Problem 1: Let $a, b \in \mathbb{C}$ and $|a| < r < |b|$ Let γ be a circle of radius r centered at the origin. Evaluate

$$\int_\gamma \frac{dz}{(z-a)(z-b)}$$

(Use only the definition of the integral but not Cauchy theorem or residues)

Solution

Problem 2: Let γ_R^+ be an upper semicircle of radius R centeret at the origin. Show that

$$\int_{\gamma_R^+} \frac{1 - e^{iz}}{z^2} dz \underset{R \to 0}{\to} 0$$

Solution

Problem 3: Recall that an open set $\Omega \subset \mathbb{C}$ is called connected if it cannot be expressed as a union of disjoing non-empty open sets. Show that Ω is connected if and only if every two points $z_1, z_2 \in \Omega$ can be connected by a polygonal path γ, i.e. a piece-wise smooth curve that consists of finitely many straight line segments.

Solution

$\Longleftarrow$: Assume that we can find two disjoint non-empty open sets Ω_1 and Ω_2 that satisfy $\Omega = \Omega_1 \cup \Omega_2$.

Problem 4: Suppose f is holomorphic in $\Omega \in \mathbb{C}$ and $Re(f)$ is constant. Prove that f is locally constant. Is it necessarily constant?

Solution

Problem 5: Let $\mathbb{D}$ be a the (open) unit disc and fix $w \in \mathbb{D}$. Consider the function $F(z) = \dfrac{w - z}{1 - \bar{w}z}$. Prove that F is a bjiective holomorphic function $\mathbb{D} \to \mathbb{D}$.

Solution

Problem 6: (a) Show that the Cauchy-Riemann equations take the following form in polar coordinates:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(b) Use (a) to show that the logarithm function defined as $\log(z) = \log(r) + i\theta$ is holomorphic for $r > 0, -\pi < \theta < \pi$

Solution

Problem 7: Let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the Laplacian. Show that $\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$

Solution

Problem 8: (a) Let α_n be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = L$.
Prove: $\lim_{n\to\infty} a_n^{1/n} = L$
SS: In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.
(b) Use (a) to compute radius of convergence of hypergeometric series

$$1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(a+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n$$

Here $\alpha, \beta, \gamma \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \ldots$

Solution

Problem 9: Prove that
(a) $\sum_{n\geq 0} nz^n$ does not converge at any point of the unit circle
(b) $\sum_{n\geq 1} \frac{z^n}{n^2}$ converges at every point of the unit circle

Solution

Problem 10: Let f be a power seies centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

Solution