Simple lifecycle model with mental health

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The model

Workers live T periods and face a discrete choice of whether to work or not. But in the last period nobody works. Periods are denoted by t. The labor force participation decision is denoted $n_t \in \{0, 1\}$.

Individuals derive period utility from consumption and leisure

$$u\left(c_t, l_t\right) = \frac{\left(c_t^{\alpha} l_t^{1-\alpha}\right)^{1-\sigma}}{1-\sigma}$$

where

$$l_t = 1 - \phi_n n_t - \phi_H \mathbf{1}_{H_t}$$

The parameter ϕ_n represents a time cost from labor decision n_t. ϕ_H represents a time cost associated with being in one of four composite health states $H = \{mf\} \in \{BB, BG, GB, GG\}$.

Consumption is constrained in the following way

$$c_t + a_{t+1} = z_t^H n_t + a_t \left(1 + r\right); \forall t$$

$$a_t > -\kappa; a_0 = a_{J+1} = 0; c_t, n_t \geq 0; \forall t$$

Where z_t^H is labor productivity and returns to labor which takes the following form

$$z_t^H = \lambda_t^H \exp\left(\nu_t\right) \exp\left(\gamma\right).$$

Where λ_t^H is a deterministic component that depends on age and health status. ν_t is a persistent $AR\left(1\right)$ shock that takes the form

$$\nu_t = \rho_\nu \nu_{t-1} + \varepsilon_t; \varepsilon_t \sim \mathcal{N}\left(0, \sigma_\varepsilon\right).$$

There will likely be a discrete (Markov?) implementation of this. Finally, γ is a fixed effect determined at birth.

The composite health state H is made up of a mental health component m and a physical health component f. Each of these health components can take on one of two binary states, that is good G or bad B. With a slight abuse of notations $m, f \in \{B, G\}$. Thus, the composite health state H can take on four possible values: $H \in \{BB, BG, GB, GG\}$. This health state evolves according to the one period Markov probability matrix Λ_t. Which for now is just conditional on age and the previous health status.

$$H' \sim \Lambda_t \left(H'|H\right); H_0 \sim \Lambda_0 \left(H\right)$$

From this set up we can derive equations that fully characterize the solution. These equations are:

$$\text{The Euler Eq.: } u_{c_t} = \mathbb{E}\, \beta \left(1 + r\right) u_{c_{t+1}} \tag{1}$$

$$\text{The static condition: } \phi_n u_{l_t} = z_t^H u_{c_t} \tag{2}$$

$$\text{The budget constraint: } c_t + a_{t+1} = z_t^H n_t + \left(1 + r\right) a_t \tag{3}$$

The sequential problem

Knowing that the period budget constraint is as above we can write the problem of the individual over his lifecycle in the following way

$$\max_{\{c_t, n_t, a_{t+1}\}} \sum_{t=0}^{t=T} \mathbb{E}\, \beta^t u \left(c_t, 1 - \phi_n n_t - \phi_H \mathbf{1}_{H_t}\right)$$

s.t.

$$\sum_{t=0}^{t=T} \frac{c_t - z_t^H n_t}{\left(1 + r\right)^t} = 0$$

or equivalently

$$c_t + a_{t+1} = z_t^H n_t + a_t \left(1 + r\right); \forall t$$

$$a_0 = a_{J+1} = 0;$$

and

$$a_t > -\kappa, \forall t; c_t, n_t \geq 0$$

$$H' \sim \Lambda_t \left(H'|H\right); H_0 \sim \Lambda_0 \left(H\right)$$

The recursive problem

Using the Bellman equation we are able to write this problem recursively. Let the state space be written as $S_t = \left\{a_t, z_t^H, H_t\right\}$. Which are all factors known to to the individual at time t which effect current utility or the probability distribution of future utilities. Every individual enters at $t=1$ and a_0. Health status H_0 is drawn from the initial distribution $\Lambda_0(H)$ and labor productivity z_0^H is drawn based on the shock process described above. The recursive problem is thus

$$V_t\left(a, z^H, H\right) = \max_{c,n,a'} \left\{ u\left(c, 1-\phi_n n - \phi_H \mathbf{1}_H\right) + \beta\, \mathbb{E}_{z',H'} V_{t+1}\left(a', z'^{H'}, H'\right) \right\}$$

s.t

$$c + a' = z^H n + a\left(1+r\right); \forall t$$

$$a_0 = a_{J+1} = 0;$$

and

$$a_t > -\kappa, \text{ and } c_t, n_t \geq 0 \forall t$$

$$H' \sim \Lambda_t\left(H'|H\right); H_0 \sim \Lambda_0\left(H\right)$$

and z_t^H follows the process described above.

Some derivatives

Recall

$$u\left(c_t, l_t\right) = \frac{\left(c_t^{\alpha} l_t^{1-\alpha}\right)^{1-\sigma}}{1-\sigma}$$

where

$$l_t = 1 - \phi_n n_t - \phi_H \mathbf{1}_{H_t}.$$

So the relevant derivatives of the utility function can be written as

$$\frac{\partial\left(u\right)}{\partial\left(c_t\right)} = \alpha c_t^{\alpha-1} l_t^{1-\alpha} \left(c_t^{\alpha} l_t^{1-\alpha}\right)^{-\sigma}$$

$$\frac{\partial\left(u\right)}{\partial\left(l_t\right)} = (1-\alpha) c_t^{\alpha} l_t^{-\alpha} \left(c_t^{\alpha} l_t^{1-\alpha}\right)^{-\sigma}$$

$$\frac{\partial\left(u\right)}{\partial\left(n_t\right)} = -\phi_n \frac{\partial\left(u\right)}{\partial\left(l_t\right)} = -\phi_n (1-\alpha) c_t^{\alpha} l_t^{-\alpha} \left(c_t^{\alpha} l_t^{1-\alpha}\right)^{-\sigma}$$

by the chain rule.

Using the static condition

We have the within period optimality condition

$$\phi_n u_l = z^H u_c$$

plugging in our derivatives from above gives

$$\phi_n(1-\alpha)c^{\alpha}l^{-\alpha}\left(c^{\alpha}l^{1-\alpha}\right)^{-\sigma} = z^H\alpha c^{\alpha-1}l^{1-\alpha}\left(c^{\alpha}l^{1-\alpha}\right)^{-\sigma}$$

$$\implies \frac{\phi_n}{z^H} = \frac{\alpha c^{\alpha-1}l^{1-\alpha}\left(c^{\alpha}l^{1-\alpha}\right)^{-\sigma}}{(1-\alpha)c^{\alpha}l^{-\alpha}\left(c^{\alpha}l^{1-\alpha}\right)^{-\sigma}} = \frac{\alpha}{(1-\alpha)} \times \frac{l}{c}$$

$$\implies \frac{\phi_n}{z^H}\frac{(1-\alpha)}{\alpha} * c = l$$

this gives us a simple relationship between current period consumption and leisure (and thus labor also).

Using the consumption Euler equation

The Euler equation is

$$u_c(c,l) = \beta(1+r)\mathbb{E}_{H',z'}\left\{u_{c'}\left(c',l'\right)\right\}$$

from the above static condition we know we can rewrite both sides as functions of only c

$$u_c(c,l(c)) = \beta(1+r)\mathbb{E}_{H',z'}\left\{u_{c'}\left(c',l'\left(c'\right)\right)\right\}$$

where

$$l(c) = \frac{\phi_n}{z^H}\frac{(1-\alpha)}{\alpha} * c$$

we thus have a relationship between this periods consumption c and next periods consumption c' only in terms of consumption and parameters.

Some inverse derivatives

Inverse marginal utility of consumption A useful function in solving the problem numerically is the inverse of $\frac{\partial u}{\partial c} \equiv \left(\frac{\partial u}{\partial c}\right)^{(-1)} \equiv (u_c)^{(-1)}$. Let

$$x = u_c = \alpha c^{\alpha-1}l^{1-\alpha}\left(c^{\alpha}l^{1-\alpha}\right)^{-\sigma}$$

$$\implies \frac{c^{\alpha-1}l^{1-\alpha}}{\left(c^{\alpha}l^{1-\alpha}\right)^{\sigma}} = \frac{x}{\alpha}$$

Which Mathematica solves for c as

$$Solve\left[\frac{c^{\alpha-1}l^{1-\alpha}}{\left(c^{\alpha}l^{1-\alpha}\right)^{\sigma}} == \frac{x}{\alpha}, c\right]$$

$$\implies c = \left(\frac{xl^{-((1-\alpha)(1-\sigma))}}{\alpha}\right)^{\frac{1}{\alpha(-\sigma)+\alpha-1}}$$

so

$$(u_c)^{-1} = \left(\frac{xl^{-((1-\alpha)(1-\sigma))}}{\alpha}\right)^{\frac{1}{\alpha(-\sigma)+\alpha-1}}$$

Which is confirmed analytically. Transform for easy reading into python

$$c = \left(\frac{x}{a * l^{(1-a)*(1-s)}}\right)^{\frac{1}{a-1-s*a}}$$

confirming that python reads it in properly:

$$\left(\frac{x}{al^{(1-a)(1-s)}}\right)^{1\cdot\frac{1}{-as+a-1}}$$

And since $\frac{\phi_n}{z^H}\frac{(1-\alpha)}{\alpha} * c = l$ we have

$$c = \left(\frac{x\left(\frac{\phi_n}{z^H}\frac{(1-\alpha)}{\alpha} * c\right)^{-((1-\alpha)(1-\sigma))}}{\alpha}\right)^{\frac{1}{\alpha(-\sigma)+\alpha-1}}$$

again solving for c

$$Solve\left[c == \left(\frac{x\left(\frac{p}{z}\frac{(1-\alpha)}{\alpha} * c\right)^{-((1-\alpha)(1-\sigma))}}{\alpha}\right)^{\frac{1}{\alpha(-\sigma)+\alpha-1}}, c\right]$$

$$\left\{\left\{c \to \left(\frac{\alpha\left((1-\alpha)^{\alpha\sigma-\alpha-\sigma}\alpha^{\alpha(-\sigma)+\alpha+\sigma-1}\phi^{\alpha\sigma-\alpha-\sigma+1}z^{\alpha(-\sigma)+\alpha+\sigma-1} - (1-\alpha)^{\alpha\sigma-\alpha-\sigma}\alpha^{\alpha(-\sigma)+\alpha+\sigma}\phi^{\alpha\sigma-\alpha-\sigma+1}\right.}{x}\right.\right.\right.$$

or let $K = \frac{\phi_n}{z^H}\frac{(1-\alpha)}{\alpha}$

$$Solve\left[\frac{c^{\alpha-1}(K * c)^{1-\alpha}}{\left(c^{\alpha}(K * c)^{1-\alpha}\right)^{\sigma}} == \frac{x}{\alpha}, c\right]$$

$$\left\{\left\{c \to \left(\frac{xK^{\alpha(-\sigma)+\alpha+\sigma-1}}{\alpha}\right)^{-1/\sigma}\right\}\right\}$$

so

$$(u_c)^{-1} = \left(\frac{xK^{\alpha(-\sigma)+\alpha+\sigma-1}}{\alpha}\right)^{-1/\sigma}$$

where $K = \frac{\phi_n}{z^H}\frac{(1-\alpha)}{\alpha}$.

Inverse marginal utility of leisure Similarly we have

$$x = u_l = (1-\alpha)c^{\alpha}l^{-\alpha}\left(c^{\alpha}l^{1-\alpha}\right)^{-\sigma}$$

Setting up the equation to solve gives

$$Solve\left[\frac{x}{(1-\alpha)} == c^{\alpha}l^{-\alpha}\left(c^{\alpha}l^{1-\alpha}\right)^{-\sigma}, l\right]$$

$$\implies l = \left(\frac{xc^{-\alpha(1-\sigma)}}{1-\alpha}\right)^{\frac{1}{\alpha\sigma-\alpha-\sigma}}$$

so

$$(u_l)^{-1} = \left(\frac{xc^{-\alpha(1-\sigma)}}{1-\alpha}\right)^{\frac{1}{\alpha\sigma-\alpha-\sigma}}$$

For python

$$\left(\frac{x * c^{-\alpha*(1-\sigma)}}{1-\alpha}\right)^{\frac{1}{\alpha*\sigma-\alpha-\sigma}}$$

from python

$$\left(\frac{c^{-a(1-s)}x}{1-a}\right)^{1\cdot\frac{1}{as-a-s}}$$

Analytical solutions

Analytically solving for the consumption, leisure and therefore labor decisions given the current asset stock a_t and the choice of future assets a_{t+1} yields

$$c_t = \alpha\left[\frac{z_t^H}{\phi_n}\left(1-\phi_{H_t}\right) + (1+r)\,a_t - a_{t+1}\right]$$

$$l_t = (1-\alpha)\left[\left(1-\phi_{H_t}\right) + \frac{\phi_n}{z_H^t}\left((1+r)\,a_t - a_{t+1}\right)\right]$$

and thus

$$n_t = \frac{\alpha}{\phi_n}\left(1-\phi_{H_t}\right) + \frac{\alpha-1}{z_H^t}\left((1+r)\,a_t - a_{t+1}\right)$$

Testing some python latex input output Input

$$\frac{\partial\,(u)}{\partial\,(c_t)} = \alpha * c_t^{\alpha-1}l_t^{1-\alpha}\left(c_t^{\alpha}l_t^{1-\alpha}\right)^{-\sigma}$$

output

$$ac^{a-1}l^{1-a}\left(c^{a}l^{1-a}\right)^{-s}$$

Input

$$\frac{\partial\left(u\right)}{\partial\left(l_t\right)} = \left(1-\alpha\right) * c_t^{\alpha} l_t^{-\alpha}\left(c_t^{\alpha} l_t^{1-\alpha}\right)^{-\sigma}$$

output

$$c^{a} l^{-a}\left(c^{a} l^{1-a}\right)^{-s}\left(1-a\right)$$