Dynamic programming example

February 20, 2017

The model

Workers live T periods and face only a discrete choice of whether to work or not each period except the last one, when nobody works. Periods are denoted by t. Denote the participation decision with $d_t = \{0, 1\}$.

Individuals derive period utility from consumption and leisure:

$$u_t(d_t) = ln(c_t(d_t)) + \theta d_t + \eta_t d_t$$

where θ is a parameter representing disutility from working, and $\eta_t \sim N(0, \sigma_d)$ a shock to such disutility. Consumption is constrained by the sum of labor and non-labor income:

$$c_t(d_t) = w_t(d_t) + n_t(d_t).$$

Saving or borrowing is not allowed.

Non-labor income n_t depends linearly on education, which takes three discrete values $e_t \epsilon \{0, 1, 2\}$, work participation d_t, and marital status $m_t \epsilon \{0, 1\}$:

$$\ln(n_t(d_t)) = \gamma_0 + \gamma_1 I(e_i = 1) + \gamma_2 I(e_i = 2) + \gamma_3 d_t + \gamma_4 m.$$

If an individual receives a job offer, that job comes with annual log earnings that depend on education, experience $x_t \epsilon \{0, 1, ..., T-1\}$, whether they worked last period, and marital status, plus a normally distributed wage shock $\epsilon_t \sim N(0, \sigma_w)$:

$$\ln(w_t(1)) = \alpha_0 + \alpha_1 I(e_i = 1) + \alpha_2 I(e_i = 2) + \alpha_3 x_t + \alpha_4 x_t^2 + \alpha_5 (1 - d_{t-1}) + \epsilon_t$$

Parameter α_5 represents a cost to re-entering the labor market (if less than zero). Labor income is zero if the person does not receive a job offer or rejects the offer

$$w_t(0) = 0$$

Agents that worked in period $t-1$ receive with certainty a wage offer w_t. Agents that did not work in $t-1$ receive a wage offer with a probability $p_t < 1$ that depends on education and marital status specified by the following equation:

$$\begin{aligned} p_t &= \frac{exp(\lambda_0 + \lambda_1 I(e_i = 1) + \lambda_2 I(e_i = 2) + \lambda_3 m)}{1 + exp(\lambda_0 + \lambda_1 I(e_i = 1) + \lambda_2 I(e_i = 2) + \lambda_3 m)}, \ t < T \\ p_T &= 0 \end{aligned}$$

Note that the logit formulation is convenient to make this number strictly between zero and one. Nobody works in the last period.

We can write lifetime utility recursively using the Bellman equation. Denote the state space vector with $S_t = \{x_t, e_t, d_{t-1}, m_t, \epsilon_t, \eta_t\}$, which are all factors known to the individual at t affecting current utility or the probability distribution of future utilities. Each individual enters $t = 1$ with $x_1 = 0$ and $d_0 = 0$. The value function is

$$V(S_t) \quad = \quad \begin{cases} max_{d_t} E\left[\sum_{\tau=1}^{T} \beta^{\tau-1} u_\tau(d_\tau)|S_\tau\right] & t < T \\ u_t(0) & t = T \end{cases}$$

All deterministic variables of the state space are pre-determined exogenously at time $t = 1$, and ony x_t evolves according to:

$$x_t = x_{t-1} + d_{t-1}.$$

The value function can be written as the maximum over two alternative-specific value functions $V_t^d(S_t)$,$d = \{0, 1\}$:

$$V(S_t) \quad = \quad \begin{cases} max\left(V_t^0(S_t), V_t^1(S_t)\right) & \text{if an offer is received} \\ V_t^0(S_t) & \text{otherwise} \end{cases}$$

where

$$V_t^d(S_t) = \begin{cases} u_t(d) + \beta E\left[V(S_{t+1})|S_t, d\right] & \text{for } t < T \\ u_t(0) & \text{for } t = T \end{cases} \tag{1}$$

The expectation in (1) is taken over the random components of the state space at $t+1$ conditional on the state space elements at t. Equation (1) defines a latent variable v_t^* that we can use to compute the probability of choosing to work

$$v_t^*(S_t) \quad = \quad V_t^1(S_t) - V_t^0(S_t)$$

that is:

$$v^*(S_t) = u_t(1) - u_t(0) + \beta\left[E\left[V(S_{t+1})|S_t, d_t = 1\right] - E\left[V(S_{t+1})|S_t, d_t = 0\right]\right]$$

and we can use normality to write the distribution of the latent variable [check??]:

$$v_t^*(\bar{S}_t) \sim N(ln\left(w_t(1) + n_t(1)\right) + \theta - ln\left(n_t(0)\right), exp(\bar{w})^2\sigma_\epsilon^2 + \sigma_\eta^2)$$

where $\bar{S}_t$ is the vector of deterministic components of the steady state, and $\bar{w}_t$ is the deterministic part of the wage function.