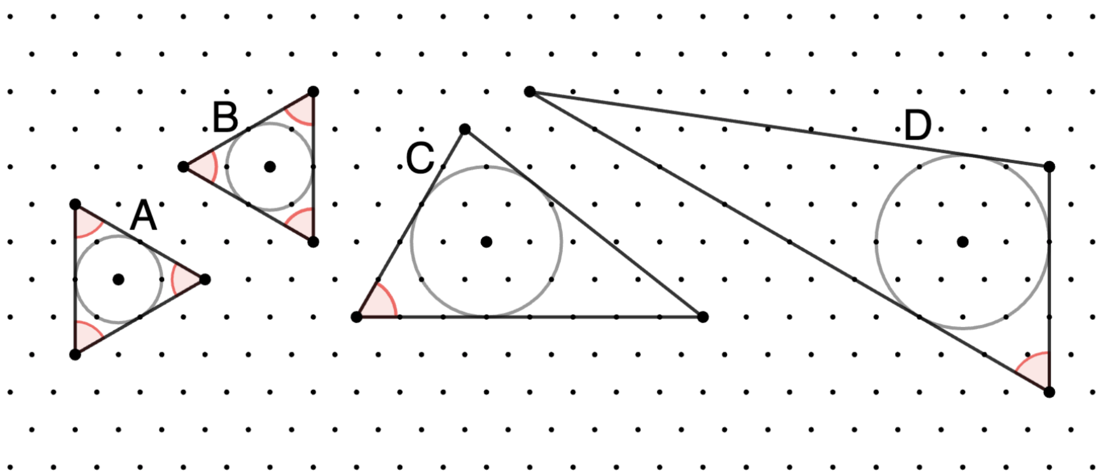
Problem 883

In this problem we consider triangles drawn on a **hexagonal lattice**, where each lattice point in the plane has six neighbouring points equally spaced around it, all distance 1 away. We call a triangle *remarkable* if

- All three vertices and its **incentre** lie on lattice points
- At least one of its angles is $60°$



Above are four examples of remarkable triangles, with $60°$angles illustrated in red. Triangles A and B have inradius 1; C has inradius 3; D has inradius 2. Define $T(r)$ to be the number of remarkable triangles with inradius $\leq r$. Rotations and reflections, such as triangles A and B above, are counted separately; however direct translations are not. That is, the same triangle drawn in different positions of the lattice is only counted once.

You are given $T(0.5) = 2$, $T(2) = 44$, and $T(10) = 1302$.

Find $T(106)$.

Solution

We start by getting a grasp of how many radii are possible for a given r.

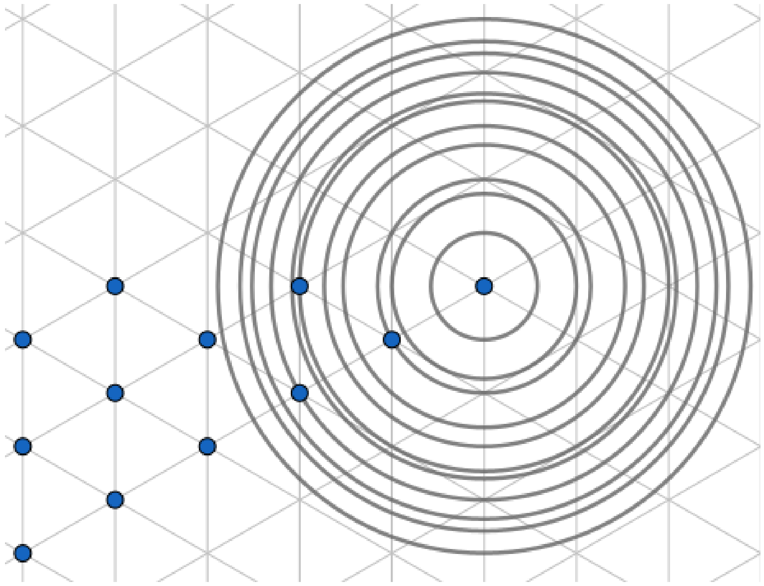


Figure 1: Example of all possible inner circles for $r \leq 2.5$, derived from the blue dots that represent a corner with a 60°angle.

Reasoning from the above illustration, we can see that the number of possible radii for a given r is determined by the amount of blue dots that are at most $2r$ distance from the center of the inner circle. Is we know the position of all (unique) possible blue dots, we know all possible inner inradii.

By finding a formula for all the possible blue dots, we can find a formula for the number of possible inradii. One can show that the set of possible raddii equals

$$\bigcup_{i \in \mathbb{N}^+} \left\{ \frac{1}{2}\sqrt{i^2 - ik + k^2} : k \in \mathbb{N}, k \leq \frac{i}{2} \right\}$$

Note that if one wants to find a set of all radii smaller than a given r, R_r, one can prove that

$$R_r \subseteq \bigcup_{i=1}^{\frac{1}{2}i\sqrt{3} \leq r} \left\{ \frac{1}{2}\sqrt{i^2 - ik + k^2} : k \in \mathbb{N}, k \leq \frac{i}{2} \right\}$$