Optimal Portfolio Weights I: Mean-Variance Optimization

In [1]:

```
# Working with data:
import numpy  as np                                      # For scientific computing
import pandas as pd                                      # Working with tables.

# Downloading files:
import requests, zipfile, io                                   # To access websites

# Specific data providers:
from tiingo import TiingoClient                          # Stock prices.
import quandl                                            # Economic data, futures p

# API keys:
tiingo = TiingoClient({'api_key':'XXXX'})
quandl.ApiConfig.api_key = 'YYYY'

# Plotting:
import matplotlib.pyplot as plt                           # Basic plot library.
plt.style.use('ggplot')                                   # Make plots look nice
```

Get data

Get ETF prices and returns (TLT: 20+ year treasuries, IEF: 7-10 year treasuries, SHY: 1-3 year treasuries:

In [2]:

```
# start in 2003 since Treasury ETFs not available earlier
PRICE = tiingo.get_dataframe(['SPY','TLT','IEF','SHY'], '2003-1-1', metric_name=
PRICE.index = pd.to_datetime(PRICE.index).tz_convert(None)

RET = PRICE.pct_change()
RET[:3]
```

Out[2]:

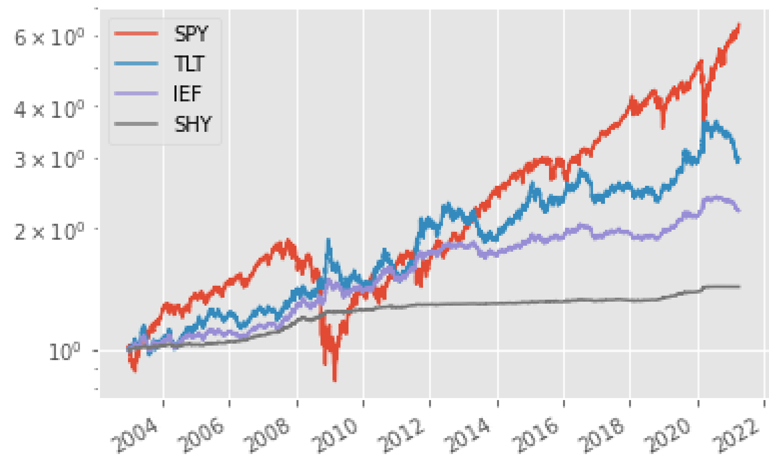
	SPY	TLT	IEF	SHY
2003-01-02	NaN	NaN	NaN	NaN
2003-01-03	0.003075	0.002318	0.001651	0.000122
2003-01-06	0.017625	-0.002660	-0.002473	-0.000610

Compare stock and treasury compond returns:

In [3]:

```
RET.add(1).cumprod().plot(logy=True)
```

Out[3]: <AxesSubplot:>



Get federal funds rate and treasury yields:

In [4]:
```
RATES = quandl.get(['FRED/FEDFUNDS','FRED/DGS1','FRED/DGS5','FRED/DGS10','FRED/D
RATES.columns = ['FedFunds','Treasury_1', 'Treasury_5', 'Treasury_10', 'Treasury
RATES
```

Out[4]:

Date	FedFunds	Treasury_1	Treasury_5	Treasury_10	Treasury_30
1954-07-01	0.0080	NaN	NaN	NaN	NaN
1954-08-01	0.0122	NaN	NaN	NaN	NaN
1954-09-01	0.0107	NaN	NaN	NaN	NaN
1954-10-01	0.0085	NaN	NaN	NaN	NaN
1954-11-01	0.0083	NaN	NaN	NaN	NaN
...	...	...	...	...	...
2021-03-29	NaN	0.0006	0.0089	0.0173	0.0243
2021-03-30	NaN	0.0006	0.0090	0.0173	0.0238
2021-03-31	NaN	0.0007	0.0092	0.0174	0.0241
2021-04-01	NaN	0.0006	0.0090	0.0169	0.0234
2021-04-02	NaN	0.0007	0.0097	0.0172	0.0235

15146 rows × 5 columns

Calculate margin rate:

In [5]:
```
RET = RET.join(RATES.FedFunds.rename('MarginRate'), how='outer')
RET['MarginRate'] = RET.MarginRate.ffill()/252 + 0.01/252          # Assume mar
RET = RET.dropna(subset=['SPY'])
RET
```

Out[5]:

	SPY	TLT	IEF	SHY	MarginRate
2003-01-03	0.003075	0.002318	0.001651	0.000122	0.000089
2003-01-06	0.017625	-0.002660	-0.002473	-0.000610	0.000089

	SPY	TLT	IEF	SHY	MarginRate
2003-01-07	-0.002474	0.003594	0.003187	0.000732	0.000089
2003-01-08	-0.014451	0.004968	0.002353	0.000853	0.000089
2003-01-09	0.015538	-0.019198	-0.012091	-0.001827	0.000089
...	...	...	...	...	...
2021-03-29	-0.000505	-0.008488	-0.003434	-0.000232	0.000042
2021-03-30	-0.002653	0.005240	-0.000883	0.000000	0.000042
2021-03-31	0.004053	-0.005580	-0.001415	-0.000348	0.000042
2021-04-01	0.010799	0.016575	0.004425	-0.000008	0.000042
2021-04-05	0.014353	-0.004363	-0.002823	-0.000232	0.000042

4594 rows × 5 columns

Stocks vs Bonds

Compare annual returns of SPY and TLT:

In [6]:

```
# Compound daily returns within each year:
RET[['SPY','TLT']].add(1).groupby(RET.index.year).prod().sub(1)  # Or use "resam
```

Out[6]:

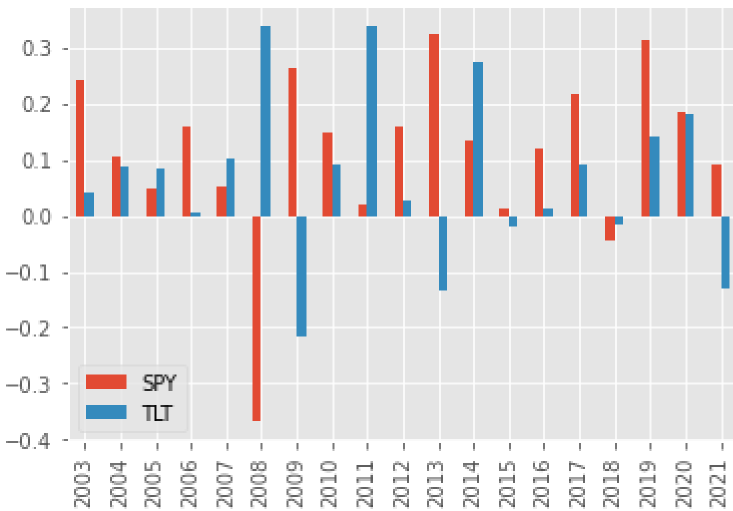
	SPY	TLT
2003	0.241793	0.043166
2004	0.107028	0.087111
2005	0.048258	0.086066
2006	0.158482	0.007108
2007	0.051356	0.102911
2008	-0.368069	0.339240
2009	0.263661	-0.218006
2010	0.150577	0.090460
2011	0.018879	0.339593
2012	0.159917	0.026311
2013	0.323067	-0.133718
2014	0.134621	0.272975
2015	0.012523	-0.017872
2016	0.120013	0.011795
2017	0.217003	0.091814
2018	-0.045571	-0.016075
2019	0.312217	0.141207
2020	0.183732	0.181521

	SPY	TLT
2021	0.090439	-0.128765

Plot this:

In [7]:
```
RET[['SPY','TLT']].add(1).groupby(RET.index.year).prod().sub(1).plot.bar()
```

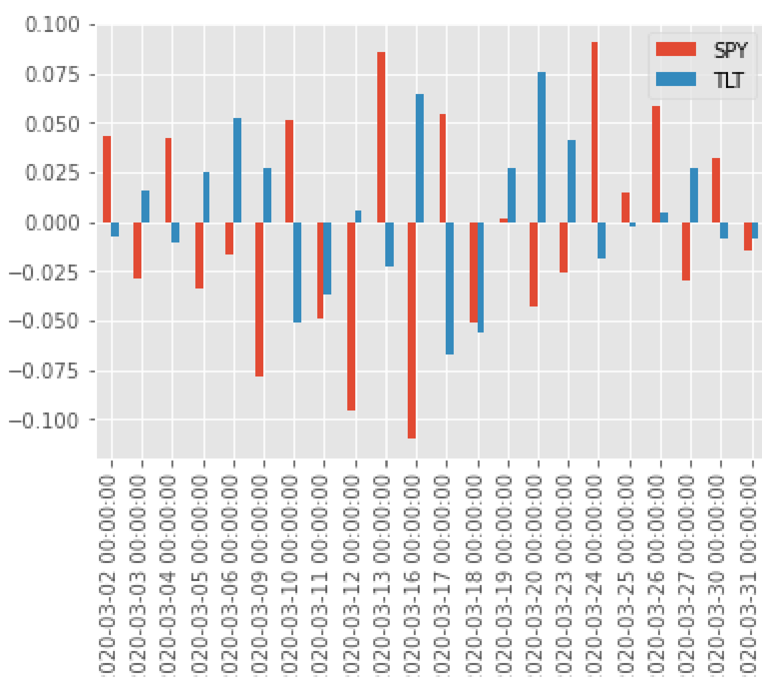
Out[7]: <AxesSubplot:>



Compare daily returns in March 2020:

In [9]:
```
RET.loc['2020-3',['SPY','TLT']].plot.bar()
```

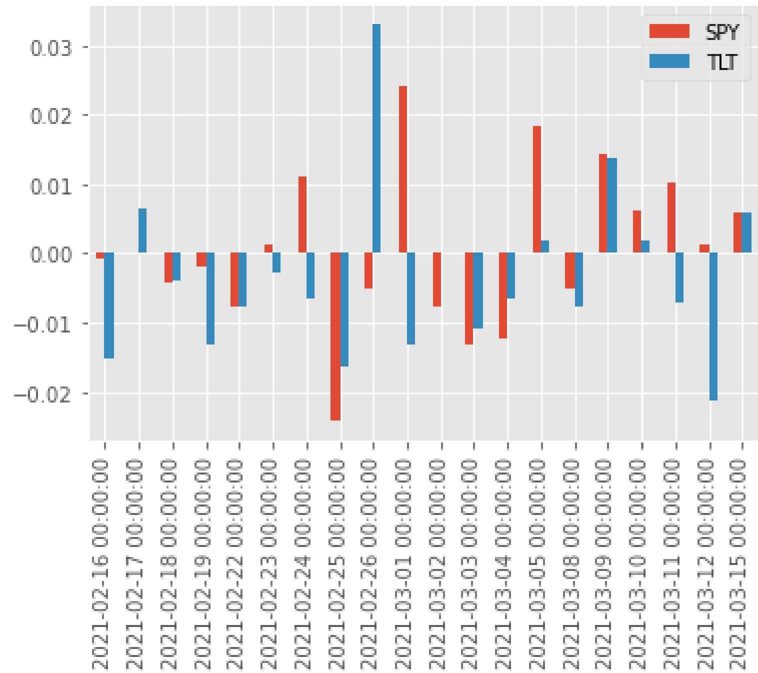
Out[9]: <AxesSubplot:>



Daily return February 15 to March 15 2021:

In [10]:
```
RET.loc['2021-2-15':'2021-3-15',['SPY','TLT']].plot.bar()
```

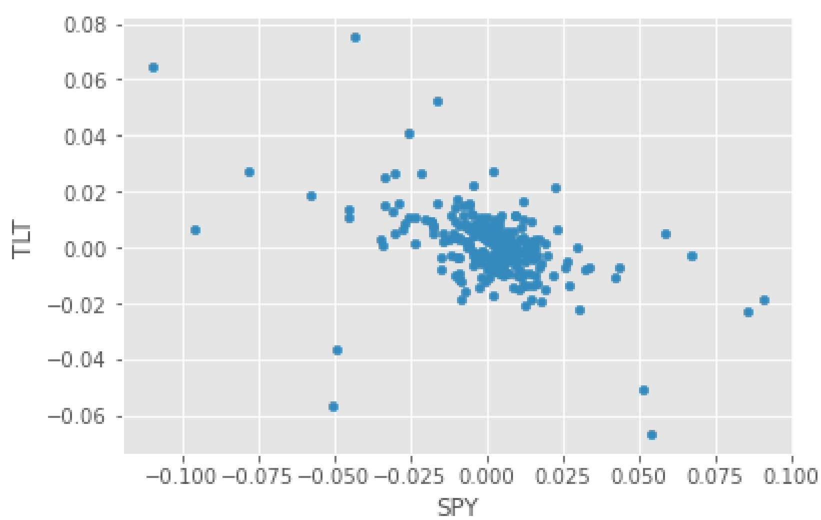
Out[10]: <AxesSubplot:>



Scatter plot of daily returns in 2020:

In [11]:
```
RET.loc['2020'].plot.scatter('SPY','TLT')
```

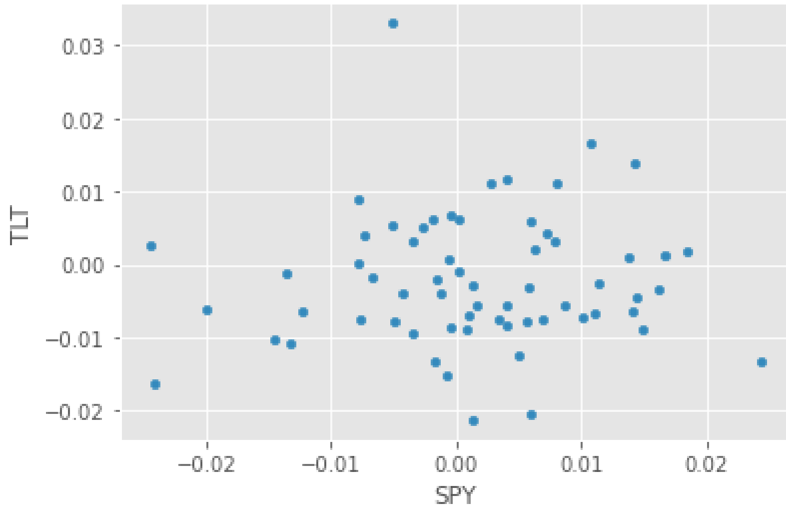
Out[11]: <AxesSubplot:xlabel='SPY', ylabel='TLT'>



Same graph for 2021:

In [13]:
```
RET.loc['2021'].plot.scatter('SPY','TLT')
```

Out[13]: <AxesSubplot:xlabel='SPY', ylabel='TLT'>



Correlation of returns for entire sample:

In [14]:
```
RET.corr()
```

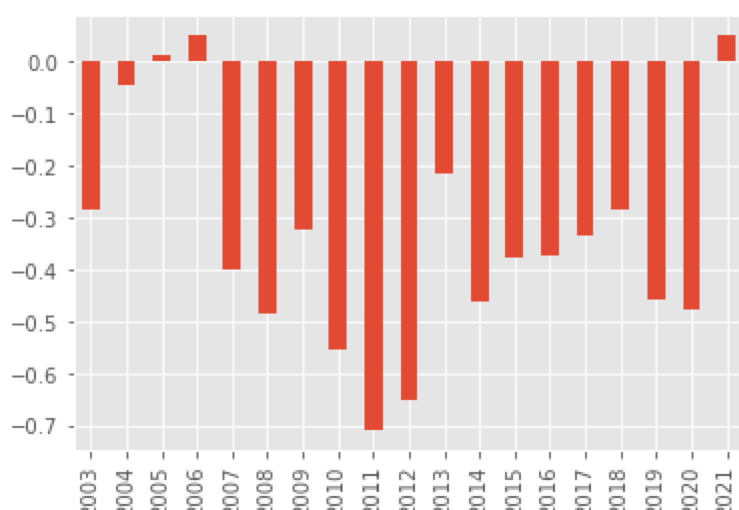
Out[14]:

	SPY	TLT	IEF	SHY	MarginRate
SPY	1.000000	-0.402390	-0.390339	-0.331498	-0.012482
TLT	-0.402390	1.000000	0.911492	0.571435	0.004984
IEF	-0.390339	0.911492	1.000000	0.744602	0.011905
SHY	-0.331498	0.571435	0.744602	1.000000	0.069698
MarginRate	-0.012482	0.004984	0.011905	0.069698	1.000000

Correlation for each year:

In [15]:
```
RET[['SPY','TLT']].groupby(RET.index.year).corr().unstack().SPY.TLT.plot.bar()
```

Out[15]: <AxesSubplot:>



Mean-variance optimization

Annual returns:

In [16]:
```
r_annual = RET[:'2020'].add(1).resample('A').prod().sub(1)
r_annual
```

Out[16]:

	SPY	TLT	IEF	SHY	MarginRate
2003-12-31	0.241793	0.043166	0.037300	0.021914	0.021391
2004-12-31	0.107028	0.087111	0.041268	0.006631	0.023832
2005-12-31	0.048258	0.086066	0.026422	0.015310	0.043118
2006-12-31	0.158482	0.007108	0.025180	0.038910	0.061250
2007-12-31	0.051356	0.102911	0.103745	0.073509	0.061777
2008-12-31	-0.368069	0.339240	0.179071	0.066157	0.029747
2009-12-31	0.263661	-0.218006	-0.065888	0.003526	0.011663
2010-12-31	0.150577	0.090460	0.093724	0.022779	0.011827
2011-12-31	0.018879	0.339593	0.156441	0.014405	0.011074
2012-12-31	0.159917	0.026311	0.036634	0.002762	0.011376
2013-12-31	0.323067	-0.133718	-0.060875	0.002151	0.011133
2014-12-31	0.134621	0.272975	0.090651	0.004467	0.010954
2015-12-31	0.012523	-0.017872	0.015097	0.004293	0.011396
2016-12-31	0.120013	0.011795	0.010058	0.008205	0.014051
2017-12-31	0.217003	0.091814	0.025545	0.002615	0.020162
2018-12-31	-0.045571	-0.016075	0.009891	0.014642	0.028623
2019-12-31	0.312217	0.141207	0.080295	0.033802	0.032078
2020-12-31	0.183732	0.181521	0.100067	0.030343	0.013801

Calculate annual excess returns (subtract 1-year treasury rates):

In [17]:
```
r_annual_Tbill = RATES.Treasury_1.resample('A').first()  # Yield at the beginnin
r_annual_Tbill
```

Out[17]:
```
Date
1954-12-31       NaN
1955-12-31       NaN
1956-12-31       NaN
1957-12-31       NaN
1958-12-31       NaN
               ...
2017-12-31    0.0089
2018-12-31    0.0183
2019-12-31    0.0260
2020-12-31    0.0156
2021-12-31    0.0010
Freq: A-DEC, Name: Treasury_1, Length: 68, dtype: float64
```

In [18]:
```
rx_annual = r_annual.sub(r_annual_Tbill, 'rows').dropna() # Subtract a series fr
```

```
rx_annual
```

Out[18]:

	SPY	TLT	IEF	SHY	MarginRate
2003-12-31	0.227593	0.028966	0.023100	0.007714	0.007191
2004-12-31	0.093928	0.074011	0.028168	-0.006469	0.010732
2005-12-31	0.020358	0.058166	-0.001478	-0.012590	0.015218
2006-12-31	0.114682	-0.036692	-0.018620	-0.004890	0.017450
2007-12-31	0.001356	0.052911	0.053745	0.023509	0.011777
2008-12-31	-0.399769	0.307540	0.147371	0.034457	-0.001953
2009-12-31	0.259661	-0.222006	-0.069888	-0.000474	0.007663
2010-12-31	0.146077	0.085960	0.089224	0.018279	0.007327
2011-12-31	0.015979	0.336693	0.153541	0.011505	0.008174
2012-12-31	0.158717	0.025111	0.035434	0.001562	0.010176
2013-12-31	0.321567	-0.135218	-0.062375	0.000651	0.009633
2014-12-31	0.133321	0.271675	0.089351	0.003167	0.009654
2015-12-31	0.010023	-0.020372	0.012597	0.001793	0.008896
2016-12-31	0.113913	0.005695	0.003958	0.002105	0.007951
2017-12-31	0.208103	0.082914	0.016645	-0.006285	0.011262
2018-12-31	-0.063871	-0.034375	-0.008409	-0.003658	0.010323
2019-12-31	0.286217	0.115207	0.054295	0.007802	0.006078
2020-12-31	0.168132	0.165921	0.084467	0.014743	-0.001799

Let's calculate the risk premium and volatility of a portfolio of SPY and TLT.

Risk premiums for these assets:

In [19]:

```
meanx = rx_annual[['SPY','TLT']].mean()
meanx
```

Out[19]:

```
SPY     0.100888
TLT     0.064561
dtype: float64
```

Example weights:

In [20]:

```
w = pd.Series({'SPY':0.7, 'TLT':0.3})
w
```

Out[20]:

```
SPY     0.7
TLT     0.3
dtype: float64
```

Calculate portfolio risk premium:

$$\text{portfolio risk premium} \quad = \quad \mathbf{w} \cdot E[\mathbf{r}^x] \quad = \quad \underbrace{\begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \times \begin{bmatrix} 0.100888 \\ 0.064561 \end{bmatrix}}_{\text{"dot product"}} \quad = \quad \underline{0.7 \times 0.10}$$

Implement this calculation:

In [21]:
```
(w * meanx).sum()          # weighted average of individual risk premiums
```

Out[21]: 0.08999016538623958

Or use "dot" method:

In [22]:
```
w.dot(meanx)
```

Out[22]: 0.08999016538623958

Note: to use the "dot" method, you need to have the same indexes in "w" and in "meanx" (here: SPY, TLT). If the indexes are not the same, you get a "matrixes not aligned" error. For example:

In [23]:
```
m = rx_annual[['SPY','TLT', 'IEF']].mean()

w.dot(m)
```

```
---------------------------------------------------------------------------
ValueError                                Traceback (most recent call last)
<ipython-input-23-8c0012b420a0> in <module>
      1 m = rx_annual[['SPY','TLT', 'IEF']].mean()
      2 
----> 3 w.dot(m)

~/opt/anaconda3/lib/python3.7/site-packages/pandas/core/series.py in dot(self, o
ther)
   2539             common = self.index.union(other.index)
   2540             if len(common) > len(self.index) or len(common) > len(other.
index):
-> 2541                 raise ValueError("matrices are not aligned")
   2542 
   2543             left = self.reindex(index=common, copy=False)

ValueError: matrices are not aligned
```

The portfolio variance is the dot product between the portfolio weights and the covariance matrix:

$$\text{portfolio variance} \; = \text{weights} \; \times \; \text{covariance matrix} \; \times \; \text{weights}$$

Implement this:

In [24]:
```
# Covariance matrix:
cov = RET[['SPY','TLT']].cov() * 252       # multiply by 252 to annualize the cov
cov
```

Out[24]:

	SPY	TLT
SPY	0.036210	-0.010649

	SPY	TLT
TLT	-0.010649	0.019343

Portfolio volatility:

In [25]:
```
w.dot(cov).dot(w) ** 0.5        # here we use the annualized cov (otherwise we ne
```

Out[25]: 0.12252024425183612

Repeat this calculation for multiple weights:

In [26]:
```
t = pd.DataFrame(index=range(11))

for i in t.index:
    w                        = pd.Series({'SPY':i/10, 'TLT':1-i/10})
    t.loc[i,'SPY']           = w.SPY
    t.loc[i,'TLT']           = w.TLT
    t.loc[i,'Risk_premium'] = w.dot(meanx)
    t.loc[i,'Volatility']   = w.dot(cov).dot(w) ** 0.5

t['Sharpe'] = t.Risk_premium / t.Volatility
t
```

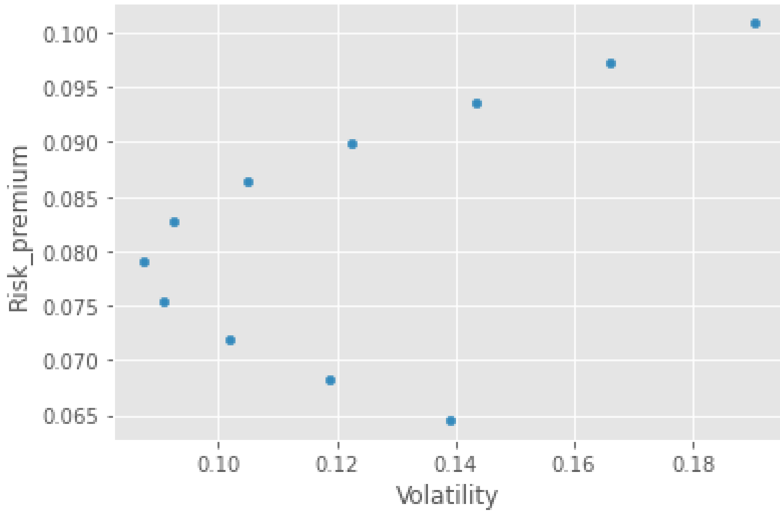
Out[26]:

	SPY	TLT	Risk_premium	Volatility	Sharpe
0	0.0	1.0	0.064561	0.139079	0.464207
1	0.1	0.9	0.068194	0.118798	0.574033
2	0.2	0.8	0.071827	0.102079	0.703640
3	0.3	0.7	0.075459	0.090908	0.830066
4	0.4	0.6	0.079092	0.087438	0.904551
5	0.5	0.5	0.082725	0.092540	0.893937
6	0.6	0.4	0.086357	0.104971	0.822680
7	0.7	0.3	0.089990	0.122520	0.734492
8	0.8	0.2	0.093623	0.143320	0.653244
9	0.9	0.1	0.097256	0.166153	0.585336
10	1.0	0.0	0.100888	0.190290	0.530181

Plot risk vs return:

In [27]:
```
t.plot.scatter('Volatility','Risk_premium')
```

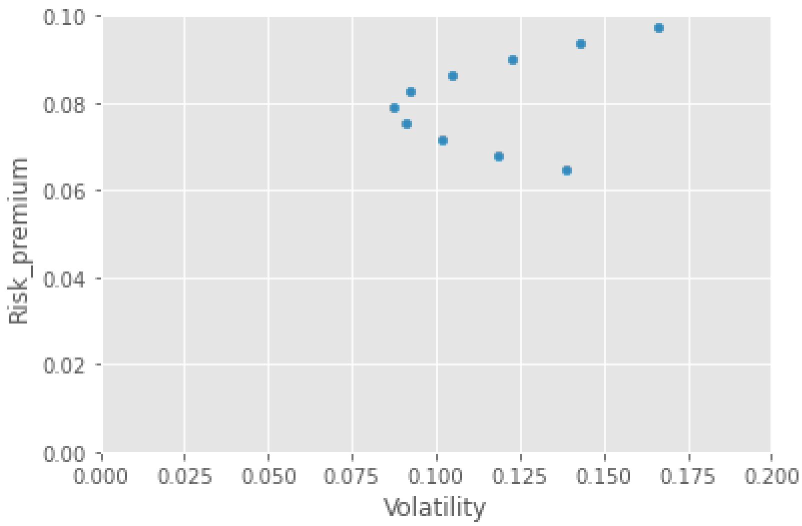
Out[27]: <AxesSubplot:xlabel='Volatility', ylabel='Risk_premium'>



The Sharpe ratio is the slope of a line that goes from through the point of the specific SPY/TLT combination:

In [28]:
```
t.plot.scatter('Volatility','Risk_premium', ylim=(0,0.1), xlim=(0,0.2))
```

Out[28]: <AxesSubplot:xlabel='Volatility', ylabel='Risk_premium'>



Maximum Sharpe ratio weights:

$$\mathbf{w} = \frac{\mathbf{\Sigma}^{-1}E[\mathbf{r}^x]}{\mathrm{Sum}\left(\mathbf{\Sigma}^{-1}E[\mathbf{r}^x]\right)}$$

($\Sigma =$ covariance matrix).

Calculate the inverse of the covariance matrix:

In [29]:
```
np.linalg.inv(cov)
```

Out[29]:
```
array([[32.95187294, 18.14189977],
       [18.14189977, 61.68663144]])
```

Make this easier to read:

In [30]:
```
cov_inv = pd.DataFrame(np.linalg.inv(cov), columns=cov.columns, index=cov.index)
cov_inv
```

Out[30]:

	SPY	TLT
SPY	32.951873	18.141900
TLT	18.141900	61.686631

Maximum Sharpe ratio weights:

In [31]:
```
w_maxSharpe = cov_inv.dot(meanx) /  cov_inv.dot(meanx).sum()
w_maxSharpe
```

Out[31]:
```
SPY     0.436114
TLT     0.563886
dtype: float64
```

Minimum volatility portfolio weights:

$$\mathbf{w}_{\mathbf{minvol}} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathrm{Sum}(\mathbf{\Sigma}^{-1}\mathbf{1})}$$

where $\mathbf{1}$ is a vector of ones and $\mathbf{\Sigma}^{-1}\mathbf{1}$ is row or column sum of the inverse covariance matrix.

Implement this:

In [32]:
```
w_minVol = cov_inv.sum() / cov_inv.sum().sum()
w_minVol
```

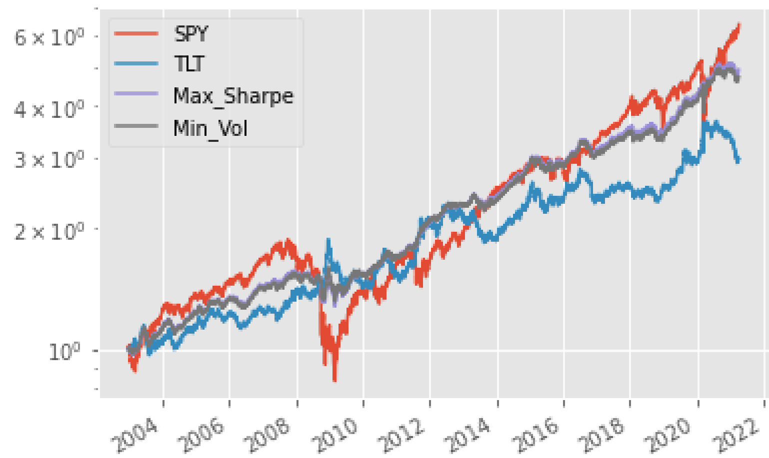
Out[32]:
```
SPY     0.39026
TLT     0.60974
dtype: float64
```

Compound returns of optimal portfolios

In [33]:
```
t = pd.DataFrame()
t['SPY']          = RET.SPY
t['TLT']          = RET.TLT
t['Max_Sharpe'] = RET.multiply(w_maxSharpe).sum('columns')
t['Min_Vol']     = RET.multiply(w_minVol).sum('columns')

t.add(1).cumprod().plot(logy=True)
```

Out[33]:
```
<AxesSubplot:>
```


SPY
TLT
Max_Sharpe
Min_Vol
6×10^0
4×10^0
3×10^0
2×10^0
10^0
2004
2006
2008
2010
2012
2014
2016
2018
2020
2022