Hw3 es 2 point 4

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3) $\|A-A_k\|_2^2 =$ given the definition of 2 norm of a matrix $\max\limits_{\|x\|_2 \neq 0} \frac{\|(A-A_k)x\|_2^2}{\|x\|_2^2} = \max\limits_{\|x\|_2=1} \|(A-A_k)x\|_2^2$

$$A-A_k = \sum_{i=1}^{n} \sigma_i u_i v_i^T - \sum_{i=1}^{k} \sigma_i u_i v_i^T = \sum_{i=k+1}^{n} \sigma_i u_i v_i^T = U\Sigma V^T$$

we want to evaluate $\max\limits_{\|x\|_2=1} \|U\Sigma V^T x\|_2^2$ where $U = \begin{pmatrix} | & & | \\ u_1 & - & u_n \\ | & & | \end{pmatrix}$, $V^T = \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & \end{pmatrix}$, $\Sigma = \begin{pmatrix} 0 \cdots 0 & & \cdots & 0 \\ \vdots & \sigma_{k+1} & & \vdots \\ 0 \cdots & & \ddots & \sigma_n \end{pmatrix}$

$\Rightarrow \max\limits_{\|x\|_2=1} \|(U\Sigma V^T x)^T (U\Sigma V^T x)\|_2 = \max\limits_{\|x\|_2=1} \|x^T V \Sigma^T U^T U \Sigma V^T x\|_2 \Rightarrow$ since U is orthogonal $\max\limits_{\|x\|_2=1} \|x^T V \Sigma^T \Sigma V^T x\|_2$

$\Rightarrow \max\limits_{\|x\|_2=1} \|\Sigma V^T x\|_2^2$ then we define $z = V^T x$ $\|z\| = \|x\| = 1$ because V is orthogonal $\|z\|_2^2 = \|V^T x\|_2^2 = \|x^T V V^T x\|_2 = \|x^T x\| = \|x\|_2^2 = \|x\|_2 = 1$

$\Rightarrow \max\limits_{\|z\|=1} \|\Sigma z\|_2^2 \Rightarrow$ since Σ is diagonal $\begin{pmatrix} 0 \cdots 0 & & \cdots & 0 \\ & \sigma_{k+1} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \sigma_n \end{pmatrix} \Rightarrow$ the max is σ_{k+1}^2 when z is $(1\ 0 \cdots 0)$

given a diagonal matrix the max of the l2 norm is the maximum value of its diagonal because we can set to 1 the component of the vector z that multiplies the biggest element in the matrix in our case the sigma1 and the others to 0 given that z has to have norm = 1. We want the square of this value so sigma 1 squared

$$\text{so } \|A_k\|_2^2 = \max_{\|z\|=1} \|\Sigma z\|_2^2 = \sigma_{k+1}^2$$