hw3 es3 point 1

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we have to prove that $\sigma_k \leq \frac{\|A\|_F}{\sqrt{k}} \Rightarrow \|A\|_F^2 \geq k\,\sigma_k^2$

we know from the es 2 point 1 that $\|A\|_F^2 = \sum_{i=1}^{m} \sigma_i^2$ where σ_i are the singular values $\Rightarrow \sum_{i=1}^{m} \sigma_i^2 \geq k\,\sigma_k^2$

knowing that the singular values are ordered from the biggest to the lowest we can write $\sum_{i=1}^{m} \sigma_i^2 = \sum_{i=1}^{k} \sigma_i^2 + \sum_{i=k+1}^{m} \sigma_i^2 \geq k\,\sigma_k^2$

we know that $\sum_{i=1}^{k} \sigma_i^2 \geq k\,\sigma_k^2$ because the $\sigma_i \geq \sigma_k$ when $i \leq k$ because of the SVD decomposition $A = U \Sigma V^T$

$$\Sigma = \begin{pmatrix} \sigma_1 & \cdots & \cdots & 0 \\ \vdots & \sigma_k & & \vdots \\ 0 & \cdots & \cdots & \sigma_m \end{pmatrix} \quad \sigma_{i+1} \geq \sigma_i \ \forall i \text{ in } 0,\dots,m-1 \text{ and } \sigma_i \geq 0 \ \forall i \text{ in } 0\dots m$$

so $\sum_{i=1}^{k} \sigma_i^2 \geq k \cdot \sigma_k^2 \Rightarrow \sum_{i=1}^{k} \sigma_i^2 + \sum_{i=k+1}^{m} \sigma_i^2 \geq k\,\sigma_k^2 \Rightarrow \|A\|_F^2 \geq k\,\sigma_k^2 \Rightarrow$

$$\|A\|_F \geq \sqrt{k}\,\sigma_k \Rightarrow \sigma_k \leq \frac{\|A\|_F}{\sqrt{k}}$$