Hw3 es2 point 2

23 November 2022 10:34

We could use the proof done in the lecture or doing it a little bit differently

2) $$\|A - A_k\|_F^2 = \left\| \sum_{i=1}^{m} \sigma_i u_i v_i^T - \sum_{i=1}^{k} \sigma_i u_i v_i^T \right\|_F^2 = \left\| \sum_{i=k+1}^{m} \sigma_i u_i v_i^T \right\|_F^2 = \|U \Sigma V^T\|_F^2$$

where $U = \begin{pmatrix} | & & | \\ u_i & \cdots & u_m \\ | & & | \end{pmatrix}$, $V^T = \begin{pmatrix} - v_i - \\ \vdots \\ - v_m - \end{pmatrix}$, $\Sigma = \begin{pmatrix} 0 & & & \\ & \ddots 0 & & \\ & & \sigma_{k+1} & \\ & & & \ddots \sigma_m \end{pmatrix}$

now $\|A\|_F^2 = Tr(A^T A) \Rightarrow Tr\left((U \Sigma V^T)^T (U \Sigma V^T)\right) = Tr(V \Sigma^T U^T U \Sigma V^T) \Rightarrow$

Since U and V are orthonormal basis then $UU^T = U^T U = V^T V = VV^T = I$

$\Rightarrow Tr(V \Sigma^T \Sigma V^T) \Rightarrow$

Now the trace of the product is cyclic so $\Rightarrow Tr(V^T V \Sigma^T \Sigma) = Tr(\Sigma^T \Sigma) = \|\Sigma\|_F^2$

Since Σ is a matrix done in this way $\Sigma = \begin{pmatrix} 0 & & & \\ & \ddots 0 & & \\ & & \sigma_{k+1} & \\ & & & \ddots \sigma_m \end{pmatrix}$ Then the frobineaus norm is $\|\Sigma\|_F^2 = \sum_{i=k+1}^{m} \sigma_i^2$

so $$\|A - A_k\|_F^2 = \|\Sigma\|_F^2 = \sum_{i=k+1}^{m} \sigma_i^2$$