hw3 es 3 point 2

25 November 2022 10:16

we have to prove that there exists a matrix B of rank at most k such that $\|A-B\|_2 \le \frac{\|A\|_F}{\sqrt{k}} \Rightarrow$

$\|A\|_F^2 \ge k\,\|A-B\|_2^2$ from point 2.1 we know that we can rewrite $\|A\|_F^2$ as $\sum_{i=1}^{n} \sigma_i^2$

and if we take $B = A_k$ we can write $\sum_{i=1}^{n} \sigma_i^2 \ge k\,\|A-A_k\|_2^2 \Rightarrow \|A-A_k\|_2^2$ from point 2.4 is $\sigma_{k+1}^2 \Rightarrow$

$\sum_{i=1}^{n} \sigma_i^2 \ge k\,\sigma_{k+1}^2$ and this relation is verified because the eigenvalues are ordered and greater than 0. Note also that B is a matrix with rank k for the construction using SVD.

$$\Downarrow$$

$$\sum_{i=1}^{k} \sigma_i^2 + \sum_{i=k+1}^{n} \sigma_i^2 \ge k\,\sigma_{k+1}^2 \Rightarrow \sum_{i=1}^{k} \sigma_i \ge k\,\sigma_{k+1} \quad \text{so} \quad \sum_{i=1}^{k} \sigma_i^2 + \sum_{i=k+1}^{n} \sigma_i^2 \ge k\,\sigma_{k+1}^2$$

$$\Rightarrow \sum_{i=1}^{n} \sigma_i^2 \ge k\,\sigma_{k+1}^2 \Rightarrow \|A\|_F^2 \ge k\,\|A-A_k\|_2^2 \Rightarrow \|A-A_k\|_2 \le \frac{\|A\|_F}{\sqrt{k}}$$

so there exist a matrix B= Ak of rank k such that the relationship is verified