

Hw3 es 2 point 3

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$$3) \|A_K\|_2^2 = \max_{\|x\|_2=1} \|A_K x\|_2^2 \quad \text{given the definition of 2 norm of a matrix} \quad \frac{\max_{\|x\|_2=1} \|A_K x\|_2^2}{\|x\|_2^2} = \max_{\|x\|_2=1} \|A_K x\|_2^2 = \max_{\|x\|_2=1} \|U \Sigma V^T x\|_2^2 \Rightarrow$$

where $U = \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, V^T = \begin{pmatrix} -v_1 & & \\ & \ddots & \\ & & -v_m \end{pmatrix}, \Sigma = \begin{pmatrix} b_1 & & & 0 \\ & \ddots & & \\ & & b_k & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$

$$\Rightarrow \max_{\|x\|_2=1} \|(U \Sigma V^T)^T (U \Sigma V^T)\|_2 = \max_{\|x\|_2=1} \|x^T V \Sigma^T U^T U \Sigma V^T x\|_2 \Rightarrow \text{since } U \text{ is orthogonal } \max_{\|x\|_2=1} \|x^T V \Sigma^T \Sigma V^T x\|_2$$

$$\Rightarrow \max_{\|x\|_2=1} \|\Sigma V^T x\|_2^2 \quad \text{then we define } z = V^T x \quad \|z\| = \|x\| = 1 \quad \text{because } V \text{ is orthogonal} \quad \|z\|_2^2 = \|V^T x\|_2^2 = \|\bar{x} V V^T x\|_2 = \|\bar{x}^T x\| = \|x\|_2^2 = \|x\|_2^2 = 1$$

$$\Rightarrow \max_{\|z\|=1} \|\Sigma z\|_2^2 \Rightarrow \text{since } \Sigma \text{ is diagonal } \begin{pmatrix} b_1 & & & \\ & \ddots & & \\ & & b_k & \\ & & & 0 \dots 0 \end{pmatrix} \Rightarrow \text{the max is } b_1^2 \quad \text{when } z \text{ is } \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$$

given a diagonal matrix the max of the l2 norm is the maximum value of its diagonal because we can set to 1 the component of the vector z that multiplies the biggest element in the matrix in our case the sigma1 and the others to 0 given that z has to have norm = 1. We want the square of this value so sigma 1 squared

$$\text{so } \|A_K\|_2^2 = \max_{\|z\|=1} \|\Sigma z\|_2^2 = b_1^2$$