<u>Theorem</u> : If Σ_{22} is nonsingular, then

$$\underset{\sim}{Y}_1 \mid \underset{\sim}{Y}_2 = \underset{\sim}{y}_2 \sim N\left(\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{y}_2 - \underset{\sim}{\mu}_2),\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

proof : Let $\underset{p\times 1}{\underset{\sim}{X}} = (\underset{\sim}{Y}_1 - \underset{\sim}{\mu}_1) - \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{Y}_2 - \underset{\sim}{\mu}_2) = B\underset{\sim}{Y} + \underset{\sim}{b}$. Then consider

$$\begin{bmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} ?^{(1)} \\ \underset{\sim}{\mu}_2 \end{bmatrix}, \left[\begin{array}{c|c} ?^{(3)} & ?^{(2)} \\ \hline ?^{(2)} & \Sigma_{22} \end{array}\right] \right).$$

Note that $E(\underset{\sim}{X}) = \underset{\sim}{0}$. (1)

Also, $\mathrm{cov}(\underset{\sim}{X}, \underset{\sim}{Y}_2) = E[(\underset{\sim}{X} - \underset{\sim}{0})(\underset{\sim}{Y}_2 - \underset{\sim}{\mu}_2)']$

$$= E[\underset{\sim}{X}(\underset{\sim}{Y}_2 - \underset{\sim}{\mu}_2)'] = E[(\underset{\sim}{Y}_1 - \underset{\sim}{\mu}_1)(\underset{\sim}{Y}_2 - \underset{\sim}{\mu}_2)'] - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}$$

$$= \Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}$$

$$= \mathbf{0} \quad (2)$$

so $\underset{\sim}{X}$ and $\underset{\sim}{Y}_2$ are independent.

Now $\mathrm{cov}(\underset{\sim}{X}) = E\left[\left((\underset{\sim}{Y}_1 - \underset{\sim}{\mu}_1) - \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{Y}_2 - \underset{\sim}{\mu}_2)\right)\left((\underset{\sim}{Y}_1 - \underset{\sim}{\mu}_1) - \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{Y}_2 - \underset{\sim}{\mu}_2)\right)'\right]$

$$= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} + \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}\Sigma_{22}^{-1}\Sigma_{21}$$

$$= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \ . \quad (3)$$

Note that

$$\underset{\sim}{Y}_1 = \underset{\text{Independent of } \underset{\sim}{Y}_2}{\underset{\sim}{X}} + \underbrace{\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{Y}_2 - \underset{\sim}{\mu}_2)}_{\text{Fixed when } \underset{\sim}{Y}_2 = \underset{\sim}{y}_2.} \ .$$

It follows that

$$\underset{\sim}{Y}_1 \mid \underset{\sim}{Y}_2 = \underset{\sim}{y}_2 \sim N\left(\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{y}_2 - \underset{\sim}{\mu}_2),\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right).$$

<u>Remark</u> : If $\Sigma_{12} = 0$, then $\underset{\sim}{Y}_1 \mid \underset{\sim}{Y}_2 = \underset{\sim}{y}_2 \sim N(\underset{\sim}{\mu}_1, \Sigma_{11})$.

See Handout # 3 or Theorem 4.4d in textbook.