EXTRA

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \right). \qquad J = \begin{vmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{vmatrix} = \sigma_x\sigma_y$$

Let $U = \frac{X-\mu_x}{\sigma_x}$, $V = \frac{Y-\mu_Y}{\sigma_Y}$. Then $x = \mu_x + \sigma_x u$, $y = \mu_y + \sigma_y v$ and

$$f_{U,V}(u,v) = f_{X,Y}(\mu_x+\sigma_x u, \mu_y+\sigma_y v)\cdot\sigma_x\sigma_y \; I_{(-\infty,\infty)}(u)\, I_{(-\infty,\infty)}(v)$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(u^2+v^2-2\rho uv)} I_{(-\infty,\infty)}(u)\, I_{(-\infty,\infty)}(v).$$

Now $M_{X,Y}(s,t) = E(e^{sX}e^{sY}) = E\left(e^{s(\mu_x+\sigma_x U)} e^{t(\mu_y+\sigma_y V)}\right)$

$$= e^{s\mu_x+t\mu_y} E\left(e^{(s\sigma_x)U} e^{(t\sigma_y)V}\right)$$

$$= e^{s\mu_x+t\mu_y} M_{U,V}(s\sigma_x, t\sigma_y).$$

Note that $M_{U,V}(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{su} e^{tv} f_{U,V}(u,v)\,du\,dv$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{su+tv} e^{-\frac{1}{2(1-\rho^2)}(u^2+v^2-2\rho uv)}\,du\,dv$$

completing the square wrt u

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{tv-\frac{1}{2(1-\rho^2)}\left[v^2-(\rho v+(1-\rho^2)s)^2\right]} e^{-\frac{1}{2(1-\rho^2)}\left[u-(\rho v+(1-\rho^2)s)\right]^2}\,du\,dv$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tv-\frac{1}{2(1-\rho^2)}\left[v^2-(\rho v+(1-\rho^2)s)^2\right]} \underbrace{\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[u-(\rho v+(1-\rho^2)s)\right]^2}\,du\right)}_{=1} dv$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tv-\frac{1}{2(1-\rho^2)}\left[v^2-(\rho v+(1-\rho^2)s)^2\right]}\,dv$$

$$\vdots$$

$$= e^{\frac{1}{2}(s^2+t^2+2\rho st)} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(v-(\rho s+t))^2}\,dv}_{=1}$$

$$= e^{\frac{1}{2}(s^2+t^2+2\rho st)}$$

$\therefore$ $M_{X,Y}(s,t) = e^{s\mu_x+t\mu_y} M_{U,V}(s\sigma_x, t\sigma_y)$

$$= e^{s\mu_x+t\mu_y} e^{\frac{1}{2}((s\sigma_x)^2+(t\sigma_y)^2+2\rho\sigma_x\sigma_y st)}$$

$$= e^{\mu_x s+\mu_y t + \frac{1}{2}(\sigma_x^2 s^2+\sigma_y^2 t^2+2\rho\sigma_x\sigma_y st)}.$$