Hw3 es 2 point 3

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3) $\|A_k\|_2^2 =$ given the definition of 2 norm of a matrix $\max_{\|x\|_2 \neq 0} \frac{\|A_k x\|_2^2}{\|x\|_2^2} = \max_{\|x\|_2=1} \|A_k x\|_2^2 = \max_{\|x\|_2=1} \|U \Sigma V^T x\|_2^2 \Rightarrow$

where $U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{pmatrix}$, $V^T = \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_m & \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sigma_1 & \cdots & & 0 \\ \vdots & \sigma_k & & \vdots \\ & & 0 & \\ 0 & \cdots & & 0 \end{pmatrix}$

$\Rightarrow \max_{\|x\|_2=1} \|(U\Sigma V^T x)^T (U \Sigma V^T x)\|_2 = \max_{\|x\|_2=1} \|x^T V \Sigma^T U^T U \Sigma V^T x\|_2 \Rightarrow$ since U is orthogonal $\max_{\|x\|_2=1} \|x^T V \Sigma^T \Sigma V^T x\|_2$

$\Rightarrow \max_{\|x\|_2=1} \|\Sigma V^T x\|_2^2$ then we define $z = V^T x$ $\|z\| = \|x\| = 1$ because V is orthogonal $\|z\|_2^2 = \|V^T x\|_2^2 = \|x^T V V^T x\| = \|x^T x\| = \|x\|_2^2 = \|x\|_2 = 1$

$\Rightarrow \max_{\|z\|=1} \|\Sigma z\|_2^2 \Rightarrow$ since Σ is diagonal $\begin{pmatrix} \sigma_1 & \cdots & & 0 \\ & \sigma_k & & \vdots \\ \vdots & & 0 & \\ 0 & \cdots & & 0 \end{pmatrix} \Rightarrow$ the max is σ_1^2 when z is $(1 \ 0 \cdots 0)$

given a diagonal matrix the max of the l2 norm is the maximum value of its diagonal because we can set to 1 the component of the vector z that multiplies the biggest element in the matrix in our case the sigma1 and the others to 0 given that z has to have norm = 1. We want the square of this value so sigma 1 squared

so $\|A_k\|_2^2 = \max_{\|z\|=1} \|\Sigma z\|_2^2 = \sigma_1^2$