Example of Convergence in Probability for a Nondegenerate Case

Let $X \sim f(x) = \dfrac{1}{\pi(1+x^2)} I_{(-\infty,\infty)}(x)$ so $X \sim \text{Cauchy}(0,1)$.

Let $Y_n \sim N(\mu, \frac{1}{n})$ independent of X.

Define the sequence of rvs $\{X_n\}_{n=1}^{\infty}$ by $X_n = X + Y_n$.

Claim: $X_n \xrightarrow{P} X + \mu$.

$$|X_n - (X+\mu)| = |Y_n - \mu| = |Z_n| \quad \text{where } Z_n \sim N(0, \tfrac{1}{n}).$$

Let $\epsilon > 0$. Then

$$\begin{aligned}
P(|X_n - (X+\mu)| < \epsilon) &= P(|Z_n| < \epsilon) \\
&= P(-\epsilon < Z_n < \epsilon) \\
&= P\left(-\epsilon\sqrt{n} < \frac{Z_n - 0}{\sqrt{\frac{1}{n}}} < \epsilon\sqrt{n}\right) \\
&= \Phi(\epsilon\sqrt{n}) - \Phi(-\epsilon\sqrt{n})
\end{aligned}$$

where Φ is the cdf of $Z \sim N(0,1)$.

Then
$$\begin{aligned}
\lim_{n\to\infty} P(|X_n - (X+\mu)| < \epsilon) &= \lim_{n\to\infty} \left(\Phi(\epsilon\sqrt{n}) - \Phi(-\epsilon\sqrt{n})\right) \\
&= 1 - 0 \\
&= 1.
\end{aligned}$$

Therefore $X_n \xrightarrow{P} X + \mu$ where $X + \mu \sim \text{Cauchy}(\mu, 1)$.