Handout #4

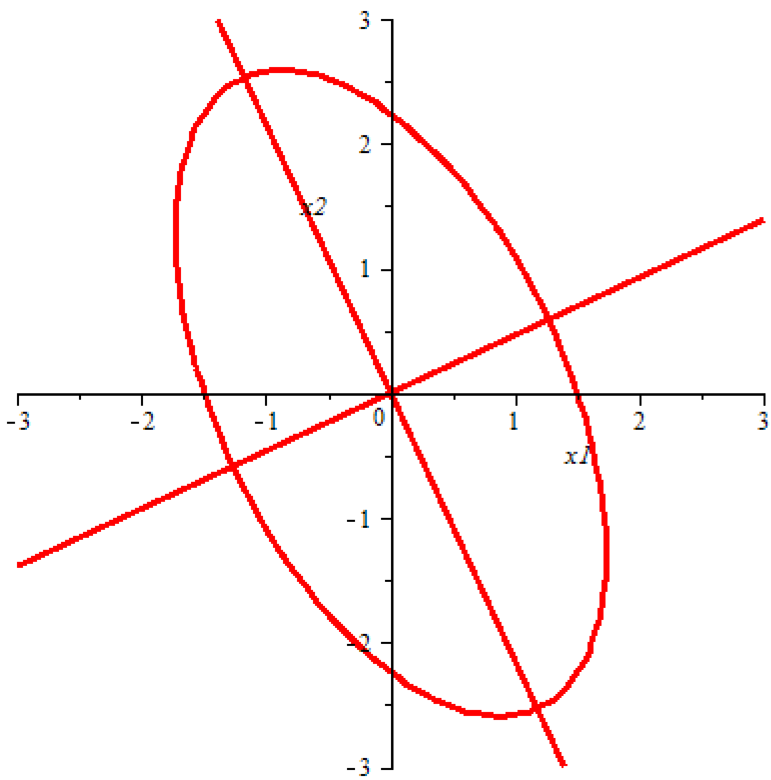
Example of Constant Statistical Distance using Eigenvalues and Eigenvectors

Consider the squared statistical distance measured from the origin **O = (0, 0)** given in problem 1.8 from the textbook (also see Homework #2):

$$d^2(P,\mathbf{O}) = \tfrac{1}{3}x_1^2 + \tfrac{2}{9}x_1x_2 + \tfrac{4}{27}x_2^2,$$

for the point $P=(x_1,x_2)$. This can be written in the form $d^2(P,\mathbf{O})=\mathbf{x}'\mathbf{A}\mathbf{x}$ where now $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{bmatrix}$. We would like to determine the set of all points that are 1 unit of squared statistical distance from the origin, by determining the eigenvalues and eigenvectors of the matrix **A**.

It can be shown that the eigenvalues are $\lambda_1=\frac{13+\sqrt{61}}{54}\approx 0.3854$ and $\lambda_2=\frac{13-\sqrt{61}}{54}\approx 0.0961$ with corresponding normalized eigenvectors $\mathbf{e}_1=\begin{bmatrix} 0.91 \\ 0.42 \end{bmatrix}$ and $\mathbf{e}_2=\begin{bmatrix} -0.42 \\ 0.91 \end{bmatrix}$. Since $1/\sqrt{\lambda_1}\approx 1.61$ is less than $1/\sqrt{\lambda_2}\approx 3.23$, the $y_2=\mathbf{x}'\mathbf{e}_2$ axis of the ellipse is longer than the $y_1=\mathbf{x}'\mathbf{e}_1$ axis. The actual **x**-coordinates of the endpoints of the ellipse's axes are given by $\mathbf{x}=\frac{1}{\sqrt{\lambda_1}}\mathbf{e}_1=\pm\begin{bmatrix} 1.46 \\ 0.68 \end{bmatrix}$ and $\mathbf{x}=\frac{1}{\sqrt{\lambda_2}}\mathbf{e}_2=\pm\begin{bmatrix} -1.37 \\ 2.92 \end{bmatrix}$, as graphed below. (All calculations were done to four decimal places and then rounded for presentation.) Also note that the correlation between variables x_1 *and* x_2 is negative.



(Courtesy of Dr. Roy St. Laurent)