2102 509 – Introduction to Optimization Techniques
Homework # 2: Due Date Wed 22nd October 2025 by 11 am.
Submit your homework (hard copy) in the classroom.
LATE homework will not be accepted. Duplication of homework is strictly forbidden.

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1. (10 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b}$, where $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{Q}$ is a real symmetric positive definite $n \times n$ matrix. Consider the algorithm

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \beta\alpha_k\boldsymbol{g}^{(k)},$$

where $\boldsymbol{g}^{(k)} = \boldsymbol{Q}\boldsymbol{x}^{(k)} - \boldsymbol{b}$, $\alpha_k = \boldsymbol{g}^{(k)T}\boldsymbol{g}^{(k)}/\boldsymbol{g}^{(k)T}\boldsymbol{Q}\boldsymbol{g}^{(k)}$, and $\beta \in \mathbb{R}$ is a given constant. (Note that the above reduces to the steepest descent algorithm if $\beta = 1$.) Show that $\{\boldsymbol{x}^{(k)}\}$ converges to $\boldsymbol{x}^* = \boldsymbol{Q}^{-1}\boldsymbol{b}$ for any initial condition $\boldsymbol{x}^{(0)}$ if and only if $0 < \beta < 2$.

2. (10 points) Consider the problem

$$\text{minimize} \quad f(x_1, x_2) = x_1^2 + 2x_2^2.$$

If the starting point is $x_0 = (2, 1)^T$, use mathematical induction to show that the sequence of points generated by the steepest-descent algorithm with an exact line search is given by

$$x_k = \left(\frac{1}{3}\right)^k \begin{bmatrix} 2 \\ (-1)^k \end{bmatrix}.$$

3. (10 points) Consider "Rosenbrock's Function": $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$, where $x = [x_1, x_2]^T$ (know to be a "nasty" function and often used as a benchmark for testing algorithms). This function is also known as the banana function because of the shape of its level sets. Determine all the local minimizers by using the second-order conditions. Are the obtained results the global minimizer of f? Explain.

4. (20 points) Write your MATLAB function for solving the n-dimensional minimization problem using the modified Newton's method, where the line-search subproblem is solved using golden section method.

When the Newton search direction $s_k = -\left[\nabla^2 f(x_k)\right]^{-1}\nabla f(x_k)$ does not have descent property, replace it with the steepest descent direction; that is to say, set $s_k = -\nabla f(x_k)$. As a result of this simple modification, the step length λ_k takes positive values for every iteration.

The format of Newton are as follows.

function [x_{min},f_{min},Xk,Fk,Gk,nF,nG,nH,CHN,IFLAG] = Newton(Fcn,x_0,epsilon,ε_{rel},ε_{abs},itmax).

- Fcn is the name of the function providing the values of $f(x)$, $\nabla f(x)$ and $\nabla^2 f(x)$ for $x \in \mathbb{R}^n$.
- x_{min} is the estimate of the minimizer of f.
- f_{min} is value of $f(x_{min})$.
- Xk is the array containing iterates x_k ($k = 0, 1, 2, \ldots$).

- Fk and Gk are the arrays containing $f(x_k)$ and $\nabla f(x_k)$ $(k = 0, 1, 2, \ldots)$, respectively.
- nF, nG and nH are the arrays containing the numbers of evaluation of the objective function f, the gradient ∇f and the Hessian matrix $\nabla^2 f$, respectively, in each iteration.
- CHN is an array containing σ_k $(k = 0, 1, 2, \ldots)$ where $\sigma_k = 0$ if s_k is a Newton direction with descent property and $\sigma_k = 1$ if s_k is replaced with the steepest direction.
- IFLAG is a flag variable that indicates the status of return from calling Newton. For example, if the iteration converges successfully, set IFLAG = 0; if the iteration does not converge with itmax iterations, set IFLAG = 1; etc.
- x_0 is the given starting point.
- epsilon is the parameter used in the stopping criterion for Newton.
- itmax denotes the maximum number of the iterations allowed.
- ε_{rel} and ε_{abs} are the parameters used in the stopping criterion for line search, i.e., the line search terminates if the following is satisfied.

$$\left|s_k^T \nabla f(x_{k+1})\right| \leq \left|s_k^T \nabla f(x_k)\right| \varepsilon_{\text{rel}} + \varepsilon_{\text{abs}} \quad \text{and} \quad f(x_{k+1}) < f(x_k).$$

Notice that if the parameters ε_{rel} and ε_{abs} are set to be relatively small, then the line search becomes exact. Write proper comments in your code so that it can be examined.

Use the above program to determine the minimum of the Rosenbrock's function with

$$\varepsilon_{\text{rel}} = 0.05 \quad \text{and} \quad \varepsilon_{\text{abs}} = 10^{-3}.$$

Run the program with three different starting points.

- Case 1: $x_0 = [-1.2,\ 1]^T$;
- Case 2: $x_0 = [10,\ 10]^T$;
- Case 3: $x_0 = [-1,\ -1]^T$.

Locate the value of x^* with at least 4 significant digits. Explain how you choose the parameter epsilon in solving the problem. For each case, tabulate the following results: k, x_k, $f(x_k)$, $||\nabla f(x_k)||$, nF_k, nG_k, nH_k, σ_k . Compare the obtained results and make a discussion.

The format of Fcn is as follows:

function [f, gradient, Hessian] = Fcn(x, options).

If options = 1, then only f is computed. If options = 2, then f and gradient are computed. If options = 3, then f, gradient and Hessian are computed.

Suchin Arunsawatwong, 15th October 2025.