

Algorithms Test 1 Review

Benjamin Boudra

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1 EX 1

1.1 Prompt

1.2 Answer

2 EX 2

2.1 Prompt

2.2 Answer

3 EX 3

3.1 Prompt

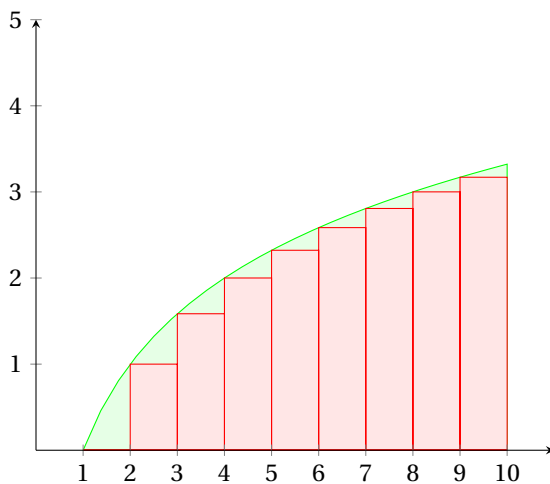
Use the technique of bounding definite integrals to find the Θ category for the function.

$$A(n) = \log_2(1) + \log_2(2) + \log_2(3) + \dots + \log_2(n-1) + \log_2(n) \quad (1)$$

Actually, you should use the integral bound technique for one equality, and use trivial analysis for the other.

3.2 Answer

3.2.1 Step 1: Draw a graph of the summation and Integral



3.2.2 Step 2: Deduce the Graph's implications

Following from the facts that:

1. The integral is the area under the function $\log_2(x)$ from 1 to n .
2. If we chose to view the integral as the summation of all of its length one segments from 1 to n plus the integral of whatever is left over (if n is not an integer value), the resulting integral's value will be unaffected.
3. The value of each integral segment is greater than the value of its corresponding series segment because:
 - (a) the value of the function and the series are equal at integer values
 - (b) the value of the function increases between integers and the value of the series does not.

$\int_1^n \log_2(x)$ is greater than $\sum_{i=1}^n \log_2(i)$ for any n greater than 1. Thus, to find the upper bound or O of $\sum_{i=1}^n \log_2(i)$ we merely need to find the integral of $\int_1^n \log_2(x)$

3.2.3 Step 3: Calculate the Definite integral

So now I will calculate $\int_1^n \log_2(x)$

1. Recognize that to take the integral of a logarithm, we will have to perform integration by parts. so we must chose u and dv values.

$$u = \log_2(x) \quad du = 1/x \ln(2) \quad dv = 1 \quad v = x \quad (2)$$

2. solve for the indefinite integral

$$\int_1^n \log_2(x) = \frac{\ln(x) * x}{\ln(2)} \Big|_1^n - \int_1^n \frac{x}{x \ln(2)} = \frac{\ln(x) * x}{\ln(2)} \Big|_1^n - \int_1^n \frac{1}{\ln(2)} = \frac{\ln(x) * x}{\ln(2)} - \frac{x}{\ln(2)} \Big|_1^n = \frac{\ln(x) * x - x}{\ln(2)} \Big|_1^n \quad (3)$$

3. solve for the definite integral

$$\frac{\ln(x) * x - x}{\ln(2)} \Big|_1^n = \frac{\ln(n) * n - n}{\ln(2)} - \left(\frac{\ln(1) * 1 - 1}{\ln(2)} \right) = \frac{\ln(n) * n - n}{\ln(2)} + \left(\frac{1}{\ln(2)} \right) = \frac{\ln(n) * n - n + 1}{\ln(2)}$$

Thus, The equation:

$$\frac{\ln(n) * n - n + 1}{\ln(2)}$$

Acts as an upper bound for the summation.

3.2.4 Step 4: Prove Upper Bound is in $\Theta(n \log(n))$

According to our notes, an equation has the time complexity category of $\Theta(n)$ if it satisfies the following definition:

Definition 1. For functions f and g , we can say that $f \in \Theta(g) \iff$ there is a positive integers N, M and positive real numbers n, m such that the following equations are satisfied:

$$\forall_{n > N}, f(n) \leq cg(n) \quad \text{also known as } O(n)$$

and

$$\forall_{m > M}, f(m) \geq cg(m) \quad \text{also known as } \Omega(n)$$

1. prove equation is $O(n \log(n))$.

(a) chose c value, $c = 1/\ln(2)$

(b) solve for n .

$$\frac{\ln(n) * n - n + 1}{\ln(2)} \leq \frac{n * \ln(n)}{\ln(2)}$$

$$0 \leq \frac{n * \ln(n)}{\ln(2)} - \frac{\ln(n) * n - n + 1}{\ln(2)} = 0 \leq \frac{n * \ln(n)}{\ln(2)} - \frac{\ln(n) * n - n + 1}{\ln(2)} = 0 \leq \frac{n * \ln(n) - n * \ln(n) + n - 1}{\ln(2)} = 0 \leq \frac{n - 1}{\ln(2)} = 1 \leq n$$

(c) Thus we shall chose $n = 1$ and we have proven:

$$\frac{\ln(n) * n - n + 1}{\ln(2)} \in O(n \ln(n))$$

2. Prove equation is $\Omega(n \log(n))$.

(a) chose c value, $c = \frac{1}{2 \ln(2)}$

(b) solve for n

$$\frac{\ln(n) * n - n + 1}{\ln(2)} \geq \frac{n * \ln(n)}{2\ln(2)}$$

$$0 \geq \frac{n * \ln(n)}{2\ln(2)} - \frac{\ln(n) * n - n + 1}{\ln(2)} = 0 \geq \frac{n * \ln(n)}{\ln(2)} - \frac{2 * (\ln(n) * n - n + 1)}{\ln(2)} = 0 \geq \frac{n * \ln(n) - 2\ln(n) * n + 2n - 2}{\ln(2)} = 0 \geq \frac{-\ln(n) * n + 2n - 2}{\ln(2)}$$

Use calculator to solve for n in equation and you get:

$$n \geq 1$$

Thus we shall choose $n=1$ and the equation is true, thus we have proven:

$$\frac{\ln(n) * n - n + 1}{\ln(2)} \in \Omega(n \ln(n)) \quad (4)$$

(c) Prove equation is in $\Theta(n * \log(n))$.

According to definition 1 above: A function is in $\Theta(n \log(n))$ if it has a $O(n \log(n))$ and a $\Omega(n * \log(n))$. We have proven that it has both. Therefore the function:

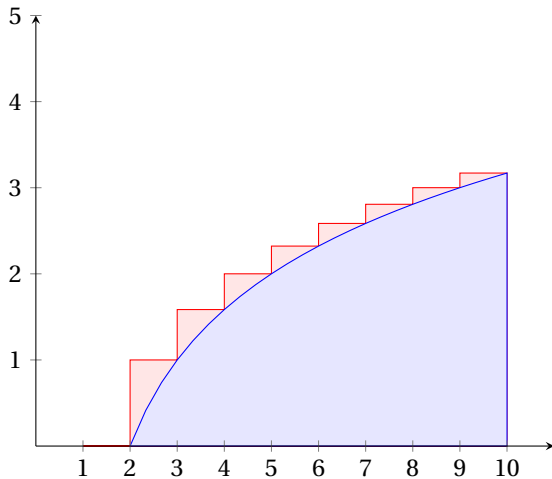
$$\frac{\ln(n) * n - n + 1}{\ln(2)} \in \Theta(n \ln(n))$$

3.2.5 Step 5: Find Lower Bound

If we shift the original function $\log_2(x)$ over one position to the right by adding 1 to the x coordinate to get the function $\log_2(x-1)$. The summation function $\sum_1^n \ln(n)$ will be larger than $\int_1^n \ln(n)$ because if we break up the integral into a sum of smaller integrals like we did in step 2, each summation segment will be larger than its corresponding integral segment for the following reasons:

- At the beginning of each integer, the integral segment's value is smaller than the series equivalent.
- Between integers the value of the integral segment will approach the value of the summation segment. However, it will never pass it.
- Due to the fact that the function is always increasing in size. This pattern will continue infinitely.

See the graph below for a visual reference:



3.2.6 Find Integral of $\log_2(x-1)$

For simplicity:

$$\log_2(x-1) = \frac{\ln(x-1)}{\ln(2)}$$

1. Choose t , u , and dv

$$t = x - 1 \quad u = \ln(t) \quad du = \frac{1}{t} * dt \quad dv = 1 \quad v = t \quad (5)$$

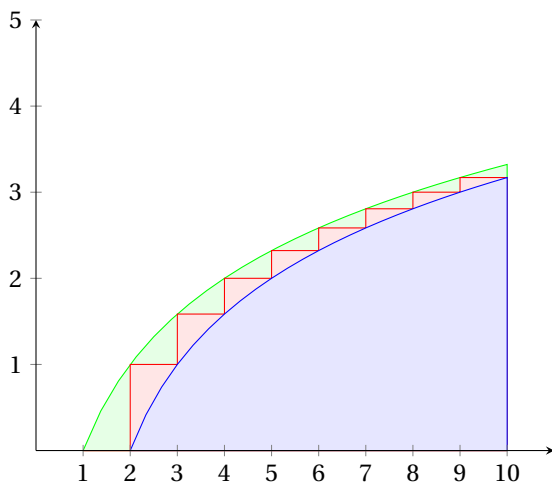
$$\int_1^n \log_2(x) = \frac{\ln(t) * t}{\ln(2)} \Big|_1^n - \frac{1}{\ln(2)} \int_1^n \frac{1}{t} * t * dt$$

$$\int_1^n \log_2(x) = \frac{\ln(t) * t}{\ln(2)} \Big|_1^n - \frac{1}{\ln(2)} \int_1^n 1 * dt$$

$$\int_1^n \log_2(x) = \frac{\ln(t) * t}{\ln(2)} - \frac{t}{\ln(2)} \Big|_1^n$$

$$\int_1^n \log_2(x) = \frac{\ln(t) * t - t}{\ln(2)} \Big|_1^n$$

$$\int_1^n \log_2(x) = \frac{(x-1) * (\ln(x-1) - 1)}{\ln(2)} \Big|_1^n$$



4 EX 4

4.1 Prompt

4.2 Answer

5 EX 5

5.1 Prompt

5.2 Answer

6 EX 6

6.1 Prompt

6.2 Answer

7 EX 7

7.1 Prompt

7.2 Answer