Algorithms Test 1 Review

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3.1 Prompt

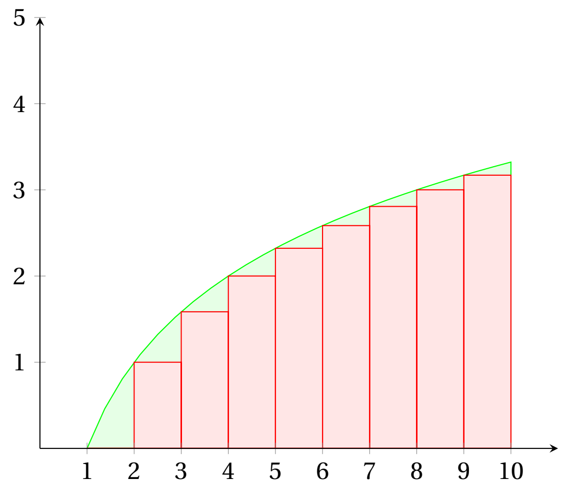
Use the technique of bounding definite integrals to find the Θ category for the function.

$$A(n) = log_2(1) + log_2(2) + log_2(3) + \ldots + log_2(n-1) + log_2(n) \tag{1}$$

Actually, you should use the integral bound technique for one equality, and use trivial analysis for the other.

3.2 Answer

3.2.1 Step 1: Draw a graph of the summation and Integral



3.2.2 Step 2: Deduce the Graph's implications

Following from the facts that:

1. The integral is the area under the function $log_2(x)$ from 1 to n.
2. If we chose to view the integral as the summation of all of its length one segments from 1 to n plus the integral of whatever is left over (if n is not an integer value), the resulting integral's value will be unaffected.
3. The value of each integral segment is greater than the value of its corresponding series segment because:
   (a) the value of the function and the series are equal at integer values
   (b) the value of the function increases between integers and the value of the series does not.

$\int_1^n log_2(x)$ is greater than $\sum_{i=1}^n log_2(i)$ for any n greater than 1. Thus, to find the upper bound or O of $\sum_{i=1}^n log_2(i)$ we merely need to find the integral of $\int_1^n log_2(x)$

3.2.3 Step 3: Calculate the Definite integral

So now I will calculate $\int_1^n log_2(x)$

1. Recognize that to take the integral of a logarithm, we will have to perform integration by parts. so we must chose u and dv values.
$$u = log_2(x) \qquad du = 1/xln(2) \qquad dv = 1 \qquad v = x \tag{2}$$

2. solve for the indefinite integral
$$\int_1^n log_2(x) = \frac{ln(x)*x}{ln(2)}\bigg|_1^n - \int_1^n \frac{x}{xln(2)} = \frac{ln(x)*x}{ln(2))}\bigg|_1^n - \int_1^n \frac{1}{ln(2)} = \frac{ln(x)*x}{ln(2)} - \frac{x}{ln(2)}\bigg|_1^n = \frac{ln(x)*x-x}{ln(2)}\bigg|_1^n \tag{3}$$

3. solve for the definite integral
$$\frac{ln(x)*x-x}{ln(2)}\bigg|_1^n = \frac{ln(n)*n-n}{ln(2)} - \left(\frac{ln(1)*1-1}{ln(2)}\right) = \frac{ln(n)*n-n}{ln(2)} + \left(\frac{1}{ln(2)}\right) = \frac{ln(n)*n-n+1}{ln(2)}$$

Thus, The equation:
$$\frac{ln(n)*n-n+1}{ln(2)}$$
Acts as an upper bound for the summation.

3.2.4 Step 4: Prove Upper Bound is in $\Theta(nlog(n))$

According to our notes, an equation has the time complexity category of $\Theta(n)$ if it satisfies the following definition:

Definition 1. For functions f and g, we can say that $f \in \Theta(g) \iff$ there is a positive integers N, M and positive real numbers n, m such that the following equations are satisfied:
$$\forall_{n>N}, f(n) \le cg(n) \qquad \text{also known as O(n)}$$
and
$$\forall_{m>M}, f(m) \ge cg(m) \qquad \text{also known as } \Omega \text{ (n)}$$

1. prove equation is $O(nlog(n))$.

   (a) chose c value, $c = 1/ln(2)$

   (b) solve for n.
$$\frac{ln(n)*n-n+1}{ln(2)} \le \frac{n*ln(n)}{ln(2)}$$
$$0 \le \frac{n*ln(n)}{ln(2)} - \frac{ln(n)*n-n+1}{ln(2)} = 0 \le \frac{n*ln(n)}{ln(2)} - \frac{ln(n)*n-n+1}{ln(2)}$$
$$= 0 \le \frac{n*ln(n)-n*ln(n)+n-1}{ln(2)} = 0 \le \frac{n-1}{ln(2)} = 1 \le n$$

   (c) Thus we shall chose n = 1 and we have proven:
$$\frac{ln(n)*n-n+1}{ln(2)} \in O(nln(n))$$

2. Prove equation is $\Omega(nlog(n))$.

   (a) chose c value, $c = \frac{1}{2ln(2)}$

(b) solve for n

$$\frac{ln(n)*n-n+1}{ln(2)} \geq \frac{n*ln(n)}{2ln(2)}$$

$$0 \geq \frac{n*ln(n)}{2ln(2)} - \frac{ln(n)*n-n+1}{ln(2)} = 0 \geq \frac{n*ln(n)}{ln(2)} - \frac{2*(ln(n)*n-n+1)}{ln(2)} = 0 \geq \frac{n*ln(n)-2ln(n)*n+2n-2}{ln(2)}$$
$$= 0 \geq \frac{-ln(n)*n+2n-2}{ln(2)}$$

Use calculator to solve for n in equation and you get:

$$n \geq 1$$

Thus we shall choose n=1 and the equation is true, thus we have proven:

$$\frac{ln(n)*n-n+1}{ln(2)} \in \Omega(nln(n)) \tag{4}$$

(c) Prove equation is in $\Theta(n*log(n))$.
According to definition 1 above: A function is in $\Theta(nlog(n))$ if it has a $O(nlog(n))$ and a $\Omega(n*log(n)$. We have proven that it has both. Therefore the function:

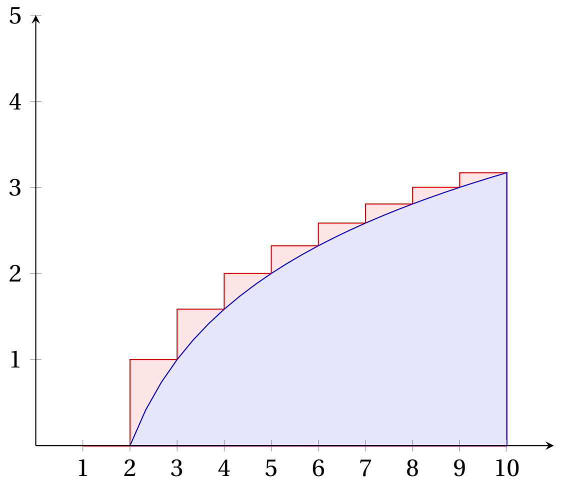
$$\frac{ln(n)*n-n+1}{ln(2)} \in \Theta(nln(n))$$

3.2.5 Step 5: Find Lower Bound

If we shift the original function $log_2(x)$ over one position to the right by adding 1 to the x coordinate to get the function $log_2(x-1)$. The summation function $\sum_1^n ln(n)$ will be larger than $\int_1^n ln(n)$ because if we break up the integral into a sum of smaller integrals like we did in step 2, each summation segment will be larger than its corresponding integral segment for the following reasons:

- At the beginning of each integer, the integral segment's value is smaller than the series equivalent.
- Between integers the value of the integral segment will approach the value of the summation segment. However, it will never pass it.
- Due to the fact that the function is always increasing in size. This pattern will continue infinitely.

See the graph below for a visual reference:

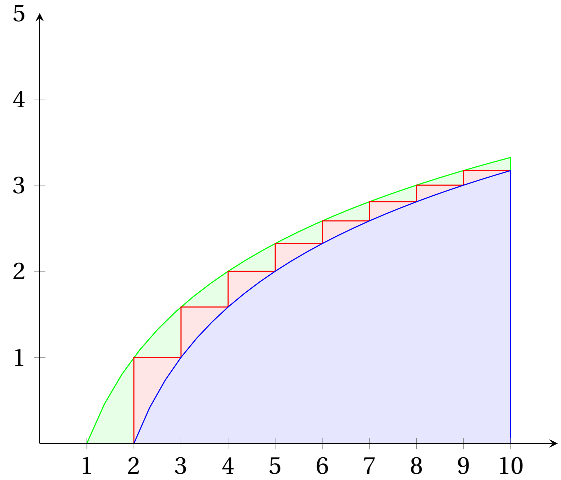


3.2.6 Find Integral of $log_2(x-1)$

For simplicity:

$$log_2(x-1) = \frac{ln(x-1)}{ln(2)}$$

beginenumerate



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4.1 Prompt

4.2 Answer

5 EX 5

5.1 Prompt

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6 EX 6

6.1 Prompt

6.2 Answer

7 EX 7

7.1 Prompt

7.2 Answer