## Chapter 4: Completeness Axiom

**Definition 1** (Max and Min). Let $S$ be a nonempty subset of $\mathbb{R}$.

1. If $S$ contains a largest element $s_0$ (that is, $s_0$ belongs to $S$ and $s \leq s_0$ for all $s \in S$, then we call $s_0$ the *maximum* of $S$ and write $s_0 = \max S$.

2. If $S$ contains a smallest element, then we call the smallest element the *minimum* of $S$ and write it as $\min S$

**Definition 2** (Bounds). Let $S$ be a nonempty subset of $\mathbb{R}$.

1. If a real number $M$ satisfies $s \leq M$ for all $s \in S$, then $M$ is called an *upper bound of $S$* and the set $S$ is said to be *bounded above*

2. If a real number $m$ satisfies $m \leq s$ for all $s \in S$, then $m$ is called an *lower bound of $S$* and the set $S$ is said to be *bounded below*

3. The set $S$ is said to be *bounded* if it is bounded above and below. Thus $S$ is bounded if there exist real numbers $m$ and $M$ such that $S \subseteq [m, M]$.

**Definition 3** (sup and inf). Let $S$ be a nonempty subset of $\mathbb{R}$.

1. If $S$ is bounded above and $S$ has a least upper bound, then we will call it the *supremum of $S$* and denote it by $\sup S$.

2. if $S$ is bounded below and $S$ has a greatest lower bound, then we will call it the *infimum of $S$* and denote it by $\inf S$.

**Theorem 4** (Completeness Axiom). *Every nonempty subset $S$ of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.*

**Corollary 5.** *Every nonempty subset $S$ of $\mathbb{R}$ that is bounded below has a greatest lower bound $\inf S$.*

**Theorem 6** (Archimedean Property). *If $a > 0$ and $b > 0$, then for some positive integer $n$, we have $na > b$*

**Theorem 7** (Density of $\mathbb{Q}$). *If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.*

## Chapter 7: Limits of Sequences

**Definition 8** (Convergence). A sequence $(s_n)$ of real numbers is said to converge to the real number $s$ provided that for each $\epsilon > 0$ there exists a number $N$ such that $n > N$ implies $|s_n - s| \leq \epsilon$. A sequence that does not converge to some real number is said to diverge.

## Chapter 8: Proofs

All that you need to know here is the idea of computationally deriving your epsilon to use in your proof.

## Chapter 9: Limit Theorems

**Theorem 9.** *Convergent sequences are bounded*

**Theorem 10.** *If the sequence $(s_n)$ converges to $s$ and $k$ is in $\mathbb{R}$, then the sequence $(ks_n)$ converges to $ks$. That is, $\lim(ks_n) = k \cdot \lim(s_n)$*

**Theorem 11.** *If $(s_n)$ converges to $s$ and $(t_n)$ converges to $t$, then $\lim(s_n + t_n)$ converges to $s + t$. That is, $\lim(s_n + t_n) = \lim s_n + \lim t_n$*

**Lemma 12.** *If $(s_n)$ converges to $s$, if $s_n \neq 0$ for all $n$, and if $s \neq 0$, then $(1/s_n)$ converges to $1/s$.*

**Theorem 13.** *Suppose $(s_n)$ converges to $s$ and $(t_n)$ converges to $t$. If $s \neq 0$ and $s_n \neq 0$ for all $n$, then $(t_n/s_n$ converges to $t/s/$*

**Theorem 14.** *Basic Examples*
*(a) $\lim_{n \to \infty}(\frac{1}{n^p}) = 0$ for $p > 0$*

*(b)* $\lim_{n \to \infty} a^n = 0$ *if $|a| < 1$.*
*(c)* $\lim(n^{1/n}) = 1$.
*(d)* $\lim_{n \to \infty}(a^{1/n}) = 1$ *for $a > 0$*

**Definition 15.** For a sequence $(s_n)$, we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N$ implies $s_n > M$.

**Theorem 16.** *Let $(s_n)$ and $(t_n)$ be sequences such that $\lim s_n = +\infty$ and $\lim t_n > 0$ [$\lim t_n$ can be finite or $+\infty$]. Then $\lim s_n t_n = +\infty$.*

**Theorem 17.** *For a sequence $(s_n)$ of positive real numbers, we have $\lim s_n = +\infty$ if and only if $\lim(\dfrac{1}{s_n} = 0$*

## Chapter 10: Monotonic & Cauchy Sequences

**Definition 18.** A sequence $(s_n)$ of real numbers is called an **increasing sequence** if $s_n \leq s_{n+1}$ for all $n$, and is called a **decreasing sequence** if $s_n \geq s_{n+1}$ for all $n$

**Theorem 19.** *All bounded monotone sequences converge*

**Theorem 20.** *Unbounded increasing sequences have a limit of $s_n = +\infty$, and unbounded decreasing has $\lim s_n = -\infty$*

**Corollary 21.** *If $(s_n)$ is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or diverges to $-\infty$. Thus $\lim s_n$ is always meaningful for monotone sequences.*

**Definition 22.** Let $(s_n)$ be any sequence of real numbers. We define

$$\lim\sup s_n = \lim N \to \infty \sup\{s_n : n > N\} = \sup S$$

and

$$\lim\inf s_n = \lim N \to \infty \inf\{s_n : n > N\} = \inf S$$

**Theorem 23.** *Let $(s_n)$ be a sequence in $\mathbb{R}$.*
*(i) If $\lim s_n$ is defined [as a real number, $+\infty$, or $-\infty$], then $\lim\inf s_n = \lim s_n = \lim\sup s_n$*
*(ii) If $\lim\inf s_n = \lim\sup s_n$, then $\lim s_n$ is defined and $\lim s_n = \lim\inf s_n = \lim\sup s_n$*

**Definition 24.** A sequence $(s_n)$ or real numbers is called a Cauchy sequence if for each $\epsilon > 0$ there exists a number $N$ such that $m, n > N$ implies $|s_n - s_m| < \epsilon$

**Lemma 25.** *Convergent sequences are Cauchy sequences.*

**Lemma 26.** *Cauchy sequences are bounded*

**Theorem 27.** *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

## Chapter 11: Subsequences

**Definition 28.** A sequence $(s_n)$ of real numbers is called an *increasing sequence* if $s_n \leq s_{n+1}$ for all $n$, and $(s_n)$ is called *decreasing sequence* if $s_n \geq s_{n+1}$ for all $n$. Note that if $(s_n)$ is increasing, then $s_n \leq s_m$ whenever $n < m$. A sequence that is increasing or decreasing will be called a *monotone sequence* or a *monotonic sequence*.

**Theorem 29.** *All bounded monotone sequence converge.*

**Theorem 30.** *(i) If $(s_n)$ is an unbounded increasing sequence, then $\lim s_n = +\infty$*
*(ii) If $(s_n)$ is an unbounded decreasing sequence, then $\lim s_n = -\infty$*

**Corollary 31.** *If $(s_n)$ is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or diverges to $-\infty$. Thus $\lim s_n$ is always meaningful for monotone sequences.*

**Definition 32.** Let $(s_n)$ be a sequence in $\mathbb{R}$. We define

$$\lim\sup s_n = \lim_{N \to \infty} \sup\{s_n : n > N\}$$

and

$$\lim\inf s_n = \lim_{N \to \infty} \inf\{s_n : n > N\}$$

**Theorem 33.** *Let $(s_n)$ be a sequence in $\mathbb{R}$.*
*(i) If $\lim s_n$ is defined [as a real number, $+\infty$ or $-\infty$], then $\liminf s_n = \lim s_n = \limsup s_n$.*
*(ii) If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$.*

**Definition 34.** A sequence $(s_n)$ of real numbers i called a *Cauchy sequence* if
for each $\epsilon > 0$ there exists a number $N$ such that $m, n > N$ implies $|s_n - s_m| < \epsilon$.

**Lemma 35.** *Convergent sequences are Cauchy sequences.*

**Lemma 36.** *Cauchy sequences are bounded.*

**Theorem 37.** *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

## Chapter 12: $\limsup$ and $\liminf$

**Definition 38.** Let $(s_n)$ be any sequence of real numbers, and let $S$ be the set of subsequential limits of $(s_n)$.
Recall
$$\limsup s_n = \lim N \to \infty \sup\{s_n : n > N\} = \sup S$$
and
$$\liminf s_n = \lim N \to \infty \inf\{s_n : n > N\} = \inf S$$

**Theorem 39.** *If $(s_n)$ converges to a positive real number $s$ and $(t_n)$ is any sequence, then*

$$\limsup s_n t_n = s \cdot \limsup t_n$$

*Here we allow the conventions $s \cdot (+\infty) = +\infty$ and $s \cdot (-\infty) = -\infty$ for $s > 0$.*

Proof sketch: handle the three cases separately using subsequences, doing the inequality left to right, then right to left using $\lim \dfrac{1}{s_n} = \dfrac{1}{s}$

**Theorem 40.** *Let $(s_n)$ be any sequence of nonzero real numbers. Then we have $\liminf \left| \dfrac{s_{n+1}}{s_n} \right| \leq \liminf |s_n|^{1/n} \leq$*
*$\limsup |s_n|^{1/n} \leq \limsup \left| \dfrac{s_{n+1}}{s_n} \right|$*

Middle is obvious, first and third have similar proofs

**Corollary 41.** *If $\lim \left| \dfrac{s_{n+1}}{s_n} \right|$ exists [and equals L], then $\lim |s_n|^{1/n}$ exists [and equals L].*

## Chapter 14: Series

**Definition 42.** We say a series $\sum a_n$ satisfies the **Cauchy criterion** if its sequence $(s_n)$ of partial sums is a Cauchy sequence.

**Theorem 43.** *A series converges if and only if it satisfies the Cauchy criterion.*

**Corollary 44.** *If a series $\sum a_n$ converges, then $\lim a_n = 0$*

**Definition 45** (Comparison Test)**.** Let $\sum a_n$ be a series where $a_n geq 0$ for all $n$.
(i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all $n$, then $\sum b_n$ converges.
(ii) If $\sum a_n = +\infty$ and $b_n \geq a_n$ for all $n$, then $\sum b_n = +\infty$.

**Corollary 46.** *Absolutely convergent series are absolutely convergent*

**Definition 47** (Ratio Test)**.** A series $\sum a_n$ of nonzero terms
(i) converges absolutely if $\limsup \left| \dfrac{a_{n+1}}{a_n} \right| < 1$
(ii) diverges if $\liminf \left| \dfrac{a_{n+1}}{a_n} \right|$
(iii) Otherwise $\liminf \left| \dfrac{a_{n+1}}{a_n} \right| \leq 1 \leq \limsup \left| \dfrac{a_{n+1}}{a_n} \right|$ and the test gives no information.

**Definition 48** (Root Test). Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$
(i) converges absolutely if $\alpha < 1$,
(ii) diverges if $\alpha > 1$.
(iii) Otherwise $\alpha = 1$ and the test gives no information.

## CHAPTER 15: ALTERNATING SERIES AND INTEGRAL TEST

**Theorem 49.** $\sum \dfrac{1}{n^p}$ *converges if and only if $p > 1$*

**Theorem 50** (Alternating Series)**.** *If $a_1 \geq a_2 \geq \ldots a_n \leq \ldots \leq 0$ and $\lim a_n = 0$, then the alternating series $\sum (-1)^n + 1 a_n$ converges. Moreoever, the partial sums $s_n$ from $k = 1$ to $n$ satisfy $|s - s_n| \leq a_n$ for all $n$.*

## CHAPTER 17: CONTINUITY

**Definition 51.** Let $f$ be a real-valued function whose domain is a subset of $\mathbb{R}$. The function $f$ is *continuous at* $x_0$ if, for every sequence $(x_n)$ in $\text{dom}(f)$ converging to $x_0$, we have $\lim_n f(x_n) = f(x_0)$. If $f$ is continuous at each point of a set $S \subseteq \text{dom}(f)$, then $f$ is said to be *continuous on $S$*. THen function $f$ is said to be *continuous* if it is continuous on $\text{dom}(f)$.

**Theorem 52.** *Let $f$ be a real-valued function whose domain is a subset of $\mathbb{R}$. Then $f$ is continuous at $x_0 \in dom\,(f)$ if and only if:*

*for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in dom\,(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$.*

**Theorem 53.** *Let $f$ be a real-valued function with $dom\,(f) \subseteq \mathbb{R}$. If $f$ is continuous at $x_0$ in $dom\,(f)$, then $|f|$ and $kf, k \in \mathbb{R}$, are continuous at $x_0$.*

**Theorem 54.** *Let $f$ and $g$ be real-valued functions that are continuous at $x_0$ in $\mathbb{R}$.*
*Then*
*(i) $f + g$ is continuous at $x_0$*
*(ii) $fg$ is continouos at $x_0$*
*(iii) $f/g$ is continuous at $x_0$ if $g(x_0) \neq 0$*

**Theorem 55.** *If $f$ is continuous at $x_0$ and $g$ is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at $x_0$.*

## CHAPTER 20: LIMITS OF FUNCTIONS

**Definition 56.** Let $S$ be a subset of $\mathbb{R}$, let $a$ be a real number or symbol $\infty$ or $-\infty$ that is the limit of some sequence in $S$, and let $L$ be a real number or symbol $+\infty$ or $-\infty$. We write $\lim_{x \to a^s} f(x) = L$ if

$$f \text{ is a function defined on } S,$$

and

$$\text{for every sequence } (x_n) \text{ in } S \text{ with limit } a, \text{ we have } \lim_{n \to \infty} f(x_n) = L.$$

*Remark* 57.
(a) From the first definition of the continuity chapter, we see that a function $f$ is continuous at $a$ in $\text{dom}\,(f) = S$ if and only if $\lim_{x \to a^s} = f(a)$.
(b) Observe that limits, when they exist, are unique. This follows from the second statement in the previous definition, since limits of sequences are unique.

**Definition 58.** .

1. For $a \in \mathbb{R}$ and a function $f$ we write $\lim_{x \to a} f(x) = L$ provided $\lim_{x \to a^s} f(x) = L$ for some set $S = J \setminus \{a\}$ where $J$ is an open interval containing $a$. $\lim_{x \to a} f(x)$ is called the [*two-sided*] *limit of $f$ at $a$*. Note $f$ need not be defined at $a$ and, even if $f$ is defined at $a$, the value $f(a)$ need not equal $\lim_{x \to a} f(x)$. In fact, $f(a) = \lim_{x \to a} f(x)$ iff $f$ is defined on an open interval containing $a$ and $f$ is continuous at $a$.

2. For $a \in \mathbb{R}$ and a function $f$ we write $\lim_{x \to a^+} f(x) = L$ provided $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (a, b)$. $\lim_{x \to a^+} f(x)$ is the *right-hand limit of $f$ at $a$*. Again $f$ need not be defined at $a$.

3. For $a \in \mathbb{R}$ and a function $f$ we write $\lim_{x \to a^-} f(x) = L$ provided $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (c, a)$. $\lim_{x \to a^-} f(x)$ is the *right-hand limit of $f$ at $a$*.

4. For a function $f$ we write $\lim_{x \to \infty} f(x) = L$ provided $\lim_{x \to \infty^S} f(x) = L$ for some interval $S = (c, \infty)$. Likewise, we write $\lim_{x \to -\infty} f(x) = L$ provided $\lim_{x \to -\infty^S} f(x) = L$ for some interval $S = (-\infty, b)$

**Theorem 59.** *Let $f_1$ and $f_2$ be functions for which the limits $L_1 = \lim_{x \to a^S} f_1(x)$ and $L_2 = \lim_{x \to a^S} f_2(x)$ exist and are finite. Then*

1. $\lim_{x \to a^S}(f_1 + f_2)(x)$ *exists and equals $L_1 + L_2$*

2. $\lim_{x \to a^S}(f_1 f_2)(x)$ *exists and equals $L_1 L_2$*

3. $\lim_{x \to a^S}(f_1/f_2)(x)$ *exists and equals $L_1/L_2$ provided $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$*

**Theorem 60.** *Let $f$ be a function for which the limit $L = \lim_{x \to a^S} f(x)$ exists and is finite. If $g$ is a function defined on $\{f(x) : x \in S\} \cup \{L\}$ that is continuous at $L$, then $\lim_{x \to a^S} g \circ f(x)$ exists and equals $g(L)$.*

**Corollary 61.** *Let $f$ be a function defined on $J \setminus \{a\}$ for some open interval $J$ containing $a$, and let $L$ be a real number. Then $\lim_{x \to a} f(x) = L$ if and only if*

> *for each $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$*

**Corollary 62.** *Let $f$ be a function defined on some interval $(a, b)$, and let $L$ be a real number. Then $\lim_{x \to a^+} f(x) = L$ if and only if*

> *for each $\epsilon > 0$ there exists $\delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - L| < \epsilon$*

**Theorem 63.** *Let $f$ be a function defined on $J \setminus a$ for some open interval $J$ containing $a$. Then $\lim_{x \to a} f(x)$ exists if and only if the limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist and are equal, in which case all three limits are equal.*