1 Singularities

Theorem 1 (Riemann's Theorem on removeable singularities)**.** *$f(z)$ has an isolated singularity at z_0, bounded in punctured neighborhood of z_0, then f can be exteneded to a holomorphic function at z_0.*

Lemma 2. *If $F(s,z)$ is continuous in $[0,1]\times\Omega$, such that $\Omega\subset\mathbb{C}$ open, and holomorphic in z, then $\int_0^1 F(s,z)ds$ is holomorphic in Ω*

One argument: write integral as a limit of Riemann sums (obviously holomorphic in z) $\implies$ the limit is holomorphic if we can show uniform convergence on compact subsets of Ω

Another argument:
(1) $\int_0^1 F(s,z)dz$ continuous in Ω. Choose $\overline{\mathbb{D}}$ such that $z\in\overline{\mathbb{D}}\subset\Omega$. $F(s,z)$ is uniformly continuous on $[0,1]\times\Omega$. $\forall\epsilon\exists\delta$ such that if $|z'-z|<\delta$ then $|F(s,z')-F(s,z)|<\epsilon\ \forall s\in[0,1] \implies \left|\int_0^1 F(s,z')-F(s,z)dz\right| \leqslant \int_0^1 |F(s,z')-F(s,z)|ds < \int_0^1 dz\epsilon = \epsilon - 0 = \epsilon \implies \int_0^1 F(s,z)ds$ continuous in z.

(2) By Morera's Theorem, it suffices to check $\int_T \left(\int_0^1 F(s,z)ds\right)dz = 0, T\subset\Omega$ with its interior. By Frobinius $= \int_0^1 \left(\int_T F(s,z)dz\right)ds = 0$. Interior integral is zero by Cauchy or Goursat, so entire integral is 0, and f is holomorphic by Morera's.

Corollary 3. *z_0 is a pole of $f(z) \iff \lim_{z\to z_0}|f(z)| = \infty$*

z_0 is a pole of $f(z) \implies z_0$ is a zero of $\frac{1}{f(z)} \implies \lim_{z\to z_0}\frac{1}{f(z)} = 0 \implies \lim_{z\to z_0}|f(z)| = \infty$.
If $\lim_{z\to z_0}|f(z)| = \infty \implies \lim_{z\to z_0}\left|\frac{1}{f(z)} = 0\right| \implies \frac{1}{f(z)}$ is bounded near $z_0 \implies \frac{1}{f(z)}$ can be extended to a holomorphic function at z_0, all if $g(z), g(z_0) = \lim_{z\to z_0} g(z) = 0 \implies f(z) = \frac{1}{g(z)}$ has a pole at z_0.

Definition 4. Refined classification of isolated singularities at z_0.

(1) removable $\iff f(z)$ is bounded near $z_0 \iff f(z)$ is holomorphic at z_0 (last by Riemann)

(2) pole $\iff f(z) = \frac{1}{g(z)}, g(z)$ is holomorphic, $g(z_0) = 0 \iff \lim_{z\to z_0}|f(z)| = \infty$

(3) Essential singularities? (holo on deleted neighborhood but not remov sing. or pole)

Example 5. $e^{1/z}$ at $z = 0$

Theorem 6 (Casorati-Weierstrass)**.** *z_0 is an essential singularity of $f(z) \implies f(0<|z-z_0|<r)$ dense in $\mathbb{C}\forall r$.*

Argue by contradiction: suppose $\exists w_0$ and R such that $|f(z)-w_0| > R\forall z$ such that $0<|z-z_0|<r$ $\frac{1}{f(z)-w_0} < R$ is bounded in same annulus. So by Riemann's Theorem, $= g(z)$, which is holomorphic for $|z-z_0|<r$ (by Riemann's Theorem). So $f(z) = w_0 + \frac{1}{g(z)}$. If $g(z_0)\neq 0$, then f is holomorphic at z_0. If $g(z_0) = 0 \implies w_0 + \frac{1}{g(z)}$ has a pole at z_0, which is a contradiction

Theorem 7 (Picard's Theorem)**.** *Every $\alpha\in\mathbb{C}$, with at most one exception, belongs to the image $f(0<|z-z_0|<r)\forall r$, and occurs infinitely many times*

Covers all for given r, shrink r, covers all by Picard's, shrink, etc.

"Singularity at ∞": in book

"Riemann Sphere" $S^2 = \mathbb{CP}^1 = \mathbb{C}$ disjoint union $\{\infty\}$.

$f(z)$ has an isolated singularity at $\infty \iff f(\frac{1}{z})$ is holomorphic for $0 < \left|\frac{1}{z}\right| < \frac{1}{R} \iff f(z)$ is holomorphic for $|z| > R$ for some R.

$f(z)$ has a removable singularity at $\infty \iff f(z)$ is bounded for $|z| > R \iff f(\frac{1}{z})$ is holomorphic at $z = 0 \iff f(\frac{1}{z}) = \sum_{n \geqslant 0} a_n \left(\frac{1}{z}\right)^n$ converges for $\frac{1}{z} < r$

$f(z)$ has a pole at $\infty \iff f(\omega)$ has a pole at $0, \omega = 1/z \iff \lim \omega \to 0 |f(\omega)| = \infty \iff \lim_{z\to\infty} |f(z)| = \infty$.

$f(\omega) = \frac{a_{-n}}{\omega^n} + \ldots + \frac{a_{-1}}{\omega} + H(\omega)$ holo at $\omega \iff$ bounded near 0. $f(z) = a_{-n}z^n + \ldots + a_{-1}z + H(\frac{1}{z})$ bounded at ∞ (for $|z| > R$ for some R)

Theorem 3.4.

argument principle

Theorem 8 (Roche Theorem). *Suppose $f(z)$ and $g(z)$ are holomorphic in Ω, which contains a simple closed curve γ and its interior. Supppose $|f(z)| > |g(z)| \forall z \in \gamma$. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros (counted with multiplication) inside γ.*

Let $f_s(z) = f(z) + sg(z), s \in [0,1]$. $f_s(z)$ is holomorphic in $\Omega \forall s \in [0,1]$. $\forall z \in \gamma$ $|f_s(z)| = |f(z) + sg(z)| \geqslant |f(z) - s|g(z)| \geqslant 0 \implies f_s(z)$ doesn't vanish along $\gamma \implies$ the number of zeros of $f_s(z)$ inside γ (with multiplication) is equal to $= \frac{1}{2\pi}\int_\gamma \frac{f'(z) + sg'(z)}{f(z) + sg(z)}dz$. The integral is a continuous function of (z, s) on $\gamma \times [0,1]$ (compact) $\implies \frac{1}{2\pi}\int_\gamma \frac{f'(z) + sg'(z)}{f(z) + sg(z)}dz$ is continuous in S. But it takes integer values $\implies \frac{1}{2\pi}\int_\gamma \frac{f'(z) + sg'(z)}{f(z) + sg(z)}dz$ is constant in S. In particular, the same for $f(z)$ (where $s = 0$)and $f(z) + g(z)$ (where $s = 1$)

Example 9. Find the number of zeros inside $|z| < 1$ of $z^{100} + 4z^3 - z + 1 = f(z) + g(z)$. $f(z) = 4x^3$, with 3 zeros with multiplication, and $g(x) = z^{100} - z + 1$. $|4z^3| = 4$. $|g(x)| \leqslant 3 < 4$.

Theorem 10 (Open Mapping Theorem). *If $f(z)$ is holomorphic in connected open Ω and nonconstant $\implies f : \Omega \to f(\Omega)$ is an open map (sends open sets to open sets).*

Remark 11. We want connected to avoid something like HW1P4

It suffices to show that $f(\Omega)$ is open. Say $f(z_0) = \omega_0$. Show the image of f contains some neighborhood of ω_0, or $\exists r > 0$ such that $\{|\omega - \omega_0| < r\} \subset f(\Omega)$. Equivalent to sayint that $f(z) - \omega$ has a root in Ω $\forall\omega$ such that $|\omega - \omega_0| < r$. $f(z) - \omega$ has a solution z_0. Chose a circle $|z - z_0| = \delta$. $f(z) - w_0$ has a zero inside the circle. Apply Rouche theorem.

We know $\exists\delta$ such that $f(z) - \omega_0 \neq 0$ for some $z, |z - z_0| = \delta$ because roots form discrete set. Take $r = \min |f(z) - \omega_0|$, chain inequaliities, apply Rouche Theorem

2 Riemann Zeta function

Say $s > 1$, and the sum $\sum \frac{1}{n^s} = \zeta(s)$ converges for $\forall s > 1$. In Calculus, you prove this converges with the integral test.

Theorem 12. *$\zeta(s)$ admits an analytic continutation to $\mathbb{C} \setminus 1$ where it has a simple pole*

Theorem 13. *The only zeros of $\zeta(s)$ outside the "critical strip" $0 \leqslant Re(s) \leqslant 1$ are "trivial zeros": -2, -4,. . .. (simple zeros of Gamma function)*

Theorem 14 (Hadamard, Valle Poussin). *No zeros exist on the boundary of the critial strip*

Theorem 15 (Riemann Hypothesis). *All zeros are in the middle of the strip, along $Re(s) = 1/2$.*

Definition 16 (Tchebyshev's ψ-function). $\psi(x) = \sum_{p^m \leqslant X} \log p$, p is prime, $x \in \mathbb{R}, x > 0$.

Definition 17 (Riemann's Explicit Formula). $\psi(x) = x - \sum_{\zeta(\rho)=0} \frac{x^\rho}{\rho} - \ln(2\pi) - \frac{1}{2}\ln(1 - x^{-2}), 0 < Re(\rho) < 1$

Definition 18 (Wacky Prime Number Theorem). As $x \to \infty$, the last terms just goes away. $\psi(x) \sim X$, or $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$

If you believe the Riemann hypothesis, $\rho = 1/2$

$\zeta(s) = \sum \frac{1}{n^s}, \mathrm{Re}(s) > 1$

Definition 19 (Euler's product expansion). $\zeta(s) = \prod_p \frac{1}{1-\frac{1}{p^s}}, \mathrm{Re}(s) > 1$ for p prime

Corollary 20. *$\zeta(s)$ has no zeros in $\mathrm{Re}(s) > 1$*

Idea: $\frac{1}{1-\frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots$

$\implies \prod_p \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{p_{i_1}^{k_{i_1}s} \cdots p_{i_r}^{k_{i_r}}} = \sum_{n\geqslant 1} \frac{a(n)}{n^s}$ where $a(n)$ is the number of ways to write n

Function is holomorphic unless $e^{s \ln p} = 1$, but can't happen since $\mathrm{Re}(s) > 1$. Also, cannot vanish. Product of non-vanishing factors can't vanish (?)

Caution: $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \to 0$, so need to justify

$\prod_{p<N} (1 + \frac{1}{p^s} + \ldots + \frac{1}{p^{Ms}}) < 1 + \frac{1}{2^s} + \ldots = \zeta(s)$
send M to infinity, holds, send N to infinity, holds.

Claim: $\prod_p \frac{1}{1-\frac{1}{p^s}}$ is holomorphic for $\mathrm{Re}(s) > 1$

Proposition 21. *Suppose $F_n(z)$ holom. in Ω, $|F_n(z)| \leqslant c_n$ in Ω and $x \sum c_n < \infty$. Then $\prod_{n\geqslant 1}(1 + F_n(z))$ converges uniformly in Ω to a holomorphic function in Ω, which is equal to 0 only at zeros of its factors.*

As the limit of a sequence of holom. functions converges uniformly on compact sets

In our case: need to show that $\prod(1 - \frac{1}{p^s})$ is holomorphic.
$\left|\frac{1}{p^s}\right| = \frac{1}{p^{\mathrm{Re}(s)}}$.
$\sum \frac{1}{p^{\mathrm{Re}}(s)} < \sum \frac{1}{n^{\mathrm{Re}(s)}} < \infty$ since $\mathrm{Re}(s) > 1$
$\implies \prod(1 = \frac{1}{p^s})$ is holomorphic for $\mathrm{Re}(s) < 1$
doesn't vanish $\implies \prod \frac{1}{1-\frac{1}{p^s}}$ is holomorphic for $\mathrm{Re}(s) > 1$

For $n >> 0$, $|F_n(z)| < \frac{1}{2}$. Let's assume that this is true $\forall n$
$\implies \mathrm{Re}(1 + F_n(z)) > 0$
$\implies \log(1 + F_n(z))$ is defined, where log is standard branch.

Let's show that $\sum \log(1 + F_n(z))$ converges absolutely and uniformly

$|\log(1 + w)|$ for $|w| < \frac{1}{2}$
$\log(1 + w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \cdots$
$\implies \log(1 + w) = w[1 - \frac{w}{2} + \frac{w^2}{3} - \cdots]$
$\implies |\log(1 + w)| \leqslant |w|[1 + \frac{|w|}{2} + \frac{|w|^2}{3} + \cdots]$

$\implies |\log(1+w)| \leqslant |w|[1+\frac{|w|}{2}+\frac{|w|^2}{3}+\cdots]$
$\implies |\log(1+w)| \leqslant |w|[1+|w|+|w|^2+\cdots]$
$\implies |\log(1+w)| \leqslant 2|w|$

$\prod \frac{1}{1-\frac{1}{p}} = \infty$ because $\leqslant \sum \frac{1}{n} = \infty \implies \prod(1-\frac{1}{p})0$

Why doesn't this contradict the previous theorem?

$\implies \sum \frac{1}{p} = \infty$ (b.c otherwise the previous theorem applies as the terms are no zeros)
$\implies$ there are infinitely many primes

Theorem 22. *$\zeta(s)$ has no zeros for $Re(s) = 1$ (assuming $\zeta(s)$ has an analytic continutation into $Re(s) > 0$ with simple pole at $s = 1$ and no other poles)*

Argue by contradiction, suppose that $\zeta(1+it) = 0$

Remark 23. $|\zeta^3(s)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \to 0$ as $\sigma \to 1^+$

Remark 24. $|\zeta^3(s)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geqslant 1$ for $\sigma > 1$

Past two remarks give the proof

Remark 25. $\zeta(s)$ has a simple pole at $s = 1 \implies \zeta(s) = \frac{1}{s-1}+$ holomorphic near $s = 1$
$\implies |\zeta^3(\sigma)| = \frac{|a|^3}{|\sigma-1|^3}+$ bounded near $\sigma = 1$

Last time: if $\zeta(1+it) = 0 \implies |\zeta^3(s)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \to 0$ as $\sigma \to 1^+$ assuming $\zeta(s)$ has an analytic continutation to $\mathrm{Re}(s) > 0$ with a simple pole at 1 and no other poles along $\mathrm{Re}(s) = 1$

$|\zeta^3(s)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geqslant 1$ for $\sigma > 1$

$3\ln|\zeta(s)| + 4\ln|\zeta^4(\sigma+it)| + \ln|\zeta(\sigma+2it)| \geqslant 0$ (regular natural logarithm for real numbers)

$\mathrm{Re}(1) > 1 \implies \zeta(s) = \prod_{\text{primes}} \frac{1}{1-p^{-s}}$, take absolute value of both sides, take logarithm is $\sum_{\text{primes}} -\ln|1-p^{-s}|$

$|p^{-s}| < 1 \implies$ we can compute principal branch $\log(1-p^{-s}) \implies -\ln|1-p^{-s}| = \mathrm{Re} - \log(1-p^{-s})$

$|z| < 1 \implies -\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots \implies -\log(1-p^{-s}) = p^{-s} + \frac{p^{-2s}}{2} + \ldots$

$\mathrm{Re}(\sum$ primes ...) [aka $\ln|\zeta(s)|$]= $\sum_{n\geqslant 1} \frac{c_n}{n^s} c_n \geqslant 0$

2.1 Schwarz function idk where this is going

$f(x) \in C^\infty(\mathbb{R})$ is a Schwarz function if f and all its derivatives decay faster than any 1/polynomial function.

$|\frac{\partial f}{\partial x^m} x^n < c_{n,m}|$ for some $c_{n,m}$ Basic example: $e^{-\pi x^2}$. At some point we will assume that $f(x)$ is also even.

Definition 26. An associated θ function $\theta_f(y) = \sum_{n\in\mathbb{Z}} f(y \cdot n)$ $y > 0$

C^∞ function

Definition 27. Asn associated gamma function $\Gamma_f(s) = \int_0^\infty t^s f(t) \frac{dt}{t}$.

This function will be analytic if $\mathrm{Re}(s) > 0$

If $f(x) = e^{-\pi x^2}$ then $\theta_f(y) = \sum_{n\in Z} e^{-\pi n^2 y^2}$

Recall Jacobi theta function $\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$, $z \in \mathbb{C}$.

$\theta_f(y) = \theta(iy^2)$

"Gamma factor" $\Gamma_f(s) = \int_0^\infty t^s e^{\pi t^2} \frac{dt}{t}$
$x = \pi t^2, = x^{1/2}\pi^{-1/2}, dx/x = 2\pi t dt/(\pi t^2) = 2dt/t$
$\implies \int_0^\infty x^{s/2}\pi^{-s/2}e^{-x}$
$= \frac{1}{2}\pi^{-s/2}\int_0^\infty x^{s/2}e^{-x}\frac{dx}{x} = \frac{1}{2}\pi^{s/2}\Gamma(s/2)$

Lemma 28. $\theta_f(y) \in C^\infty(\mathbb{R})$ *and derivatives can be computed term by term.*

Just need to show differentiability (f' is also Schwarz)

Check uniform and absolute convergence (on compact sets)

$|f(yn)| \leqslant \frac{C}{(yn)^2} = \frac{C}{y^2n^2}$

$\sum |f(yn)| \leqslant \frac{C}{y^2}\sum \frac{1}{n^2}$. Absolute convergence