tilde{f} and \tilde{g} . For any space X such that there exist two Reeb quotient maps $p_f : X \rightarrow R_f$ and $p_g : X \rightarrow R_g$, the diagram



is a Reeb zigzag diagram with limit object X . The cost of this Reeb zigzag diagram is exactly $\|f - g\|_\infty$. Hence, $\delta_{ePL}(R_f, R_g) \leq \|f - g\|_\infty$. \square

Our proof of universality of the edit distance is similar to previous universality proofs for the bottleneck distance [5] and for the interleaving distance [10].

Proposition 4.5. δ_{ePL} is a universal distance.

Proof. Let R_f, R_g be Reeb graphs with Reeb functions \tilde{f} and \tilde{g} . Let $\delta_{ePL}(R_f, R_g) = d$. Hence, for any $\varepsilon > 0$, there is a Reeb zigzag diagram Z between $R_f = R_1$ and $R_g = R_n$, with limit L and functions f_i as in Definition 4.1, having cost

$$c_Z = \|s^L\|_\infty = \|\max_i f_i - \min_j f_j\|_\infty \leq d + \varepsilon.$$

Let $p_f : L \rightarrow R_f$ and $p_g : L \rightarrow R_g$ be the induced Reeb quotient maps. If d_S is any other stable distance (cf. Definition 3.1) between R_f and R_g , we have

$$d_S(R_f, R_g) \leq \|\tilde{f} \circ p_f - \tilde{g} \circ p_g\|_\infty \leq \|\max_i f_i - \min_j f_j\|_\infty \leq d + \varepsilon.$$

Since the above holds for all $\varepsilon > 0$, we have $d_S(R_f, R_g) \leq d = \delta_{ePL}(R_f, R_g)$. \square

5 Stability and universality of the Reeb graph edit distance

We now turn to the proof of stability and universality for the Reeb graph edit distance. Recall that, in the case of δ_{eGraph} , the admissible Reeb zigzag diagrams are PL zigzags of finite topological graphs. As mentioned above, the distance δ_{eGraph} is applicable to Reeb graphs of compact triangulable spaces.

Lemma 5.1. *Let $X = |K|$ and let V be the vertex set of K . Let $f, g : X \rightarrow \mathbb{R}$ be PL functions, simplexwise linear on K . Let $\chi : \text{im } f \rightarrow \text{im } g$ be a weakly order preserving PL surjection such that $\chi \circ f(v) = g(v)$ for every vertex $v \in V$. Then there is a Reeb quotient map $X/\sim_f \rightarrow X/\sim_g$.*

Proof. For simplicity, we write $R_f = X/\sim_f$, $R_g = X/\sim_g$, and $R_h = X/\sim_h$, where $h = \chi \circ f$. Applying Proposition 2.2, f can be factorized as $f = \tilde{f} \circ q_f$, where $q_f : X \rightarrow R_f$ is the canonical projection and $\tilde{f} : R_f \rightarrow \mathbb{R}$ is a Reeb function. Analogously, we obtain $g = \tilde{g} \circ q_g$ and $h = \tilde{h} \circ q_h$. We show that there is a Reeb quotient map $k : X \rightarrow R_h$ making the following diagram commute:

$$\begin{array}{ccccc} \text{im } f & \xrightarrow{\chi} & \text{im } g \\ \uparrow \tilde{f} & \nearrow q_h & \uparrow \tilde{g} \\ R_f & R_h & R_g \\ \uparrow q_f & \nearrow q_h & \uparrow q_g \\ X & X & X \end{array}$$

The claim then follows by applying Lemma 2.7 to obtain Reeb quotient maps $R_f \rightarrow R_h$ and $R_h \rightarrow R_g$, which compose to the desired map $R_f \rightarrow R_g$.

In order to prove the existence of such a Reeb quotient map k , we define the relation

$$k = q_h \circ ((h^{-1} \circ g) \cap \text{st}_K)$$

on $X \times R_h$. Here st_K denotes the open star on $X = |K|$, defined as

$$\text{st}_K(x) = \{y \in X \mid \sigma \in K, y \in \sigma^\circ, x \in \sigma\}.$$

Note that the converse relation to the open star is the (*closed*) carrier, $\text{st}_K^{-1} = \text{carr}_K$, where $\text{carr}_K(A)$ is the underlying space of the smallest subcomplex of K containing $A \subseteq X$. We will also use the *open carrier*

relation carr_K° , where $\text{carr}_K^\circ(A)$ is the smallest union of open simplices of K covering A . Note that the open carrier relation is symmetric, i.e., $(\text{carr}_K^\circ)^{-1} = \text{carr}_K^\circ$. Moreover, we have $\text{carr}_K^\circ \subseteq \text{st}_K$.

The remainder of the proof is split into several lemmas. Lemma 5.2 describes the behaviour of the functions h and g on the simplices of K . Lemma 5.3 shows that k is a continuous surjection, and Lemma 5.4 shows that k has connected fibers. Since $\tilde{h} \circ k = g$, we conclude that k is PL. Thus, k is a Reeb quotient map, and the claim follows from Lemma 2.7. \square

Lemma 5.2. *For every simplex σ in K , $g(\sigma) = h(\sigma)$ and $g(\sigma^\circ) \subseteq h(\sigma^\circ)$.*

Proof. We have $h(\sigma) = g(\sigma)$ because h is equal to g on the vertices of K , and $h = \chi \circ f$ with f linear on σ and χ a weakly order preserving surjection.

To show that $g(\sigma^\circ) \subseteq h(\sigma^\circ)$, note that since g is linear on σ , either g is constant on σ and so $g(\sigma^\circ) = g(\sigma) = h(\sigma)$, or $g(\sigma^\circ) = (g(v), g(w))$ for some vertices v, w of σ . In the latter case, since h and g coincide on the vertices, we have $g(\sigma^\circ) = g(\sigma)^\circ = h(\sigma)^\circ$. Finally, since $h(\sigma^\circ) \subseteq h(\sigma) \subseteq \bar{h}(\sigma^\circ)$ are nested intervals, we have $h(\sigma)^\circ \subseteq h(\sigma^\circ)$ and the claim follows. \square

Lemma 5.3. *k is a continuous surjection.*

Proof. We first show that k is right-unique, i.e., for any $x \in X$ and $y, y' \in k(x)$, we have $y = y'$. To see this, let $t = g(x)$ and note that $\tilde{h}(y) = \tilde{h}(y') = t$. Let $\sigma \in K$ be such that $x \in \sigma^\circ$. By Lemma 5.2 there is a point $\zeta \in \sigma^\circ$ with $h(\zeta) = g(x) = t$; in particular, $\zeta \in h^{-1}(t) \cap \text{st}_K(x)$. Furthermore, there are points $\xi, \xi' \in h^{-1}(t) \cap \text{st}_K(x)$ with $\xi \in q_h^{-1}(y)$ and $\xi' \in q_h^{-1}(y')$. But since $h^{-1}(t) \cap \tau$ is necessarily connected for every simplex τ , we know that ζ lies in the same connected component of $h^{-1}(t) \cap \text{st}_K(x)$ as both ξ and ξ' , and so we have $y = q_h(\xi) = q_h(\xi') = y'$ as claimed.

To show that k is left-total, we need to show that for every $x \in X$, $k(x) \neq \emptyset$. It suffices to show that for every $x \in X$, $\text{st}_K(x)$ contains a point x' with $h(x') = g(x)$. This follows by considering the simplex $\sigma \in K$ with $x \in \sigma^\circ$. Now by Lemma 5.2, there is a point $x' \in \sigma^\circ \subseteq \text{st}_K(x)$ with $h(x') = g(x)$ as claimed.

To show that k is right-total, we show that for every $y \in R_h$, there is some

$$x \in k^{-1}(y) = (\text{carr}_K \circ q_h^{-1})(y) \cap (g^{-1} \circ \tilde{h})(y),$$

or equivalently, there is some $x \in \text{carr}_K \circ q_h^{-1}(y)$ such that $g(x) = \tilde{h}(y)$. If $q_h^{-1}(y)$ contains some vertex v of K , choose $x = v$. Otherwise, let $\xi \in q_h^{-1}(y)$, and let $\sigma \in K$ be such that $\xi \in \sigma^\circ$. Now by Lemma 5.2 there is a point $x \in \sigma \subseteq \text{carr}_K \circ q_h^{-1}(y)$ with $g(x) = h(\xi) = \tilde{h}(y)$.

Finally, to show that k is continuous, we show that for every closed subset L of R_h , the preimage $k^{-1}(L)$ is closed. Since $k^{-1} = (\text{carr}_K \circ q_h^{-1}) \cap (g^{-1} \circ \tilde{h})$, it is sufficient to show that both $\text{carr}_K \circ q_h^{-1}(L)$ and $g^{-1} \circ \tilde{h}(L)$ are closed in X . First note that $\text{carr}_K \circ q_h^{-1}(L)$ is closed as a subcomplex of K . Furthermore, the image $\tilde{h}(L)$ is closed by the closed map lemma. By continuity of g it follows that $g^{-1} \circ \tilde{h}(L)$ is closed in X . \square

Lemma 5.4. *The fibers of k are connected.*

Proof. Let $y \in R_h$ be a point in the Reeb graph with value $t = \tilde{h}(y)$, and $C = q_h^{-1}(y) \subseteq h^{-1}(t)$ the corresponding component of the level set of h . Let $U = \text{carr}_K(C)$, and let L be the corresponding subcomplex of K . Writing $D = k^{-1}(y)$, we have $C = U \cap h^{-1}(t)$ and $D = U \cap g^{-1}(t)$. To prove that D is connected, it is sufficient to show that C and D have finite closed covers with isomorphic nerves; since C is connected, both nerves and hence also D are then connected too.

The cover of C is given by $\{\sigma \cap C \mid \sigma \in L\}$, and similarly the cover of D is $\{\sigma \cap D \mid \sigma \in L\}$. Observe that any two cover elements of C , say $\sigma \cap C$ and $\tau \cap C$, have a nonempty intersection $(\sigma \cap C) \cap (\tau \cap C) = (\sigma \cap \tau) \cap C$ if and only if $t \in h(\sigma \cap \tau)$. Similarly, $\sigma \cap D$ and $\tau \cap D$ have nonempty intersection if and only if $t \in g(\sigma \cap \tau)$. But $g(\sigma \cap \tau) = h(\sigma \cap \tau)$ by Lemma 5.2, and so the nerves of both covers are isomorphic as claimed. \square

We thus have shown the existence of the Reeb quotient map k . This completes the proof of Lemma 5.1. We will now apply Lemma 5.1 to construct Reeb graph edit zigzags from straight line homotopies.

Lemma 5.5. Let $X = |K|$ be a compact triangulable space, with PL functions $f, g : X \rightarrow \mathbb{R}$, simplexwise linear on K . Consider the straight line homotopy $f_\lambda = (1 - \lambda)f + \lambda g$, with $0 \leq \lambda \leq 1$. Then there exists a partition $0 = \lambda_1 < \dots < \lambda_n = 1$ such that for every $1 \leq i < n$ and $\rho \in (\lambda_i, \lambda_{i+1})$, there exist weakly order preserving PL surjections $\chi_i : \text{im } f_\rho \rightarrow \text{im } f_{\lambda_i}$ and $\xi_{i+1} : \text{im } f_\rho \rightarrow \text{im } f_{\lambda_{i+1}}$ with

$$\chi_i \circ f_\rho(v) = f_{\lambda_i}(v) \text{ and } \xi_{i+1} \circ f_\rho(v) = f_{\lambda_{i+1}}(v)$$

for every vertex v in K .

Proof. Consider the set of values $0 < \lambda < 1$ such that there exist vertices $v, w \in K$ with

$$f_\lambda(v) = f_\lambda(w), \text{ but } f_\rho(v) \neq f_\rho(w) \text{ for every } \rho \neq \lambda.$$

This set is finite because the function $\lambda \mapsto f_\lambda(v) - f_\lambda(w)$ is linear and K has a finite number of vertices. Let $\{\lambda_i\}_{1 \leq i \leq n}$ be this set together with 0 and 1, indexed in ascending order. By the linearity of f_λ with respect to the parameter λ , we also see that the order induced by f_ρ on the vertices is the same for every $\rho \in (\lambda_i, \lambda_{i+1})$. Indeed, if there exist two distinct vertices v, w of K such that $f_\rho(v) = f_\rho(w)$ for some $\rho \in (\lambda_i, \lambda_{i+1})$, then $f_\lambda(v) = f_\lambda(w)$ for every $\lambda \in [0, 1]$. By continuity, the order is still weakly preserved along $[\lambda_i, \lambda_{i+1}]$.

Therefore, the function $f_\rho(v) \mapsto f_{\lambda_i}(v)$ is well-defined and can be extended to a piecewise linear function χ_i satisfying the claim. The function ξ_{i+1} can be defined similarly. \square

Theorem 5.6. δ_{eGraph} is a stable distance.

Proof. Let $X = |K|$ be a compact triangulable space and $f, g : X \rightarrow \mathbb{R}$ be PL functions, simplexwise linear on K . Consider the straight line homotopy $f_\lambda = (1 - \lambda)f + \lambda g$, with $0 \leq \lambda \leq 1$, and take values $\lambda_i \in [0, 1]$, $1 \leq i \leq n$, as in Lemma 5.5. Set $\rho_i = (\lambda_i + \lambda_{i+1})/2$.

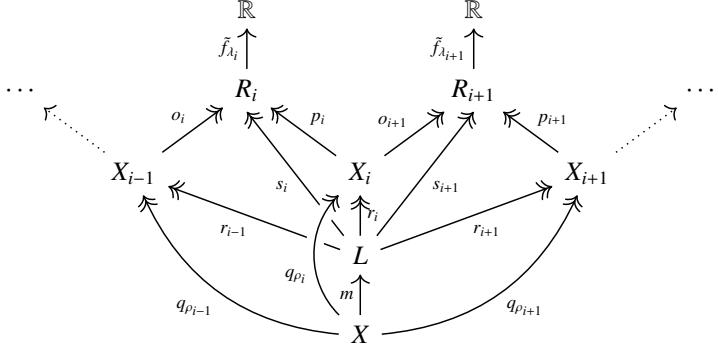
Consider the Reeb cone (2), with $V = X$, $R_i = X/\sim_{f_{\lambda_i}}$, $i = 1, \dots, n$, and $X_i = X/\sim_{f_{\rho_i}}$, $i = 1, \dots, n-1$. The canonical projections $q_{\rho_i} : X \rightarrow X_i$ and $q_{\lambda_i} : X \rightarrow R_i$ are Reeb quotient maps, and the Reeb functions $R_i \rightarrow \mathbb{R}$ are induced by f_{λ_i} as in Proposition 2.2. We show that there are Reeb quotient maps $p_i : X/\sim_{f_{\rho_i}} \rightarrow X/\sim_{f_{\lambda_i}}$ and $o_{i+1} : X/\sim_{f_{\rho_i}} \rightarrow X/\sim_{f_{\lambda_{i+1}}}$ that make the following diagram commute:

$$\begin{array}{ccc} R_i = X/\sim_{f_{\lambda_i}} & & R_{i+1} = X/\sim_{f_{\lambda_{i+1}}} \\ \nearrow q_{\lambda_i} \quad \swarrow p_i & & \nearrow q_{\lambda_{i+1}} \quad \swarrow o_{i+1} \\ X_i = X/\sim_{f_{\rho_i}} & & \\ \uparrow q_{\rho_i} & & \end{array}$$

We prove the existence of p_i , that of o_{i+1} being analogous. By Lemma 5.5, there is a weakly order preserving PL surjection $\chi_i : \text{im } f_{\rho_i} \rightarrow \text{im } f_{\lambda_i}$ such that $\chi_i \circ f_{\rho_i} = f_{\lambda_i}$. Hence, Lemma 5.1 provides the desired Reeb quotient map $p_i : X/\sim_{f_{\rho_i}} \rightarrow X/\sim_{f_{\lambda_i}}$.

Now consider the limit L over the Reeb zigzag diagram consisting of the maps p_i and o_i , with maps $r_i : L \rightarrow X_i$ and $s_i : L \rightarrow R_i$. Since the maps from X in the above Reeb cone factor through L , we obtain

the commutative diagram



We have $f_{\lambda_i} = f_{\lambda_i}^L \circ m$ for $1 \leq i \leq n$, with $f_{\lambda_i}^L = \tilde{f}_{\lambda_i} \circ s_i$. Hence, for every $\ell \in L$,

$$s^L(\ell) = \max_j f_{\lambda_j}^L(\ell) - \min_k f_{\lambda_k}^L(\ell) \leq \sum_{i=1}^{n-1} |f_{\lambda_{i+1}}^L(\ell) - f_{\lambda_i}^L(\ell)|.$$

By the surjectivity of q_{ρ_i} , for every i there is $x_{\ell,i} \in X$ such that $q_{\rho_i}(x_{\ell,i}) = r_i(\ell)$. Thus,

$$|f_{\lambda_{i+1}}^L(\ell) - f_{\lambda_i}^L(\ell)| = |f_{\lambda_{i+1}}(x_{\ell,i}) - f_{\lambda_i}(x_{\ell,i})| \leq (\lambda_{i+1} - \lambda_i) \cdot \|f - g\|_\infty.$$

In conclusion, for every $\ell \in L$,

$$s^L(\ell) \leq \sum_{i=1}^{n-1} (\lambda_{i+1} - \lambda_i) \cdot \|f - g\|_\infty = \|f - g\|_\infty.$$

□

Corollary 5.7. δ_{eGraph} is a universal distance.

Proof. The claim is a direct consequence of inequality (3) together with Theorem 5.6 and Propositions 4.4 and 4.5. □

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