Chapter 8 - Hypothesis Testing

Def (8.1.1) A <u>hypothesis</u> is a statement about a population parameter.

Def (8.1.2) A hypothesis test involves choosing between two complementary hypotheses: the <u>null hypothesis</u> (H_0) and the <u>alternative hypothesis</u> (H_1).

Population parameter: θ

Parameter Space: Θ so $\theta \in \Theta$.

1. $H_0: \theta \in \Theta_0$ where Θ_0 is a subset of Θ
2. $H_1: \theta \in \Theta_0^c$ where $\Theta_0^c = \Theta_1$ is the complement of Θ_0 in Θ.

Ex: Initial claim is that less than half of adult Americans can name at least one U.S. Supreme Court justice. Survey $n=1000$ adults and record

$$X_i = \begin{cases} 1, & \text{if can name at least one justice} \\ 0, & \text{if cannot " " " " "} \end{cases}$$

Suppose $X_i \overset{iid}{\sim} \text{Bernoulli}(\theta)$ where $f(x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i} I_{\{0,1\}}(x_i)$ and $\theta = P(X_i = 1)$.

Consider

$$\Theta = ? = [0,1] \quad , \quad \begin{aligned} H_0&: \theta \in [0, \tfrac{1}{2}) = \Theta_0 \\ H_1&: \theta \in [\tfrac{1}{2}, 1] = \Theta_0^c = \Theta_1 \end{aligned}$$

In this case we have a composite H_0 and a composite H_1, where <u>composite</u> means that the underlying distribution is not completely specified in the hypothesis.

Ex: $\Theta = \{0,1\}$

$H_0: \theta = 0$

$H_1: \theta = 1$

$$f(x|\theta=0) = \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{1}{2\sigma_0}(x-\mu)^2} \quad \text{where } \mu \in \mathbb{R},\ \sigma_0^2 > 0 \text{ are known}$$

$$f(x|\theta=1) = \frac{1}{2\sigma_1} e^{-\frac{|x-\mu|}{\sigma_1}} \quad \text{, where } \mu \in \mathbb{R},\ \sigma_1^2 > 0 \text{ are known}$$

In this case we have a simple H_0 and a simple H_1, where <u>simple</u> means that the underlying distribution is completely specified in the hypothesis.

Def (8.1.3) A <u>hypothesis testing procedure</u> is a rule that specifies;

1. for which sample values, the decision is made to accept H_0 as true.

and

2. for which sample values the decision is made to reject H_0 and thus H_1 is accepted as true.

<u>Remarks</u> :

1. In 1. above, the corresponding subset of the sample space is called the "<u>acceptance region</u>".
2. In 2. above, the corresponding subset of the sample space is called the <u>rejection region</u>.
3. Practitioners use the phrase "fail to reject H_0" in lieu of "accept H_0" when H_1 is viewed as the researcher's hypothesis.

<u>Ex</u>:

$H_0: \theta \in [0, \frac{1}{2})$
$H_1: \theta \in [\frac{1}{2}, 1]$

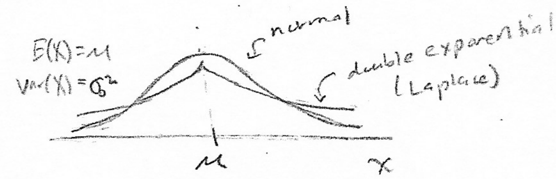
Sample size $= n$. Reject H_0 when $\hat{\theta} = \bar{X} \geq 0.5$

<u>Ex</u>

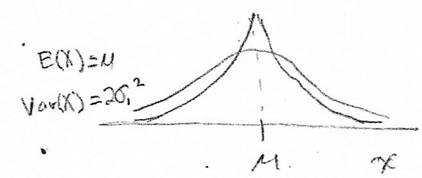
$H_0: \theta = 0$
$H_1: \theta = 1$

sample size $= 1$. Reject H_0 when ? (not so clear).

case 1:



case 2:



<u>Methods of Finding Tests</u> (Section 8.2)

<u>Likelihood Ratio Tests</u> (Section 8.2.1)

Recall that $L(\theta | \underset{\sim}{x}) \propto f(\underset{\sim}{x} | \theta) = \prod_{i=1}^{n} f(x_i | \theta)$ when $X_1, \dots, X_n \overset{iid}{\sim} f(x | \theta)$.

Def (8.2.1) Consider the test $H_0: \theta \in \Theta_0$ versus $H_1: \theta \in \Theta_1$ where $\Theta_1 = \Theta_0^c$.
The <u>likelihood ratio test statistic</u> is

$$\lambda(\underset{\sim}{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta | \underset{\sim}{x})}{\sup_{\theta \in \Theta} L(\theta | \underset{\sim}{x})} = \frac{\sup_{\theta \in \Theta_0} L(\theta | \underset{\sim}{x})}{\sup_{\theta \in \Theta_0 \cup \Theta_1} L(\theta | \underset{\sim}{x})}$$

$$= \begin{cases} 1 & \text{, if } \sup_{\theta \in \Theta_0} L(\theta | \underset{\sim}{x}) \geq \sup_{\theta \in \Theta_1} L(\theta | \underset{\sim}{x}) \\ \dfrac{\sup_{\theta \in \Theta_0} L(\theta | \underset{\sim}{x})}{\sup_{\theta \in \Theta_1} L(\theta | \underset{\sim}{x})} < 1 & \text{, otherwise} \end{cases}$$

Remarks: 1. $0 \le \lambda(\underline{x}) \le 1$ and values of $\lambda(\underline{x})$ much smaller than 1 indicates that H_1 is better supported by the data than is H_0.

2. $\sup_{\theta \in \Theta} L(\theta|\underline{x})$ is the $L(\theta|\underline{x})$ evaluated at the MLE of θ.

Def: A likelihood ratio test (LRT) is any test that has a rejection region of the form $\{\underline{x} : \lambda(\underline{x}) \le c\}$ for some $c \in [0,1]$.

Ex: $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(\theta)$, $0 \le \theta \le 1$.

$H_0: \theta = \frac{1}{4}$
$H_1: \theta = \frac{3}{4}$

(omit)

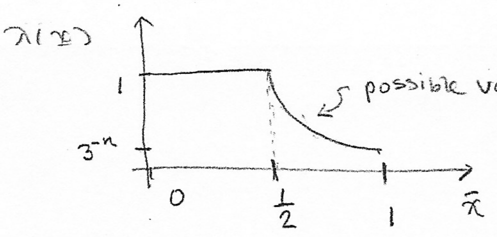
Under H_0, $L_0(\theta|\underline{x}) = \left(\frac{1}{4}\right)^{\Sigma x_i}\left(\frac{3}{4}\right)^{n-\Sigma x_i} = \left(\frac{1}{4}\right)^n 3^{n-\Sigma x_i} = \left(\frac{1}{4}\right)^n 3^{n(1-\bar{x})}$.

Under H_1, $L_1(\theta|\underline{x}) = \left(\frac{3}{4}\right)^{\Sigma x_i}\left(\frac{1}{4}\right)^{n-\Sigma x_i} = \left(\frac{1}{4}\right)^n 3^{\Sigma x_i} = \left(\frac{1}{4}\right)^n 3^{n\bar{x}}$

(omit)

$$\lambda(\underline{x}) = \frac{L_0(\theta|\underline{x})}{\max\{L_0(\theta|\underline{x}), L_1(\theta|\underline{x})\}}$$

$$= \begin{cases} 1, & \text{if } (L_0(\theta|\underline{x}) \ge L_1(\theta|\underline{x}) \text{ iff}) \ \bar{x} \le \frac{1}{2} \\ 3^{n(1-2\bar{x})}, & \text{if } \bar{x} > \frac{1}{2} \end{cases}$$



possible values are a set of discrete points along this curve.

Now $\lambda(\underline{x}) \le c \iff 3^{n(1-2\bar{x})} \le c$

$$\iff \bar{x} \ge \frac{1}{2}\underbrace{\left(1 - \frac{\log c}{n \log 3}\right)}_{>1} > \frac{1}{2} \cdot \begin{cases} (\frac{1}{2}, \infty) & \text{for } c \in (0,1) \\ (\frac{1}{2}, 1) & \text{for } c \in (3^{-n}, 1) \end{cases}$$

So the LRT rejects $H_0: \theta = \frac{1}{4}$ in favor of $H_1: \theta = \frac{3}{4}$ for large values of $\bar{X}$, the sample proportion.

Ex: $X_1, \ldots, X_n \overset{iid}{\sim} f(x_i|\theta)$ where $f(x_i|\theta) = \frac{1}{\theta} e^{-\frac{x_i}{\theta}} I_{(0,\infty)}(x_i)$, $\theta > 0$.

i.e., $\Theta = \{\theta : \theta > 0\}$.

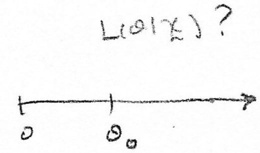
Consider $H_0: \theta \le \theta_0$ so $\Theta_0 = \{\theta : 0 < \theta \le \theta_0\}$

$H_1: \theta > \theta_0$ so $\Theta_1 = \{\theta : \theta > \theta_0\}$

$$\sup_{\theta \in \Theta} L(\theta|\underline{x}) = \sup_{\theta > 0} \frac{1}{\theta^n} e^{-\frac{n\bar{x}}{\theta}}$$ occurs at the MLE of $\theta = \hat{\theta} = \bar{X}$

$$= \frac{1}{\bar{x}^n} e^{-n}$$

$$\sup_{\theta \in \Theta_0} L(\theta|\underline{x}) = \sup_{0 < \theta \le \theta_0} \frac{1}{\theta^n} e^{-\frac{n\bar{x}}{\theta}} = \begin{cases} \frac{1}{\bar{x}^n} e^{-n}, & \text{if } \bar{x} \le \theta_0 \\ \frac{1}{\theta_0^n} e^{-\frac{n\bar{x}}{\theta_0}}, & \text{if } \bar{x} > \theta_0 \end{cases}$$

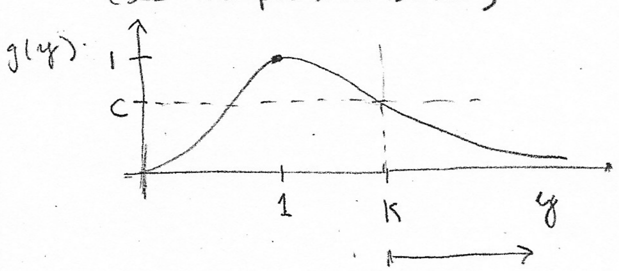


$$\therefore \lambda(\underline{x}) = \begin{cases} 1, & \text{if } \bar{x} \le \theta_0 \\ \frac{\bar{x}^n e^{-n\bar{x}/\theta_0}}{\theta_0^n e^{-n}}, & \text{if } \bar{x} > \theta_0 \end{cases}$$

For some $c \in (0,1)$, the LRT rejects H_0 if $\bar{x} > \theta_0$ and $\frac{\bar{x}^n}{\theta_0^n} \frac{e^{-n\bar{x}/\theta_0}}{e^{-n}} \le c$

$\Leftrightarrow$ $\frac{\bar{x}}{\theta_0} > 1$ and $\left(\frac{\bar{x}}{\theta_0}\right)^n e^{-n\left(\frac{\bar{x}}{\theta_0} - 1\right)} \le c$.

Let $y = \frac{\bar{x}}{\theta_0}$. Hence reject H_0 if $y > 1$ and $g(y) = y^n e^{-n(y-1)} \le c$.
(see the picture below)



i.e., Reject H_0 if $y > K$ for some $K \in (1, \infty)$

$\Leftrightarrow$ Reject H_0 if $\bar{x} > K\theta_0$ where $1 < K < \infty$.

Remark: The initial form of the rejection region is messy, yet after some manipulation the rejection region reduces to a very simple form.