Problem 1: Describe all homomorphisms from a given ring R to a given ring S explicitly, i.e. say where every element $r \in R$ goes to in S. Prove that your functions are indeed homomorphisms and that there are no other homomorphisms.

- $R = \mathbb{Z}$, $S = \mathbb{Z} \times \mathbb{Z}$
- $R = \mathbb{Z}_5, S = \mathbb{Q}$
- $R = \mathbb{Z}_2 \times \mathbb{Z}_2, S = \mathbb{Z}_2$

Solution
Preface: A general strategy that I will be using in my solutions is using the fundamental theorem on homomorphisms: (a)

(b)

(c) Define $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2$ to be

$$\begin{aligned}(0,0) &\to 0\\ (0,1) &\to 1\\ (1,0) &\to 1\\ (1,1) &\to 0\end{aligned}$$

Which looks a lot like an xor gate. In other words, $\phi((a,b)) = a + b \in \mathbb{Z}_2$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2$ are commutative rings (since both are abelian under both operations from 411), so both $+$ and $\cdot$ are associative and commutative in our ring (axiom)

$$\begin{aligned}\phi((a,b) + (a',b')) &= \phi((a+a', b+b'))\\ &= (a+a') + (b+b')\\ &= a + (a'+b) + b'\\ &= a + (b+a') + b'\\ &= (a+b) + (a'+b')\\ &= \phi((a,b)) + \phi((a',b'))\end{aligned}$$

$$\begin{aligned}\phi((a,b)(a',b')) &= \phi((aa', bb'))\\ &= (aa')(bb')\\ &= a(a'b)b'\\ &= a(ba')b'\\ &= (ab)(a'b)'\\ &= \phi((a,b))\phi((a',b))'\end{aligned}$$

Problem 2: For a given subset S if a given ring R, decide whether S is a subring or not (with proof)

- $S = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}, R = \mathbb{R}$
- $S = \{f(x) | f'(3) = 0\}, R = \{f : \mathbb{R} \to \mathbb{R}\}$

Solution

Problem 3: Describe all units in a given ring R explicitly

- $R = \mathbb{Z}_4 \times \mathbb{Z}_4$
- $R = Mat_2(Z_2)$

Solution

(a) the unity of Z_4 is $(1, 1)$, because 1 is the multiplicative identity of $\mathbb{Z}_4$. Therefore, the units of R are the pairs with entries that are invertible in $\mathbb{Z}_4$, more specifically 1 and 3. There does not exist any element x such that $2x = 1 \in \mathbb{Z}_4$, because $\gcd(2, 4) = 2$, so the Diophantine equation equivalent to this congruence cannot equal any positive number strictly less than 2. Therefore, the units in R are $(1, 1), (1, 3), (3, 1)$, and $(3, 3)$.

(b)

Problem 4: Given an example of a ring with unit $1 \neq 0$ that has a subtring with a non-zero unity $e \neq 1$

Solution

Problem 5: Let U be a collection of all units in a ring $(R, +, \cdot)$ with unity, Prove that $(U, \cdot)$ is a group

Solution

$Associa(R, +, \cdots)$ being a ring $\implies \cdot$ is associative for all elements in R. Therefore, because all elements in U are also in R, they must all satisfy associativity under multiplication.

Say R has unity 1. $1 \in U$ because $\forall a \in R : a \cdot 1 = a \implies 1 \cdot 1 = 1$, which is the definition of a unit. Hence, $1 \in U$. Because all other units in U are also in R, the above property of unity (or identity for groups) is satisfied, and $1 \in R$ is the identity element of U.

Problem 6: Let X be the collection of all rings. Prove that isomorphism of rings gives an equivalence relation on X

Solution

Problem 7: An element x of a ring R is called nilpotent if $x^n = 0$ for osme $n > 0$.

- Find all nilpotents in $\mathbb{Z}_{2022}$
- Give an example of a ring with 2 nilpotents
- Let R be a commutative ring with nilpotents x, y. Show that $x + y$ is also nilpotent

Solution

Problem 8: Find all solutions of the equation $x^2 + 2x + 2 = 0$ in a given ring R

- $R = \mathbb{Z}_5$
- $R = \mathbb{Z}_7$
- $R = \mathbb{Z}_8$

Solution

Problem 9: Show that the characteristic of an integral domain is either 0 or a prime number p

Solution

Problem 10: For each of the following rings R decide (with proof) whether R is a field of whether R is an integral domain:

- $R = \mathbb{Z}_{2021}$
- $R = \{evenintegers\}$
- $R = \{\text{polynomials with } x \text{ with coeffiecients in } \mathbb{R}\}$ $(\mathbb{R}[X])$
- $R = \mathbb{C}$

Solution