1 SINGULARITIES

Theorem 1 (Riemann's Theorem on removeable singularities)**.** *$f(z)$ has an isolated singularity at z_0, bounded in punctured neighborhood of z_0, then f can be exteneded to a holomorphic function at z_0.*

Lemma 2. *If $F(s,z)$ is continuous in $[0,1]\times\Omega$, such that $\Omega\subset\mathbb{C}$ open, and holomorphic in z, then $\int_0^1 F(s,z)ds$ is holomorphic in Ω*

One argument: write integral as a limit of Riemann sums (obviously holomorphic in z) $\implies$ the limit is holomorphic if we can show uniform convergence on compact subsets of Ω

Another argument:
(1) $\int_0^1 F(s,z)dz$ continuous in Ω. Choose $\overline{\mathbb{D}}$ such that $z\in\overline{\mathbb{D}}\subset\Omega$. $F(s,z)$ is uniformly continuous on $[0,1]\times\Omega$. $\forall\epsilon\exists\delta$ such that if $|z'-z|<\delta$ then $|F(s,z')-F(s,z)|<\epsilon\ \forall s\in[0,1] \implies \left|\int_0^1 F(s,z')-F(s,z)dz\right|\leqslant\int_0^1|F(s,z')-F(s,z)|ds < \int_0^1 dz\epsilon=\epsilon-0=\epsilon \implies \int_0^1 F(s,z)ds$ continuous in z.

(2) By Morera's Theorem, it suffices to check $\int_T\left(\int_0^1 F(s,z)ds\right)dz=0, T\subset\Omega$ with its interior. By Frobinius $=\int_0^1\left(\int_T F(s,z)dz\right)ds=0$. Interior integral is zero by Cauchy or Goursat, so entire integral is 0, and f is holomorphic by Morera's.

Corollary 3. *z_0 is a pole of $f(z)$* $\iff \lim_{z\to z_0}|f(z)|=\infty$

z_0 is a pole of $f(z) \implies z_0$ is a zero of $\frac{1}{f(z)} \implies \lim_{z\to z_0}\frac{1}{f(z)}=0 \implies \lim_{z\to z_0}|f(z)|=\infty$.
If $\lim_{z\to z_0}|f(z)|=\infty \implies \lim_{z\to z_0}\left|\frac{1}{f(z)}=0\right| \implies \frac{1}{f(z)}$ is bounded near $z_0 \implies \frac{1}{f(z)}$ can be extended to a holomorphic function at z_0, all if $g(z), g(z_0)=\lim_{z\to z_0}g(z)=0 \implies f(z)=\frac{1}{g(z)}$ has a pole at z_0.

Definition 4. Refined classification of isolated singularities at z_0.

(1) removable $\iff f(z)$ is bounded near $z_0 \iff f(z)$ is holomorphic at z_0 (last by Riemann)

(2) pole $\iff f(z)=\frac{1}{g(z)}, g(z)$ is holomorphic, $g(z_0)=0 \iff \lim_{z\to z_0}|f(z)|=\infty$

(3) Essential singularities? (holo on deleted neighborhood but not remov sing. or pole)

Example 5. $e^{1/z}$ at $z=0$

Theorem 6 (Casorati-Weierstrass)**.** *z_0 is an essential singularity of $f(z) \implies f(0<|z-z_0|<r)$ dense in $\mathbb{C}\forall r$.*

Argue by contradiction: suppose $\exists w_0$ and R such that $|f(z)-w_0|>R\forall z$ such that $0<|z-z_0|<r$ $\frac{1}{f(z)-w_0}<R$ is bounded in same annulus. So by Riemann's Theorem, $=g(z)$, which is holomorphic for $|z-z_0|<r$ (by Riemann's Theorem). So $f(z)=w_0+\frac{1}{g(z)}$. If $g(z_0)\neq 0$, then f is holomorphic at z_0. If $g(z_0)=0 \implies w_0+\frac{1}{g(z)}$ has a pole at z_0, which is a contradiction

Theorem 7 (Picard's Theorem)**.** *Every $\alpha\in\mathbb{C}$, with at most one exception, belongs to the image $f(0<|z-z_0|<r)\forall r$, and occurs infinitely many times*

Covers all for given r, shrink r, covers all by Picard's, shrink, etc.

"Singularity at ∞": in book

"Riemann Sphere" $S^2=\mathbb{CP}^1=\mathbb{C}$ disjoint union $\{\infty\}$.

$f(z)$ has an isolated singularity at $\infty \iff f(\frac{1}{z})$ is holomorphic for $0<\left|\frac{1}{z}\right|<\frac{1}{R} \iff f(z)$ is holomorphic for $|z|>R$ for some R.

$f(z)$ has a removable singularity at $\infty \iff f(z)$ is bounded for $|z| > R \iff f(\frac{1}{z})$ is holomorphic at $z = 0 \iff f(\frac{1}{z}) = \sum_{n \geqslant 0} a_n \left(\frac{1}{z}\right)^n$ converges for $\frac{1}{z} < r$

$f(z)$ has a pole at $\infty \iff f(\omega)$ has a pole at $0, \omega = 1/z \iff \lim \omega \to 0 |f(\omega)| = \infty \iff \lim_{z \to \infty} |f(z)| = \infty$.

$f(\omega) = \frac{a_{-n}}{\omega^n} + \ldots + \frac{a_{-1}}{\omega} + H(\omega)$ holo at $\omega \iff$ bounded near 0. $f(z) = a_{-n}z^n + \ldots + a_{-1}z + H(\frac{1}{z})$ bounded at ∞ (for $|z| > R$ for some R)

Theorem 3.4.

argument principle

Theorem 8 (Roche Theorem)**.** *Suppose $f(z)$ and $g(z)$ are holomorphic in Ω, which contains a simple closed curve γ and its interior. Supppose $|f(z)| > |g(z)| \forall z \in \gamma$. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros (counted with multiplication) inside γ.*

Let $f_s(z) = f(z) + sg(z), s \in [0,1]$. $f_s(z)$ is holomorphic in $\Omega \forall s \in [0,1]$. $\forall z \in \gamma$ $|f_s(z)| = |f(z) + sg(z)| \geqslant |f(z) - s|g(z)| \geqslant 0 \implies f_s(z)$ doesn't vanish along $\gamma \implies$ the number of zeros of $f_s(z)$ inside γ (with multiplication) is equal to $= \frac{1}{2\pi} \int_\gamma \frac{f'(z) + sg'(z)}{f(z) + sg(z)} dz$. The integral is a continuous function of (z, s) on $\gamma \times [0,1]$ (compact) $\implies \frac{1}{2\pi} \int_\gamma \frac{f'(z) + sg'(z)}{f(z) + sg(z)} dz$ is continuous in S. But it takes integer values $\implies \frac{1}{2\pi} \int_\gamma \frac{f'(z) + sg'(z)}{f(z) + sg(z)} dz$ is constant in S. In particular, the same for $f(z)$ (where $s = 0$)and $f(z) + g(z)$ (where $s = 1$)

Example 9. Find the number of zeros inside $|z| < 1$ of $z^{100} + 4z^3 - z + 1 = f(z) + g(z)$. $f(z) = 4x^3$, with 3 zeros with multiplication, and $g(x) = z^{100} - z + 1$. $|4z^3| = 4$. $|g(x)| \leqslant 3 < 4$.

Theorem 10 (Open Mapping Theorem)**.** *If $f(z)$ is holomorphic in connected open Ω and nonconstant $\implies f : \Omega \to f(\Omega)$ is an open map (sends open sets to open sets).*

Remark 11. We want connected to avoid something like HW1P4

It suffices to show that $f(\Omega)$ is open. Say $f(z_0) = \omega_0$. Show the image of f contains some neighborhood of ω_0, or $\exists r > 0$ such that $\{|\omega - \omega_0| < r\} \subset f(\Omega)$. Equivalent to sayint that $f(z) - \omega$ has a root in Ω $\forall \omega$ such that $|\omega - \omega_0| < r$. $f(z) - \omega$ has a solution z_0. Chose a circle $|z - z_0| = \delta$. $f(z) - w_0$ has a zero inside the circle. Apply Rouche theorem.

We know $\exists \delta$ such that $f(z) - \omega_0 \neq 0$ for some $z, |z - z_0| = \delta$ because roots form discrete set. Take $r = \min |f(z) - \omega_0|$, chain inequaliities, apply Rouche Theorem