# CSE584 HOMEWORK 4

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## 1. Question 1

According to lectures, we need to compute $\Lambda(q,\theta)$ First.

For this question, we have approximation: $q(\beta_2,\mu_2,\sigma_2^2)=\dfrac{\beta_2^k}{\Gamma(k)}\tau^{k-1}e^{-\beta_2\tau}\dfrac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}$

$$
\begin{aligned}
\Lambda(q,\theta) &= \Lambda(q(\beta_2,\mu_2,\sigma_2^2),\beta,\mu,\sigma)\\
&= \int_0^\infty\int_{-\infty}^\infty \frac{\beta_2^k}{\Gamma(k)}\tau^{k-1}e^{-\beta_2\tau}\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}\log\frac{(\prod_{i=1}^n\sqrt{\frac{\tau}{2\pi}}e^{-\frac{\tau(x_i-y)^2}{2}})\frac{\beta^k}{\Gamma(k)}\tau^{k-1}e^{-\beta\tau}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\frac{\beta_2^k}{\Gamma(k)}\tau^{k-1}e^{-\beta_2\tau}\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}}dyd\tau\\
&= \int_0^\infty\int_{-\infty}^\infty \frac{\beta_2^k}{\Gamma(k)}\tau^{k-1}e^{-\beta_2\tau}\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}\\
&\left[\frac{n}{2}\log\tau-\frac{n}{2}\log 2\pi-\sum_{i=1}^n\frac{(x_i-y)^2\tau}{2}+\log\sigma_2-\log\sigma+k\log\beta-k\log\beta_2+\frac{(y-\mu_2)^2}{2\sigma_2^2}-(\beta-\beta_2)\tau-\frac{(y-\mu)^2}{2\sigma^2}\right]dyd\tau\\
&= \frac{n}{2}\psi(k)-\frac{n}{2}\log\beta-\frac{n}{2}\log 2\pi-\frac{nk}{2\beta}\mu_2^2-\frac{nk}{2\beta}\sigma_2^2-\frac{k}{2\beta}\sum_{i=1}^n x_i^2+\frac{k}{\beta}\mu_2\sum_{i=1}^n x_i+\log\sigma_2-\log\sigma+k\log\beta-k\log\beta_2\\
&+\frac{1}{2}-(\beta-\beta_2)\frac{k}{\beta}-\frac{\mu_2^2}{2\sigma^2}-\frac{\sigma_2^2}{2\sigma^2}-\frac{\mu^2}{2\sigma^2}+\frac{\mu\mu_2}{\sigma^2}\\
&= -\frac{n}{2}\log\beta-\frac{nk}{2\beta}\mu_2^2-\frac{nk}{2\beta}\sigma_2^2-\frac{k}{2\beta}\sum_{i=1}^n x_i^2+\frac{k}{\beta}\mu_2\sum_{i=1}^n x_i+\log\sigma_2-\log\sigma+k\log\beta-k\log\beta_2+\frac{\beta_2k}{\beta}\\
&-\frac{\mu_2^2}{2\sigma^2}-\frac{\sigma_2^2}{2\sigma^2}-\frac{\mu^2}{2\sigma^2}+\frac{\mu\mu_2}{\sigma^2}+C,\text{where C is some constant}
\end{aligned}
$$

Since we want to minimize $\Lambda(q,\theta)$, so that constant $C$ does not matter and can be removed.
Next, take derivative to update $\beta,\mu,\sigma,\beta_2,\mu_2,\sigma_2$.
(1)

$$
\begin{aligned}
\frac{d\Lambda}{d\beta} &= (k-\frac{n}{2})\frac{1}{\beta}-\frac{1}{\beta^2}(-\frac{nk\mu_2^2}{2}-\frac{nk\sigma_2^2}{2}-\frac{k}{2}\sum_{i=1}^n x_i^2+k\mu\sum_{i=1}^n x_i+\beta_2k)=0\\
\beta &\leftarrow \frac{-\frac{nk\mu_2^2}{2}-\frac{nk\sigma_2^2}{2}-\frac{k}{2}\sum_{i=1}^n x_i^2+k\mu\sum_{i=1}^n x_i+\beta_2k}{k-\frac{n}{2}}
\end{aligned}
$$

(2)

$$
\begin{aligned}
\frac{d\Lambda}{d\mu} &= -\frac{\mu}{\sigma^2}+\frac{\mu_2}{\sigma^2}=0\\
\mu &\leftarrow \mu_2
\end{aligned}
$$

(3) Let $t = \sigma^2$

$$\frac{d\Lambda}{dt} = -\frac{1}{t} - \frac{1}{t^2}(-\frac{\mu_2^2}{2} - \frac{\sigma_2^2}{2} - \frac{\mu^2}{2} + \mu\mu_2) = 0$$
$$t = \sigma^2 \leftarrow \frac{\mu_2^2}{2} + \frac{\sigma_2^2}{2} + \frac{\mu^2}{2} - \mu\mu_2$$

(4)

$$\frac{d\Lambda}{d\beta_2} = -\frac{k}{\beta_2} + \frac{k}{\beta} = 0$$
$$\beta_2 \leftarrow \beta$$

(5)

$$\frac{d\Lambda}{d\mu_2} = (-\frac{nk}{2\beta} - \frac{1}{2\sigma^2})\mu_2 + \frac{k}{\beta}\sum_{i=1}^{n} x_i + \frac{\mu}{\sigma^2} = 0$$
$$\mu_2 \leftarrow \frac{\frac{k}{\beta}\sum_{i=1}^{n} x_i + \frac{\mu}{\sigma^2}}{\frac{nk}{2\beta} + \frac{1}{2\sigma^2}}$$

(6) Let $t_2 = \sigma_2^2$.

$$\frac{d\Lambda}{dt_2} = -\frac{nk}{2\beta} + \frac{1}{t} - \frac{1}{2\sigma^2} = 0$$
$$t_2 = \sigma_2^2 \leftarrow \frac{2\beta\sigma^2}{nk\sigma^2 + \beta}$$