Long Homework 3 (due in 2 weeks)

① Let $f(z)$ be holomorphic on the unit disc $\mathbb{D}$ and continuous along the arc $I = \{e^{i\theta} : \alpha < \theta < \beta\}$. Furthermore, suppose $|f(z)| = 1 \quad \forall \quad z \in I$. Show that $f(z)$ can be analytically continued to an open set that contains $\mathbb{D} \cup I$.

② Let $f(z)$ be holomorphic on the unit disc $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose $|f(z)| = 1$ for every z such that $|z| = 1$. Prove that $f(z)$ is constant on $\overline{\mathbb{D}}$

③ Let $f(z)$ be a continuous complex-valued function on an open set $\Omega \subset \mathbb{C}$ with continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Suppose $\int_C f(z)\,dz = 0$ for every circle $C < \Omega$. Use Green's Theorem to show that $f(z)$ satisfies Cauchy-Riemann equations and, therefore, is holomorphic.

4. Let $f(z)$ be holomorphic in the disc $|z-z_0|<R$. Show that, $\forall\, 0<r<R$,
$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta$$

5. Let $f(z)$ be a holomorphic function without zeros in a simply-connected open set Ω. Show that there exists a holomorphic function $g(z)$ on Ω such that $f(z)=e^{g(z)}$.

The remaining exercises are from Chapter 3 of Stein-Shakarchi (Section 8)

6. Problem 1
7. Problem 2
8. Problem 4
9. Problem 8

10 Problem 9

11 Problem 12

12 Problem 13

13 Problem 15(b)

14 Problem 15(c)

15 Problem 16