Problem 1: Let $a, b \in \mathbb{C}$ and $|a| < r < |b|$ Let γ be a circle of radius r centered at the origin. Evaluate

$$\int_\gamma \frac{dz}{(z-a)(z-b)}$$

(Use only the definition of the integral but not Cauchy theorem or residues)

Solution
First, note that because $|a| < r$ and $|z| = r$, $|z| > 0$. Using partial fraction decomposition of the integrand,

$$\begin{aligned}
\frac{1}{(z-a)(z-b)} &= \frac{A}{z-a} + \frac{B}{z-b} \\
Az - Ab + Bz - ab &= 1 \\
A + B = 0, A &= -B \\
B(b-a) &= 1 \\
\frac{1}{(z-a)(z-b)} &= \frac{1}{(a-b)(z-a)} - \frac{1}{(a-b)(z-b)}
\end{aligned}$$

Taking the integral

$$\begin{aligned}
\int_\gamma \frac{dz}{(z-a)(z-b)} &= \int_\gamma \frac{1}{(a-b)(z-a)} - \frac{1}{(a-b)(z-b)} dz \\
&= \frac{1}{a-b}\left[\int_\gamma \frac{1}{z-a} dz - \int_\gamma \frac{1}{z-b} dz\right] \\
&= \frac{1}{a-b}\left[\int_\gamma \frac{1}{z}\frac{1}{1-a/z} dz - \int_\gamma -\frac{1}{b}\frac{1}{1-z/b} dz\right] \\
\sum_{n\geq 0} ar^n &= \frac{a}{1-r}, |r| < 1 \\
|a| < r \implies |a| < |z| &\implies \frac{|a|}{|z|} < 1 \\
r < |b| \implies |z| < |b| &\implies \frac{|z|}{|b|} < 1 \\
\int_\gamma \frac{dz}{(z-a)(z-b)} &= \frac{1}{a-b}\left[\int_\gamma \frac{1}{z}\sum_{n\geq 0}(\frac{a}{z})^n dz + \int_\gamma \frac{1}{b}\sum_{n\geq 0}(\frac{z}{b})^n dz\right] \\
&= \frac{1}{a-b}\left[\int_\gamma \sum_{n\geq 0}\frac{1}{z}(\frac{a}{z})^n dz + \int_\gamma \sum_{n\geq 0}\frac{1}{b}(\frac{z}{b})^n dz\right]
\end{aligned}$$

From the third to last and second to last lines, the series both converge absolutely, so the order of summation and integration can be interchanged.

$$\begin{aligned}
\int_\gamma \frac{dz}{(z-a)(z-b)} &= \frac{1}{a-b}\left[\sum_{n\geq 0}\int_\gamma \frac{1}{z}(\frac{a}{z})^n dz + \sum_{n\geq 0}\int_\gamma \frac{1}{b}(\frac{z}{b})^n dz\right] \\
&= \frac{1}{a-b}\left[\sum_{n\geq 0} a^n \int_\gamma \frac{1}{z^{n+1}} dz + \sum_{n\geq 0}\frac{1}{b^{n+1}}\int_\gamma z^n dz\right]
\end{aligned}$$

$f(z) = z^n$ has primitive $F(z) = \dfrac{z^{n+1}}{n+1}$, where $n \neq -1$. By Corollary 3.3 in Stein Shakarchi:

Corollary 3.3 *If γ is a closed curve in an open set Ω, and f is continuous and has a primitive in Ω, then*

$$\int_\gamma f(z)dz = 0$$

The second integral will always be continuous and have a primitive, and therefore evaluates to 0 for all $n \geq 0$, so the second sum evaluates to 0. Additionally, because $|z| > 0$, $f(z)$ is continuous in its defined set Ω, and the first integral is also continuous with a primitive where $n > 0$ (n in the context of the first summation, which results in the exponent of z being -1). Therefore, all terms of the first summation for $n > 0$ are 0, leaving just

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b}\int_\gamma \frac{1}{z}dz$$

Using the parameterization $z = re^{i\theta}$, we can evaluate this exactly as

$$\begin{aligned} \int_\gamma \frac{dz}{(z-a)(z-b)} &= \frac{1}{a-b}\int_0^{2\pi} \frac{1}{re^{i\theta}} ire^{i\theta} d\theta \\ &= \frac{i}{a-b}\int_0^{2\pi} d\theta \\ &= \frac{2\pi i}{a-b} \end{aligned}$$

Problem 2: Let γ_R^+ be an upper semicircle of radius R centeret at the origin. Show that

$$\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2} dz \underset{R\to 0}{\to} 0$$

Solution
Using the regular $z = Re^{i\theta}$ polar substitution:

$$\begin{aligned}\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2} dz &= \int_0^\pi \frac{1-e^{iRe^{i\theta}}}{(Re^{i\theta})^2} iRe^{i\theta} dz \\ &= \int_0^\pi \frac{1-e^{iRe^{i\theta}}}{Re^{i\theta}} idz \\ &= \frac{i}{R}\int_0^\pi \frac{1-e^{iRe^{i\theta}}}{e^{i\theta}} dz\end{aligned}$$

Now, all three terms in the integrand are bounded over a finite arc length, so we can conclude that the integral is bounded. As $R \to \infty$, $(\frac{i}{R} \cdot \text{bounded}) \to 0$. Therefore $\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2} dz \to 0$ as $R \to \infty$

Problem 3: Recall that an open set $\Omega \subset \mathbb{C}$ is called connected if it cannot be expressed as a union of disjoing non-empty open sets. Show that Ω is connected if and only if every two points $z_1, z_2 \in \Omega$ can be connected by a polygonal path γ, i.e. a piece-wise smooth curve that consists of finitely many straight line segments.

Solution
I am assuming that the pathwise connecting curve γ has to be entirely contained in Ω as it is defined in Stein Shakarchi, else this biconditional is not true. Otherwise, you could just take any two points in two distant, disjoint, non-empty subsets and join them with a curve that goes through $\mathbb{C}$. Two points in a connected set $\Omega \subset \mathbb{C}$ can be pathwise connected, as is proved below, so the statement to be proved would only be a left to right implication.

$\Longleftarrow$:
Assuming both that $\Omega \subset \mathbb{C}$ is open and that every two points $z_1, z_2 \in \Omega$ can be connected by a polygonal path γ, i.e. a piece-wise smooth curve that consists of finitely many straight line segments, we claim that there exist disjoint, non-empty open sets $\Omega_1, \Omega_2 \subset \mathbb{C}$ such that $\Omega_1 \cup \Omega_2 = \Omega$, and will arrive at a contradiction.

Fix two arbitrary points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$. Define $\gamma : [0, 1] \to \Omega$ to be such a polygonal path that connects w_1 and w_2 such that $\gamma(0) = w_1$ and $\gamma(1) = w_2$. Since γ is smooth, each point is uniquely defined by a value in [0, 1], and since [0, 1] is continuous, every point along the path has a value that maps to it, meaning γ has an inverse. Define the intervals $\gamma^{-1}(\Omega_1)$ and $\gamma^{-1}(\Omega_2)$, the subintervals of $[0, 1]$ that γ maps to points in Ω_1 and Ω_2 respectively. These sets must be disjoint, since $\Omega_1 \cap \Omega_2 = \emptyset$ (and no single value in the parameterization can define two separate points). Additionally, neither interval is non-empty: $\gamma^{-1}(\Omega_1)$ contains $\gamma^{-1}(w_1)$ and $\gamma^{-1}(w_2)$. Finally, $\gamma^{-1}(\Omega_1) \cup \gamma^{-1}(\Omega_2) = [0, 1]$, because $\gamma \subset \Omega$.

Therefore the interval [0, 1] is *not* connected, which is a contradiction (since every interval in $\mathbb{R}$ is connected), and our assumption that Ω is not connected is incorrect.

$\Longrightarrow$:
Assume that Ω is an open, connected, non-empty set in $\mathbb{C}$. Fix a point $w \in \Omega$. We claim that there exists some point $v \in \Omega$ such that there is no path in Ω connecting w and v, and want to arrive at a contradiction.

Define $\Omega_1 \subset \Omega$ as the set of all points that can be connected to w by a polygonal path in Ω, and $\Omega_2 \subset \Omega$ to be all points that cannot be connected to w. Our goal is to show that Ω_2 must be empty $\implies \nexists v \in \Omega$.

Ω_1 and Ω_2 are open: assuming Ω_1 is closed, take an arbitrary boundary point x_1 (which can be connected to w by finitely many line segments) with a neighborhood including a point x_2 in Ω but not in Ω_1. There has to exist such x_1, x_2, else Ω contains all its boundary points. We could then connect x_1 to x_2 with a straight line segment $x_2 - x_1 \in \mathbb{C}$. Now x_2 can be connected to w with finitely many line segments, meaning x_2 must be in Ω_1, and w_1 is no longer a boundary point. This can be done for all boundary points of Ω_1 (so long as the diameter of the neighborhood does not exit Ω), therefore it is open. The reverse logic implies Ω_2 is open: if Ω_2 has a boundary point that cannot be connected, there is a point in some neighorhood in Ω that includes a point that *can* be connected to w, therefore we connect them by the finite line segment represented by the subtraction of the two points, and the boundary point is no longer in Ω_2, again making it open. (this conceptual gets us to conclude that $\Omega_1 = \Omega$, but we can be more through with the rest of the proof)

If Ω_1 and Ω_2 are obviously disjoint, since a possible path from w to a fixed candidate point can't both exist and not exist. All points in both Ω_1 and Ω_2 are in Ω by definition, so $\Omega_1 \cup \Omega_2 \subset \Omega$. Likewise, all points in Ω must be in either Ω_1 or Ω_2, since a fixed point can't (again) not have a path and have a path to w, $\implies \Omega = \Omega_1 \cup \Omega_2$.

$w \in \Omega_1 \implies \Omega_1$ nonempty (the trivial path connects w to itself). Ω_2 also being non-empty contradicted Ω being connected, therefore Ω_2 must be empty $\implies \Omega_1 = \Omega \implies \Omega$ is pathwise connected.

Problem 4: Suppose f is holomorphic in $\Omega \in \mathbb{C}$ and $Re(f)$ is constant. Prove that f is locally constant. Is it necessarily constant?

Solution
Define $z = x + iy$ and $f = u(x, y) + iv(u, y)$.

$Re(f) \implies u(x, y)$ constant $\implies \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0.$

f is holomorphic $\implies$ f satisfies the Cauchy Riemann equations $\implies \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \implies 0 = \dfrac{\partial v}{\partial y}$ and $0 = -\dfrac{\partial v}{\partial x} \implies \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y}$.

Because all four partials are constant, $f' = 0$, and is locally constant. Therefore, f is constant in each connected component/region, and by Corollary 3.4 in Stein Shakarchi, this is sufficient to call f constant in each connected component.

However, f isn't necessarily constant overall. In their proof of Corollary 3.4, Stein Shakarchi uses $f(w) = f(w_0)$ for fixed w in the region and w_0 arbitrary in the region as their necessary condition for "constant". Because Ω isn't ever constained to be connected, we can conceive a disconnected $\Omega = \Omega_1 \cup \Omega_2 \cup ... \cup \Omega_n$ for all non-empty open sets and all pairwise disjoint. f can be constant in each region (Corollary 3.4 definition), with the same global real value, but with each region can take different complex values, making f not constant (proof condition).

This construction never contradicts the partial derivatives of $v(x, y)$ being 0 in any isolated connected component. Such a contradiction would require us to construct a path between the two points in two different regions to show "global" change in $v(x, y)$, and likely take an integral along with the path with the two points as bounds, and obtain a non-zero result, as in the proof of 3.4. However, we now know that such a path is impossible to construct given Ω is not connected, using the condition proven in quesiton 3 of this homework.

Problem 5: Let $\mathbb{D}$ be a the (open) unit disc and fix $w \in \mathbb{D}$. Consider the function $F(z) = \dfrac{w-z}{1-\bar{w}z}$. Prove that F is a bjiective holomorphic function $\mathbb{D} \to \mathbb{D}$.

Solution
To show $F(z)$ is a bijective function $\mathbb{D} \to \mathbb{D}$, it suffices to show that $F(z)$ is its own inverse, is defined in all of $\mathbb{D}$, and actually maps elements of $\mathbb{D}$ to $\mathbb{D}$. First, showing bijection:

$$\begin{aligned}
F(F(z)) &= \frac{w - \dfrac{w-z}{1-\bar{w}z}}{1-\bar{w}\dfrac{w-z}{1-\bar{w}z}} \\
&= \frac{\dfrac{w - w\bar{w}z - w + z}{1-\bar{w}z}}{\dfrac{1-\bar{w}z-\bar{w}w+\bar{w}z}{1-\bar{w}z}} \\
&= \frac{w - w\bar{w}z - w + z}{1-\bar{w}z-\bar{w}w+\bar{w}z} \\
&= \frac{w\bar{w}z + z}{1-\bar{w}w} \\
&= \frac{z(w\bar{w}+1)}{1-\bar{w}w} \\
&= z
\end{aligned}$$

The division by $1-\bar{w}z$ on the third line is valid because, for $\bar{w}z = 1$ for variable $w, z \in \mathbb{D}$ w fixed, $|\bar{w}| = |w| < 1$, so $|\bar{w}||z| = 1 \implies |z| = \dfrac{1}{|w|} > 1$, therefore $z \notin \mathbb{D}$, so $1 - \bar{w}z \neq 0$ for $w, z \in \mathbb{D}$.

Assuming $w, z \in \mathbb{D}$ and $F(z) \notin \mathbb{D} \implies |F(z)| = \dfrac{|w-z|}{|1-\bar{w}z|} \geq 1$. Applying triangle inequality

$$\begin{aligned}
\frac{|w+(-z)|}{|1+(-\bar{w}z)|} &\leq \frac{|w|+|z|}{1+|\bar{w}z|} \\
1 &\leq \frac{|w|+|z|}{1+|w||z|} \\
1+|w||z| &\leq |w|+|z| \\
1-|w|-|z|+|w||z| &\leq 0 \\
(1-|w|)(1-|z|) &\leq 0
\end{aligned}$$

Since w is fixed in $\mathbb{D}$, we know that $|w| < 1 \implies 0 < 1-|w|$. Therefore $1-|z| \leq 0 \implies |z| \geq 1$ independant of which z and w we choose in $\mathbb{D}$. Therefore, our assumption that $|F(z)| > 1$ is incorrect, and $|F(z)| < 1 \implies F(z) \in \mathbb{D}$ for $w, z \in \mathbb{D}$.

$F(z)$ is holomophic because both the numerator, f_1, and deonminator, f_2, are holomorphic in $\mathbb{D}$.

$$\begin{aligned}
\lim_{h\to 0} \frac{w-(z+h)-(w+z)}{h} &= \lim_{h\to 0} \frac{w-z-h-w+z}{h} \\
&= \lim_{h\to 0} \frac{h}{h} \\
&= 1
\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{1-\bar{w}(z+h)-(1-\bar{w}z)}{h} &= \lim_{h\to 0}\frac{1-\bar{w}z+\bar{w}h-1+\bar{w}z}{h}\\ &= \lim_{h\to 0}\frac{\bar{w}h}{h}\\ &= \bar{w}\end{aligned}$$

$F = f_1/f_2$ is holomorphic in $\mathbb{D}$ so long as $f_2(z_0) \neq 0$ for any $z_0 \in \mathbb{D}$. As was said after the bijective proof, $1 - \bar{w}z \neq 0$ unless $z \notin \mathbb{D}$, therefore $\forall z_0 \in \mathbb{D} : f_2(z_0) \neq 0$, and F is a bijective holomorphic function $\mathbb{D} \to \mathbb{D}$.

Problem 6: (a) Show that the Cauchy-Riemann equations take the following form in polar coordinates:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(b) Use (a) to show that the logarithm function defined as $\log(z) = \log(r) + i\theta$ is holomorphic for $r > 0, -\pi < \theta < \pi$

Solution

(a) I don't know if you wanted us to plug in/switch variables to polar at an intermediary step using previous Cauchy-Riemann identities (and I missed office hours/didn't ask), so rather than risk that I chose to do the whole thing from the top.

Define $z = re^{i\theta}$ and $f(r,\theta) = u(r,\theta) + iv(r,\theta)$. Fix $z_0 \in C$ with $r >$ and move r towards r_0. Moving r along the angle of r_0,

$$\begin{aligned}
f'(z_0) = \lim_{r\to 0} \frac{f(re^{i\theta_0}) - f(r_0e^{i\theta_0})}{r} \\
\lim_{r\to 0} \frac{u(r,\theta_0) + iv(r,\theta_0) - u(r_0,\theta_0) - iv(r_0,\theta_0)}{r} \\
\lim_{r\to 0} \frac{u(r,\theta_0) - u(r_0,\theta_0)}{r} + i\frac{v(r,\theta_0) - v(r_0,\theta_0)}{r} \\
\left[\lim_{r\to 0} \frac{u(r,\theta_0) - u(r_0,\theta_0)}{r} + i\frac{v(r,\theta_0) - v(r_0,\theta_0)}{r}\right] \\
\left[\left(\lim_{r\to 0} \frac{u(r,\theta_0) - u(r_0,\theta_0)}{r}\right) + i\left(\lim_{r\to 0} \frac{v(r,\theta_0) - v(r_0,\theta_0)}{r}\right)\right] \\
\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]
\end{aligned}$$

Now fixing the radius r_0, fixing θ_0, and moving along the circle as $\theta \to 0$ gives

$$\begin{aligned}
f'(z_0) = \lim_{\theta\to 0} \frac{f(r_0e^{i\theta}) - f(r_0e^{i\theta_0})}{\theta} \\
r_0^{-1} \lim_{\theta\to 0} \frac{u(r_0,\theta) + iv(r_0,\theta) - u(r_0,\theta_0) - iv(r_0,\theta_0)}{\theta} \\
r_0^{-1} \lim_{\theta\to 0} \left[\frac{u(r_0,\theta) + iv(r_0,\theta) - u(r_0,\theta_0) - iv(r_0,\theta_0)}{\theta}\right] \\
r_0^{-1} \left[\lim_{\theta\to 0} \left(\frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta}\right) + i\left(\lim_{\theta\to 0} \frac{v(r_0,\theta) - v(r_0,\theta_0)}{\theta}\right)\right] \\
\left[\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right]
\end{aligned}$$

As in the proof for our standard basis Cauchy-Riemann equations, we set the real and imaginary parts of our two coexisting defitions of $f'(z_0)$ to be equal, and obtain $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}$

(b) $\log(z) = \log(re^{i\theta}) = \log(z) + \log(e^{i\theta}) = \log(z) + i\theta$. $r_0 < 0$ is equivalent to $r_1 = -r_0 > 0, \theta_1 = -\theta_0$.

Using the polar Cauchy-Riemann equations, $\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}[\log(r)] = \frac{1}{r}$, and $\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}[\theta] = 1$. Therefore

$\frac{1}{r} = \frac{1}{r} \cdot 1 \implies \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

Note that we further constrain $r \neq 0$, since $\frac{\partial u}{\partial r} = \frac{1}{r}$ does not exist where $r = 0$.

Further, both $\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}[\theta] = 0$ and $\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}[\log(r)] = 0$, satisfying $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Therefore, we can use the biconditional that f is holomorphic at $z_0 \iff f$ satisfies the Cauchy-Riemann equations at z_0 to say that f is holomorphic for $r > 0$ and $-\pi < \theta < \pi$.

Note the additional constraint on θ: we must constrain θ to not include π and $-\pi$ (the negative real axis). If we fix r, approaching the negative real axis counterclockwise: $\lim_{\theta \to \pi} \frac{\partial v}{\partial \theta} = \pi$, but approaching clockwise gives $\lim_{\theta \to -\pi} \frac{\partial v}{\partial \theta} = -\pi$, therefore f cannot be holomorphic along the negative real axis. This also prevents us from continuing to rotate to contradict holomorphic at other points.

Problem 7: Let $\Delta = \frac{\partial^2}{\partial x^2}$ be the Laplacian. Show that $\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$

Solution

$$\begin{aligned}
\Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\
&= \frac{\partial^2}{\partial x^2} - \frac{1}{i^2}\frac{\partial^2}{\partial y^2} \\
&= (\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y})(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}) \\
&= 4\left[\frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y})\frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y})\right] \\
&= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \\
&= 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}
\end{aligned}$$

The last line assumes continuous second partial derivatives, but we can do this because otherwise the claim of equality in the problem does not hold.

Problem 8: (a) Let α_n be a sequence of positive real numbers such that $\lim\limits_{n\to\infty} \dfrac{\alpha_{n+1}}{\alpha_n} = L$.

Prove: $\lim\limits_{n\to\infty} a_n^{1/n} = L$

SS: In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

(b) Use (a) to compute radius of convergence of hypergeometric series

$$1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(a+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n$$

Here $\alpha, \beta, \gamma \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \ldots$

Solution

(a) Because α_n is a sequence of positive reals, we can express the same limit with absolute values

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \lim_{n\to\infty} \frac{|\alpha_{n+1}|}{|\alpha_n|} = L$$
$$\forall \epsilon > 0 : \exists N : \forall n \geq N : \left| \frac{|\alpha_{n+1}|}{|\alpha_n|} - L \right| < \epsilon$$

Now note that we can reexpress the domininator $|\alpha_n|$ as

$$\begin{aligned} |a_n| &= |a_n| \left| \frac{a_{n-1}}{a_{n-1}} \right| \cdots \left| \frac{a_N}{a_N} \right| \\ &= \left| \frac{a_n}{a_{n-1}} \right| \left| \frac{a_{n-1}}{a_{n-2}} \right| \cdots \left| \frac{a_N + 1}{a_N} \right| |a_N| \end{aligned}$$

Each fractional term in this second product is a value less than $L + \epsilon$, so

$$\left| \frac{\alpha_{n+1}}{\alpha_n} \right| < L + \epsilon$$

Therefore

$$\begin{aligned} |a_n| &= \left| \frac{a_n}{a_{n-1}} \right| \left| \frac{a_{n-1}}{a_{n-2}} \right| \cdots \left| \frac{a_N + 1}{a_N} \right| |a_N| < (L+\epsilon)^{n-N} |a_N| \\ \implies |a_n|^{1/n} &< (L+\epsilon)^{1-\dfrac{N}{n}} |a_N|^{1/n} \end{aligned}$$

Taking the limit $n \to \infty$, $\dfrac{N}{n} \to 0$ and $|a_N|^{1/n} \to 1$, and $|a_n|^{1/n} = a_n^{a/n}$ since a_n positive,

$$\begin{aligned} a_n^{1/n} &< L + \epsilon \\ \implies a_n^{1/n} - L &< \epsilon \\ \implies \left| a_n^{1/n} - L \right| &< \epsilon \\ \therefore \lim_{n\to\infty} a_n^{1/n} &= L \end{aligned}$$

(b) The general form of a_n is

$$a_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)}$$

meaning the ratio a_{n+1}/a_n takes

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\dfrac{\alpha(\alpha+1)\cdots(\alpha+n-1)(\alpha+n)\beta(\beta+1)\cdots(\beta+n-1)(\beta+n)}{(n+1)!\gamma(\gamma+1)\cdots(\gamma+n-1)(\gamma+n)}}{\dfrac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)}} \\
&= \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} \\
&= \frac{(\frac{\alpha}{n}+1)(\frac{\beta}{n}+1)}{(1+\frac{1}{n})(\frac{\gamma}{n}+1)} \\
\frac{|a_{n+1}|}{|a_n|} &= \frac{\left|(\frac{\alpha}{n}+1)(\frac{\beta}{n}+1)\right|}{\left|(1+\frac{1}{n})(\frac{\gamma}{n}+1)\right|}
\end{aligned}$$

Taking the limit,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left|(\frac{\alpha}{n}+1)(\frac{\beta}{n}+1)\right|}{\left|(1+\frac{1}{n})(\frac{\gamma}{n}+1)\right|} &= 1 \\
\lim_{n\to\infty} \frac{\left|\frac{\alpha}{n}+1\right|\left|\frac{\beta}{n}+1\right|}{\left|1+\frac{1}{n}\right|\left|\frac{\gamma}{n}+1\right|} &= 1
\end{aligned}$$

Because taking the absolute value gives a sequence of positive reals (since norm is a real value and products of positive reals are positive reals), we can use part (a) to say that

$$\begin{aligned}
\lim_{n\to\infty} |a_n|^{1/n} &= 1 \\
\implies \limsup_{n\to\infty} |a_n|^{1/n} &= 1 \\
\implies \frac{1}{R} &= 1 \\
\therefore R &= 1
\end{aligned}$$

Problem 9: Prove that
(a) $\sum_{n\geq 0} nz^n$ does not converge at any point of the unit circle
(b) $\sum_{n\geq 1} \frac{z^n}{n^2}$ converges at every point of the unit circle

Solution
(a) Recall that $\sum_{n\geq 0} a_n$ converges $\iff \lim_{n\to\infty} a_n = 0$. $|z| = 1 \implies \lim_{n\to\infty} |nz^n| = n$, which does not tend towards 0. Therefore $\sum_{n\geq 0} nz^n$ diverges for all z such that $|z| = 1$

(b) Define $\sum n \geq 1 \frac{1}{n^2}$. Recall the comparison test, that if $\sum b_n$ converges, and $0 \leq a_n \leq b_n$ for sufficiently large n, then $\sum a_n$ also converges. Because $\zeta(2) = \sum_{n\geq 1} \frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$ and $|z| = 1 \implies \forall n \geq 1 :$ $|nz^n| = 1$, which is less than or equal to $|b_n| = 1$ for all such n, therefore $\sum_{n\geq 1} \frac{z^n}{n^2}$ converges.

Problem 10: Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

Solution
First, state

$$f(z) = \sum_{n \geq 0} a_n z^n$$

Taking the hint from Stein Shakarchi, we rexpressed $z = z_0 + (z - z_0)$, where z_0 is an arbitrary point in the disc of convergence of f: $|z_0| < R$. Using this substitution in the definition of f, we can expand the power term using the binomial theorem:

$$\begin{aligned}\sum_{n\geq 0} a_n z^n &= \sum_{n\geq 0} a_n (z_0 + (z - z_0))^n \\ &= \sum_{n\geq 0} a_n \sum_{0\leq k\leq n} \binom{n}{k} (z_0)^{n-k}(z - z_0)^k \\ &= \sum_{n\geq 0}\sum_{0\leq k\leq n} a_n \binom{n}{k} (z_0)^{n-k}(z - z_0)^k\end{aligned}$$

Because $\sum_{n\geq 0} a_n z^n$ is absolutely convergent, we can commute terms and/or swap summations. Observe that if we swap the summations so that we first iterate the sum over k, and then iterate the inner sum over n, our values for k will take values $k \geq 0$, and n will only take values $n \geq k$

$$\sum_{n\geq 0}\sum_{0\leq k\leq n} a_n \binom{n}{k} (z_0)^{n-k}(z - z_0)^k = \sum_{k\geq 0}\sum_{n\geq k} a_n \binom{n}{k} (z_0)^{n-k}(z - z_0)^k$$

Since $|z_0| < R$ and z must be in the disc of convergence as well, $|z - z_0| < R - |z_0|$ (geometrically, it is necessary for convergence that the component of z's norm along the vector defined by z_0 cannot go farther out than $R - |z_0|$, else $|z| > R$.) In other words, $|z - z_0| + |z_0| < R$. Therefore, to show absolute convergence,

$$\begin{aligned}\sum_{k\geq 0}\sum_{n\geq k} \left| a_n \binom{n}{k} (z_0)^{n-k}(z - z_0)^k \right| &= \sum_{k\geq 0}\sum_{n\geq k} |a_n| \left|\binom{n}{k}\right| |(z_0)^{n-k}| |(z - z_0)^k| \\ &= \sum_{k\geq 0}\sum_{n\geq k} |a_n| \binom{n}{k} (|z_0|)^{n-k}(|z - z_0|)^k \\ &= \sum_{k\geq 0} |a_n| \sum_{n\geq k} \binom{n}{k} (|z_0|)^{n-k}(|z - z_0|)^k \\ &= \sum_{k\geq 0} |a_n| (|z_0| + |z - z_0|)^k\end{aligned}$$

Because $|z_0| + |z - z_0| < R$, this series converges absolutely, and therefore converges, so $\sum_{k\geq 0}\sum_{n\geq k} a_n \binom{n}{k} (z_0)^{n-k}(z - z_0)^k$ converges for $|z_0| < R$, and f has a power series expansion around any point in its disc of convergence.