Measures on the real line

Lemma 1.1.10 Suppose μ defined on semialgebra S have $\mu(\emptyset) = 0$ satisfy finite additivity of μ on $\mathcal{S}$ i.e. if $S \in \mathcal{S}$. If $\bar{\mu}$ is the unique extension of μ on $\bar{\mathcal{S}}$, the algebra generated by $\mathcal{S}$, then $\bar{\mu}(A_i) \leq \Sigma_i \bar{\mu}(B_i)$ where $A, B_i \in \bar{\mathcal{S}}$ s.t. $A \subset \cup_{i=1}^n B_i$.
Proof of Lemma 1.1.10 Suppose if $S_i \in \bar{\mathcal{S}}$ s.t. $A = +_i S_i$, then for each S_i, $\exists S_{i,j} \in \mathcal{S} : +_j S_{i,j} = S_i$. So,

$$\bar{\mu}(A) = \Sigma_{i,j}\mu(S_{i,j}) = \Sigma_i \bar{\mu}(S_i)$$

Set $F_1 = B_1$ and $F_n = B_n - \cup_1^{n-1} B_i$ so that

$$\cup_i B_i = F_1 + \cdots F_n$$

$$A = A \cap (\cup_i B_i) = (A \cap F_1) + \cdots + (A \cap F_n)$$

So using the finite additivity and the fact above,

$$\bar{\mu}(A) = \Sigma_{k=1}^n \bar{\mu}(A \cap F_k) \leq \Sigma_{k=1}^n \bar{\mu}(F_k) = \bar{\mu}(\cup_i B_i)$$

Theorem 1.1.9 Let $\mathcal{S}$ be a semialgebra and let μ defined on $\mathcal{S}$ have $\mu(\emptyset) = 0$. Suppose μ satistfy (i) Finite Additivity, (ii) Countable Subadditivity i.e. if $S_i, S \in \mathcal{S}$ with $S = +_{i\geq 1} S_i$, then $\mu(S) \leq \Sigma_{i\geq 1}\mu(S_i)$. Then μ has a unique extension $\bar{\mu}$ that is a measure on $\bar{\mathcal{S}}$, the algebra generated by $\mathcal{S}$. If $\bar{\mu}$ is a sigma finite, then there is a unique extension ν that is a measure on $\sigma(\mathcal{S})$.
The proof of Theorem 1.1.9 is a long road which is given in Section A.1 in *Durrett, Proability Theory and Examples, 5th Ed.*
Theorem 1.1.4 Associated with each Stieltjes measure function F there is a unique measure μ on $(R, \mathcal{R})$ with $\mu((a, b]) = F(b) - F(a)$

$$\mu((a, b]) = F(b) - F(a) \tag{1}$$

When $F(x) = x$, the resulting measure is called **Lebesgue measure**.
Proof of Theorem 1.1.4 Let $\mathcal{S}$ be the semialgebra of half-open intervals $(a, b]$ with $-\infty \leq a < b \leq \infty$. To define μ on $\mathcal{S}$, we begin by observing that

$$F(\infty) = lim_{x\uparrow\infty} F(x) \text{ and } F(-\infty) = lim_{x\downarrow-\infty} F(x) \text{ exist}$$

and $\mu((a, b]) = F(b) - F(a)$ makes sense for all $-\infty \leq a < b \leq \infty$ since $F(\infty) > -\infty$ and $F(-\infty) < \infty$.
If $(a, b] = +_{i=1}^n (a_i, b_i]$, then after relabeling the intervals we must have $a_1 = a, b_n = b$ and $a_i = b + i - 1$ for $2 \leq i \leq n$ so that it satisfy the finite additivity of μ on $\mathcal{S}$ i.e. if $S \in \mathcal{S}$, is a finite disjoint union of sets $S_i \in \mathcal{S}$, then $\mu(S) = \Sigma_i \mu(S_i)$. Moreover, if μ satisfy the countable subadditivity, then by Theorem 1.1.9, there is a unique measure on $\sigma(S)$. To check the countable subadditivity, suppose first that $-\infty < a < b < \infty$ and $(a, b] \subset \cup_{i\geq 1}(a_i, b_i]$ where (without loss of generality) $\infty < a_i < b_i < \infty$. Pick $\delta > 0$ so that $F(a + \delta) < F(a) + \epsilon$ and pick η_i so that

$$F(b_i + \eta_i) < F(b_i) + \epsilon 2^{-i}$$

The open intervals $(a_i, b_i + \eta_i)$ cover $[a + \delta, b]$, so there is a finite subcover $(\alpha_j, \beta_j), 1 \leq j \leq J$. Since $(a + \delta, b] \subset \cup_{j=1}^{J} (\alpha_j, \beta_j]$, Lemma 1.1.10 implies

$$F(b) - F(a + \delta) \leq \Sigma_{j=1}^{J} F(\beta_j) - F(\alpha_j) \leq \Sigma_{i=1}^{\infty} (F(b_i + \eta_j) - F(a_i))$$

So, by the choice of δ and η_i,

$$F(b) - F(a) \leq 2\epsilon + \Sigma_{i=1}^{\infty} (F(b_i) - F(a_i))$$

and since ϵ is arbitrary, we have proved the result in the case $-\infty < a < b < \infty$. To remove the last restriction, observe that if $(a, b] \subset \cup_i (a_i, b_i]$ and $(A, B] \subset (a, b]$ has $-\infty < A < B < \infty$, then we have

$$F(B) - F(A) \leq \Sigma_{i=1}^{\infty} (F(b_i) - F(a_i))$$

Since the last result holds for any finite $(A, B] \subset (a, b]$, the desired result follows.