

MATH 621

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1 SINGULARITIES

Theorem 1 (Riemann's Theorem on removable singularities). $f(z)$ has an isolated singularity at z_0 , bounded in punctured neighborhood of z_0 , then f can be extended to a holomorphic function at z_0 .

Lemma 2. If $F(s, z)$ is continuous in $[0, 1] \times \Omega$, such that $\Omega \subset \mathbb{C}$ open, and holomorphic in z , then $\int_0^1 F(s, z) ds$ is holomorphic in Ω

One argument: write integral as a limit of Riemann sums (obviously holomorphic in z) \implies the limit is holomorphic if we can show uniform convergence on compact subsets of Ω

Another argument:

(1) $\int_0^1 F(s, z) dz$ continuous in Ω . Choose \bar{D} such that $z \in \bar{D} \subset \Omega$. $F(s, z)$ is uniformly continuous on $[0, 1] \times \Omega$. $\forall \epsilon \exists \delta$ such that if $|z' - z| < \delta$ then $|F(s, z') - F(s, z)| < \epsilon \forall s \in [0, 1] \implies \left| \int_0^1 F(s, z') - F(s, z) dz \right| \leq \int_0^1 |F(s, z') - F(s, z)| ds < \int_0^1 d\epsilon = \epsilon - 0 = \epsilon \implies \int_0^1 F(s, z) ds$ continuous in z .

(2) By Morera's Theorem, it suffices to check $\int_T \left(\int_0^1 F(s, z) ds \right) dz = 0, T \subset \Omega$ with its interior. By Fubini's theorem $= \int_0^1 \left(\int_T F(s, z) dz \right) ds = 0$. Interior integral is zero by Cauchy or Goursat, so entire integral is 0, and f is holomorphic by Morera's.

Corollary 3. z_0 is a pole of $f(z) \iff \lim_{z \rightarrow z_0} |f(z)| = \infty$

z_0 is a pole of $f(z) \implies z_0$ is a zero of $\frac{1}{f(z)} \implies \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0 \implies \lim_{z \rightarrow z_0} |f(z)| = \infty$.

If $\lim_{z \rightarrow z_0} |f(z)| = \infty \implies \lim_{z \rightarrow z_0} \left| \frac{1}{f(z)} \right| = 0 \implies \frac{1}{f(z)}$ is bounded near $z_0 \implies \frac{1}{f(z)}$ can be extended to a holomorphic function at z_0 , all if $g(z), g(z_0) = \lim_{z \rightarrow z_0} g(z) = 0 \implies f(z) = \frac{1}{g(z)}$ has a pole at z_0 .

Definition 4. Refined classification of isolated singularities at z_0 .

(1) removable $\iff f(z)$ is bounded near $z_0 \iff f(z)$ is holomorphic at z_0 (last by Riemann)

(2) pole $\iff f(z) = \frac{1}{g(z)}, g(z)$ is holomorphic, $g(z_0) = 0 \iff \lim_{z \rightarrow z_0} |f(z)| = \infty$

(3) Essential singularities? (holo on deleted neighborhood but not removable sing. or pole)

Example 5. $e^{1/z}$ at $z = 0$

Theorem 6 (Casorati-Weierstrass). z_0 is an essential singularity of $f(z) \implies f(0 < |z - z_0| < r)$ dense in $\mathbb{C} \forall r$.

Argue by contradiction: suppose $\exists w_0$ and R such that $|f(z) - w_0| > R \forall z$ such that $0 < |z - z_0| < r \frac{1}{|f(z) - w_0|} < R$ is bounded in same annulus. So by Riemann's Theorem, $= g(z)$, which is holomorphic for $|z - z_0| < r$ (by Riemann's

Theorem). So $f(z) = w_0 + \frac{1}{g(z)}$. If $g(z_0) \neq 0$, then f is holomorphic at z_0 . If $g(z_0) = 0 \implies w_0 + \frac{1}{g(z)}$ has a pole at z_0 , which is a contradiction

Theorem 7 (Picard's Theorem). *Every $\alpha \in \mathbb{C}$, with at most one exception, belongs to the image $f(0 < |z - z_0| < r) \forall r$, and occurs infinitely many times*

Covers all for given r , shrink r , covers all by Picard's, shrink, etc.

"Singularity at ∞ ": in book

"Riemann Sphere" $S^2 = \mathbb{CP}^1 = \mathbb{C}$ disjoint union $\{\infty\}$.

$f(z)$ has an isolated singularity at $\infty \iff f(\frac{1}{z})$ is holomorphic for $0 < \left|\frac{1}{z}\right| < \frac{1}{R} \iff f(z)$ is holomorphic for $|z| > R$ for some R .

$f(z)$ has a removable singularity at $\infty \iff f(z)$ is bounded for $|z| > R \iff f(\frac{1}{z})$ is holomorphic at $z = 0 \iff f(\frac{1}{z}) = \sum_{n \geq 0} a_n \left(\frac{1}{z}\right)^n$ converges for $\frac{1}{z} < r$

$f(z)$ has a pole at $\infty \iff f(\omega)$ has a pole at $0, \omega = 1/z \iff \lim_{\omega \rightarrow 0} |f(\omega)| = \infty \iff \lim_{z \rightarrow \infty} |f(z)| = \infty$.

$f(\omega) = \frac{a_{-n}}{\omega^n} + \dots + \frac{a_{-1}}{\omega} + H(\omega)$ holo at $\omega \iff$ bounded near 0 . $f(z) = a_{-n}z^n + \dots + a_{-1}z + H(\frac{1}{z})$ bounded at ∞ (for $|z| > R$ for some R)

Theorem 3.4.

argument principle

Theorem 8 (Roche Theorem). *Suppose $f(z)$ and $g(z)$ are holomorphic in Ω , which contains a simple closed curve γ and its interior. Suppose $|f(z)| > |g(z)| \forall z \in \gamma$. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros (counted with multiplication) inside γ .*

Let $f_s(z) = f(z) + sg(z), s \in [0, 1]$. $f_s(z)$ is holomorphic in $\Omega \forall s \in [0, 1]$. $\forall z \in \gamma$ $|f_s(z)| = |f(z) + sg(z)| \geq |f(z) - s|g(z)| \geq 0 \implies f_s(z)$ doesn't vanish along $\gamma \implies$ the number of zeros of $f_s(z)$ inside γ (with multiplication) is equal to $= \frac{1}{2\pi} \int_{\gamma} \frac{f'(z) + sg'(z)}{f(z) + sg(z)} dz$. The integral is a continuous function of (z, s) on $\gamma \times [0, 1]$ (compact) $\implies \frac{1}{2\pi} \int_{\gamma} \frac{f'(z) + sg'(z)}{f(z) + sg(z)} dz$ is continuous in S . But it takes integer values $\implies \frac{1}{2\pi} \int_{\gamma} \frac{f'(z) + sg'(z)}{f(z) + sg(z)} dz$ is constant in S . In particular, the same for $f(z)$ (where $s = 0$) and $f(z) + g(z)$ (where $s = 1$)

Example 9. Find the number of zeros inside $|z| < 1$ of $z^{100} + 4z^3 - z + 1 = f(z) + g(z)$. $f(z) = 4z^3$, with 3 zeros with multiplication, and $g(x) = z^{100} - z + 1$. $|4z^3| = 4$. $|g(x)| \leq 3 < 4$.

Theorem 10 (Open Mapping Theorem). *If $f(z)$ is holomorphic in connected open Ω and nonconstant $\implies f : \Omega \rightarrow f(\Omega)$ is an open map (sends open sets to open sets).*

Remark 11. We want connected to avoid something like HW1P4

It suffices to show that $f(\Omega)$ is open. Say $f(z_0) = \omega_0$. Show the image of f contains some neighborhood of ω_0 , or $\exists r > 0$ such that $\{|\omega - \omega_0| < r\} \subset f(\Omega)$. Equivalent to say that $f(z) - \omega$ has a root in $\Omega \forall \omega$ such that $|\omega - \omega_0| < r$. $f(z) - \omega$ has a solution z_0 . Chose a circle $|z - z_0| = \delta$. $f(z) - \omega_0$ has a zero inside the circle. Apply Rouché theorem.

We know $\exists \delta$ such that $f(z) - \omega_0 \neq 0$ for some $z, |z - z_0| = \delta$ because roots form discrete set. Take $r = \min |f(z) - \omega_0|$, chain inequalities, apply Rouché Theorem

2 RIEMANN ZETA FUNCTION

Say $s > 1$, and the sum $\sum \frac{1}{n^s} = \zeta(s)$ converges for $\forall s > 1$. In Calculus, you prove this converges with the integral test.

Theorem 12. $\zeta(s)$ admits an analytic continuation to $\mathbb{C} \setminus 1$ where it has a simple pole

Theorem 13. The only zeros of $\zeta(s)$ outside the "critical strip" $0 \leq \operatorname{Re}(s) \leq 1$ are "trivial zeros": $-2, -4, \dots$ (simple zeros of Gamma function)

Theorem 14 (Hadamard, Valle Poussin). No zeros exist on the boundary of the critical strip

Theorem 15 (Riemann Hypothesis). All zeros are in the middle of the strip, along $\operatorname{Re}(s) = 1/2$.

Definition 16 (Tchebyshev's ψ -function). $\psi(x) = \sum_{p^m \leq x} \log p$, p is prime, $x \in \mathbb{R}, x > 0$.

Definition 17 (Riemann's Explicit Formula). $\psi(x) = x - \sum_{\zeta(\rho)=0} \frac{x^\rho}{\rho} - \ln(2\pi) - \frac{1}{2} \ln(1 - x^{-2})$, $0 < \operatorname{Re}(\rho) < 1$

Definition 18 (Wacky Prime Number Theorem). As $x \rightarrow \infty$, the last terms just goes away. $\psi(x) \sim X$, or $\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 1$

If you believe the Riemann hypothesis, $\rho = 1/2$

$$\zeta(s) = \sum \frac{1}{n^s}, \operatorname{Re}(s) > 1$$

Definition 19 (Euler's product expansion). $\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$, $\operatorname{Re}(s) > 1$ for p prime

Corollary 20. $\zeta(s)$ has no zeros in $\operatorname{Re}(s) > 1$

Idea: $\frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots$

$$\Rightarrow \prod_p \frac{1}{1 - \frac{1}{p^s}} = \sum \frac{1}{p^{k_1 s} \dots p^{k_r s}} = \sum_{n \geq 1} \frac{a(n)}{n^s} \text{ where } a(n) \text{ is the number of ways to write } n$$

Function is holomorphic unless $e^{s \ln p} = 1$, but can't happen since $\operatorname{Re}(s) > 1$. Also, cannot vanish. Product of non-vanishing factors can't vanish (?)

Caution: $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \dots \rightarrow 0$, so need to justify

$$\prod_{p < N} (1 + \frac{1}{p^s} + \dots + \frac{1}{p^{Ms}}) < 1 + \frac{1}{2^s} + \dots = \zeta(s)$$

send M to infinity, holds, send N to infinity, holds.

Claim: $\prod_p \frac{1}{1 - \frac{1}{p^s}}$ is holomorphic for $\operatorname{Re}(s) > 1$

Proposition 21. Suppose $F_n(z)$ holom. in Ω , $|F_n(z)| \leq c_n$ in Ω and $\sum c_n < \infty$. Then $\prod_{n \geq 1} (1 + F_n(z))$ converges uniformly in Ω to a holomorphic function in Ω , which is equal to 0 only at zeros of its factors.

As the limit of a sequence of holom. functions converges uniformly on compact sets

In our case: need to show that $\prod (1 - \frac{1}{p^s})$ is holomorphic.

$$\left| \frac{1}{p^s} \right| = \frac{1}{p^{\operatorname{Re}(s)}}.$$

$$\sum \frac{1}{p^{\operatorname{Re}(s)}} < \sum \frac{1}{n^{\operatorname{Re}(s)}} < \infty \text{ since } \operatorname{Re}(s) > 1$$

$$\Rightarrow \prod (1 - \frac{1}{p^s}) \text{ is holomorphic for } \operatorname{Re}(s) < 1$$

doesn't vanish $\implies \prod \frac{1}{1 - \frac{1}{p^s}}$ is holomorphic for $\operatorname{Re}(s) > 1$

For $n \gg 0$, $|F_n(z)| < \frac{1}{2}$. Let's assume that this is true $\forall n$

$\implies \operatorname{Re}(1 + F_n(z)) > 0$

$\implies \log(1 + F_n(z))$ is defined, where \log is standard branch.

Let's show that $\sum \log(1 + F_n(z))$ converges absolutely and uniformly

$|\log(1 + w)|$ for $|w| < \frac{1}{2}$

$\log(1 + w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \dots$

$\implies \log(1 + w) = w[1 - \frac{w}{2} + \frac{w^2}{3} - \dots]$

$\implies |\log(1 + w)| \leq |w|[1 + \frac{|w|}{2} + \frac{|w|^2}{3} + \dots]$

$\implies |\log(1 + w)| \leq |w|[1 + \frac{|w|}{2} + \frac{|w|^2}{3} + \dots]$

$\implies |\log(1 + w)| \leq |w|[1 + |w| + |w|^2 + \dots]$

$\implies |\log(1 + w)| \leq 2|w|$

$\prod \frac{1}{1 - \frac{1}{p}} = \infty$ because $\leq \sum \frac{1}{n} = \infty \implies \prod (1 - \frac{1}{p}) = 0$

Why doesn't this contradict the previous theorem?

$\implies \sum \frac{1}{p} = \infty$ (b.c otherwise the previous theorem applies as the terms are no zeros)

\implies there are infinitely many primes

Theorem 22. $\zeta(s)$ has no zeros for $\operatorname{Re}(s) = 1$ (assuming $\zeta(s)$ has an analytic continuation into $\operatorname{Re}(s) > 0$ with simple pole at $s = 1$ and no other poles)

Argue by contradiction, suppose that $\zeta(1 + it) = 0$

Remark 23. $|\zeta^3(s)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \rightarrow 0$ as $\sigma \rightarrow 1^+$

Remark 24. $|\zeta^3(s)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 1$ for $\sigma > 1$

Past two remarks give the proof

Remark 25. $\zeta(s)$ has a simple pole at $s = 1 \implies \zeta(s) = \frac{1}{s-1} + \text{holomorphic near } s = 1$

$\implies |\zeta^3(\sigma)| = \frac{|a|^3}{|\sigma-1|^3} + \text{bounded near } \sigma = 1$

Last time: if $\zeta(1 + it) = 0 \implies |\zeta^3(s)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \rightarrow 0$ as $\sigma \rightarrow 1^+$ assuming $\zeta(s)$ has an analytic continuation to $\operatorname{Re}(s) > 0$ with a simple pole at 1 and no other poles along $\operatorname{Re}(s) = 1$

$|\zeta^3(s)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 1$ for $\sigma > 1$

$3 \ln |\zeta(s)| + 4 \ln |\zeta^4(\sigma + it)| + \ln |\zeta(\sigma + 2it)| \geq 0$ (regular natural logarithm for real numbers)

$\operatorname{Re}(1) > 1 \implies \zeta(s) = \prod_{\text{primes}} \frac{1}{1 - p^{-s}}$, take absolute value of both sides, take logarithm is $\sum_{\text{primes}} -\ln|1 - p^{-s}|$

$|p^{-s}| < 1 \implies$ we can compute principal branch $\log(1 - p^{-s}) \implies -\ln|1 - p^{-s}| = \operatorname{Re} - \log(1 - p^{-s})$

$|z| < 1 \implies -\log(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \dots \implies -\log(1 - p^{-s}) = p^{-s} + \frac{p^{-2s}}{2} + \dots$

$\operatorname{Re}(\sum \text{primes } \dots)$ [aka $\ln |\zeta(s)|$] = $\sum_{n \geq 1} \frac{c_n}{n^s} c_n \geq 0$

2.1 Schwarz function idk where this is going

$f(x) \in C^\infty(\mathbb{R})$ is a Schwarz function if f and all its derivatives decay faster than any $1/\text{polynomial}$ function.

$|\frac{\partial f}{\partial x^m} x^n| < c_{n,m}$ for some $c_{n,m}$ Basic example: $e^{-\pi x^2}$. At some point we will assume that $f(x)$ is also even.

Definition 26. An associated θ function $\theta_f(y) = \sum_{n \in \mathbb{Z}} f(y \cdot n)$ $y > 0$

C^∞ function

Definition 27. Asn associated gamma function $\Gamma_f(s) = \int_0^\infty t^s f(t) \frac{dt}{t}$.

This function will be analytic if $\text{Re}(s) > 0$

If $f(x) = e^{-\pi x^2}$ then $\theta_f(y) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 y^2}$

Recall Jacobi theta function $\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}$, $z \in \mathbb{C}$.

$$\theta_f(y) = \theta(iy^2)$$

"Gamma factor" $\Gamma_f(s) = \int_0^\infty t^s e^{\pi t^2} \frac{dt}{t}$

$$\begin{aligned} x &= \pi t^2, \quad x^{1/2} \pi^{-1/2}, \quad dx/x = 2\pi t dt / (\pi t^2) = 2dt/t \\ \Rightarrow \int_0^\infty x^{s/2} \pi^{-s/2} e^{-x} &= \frac{1}{2} \pi^{-s/2} \int_0^\infty x^{s/2} e^{-x} \frac{dx}{x} = \frac{1}{2} \pi^{s/2} \Gamma(s/2) \end{aligned}$$

Lemma 28. $\theta_f(y) \in C^\infty(\mathbb{R})$ and derivatives can be computed term by term.

Just need to show differentiability (f' is also Schwarz)

Check uniform and absolute convergence (on compact sets)

$$|f(y_n)| \leq \frac{C}{(y_n)^2} = \frac{C}{y^2 n^2}$$

$$\sum |f(y_n)| \leq \frac{C}{y^2} \sum \frac{1}{n^2}. \text{ Absolute convergence}$$

$$\left| \int_a^b f(t, z) dt \right| \xrightarrow{a, b \rightarrow \infty} 0 \text{ uniformly for } z \in K \subset \Omega$$

Basic case: prove absolute convergence along with uniform convergence. Need bound $|f(t, z)| \leq g(t)$ continuous for $z \in K$ and $\int_1^\infty g(t) dt < \infty$.

$$\text{Given } g(t), \left| \int_a^b f(t, z) dt \right| \leq \int_a^b g(t) dt \xrightarrow{a, b \rightarrow \infty} 0$$