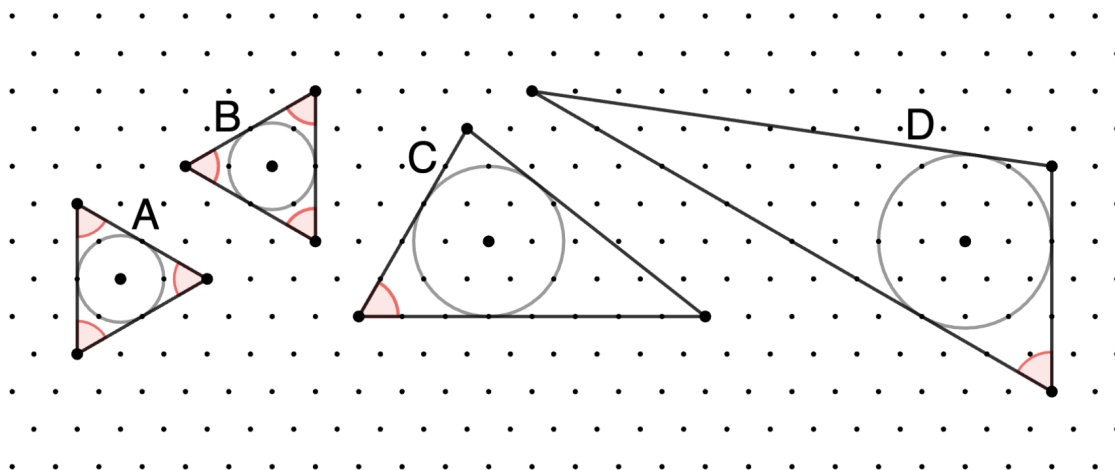


## Problem 883

In this problem we consider triangles drawn on a **hexagonal lattice**, where each lattice point in the plane has six neighbouring points equally spaced around it, all distance 1 away. We call a triangle *remarkable* if

- All three vertices and its **incentre** lie on lattice points
- At least one of its angles is  $60^\circ$



Above are four examples of remarkable triangles, with  $60^\circ$  angles illustrated in red. Triangles *A* and *B* have inradius 1; *C* has inradius 3; *D* has inradius 2. Define  $T(r)$  to be the number of remarkable triangles with inradius  $\leq r$ . Rotations and reflections, such as triangles *A* and *B* above, are counted separately; however direct translations are not. That is, the same triangle drawn in different positions of the lattice is only counted once.

You are given  $T(0.5) = 2$ ,  $T(2) = 44$ , and  $T(10) = 1302$ .

Find  $T(106)$ .

## Solution

We start by getting a grasp of how many radii are possible for a given  $r$ .

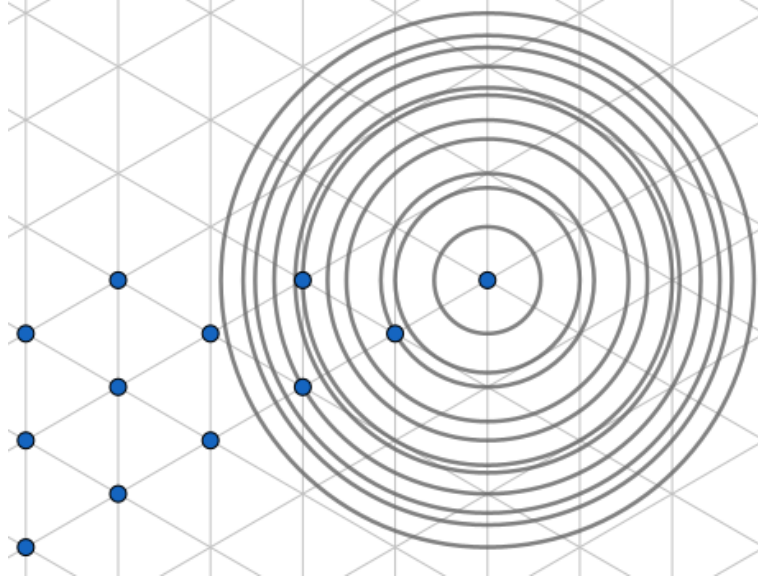


Figure 1: Example of all possible inner circles for  $r \leq 2.5$ , derived from the blue dots that represent a corner with a  $60^\circ$  angle.

Reasoning from the above illustration, we can see that the number of possible radii for a given  $r$  is determined by the amount of blue dots that are at most  $2r$  distance from the center of the inner circle. If we know the position of all (unique) possible blue dots, we know all possible inner radii.

By finding a formula for all the possible blue dots, we can find a formula for the number of possible inner radii. One can show that the set of possible radii equals

$$\bigcup_{i \in \mathbb{N}^+} \left\{ \frac{1}{2} \sqrt{i^2 - ik + k^2} : k \in \mathbb{N}, k \leq \frac{i}{2} \right\}$$

Note that if one wants to find a set of all radii smaller than a given  $r$ ,  $R_r$ , one can prove that

$$R_r \subseteq \bigcup_{i=1}^{\frac{1}{2}i\sqrt{3} \leq r} \left\{ \frac{1}{2} \sqrt{i^2 - ik + k^2} : k \in \mathbb{N}, k \leq \frac{i}{2} \right\}$$