MATH 523H, 2022 Spring

Written Homework 2

Due Friday Feb. 18

*(Exer XX) refers to the corresponding exercise number in Textbook.

Problem **1.** (Exer 9.3)

Suppose $\lim a_n = a, \lim b_n = b$, and $s_n = \dfrac{a_n^3 + 4a_n}{b_n^2 + 1}$. Prove $\lim s_n = \dfrac{a^3 + 4a}{b^2 + 1}$ carefully, using the limit theorems.

Problem **2.** (Exer 9.4 but modified)

Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{1 + s_n}$.

(a) List the first four terms of (s_n).

(b) Show that (s_n) is bounded above by 2.

Hint: Use mathematical induction.

(c) Show that (s_n) is increasing.

Hint: Use mathematical induction.

(d) By parts (b) and (c), the sequence (s_n) is bounded and increasing, hence it converses by Theorem 10.2. Using this fact and prove the limit is $\varphi = \frac{1}{2}(1 + \sqrt{5})$ (this constant is known as the *golden ratio*).

Remark: This problem justifies the infinitely nested radical expansion of the golden ratio:

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}.$$

Problem **3.** (Original, related to Exer 9.4)

Let $t_1 = 1$ and for $n \geq 1$ let $t_{n+1} = 1 + \frac{1}{t_n}$.

(a) List the first four terms of (t_n).

(b) It turns out that (t_n) converges. Assume this fact and prove the limit is $\varphi = \frac{1}{2}(1 + \sqrt{5})$.

Remark: This problem (partially) justifies the infinite continued fraction expansion of the golden ratio:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}.$$

Problem **4.** (Exer 9.12)

Assume all $s_n \neq 0$ and that the limit $L = \lim \left| \dfrac{s_{n+1}}{s_n} \right|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$.

Hint: Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.

(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.

Hint: Apply (a) to the sequence $t_n = \dfrac{1}{|s_n|}$; see Theorem 9.10.

Problem **5.** (Exer 9.13)

Show

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if} \quad |a| < 1 \\ 1 & \text{if} \quad a = 1 \\ +\infty & \text{if} \quad a > 1 \\ \text{does not exist} & \text{if} \quad a \leq -1. \end{cases}$$

Problem **6.** (Exer 9.14)

Let $p > 0$. Use Problem **4** (Exer 9.12) to show

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if} \quad |a| \leq 1 \\ +\infty & \text{if} \quad a > 1 \\ \text{does not exist} & \text{if} \quad a < -1. \end{cases}$$

Hint: For the $a > 1$ case, use Problem **4**(b).

Problem **7.** (Exer 9.15)

Show $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

Problem **8.** (Exer 9.18)

(a) Verify $1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$ for $a \neq 1$.

(b) Find $\lim_{n\to\infty} (1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

(c) Calculate $\lim_{n\to\infty} (1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n})$.

(d) What is $\lim_{n\to\infty} (1 + a + a^2 + \cdots + a^n)$ for $a \geq 1$?

Caution: Deal with the $a = 1$ case separately as part (a) does not apply to this case.