Chapter 9 - Interval Estimation

(Remark: Estimate a parameter θ by an interval of plausible values.)

Def (9.1.1) An interval estimate of a paramter θ is any pair of functions $L(\underset{\sim}{x})$ and $U(\underset{\sim}{x})$ of a sample $\underset{\sim}{x} = (x_1, \ldots, x_n) \ni$ $L(\underset{\sim}{x}) \leq U(\underset{\sim}{x})$ $\forall$ $\underset{\sim}{x} \in$ Support of $\underset{\sim}{X}$. The random interval $[L(\underset{\sim}{X}), U(\underset{\sim}{X})]$ is called an interval estimator of θ.

Remarks: 1. If $L(\underset{\sim}{x}) = -\infty$, then $(-\infty, U(\underset{\sim}{x}))$ is a one-sided upper interval estimate of θ

2. If $U(\underset{\sim}{x}) = +\infty$, then $[L(\underset{\sim}{x}), \infty)$ " " " lower " " " "

3. More generally, a confidence set $C(\underset{\sim}{x})$ is a subset of the parameter space Θ; not necessarily an interval.

Def (9.1.14). For a given $\theta \in \Theta$, the coverage probability of the random interval $[L(\underset{\sim}{X}), U(\underset{\sim}{X})]$ is the probability of the set $\{ \underset{\sim}{X} : L(\underset{\sim}{X}) \leq \theta$ and $\theta \leq U(\underset{\sim}{X}) \}$, that is, $P_\theta(\theta \in [L(\underset{\sim}{X}), U(\underset{\sim}{X})])$.

Def (9.1.5) The confidence coefficient of an interval estimator $[L(\underset{\sim}{X}), U(\underset{\sim}{X})]$ is the infimum of the coverage probabilities, i.e.,

$$\inf_{\theta \in \Theta} P_\theta(\theta \in [L(\underset{\sim}{X}), U(\underset{\sim}{X})]).$$

Ex: $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, 1)$, $\mu \in \mathbb{R}$. Recall that $\bar{X}$ is the UMVUE of μ.

Consider the interval estimator $L(\underset{\sim}{X}) = \bar{X} - \frac{k}{\sqrt{n}}$, $U(\underset{\sim}{X}) = \bar{X} + \frac{k}{\sqrt{n}}$ where $k > 0$.

$$P_\mu\left(\bar{X} - \frac{k}{\sqrt{n}} \leq \mu \text{ and } \mu \leq \bar{X} + \frac{k}{\sqrt{n}}\right) = P_\mu\left(-\frac{k}{\sqrt{n}} \leq \bar{X} - \mu \leq \frac{k}{\sqrt{n}}\right)$$

$$= P_\mu\left(-k \leq \frac{\bar{X} - \mu}{1/\sqrt{n}} \leq k\right) = \Phi(k) - \Phi(-k)$$ where Φ denotes the cdf of a $N(0,1)$ r.v.

Note that the above probability does not depend on μ and "fi" the confidence coefficient is $\Phi(k) - \Phi(-k)$.

e.g., If $P_\mu\left(\bar{X} - \frac{k}{\sqrt{n}} \leq \mu \leq \mu \leq \bar{X} + \frac{k}{\sqrt{n}}\right) = 1 - \alpha = 0.95$,

then $k = 1.96$.

(Increasing the interval width increases the confidence coefficient)

Ex: $X_1, \dots, X_n \overset{iid}{\sim}$ Bernoulli(θ), $\theta \in (0,1)$. Recall the $\bar{X}$ is the UMVUE of θ.

Consider the interval estimator $L(\underline{X}) = \bar{X} - \frac{1}{n}$, $U(\underline{X}) = \bar{X} + \frac{1}{n}$.

$$P_\theta(L(\underline{X}) \le \theta \le U(\underline{X})) = P_\theta(\bar{X} - \tfrac{1}{n} \le \theta \le \bar{X} + \tfrac{1}{n})$$

$$= P_\theta(-\tfrac{1}{n} \le \bar{X} - \theta \le \tfrac{1}{n}) = P_\theta(-1 \le \sum_{i=1}^n X_i - n\theta \le 1)$$

$$= P_\theta(n\theta - 1 \le \sum_{i=1}^n X_i \le n\theta + 1) \quad \text{when } \sum_{i=1}^n X_i \sim \text{Binomial}(n, \theta).$$

omit: Calculate this probability for each $\theta \in (0,1)$. In this case $n\theta - 1$, $n\theta + 1$ are not necessarily integers so above probability is

$$= \sum_{s = \lceil n\theta - 1 \rceil}^{\lfloor n\theta + 1 \rfloor} \binom{n}{s} \theta^s (1-\theta)^{n-s}.$$

omit: e.g., $n = 10$, $\theta = \frac{1}{10}$, $n\theta - 1 = 0$, $n\theta + 1 = 2$

omit: where $\lfloor x \rfloor$ is the largest integer not greater than x or n if $x > n$, and $\lceil x \rceil$ is the smallest integer not less than x or 0 if $x < 0$.

(see handout)

Coverage probabilities depend on θ where $\inf_{\theta \in (H)} P_\theta(\theta \in [L(\underline{X}), U(\underline{X})]) = 0.451$

Remark: It is desirable if an interval estimator of θ, $[L(\underline{X}), U(\underline{X})]$, has coverage probability = confidence coefficient $= 1 - \alpha$ which does not depend on θ.

Ex: $X \sim$ Exponential(θ), $f(x|\theta) = \frac{1}{\theta} e^{-x/\theta} I_{(0,\infty)}(x)$.

Consider the interval estimator $L(X) = aX$, $U(X) = bX$, $0 \le a \le b$.

$$P_\theta(aX \le \theta \le bX) = P_\theta(\tfrac{\theta}{b} \le X \le \tfrac{\theta}{a}) = \int_{\theta/b}^{\theta/a} f(x|\theta)\,dx \overset{?}{=} e^{-\frac{1}{b}} - e^{-\frac{1}{a}} = (1 - e^{-\frac{1}{a}}) - (1 - e^{-\frac{1}{b}})$$

$=$ confidence coefficient which does not depend on θ.

(2) Consider $a = 0$, i.e., the one-sided interval $[0, bX]$. Then $e^{-\frac{1}{b}} = 1 - \alpha = 0.95 \Rightarrow b \approx 19.4957$

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(1) If $1 - e^{-\frac{1}{a}} = 0.975$ and $1 - e^{-\frac{1}{b}} = 0.025$, then $a \approx 0.2711$, $b \approx 39.4979$.

Remark: If a $100(1-\alpha)\%$ confidence interval estimator (CIE) for θ has been determined and if $g(\cdot)$ is any monotone function, then a $100(1-\alpha)\%$ CIE for $g(\theta)$ also exists. i.e.,

$$1 - \alpha = P_\theta(L(\underline{X}) \le \theta \le U(\underline{X}))$$

$$= \begin{cases} P_\theta(g(L(\underline{X})) \le g(\theta) \le g(U(\underline{X}))), & \text{if } g \text{ is increasing in } \theta \\ P_\theta(g(U(\underline{X})) \le g(\theta) \le g(L(\underline{X}))), & \text{if } g \text{ is decreasing in } \theta \end{cases}$$

Methods of Finding Interval Estimators (section 9.2)

Inverting a Test Statistic (section 9.2.1)

Ex: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 known.

$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$

For fixed α, the UMPU {unbiased level α test} rejects H_0 if $\underset{\sim}{x} \in R = \{\underset{\sim}{x} : |\bar{x} - \mu_0| > z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\}$

when $\alpha = P_{\mu_0}(\underset{\sim}{X} \in R) = P(\text{Reject } H_0 \mid \mu = \mu_0)$. It follows that

$$1 - \alpha = P(\text{Fail to reject } H_0 \mid \mu = \mu_0)$$
$$= P_{\mu_0}\left(|\bar{X} - \mu_0| \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = P_{\mu_0}\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu_0 \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

which is true for every μ_0.

Hence $P_\mu\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$.

Theorem 9.2.2 For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a level α test of $H_0: \theta = \theta_0$. For each $\underset{\sim}{x} \in \mathcal{X}$, define $C(\underset{\sim}{x}) \subset \Theta$ by

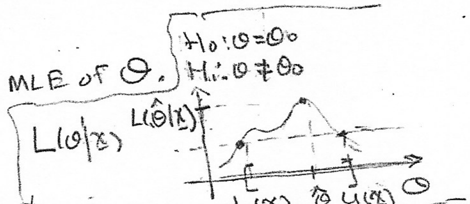
$$C(\underset{\sim}{x}) = \{\theta_0 : \underset{\sim}{x} \in A(\theta_0)\},$$

i.e., $\underset{\sim}{x} \in A(\theta_0) \Leftrightarrow \theta_0 \in C(\underset{\sim}{x})$.
Then the random set $C(\underset{\sim}{X})$ is a $1-\alpha$ confidence set for θ.

(converse also holds. see textbook).

Remark: Inverting a LRT involves the likelihood function and the MLE of θ.

$H_0: \theta = \theta_0$
$H_1: \theta \neq \theta_0$



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** Pivotal Quantities (section 9.2.2)

Def: A random quantity $Q(\underset{\sim}{X}, \theta)$ is a pivotal quantity for θ if the distribution of $Q(\underset{\sim}{X}, \theta)$ is independent of all parameters.

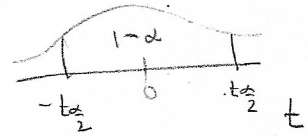
If "pivoting" can take place, then

$$1 - \alpha = P(q_1 \le Q(\underset{\sim}{X}, \theta) \le q_2)$$
"pivoting action"
$$\vdots$$
$$= P_\theta(L(\underset{\sim}{X}) \le \theta \le U(\underset{\sim}{X}))$$

Ex: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

$Q(\underset{\sim}{X}, \mu) = \dfrac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$ is a pivotal quantity for μ.

$$1 - \alpha = P\left(-t_{\frac{\alpha}{2}} \le \frac{\bar{X} - \mu}{S/\sqrt{n}} \le t_{\alpha/2}\right)$$



$$\vdots$$
$$= P_\mu\left(\bar{X} - t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right)$$

comment on expected length of interval $E\left[\left(\bar{X} + t_{\alpha/2} S/\sqrt{n}\right) - \left(\bar{X} - t_{\alpha/2}\frac{S}{\sqrt{n}}\right)\right] = 2\frac{t_{\frac{\alpha}{2}}}{\sqrt{n}} E[S] = ?$

Ex: $H_0: \theta = \theta_0$
$H_1: \theta \neq \theta_0$

$$\lambda(\underline{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta \mid \underline{x})}{\sup_{\theta \in \Theta_1} L(\theta \mid \underline{x})} = \frac{L(\theta_0 \mid \underline{x})}{L(\hat{\theta} \mid \underline{x})}.$$

Reject H_0 for $\lambda(\underline{x}) \le c$ where $\hat{\theta}$ is the MLE of $\hat{\theta}$

$\Leftrightarrow$ reject H_0 for $L(\theta_0 \mid \underline{x}) \le c\, L(\hat{\theta} \mid \underline{x})$.

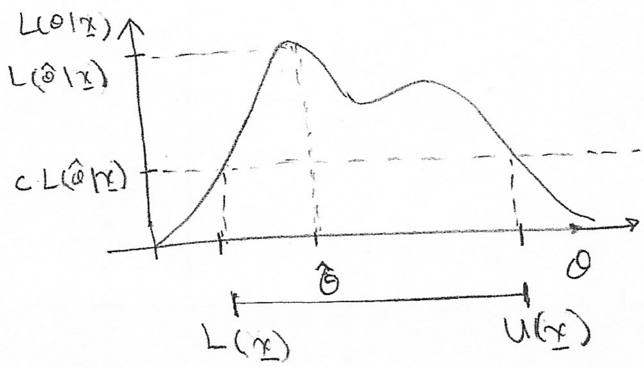
Acceptance region is

$A(\theta_0) = \{ \underline{x} : L(\theta_0 \mid \underline{x}) \ge c\, L(\hat{\theta} \mid \underline{x}) \}$ if choose $c \ni P_{\theta_0}(\underline{X} \in A(\theta_0)) = 1-\alpha$

Then

$C(\underline{x}) = \{ \theta : L(\theta \mid \underline{x}) \ge c\, L(\hat{\theta} \mid \underline{x}) \}$ is a $1-\alpha$ confidence set for θ.

(Consider the graph of $L(\theta \mid \underline{x})$).



↑ This may not actually be an interval!

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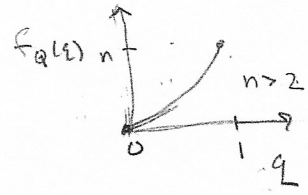
$$F_Q(q) = P\left(\frac{X_{(n)}}{\theta} \le q\right) = P(X_{(n)} \le q\theta)$$
$$= \int_0^{q\theta} \frac{n}{\theta^n} t^{n-1}\,dt = \frac{n}{\theta^n}\frac{t^n}{n}\Big|_0^{q\theta} = q^n$$
$$f_Q(q) = n q^{n-1} I_{[0,1]}(q)$$

Ex : $X_1, \dots, X_n \overset{iid}{\sim}$ Uniform $[0, \theta]$.

$X_{(n)}$ has pdf $f(t|\theta) = \frac{n}{\theta^n} t^{n-1} I_{[0,\theta]}(t)$.

Let $Q(\underset{\sim}{X}, \theta) = \frac{X_{(n)}}{\theta} = \max\left(\frac{X_1}{\theta}, \dots, \frac{X_n}{\theta}\right) \Rightarrow$ max of iid Uniform$[0,1]$.

Then $f_Q(q) = n q^{n-1} \cdot I_{[0,1]}(q)$.



$$1-\alpha = P(q_1 \le Q \le q_2) = P\left(q_1 \le \frac{X_{(n)}}{\theta} \le q_2\right)$$
$$= P_\theta\left(\frac{X_{(n)}}{q_2} \le \theta \le \frac{X_{(n)}}{q_1}\right)$$

where $\int_{q_1}^{q_2} n q^{n-1}\,dq = 1-\alpha$. e.g., choosing $q_2 = 1$ results in $1 - q_1^n = 1-\alpha$

so $q_1 = \alpha^{1/n}$ and a $100(1-\alpha)\%$ CIE for θ is $\left(X_{(n)}, \frac{X_{(n)}}{\alpha^{1/n}}\right)$.

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** Omit Section 9.2.3

Bayesian Intervals (Section 9.2.4)

Recall that the posterior distribution of θ is

$$\pi(\theta|\underset{\sim}{x}) \propto L(\theta|\underset{\sim}{x}) \cdot \pi(\theta)$$

Def : For model $f(\underset{\sim}{x}|\theta)$, $\theta \in \Theta$, a unidimensional interval, and prior $\pi(\theta)$, let $\pi(\theta|\underset{\sim}{x})$ be the posterior density. The set $A \subset \Theta$ is a $100(1-\alpha)\%$ Bayesian credible set for θ if

$$P(\theta \in A|\underset{\sim}{x}) = \int_A \pi(\theta|\underset{\sim}{x})\,d\theta = 1-\alpha.$$

Remark : Any interval (t_1, t_2) satisfying $\int_{t_1}^{t_2} \pi(\theta|\underset{\sim}{x})\,d\theta = 1-\alpha$ is a $100(1-\alpha)\%$ Bayesian credible interval or Posterior Bayes interval estimate of θ. Note that $t_1 = t_1(\underset{\sim}{x})$, $t_2 = t_2(\underset{\sim}{x})$.

Ex : $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known. Furthermore, $\theta \sim N(\mu_0, \sigma_0^2)$, μ_0, σ_0^2 known.

Recall that $$\theta|\underset{\sim}{X} = \underset{\sim}{x} \sim N\left(\frac{\frac{\sum_{i=1}^n x_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}\right).$$

Find $(t_1, t_2) \ni \int_{t_1}^{t_2} \pi(\theta|\underset{\sim}{x})\,d\theta = 1-\alpha$. The shortest $100(1-\alpha)\%$ Bayesian Credible interval is

$$t_1(\underset{\sim}{x}) = \frac{\frac{\sum_{i=1}^n x_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}} - z_{\alpha/2}\sqrt{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}}$$

$$t_2(\underset{\sim}{x}) = \frac{\frac{\sum_{i=1}^n x_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}}{\text{same}} + z_{\alpha/2}\sqrt{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}}$$

Ex: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu_1, \sigma^2)$

$Y_1, \dots, Y_m \overset{iid}{\sim} N(\mu_2, \sigma^2)$

Assume that $\underset{n\times 1}{\underline{X}}$ and $\underset{m\times 1}{\underline{Y}}$ are independent.

Find a $100(1-\alpha)\%$ CIE for $\mu_1 - \mu_2$.

From HW #10, we have that

$$\frac{(\bar{X}-\bar{Y}) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \sim t(n+m-2).$$

is a pivotal quantity for $\mu_1 - \mu_2$. Then

$$1-\alpha = P\left(-t_{\alpha/2} \le \frac{(\bar{X}-\bar{Y}) - (\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \le t_{\alpha/2} \right)$$

$$\vdots$$

$$= P\left((\bar{X}-\bar{Y}) - t_{\alpha/2} S_p\sqrt{\tfrac{1}{n}+\tfrac{1}{m}} \le \mu_1 - \mu_2 \le (\bar{X}-\bar{Y}) + t_{\alpha/2} S_p\sqrt{\tfrac{1}{n}+\tfrac{1}{m}} \right)$$

and $(\bar{X}-\bar{Y}) \pm t_{\alpha/2} S_p \sqrt{\frac{1}{n}+\frac{1}{m}}$ is the shortest $100(1-\alpha)\%$ CIE for $\mu_1 - \mu_2$.

Why is it the shortest?

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where

$$P(t_1(\underline{X}) \le \Theta \le t_2(\underline{X})) = 1-\alpha$$

for $\underline{X} = \underline{x}$ and Θ random.

Remark The corresponding confidence interval estimate

$$\left(\frac{\Sigma x_i}{n} - z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{n}} , \frac{\Sigma x_i}{n} + z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{n}} \right)$$

is always wider than the Bayesian credible interval since one ought to expect some gain with the added assumption of a known prior.