2102 505 – Introduction to Optimization Techniques
Homework #4: Due date Monday 10$^{\text{th}}$ November 2025 by 11:00 am.
Submit your homework (hard copy) in the classroom.
Late homework will not be accepted. Duplication of homework is strictly forbidden.

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1. (30 points) Write your MATLAB function for solving the n-dimensional minimization problem using conjugate gradient methods with an inexact line search satisfying the following criteria:

$$\begin{aligned} f(x_k + \lambda_k s_k) &\leq f(x_k) + \mu \lambda_k s_k^T \nabla f(x_k) && \text{(Armijo's condition)} \\ \left| s_k^T \nabla f(x_k + \lambda_k s_k) \right| &\leq -\eta s_k^T \nabla f(x_k) && \text{(Strong Wolf's condition)} \end{aligned}$$

where $0 \ll \mu < \eta < 0.5$ are specified by users. Note that as $\eta \to 0$, the line search process becomes exact.

Incorporate a watch-dog scheme to reset the search direction $s_k = -g_k$ either when the angle between the s_k and $-g_k$ is too large (say, $>$ 85 degrees) or when the direction s_k does not have descent property.

The format of the functions are as follows.

function [$x_{\min}$,$f_{\min}$,Xk,Fk,Gk,Lk,nF,nG,IFLAG,nReset] = CG(x_0,epsilon,mu,eta,itmax,option)

where: $x_{\min}$ is the estimate of the minimizer of f; $f_{\min}$ is value of $f(x_{min})$; Xk is the array containing iterates x_k ($k = 0, 1, 2, \ldots$); Fk, Gk and Lk are the arrays containing $f(x_k)$, $\nabla f(x_k)$ and λ_k ($k = 0, 1, 2, \ldots$) respectively; nF and nG are the arrays of the numbers of evaluation of the objective function f and the gradient ∇f, respectively, in each iteration; IFLAG is is a flag variable that indicates the status of return from calling CG; nReset is an integer vector recording the history of resetting s_k in each iteration where each element is set to be 0 when restart is not used, 1 when restart is used because the angle between s_k and $-g_k$ is too large, and 2 when s_k does not have descent property; x_0 is the given starting point; epsilon is the parameter used in the stopping criterion for the method; mu & eta are the parameters associated with the line search criteria; itmax denotes the maximum number of the iterations allowed; option is an integer for users to choose a method (if option = 1, then Fletcher-Reeves formula is used and if option = 2, then Polak-Ribiére formula is used).

Print out your program. Write comments in your codes so that it can be examined.

Use the above program to determine the minimum of the Rosenbrock's function, where an inexact line search is used and the starting point $x_0 = [-1.2, 1]^T$. Locate the value of x^* with at least 4 significant digits. Tabulate the following results: x_k, $f(x_k)$, $\nabla f(x_k)$, λ_k, nF, nG, the number of gradient evaluations, and whether the restart being used or not.

With $\mu = 10^{-4}$ and $\eta = 0.25$, compare the results obtained from using Fletcher-Reeves and Polak-Ribiére conjugate gradient methods. Make a discussion by taking into account the numbers of function and gradient evaluations, the frequency of restart (reset) during the computation, the number of iterations used.

The MATLAB function assumes that Rosenbrock's function has the following format:

function [f, gradient] = Rosenbrock(x, options)

If options = 1, only f is computed. If options = 2, f and gradient are computed.

2. (10 points) Consider a conjugate gradient method whose search direction s_k is of the form

$$\left.\begin{array}{rcl} s_{k+1} & = & -g_{k+1} + \beta_k s_k \\ s_0 & = & -g_0 \end{array}\right\}, \quad k = 0, 1, 2, \ldots$$

where the β_k is any real number satisfying

$$|\beta_k| \leq \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}.$$

Notice that this is not Fletcher-Reeves method but a family of conjugate gradient methods that includes Fletcher-Reeves method. Suppose that, for each k, the method chooses a step length α_k satisfying the strong Wolfe condition:

$$\left|g_{k+1}^T s_k\right| \leq -\eta g_k^T s_k.$$

Prove that the method generates descent directions s_k that satisfies

$$\frac{-1}{1-\eta} \leq \frac{g_k^T s_k}{g_k^T g_k} \leq \frac{2\eta - 1}{1-\eta} \quad \text{for all } k = 1, 2, 3, \ldots.$$

What is the maximum value of η which ensures that the method always generates search directions with descent preperty.

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