<u>Theorem 8.2.4</u> If $T(\underset{\sim}{X})$ is a sufficient statistic for θ, $\lambda(\underset{\sim}{x})$ is the LRT statistic based on $\underset{\sim}{X}$, and $\lambda^*(T(\underset{\sim}{x}))$ is the LRT statistic based on $T(\underset{\sim}{X})$, then $\lambda^*(T(\underset{\sim}{x})) = \lambda(\underset{\sim}{x})$ $\forall\, \underset{\sim}{x}$ in the sample space.

proof: see textbook

<u>Remark</u>: Using the Factorization Theorem, $\lambda(\underset{\sim}{x})$ and hence the rejection region should depend on $\underset{\sim}{x}$ only through a sufficient statistic.

<u>Ex</u>: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, σ^2 known. $\Theta = \mathbb{R}$.

$H_0: \mu = \mu_0$ $\quad \Theta_0 = \{\mu_0\}$

$H_a: \mu \neq \mu_0$ $\quad \Theta_1 = \mathbb{R} - \{\mu_0\}$

$$L(\mu \mid \underset{\sim}{x}) = e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Recall that the MLE of $\mu = \overline{X}$ which is a sufficient statistic for μ,

Then

$$\lambda(\underset{\sim}{x}) = \frac{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}}{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\overline{x})^2}}$$

$$= e^{-\frac{n}{2\sigma^2}(\overline{x}-\mu_0)^2}$$

For some $c \in (0,1)$, the LRT rejects H_0 if $e^{-\frac{n}{2\sigma^2}(\overline{x}-\mu_0)^2} \le c$

$$\Leftrightarrow \quad (\overline{x}-\mu_0)^2 \ge -\frac{2\sigma^2}{n}\log c.$$

Hence reject H_0 if $|\overline{x}-\mu_0| \ge \sqrt{-\frac{2\sigma^2}{n}\log c}$

$$\Leftrightarrow \quad \overline{x} \ge \mu_0 + \sqrt{-\frac{2\sigma^2}{n}\log c} \quad \text{or} \quad \overline{x} \le \mu_0 - \sqrt{-\frac{2\sigma^2}{n}\log c}.$$

Bayesian Tests (section 8.2.2)

$\pi(\theta)$: prior distribution of $\theta \in \Theta$

$f(\underline{x}|\theta)$: joint distribution of sample

Combine prior on θ and sample distribution to form posterior distribution of θ given $\underline{x}$

where $\pi(\theta|\underline{x}) \propto L(\theta|\underline{x})\,\pi(\theta)$.

Hypothesis test: $H_0: \theta \in \Theta_0$

$H_1: \theta \in \Theta_1 = \Theta_0^c$

In the Bayesian framework, the posterior distribution $\pi(\theta|\underline{x})$ is used to calculate the probabilities that H_0 and H_1 are true.

$$P(H_0 \text{ is true}|\underline{x}) = P(\theta \in \Theta_0 |\underline{x}) = \int_{\theta\in\Theta_0} \pi(\theta|\underline{x})\,d\theta$$

$$P(H_1 \text{ is true}|\underline{x}) = P(\theta \in \Theta_1 |\underline{x}) = 1 - P(\theta\in\Theta_0|\underline{x}) = 1 - P(H_0 \text{ is true}|\underline{x}),$$

Ex: $X_1, \dots, X_n \sim \text{Bernoulli}(\theta)$, $\Theta = \{\frac{1}{4}, \frac{3}{4}\}$.

$H_0: \theta = \frac{1}{4}$

$H_1: \theta = \frac{3}{4}$

Suppose $\pi(\theta) = \begin{cases} a, & \theta = \frac{1}{4} \\ 1-a, & \theta = \frac{3}{4}. \end{cases}$ for $0 \le a \le 1$.

i.e., $P(\theta \in \Theta_0) = a$, $P(\theta \in \Theta_1) = 1-a$,

$$\pi(\theta|\underline{x}) \propto \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}\,\pi(\theta) = (1-\theta)^n\left(\frac{\theta}{1-\theta}\right)^s \cdot \pi(\theta)$$

$$= \begin{cases} a\left(\frac{3}{4}\right)^n\left(\frac{1}{3}\right)^s, & \theta = \frac{1}{4} \\ (1-a)\left(\frac{1}{4}\right)^n 3^s, & \theta = \frac{3}{4} \end{cases}$$

$$= \begin{cases} a\left(\frac{1}{4}\right)^n 3^{n-s}, & \theta = 1/4 \\ (1-a)\left(\frac{1}{4}\right)^n 3^s, & \theta = 3/4 \end{cases}$$

One can show that

i.e., $P(\theta \in \Theta_0 \mid \underline{x}) = \dfrac{a}{a+(1-a)3^{2s-n}}$, $s = 0, 1, \dots, n$

$P(\theta \in \Theta_1 \mid \underline{x}) = \dfrac{(1-a)3^{2s-n}}{a+(1-a)3^{2s-n}}$, $s = 0, 1, \dots, n$

Some Options for rejecting Ho:

1. For given values of a, n, and $s = \sum_{i=1}^{n} x_i$ one could reject Ho if

$$\frac{P(\theta \in \Theta_1 \mid \underline{x})}{P(\theta \in \Theta_0 \mid \underline{x})} = \frac{(1-a)3^{2s-n}}{a} = \left(\frac{1-a}{a}\right)3^{n(2\bar{x}-1)} > K \quad \text{for some } K \geq 1$$

$\left\{ \hat{\theta} = \bar{x} = \frac{s}{n} = \frac{\sum_{i=1}^{n} x_i}{n} \right.$.

$\Leftrightarrow$ reject Ho for $\bar{x} > \frac{1}{2}\left(\frac{1}{n}\frac{\log\left(K\frac{a}{1-a}\right)}{\log 3} + 1\right)$.

2. Reject Ho if $P(\theta \in \Theta_1 \mid \underline{x}) > \frac{1}{2}$.
3. To guard against falsely rejecting Ho, however, consider rejecting Ho

if $P(\theta \in \Theta_1 \mid \underline{x}) > K$ for some $K >> \frac{1}{2}$.

Ex: $X_1, \dots, X_n \overset{iid}{\sim} f(x_i \mid \theta)$ where $f(x_i \mid \theta) = \frac{1}{\theta} e^{-x_i/\theta} I_{(0,\infty)}(x_i)$, $\theta > 0$,

Thus $\Theta = (0, \infty)$.

$H_0: \theta \leq \theta_0$
$H_1: \theta > \theta_0$

Suppose $\pi(\theta) = \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} \theta^{-(\alpha+1)} e^{-1/(\beta\theta)} I_{(0,\infty)}(\theta)$, i.e., $\theta \sim$ Inverse Gamma(α, β)

with α, β known.

Then $\pi(\theta \mid \underline{x}) \propto \theta^{-n} e^{-\frac{n\bar{x}}{\theta}} \cdot \pi(\theta)$

so $\pi(\theta \mid \underline{x}) \sim$ Inverse Gamma$\left(\alpha+n, (n\bar{x} + 1/\beta)^{-1}\right)$.

Then $P(\theta \in \Theta_1 \mid \underline{x}) = \int_{\theta_0}^{\infty} \pi(\theta \mid \underline{x})\, d\theta$.

EXTRA: $L(\theta \mid \underline{x}) \propto \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_i} I_{(0,\infty)}(x) = \theta^{-n} e^{-\frac{n\bar{x}}{\theta}} I_{(0,\infty)}(x)$

$\pi(\theta \mid \underline{x}) \propto \theta^{-n} e^{-\frac{n\bar{x}}{\theta}} \cdot \theta^{-(\alpha+1)} e^{-\frac{1}{\beta\theta}} I_{(0,\infty)}(\theta) = \theta^{-(n+\alpha+1)} e^{-\frac{1}{\theta}\left(n\bar{x}+\frac{1}{\beta}\right)} I_{(0,\infty)}(\theta)$