Consider a Taylor Series approximation for $M_{Z_i}(\frac{t}{\sqrt{n}})$ of order 2 about $t=0$ so that

$$M_{Z_i}\left(\tfrac{t}{\sqrt{n}}\right) = M_{Z_i}(0) + M'_{Z_i}(0)\left(\tfrac{t}{\sqrt{n}}-0\right) + M''_{Z_i}(0)\,\tfrac{1}{2}\left(\tfrac{t}{\sqrt{n}}-0\right)^2 + \overset{\text{remainder}}{R_{Z_i}\left(\tfrac{t}{\sqrt{n}}\right)}.$$

Note that $M_{Z_i}(0)=1$, $M'_{Z_i}(0)=E(Z_i)=0$, $M''_{Z_i}(0)=E(Z_i^2)=1$ so that

$$M_{Z_i}\left(\tfrac{t}{\sqrt{n}}\right) = 1 + \tfrac{1}{2}\tfrac{t^2}{n} + R_{Z_i}\left(\tfrac{t}{\sqrt{n}}\right) \quad \text{and}$$

$$M_{\sqrt{n}\bar{Z}_n}(t) = \left(1+\tfrac{1}{2}\tfrac{t^2}{n}+R_{Z_i}\left(\tfrac{t}{\sqrt{n}}\right)\right)^n = \left[1+\tfrac{1}{n}\left(\tfrac{1}{2}t^2+nR_{Z_i}\left(\tfrac{t}{\sqrt{n}}\right)\right)\right]^n.$$

From Lemma 2.3.14, p. 67, $\lim_{n\to\infty}\left(1+\frac{1}{n}a_n\right)^n = e^a$ provided $\lim_{n\to\infty} a_n = a$.

Need to show that $\frac{1}{2}t^2 + nR_{Z_i}(\frac{t}{\sqrt{n}}) \to \frac{1}{2}t^2$. i.e., show $nR_{Z_i}(\frac{t}{\sqrt{n}}) \to 0$.

When $t=0$, $M_{\sqrt{n}\bar{Z}_n}(0) = 1 = (1+R_{Z_i}(0))^n \Rightarrow R_{Z_i}(0)=0$ and thus $nR_{Z_i}(0)\to 0$.

When $t\neq 0$, $nR_{Z_i}(\frac{t}{\sqrt{n}}) = t^2 \cdot \dfrac{R_{Z_i}(t/\sqrt{n})}{(t/\sqrt{n})^2}$ and $\dfrac{R_{Z_i}(\frac{t}{\sqrt{n}})}{(t/\sqrt{n})^2} \to 0$ by Taylor's Theorem (Theorem 5.5.21) p. 241

and thus $nR_{Z_i}(t/\sqrt{n}) \to 0$ since t is fixed.

It follows that $M_{\sqrt{n}\bar{Z}_n}(t) \to e^{\frac{1}{2}t^2}$ and thus $\sqrt{n}\,\frac{(\bar{X}_n-\mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1)$.