A Utility Function for Study Sessions

Bryan Bugyi

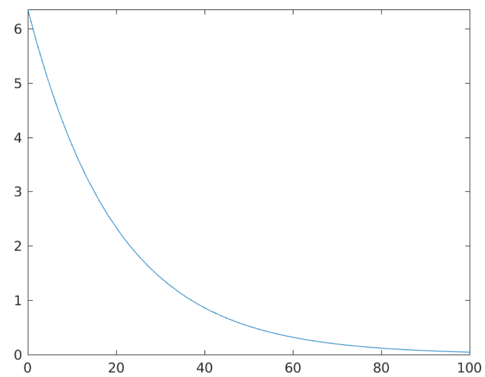
1 Introduction

As a student, it can be tempting to try to cram a week's worth of studies into a single day, or perhaps two. Any student who has tried this, however, is likely to have found that it is difficult to retain information this way. The first few hours of studying go as expected, but after that your ability to focus and actively engage with the material begins to wane. Subsequently, you don't remember as much of the material as you had hoped. In other words, you put in the hours, but it doesn't seem like they were used efficiently. This is not new information. The consequences of the law of diminishing returns are widely understood to reach far outside of their scope of modern economics; I would argue the principle has near universal relevance.

While researching the subject, I came across several mathematical models that explain the phenomena of diminishing returns, most of which were exponential functions. For example, the function $f(x) = 3e^{\frac{-(x-15)}{20}}$, the graph of which is shown in Figure 1, fits the bill quite nicely for a typical diminishing returns function. These relatively trivial exponential functions, however, do not do a good job at modeling a human being's ability to focus.

This failure stems from the fact that the appropriate utility function would not be strictly decreasing. After all, when a study session first begins there is a great deal of inefficiency which results from our need to reorient ourselves to previous material. This is especially true when we begin reading a chapter that we have already partially read, start working on a problem set that we have already done work on previously, start working on project X where project X represents a partially complete project, etc..

Figure 1: $f(x) = 3e^{\frac{-(x-15)}{20}}$



I have been able to find several existing attempts to provide mathematical representations for long run utility models of studying. Additionally, I have found several examples of short run utility models being approached experimentally (e.g. "vigilance decrement" research, though not specific to studying, approaches a very similar model), but I could not find any attempts to derive a general formula from these results. Thus, in my opinion there seems to be a small section of this research area that may have been left neglected. Namely, few attempts seem to have been made to provide a mathematical representation for a short run utility model of studying. Such a representation, if accurate, could potentially be used to optimize a student's study sessions, thus making them more efficient.

This paper explores my attempt to provide such a representation as well as my subsequent findings. Specifically, my intention in writing this paper (other than simply to organize my findings for later reference) is to demonstrate an example utility function that I believe provides a relatively accurate model of a human beings ability to study efficiently in the short run, which I will assume to be a single continuous study session. After which, I will show how such a function could be used to maximize overall utility, which is, as I will go on to show, analogous to maximizing overall productivity.

1.1 Assumptions

In the following few pages, I shall present a conjecture describing how the short run returns of studying could be modeled using an appropriate utility function.

This conjecture rests on several assumptions which I have arrived at based only on introspection. Thus, the research that this paper presents falls more appropriately within the scope of econometrics than it does cognitive theory. That is, we assume that my model is correct (a HUGE leap of faith) and focus solely on the problem of how to represent the model mathematically. My assumptions of this model are as follows:

1. Our marginal returns gained from studying (our returns in this case can be thought of as knowledge gained, better grade on test, better understanding of the material, etc.) can be measured with a utility function $u(t)$, where t is the amount of time the student has been studying and $\forall t, 0 \leq u(t) \leq 1.0$.

2. $u(t)$ is a continuous function.

3. $u(0) = 0.0$.

4. As we better orient ourselves with the material and eventually reach a state of flow, our marginal returns increase rapidly at a diminishing rate until they reach a global maximum, which we will denote as $u(t = t_{\max}) = 1.0$. (Be sure to understand that we are using $t_{\max}$ to denote the time at which the utility function u is maximized, and not to indicate a maximum time.)

5. After we have reached this point of maximum marginal returns, our marginal returns immediately begin to diminish, but at a much slower rate than the rate at which they had initially rose (i.e. the rate at which the marginal returns rose from $t = 0$ to $t = t_{\max}$).

6. As t approaches infinity, $u(t)$ approaches zero.

2 Finding an Appropriate Utility Function

After some exploration (and with some inspiration from the design of certain fuzzy logic membership functions) I stumbled upon the following function, which I have selected based solely on the fact that it adheres to our initial assumptions:

$$u(t) = t^{e - e^{\sqrt[n]{t}}} \tag{1}$$

In order to establish a value for n, let us make one final assumption:

7. $u(5 \text{ hours}) = 0.5$. That is, after studying for 5 hours the marginal return is half the value of $u(t_{\max})$.

I have separated Assumption 7 from our original six assumptions on the basis that I believe this assumption requires a much greater leap of faith. As such, I am not as confident in Assumption 7 as I am with the original six. Luckily, the validity of Assumption 7 has only a minor impact on our final result (namely, the value of n, as I will now show).

Given Assumption 7, we can set $n = 11.7$, which reduces $u(t)$ to the following:

$$u(t) = t^{e - e^{\sqrt[11.7]{t}}} \tag{2}$$

3 Exploring Our Results

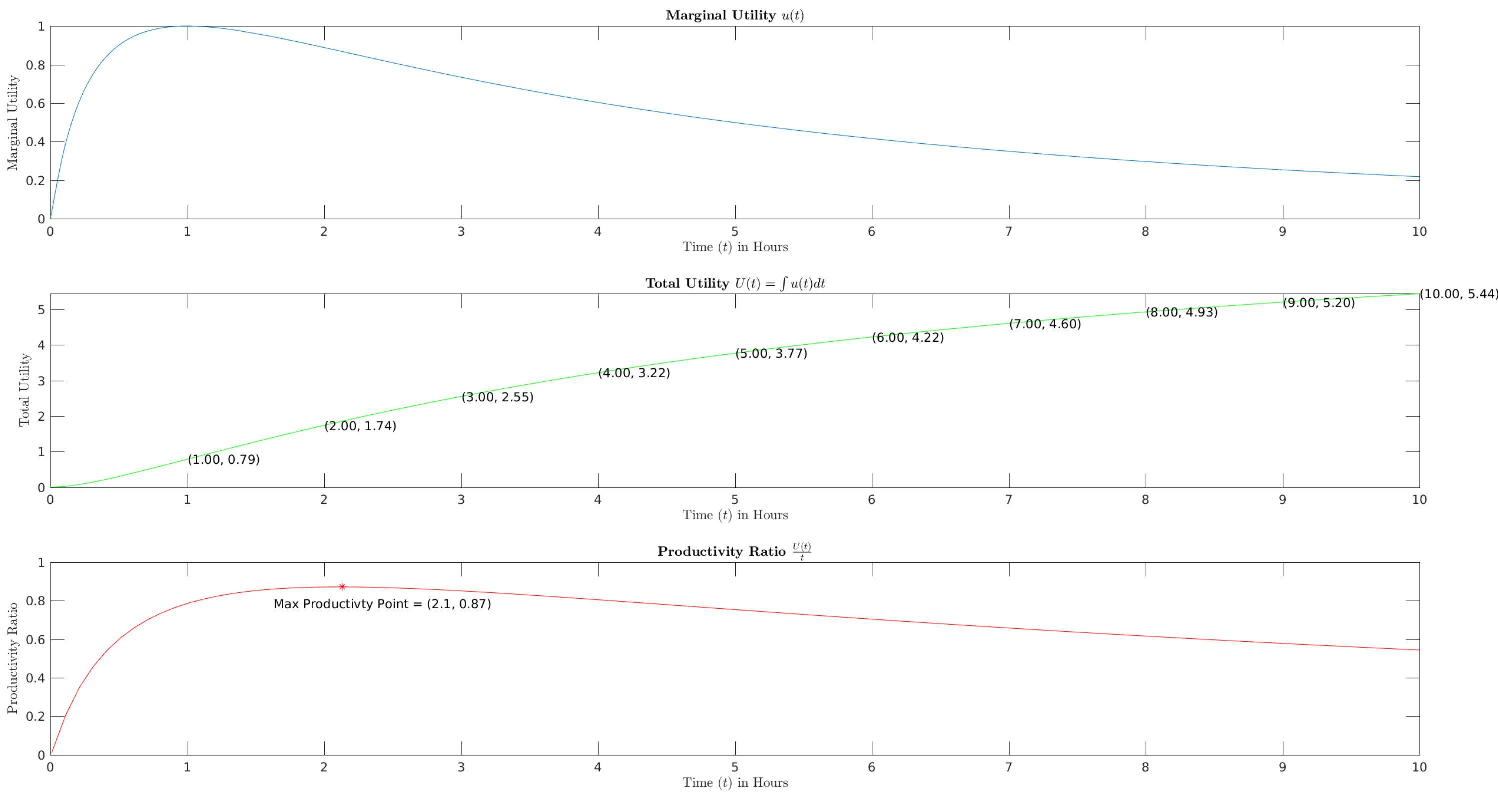
Using MATLAB, I have produced a plot of the utility function given in Equation 2 as well as a plot of $U(t) = \int_0^t u(t')dt'$. $U(t)$ represents the total utility gained after t units of time spent studying without taking a break. I have also produced a plot of the resulting productivity function, which is the ratio $\frac{U(t)}{t}$. These plots can be found in Figure 2.

4 Discussion

Let us begin by assuming that this function does indeed accurately reflect the utility one could expect to gain from a study session, which I believe it more or less does (excluding Assumption 7 which is admittedly a bit of a stretch).

If we can concede that $u(t)$ relates time to the instantaneous utility of a study session then we must also concede that its antiderivative relates time to the total utility gained in that study session, $U(t)$, plus or minus some

Figure 2: Plots of the Marginal Utility $u(t)$, the Total Utility $U(t)$, and the Productivity Ratio $\frac{U(t)}{t}$



constant. This constant of integration could perhaps represent the unique potential of the individual or could instead be a combination of various forms of "cognitive capital," such as how well rested the individual is, how well nourished they are, how healthy they are, etc.. I will not dwell on this constant, however, as the topic is beyond the scope of this project.

4.1 Maximizing Returns

For a student with a desire to make the most of her study sessions, the question then becomes how she can maximize her total returns. Our first instinct may be to presume that maximizing returns is equivalent to maximizing $U(t)$. And this presumption, within the limited scope of our aforementioned assumptions, would be correct; however, this approach would blatantly miss the bigger picture.

After all, our understanding of this utility function works under the limiting assumption that $u(t)$ is continuous (Assumption 2), but in the real-world we do not need to study in a strictly continuous manor; we are free to break our study time into discrete sessions if we so choose. Splitting your study time up in this way is, in fact, recommended. With this realization, the goal now shifts from maximizing $U(t)$ to maximizing the productivity of $U(t)$, which can be measured by the ratio $\frac{U(t)}{t}$. The graph of this ratio is shown in the third plot from the top in Figure 2.

Why is this so? The answer to this question lies in an analogy to the economist's requirements of a short run production function, a function which $u(t)$ most certainly bares resemblance to. Specifically, the "short run" from an economical viewpoint refers to a period of time in which there is at least one fixed factor input.

The fact that many of the variables that returns depend on are static in the short run is the very reason that returns diminish in the short run. The remedy I propose is simple. After you have reached the point of maximum productivity (which would be after 2.1 hours according to our model, which should not be taken too seriously from a quantitative standpoint), I recommend you take a longer break. The goal here is to replenish your mind (I would like to imagine that t has been reset to 0 after an ideal long break). After this break, you should do your best to change as many input variables as possible. Change your study location, study a different topic, and perhaps even try to study from a different medium (if you were previously reading a textbook, try now to attend a lecture or watch an educational video instead).

After the ideal study break and ideal change of input variables, my hypothesis is that the utility function $u(t)$ will once again be an appropriate model for your expected returns where t has now been reset to 0.

Under Construction (to be continued. . .)