Chapter 4: Completeness Axiom

Definition 1 (Max and Min)**.** Let S be a nonemtpy subset of $\mathbb{R}$.

1. If S contains a largest element s_0 (that is, s_0 belongs to S and $s \leqslant s_0$ for all $s \in S$, then we call s_0 the *maximum* of S and write $s_0 = \max S$.
2. If S contains a smallest element, then we call the smallest element the *minimum* of S and write it as $\min S$

Definition 2 (Bounds)**.** Let S be a nonempty subset of $\mathbb{R}$.

1. If a real number M satisfies $s \leqslant M$ for all $s \in S$, then M is called an *upper bound of* S and the set S is said to be *bounded above*
2. If a real number m satisfies $m \leqslant s$ for all $s \in S$, then m is called an *lower bound of* S and the set S is said to be *bounded below*
3. The set S is said to be *bounded* if it is bounded above and below. Thus S is bounded if there exist real numbers m and M such that $S \subseteq [m, M]$.

Definition 3 (sup and inf)**.** Let S be a nonempty subset of $\mathbb{R}$.

1. If S is bounded above and S has a least upper bound, then we will call it the *supremum of* S and denote it by $\sup S$.
2. if S is bounded below and S has a greatest lower bound, then we will call it the *infimum of* S and denote it by $\inf S$.

Theorem 4 (Completeness Axiom)**.** *Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words,* $\sup S$ *exists and is a real number.*

Corollary 5. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound* $\inf S$*.*

Theorem 6 (Archimedean Property)**.** *If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$*

Theorem 7 (Density of $\mathbb{Q}$)**.** *If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.*

Chapter 7: Limits of Sequences

Definition 8 (Convergence)**.** A sequence (s_n) of real numbers is said to converge to the real number s provided that for each $\epsilon > 0$ there exists a number N such that $n > N$ implies $|s_n - s| \leqslant \epsilon$. A sequence that does not converge to some real number is said to diverge.

Chapter 8: Proofs

All that you need to know here is the idea of computationally deriving your epsilon to use in your proof.

Chapter 9: Limit Theorems

Theorem 9. *Convergent sequences are bounded*

Theorem 10. *If the sequence (s_n) converges to s and k is in $\mathbb{R}$, then the sequence (ks_n) converges to ks. That is,* $\lim(ks_n) = k \cdot \lim(s_n)$

Theorem 11. *If (s_n) converges to s and (t_n) converges to t, then* $\lim(s_n + t_n)$ *converges to $s + t$. That is,* $\lim(s_n + t_n) = \lim s_n + \lim t_n$

Lemma 12. *If (s_n) converges to s, if $s_n \neq 0$ for all n, and if $s \neq 0$, then $(1/s_n)$ converges to $1/s$.*

Theorem 13. *Suppose (s_n) converges to s and (t_n) converges to t. If $s \neq 0$ and $s_n \neq 0$ for all n, then $(t_n/s_n$ converges to $t/s/$*

Theorem 14. *Basic Examples*
(a) $\lim_{n\to\infty}(\frac{1}{n^p}) = 0$ *for* $p > 0$
(b) $\lim_{n\to\infty} a^n = 0$ *if* $|a| < 1$.
(c) $\lim(n^{1/n}) = 1$.
(d) $\lim_{n\to\infty}(a^{1/n}) = 1$ *for* $a > 0$

Definition 15. For a sequence (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N$ implies $s_n > M$.

Theorem 16. *Let* (s_n) *and* (t_n) *be sequences such that* $\lim s_n = +\infty$ *and* $\lim t_n > 0$ *[*$\lim t_n$ *can be finite or* $+\infty$*]. Then* $\lim s_n t_n = +\infty$.

Theorem 17. *For a sequence* (s_n) *of positive real numbers, we have* $\lim s_n = +\infty$ *if and only if* $\lim(\frac{1}{s_n} = 0$

Chapter 10: Monotonic & Cauchy Sequences

Definition 18. A sequence (s_n) of real numbers is called an **increasing sequence** if $s_n \leqslant s_{n+1}$ for all n, and is called a **decreasing sequence** if $s_n \geqslant s_{n+1}$ for all n

Theorem 19. *All bounded monotone sequences converge*

Theorem 20. *Unbounded increasing sequences have a limit of* $s_n = +\infty$*, and unbounded decreasing has* $\lim s_n = -\infty$

Corollary 21. *If* (s_n) *is a monotone sequence, then the sequence either converges, diverges to* $+\infty$*, or diverges to* $-\infty$*. Thus* $\lim s_n$ *is always meaningful for monotone sequences.*

Definition 22. Let (s_n) be any sequence of real numbers. We define

$$\limsup s_n = \lim N \to \infty \sup\{s_n : n > N\} = \sup S$$

and

$$\liminf s_n = \lim N \to \infty \inf\{s_n : n > N\} = \inf S$$

Theorem 23. *Let* (s_n) *be a sequence in* $\mathbb{R}$.
(i) If $\lim s_n$ *is defined [as a real number,* $+\infty$*, or* $-\infty$*], then* $\liminf s_n = \lim s_n = \limsup s_n$
(ii) If $\lim inf s_n = \limsup s_n$*, then* $\lim s_n$ *is defined and* $\lim s_n = \lim inf s_n = \limsup s_n$

Definition 24. A sequence (s_n) or real numbers is called a Cauchy sequence if for each $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$

Lemma 25. *Convergent sequences are Cauchy sequences.*

Lemma 26. *Cauchy sequences are bounded*

Theorem 27. *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

Chapter 11: Subsequences

Definition 28. A sequence (s_n) of real numbers is called an *increasing sequence* if $s_n \leqslant s_{n+1}$ for all n, and (s_n) is called *decreasing sequence* if $s_n \geqslant s_{n+1}$ for all n. Note that if (s_n) is increasing, then $s_n \leqslant s_m$ whenever $n < m$. A sequence that is increasing or decreasing will be called a *monotone sequence* or a *monotonic sequence*.

Theorem 29. *All bounded monotone sequence converge.*

Theorem 30. *(i) If* (s_n) *is an unbounded increasing sequence, then* $\lim s_n = +\infty$
(ii) If (s_n) *is an unbounded decreasing sequence, then* $\lim s_n = -\infty$

Corollary 31. *If* (s_n) *is a monotone sequence, then the sequence either converges, diverges to* $+\infty$*, or diverges to* $-\infty$*. Thus* $\lim s_n$ *is always meaningful for monotone sequences.*

Definition 32. Let (s_n) be a sequence in $\mathbb{R}$. We define

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$$

Theorem 33. *Let (s_n) be a sequence in $\mathbb{R}$.*
(i) If $\lim s_n$ is defined [as a real number, $+\infty$ or $-\infty$], then $\liminf s_n = \lim s_n = \limsup s_n$.
(ii) If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$.

Definition 34. A sequence (s_n) of real numbers i called a *Cauchy sequence* if for each $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$.

Lemma 35. *Convergent sequences are Cauchy sequences.*

Lemma 36. *Cauchy sequences are bounded.*

Theorem 37. *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

CHAPTER 12: $\limsup$ AND $\liminf$

Definition 38. Let (s_n) be any sequence of real numbers, and let S be the set of subsequential limits of (s_n). Recall

$$\limsup s_n = \lim N \to \infty \sup\{s_n : n > N\} = \sup S$$

and

$$\liminf s_n = \lim N \to \infty \inf\{s_n : n > N\} = \inf S$$

Theorem 39. *If (s_n) converges to a positive real number s and (t_n) is any sequence, then*

$$\limsup s_n t_n = s \cdot \limsup t_n$$

Here we allow the conventions $s \cdot (+\infty) = +\infty$ and $s \cdot (-\infty) = -\infty$ for $s > 0$.

Proof sketch: handle the three cases separately using subsequences, doing the inequality left to right, then right to left using $\lim \frac{1}{s_n} = \frac{1}{s}$

Theorem 40. *Let (s_n) be any sequence of nonzero real numbers. Then we have* $\liminf \left|\frac{s_{n+1}}{s_n}\right| \leqslant \liminf |s_n|^{1/n} \leqslant \limsup |s_n|^{1/n} \leqslant \limsup \left|\frac{s_{n+1}}{s_n}\right|$

Middle is obvious, first and third have similar proofs

Corollary 41. *If $\lim \left|\frac{s_{n+1}}{s_n}\right|$ exists [and equals L], then $\lim |s_n|^{1/n}$ exists [and equals L].*

CHAPTER 14: SERIES

Definition 42. We say a series $\sum a_n$ satisfies the **Cauchy criterion** if its sequence (s_n) of partial sums is a Cauchy sequence.

Theorem 43. *A series converges if and only if it satisfies the Cauchy criterion.*

Corollary 44. *If a series $\sum a_n$ converges, then $\lim a_n = 0$*

Definition 45 (Comparison Test)**.** Let $\sum a_n$ be a series where $a_n geq0$ for all n.
(i) If $\sum a_n$ converges and $|b_n| \leqslant a_n$ for all n, then $\sum b_n$ converges.
(ii) If $\sum a_n = +\infty$ and $b_n \geqslant a_n$ for all n, then $\sum b_n = +\infty$.

Corollary 46. *Absolutely convergent series are absolutely convergent*

Definition 47 (Ratio Test)**.** A series $\sum a_n$ of nonzero terms
(i) converges absolutely if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$
(ii) diverges if $\liminf \left|\frac{a_{n+1}}{a_n}\right|$
(iii) Otherwise $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leqslant 1 \leqslant \limsup \left|\frac{a_{n+1}}{a_n}\right|$ and the test gives no information.

Definition 48 (Root Test)**.** Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$
(i) converges absolutely if $\alpha < 1$,
(ii) diverges if $\alpha > 1$.
(iii) Otherwise $\alpha = 1$ and the test gives no information.

CHAPTER 15: ALTERNATING SERIES AND INTEGRAL TEST

Theorem 49. $\sum \frac{1}{n^p}$ *converges if and only if* $p > 1$

Theorem 50 (Alternating Series)**.** *If* $a_1 \geqslant a_2 \geqslant \dots a_n \leqslant \dots \leqslant 0$ *and* $\lim a_n = 0$*, then the alternating series* $\sum(-1)^n + 1a_n$ *converges. Moreoever, the partial sums* s_n *from* $k = 1$ *to* n *satisfy* $|s - s_n| \leqslant a_n$ *for all* n*.*

CHAPTER 17: CONTINUITY

Definition 51. Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is *continuous at* x_0 if, for every sequence (x_n) in dom(f) converging to x_0, we have $\lim_n f(x_n) = f(x_0)$. If f is continuous at each point of a set $S \subseteq$ dom(f), then f is said to be *continuous on* S. THen function f is said to be *continuous* if it is continuous on dom(f).

Theorem 52. *Let* f *be a real-valued function whose domain is a subset of* $\mathbb{R}$*. Then* f *is continuous at* $x_0 \in$ *dom* (f) *if and only if:*

for each $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $x \in$ *dom* (f) *and* $|x - x_0| < \delta$ *imply* $|f(x) - f(x_0)| < \epsilon$*.*

Theorem 53. *Let* f *be a real-valued function with dom* $(f) \subseteq \mathbb{R}$*. If* f *is continuous at* x_0 *in dom* (f)*, then* $|f|$ *and* $kf, k \in \mathbb{R}$*, are continuous at* x_0*.*

Theorem 54. *Let* f *and* g *be real-valued functions that are continuous at* x_0 *in* $\mathbb{R}$*.*
Then
(i) $f + g$ *is continuous at* x_0
(ii) fg *is continouos at* x_0
(iii) f/g *is continuous at* x_0 *if* $g(x_0) \neq 0$

Theorem 55. *If* f *is continuous at* x_0 *and* g *is continuous at* $f(x_0)$*, then the composite function* $g \circ f$ *is continuous at* x_0*.*

CHAPTER 20: LIMITS OF FUNCTIONS

Definition 56. Let S be a subset of $\mathbb{R}$, let a be a real number or symbol ∞ or $-\infty$ that is the limit of some sequence in S, and let L be a real number or symbol $+\infty$ or $-\infty$. We write $\lim_{x\to a^s} f(x) = L$ if

$$f \text{ is a function defined on } S,$$

and

$$\text{for every sequence } (x_n) \text{ in } S \text{ with limit } a, \text{ we have } \lim_{n\to\infty} f(x_n) = L.$$

Remark 57.
(a) From the first definition of the continuity chapter, we see that a function f is continuous at a in dom $(f) = S$ if and only if $\lim_{x\to a^S} = f(a)$.

(b) Observe that limits, when they exist, are unique. This follows from the second statement in the previous definition, since limits of sequences are unique.

Definition 58. .

1. For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some set $S = J \setminus \{a\}$ where J is an open interval containing a. $\lim_{x\to a} f(x)$ is called the [*two-sided*] *limit of* f *at* a. Note f need not be defined at a and, even if f is defined at a, the value $f(a)$ need not equal $\lim_{x\to a} f(x)$. In fact, $f(a) = \lim_{x\to a} f(x)$ iff f is defined on an open interval containing a and f is continuous at a.
2. For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a^+} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some open interval $S = (a, b)$. $\lim_{x\to a^+} f(x)$ is the *right-hand limit of* f *at* a. Again f need not be defined at a.
3. For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a^-} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some open interval $S = (c, a)$. $\lim_{x\to a^-} f(x)$ is the *right-hand limit of* f *at* a.
4. For a function f we write $\lim_{x\to\infty} f(x) = L$ provided $\lim_{x\to\infty^S} f(x) = L$ for some interval $S = (c, \infty)$. Likewise, we write $\lim_{x\to-\infty} f(x) = L$ provided $\lim_{x\to-\infty^S} f(x) = L$ for some interval $S = (-\infty, b)$

Theorem 59. *Let* f_1 *and* f_2 *be functions for which the limits* $L_1 = \lim_{x\to a^S} f_1(x)$ *and* $L_2 = \lim_{x\to a^S} f_2(x)$ *exist and are finite. Then*

1. $\lim_{x\to a^S}(f_1 + f_2)(x)$ *exists and equals* $L_1 + L_2$
2. $\lim_{x\to a^S}(f_1 f_2)(x)$ *exists and equals* $L_1 L_2$
3. $\lim_{x\to a^S}(f_1/f_2)(x)$ *exists and equals* L_1/L_2 *provided* $L_2 \neq 0$ *and* $f_2(x) \neq 0$ *for* $x \in S$

Theorem 60. *Let* f *be a function for which the limit* $L = \lim_{x\to a^S} f(x)$ *exists and is finite. If* g *is a function defined on* $\{f(x) : x \in S\} \cup \{L\}$ *that is continuous at* L*, then* $\lim_{x\to a^S} g \circ f(x)$ *exists and equals* $g(L)$.

Corollary 61. *Let* f *be a function defined on* $J \setminus \{a\}$ *for some open interval* J *containing* a*, and let* L *be a real number. Then* $\lim_{x\to a} f(x) = L$ *if and only if*

$$\text{for each } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } 0 < |x - a| < \delta \text{ implies } |f(x) - L| < \epsilon$$

Corollary 62. *Let* f *be a function defined on some interval* (a, b)*, and let* L *be a real number. Then* $\lim_{x\to a^+} f(x) = L$ *if and only if*

$$\text{for each } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } a < x < a + \delta \text{ implies } |f(x) - L| < \epsilon$$

Theorem 63. *Let* f *be a function defined on* $J \setminus a$ *for some open interval* J *containing* a*. Then* $\lim_{x\to a} f(x)$ *exists if and only if the limits* $\lim_{x\to a^+} f(x)$ *and* $\lim_{x\to a^-} f(x)$ *both exist and are equal, in which case all three limits are equal.*